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Stochastic dependencies in derivative pricing:

**Decoupled BNS-volatility,
sequential modeling of jumps,
and extremal WWR**

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Abstract

This thesis addresses different topics of dependence modeling in derivative pricing. It covers a new stochastic volatility model for asset price returns, multi-dimensional generalizations of popular univariate jump-diffusion models, and a model-free approach to identify extremal cases of bilateral credit valuation adjustments (BCVA). Based on joint subordination, dependent multi-dimensional compound Poisson processes are constructed. The presented approach allows to introduce dependence between the components of the multivariate processes, without altering the marginal laws. This is a very convenient feature e.g. for sequentially calibrating multi-dimensional models. Moreover, the two-dimensional time-changed compound Poisson process is used to extend the popular BNS model by partly decoupling the volatility jumps from the asset price returns. Finally, worst-case dependence structures between portfolio values and the default times of the contractual parties to a derivative transaction, which lead to extremal wrong-way risk (WWR), are established. This is achieved by solving a mass-transportation problem. The tight bounds for the BCVA, which are produced by our model-free methodology, are useful measures for the model risk of WWR models.

Zusammenfassung

Diese Arbeit beschäftigt sich mit verschiedenen Themen der Abhängigkeitsmodellierung in der Derivatebewertung. Sie behandelt ein neues stochastisches Volatilitätsmodell, mehrdimensionale Erweiterungen von bekannten eindimensionalen Sprung-Diffusions-Modellen für Preisprozesse und einen modellfreien Ansatz um Extremfälle von BCVA zu ermitteln. Basierend auf gemeinsamer Subordination werden mehrdimensionale zusammengesetzte Poissonprozesse konstruiert. Die vorgelegte Methode erlaubt es, Abhängigkeit zwischen den Komponenten des mehrdimensionalen Prozesses einzuführen, ohne die Randverteilungen zu verändern. Dies ist eine sehr praktische Eigenschaft, beispielsweise für die sequentielle Kalibrierung von mehrdimensionalen Modellen. Zudem wird der zweidimensionale zusammengesetzte Poissonprozess verwendet um das bekannte BNS Modell, durch teilweises Entkoppeln der Volatilitätssprünge von den Preissprüngen, zu erweitern. Abschließend werden Worst-Case Abhängigkeitsstrukturen zwischen dem Portfoliowert und den Ausfallszeiten der Vertragspartner einer Derivatetransaktion ermittelt, die zu extremen WWR führen. Dies wird durch das Lösen eines Transportproblems erreicht. Die Schranken für das BCVA, welche durch unseren modellfreien Ansatz erzeugt werden, stellen nützliche Maße für das Modellrisiko von WWR-Modellen dar.

Acknowledgements

First and foremost, I thank my supervisor Matthias Scherer for the continuing support over the last years. Whenever I had a question, the door to his office was open and I could always rely on his valuable feedback. I am also grateful to the co-authors of my paper projects Jan-Frederik Mai and Karl Hofmann for the fruitful collaboration.

I would like to express my gratitude to Rudi Zagst for making it possible to do a doctorate at the KPMG Center of Excellence in Risk Management, which I experienced as a unique opportunity to build up practical know-how apart from doing research.

My colleagues at the chair of mathematical finance made my Ph.D study a memorable time of my life. I enjoyed the stimulating discussions, the joint conference trips, the lunch and coffee breaks, the competitive table soccer matches, and the after-work beers with them. Special thanks go to my friend Maximilian Gass for our joint mathematical journey from the first homework exercises in 2006 till our last shared days at the chair in 2015.

Finally, I thank my parents for their enduring encouragement and my wife Sandra for her loving support in all situations.

Contents

1 Introduction

Modeling financial assets and pricing derivatives are tasks having faced a drastic change over the last decades. Since pricing and risk managing of complex products increasingly came into focus, more and more stylized facts of time series of asset prices are supposed to be captured by financial models. This naturally results in more sophisticated, but also more complex, models. Seeding in the groundbreaking works of Samuelson [1965] and Black and Scholes [1973], where the asset price follows a geometric Brownian motion, many extensions and variants of the popular Black–Scholes model have been proposed. For example, Merton [1973] weakened the constant volatility assumption, resulting in a time dependent volatility. Later extensions model the volatility as a stochastic process, e.g. the local volatility model class, where the volatility is a function of time and current spot price. A popular example of that class is the model by Cox and Ross [1976]. Other proposals, like Merton [1976], add jumps to the asset price dynamics to explain sudden market movements. A wave of further enhancement followed, resulting in a zoo of models. One of those approaches, which combines both, price jumps and stochastic volatility dynamics, has been proposed by Barndorff-Nielsen and Shephard [2001]. Their model, which we will call Barndorff-Nielsen–Shepard model, or short BNS model, plays a prominent role throughout the present thesis. Such univariate models, describing one asset only, have also been generalized to sound multi-dimensional model frameworks incorporating dependencies between different assets. This is achieved by linking the stochastic drivers of the marginal asset processes. This is a straightforward task for Brownian motions, but becomes challenging for jump-driven price dynamics. Those multivariate models, however, are essential for pricing financial derivatives, which rely on more than only one underlying price process. At the latest since the collapse of Lehman Brothers in 2008 it became inevitable to recognize counterparty default risk as integral part of the valuation process of financial derivatives, which changed the classical view on derivative pricing. Doing so alters the risk neutral price by adjustments accounting for a possible loss in case of a default of one contractual party.

