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A Universal Constant for Exponential Riesz Sequences

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Abstract. The aim of this paper is to study certain correlations between lower and upper bounds of exponential Riesz sequences, in particular between sharp lower and upper bounds, where we show that the product of the sharp bounds of an exponential Riesz sequence is bounded from above by a universal constant. The result is applied to the norms of coefficient and frame operators and their inverses.

Keywords: *Sharp bounds, exponential Riesz sequences, exponential Riesz bases, exponential frames*

AMS subject classification: Primary 42 C 15, secondary 94 A 12

1. Introduction

Let H be a separable Hilbert space over $\mathbb{C}$. A sequence $\Phi = (\varphi_n)_{n\in\mathbb{Z}}$ of elements in H is called a *Riesz-Fischer sequence* or a *Bessel sequence* for H, if there is a constant $A > 0$ or $B > 0$ such that for all numbers $n \in \mathbb{N}$ and $c_{-n}, \ldots, c_n \in \mathbb{C}$

$$A \sum_{j=-n}^{n} |c_j|^2 \leq \Big\| \sum_{j=-n}^{n} c_j \varphi_j \Big\|_H^2 \tag{1}$$

or

$$\Big\| \sum_{j=-n}^{n} c_j \varphi_j \Big\|_H^2 \leq B \sum_{j=-n}^{n} |c_j|^2, \tag{2}$$

respectively. If Φ is both a Bessel and a Riesz-Fischer sequence, it is called a *Riesz sequence* for H. The constants A and B are called *lower* and *upper bounds*, respectively. It is an easy matter to check that the supremum of all lower bounds and the infimum of all upper bounds of a Riesz sequence is again a lower bound and an upper bound, which we denote by $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$, respectively. The constants $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ are called the *sharp* lower and the *sharp* upper bounds for Φ, respectively.

In this paper, we shall be concerned with *exponential* Riesz sequences for $L^2(-\sigma,\sigma)$, i.e. with Riesz sequences for $L^2(-\sigma,\sigma)$ of the form $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$, where $\sigma > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers. From (1) and (2) it follows readily that if $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$

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is a Riesz sequence for $L^2(-\sigma,\sigma)$, then the sequence $(\Im\lambda_n)_{n\in\mathbb{Z}}$ of imaginary parts must be uniformly bounded.

Young (see [9: Proposition 1/Corollary 1] and [10: Chapter 4/Propositions 2 and 3, Theorem 3 and Chapter 2/Remark following Theorem 17]) has shown that for a sequence $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ of exponentials to be a Riesz sequence for $L^2(-\sigma,\sigma)$ it is sufficient that it be a Riesz-Fischer sequence for $L^2(-\sigma,\sigma)$, provided $(\Im\lambda_n)_{n\in\mathbb{Z}}$ is uniformly bounded. We shall have a closer look at this result and the occuring bounds. In particular, if $\sigma,\tau>0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers such that

$$\sup_{n\in\mathbb{Z}}|\Im\lambda_n|\le\tau \tag{3}$$

and $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is a Riesz-Fischer sequence for $L^2(-\sigma,\sigma)$ with lower bound A, we shall construct an upper bound B for $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$, depending only on A, σ and τ (Proposition 1). From this we shall obtain a universal constant for the product of the sharp bounds of exponential Riesz sequences (Theorem 1).

The results of this paper are part of the author's doctoral thesis [5: Chapter 4].

2. Results

We need the following

Definition 1. A sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers is called *separated by* $\delta>0$, if $\inf_{n\neq m}|\lambda_n-\lambda_m|\ge\delta$. The sequence is called *separated*, if there is some $\delta>0$ such that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated by δ.

