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AN AXIOMATIC THEORY OF WELL-ORDERINGS

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Abstract. We introduce a new simple first-order framework for theories whose objects are well-orderings (lists). A system ALT (axiomatic list theory) is presented and shown to be equiconsistent with ZFC (Zermelo Fraenkel Set Theory with the Axiom of Choice). The theory sheds new light on the power set axiom and on Gödel's axiom of constructibility. In list theory there are strong arguments favoring Gödel's axiom, while a bare analogon of the set theoretic power set axiom looks artificial. In fact, there is a natural and attractive modification of ALT where every object is constructible and countable. In order to substantiate our foundational interest in lists, we also compare sets and lists from the perspective of finite objects, arguing that lists are, from a certain point of view, conceptually simpler than sets.

§1. Introduction. Sets are the dominant foundational structure of modern mathematics. In the most successful first-order theory ZFC everything is a set, and all mathematical concepts can be interpreted as sets. To date there are no other concepts and associated axiomatic systems who can rival ZFC in interpretative power, conscious or unconscious usage and acceptance of a vast number of mathematicians, and foundational interest of logicians and philosophers alike.

The purpose of this paper is to present an alternative to sets, which can claim to have the same interpretative power as ZFC, and which might be interesting from various foundational points of view, including those of philosophy and computer science.

Any new foundational theory naturally provokes critical reactions. The set theorist Woodin (2001, p. 690) writes in one of his articles on the Continuum Hypothesis:

> "Of course, for the dedicated skeptic there is always the 'widget possibility'. This is the future where it is discovered that instead of sets we should be studying widgets. Further, it is realized that the axioms for widgets are obvious and, moreover, that these axioms resolve the Continuum Hypothesis (and everything else). For the eternal skeptic, these widgets are the integers (and the Continuum Hypothesis is resolved as being meaningless)."

Our "widgets" are finite or transfinite lists (well-orderings), not finite numbers. But the most important difference to the sceptic described in this quotation is that we by no means want to indicate that one should study and conceptually analyze lists "instead of sets." We simply argue that lists are worth of study and that set theory might not be the eternal or only answer. There are interesting and natural worlds of mathematical thought closely related to set theory where prominent foundational issues look different. We want to enrich the foundational discussion, without intentions to denigrate set theory. (The author considers himself to be a set theorist, but one with a pluralistic view of foundations.)

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Historically, the strong position of axiomatic set theory is the outcome of a complicated period spanning almost a century. Cantor discovered infinite powers and infinite numbers in the 1870s. He used and developed these two concepts in his analysis of "sets of reals" ("lineare Punktmannigfaltigkeiten") in the early 1880s. Cantor's initial notion of a set was neither foundational nor iterative, and it leads to topology, measure theory, and order theory, but not immediately to logic and axiomatics. Borel, Hausdorff, and Lusin are Cantor's heirs here, while Zermelo's work can be seen as the continuation of Cantor's unpublished discoveries of the 1890s, including discussions of inconsistencies and of set theoretic axioms. (See Cantor, 1991, p. 387ff.) Solving a major open problem of the pure theory, Zermelo proved the well-ordering theorem in 1904. His axiom system of 1908 was designed to make his proof work—and that is why the axiom of choice is there while the replacement scheme is missing; "making his proof work" includes avoiding the Russell paradox. It then took several decades until Zermelo's system was completed, formalized, and understood and widely accepted as a foundational theory for all of mathematics. (See, e.g., Ebbinghaus, 2007; Ferreirós, 1999; Kanamori, 1996, 2003; Deiser, 2009) for the development of set theory and some of the initial conceptual difficulties; see, for example, Blass & Gurevich (2004) for an informal discussion of the foundational role of sets.

The full story behind these remarks reveals how ZFC depends on historical circumstances, and it shows us the forge our modern notion of a set went through. The author thinks that it is very helpful to be aware of the history of set theory in order to avoid overhasty and biased reactions to alternative foundational approaches (as well as to ZFC itself and its modifications and extensions). In particular this concerns questions of acceptance, intuitiveness, evidence of axioms, ease of use, and teachability. So the reader is asked to judge our argumentation critically, but to judge at the same time if his objections rely on views shaped by 100 years of ZFC, or by sets as foundational objects in general. Of course, the reader who is convinced that ZFC is something like an "absolute truth" or that history has simply revealed it as the foundational theory par excellence will raise objections to such a historically minded judgement.

The present theory of lists in fact emerged from studies in the history of set theory, from an attempt to understand and formalize Cantorian ordinals and cardinals, which are no sets. Our definition of the transfinite numbers will be in line with Cantor's views, which on their part are akin to Euclid's traditional definition of number as a system of indistinguishable units. On the other hand there are no Cantorian sets in our theory of lists, and therefore we do not claim nor do we intend to present a formalized version of Cantor's mathematical intuitions.

Given the key role of well-orderings in Cantor's theory as well as in ZFC, they are certainly a natural candidate for a comprehensive infinitary concept, and therefore they do not need an introductory motivation. But after presenting our system we will look at finitary lists and sets more closely, and this will yield conceptually independent arguments favoring lists.

A new theory developing everything from scratch comes along with many definitions and elaborations of the basic consequences of its axioms, and this cannot be done within the scope of a journal paper like this. (A full exposition of axiomatic list theory can be found in Deiser, 2006. This treatise moreover develops an axiomatic system for multisets or aggregates.) On the other hand, a passably self-contained outline omitting some details and proofs but including the discussion of foundational aspects might be a good way to introduce the theory and to open it for mathematical as well as philosophical discussion.

The paper is arranged as follows. After a short informal discussion of lists we introduce our first-order language (Section 2). In this language we develop a theory called ALT for “axiomatic list theory” (Sections 3–7). The language and the theory are intended to be simple and intuitive. The reader who knows axiomatic set theory will find most of the material discussed here easy reading, after he acquainted himself with the basic definitions and concepts. We consider this to be a strength of our approach. ZFC is a simple theory, too. The theory ALT is equiconsistent with ZFC, and the proof of this theorem is outlined in Section 8. In Section 9 we introduce cardinals in order to give an example of how mathematics can be further developed in a universe of lists. In Section 10 we look at hierarchies and a list-theoretic analogue of Gödel’s axiom of constructibility.

In the last two sections of the paper we step back and discuss the conceptual foundations of lists, aiming at a more profound “apology” for list theory and at a “liberation” of ALT from inadequate set theoretic influences:

In Section 11 we compare (hereditarily) finite lists and sets. We argue that finite lists are canonical objects and that they are conceptually simpler than finite sets. This perspective is antipodal to the point of view shaped by set theory, but it is supported by the experiences of computer science. In a whole, the discussion yields an, in our opinion, important argument for basing an infinitary theory on the notion of a list. It also shows that the notion of a well-ordering does not conceptually depend on the notion of a set (while it historically evolved from Cantor’s study of real numbers and sets of real numbers in the 1870s and 1880s).