In derivatives pricing, stochastic dependencies can play a crucial role. Basically, there are three areas where modeling dependence might have a huge impact on the resulting derivative values. *First*, dependence can occur directly as model parameter in univariate models for financial assets. For example, Heston [1993] proposed a diffusion-type model with stochastic volatility. Both, the asset value and its variance process are driven by a Brownian motion each, which do not necessarily coincide, but are stochastically dependent. Obviously, the correlation between the two Brownian motions represents one out of six parameters of the Heston model. Another popular example is the model by Stein and Stein [1991]. Their model dynamics also depend on the correlation of two Brownian motions.[1] *Secondly*, in multi-dimensional models, stochastic dependence is essential to recognize interactions between assets and to model similar (or converse) movements. Pricing multi-underlying derivatives therefore relies on sensible joint modeling of their ingredients, which can be, for example, stock prices, interest rates, foreign exchange rates, or default times. Some derivatives rely only on underlying of the same kind, like basket options (stocks) or collateral debt obligations (survival processes).[2] Pricing those derivatives therefore necessitates a multivariate framework with similar marginal processes. On the contrary, a cross-asset model is required to price derivatives like quanto options (FX-equity).[3] *Thirdly*, price adjustments due to the possibility of default events can heavily be affected by the dependence between the value of the derivative and the credit worthiness of the contractual parties. In these cases, we speak of wrong-way risk (WWR). So, valuing WWR also requires a sound multivariate setup for modeling the underlying processes of the derivatives and the default times of the two counterparts to a derivative transaction.

In the present thesis, we contribute new approaches to all of those three fields. *First*, we introduce a new model, the so-called weak-link Γ-OU-BNS model, which generalizes the popular univariate Γ-OU-BNS model[4] by introducing a new parameter, which drives the dependence between asset price jumps and volatility jumps. The BNS model class

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[1] Many other univariate models exist having parameter representing the dependence between stochastic drivers, e.g. extensions of Heston's model, like Schöbel and Zhu [1999].

[2] A bunch of models exist in the literature for pricing those derivatives. Just to give a few examples, Milevsky and Posner [1998] and Brooks et al. [1994] among others price basket options in a multi-dimensional Black-Scholes framework. Vasicek [1987] proposed a multivariate generalization of Merton's structural model (Merton [1974]), which can be used to value collateral debt obligations.

[3] Again, a vast number of model proposals exist for cross-asset derivatives, for example Derman et al. [1990] use a two-dimensional Black–Scholes model for Quanto option pricing, whereas Dimitroff et al. [2009] propose a multi-dimensional Heston model.

[4] A tractable example of the BNS model class, which we will present in Section 4.1.

imposes a Lévy subordinator driven Ornstein–Uhlenbeck structure for the squared volatility process. Furthermore, in the extended notion according to Nicolato and Venardos [2003], upward jumps in the squared volatility process are accompanied by downward jumps in the asset price. There is, however, empirical evidence (e.g. Jacod and Todorov [2010]) that asset prices and volatility do not always jump together, but there are separate jumps in both processes, which cannot be captured by the classical BNS model class. We therefore extend the BNS model class in a generic way, accounting for jumps in the asset price as well as the squared volatility process which do not necessarily have to occur simultaneously. We employ a two-dimensional Lévy process to account for the jumps in the squared volatility process and the asset price process, where the coordinate processes can have any possible dependence structure. One tractable example of this model class is the weak-link Γ-OU-BNS model. Here, the jumps are driven by dependent compound Poisson processes constructed by a time-change construction. For this model, the characteristic function of the log price process can be calculated in closed form, which is essential for fast derivatives pricing via Fourier methods.