Proposition 1. *Let $\tau\ge 0,\sigma>0,A>0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers satisfying* (3)*, such that $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is a Riesz-Fischer sequence for $L^2(-\sigma,\sigma)$ with lower bound A. Then there holds:*

1. *$(\lambda_n)_{n\in\mathbb{Z}}$ is separated by $\delta=\delta(A,\sigma,\tau)$, where*

$$\delta(A,\sigma,\tau)=\frac{1}{\sigma}\log\left(1+e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\right).$$

2. *$(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma,\sigma)$ with upper bound $B=B(A,\sigma,\tau)$, where*

$$\begin{aligned}B(A,\sigma,\tau)&=\frac{2}{\sigma}(e^{2\sigma(\tau+1)}-1)\Big(1+\frac{2}{\delta}\Big)^2\\&=\frac{2}{\sigma}(e^{2\sigma(\tau+1)}-1)\left(1+\frac{2\sigma}{\log\left(1+e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\right)}\right)^2.\end{aligned}\tag{4}$$

Remark 1. Proposition 1 states more explicitly a result of Young (see [9: Proposition 1, Corollary 1] and [10: Chapter 4/Propositions 3 + 2, Theorem 3 and Chapter 2/Remark following Theorem 17]) who proved that, under the assumptions of Proposition 1, the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ must be separated and that $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence.

From Proposition 1 we shall obtain

Theorem 1. *For every $\sigma > 0$ and every $\tau \geq 0$, there exists a constant $C(\sigma,\tau) > 0$ such that the following holds:*

If $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers satisfying (3) *and such that $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma,\sigma)$, then the product of the sharp bounds $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ is bounded from above by $C(\sigma,\tau)$:*

$$A_{opt}(\Phi)\, B_{opt}(\Phi) \leq C(\sigma,\tau). \tag{5}$$

The constant $C(\sigma,\tau)$ can be chosen as

$$C(\sigma,\tau) = 256\, e^{4\sigma\tau+2\sigma}(\sigma+\tfrac{1}{8})^2. \tag{6}$$

Remark 2. There is no universal constant $D(\sigma,\tau) > 0$ such that $A_{opt}(\Phi)\, B_{opt}(\Phi) \geq D(\sigma,\tau)$ for all exponential Riesz sequences $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ for $L^2(-\sigma,\sigma)$ satisfying (3).

Counterexample. For $0 < \varepsilon < 1$ we define

$$\lambda_n^\varepsilon = \begin{cases} n & \text{for } n \in \mathbb{Z}\backslash\{0\} \\ 1-\varepsilon & \text{for } n = 0 \end{cases} \qquad \text{and} \qquad \Phi^\varepsilon = (e^{i\lambda_n^\varepsilon \bullet})_{n\in\mathbb{Z}}.$$

From the orthonormality of $(\frac{1}{\sqrt{2\pi}}e^{in\bullet})_{n\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$ it follows that, for $0 < \varepsilon < 1$, Φ^ε is a Riesz sequence for $L^2(-\pi,\pi)$ and $B_{opt}(\Phi^\varepsilon) \leq 4\pi$. However, from $\|e^{i\lambda_0^\varepsilon \bullet} - e^{i\lambda_1^\varepsilon \bullet}\|^2_{L^2(-\pi,\pi)} \to 0$ for $\varepsilon \to 0$ we conclude $A_{opt}(\Phi^\varepsilon) \to 0$ for $\varepsilon \to 0$. This shows $A_{opt}(\Phi^\varepsilon)\, B_{opt}(\Phi^\varepsilon) \to 0$ for $\varepsilon \to 0$.

Definition 2. A sequence $\Phi = (\varphi_n)_{n\in\mathbb{Z}}$ in a separable Hilbert space H over $\mathbb{C}$ is called a *frame* for H (cf. Duffin and Schaeffer [3: Section 3]), if there exist constants $A > 0$ and $B > 0$ such that for all $f \in H$

$$A\,\|f\|_H^2 \leq \sum_{n\in\mathbb{Z}} |(f,\varphi_n)_H|^2 \leq B\,\|f\|_H^2.$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. The operators

$$T_\Phi : H \to l^2(\mathbb{Z}), \quad f \mapsto ((f,\varphi_n)_H)_{n\in\mathbb{Z}}$$

and

$$S_\Phi : H \to H, \quad f \mapsto \sum_{n\in\mathbb{Z}} (f,\varphi_n)_H \varphi_n$$

are called the *coefficient operator* and the *frame operator*, respectively, corresponding to the frame Φ.