In Section 12 we discuss the theory ALT from a purely list-theoretic point of view. This gives some new perspectives on two pivotal principles: Gödel’s axiom of constructibility and the analogon of the set theoretical power set axiom. We argue that Gödel’s axiom is much more convincing in list theory than in set theory, while the analogon of the power set axiom (which is a member of ALT) looks artificial. These considerations lead to an attractive modification of ALT, called NEU, where every lists is constructible as well as countable, and where there are as many reals as ordinals. So finally our point of view is the following: ALT is a reasonable axiomatic theory for lists as long as we take ZFC as our guide, and it is a reasonable theory for lists to start with. But if we remove the set theoretic guideline, then we arrive at a much more intrinsic theory like NEU. The analogon of the power set axiom is dismissed here, while a list-theoretic version of Gödel’s axiom of constructibility has a strong axiomatic position—one which it never reached in set theory.

§2. Lists and the iota-language. Let us start with an informal description of *lists*. We also want to introduce some basic notions which will play an important role later.

A finite list x has the form

$$x = \langle a_0, a_1, \dots, a_{n-1} \rangle.$$

The object a_0 is the first *entry* of x, a_1 the second entry of x, and so on. The object a_{n-1} is the *last entry* of x. Our lists are hereditary, and so the entries of x are again lists, etc.

Every entry of a list has one or more *positions* in the list. The number 2 *appears* in the list $\langle 3, 2, 1, 2 \rangle$ at two different positions.

Two lists are equal if they have the same positions, and if their entries agree at all of their positions. Thus $\langle a \rangle \neq \langle a, a \rangle$ for all a and $\langle a, \langle a, b \rangle \rangle \neq \langle a, \langle b, a \rangle \rangle$ if and only if $a \neq b$.

There are at least two questions arising at this point. Firstly:
What exactly is a position? Or more technically: How can we speak about lists as directly as possible, without the use of sets and without the use of natural numbers?

And secondly:
What is the correct notion generalizing finite lists to include infinite ones?

Our answer to the first question is that positions are certain distinguished lists. They are "neutral," "unfilled" lists. The second position of any list will be the list $\langle\langle\rangle, \langle\rangle\rangle$, where $\langle\rangle$ denotes the empty list. The reader will see below how this can be formulated axiomatically in an intuitive and noncircular way. There is indeed a simple, flexible, and powerful first-order language for the description of lists (and this seems to be one of the achievements of this work).

Our answer to the second question is that arbitrarily long well-orderings are the canonical objects generalizing finite lists. The most basic and important constructor of lists is *end extension by one element,* as can be seen by writing down any list. (This view is also strongly supported by computer science.) And if we allow infinitely iterated end extensions by one element, we arrive at the notion of a well-ordering.

We now present the language and the axioms. We work in first-order logic with equality and one ternary relation symbol ι (iota). Instead of $\iota(x, y, z)$ we shall write $x\ \iota_z\ y$; we call this the *index-notation* for a ternary relation symbol.

The objects of the theory are called *lists* or *well-orderings*. As everything is a set in ZFC, everything is a list in our theory.

The intuitive meaning of the iota-relation is:
$x\ \iota_z\ y$ *means that* x appears in y at position z.

The small Greek ι is for "in." It was also chosen because it is not overloaded in mathematics as many other symbols are.

We also say that x is an *entry* of y at position z, if $x\ \iota_z\ y$ holds.

The most important definition of our theory is:

DEFINITION 2.1 (position and position of a list, $z < y$). *A list z is called a* position, *if there are x and y such that $x\ \iota_z\ y$. A position z is a* position of y, *in symbols $z < y$, if there is an x such that $x\ \iota_z\ y$.*

The original German term for *position* is "Ort" meaning "location, place, position, site, spot" (as well as "small city").

The idea is that only special lists can appear in the index of the ι-relation. In the following, small Greek letters $\alpha, \beta, \gamma, \dots$ will always denote positions. Thus:
$\forall\alpha\phi$ is the formula $\forall\alpha(\alpha$ is a position $\rightarrow \phi)$, and
$\exists\alpha\phi$ is the formula $\exists\alpha(\alpha$ is a position $\wedge\ \phi)$.

That Greek letters are the symbols of choice to represent positions will be made clear by our axioms (see the axiom of positions (O) below).

The most simple object of our theory is the empty list. A list y is called *empty* if y has no positions. An empty list is denoted by $\langle\rangle$ or 0. It will follow from our axioms that there is exactly one empty list, but we do not need this to define the notion of an ordinal:

DEFINITION 2.2 (ordinal or order-type). *A list x is called an* ordinal *or* (order-) type, *if every entry of x is empty. An ordinal x is called the* type *of a list y, in symbols $x = type(y)$, if x and y have the same positions.*

It will indeed follow from our axioms that the type of a list uniquely exists. The type results from x by replacing every entry of x by the unit 0, and we will allow arbitrary replacements of the entries of x.

Since Thales and Euclid natural numbers have been defined as systems of units. Euclid writes in his Elements (VII.2): "Number is a multitude composed of units" (see also Deiser, 2010; Gericke, 1970; Heath, 2003). Cantor, following these ancient traditions, understood order-types of linear orders as ordered systems of units: They emerge from an ordered structure by abstracting from the nature of its elements while keeping their order. The act of abstraction transforms the elements of the ordering into units. Cardinals are then obtained by a second step abstracting from the order, too. The Cantorian ordinals are the order-types of well-orderings won by this act of abstraction (see, e.g., Cantor, 1890; Cantor, 1897). Cantor never tried to make this traditional and intuitive idea precise, though it was strongly criticized by Frege (1967, pp. 163–166, 1983, p. 79 f), who regarded the involved act of abstraction as impossible or at least vague. Frege (1986, sec. III, §29–§54) himself discussed "unit and one" at length in his "Grundlagen der Arithmetik" of 1884.

The above definition of ordinals, together with the axioms below, is intended to be a faithful and direct formal realization of the Cantorian ordinals-with the conceptual difference that zeros play the role of Cantor's units ("Ones" or "Einsen"). To give a formal definition of this kind was the primordial motivation for a theory of lists.

We next define:

DEFINITION 2.3 (initial segment, $x \trianglelefteq y$). *x is an* initial segment *of y, in symbols $x \trianglelefteq y$, if for all z, α we have: $z \iota_\alpha x$ implies $z \iota_\alpha y$.*

The axioms below will justify the term "initial segment," instead of "part" or "sublist," which would be certainly more appropriate at this stage.

§3. The axioms of extensionality. We start with two axioms of extensionality, which might be regarded as part of our language. The first one is certainly no surprise:

(Ext 1) First axiom of extensionality

A list is determined by its positions and entries.

$\forall x, y. x \trianglelefteq y \wedge y \trianglelefteq x \rightarrow x = y.$

The second axiom of extensionality postulates that there is never more than one entry at a position of a list:

(Ext 2) Second axiom of extensionality

An entry at a position of a list is unique.

$\forall x, y, z, \alpha. x \iota_\alpha z \wedge y \iota_\alpha z \rightarrow x = y.$

By this axiom we can define:

DEFINITION 3.1 (the entry $x(\alpha)$). *If $\alpha < x$, then we denote the unique entry of x at position α by $x(\alpha)$.*

Thus for all x, y, and α we have: $y = x(\alpha)$ iff $y \iota_\alpha x$.

As braces "{" and "}" are used in set theory, we use brackets "⟨" and "⟩" for lists, following the set theoretic notation for sequences. Thus

$x = \langle x(\alpha) \mid \alpha < x \rangle,$

$type(x) = \langle 0 \mid \alpha < x \rangle$, etc.