Secondly, new tractable multi-dimensional jump-diffusion models based on univariate models by Kou [2002], Bannör and Scherer [2013], and the Γ-OU-BNS model are presented. Here, we use a bottom-up approach. That means, we start with d univariate models and merge them to one multivariate model by adding a certain dependence structure. This is achieved by introducing dependence to the univariate jump processes in a way that does not alter the marginal laws; a very convenient feature e.g. for a sequential calibration of the model's parameters to market quotes. Usually, the number of parameters in multi-dimensional models is very high, which is often the bottleneck of these models when it comes to practical applications, since it is difficult to capture so many model parameters from market quotes. In our model framework, we aim at keeping the number of parameters, which cannot be calibrated to market prices of plain vanilla options data, as low as possible to ensure practical tractability.

Thirdly, we investigate extremal cases of wrong-way risk by detecting worst-case dependence structures between derivative prices and default times, leading to maximal and minimal bilateral credit valuation adjustments (BCVA). In order to calculate BCVA recognizing wrong-way risk, a sound model for the dependence structure between three quantities is required. In particular, the crucial building blocks are: the default times of the two contractual parties to the derivative transaction and the derivative value at the first of the two default times. In the literature, there exist various proposals

on how this dependence structure should be modeled. However, no market consensus emerged. In practice, independence between all three, or at least two of the quantities is still a popular choice, although it is a over-simplification and completely misses the root of WWR. Moreover, many WWR methodologies depend strongly on the marginal models for the default times and the model for the underlying of the derivative. In any case, specifying the dependence structure imposes one to model risk. Even within some parametric model one typically obtains a considerable interval of BCVA values when the dependence parameters are taken to the extremes. Here, we present an approach to identify model-free bounds for BCVA. These tight bounds can be used as some model risk measure on how flexible a given parametric model is with respect to explaining the range of possible adjustments.

The remainder of the thesis is structured as follows: The foundation for the studies is laid in Chapter 2, which addresses the mathematical background and introduces the notation used throughout the thesis. Chapter 3 presents a new useful construction of dependent compound Poisson processes with exponentially distributed jump sizes. We investigate the implied dependence structure and point out, why this construction is a tractable tool for modeling dependent asset price jumps. Chapter 4 and Chapter 5 utilize this construction extensively. In Chapter 4 we present the weak-link Γ-OU-BNS model, a new stochastic volatility model with decoupled jumps. Chapter 5 addresses multi-dimensional versions of popular univariate jump-diffusion models. Chapter 6 elaborates the investigation of extremal dependence structures within wrong-way risk models for BCVA calculations. Finally, Chapter 7 summarizes the main results of the present thesis. The four main chapters (3 to 6) are structured likewise. Each starts with an introductory section on some fundamentals and concludes with a section on applications. If not stated otherwise, we claim the content of these chapters, excluding the fundamental sections, to be our own work based on four peer-reviewed articles published prior to this thesis, namely:

[Bannör et al., 2015] Bannör, K. F., Scherer, M., and **Schulz, T.** (2015). A two-sided BNS model for multicurrency FX markets. In Glau, K., Scherer, M., and Zagst, R., editors, *Innovations in Quantitative Risk Management*, pages 93–107. Springer International Publishing, Cham.

[Hofmann and Schulz, 2016] Hofmann, K. F. and **Schulz, T.** (2016). A general Ornstein-Uhlenbeck stochastic volatility model with Lévy jumps. *International Journal of Theoretical and Applied Finance*, 19(08):1–23.

[Mai et al., 2014] Mai, J.-F., Scherer, M., and **Schulz, T.** (2014). Sequential modeling of dependent jump processes. *Wilmott Magazine*, 2014(70):54–63.

[Scherer and Schulz, 2016] Scherer, M. and **Schulz, T.** (2016). Extremal dependence for bilateral credit valuation adjustments. *International Journal of Theoretical and Applied Finance*, 19(07):1–21

In particular, Chapter 3 contains results from Mai et al. [2014], Chapter 4 from Hofmann and Schulz [2016], Chapter 5 from Bannör et al. [2015] and Mai et al. [2014], and Chapter 6 from Scherer and Schulz [2016].

2 Mathematical prerequisites

In this preliminary chapter we provide an overview of the nomenclature and a brief introduction of the underlying mathematical theory required by our study. We start with some basic distributions and the definition of the Laplace and Fourier transform, followed by an introduction to Lévy processes with focus on compound Poisson processes, and a brief introduction on copula theory.

Definition 2.1 (Normal distribution)
Let $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$, and let X be a random variable with density of the form

$$f(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}, \quad \forall x \in \mathbb{R}.$$

Then, the law of X is called *normal distribution with mean μ and variance σ^2*, abbreviated by $X \sim \mathcal{N}\left(\mu,\,\sigma^2\right)$.

