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The smallest sets of points not determined by their X-rays

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Abstract

Let F be an n-point set in $\mathbb{K}^d$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{Z}\}$ and $d \geqslant 2$. A (discrete) X-ray of F in direction s gives the number of points of F on each line parallel to s. We define $\psi_{\mathbb{K}^d}(m)$ as the minimum number n for which there exist m directions $s_1, \ldots, s_m$ (pairwise linearly independent and spanning $\mathbb{R}^d$) such that two n-point sets in $\mathbb{K}^d$ exist that have the same X-rays in these directions. The bound $\psi_{\mathbb{Z}^d}(m) \leqslant 2^{m-1}$ has been observed many times in the literature. In this note, we show $\psi_{\mathbb{K}^d}(m) = O(m^{d+1+\varepsilon})$ for $\varepsilon > 0$. For the cases $\mathbb{K}^d = \mathbb{Z}^d$ and $\mathbb{K}^d = \mathbb{R}^d$, $d > 2$, this represents the first upper bound on $\psi_{\mathbb{K}^d}(m)$ that is polynomial in m. As a corollary, we derive bounds on the sizes of solutions to both the classical and two-dimensional Prouhet–Tarry–Escott problem. Additionally, we establish lower bounds on $\psi_{\mathbb{K}^d}$ that enable us to prove a strengthened version of Rényi's theorem for points in $\mathbb{Z}^2$.

1. *Introduction*

The problem of reconstructing point sets from their X-rays has a long history; perhaps the 1952 paper [**20**] by Rényi represents one of the first works in this field. Of special interest are questions of uniqueness. Two sets with the same X-rays are said to be *tomographically equivalent* [**8**, **9**]; the sets are also commonly referred to as *switching components* [**13**, **22**] or *ghosts* [**12**, Section 15.4]. In [**15**], Matoušek, Přívětivý, and Škovroň show that almost all sets of m directions (in the sense of measure) allow for a unique reconstruction of $2^{Cm/\log(m)}$-point sets in the real plane (here $C > 0$ is a constant and the result holds for large m). For almost all choices of m directions, there thus exist only superpolynomial size switching components. By a careful selection of directions, however, we can reduce them to a polynomial size.

To make this precise, let F be an n-point set in $\mathbb{K}^d$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{Z}\}$ and $d \geqslant 2$. A (discrete) X-ray of F in direction s gives the number of points of F on each line parallel to s. We define $\psi_{\mathbb{K}^d}(m)$ as the minimum number n for which there exist m directions $s_1, \ldots, s_m$ (pairwise linearly independent and spanning $\mathbb{R}^d$) such that two different n-point sets in $\mathbb{K}^d$ exist that have the same X-rays in these directions. We derive lower and upper bounds on $\psi_{\mathbb{K}^d}$.

Two constructions are known to yield upper bounds on $\psi_{\mathbb{K}^d}$. The first construction is based on regular polygons. The two disjoint m-point sets of alternate vertices of a regular $2m$-gon in $\mathbb{R}^2$ yield $\psi_{\mathbb{R}^2}(m) \leqslant m$. This cannot be transferred to $\mathbb{Z}^d$ as any (planar) regular polygon with integer vertices must have 3, 4, or 6 vertices [**3**, **21**]. The functions $\psi_{\mathbb{R}^2}$ and $\psi_{\mathbb{Z}^2}$ are, in fact, different functions as we show $\psi_{\mathbb{Z}^2}(m) \geqslant m + 1$ if $m = 5$ or $m > 6$ (see Theorem 2.2). From this, we derive a strengthened version of Rényi's theorem (see Theorem 2.1 and Corollary 2.3 in Section 2).