Remark 3. From Definition 2 it follows easily that the coefficient operator corresponding to a frame is an injective, bounded linear operator, with bounded inverse on its range. Furthermore, it can be shown that S_Φ is a well-defined, bijective, bounded linear map (cf. Duffin and Schaeffer [3: Section 3]; the sum converges in the norm of H).

Definition 3. A frame in a separable Hilbert space over $\mathbb{C}$ is called an *exact* frame (or a *Riesz basis*), if it is no longer a frame after any of its elements is removed.

Remark 4. Every exact frame is a Riesz sequence with the same bounds (cf. Duffin and Schaeffer [3: Lemma X] and Kölzow [4: Section II.1/Corollary 1 to Theorem 8]).

As a consequence of Theorem 1, we have

Corollary 1. *For the constant $C(\sigma,\tau)$ of Theorem* 1 *there holds:*

If $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers satisfying (3) *and such that $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, then we have the following inequalities for the norms of the coefficient and frame operator and their inverses:*

$$\|T_\Phi\| \le \sqrt{C(\sigma,\tau)}\,\|T_\Phi^{-1}\|$$
$$\|S_\Phi\| \le C(\sigma,\tau)\,\|S_\Phi^{-1}\|$$
$$\|T_\Phi\|^2 \le C(\sigma,\tau)\,\|S_\Phi^{-1}\|.$$

Remark 5. A statement analog to Corollary 1 does not hold for arbitrary (non-exact) exponential frames.

Counterexample. We define the sequence $\Phi_m = (e^{i\frac{n}{m}\bullet})_{n\in\mathbb{Z}}$ where $m \in \mathbb{N}$. It can be shown that Φ_m is a frame for $L^2(-\pi,\pi)$, but $\|T_{\Phi_m}\|^2 = \|T_{\Phi_m}^{-1}\|^{-2} = \|S_{\Phi_m}\| = \|S_{\Phi_m}^{-1}\|^{-1} = 2\pi m$.

Definition 4. Denote by PW_σ^2 the *Paley-Wiener space*, consisting of all entire functions of exponential type at most σ, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. The norm on PW_σ^2 is the usual $L^2(\mathbb{R})$-norm.

For a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, the operator I_Λ is defined by

$$I_\Lambda : PW_\sigma^2 \to \mathbb{C}^{\mathbb{Z}}, \quad F \mapsto (F(\lambda_n))_{n\in\mathbb{Z}}.$$

Remark 6. If, for $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$, $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, then I_Λ defines a bijective, bounded linear operator from PW_σ^2 onto $l^2(\mathbb{Z})$ (i.e. Λ is a *complete interpolating sequence*, cf. Young [10: Chapter 4/Theorem 9]).

From Corollary 1 we obtain

Corollary 2. *For the constant $C(\sigma,\tau)$ of Theorem 1 there holds:*

If $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers satisfying (3) *and such that $(e^{i\lambda_n\bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, and if $I_\Lambda : PW_\sigma^2 \to l^2(\mathbb{Z})$ is the bijection considered above, then*

$$\|I_\Lambda\| \le \frac{\sqrt{C(\sigma,\tau)}}{2\pi}\,\|I_\Lambda^{-1}\|.$$

3. Proofs

We shall need the following

Lemma 1 (see [6: Lemma 1]). *Let $\delta,\sigma > 0$, $\tau \ge 0$, and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, separated by δ and satisfying* (3). *Then for all functions F of the Paley-Wiener space PW_σ^2 the inequality*

$$\sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \le \frac{e^{2\sigma(\tau+1)}-1}{\pi\sigma}\Big(1+\frac{2}{\delta}\Big)^2 \int_{\mathbb{R}} |F(x)|^2 dx \tag{7}$$

holds.

It should be noted that for separated real sequences the first inequality of type (7) (with a different constant) was given by Plancherel and Pólya [8: p. 126].

Proof of Proposition 1.