§4. The axioms of positions. The second group of axioms consists of three postulates clarifying the notion of a position.

The first one is:

(Trans) Transitivity of positions

The positions of the positions of a position are positions of the position.

$\forall \alpha, \beta, \gamma . \alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma .$

The next member of this group is an axiom scheme which expresses that we are describing well-founded structures:

(Min) Minimality scheme

For every formula $\phi(x)$ (with parameters):
$\exists\alpha\phi(\alpha) \to \exists\alpha.\, \phi(\alpha) \wedge \forall\beta < \alpha\, \neg\phi(\beta)$.

This axiom provides us with proofs by induction on positions:

THEOREM 4.1 (induction on positions).
(a) Let ϕ be a formula such that $\forall\alpha.\forall\beta < \alpha\, \phi(\beta) \to \phi(\alpha)$. Then $\phi(\alpha)$ holds for all positions α.
(b) Let ϕ be a formula and α be a position such that $\forall\beta < \alpha.\forall\gamma < \beta\, \phi(\gamma) \to \phi(\beta)$. Then $\phi(\beta)$ holds for all $\beta < \alpha$.

Proof. For (a): *Suppose not.* Then there exists a $\neg\phi$-minimal position α by (Min). Then $\phi(\beta)$ holds for all $\beta < \alpha$. Using the hypothesis we get that $\phi(\alpha)$ holds, *contradiction.*

For (b): *Suppose not.* Let $\psi(\beta) = \neg\phi(\beta) \wedge \beta < \alpha$. By (Min) there exists a ψ-minimal position β. Then $\neg\phi(\beta)$, $\beta < \alpha$, and for all $\gamma < \beta$ we have:

$\phi(\gamma)$ *or* γ is not a position of α.

But every position γ of β is also a position of α by (Trans). Thus $\phi(\gamma)$ holds for all $\gamma < \beta$. Thus $\phi(\beta)$ holds by hypothesis, *contradiction.* □

Finally, the third axiom of this group describes the nature of positions fully, not only properties of them. We defined ordinals as lists of units, following Cantor's definition by abstraction, with the twist that the empty list figures as our unit. Now positions themselves are certain lists, and intuitively they are "neutral with respect to entries" or "unfilled." No position of a list is really unfilled in our language, so the best we can do is to equate "unfilled" with "filled with the empty list." This leads to the identification of ordinals and positions:

(O) Characterization of the positions

The positions are exactly the ordinals. $\forall z.\exists x, y\; x\, \iota_z\, y \leftrightarrow \forall\alpha < z\; z(\alpha) = \langle\rangle$.

The "O" here is for "ordinal axiom" (and for "Ortsaxiom" as well as for "Ordinalzahlaxiom" in German).

Our convention to denote positions with small Greek letters is now just the set theoretic convention to denote ordinals with such letters.

It follows from (O) that the empty list exists: Let x be arbitrary. If x has no position at all, then $x = \langle\rangle$. Thus let α be a position of x. If α has no position at all, then $\alpha = \langle\rangle$. Thus let β be a position of α. Then $\alpha(\beta) = \langle\rangle$, since α is an ordinal. Thus $\langle\rangle$ exists.

A basic consequence of the axioms (Trans), (Min), and (O) is that the positions are linearly ordered. We give the full proof of this fact in order to illustrate these axioms.

THEOREM 4.2 (linearity theorem). *The positions are linearly ordered by $<$, that is, by $\alpha < \beta$* if *α is a position of β.*

Proof. The relation $<$ is transitive by the axiom (Trans).

Suppose that $<$ is not irreflexive. Then by (Min) there is a minimal α such that $\alpha < \alpha$. Then *non*$(\beta < \beta)$ holds for all $\beta < \alpha$ by minimality of α. But $\alpha < \alpha$, and therefore *non*$(\alpha < \alpha)$, *contradiction.*

Suppose that $<$ is not linear. Then by (Min) there is a minimal α such that:

$\exists\beta.\, \neg\, \alpha < \beta \wedge \alpha \neq \beta \wedge \neg\, \beta < \alpha$.

We now use (Min) again, with parameter α, to get a minimal β such that:

$\neg\, \alpha < \beta \wedge \alpha \neq \beta \wedge \neg\, \beta < \alpha$.

Then $\delta < \beta$ holds for all $\delta < \alpha$: for if we had $non(\delta < \beta)$ for some $\delta < \alpha$, then $\beta \leq \delta$ by minimality of α, and thus $\beta < \alpha$ would hold by (Trans). In the same way we have $\delta < \alpha$ for all $\delta < \beta$ by minimality of β. Thus α and β have the same positions, and therefore $\alpha = \beta$ by axiom (O), which is a *contradiction.* □

The proof shows that for linearity it is enough to assume the so-called third axiom of extensionality instead of (O):

"A position is determined by its positions."

With respect to (Ext1), (Ext2), (Trans), and (Min), the linearity theorem is in fact equivalent to this weakening of the axiom (O).

§5. Richness of the universe. The members of the third group of axioms postulate that our universe is rich in objects. Two axioms claim that there are many positions, while the third one claims that we can arbitrarily replace the entries of a list by other entries.

An ordinal λ is a *limit,* if $\lambda \neq 0$ and for every $\alpha < \lambda$ there is a β such that $\alpha < \beta < \lambda$. We postulate:

(In) Axiom of infinity.

There exists a limit.

The least limit ordinal is denoted by ω. A position α is a *natural number* if $\alpha < \omega$. Every metamathematical natural number n also denotes an object of the theory—which is also denoted by n – via $n + 1 =$ "the least ordinal greater than n." Thus, for example, $3 = \langle 0, 0, 0\rangle$. A list x is a *positional list,* if all entries of x are positions. We postulate:

(Sup) Axiom of suprema

The entries of a positional list are bounded by a position.

$\forall x.\, x$ is a positional list $\rightarrow \exists\gamma\, \forall\alpha < x\; x(\alpha) \leq \gamma$.

An *L-function* $F(\alpha, x)$ is a formula $\phi(\alpha, x, y)$ such that for every α and every x there is exactly one y such that $\phi(\alpha, x, y)$ holds. We write $y = F(\alpha, x)$ in this case. Thus an L-function is a "functional class." Our "L" here is for "language." The formula may also contain parameters, which are not displayed.

Now let $F(\alpha, x)$ be an L-function, and let x,y be lists. The list y is called the *F-replacement* of the entries of x, in symbols $y = F \circ x$, if $type(x) = type(y)$ and $y(\alpha) = F(\alpha, x(\alpha))$ for all $\alpha < x$.

Our second scheme of axioms is:

(Rep) Replacement scheme

For every L-function $F(\alpha, x)$: For all x there exists $F \circ x$.

An immediate consequence is the existence of types: Let $F(\alpha, x) = 0$ for all α and x. Then for every x the list $F \circ x$ is the type of x.

Another simple consequence of the scheme (Rep) is the existence of the *restriction* $y \mid \beta = \langle y(\alpha) \mid \alpha < \beta\rangle$ for all $\beta < y$. Here we let $F(\alpha, x)$ be any L-function such that $F(\alpha, 0) = y(\alpha)$ for all $\alpha < y$. Then $y \mid \beta = F \circ \beta$.