For $\mu \in \mathbb{R}^d$, $d \in \mathbb{N}$, and a positive definite matrix $\Sigma \in \mathbb{R}_+^{d \times d}$, we call the law of an $\mathbb{R}^d$-valued random variable X with density of the form

$$f(x) = \frac{1}{\sqrt{(2\,\pi)^d\,\det(\Sigma)}}\, e^{-\frac{1}{2}\,(x-\mu)^\top \Sigma^{-1}(x-\mu)}, \quad \forall x \in \mathbb{R}^d,$$

a *d-dimensional normal distribution with mean μ and covariance matrix Σ*.

Definition 2.2 (Exponential distribution)
Let $\eta > 0$ and let X be a nonnegative random variable with density of the form $f(x) = \eta\,e^{-\eta\,x}$ for all $x \geq 0$. Then, the law of X is called *exponential distribution with parameter η*, abbreviated by $X \sim \mathrm{Exp}(\eta)$.

Definition 2.3 (Erlang distribution)
Let $d \in \mathbb{N}$ and $X_1, X_2, \ldots, X_d$ be mutually independent $\mathrm{Exp}(\eta)$-distributed random variables. Then, it follows directly by applying $d-1$ times the convolution

formula on the density of an exponential distribution that the density of the sum of X_1, X_2, ..., X_d is given by

$$f(x) = \frac{\eta^d \, x^{d-1} \, e^{-\eta \, x}}{(d-1)!}, \quad \forall x \in \mathbb{R}.$$

This law is called *Erlang distribution*, abbreviated by Erlang(d, η).

While the normal distribution is used to describe diffusion-type price movements, we utilize the exponential distribution to describe jump magnitudes of stochastic processes, e.g. jumps in stock price process. The sum of several jumps is hence Erlang-distributed. The number of jumps in a finite time interval can be modeled by a Poisson distribution.

Definition 2.4 (Poisson distribution)
Let $c \geq 0$ and let N be an $\mathbb{N}_0$-valued random variable fulfilling $\mathbb{P}(N = n) = e^{-c} \frac{c^n}{n!}$ for all $n \in \mathbb{N}_0$. Then, the law of N is called *Poisson distribution with parameter c*, abbreviated by $N \sim \text{Poi}(c)$.

The Geometric distribution is the discrete analogue to the exponential law and can be interpreted as the number of failures of a repeated Bernoulli experiment before the first success.

Definition 2.5 (Geometric distribution)
Let $0 < p \leq 1$ and let N be an $\mathbb{N}_0$-valued random variable fulfilling $\mathbb{P}(N = n) = p \, (1 - p)^n$ for all $n \in \mathbb{N}_0$. Then, the law of N is called *geometric distribution with parameter p*, abbreviated by $N \sim \text{Geo}(p)$.

The parameter p of the geometric distribution coincides with the success probability of the constructing Bernoulli experiment. The expectations of the presented distributions provide a natural interpretation of their parameters.

Theorem 2.6 (First moments of some basic distributions)
  (i) Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. Then,

$$\mathbb{E}\left[X\right] = \mu, \qquad \mathbb{E}\left[X^2\right] = \mu^2 + \sigma^2, \qquad \text{Var}\left[X\right] = \sigma^2.$$

(ii) Let $X \sim \mathrm{Exp}(\eta)$. Then,

$$\mathbb{E}[X] = \frac{1}{\eta}, \qquad \mathbb{E}[X^2] = \frac{2}{\eta^2}, \qquad \mathrm{Var}[X] = \frac{1}{\eta^2}.$$

(iii) Let $N \sim \mathrm{Poi}(c)$. Then,

$$\mathbb{E}[N] = c, \qquad \mathbb{E}[N^2] = c^2 + c, \qquad \mathrm{Var}[N] = c.$$

(iv) Let $N \sim \mathrm{Geo}(p)$. Then,

$$\mathbb{E}[N] = \frac{1-p}{p}, \qquad \mathbb{E}[N^2] = \frac{(1-p)^2 + 1 - p}{p^2}, \qquad \mathrm{Var}[N] = \frac{1-p}{p^2}.$$

Proof

The first two moments of these basic distributions can easily be calculated, as it is stated in standard textbooks on probability theory like Billingsley [1995], Kallenberg [2002], or Klenke [2007]. □

Alternatively, these moments can be derived from the derivatives of their moment generating function (cf. Theorem 2.8 (iv)), which is defined in the sequel.