1. For the following proof that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated we use similar arguments as Pavlov did in [7: Theorem 1]. Let $k,n \in \mathbb{Z}$, where $k \neq n$. Then we have from (1)

$$\begin{aligned} 2A &= A(|1|^2 + |-1|^2) \\ &\le \int_{-\sigma}^{\sigma} |e^{i\lambda_k x} - e^{i\lambda_n x}|^2 dx \\ &= \int_{-\sigma}^{\sigma} |e^{i\lambda_k x}|^2 |1 - e^{i(\lambda_n - \lambda_k)x}|^2 dx \\ &\le e^{2\sigma\tau} \int_{-\sigma}^{\sigma} |1 - e^{i(\lambda_n - \lambda_k)x}|^2 dx. \end{aligned} \tag{8}$$

For $x \in (-\sigma, \sigma)$ we have

$$\begin{aligned} |1 - e^{i(\lambda_n - \lambda_k)x}| &= \left| \sum_{m=1}^{\infty} \frac{(i(\lambda_n - \lambda_k)x)^m}{m!} \right| \\ &\le \sum_{m=1}^{\infty} \frac{(|\lambda_n - \lambda_k|\,|\sigma|)^m}{m!} \\ &= e^{|\lambda_n - \lambda_k|\sigma} - 1. \end{aligned}$$

Hence we conclude from (8)

$$2A \le e^{2\sigma\tau} 2\sigma (e^{|\lambda_n - \lambda_k|\sigma} - 1)^2.$$

From this it follows easily that

$$|\lambda_n - \lambda_k| \ge \frac{1}{\sigma} \log\Big(1 + e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\Big) = \delta(A, \sigma, \tau),$$

i.e. $(\lambda_n)_{n\in\mathbb{Z}}$ is separated by $\delta = \delta(A,\sigma,\tau)$.

2. From Lemma 1 and assertion 1 we derive (7) with $\delta = \delta(A,\sigma,\tau)$. By the Paley-Wiener theorem, this is equivalent to

$$\sum_{n\in\mathbb{Z}} |(f, e^{i\lambda_n \bullet})|^2 \le \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1)\Big(1 + \frac{2}{\delta}\Big)^2 \|f\|^2$$

for all $f \in L^2(-\sigma,\sigma)$. By a theorem of Boas [2: Theorem 1] (cf. Young [10: Chapter 4/Theorem 3]) the latter inequality is equivalent to $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ being a Bessel sequence with upper bound $B = B(A,\sigma,\tau)$, as defined by (4) ∎

Proof of Theorem 1. Let $\sigma > 0, \tau \ge 0$, and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers satisfying (3) and such that $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence with sharp bounds $A = A_{opt}(\Phi)$ and $B_{opt}(\Phi)$. From Proposition 1 we conclude

$$B_{opt}(\Phi) \le \frac{2(e^{2\sigma(\tau+1)} - 1)}{\sigma} \left(1 + \frac{2\sigma}{\log\big(1 + e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\big)}\right)^2. \tag{9}$$

From

$$A \le \|e^{i\lambda_n \bullet}\|^2_{L^2(-\sigma,\sigma)} \le 2\sigma\, e^{2\sigma\tau} \qquad (n \in \mathbb{Z})$$

we derive

$$\frac{1}{\sqrt{8}}\, e^{-\sigma\tau} \sqrt{\frac{A}{\sigma}} \le \frac{1}{2}. \tag{10}$$

Using $\log(1+x) \ge \frac{x}{2}$ for $x \in [0, \frac{1}{2})$, we thus obtain

$$\log\left(1 + e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\right) \ge \log\left(1 + \frac{1}{\sqrt{8}}\, e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}\right) \ge \frac{1}{2}\,\frac{1}{\sqrt{8}}\, e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}}.$$

Using (9) and (10), we conclude

$$\begin{aligned} B_{opt}(\Phi) &\le \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1)\left(1 + 4\sqrt{8}\,\sigma\, e^{\sigma\tau}\sqrt{\frac{\sigma}{A}}\right)^2 \\ &\le \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1)\left(\frac{1}{8} + \sigma\right)^2\left(4\sqrt{8}\, e^{\sigma\tau}\sqrt{\frac{\sigma}{A}}\right)^2 \\ &\le 256\, e^{4\sigma\tau + 2\sigma}\left(\frac{1}{8} + \sigma\right)^2 \frac{1}{A}. \end{aligned}$$