As expected, using the replacement scheme we can prove a recursion theorem for L-functions:

THEOREM 5.1 (recursion theorem). *Let $F(x)$ be a L-function. Then there is exactly one positional function $G(\alpha)$ such that $G(\alpha) = F(G \mid \alpha)$ for all α.*

Proof. We prove by induction on α that there is exactly one x such that $\alpha = type(x)$ and $x(\beta) = F(x \mid \beta)$ for all $\beta < \alpha$. We call x the α-approximation (w.r.t., F). $\langle\rangle$ is the unique 0-approximation . In the successor step, let x be the unique α-approximation. Then $x + \langle F(x)\rangle$, the end-extension of x by $F(x)$, is the unique $(\alpha + 1)$-approximation. For the limit step λ, we let H be a positional function such that:

$H(\alpha) = F$(“the unique α-approximation”) for all $\alpha < \lambda$.

Then $x = H \mid \lambda$ is the unique λ-approximation.

We now let $G(\alpha) = F$(“the unique α-approximation”) for all α. Then G is as desired, as is shown by induction on α. □

A special case of (Rep) is the following scheme of definability, claiming that we can fill the “empty” entries of any position with arbitrary objects:

(Def) Definability scheme

For any L-function $F(\alpha)$: For all β there exists $y = F|\beta$ $(= \langle F(\alpha) \mid \alpha < \beta\rangle)$.

If (T) is the axiom stipulating that every list has a type, then (Rep) is easily seen to be equivalent to (Def) + (T) with respect to the other axioms.

Already (T) provides us with successor ordinals: If α is ordinal, then α is a position by (O). So there is an x such that $\alpha < x$. But then $\alpha < type(x)$, and therefore there is a least ordinal $\beta > \alpha$ by (Min). As usual β is denoted by $\alpha + 1$.

If x is a list and p is a positional list with $p(\alpha) < x$ for all $\alpha < p$, then we let

$x \circ p = \langle x(p(\alpha)) \mid \alpha < p\rangle$.

This *composition* $x \circ p$ exists by (Rep).

We can now define the important concept of a sieving.

DEFINITION 5.2 (sieving). *y is a* sieving *of x, if there is a positional list p such that: $p(\alpha) < p(\beta)$ for all $\alpha < \beta < p$* and $y = x \circ p$.

For applications sievings defined by a formula are often useful. If x is a list and $\phi(y)$ is a formula, then we define $sieve(x, \phi)$ to be the list which results by collecting, in order of their appearance, all entries y of x for which $\phi(y)$ is true. This can be made precise, but we content ourselves with some examples here. For this we let $x = \langle 1, 2, 3, 2, 5, 4\rangle$. Then $sieve(x,$ “the entry is even”) is the list $\langle 2, 2, 4\rangle$, and $sieve(x,$ “the entry appears at an even position”) is the list $\langle 1, 3, 5\rangle$, because 0, 2, 4 are the even positions of y and 1, 3, 5 the corresponding entries. For every list x, we define

$delrep(x) = sieve(x,$ “the entry has not appeared so far”).

Then $delrep(x)$ is the list x with all repetitions deleted. For x as above we have $delrep(x) = \langle 1, 2, 3, 5, 4\rangle$. A list x is *injective,* if $x = delrep(x)$.

In the same way we can sieve L-functions $F(\alpha)$. The result is either a list or else an L-function $G(\alpha)$. For an example, we let $Id(\alpha) = \alpha$ for every α. Then $sieve(Id,$ “the entry is 0 or a limit”) is the L-function $G(\alpha)$ such that $G(\alpha) = \omega \times \alpha$ for all α, granted that $\omega \times \alpha$ exists for all α, which is true in our theory.

§6. The axiom of regularity.

In list theory, the most natural formulation of regularity involves the following notion:

DEFINITION 6.1 (downward transitive). *y is called* downward transitive, *if $y(\alpha)$ is a sieving of $y \mid \alpha$ for every $\alpha < y$.*

So a downward transitive list y is "built" according to the following rule: We may only add z as a new last entry to the list constructed so far, if all entries of z have already been added to the list in the order in which they appear in z.

We now postulate:

(Reg) Axiom of regularity

For all x there exists a downward transitive y in which x appears.

So every list is backed up by a list which is built according to the above rule. The motivation and role of this axiom is very similar to that of the axiom of regularity in ZFC. Indeed it can be shown that (Reg) is equivalent to the following principle, which more resembles the set-theoretic axiom: "For all $x \neq \langle\rangle$ there exists an $\alpha < x$ such that: $x(\beta)$ does not appear in $x(\alpha)$ for any $\beta < x$."

Using the axiom of regularity, we can justify inductions and recursions on entries. One theorem here reads:

THEOREM 6.2 (recursion on entries). *Let $H(x, y)$ be an L-function. Then there is exactly one L-function $G(x)$ such that $G(x) = H(x, G \circ x)$ for all x.*

A more general version allows that in order to define $G(x)$ we may assume that $G(y)$ has already been defined for all $y \neq x$ appearing in a downward transitive list with last entry x. Theorems of this kind are a little harder to prove than the corresponding theorems in set theory. We omit the proof of Theorem 6.2 here.

There is a canonical L-function $Tr^*(x)$ mapping any x to a downward transitive list with last entry x. This L-function is uniquely defined by the equation:

$Tr^*(x) = (\Sigma Tr^* \circ x) + \langle x\rangle$,

where $z + \langle x\rangle$ again denotes the end-extension of z by one element x and Σy denotes the catenation of all entries of y. Thus, for example,

$\langle 1, 2\rangle + \langle 1\rangle = \langle 1, 2, 1\rangle$, and

$\Sigma\langle\langle 1, 2\rangle, \langle\rangle, \langle 2, \langle 3\rangle\rangle\rangle = \langle 1, 2, 2, \langle 3\rangle\rangle$.

The existence of Tr^* follows from the recursion theorem. But in fact Tr^* or a similar function is used to prove the above theorem about recursion on entries. But this is a mere technical point, and the recursive definition expresses Tr^* most clearly.

We finally define an L-function Com by:

$Com(x) = type(Tr^*(x))$ for every x.

The L-function $Com(x)$ is called the *(natural) complexity* of the list x.

§7. The axiom (SL). The following axiom corresponds to the power set axiom of ZFC. It does not appear to be very natural in the context of lists. We will discuss this in Section 12 in more detail.

(SL) Listings of all sievings

For every list x there exists a list y such that every sieving of x appears in y.

A nontrivial consequence of (SL) is the following combination of the axiom of choice and the collection scheme:

(CC) Scheme of collective choice

For every formula $\phi(\beta, x)$ (with parameters): Let α be a position such that for all $\beta < \alpha$ there is an x such that $\phi(\beta, x)$. Then there is a list y of type α such that $\phi(\beta, y(\beta))$ holds for all $\beta < \alpha$.

$\forall\alpha.\forall\beta < \alpha\exists x\phi(\beta, x) \rightarrow \exists y.type(y) = \alpha \wedge \forall\beta < \alpha\phi(\beta, y(\beta))$.