Definition 2.7 (Characteristic function and Laplace transform)

The *characteristic function* φ_X of an $\mathbb{R}^d$-valued random variable X, $d \in \mathbb{N}$, is defined by

$$\varphi_X(u) := \mathbb{E}[e^{\mathrm{i}\, u^\top X}]$$

for any $u \in \mathbb{R}^d$.

Moreover, for any $\mathbb{R}_+$-valued random variable X, the *Laplace transform*[1] $\tilde{\varphi}_X$ is defined by

$$\tilde{\varphi}_X(u) := \mathbb{E}[e^{-u\, X}]$$

for any $u \in \mathbb{R}_+$.

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[1] In general, for an $\mathbb{R}$-valued random variables X, the function $f(u) := \mathbb{E}[e^{u\, X}]$, $u \in \mathbb{R}$ is called moment generating function. Note that in contrast to that function, the Laplace transform, which we only consider for non-negative random variables, always exists.

In some textbooks, the characteristic function is also called *Fourier transform*. The next theorem lists some useful properties of Fourier and Laplace transforms.

Theorem 2.8 (Properties of Fourier and Laplace transforms)

   (i) The law of a random variable X on $\mathbb{R}^d$ is uniquely determined by its characteristic function φ_X.

  (ii) The law of a random variable X on $\mathbb{R}_+$ is uniquely determined by its Laplace transform $\tilde{\varphi}_X$.

 (iii) Let $\tilde{\varphi}_X : \mathbb{R}_+ \to [0,1]$ be the Laplace transform of an $\mathbb{R}_+$-valued random variable X. If an explicit expression for $\tilde{\varphi}_X : u \mapsto \tilde{\varphi}_X(u)$ is given, which is analytic[2] on the set $\mathbb{C}_+ := \{w : \mathrm{Re}(w) \geq 0\}$, then the function $\tilde{\varphi}_X$ can be continued on $\mathbb{C}_+$ and it holds that $\mathbb{E}[e^{-w\,X}] = \tilde{\varphi}_X(w), \forall w \in \mathbb{C}_+$. In particular, $\varphi_X(u) = \tilde{\varphi}_X(-\mathrm{i}\,u)$.

 (iv) Let $\tilde{\varphi}_X$ be the Laplace transform of X and let $n \geq 1$. Then, X has a finite n-th moment if and only if $\tilde{\varphi}_X^{(n)}(0)$ exists, where $\tilde{\varphi}_X^{(n)}$ denotes the n-th derivative of $\tilde{\varphi}_X$. Moreover, it holds that

$$\mathbb{E}\,[X^n] = (-1)^n\,\tilde{\varphi}_X^{(n)}(0).$$

Proof

   (i) See, for example, [Klenke, 2007, Theorem 15.8].

  (ii) See, for example, [Sato, 1999, Proposition 2.6].

 (iii) This statement can be shown by using the uniqueness theorem for analytic functions, as it is done, for example, in the proof of [Sato, 1999, Theorem 24.11].

 (iv) See, for example, [Feller, 1971, XIII.2(ii)]. $\qquad\qquad\square$

Example 2.9 (Transforms of some basic distributions)

   (i) Let $G \sim \mathcal{N}\left(\mu,\,\sigma^2\right)$. Then, $\varphi_G(u) = e^{\mathrm{i}\,u\,\mu - \frac{1}{2}\,\sigma^2\,u^2}$. In general, a d-dimensional normal distributed random variable G with mean μ and covariance matrix Σ is given by $\varphi_G(u) = e^{\mathrm{i}\,u^\top\mu - \frac{1}{2}\,u^\top\Sigma\,u^2}$.

  (ii) Let $X \sim \mathrm{Exp}(\eta)$. Then, $\tilde{\varphi}_X(u) = \frac{\eta}{\eta+u}$.

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[2]A function is called analytic on a region C if it is complex differentiable in every point in C.

(iii) Let $N \sim \text{Poi}(c)$. Then, $\tilde{\varphi}_N(u) = \exp\left(c\left(e^{-u} - 1\right)\right)$.

(iv) Let $M \sim \text{Geo}(p)$. Then, $\tilde{\varphi}_M(u) = \frac{p}{1-(1-p)e^{-u}}$.

For a proof, see for example, [Klenke, 2007, Theorem 15.12]. Note that these expressions for the Laplace transforms of X, N, and M are analytic on $\mathbb{C}_+$. Thus, using Theorem 2.8 (iii)), the characteristic functions are given by $\varphi_X(u) = \tilde{\varphi}_X(-\mathrm{i}\,u)$, $\varphi_N(u) = \tilde{\varphi}_N(-\mathrm{i}\,u)$, and $\varphi_M(u) = \tilde{\varphi}_M(-\mathrm{i}\,u)$. As mentioned above, the moments can now be calculated by computing the derivatives of the Laplace transforms and using Theorem 2.8 (iv).