This shows $A_{opt}(\Phi)\, B_{opt}(\Phi) \le C(\sigma,\tau)$ for $C(\sigma,\tau)$ defined by (6) ∎

Proof of Corollary 1. Let $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ be an exact frame for $L^2(-\sigma,\sigma)$. Since the frame bounds of the exact frame Φ coincide with the bounds of Φ as a Riesz sequence (Remark 4), we conclude that for all $f \in L^2(-\sigma,\sigma)$

$$A_{opt}(\Phi)\, \|f\|^2 \le \sum_{n\in\mathbb{Z}} |(f, e^{i\lambda_n \bullet})|^2 \le B_{opt}(\Phi)\, \|f\|^2 \tag{11}$$

and that $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ are the best possible constants in this inequality. From this we derive

$$\|T_\Phi\| = \sqrt{B_{opt}(\Phi)} \qquad \text{and} \qquad \|T_\Phi^{-1}\| = \frac{1}{\sqrt{A_{opt}(\Phi)}}. \tag{12}$$

The norms of S_Φ and S_Φ^{-1} also can be expressed by the best constants occuring in (11). It holds

$$\|S_\Phi\| = B_{opt}(\Phi) \qquad \text{and} \qquad \|S_\Phi^{-1}\| = \frac{1}{A_{opt}(\Phi)} \tag{13}$$

(cf. Benedetto and Walnut [1: Theorem 3.2/a]). Thus the result follows from inequalities (5), (12) and (13) ∎

Proof of Corollary 2. It follows from Corollary 1 and the Paley-Wiener theorem (cf. Young [10: Chapter 2/Theorem 18]) ∎

Open questions.

1. For what classes of exact frames (not of exponential type) do analogues to Theorem 1 and Corollary 1 exist? It can be thought, e.g., of the class of all exact Gabor frames with discretisation parameters t_0 and ω_0 such that $t_0\,\omega_0 = 1$.

2. What is the best constant $C_{opt}(\sigma,\tau)$ fulfilling inequality (5)? We conjecture that $C_{opt}(\pi,0) = 4\pi^2$.

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References

[1] Benedetto, J. J. and D. F. Walnut: *Gabor frames for L^2 and related spaces.* In: Wavelets: Mathematics and Application (eds.: J. J. Benedetto and M. W. Frazier). Boca Raton: CRC Press 1994, pp. 97 – 162.

[2] Boas, J. R., Jr.: *A general moment problem.* Amer. J. Math. 63 (1941), 361 – 370.

[3] Duffin, R. J. and A. C. Schaeffer: *A class of nonharmonic Fourier series.* Trans. Amer. Math. Soc. 72 (1952), 341 – 366.

[4] Kölzow, D.: *Wavelets. A Tutorial and a Bibliography.* Rend. Ist. Matem. Trieste 26, Suppl. 1994.

[5] Lindner, A.: *Über exponentielle Rahmen, insbesondere exponentielle Riesz-Basen.* Dissertation. Erlangen-Nürnberg: Universität 1999.

[6] Lindner, A. M.: *On lower bounds of exponential frames.* J. Fourier Anal. Appl. 5 (1999), 187 – 194.

[7] Pavlov, B. S.: *Basicity of an exponential system and Muckenhoupt's condition* (in Russian). Dokl. Akad. Nauk SSSR 247 (1979), 37 – 40; English transl.: Soviet Math. Dokl. 20 (1979), 655 – 659.

[8] Plancherel, M. and G. Pólya: *Fonctions entières et intégrales de Fourier multiples.* Seconde partie. Comment. Math. Helv. 10 (1938), 110 – 163.

[9] Young, R. M.: *Interpolation in a classical Hilbert space of entire functions.* Trans. Amer. Math. Soc. 192 (1974), 97 – 114.

[10] Young, R. M.: *An Introduction to Nonharmonic Fourier Series.* New York: Acad. Press 1980.

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