THEOREM 7.1. *(CC) is provable from the present axioms.*

We sketch a proof of this theorem. One first proves by induction on γ:

PROPOSITION 7.2 (boundedness lemma). *Let γ a position. Then there exists a position η such that every injective list x, in which only lists of complexity $< \gamma$ appear, is shorter than η.*

The boundedness lemma is now used to prove the following theorem about complexity lists:

THEOREM 7.3 (existence of complexity lists). *Let γ be a position. Then there exists a complexity list k for γ, that is,*
(i) k is injective,
(ii) for all $\alpha < k$ we have $Com(k(\alpha)) < \gamma$,
(iii) for all x such that $Com(x) < \gamma$ we have: x appears in k.

We can now easily prove the principle of collective choice:

THEOREM 7.4. *(CC) holds.*

Proof. Let $\phi(\beta, x)$ and α be as in (CC). For all $\beta < \alpha$ we let
$x(\beta) =$ "the least complexity of an x such that $\phi(\beta, x)$."
Let γ be greater than all $x(\beta)$, $\beta < \alpha$, and let k be a complexity list for γ. We now set for all $\beta < \alpha$:
$y(\beta) = k$("the least $\delta < k$ such that $\phi(\beta, k(\delta))$").
Then y is as desired. □

Thus in the presence of (SL) no version of the axiom of choice is needed in list theory.

§8. The theory ALT.

DEFINITION 8.1. *The theory ALT consists of the 10 axioms (Ext1), (Ext2), (Trans), (Min), (O), (In), (Sup), (Rep), (Reg), and (SL).*

Here ALT is for *axiomatic list theory.* The basic result about this theory is:

THEOREM 8.2. *The ι-theory ALT and the $\in$-theory ZFC are equiconsistent.*

We sketch the somewhat lengthy proof of this theorem (see Deiser, 2006, for the full proof). First we show that the consistency of ZFC implies the consistency of ALT. We work in ZFC and construct a class model of ALT. We let:

$OF = \{f \mid f : \alpha \rightarrow V, \alpha \in On\}$.

Here OF is for "ordinal functions." We then let HOF be the class of hereditarily ordinal functions (with respect to their range) which can be uniquely defined by:

$HOF = \{f \in OF \mid rng(f) \subseteq HOF\}$.

Basic elements of HOF are the functions of the form $f : \alpha \rightarrow \{0\}$ for $\alpha \in On$. They play the role of positions in the model, of course. For all $\alpha \in On$ we let:

$\alpha^* =$ "the function $f : \alpha \rightarrow \{0\}$" (in particular $0^* = 0 = \emptyset$).

Next we define a three-place relation ι on HOF in index notation. We set, for all $x, y, z \in HOF$:

$x \iota_z y$, *if* with $\alpha = dom(z)$ we have: $z = \alpha^*$, $\alpha < dom(y)$, and $y(\alpha) = x$.

One now shows that all axioms of ALT are true in the proper class model $\langle HOF, \iota\rangle$. As usual this proves that the consistency of ZFC implies the consistency of ALT.

The structure of the other direction, that is, that the consistency of ALT implies the consistency of ZFC, is different, since there is no easy definition of "set" in ALT, unless further axioms are assumed (we will discuss this later). We will not construct a model, but we can nevertheless carry out the argument. We first define:

DEFINITION 8.3 (ι-relation, ι-equality). *For all x, y we set:*
(i) $x \iota y$, if *x appears in y (at some position α),*
(ii) $x \subseteq_\iota y$, if *for all $z \iota x$ we have $z \iota y$.*
(iii) $x =_\iota y$, if *$x \subseteq_\iota y$ and $y \subseteq_\iota x$.*

The relation $=_\iota$ is an equivalence relation, but not a congruence relation with respect to the ι-relation: If $x_1 =_\iota x_2$ and $x_1 \iota y$, then $x_2 \iota y$ does not hold in general.

Thus the ι-relation (without an index) is only a first approximation of a set-theoretic $\in$-relation, and $=_\iota$ is only a first approximation of set theoretic equality. But using recursion on entries, we can define:

DEFINITION 8.4 ($\in$-relation, $\in$-equality). *We recursively define for all x, y:*
*$x \in y$,*if *$\exists z \iota y\ x =_\in z$,*
$x =_\in y$, if *$\forall z \iota x\ z \in y \wedge \forall z \iota y\ z \in x$.*

If $x \in y$ holds, then we say: x is an element of *y.*
If $x =_\in y$ holds, then x and y are called set-theoretically equal *or simply* $\in$-equal.

It is not hard to see that for all x and y we have that $x \iota y$ implies $x \in y$, and that $x =_\iota y$ implies $x =_\in y$. Since $x =_\iota x$ holds, we get that $x =_\in x$ is true for all x. Transitivity of $\in$-equality follows from the following general theorem which establishes the desired congruence properties.

THEOREM 8.5 (congruence theorem). *For all x, y, z we have:*
(i) $x =_\in y$ and $y =_\in z$ implies *$x =_\in z$,*
(ii) $x =_\in y$ and $y \in z$ implies *$x \in z$,*
(iii) $x =_\in y$ and $z \in y$ implies *$z \in x$.*

Now for every ZFC-formula ϕ we let ϕ^* be the formula of list theory which emerges from ϕ by replacing all prime formulae $x = y$ and $x \in y$ of ϕ by the ι-formulae defining the relations $x =_\in y$ and $x \in y$, respectively. One now shows:

ALT $\vdash \phi^*$ for every axiom ϕ of ZFC.

Using, for example, Hilbert calculus, we can prove: If d is a formal proof of the set theoretic formula ϕ using the axioms of ZFC, then d can be transformed effectively to a formal proof d^* of the list theoretic formula ϕ^* using the axioms of ALT. It follows that the consistency of ALT implies the consistency of ZFC.

§9. Cardinals. In order to give some idea how list theory works (and differs in character from ZFC), we define some cardinality notions and prove the Cantor–Bernstein theorem in ALT. (The reader not interested in this can proceed to Section 10.)

Suppose that y and z are injective lists, and let $\alpha = type(y)$ and $\beta = type(z)$. We want to express that there exists a one-one correspondence between the entries of z and y. This can be done without introducing functions, since we can equivalently talk about an injective positional list x of length β in which all $\gamma < \alpha$ appear. If x is such a list, then "map $z(\delta)$ to $y(x(\delta))$ for all $\delta < \beta$" is a correspondence between the entries of z and y, and conversely any such correspondence induces such a list x: Let $x(\delta)$ be the position of the image of $z(\delta)$ in y, for all $\delta < \beta$. We define:

DEFINITION 9.1 (rearrangement, $\alpha \equiv \beta$). *A positional list x is a β-*rearrangement *of α, if: (i) x is injective, (ii) $type(x) = \beta$, (iii) every $\gamma < \alpha$ is an entry of x.*
We set $\alpha \equiv \beta$, if there is a β-rearrangement of α.

It is straightforward to prove that the relation $\alpha \equiv \beta$ is reflexive, symmetric, and transitive.

It is now natural to define cardinals as follows:

DEFINITION 9.2 (cardinality of a position, cardinal position). *(i) For a position α we let $|\alpha| =$ "the least β such that $\alpha \equiv \beta$." The position $|\alpha|$ is called the* cardinality of α.
(ii) A position α is called a cardinal position *or* cardinal (number), *if $\alpha = |\alpha|$.*

It is easy to see that for all α, β we have $\alpha \equiv \beta$ *iff* $|\alpha| = |\beta|$. Moreover, $|\alpha|$ is a cardinal number for all α.