The second well-known construction for upper bounds on $\psi_{\mathbb{K}^d}$ is based on two-colourings of the unit cube $[0, 1]^m$ in $\mathbb{Z}^m$. More precisely, two different sets with equal X-rays in coordinate directions are obtained as the two disjoint sets of 2^{m-1} alternate vertices of $[0, 1]^m$. By

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projecting into $\mathbb{Z}^d$, the bound $\psi_{\mathbb{Z}^d}(m) \leqslant 2^{m-1}$ is obtained. This construction seems to be due to Lorentz [14]; see also [2; 5, Lemma 2.3.2; 7, Theorem 4.3.1]. As $\mathbb{Z}^d \subseteq \mathbb{R}^d$, this, of course, yields also $\psi_{\mathbb{R}^d}(m) \leqslant 2^{m-1}$.

Our main observation is contained in the statement of Theorem 3.3, where we prove $\psi_{\mathbb{Z}^d}(m) = O(m^{d+1+\varepsilon})$ for $\varepsilon > 0$. This is, to our knowledge, the first upper bound on $\psi_{\mathbb{K}^d}(m)$ that is polynomial in m. Our proof is non-constructive.

We conclude in Section 4 by stating some remarks and consequences that relate our bounds to the *Prouhet–Tarry–Escott* problem (PTE$_r$) from number theory (see, for example, [10, Section 21.9]).

Throughout the paper, ζ is the Riemann zeta function, m and n denote natural numbers, and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{N} = \{1, 2, \dots\}$ are, respectively, the sets of integers, reals, and natural numbers. We use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[n] = \{1, \dots, n\}$, and $2\mathbb{Z} = \{2z \, : \, z \in \mathbb{Z}\}$. With $G_n^d = [n]^d$, we denote the set of d-tuples of positive integers less than or equal to n. If $\xi \in \mathbb{R}$, then $\lceil \xi \rceil$ denotes the smallest integer greater than or equal to ξ. The symbol O has the usual meaning: $f(m) = O(g(m))$ means that $f(m)/g(m)$ is bounded as $m \to \infty$. A property is said to hold for large m if that property holds for all m larger than some m_0.

2. *Lower bounds*

In this section, we derive lower bounds on $\psi_{\mathbb{K}^d}$. The key ideas are not new, but appear scattered and isolated in different contexts in the literature (see [20] and [1, Proof of Theorem 2.2]).

THEOREM 2.1. *For every $d \geqslant 2$, we have $\psi_{\mathbb{K}^d}(m) \geqslant m$.*

Proof. This is a reformulation of Rényi's theorem (proved in [20] and generalized to arbitrary dimensions by Heppes [11]), which states that any n-point set in $\mathbb{K}^d$ is uniquely determined by its X-rays from $n+1$ different directions. For completeness, we reproduce a short proof. Suppose that there are two sets F, F' with equal X-rays in $m+1$ directions, each set containing at most m points. Without loss of generality, there exists a point $p \in F \setminus F'$. Since F and F' have equal X-rays, there needs to be a point of F' on each of the $m+1$ lines through p. This implies that F' contains at least $m+1$ points, a contradiction. □

The bound is tight for $m \in \{1, 2, 3, 4, 6\}$ and $\mathbb{K}^d = \mathbb{Z}^2$; examples showing this for $m = 1, 2, 3, 4, 6$ are, respectively, provided by any two 1-point sets in $\mathbb{Z}^2$, two-colourings of the unit cube in $\mathbb{Z}^2$, the sets $F = \{(0,0), (1,2), (2,1)\}$, $F' = \{(1,0), (0,1), (2,2)\}$, and the examples shown in [7, Figures 4.3 and 4.5]. For the remaining cases, however, we can improve the bound as stated in the following result.