Now, we go over from random variables to stochastic processes, which are families of random variables. Stochastic processes can be used to describe the random behavior of asset prices over time. We assume that the stochastic processes live on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F}$ fulfills the usual hypothesis.[3] A *filtration* is a family of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that is increasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $s \leq t$.

Definition 2.10 (Stochastic process)

(i) A family of random variables $X = \{X_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$, $d \in \mathbb{N}$, is called (d-dimensional) *stochastic process*.

(ii) For every $\omega \in \Omega$, we say that the mapping $[0, \infty) \to \mathbb{R}$, $t \mapsto X_t(\omega)$ is a *path* of X.

(iii) A stochastic process X is called $\mathbb{F}$-*adapted* w.r.t. a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(iv) An $\mathbb{F}$-adapted stochastic process with $E[|X_t|] < \infty$ for all $t \geq 0$ is called $\mathbb{F}$-*martingale* if
$$E[X_t | \mathcal{F}_s] = X_s \text{ for all } 0 \leq s \leq t.$$

In the following we introduce several types of stochastic processes. Firstly, we have a look at the Brownian motion, a diffusion-type process, and at the Poisson process, which is an increasing counting process taking only natural numbers.

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[3]For detailed information on this setup, we refer to Protter [2005].

Definition 2.11 (Brownian motion)

An $\mathbb{F}$-adapted stochastic process $B = (B_t)_{0 \leq t \leq \infty}$ with a.s. continuous paths taking values in $\mathbb{R}^d$ and starting in zero is called a *d-dimensional Brownian motion with correlation matrix* Σ if

(i) for $0 \leq s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$,

(ii) for $0 < s < t$, $B_t - B_s$ is a normally distributed random variable with mean zero and covariance matrix equal to $(t-s)\Sigma$, where $\Sigma \in [-1, 1]^{d \times d}$ denotes a positive semidefinite correlation matrix.

Definition 2.12 (Poisson process)

An $\mathbb{N}_0$-valued stochastic process N fulfilling

(i) $N_0 = 0 \quad \mathbb{P}$-a.s.,

(ii) the paths of N are $\mathbb{P}$-a.s. càdlàg, i.e. right-continuous with existing left limits,

(iii) for any $n \in \mathbb{N}$ and for any $0 = t_0 < t_1 < \cdots < t_n$ the family $\{N_{t_i} - N_{t_{i-1}}\}_{i=1,\ldots,n}$ is independent.

(iv) $N_t - N_s \sim \text{Poi}(c\,(t-s))$, for any $t > s \geq 0$,

is called *Poisson process with intensity* $c \geq 0$.

Definition 2.12 seems a bit technical on first sight. However, the next theorem yields an intuitive characterization of Poisson processes and shows how such a process can be constructed.

Theorem 2.13 (Construction of a Poisson process)

(i) Let $c > 0$ and $E_1, E_2, \ldots$ be a sequence of independent $\text{Exp}(c)$-distributed random variables. Define

$$T_n = \sum_{i=1}^{n} E_i, \quad \forall n \geq 1. \tag{2.1}$$

Then, the stochastic process defined by

$$N_t = \# \{n \geq 1 : T_n \leq t\}, \quad t \geq 0, \tag{2.2}$$

is a Poisson process with intensity c.

(ii) Let N be a Poisson process with intensity c. Then, N fulfills Equation (2.2) with $T_1, T_2, \ldots$ given by Equation (2.1), where $E_1, E_2, \ldots$ is a sequence of independent $\mathrm{Exp}(c)$-distributed random variables.

Proof

See, for example, [Mikosch, 2009, Theorem 2.1.6]. $\qquad\square$

The random variables $T_1, T_2, \ldots$ are called *arrival times* or *jump times* of the Poisson process. Hence, a Poisson process can be seen as a sequence of stochastic arrivals and the waiting time between two consecutive arrival times is exponentially distributed. By mapping independent and identically distributed random variables to each arrival time, we can construct a compound Poisson process, which has, in contrast to a Poisson process, stochastic increments.