Let α, β be such that $|\alpha| < \beta < \alpha$. Then we should have $|\alpha| = |\beta|$. Indeed, overlappings of the form $|\beta| < |\alpha| < \beta < \alpha$ as well as inclusions of the form $|\alpha| < |\beta| < \beta < \alpha$ are ruled out by the following observation:

PROPOSITION 9.3 (monotonicity of the cardinality operation). *Let α, β be positions such that $\alpha \leq \beta$. Then $|\alpha| \leq |\beta|$.*

Proof. Let $\kappa = |\beta|$, and let x be a κ-rearrangement of β. We set:

$y = sieve(x,$ "the entry is less than α").

Then $type(y) \leq type(x)$, since y is a sieving of x. Moreover, y is a $type(y)$- rearrangement of α. Thus we have:

$|\alpha| \leq type(y) \leq type(x) = \kappa = |\beta|$. □

COROLLARY 9.4. *Let α, β be positions such that $|\alpha| \leq \beta \leq \alpha$. Then $|\alpha| = |\beta|$.*

Proof. We have $||\alpha|| \leq |\beta|$ and $|\beta| \leq |\alpha|$ by monotony. But $||\alpha|| = |\alpha|$, and therefore $|\alpha| \leq |\beta| \leq |\alpha|$. □

Thus the rearrangement-relation divides the positions into intervals (the Cantorian "number classes").

COROLLARY 9.5 (Cantor–Bernstein, inclusion form). *Assume that $\alpha \leq \beta \leq \gamma$ and $\alpha \equiv \gamma$. Then $\alpha \equiv \beta \equiv \gamma$.*

Proof. We have $|\alpha| \leq |\beta| \leq |\gamma|$ by monotony, and $|\gamma| = |\alpha|$, since $\alpha \equiv \gamma$. Thus $|\alpha| = |\beta| = |\gamma|$, and therefore $\alpha \equiv \beta \equiv \gamma$. □

§10. Hierarchies and L. A hierarchy is a linguistically defined list in which all lists appear:

DEFINITION 10.1 (hierarchy). *An L-function $F(\alpha)$ is a* hierarchy, *if:*
For all x there is an α such that $F(\alpha) = x$.

A natural attempt to define a hierarchy is to close the list $\langle 0\rangle$ under simple operations, and we can indeed define a candidate for a hierarchy in this way. We call this candidate L, since it is an analogon of Gödel's constructible universe. We start with $L(0) = 0$. We now recursively close $L|\alpha$ under two operations. The first one is "end-extension by one element." If x and y appear in the list as constructed so far, but $x + \langle y\rangle$ does not, then we extend our list by $x + \langle y\rangle$. We repeat this until we have produced a list $L|\lambda$ which is closed under end-extension by one element. Technically this can be canonically achieved by using (the inverse of) Gödel's pairing function Γ, which is an easily defined L-function such that $\Gamma(\alpha) = \langle \beta, \gamma\rangle$ runs through all positional list of length 2 when α runs through all positions. The limit positions λ for which $L|\lambda$ is closed under end-extension by one element are then exactly the ordinals of the form ω^{ω^α}, $\alpha \geq 0$, which we call *special*. At these special positions we choose a certain definable sieving of the list $L|\lambda$, which does not appear in $L|\lambda$; this "certain new sieving" can also be defined in a uniform way, though the definition is a little more involved. Then we close again under end-extension by one element until we reach the next special limit, add a new sieving, etc.

Ignoring the interesting details of this construction, we consider the following principle:

(L) Principle of constructibility

The L-function $L(\alpha)$ is a hierarchy.

The construction of the L-function $L(\alpha)$ can be executed using the axioms (Ext1), (Ext2), (Trans), (Min), (O), (In), (Sup), and (Rep). These axioms are then also true in L. Moreover, the axioms (Reg), (CC), (L) are true in L, too, and $L(\alpha)$ constructed inside L is again the L-function $L(\alpha)$. If (SL) is true in the universe, then (SL) holds in L, too. (Here "ϕ is true in L" means that the list theoretic formula ϕ^L holds, where ϕ^L emerges from ϕ when we replace all "$\forall x\,\psi$" in ϕ by "$\forall x.\exists\alpha\ x = L(\alpha) \to \psi$" and all "$\exists x\,\psi$" in ϕ by "$\exists x.\exists\alpha\ x = L(\alpha) \wedge \psi$.")

In L—as in any other hierarchy—we can easily define the notion of a set:

Definition 10.2 (sets in L). *Let* $x = L(\alpha)$ *for some* α. *Then* x *is a* set (in L), *if* $L(\beta) \neq_\in x$ *for all* $\beta < \alpha$.

We can then, working in ALT, define the L-function

$S = sieve(L,$ "the entry is a set").

One can now show S together with the $\in$-relation is a model of ZFC.

§11. Finitary lists and sets. Though ZFC and ALT are equiconsistent, we think that there are important differences going beyond the fact that some concepts and theorems are little easier to develop and prove in one theory than in the other.

All theories dealing with infinite objects are based on hereditarily finite structures. So let us compare the "real" lists and sets in this way.

To do this, we consider the language with one binary function symbol f and one constant c. Instead of $f(s,t)$ we write (st), where s and t are terms. Thus the simplest terms of our language are c, (cc), $c(cc)$, $(cc)c$, $(cc)(cc)$, and so on.

The following facts are easy to verify:

"Hereditarily finite lists" is the free structure generated by f and c without further laws.

"Hereditarily finite sets" is the free structure generated by f and c with the additional laws:

(L1) $((xy)z) = ((xz)y)$ *(commutativity)*,

(L2) $((xy)y) = (xy)$ *(absorption)*.

The list theoretical reading of the constant term (ts) is $t + \langle s \rangle$,the end-extension of t by the new last entry s. Thus the constant c represents the empty list $\langle\rangle = 0$, and we have, for example,

$(cc) = \langle 0 \rangle$,$(c(cc)) = \langle\langle 0 \rangle\rangle$,$((cc)c) = \langle 0, 0 \rangle$,etc.

With this reading the commutativity law describes the identification of lists which can be recursively obtained from another by changing the order of appearance of entries, and the absorption law allows us to delete repetitions. If we just allow the law (L1), we obtain the hereditarily finite multisets.

When one explains the notion of a set to beginners, then one emphasizes precisely the two "abstractions" (L1) and (L2). One argues:

"The set $\{c, b, a, b\}$ is equal to the set $\{a, b, c\}$, since the order of appearance and repetitions of the same object does not matter. Moreover, $\{a, b\}$ is equal to $\{a, b'\}$, when $b = \{c, d\}$ and $b' = \{d, c, d\}$, since we agreed that $b = b'$, etc."

This is precisely the recursive definition of $\in$ and $=_\in$ in list theory, and this is not simple, and it does not give us sets as objects. Every mathematical notion is abstract (numbers are, and lists are, too), but sets involve an additional abstraction in identifying different abstract objects. This does not imply that the concept of a set is defective, of course, as more abstraction can lead to more flexibility. Comparative considerations like this simply support the development of alternatives which allow to see mathematics (including sets) from a perspective different than sets.