THEOREM 2.2. *If $m = 5$ or $m > 6$, then $\psi_{\mathbb{Z}^2}(m) \geqslant m + 1$.*

Proof. Let $m = 5$ or $m > 6$, and suppose that there exist different n-point sets $F, F' \subseteq \mathbb{Z}^2$ with equal X-rays in $m \geqslant n$ directions. Without loss of generality, we can assume that $F \cap F' = \emptyset$. The convex hull P of $F \cup F'$ is a non-degenerate polygon with at most $2n$ vertices. Parallel to each of the m directions, there are two lines that support P with each line containing a single point from F and F', respectively (since otherwise one of F and F' contains more than n points). Since this implies that P has at least $2m$ edges, we conclude that at least $2m$ of the elements of $F \cup F'$ are vertices of P (that is, $n = m$), proving that $F \cup F'$ is the set of vertices of the non-degenerate convex $2m$-gon P. Since F and F' have the same X-rays, P has the property that any line through a vertex of P in any of the m directions meets another vertex of P. Such polygons are known as lattice U-gons with U denoting the set of m directions.

They, however, do not exist for $m > 6$ (see [**6**, Theorem 4.5]). As is shown in [**6**, Proof of Theorem 4.5] or (more simply) in [**1**, Theorem 6], there are also no lattice U-gons for exactly five directions. In other words, we have $\psi_{\mathbb{Z}^2}(m) > m$ for $m = 5$ or $m > 6$. □

The bound is tight for $m = 5$. For this, consider the 6-point sets

$$F = \{(0,2),(1,4),(2,2),(3,0),(4,3),(5,1)\} \text{ and } F' = \{(0,3),(1,1),(2,4),(3,2),(4,0),(5,2)\}.$$

It is easily verified that F and F' have the same X-rays in the five directions

$$S = \{(1,0),(0,1),(1,1),(1,-1),(-2,1)\}.$$

A reformulation of Theorem 2.2 provides a strengthened version of Rényi's theorem for $\mathbb{Z}^2$.

Corollary 2.3. *Any n-point set in $\mathbb{Z}^2$ with $n = 5$ or $n > 6$ is uniquely determined by its X-rays taken from at least n different directions.*

3. *Upper bounds*

In this section, we prove a polynomial upper bound on $\psi_{\mathbb{K}^d}$. As a prelude, we prove an upper bound on the number of lines parallel to a given direction that intersect points of G_n^d. This is followed by a lemma that asserts the existence of certain coverings of a specified finite part of the integer lattice by m families of parallel lines.

Lemma 3.1. *For any relatively prime d-tuple $s = (\sigma_1, \ldots, \sigma_d) \in \mathbb{N}_0^d \setminus \{0\}$ with $d \geqslant 2$, there are at most $dn^{d-1} \cdot \max\{\sigma_1, \ldots, \sigma_d\}$ lines parallel to s that intersect G_n^d.*

Proof. For each line ℓ parallel to $s = (\sigma_1, \ldots, \sigma_d)$ that intersects G_n^d, there is a unique point $p \in \ell \cap G_n^d$ for which $p - s \notin G_n^d$. The point $p - s$ needs to have a non-positive component, that is,

$$p \in V_i = \{(\xi_1, \ldots, \xi_d) \in G_n^d \,:\, 1 \leqslant \xi_i \leqslant \sigma_i\}$$

for an $i \in [d]$. As the number of points in $\bigcup_{i=1}^d V_i$ is clearly bounded by $dn^{d-1} \cdot \max\{\sigma_1, \ldots, \sigma_d\}$, we obtain the claimed result. (Tight bounds can be obtained similarly via the inclusion–exclusion principle, but they are not needed in the present context.) □

Lemma 3.2. *Let $\varepsilon > 0$, $m \in \mathbb{N}$, $d \geqslant 2$, and $n \in \{\lceil m^{1+(1+\varepsilon)/d} \rceil, \lceil m^{1+(1+\varepsilon)/d} \rceil + 1\}$. Then, for large m there is a set $S = \{s_1, \ldots, s_m\} \subseteq \mathbb{Z}^d$ with the property that*

- (i) *the elements of S are pairwise linearly independent spanning $\mathbb{R}^d$;*
- (ii) *the total number l of lines that are parallel to a direction in S and intersect G_n^d is bounded from above by $2^{1+1/d} dn^{d-1} m^{1+1/d}$.*

Proof. For the number $R(p,d)$ of relatively prime d-tuples in G_p^d, $p \in \mathbb{N}$, it holds by [**17**] that

$$\lim_{p \to \infty} \frac{R(p,d)}{p^d} = \frac{1}{\zeta(d)}.$$

As ζ decreases for values larger than 1 and since $\zeta(2) = \pi^2/6 < 2$, we have

$$R(p,d) > p^d/2$$

for large p.