Definition 2.14 (Compound Poisson process)

Let N be a Poisson process with intensity c and let $J_1, J_2, \ldots$ be a sequence of independent and identically distributed random variables for some distribution D on $\mathbb{R}^d$. Assume that N and $\{J_i\}_{i\in\mathbb{N}}$ are independent. Then, the stochastic process Z defined by

$$Z_t = \sum_{i=1}^{N_t} J_i, \quad \text{for all } t \geq 0,$$

is called *compound Poisson process with intensity c and jump size distribution D*.
In the special case of $\mathrm{Exp}(\eta)$-distributed random variables $J_1, J_2, \ldots$ for some $\eta > 0$, we use the notation $Z \sim \mathrm{CPP}_{\mathrm{Exp}}(c, \eta)$ or we say Z is a $\mathrm{CPP}_{\mathrm{Exp}}(c, \eta)$.

The sum of N i.i.d. random variables with $N \sim \mathrm{Poi}$ follows a *compound Poisson distribution*. Therefore, a compound Poisson process $N = \{N_t\}_{t\geq 0}$ is compound Poisson distributed for all $t > 0$. Compound Poisson processes are used throughout the thesis to model jumps in asset price processes. In Chapter 3, we present a comprehensive overview of typical fields of applications. Brownian motions, Poisson processes, and also compound Poisson processes belong to a broader class of stochastic processes, the class of so-called Lévy processes.

Definition 2.15 (Lévy process)

An $\mathbb{R}^d$-valued stochastic process X fulfilling

(i) $X_0 = 0 \quad \mathbb{P}$-a.s.,

(ii) the paths of X are $\mathbb{P}$-a.s. càdlàg,

(iii) for any $n \in \mathbb{N}$ and for any $0 = t_0 < t_1 < \cdots < t_n$ the family $\{X_{t_i} - X_{t_{i-1}}\}_{i=1,\ldots,n}$ is independent.

(iv) $N_t - N_s \stackrel{d}{=} N_{t-s}$, for any $t > s \geq 0$,

(v) for all $t \geq 0$ and $\varepsilon > 0$, $\lim\limits_{s \to t} \mathbb{P}\left(||X_t - X_s|| > \varepsilon\right) = 0$,

is called (d-dimensional) *Lévy process*.[4]

Obviously, each compound Poisson process is a Lévy process. The next theorem characterizes this subclass of Lévy processes.

Theorem 2.16 (Lévy processes with piecewise constant paths)

A stochastic process is a compound Poisson process if and only if it is a Lévy process and its paths are piece-wise constant.

Proof

See, for example, [Tankov, 2004, Proposition 3.3]. □

Remark 2.17 (Components of a Lévy process)

(i) Let X be a d-dimensional Lévy process and let M be an $n \times d$ matrix. Then Y, defined by $Y_t = M X_t$ for all $t \geq 0$, is a Lévy process on $\mathbb{R}^n$, cf. [Tankov, 2004, Theorem 4.1]. In particular, each component of a d-dimensional Lévy process is a one-dimensional Lévy process.

(ii) Let $X_1, \ldots, X_d$ be d independent one-dimensional Lévy processes. Then, it is easy to see, that the process $X := (X_1, \ldots, X_d)$ is a d-dimensional Lévy process.

(iii) A d-dimensional Lévy process with each component being a one-dimensional compound Poisson process is a d-dimensional compound Poisson process by Theorem 2.16.

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[4]By $|| \cdot ||$, we denote the Euclidean norm.

Definition 2.18 (Lévy subordinator)

A one-dimensional Lévy process with $\mathbb{P}$-a.s. non-negative paths is called *Lévy subordinator*.

Naturally, every Poisson process is a Lévy subordinator and every compound Poisson process with non-negative jump size distribution is a Lévy subordinator as well. For example, a $\mathrm{CPP}_{\mathrm{Exp}}$ is a Lévy subordinator.

Theorem 2.19 (Characteristic functions and Lévy processes)

Let X be a Lévy process on $\mathbb{R}^d$ and let φ_{X_1} be the characteristic function of X_1. Then, the following statements hold.

(i) The characteristic function of X_t is given by $\varphi_{X_t} = (\varphi_{X_1})^t$ for all $t > 0$. If X is a Lévy subordinator, then the Laplace transform of X_t is given by $\tilde{\varphi}_{X_t} = (\tilde{\varphi}_{X_1})^t$ for all $t > 0$.

(ii) Let $u \in \mathbb{R}^d$. Then the characteristic function of X_1 can uniquely be represented by.

$$\varphi_{X_1}(u) = \exp\left(-\frac{1}{2}\, u^\top A\, u + \mathrm{i}\, \gamma^\top u + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\, u^\top x} - 1 - \mathrm{i}\, u^\top x\, \mathbb{1}_{\{|x| \le 1\}}\right) \nu(\mathrm{d}x)\right),$$

where A is a symmetric non-negative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ fulfilling $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) \nu(\mathrm{d}x) < \infty$, and $\gamma \in \mathbb{R}^d$.