The free structure analysis shows that from a very natural point of view lists are conceptually simpler than multisets, and that multisets are conceptually simpler than sets. This analysis is a key element in a case for lists. (A typical reaction to the claim that lists are from a certain natural point of view simpler than sets is that this is nonsense, since well-orderings obviously carry more structure than sets and are therefore more complicated. This reaction shows how deeply set theoretic thinking has permeated even our finitary intuitions.)

A mathematical consequence of the very simple nature of hereditarily finite lists is that there is an easy and canonical enumeration of length ω of these lists. Let pair be the list of type ω enumerating, in the usual Cantorian way, all positional lists $\langle \beta, \gamma \rangle$ with $\beta, \gamma < \omega$. We now recursively define a list fin of type ω by:

$\text{fin}(0) = \langle\rangle$,

$\text{fin}(\alpha + 1) = \text{fin}(pair(\alpha)(0)) + \langle \text{fin}(pair(\alpha)(1)) \rangle$ for all $\alpha < \omega$.

Then fin lists all hereditarily finite lists without repetitions:

THEOREM 11.1. *The list* fin *is injective and all hereditarily finite lists appear in fin.*

Proof. We first prove injectivity. *Suppose not.* It follows immediately from the definition that the empty appears in fin only at position 0. Thus let $\alpha+1$ and $\beta+1$ be minimal such that $\alpha \neq \beta$ and $\text{fin}(\alpha+1) = \text{fin}(\beta+1)$. But then $pair(\alpha)(0) = pair(\beta)(0)$ and $pair(\alpha)(1) = pair(\beta)(1)$ by induction hypothesis and definition of $\text{fin}(\alpha + 1)$ and $\text{fin}(\beta + 1)$. Therefore

$$pair(\alpha) = \langle pair(\alpha)(0), pair(\alpha)(1) \rangle = \langle pair(\beta)(0), pair(\beta)(1) \rangle = pair(\beta).$$

Thus $\alpha = \beta$, since pair is injective, *contradiction.*

We now show that every hereditarily finite list appears in fin. *Suppose not.* Le x be of minimal complexity $Com(x)$ such that $non(x \imath fin)$. Trivially we have that $x \neq \langle\rangle$. We thus write x as: $x = y + \langle z \rangle$ ($y = \langle\rangle$ is possible here). By minimality of $Com(x)$ we have $y\iota$fin and $z\iota$fin, because $Com(y), Com(z) < Com(x)$. Thus let $\beta, \gamma < \omega$ such that $y = \text{fin}(\beta)$ and $z = \text{fin}(\gamma)$. Further let $\delta < \omega$ such that $pair(\delta) = \langle \beta, \gamma \rangle$. But then $\text{fin}(\delta + 1) = \text{fin}(\beta) + \langle \text{fin}(\gamma) \rangle = y + \langle z \rangle = x$, *contradiction.* □

Instead of the Cantorian pairing other pairings like the Γ-pairing can be used, but otherwise the enumeration is free of arbitrariness, and we might sum up the situation by saying: "The finite part of the universe of lists is a list." This already anticipates arguments in favor of the principle (L), which will be discussed below.

The picture that lists are simpler than sets, that they are a primitive concept while sets are not is strongly supported by computer science. Sets are not very popular as a foundational structure or data-type there. They are cumbersome to implement and slow in performance. The reasons are exactly the laws (L1) and (L2). Axiomatic list theory might help to give a foundation of mathematics which is by its very nature more appealing to computer science than set theory. The same is true for linguistics.

Interestingly, considerations of this sort have been made before the advent of computer science. When Hausdorff (1914, p. 32) gave the first set theoretic interpretation of the notion of an ordered pair in 1914, he noted (in my translation):

> "The double-indices (i, k) at elements of a determinant, the orthogonal coordinates (x, y) of points of the plane are ordered pairs of numbers. Thus this notion is fundamental in mathematics, and psychology would add that ordered, unsymmetric, selective composition of two things is even more primal than unordered, symmetric, collective. Thinking, speaking, reading, and writing are bound to chronological order, which is forced on us before we can abandon it ("absehen können"). The word is there before the set of its letters, the ordered pair (a, b) before the pair $\{a, b\}$."

Since lists are a distinguished structure in the finite world, their axiomatic extension into the infinite is a very natural enterprise, if one is interested at all in "ideal" objects. Now this extension of finite lists into the infinite seems to be the general concept of a well-ordering (and not of a mere linear ordering), since the basic constructor in finite list theory is "end-extension by one element", that is, counting, and counting through infinity leads to well-orderings.

The discussion also shows that well-orderings do not conceptually depend on sets. Historically, Cantor developed his transfinite numbers when he studied sets of reals. With his derivation $P' = \{x \in \mathbb{R} \mid x \text{ is an accumulation point of P}\}$ he found an operation which in general needs to be applied transfinitely often to a closed set P of reals in order to reach a $Q \subseteq P$ such that $Q' = Q$ (the perfect kernel of P). Thus transfinite numbers and more generally well-orderings emerged from studying sets. But now Cantorian ordinals continue the tradition of Euclid as "systems of units," and Cantor had no intentions at all to interpret his ordinals as sets as we understand them. In this way a historically minded argumentation does not refute, but supports our claim about the conceptual independence of transfinite lists from sets. That order and in particular ordinals can be interpreted as sets is a great achievement of later set theory. It was accomplished in the first three decades of the 19th century, when a greater level of precision was sought for and when sets were recognized of being able to interpret numbers, ordered pairs, relations, functions, and so on. But these interpretations do no not suggest that well-orderings are sets, depend on sets, or that sets are more basic than well-orderings. The same holds for any list theoretic interpretation of sets, of course.

If any version of transfinite list theory would have been found without set theory is a different question. As remarked in the Introduction, our theory emerged from studying Cantor's work, and as far the author understands 19th century mathematics, list theory does not belong there. But the question whether a strong foundational axiomatic theory can be based on the concept of a list or a multiset has been raised when looking at sets from computer science (see Blass & Gurevich, 2004), and experiences from computer science seem to indicate a possible way to discover a theory of lists. The free structures discussed above can be seen as a theoretical formulation of these experiences.

Philosophically, list theory continues the tradition that "everything is number." This "Pythagorean doctrine" is already present in the concept of a list, since in a list we count and string certain objects (which are again lists). But it is substantiated when we include Gödel's axiom of constructibility into our system of axioms. The view that Gödel's axiom can be seen as a continuation of the Pythagorean doctrine was put forward in set theory by Jensen (1995). In list theory this view seems to be even more impressive, since the objects of the theory are more akin to numbers than sets are. This brings us to the next section.

§12. Gödel's axiom of constructibility and the power set axiom. Gödel's axiom of constructibility is a basic and natural principle in set theory as well as in list theory, but as a postulated and accepted member of a foundational axiom system it is much more convincing in list theory than in set theory. As the universe of sets can be seen as a "set which is too large to be one," the universe of lists can be seen as a "list which is too long to be one." And if the universe of lists is to resemble its objects, then we have to assume the existence of a hierarchy. And there is, up to fine-structural organization, only one definable listing of the universe, unless further assumptions are made. To put it differently: The list theoretic analogon to the trivial set theoretic information $V = \{x \mid x = x\}$ is the existence of a hierarchy $F = \langle F(\alpha) \mid \alpha \text{ is a position}\rangle$, and this leads inevitably to the axiom of constructibility. There is also an immediate interest in the fine-structural organization of this hierarchy. From the very beginning, one is looking for a canonical and convincingly built list of all lists.