Setting $q = \lceil (2m)^{1/d} \rceil$, we note that $q \leqslant 2(2m)^{1/d}$ and $q \leqslant n$ for $m \geqslant 2$. For large m, we have

$$R(q,d) > q^d/2 \geqslant m,$$

so for our set S we can select m elements from $G_q^d \subseteq G_n^d$. We can assume that the elements of S span $\mathbb{R}^d$ since otherwise we replace d of the directions by the standard unit vectors. Property (i) is thus fulfilled (note that the elements of S are relatively prime d-tuples).

The entries of the elements in S are bounded by q, so by Lemma 3.1 we have at most

$$mdn^{d-1}q \leqslant 2^{1+1/d}dn^{d-1}m^{1+1/d}$$

lines parallel to a direction in S that intersect G_n^d. □

Theorem 3.3. *For every $\varepsilon > 0$ and $d \geqslant 2$, it holds that $\psi_{\mathbb{Z}^d}(m) = O(m^{d+1+\varepsilon})$.*

Proof. We assume that m is large enough that the set S from Lemma 3.2 exists. We set $n = \{\lceil m^{1+(1+\varepsilon)/d}\rceil, \lceil m^{1+(1+\varepsilon)/d}\rceil + 1\} \cap 2\mathbb{Z}$ and $k = \frac{1}{2}n^d$. Note that $k \in \mathbb{N}$, $k = O(m^{d+1+\varepsilon})$, and that we can assume that $n \geqslant 4$.

Let l_i, $i \in [m]$, denote the number of lines parallel to s_i that intersect G_n^d. The X-ray in direction s_i of a set in G_n^d with cardinality k gives a *weak k-composition of l_i*, that is, a solution to $\xi_1 + \cdots + \xi_{l_i} = k$ in non-negative integers [**23**, p. 15]. (The converse is generally false, because the corresponding X-ray lines may intersect G_n^d in fewer points than provided by a weak k-composition of l_i.) The number of weak k-compositions of l_i is given by

$$N(k, l_i) = \binom{k + l_i - 1}{l_i - 1}$$

and thus represents an upper bound for the number of different X-rays of k-point subsets of G_n^d in the direction s_i.

With $l = l_1 + \cdots + l_m$, we thus obtain the following upper bound on the number of different X-rays (for the directions in S) that can originate from a subset of G_n^d with cardinality k:

$$\prod_{i=1}^{m} N(k, l_i) \leqslant \prod_{i=1}^{m} \binom{n^d/2 + l_i}{l_i} \leqslant \prod_{i=1}^{m} \left(\frac{(n^d/2 + l_i)e}{l_i}\right)^{l_i} = \prod_{i=1}^{m} \left(\frac{n^d e}{2l_i} + e\right)^{l_i} \leqslant (ne + e)^l \leqslant n^{2l};$$

here the inequalities (from left to right) follow from $N(k, l_i) \leqslant N(k, l_i + 1)$, a standard inequality for binomial coefficients (see, for example, [**18**, Equation (4.9)]), $l_i \geqslant n^{d-1}$, and $n \geqslant 4$, respectively.

There are

$$\binom{n^d}{n^d/2} \geqslant 2^{n^d/2}$$

subsets of cardinality k in G_n^d. We claim that

$$n^{2l} < 2^{n^d/2}$$

holds for large m, which, by the pigeonhole principle, concludes the proof as it implies the existence of two sets in G_n^d with cardinality k and equal X-rays in the directions in S.