This picture is again supported by the hereditarily finite lists. As we have seen above, there is a canonical listing of the finite part of the universe of lists. The principle (L) can be read as the statement that this important property of finite lists—which is a structural, not a combinatorial one—carries over to the infinite. "As if it were finite" is also a leitmotiv of axiomatic set theory.

Seen this way, the axiom (L) is an "adequate" or "correct" axiom about lists. This statement is neutral about the question wether (L) should be abandoned when large cardinals axioms are studied in list theory. If so, then principles (K) saying that the universe is a certain core model replace (L) but maintain many of the arguments given in favor of (L).

A possible argument against (L) as a basic list theoretic axiom could be that this axiom is actually very complicated to write down as a sentence in the iota-language. But the length of this sentence is a mere technical point, the axiom has a clear meaning which everybody studying the theory can grasp. Of course, the axiom is different than all others, and it certainly poses a challenge in explaining the theory to nonlogicians.

If we look at set theory and list theory as of equal foundational rank, then we can see some important analogous principles in a new light: (1) The principle that there is a definable well-ordering of the universe has, as we argued, axiomatic status in list theory, while it is an interesting principle in set theory which is for many not convincing as a basic axiom. It belongs to the world of lists, from where it can be exported to set theory. (2) The axiom of choice is a genuinely set theoretical principle which does not need any analogon in list theory in the presence of a hierarchy, since then the principle (CC) of collective choice is easily provable. (3) For the power set principle the situation is opposite to the existence of a definable well-ordering of the universe. This principle belongs to set theory, and when postulated in a form like (SL) it looks like an interesting foreign body inside list theory. We shall look at this principle more closely.

The power set axiom of ZFC has been criticized from the very beginning as being impredicative, but it is nevertheless a natural axiom about sets, though certainly a bold one with respect to consistency. But the list theoretic principle (SL) as presented above is not natural at all. The power set of a set is unique, while the list (SL) claims to exists is not. The principle (SL) shares this indefiniteness with the axiom of choice in set theory, but indefiniteness has no place in list theory. Moreover: Why should a list of all sievings of an infinite list exist? This is certainly no "basic truth," and it does also not seem to be an adequate principle we formulate as an axiom after we studied lists for a while. The axiom (SL) has been imported from a different, set theoretic world and its history, where these questions existed but vanished to a large extent. (Cantor formulated the power set principle in a letter to Hilbert on October 10, 1898, but already two days later he wrote about doubts that the power set of a set is a "completed set" or "consistent multitude"; see Cantor, 1991.) Keeping (SL) as a reasonable axiom is again connected to the existence of a hierarchy. If we assume (L), then the following axiom (SL^*) is equivalent to (SL):

"For every stage α there is a stage β such that no new sieving of $L(\alpha)$ is constructed after stage β, i.e., if x is a sieving of $L(\alpha)$, then there is a $\gamma \leq \beta$ such that $x = L(\gamma)$."

If we accept this statement as a richness axiom about the universe of lists, than (SL) is acceptable as a consequence (SL^*). Without a hierarchy, the principle (SL) is simply out of place in list theory.

But even in the presence of the axiom of constructibility the axiom (SL^*) has a strong competitor in list theory. We might look at the axiom (EC) saying that "everything is countable" or "everything finally collapses to ω": "for every stage α there is a stage $\beta > \alpha$ such that $L(\beta)$ is a new sieving of $\langle n \mid n < \omega \rangle$" or "for every infinite stage α there is a stage β such that $L(\beta)$ is an ω-rearrangement of α." There is no reason why this cannot be read as a richness claim about the universe, too. It is just a richness claim contradicting (SL^*) and other richness claims. It continues the argumentation, held by some set theorists, that "V = L" was not shown to be "wrong" by large cardinal axioms and inner model theory. The combination (L) + "the reals are unbounded in the L-hierarchy" seems to be an attractive replacement of (SL) in ALT. This theory is called NEU for "never (for)ever uncountable." There is a long tradition in mathematics and philosophy that the continuum is inexhaustible and not fully mathematically seizable by means of an atomistic theory. (See Deiser, 2008, p. 129f.) The theory NEU would revive this tradition. Considerations that the reals might be conceived as "incredible rich" have also been made in connection with the independence of the Continuum Hypothesis and the arbitrary high value of the cardinality of $\mathbb{R}$ in models constructed by forcing (see Cohen 1966, p. 151). The outermost interpretation of "incredibly rich" would be that the reals do not form an object of the theory, because there are too many of them.

Thus in summary my point of view is the following: The first task in developing a theory of lists is to find a flexible and elegant language which is capable of a list theoretic interpretation of mathematical concepts. The iota-language with "ordinal-indexing" works fine here. Then we take a look at ZFC, build an analogous theory with the principal aim to get an equiconsistency result. The theory ALT is the result here. The next step is to look at the axioms of ALT more closely in order to free them from artificial set theoretic influences. Here the analogon of the power set axiom is the only member of ALT which seems to have no place in an axiomatic system originating from uninfluenced list theoretic thinking. On the other hand the axiom of constructibility, which did get much attention in set theory but did not reach the estimation of a basic axiom, has axiomatic status in list theory. List theory needs a hierarchy. Thus we adopt (L) as an axiom. Principles like (CC) have a trivial proof now, we have an easy interpretation of the concept of a set, and most importantly we have a definable universal list. The list theoretic axiom of constructibility does not share the restrictive character often felt by set theorists even before large cardinal axioms contradicting this principle became a focus of interest. Thus ALT - (SL) + (L) is a fragment of the theory we are looking for. But this fragment does not answer the question: Are there arbitrary large stages where a new real is constructed? Here our system branches. On the first branch, we can regain an list theoretic analogon of the power set axiom by reading it as a richness claim about the universe. On the second branch we look at another richness axiom which claims that everything is countable, answering the above question with "yes." To the author, the second branch—the theory NEU—looks to be the most interesting foundational theory which tries to interpret mathematical thinking in the language of lists. (An interesting task would be to analyze large cardinal principles in this theory. Though everything finally collapses to ω, one can imagine stages of the construction with large cardinals.)

Whatever branches of list theory may turn out to be fruitful and most convincing, already the theory ALT shows that the iota-language together with its basic notions and intuitions is apt for a foundational theory. We do not think that it is favorable that there is just one dominant foundational theory, since this might support an oversimplified and moreover pragmatic notion of truth in mathematics. Any discussion about the meaning of "table" as well as "truth" is limited if it is restricted to one natural language like Chinese, English, or Greek. The world is better reflected in several languages than in one, because every language develops its own cultural dynamics and its own perspectives. In list theory we are forced to be tidy, since we have to arrange everything in a well-ordered way. We have a sharp view, since we do not identify "real" as well as "ideal" objects under a recursive equivalence relation. In a whole, list theory might enrich our mathematical intuitions, and I think it adds some new aspects to the foundational discussion.

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