For the claim, we first note that

$$m^{1+(1+\varepsilon)/d} \leqslant n \leqslant 3m^{1+(1+\varepsilon)/d} \tag{1}$$

holds as $m^{1+(1+\varepsilon)/d} \geqslant 1$. It is easy to see that $\lim_{x\to\infty} x^a/2^{x^b} = 0$ for $a, b > 0$. Thus for large m and $C = 2^{3+1/d}d$, we have

$$3^C m^{C(1+(1+\varepsilon)/d)} < 2^{m^{\varepsilon/d}},$$

which, by (1) and Property (ii) of Lemma 3.2, gives

$$n^C < 2^{m^{\varepsilon/d}} \Rightarrow n^{Cm^{1+1/d}} < 2^n \Rightarrow n^{Cn^{d-1}m^{1+1/d}} < 2^{n^d} \Rightarrow n^{4l} < 2^{n^d},$$

proving the claim. □

4. *Remarks and consequences*

The previously mentioned regular $2m$-gon construction in $\mathbb{R}^2$, together with the inequality $\psi_{\mathbb{R}^d}(m) \leqslant \psi_{\mathbb{Z}^d}(m)$ for $d \geqslant 2$, yields the following corollary to Theorem 3.3.

COROLLARY 4.1. *For every* $\varepsilon > 0$ *and* $d \in \mathbb{N}$, *it holds that*

$$\psi_{\mathbb{R}^d}(m) = \begin{cases} m & \text{if } d = 2, \\ O(m^{d+1+\varepsilon}) & \text{if } d > 2. \end{cases}$$

In [**1**], the *general PTE*$_r$ problem was introduced: Given k, n, $r \in \mathbb{N}$, find two different multisets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\} \subseteq \mathbb{Z}^r$ where $x_i = (\xi_{i1}, \ldots, \xi_{ir})$, $y_i = (\eta_{i1}, \ldots, \eta_{ir})$ for $i \in [n]$ such that

$$\sum_{i=1}^{n} \xi_{i1}^{j_1} \xi_{i2}^{j_2} \cdots \xi_{ir}^{j_r} = \sum_{i=1}^{n} \eta_{i1}^{j_1} \eta_{i2}^{j_2} \cdots \eta_{ir}^{j_r}$$

for all non-negative integers $j_1, \ldots, j_r$ with $j_1 + \cdots + j_r \leqslant k$. The parameter k is called the *degree* and n the *size* of the solution. Tracing back to works of Euler and Goldbach [**4**, p. 705], the Prouhet–Tarry–Escott problem (PTE_1) is an old and largely unsolved problem in Diophantine analysis. The following corollary sharpens the bound of [**1**, Theorem 12] on the size of solutions, which for (PTE_1) is due to Prouhet [**19**].

COROLLARY 4.2. *For every* $\varepsilon > 0$, *there exists a constant* $C > 0$ *such that there are solutions of* (PTE_2) *of degree* k *and size bounded by* $Ck^{3+\varepsilon}$.

Proof. In [**1**, Theorem 8], it was shown that tomographically equivalent sets in $\mathbb{Z}^2$ for m directions yield (PTE_2) solutions of degree $m - 1$. This and Theorem 3.3 for $d = 2$ imply the statement of this corollary. □

REMARK 4.3. As the products cancel, it is evident that solutions of (PTE_1) can be obtained by applying to (PTE_2) solutions a suitable linear functional that maps $(\xi_1, \xi_2) \in \mathbb{Z}^2$ to $\alpha_1\xi_1 + \alpha_2\xi_2$ where $\alpha_1, \alpha_2 \in \mathbb{Z}$ are suitably chosen. The current best bounds for (PTE_1) are quadratic in k (see [**16**, **24**]); the bound from Theorem 3.3 is in this case weaker.

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