If the additional condition $\int_{|x| \le 1} |x|\, \nu(\mathrm{d}x) < \infty$ holds, then φ_{X_1} can uniquely be represented by

$$\varphi_{X_1}(u) = \exp\left(-\frac{1}{2}\, u^\top A\, u + \mathrm{i}\, \gamma_0^\top u + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\, u^\top x} - 1\right) \nu(\mathrm{d}x)\right), \tag{2.3}$$

where $\gamma_0 \in \mathbb{R}^d$.

(iii) Let $d = 1$. Then, X is a Lévy subordinator, if and only if the unique representation (2.3) satisfies $A = 0$, $\int_{-\infty}^0 \nu(\mathrm{d}x) = 0$, $\int_0^1 x\, \nu(\mathrm{d}x) < \infty$, and $\gamma_0 \ge 0$. Moreover, the Laplace transform of X_1 is given by

$$\tilde{\varphi}_{X_1}(u) = \exp\left(\int_0^\infty \left(e^{-u\, x} - 1\right) \nu(\mathrm{d}x) - \gamma_0\, u\right), \quad \text{for } u \ge 0.$$

Proof

(i) See, for example, [Sato, 1999, Theorem 7.10].

(ii) See, for example, [Sato, 1999, Theorem 8.1].

(iii) See, for example, [Sato, 1999, Theorem 21.5]. □

Theorem 2.19 (i) points out a nice property of Lévy processes. Since the characteristic function of a random variable uniquely determines their distribution, the distribution of X_t for some arbitrary $t > 0$ already defines the law of the whole process. This combined with the fact that the characteristic function is of exponential form, motivates Definition 2.20 of the characteristic exponent, respectively the Laplace exponent in case of Lévy subordinators. The formula for the characteristic function of Lévy processes in Theorem 2.19 (ii) is called *Lévy–Khintchine formula* and the measure ν is called *Lévy measure*. In a nutshell, ν measures the mean amount of jumps. Particularly, for a set $B \subseteq \mathbb{R}$, $\nu(B)$ gives the average number of jumps with jump size in B within a unit time interval. Note that for a compound Poisson processes it holds that $\nu(\mathbb{R}) < \infty$, i.e. in a bounded time interval the number of jumps of a compound Poisson process is a.s. finite. Lévy processes fulfilling that property are called Lévy processes with finite activity. Let us have a closer look at the Lévy–Khintchine formula. The first part $\exp\left(-\frac{1}{2} u^\top A\, u + \mathrm{i}\, \gamma^\top u\right)$ is the characteristic function of a d-dimensional normal distribution. Hence, each Lévy process can be decomposed into a Brownian motion with drift and a pure jump process. A Lévy process is therefore characterized by its so-called *Lévy triplet* $(\gamma,\, A,\, \nu)$. By Theorem 2.19 (iii) we hence know, that Lévy subordinators have no Brownian part, they rather consist only of a deterministic drift and a pure jump process with positive jump sizes.

Definition 2.20 (Characteristic exponent and Laplace exponent)
The *characteristic exponent* ψ_X of a d-dimensional Lévy process X, $d \in \mathbb{N}$, is defined by

$$\psi_X(u) := \log\left(\varphi_{X_1}(u)\right)$$

for any $u \in \mathbb{R}^d$. If X is a Lévy subordinator, then the *Laplace exponent* $\tilde{\psi}_X$ of X is defined by

$$\tilde{\psi}_X(-u) := \log\left(\tilde{\varphi}_{X_1}(u)\right)$$

for any $u \in \mathbb{R}_+$.

Remark 2.21 (On the characteristic exponent and Laplace exponent)
By Theorem 2.8 and Theorem 2.19(i), the distribution of a Lévy process is uniquely determined by its characteristic exponent. Furthermore, the distribution of a Lévy subordinator is uniquely determined by its Laplace exponent.

Similar to the continuation theorem of the Laplace transform (cf. 2.8 (iii)), the characteristic exponent and the Laplace exponent can be continued, which we state in the next theorem.

Theorem 2.22 (Continuation of characteristic and Laplace exponents)
(i) Let X be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Define

$$C := \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_i) \geq 0, \, \forall \, 1 \leq i \leq d \right\}. \qquad (2.4)$$

Furthermore, assume an explicit expression for the characteristic exponent $\psi_X : \mathbb{R}^d \to \mathbb{C}$ to be given, which is analytic on C. Then the function ψ_X can be continued on C and it fulfills $\mathbb{E}\left[e^{\mathrm{i}\, z^\top X_1}\right] = e^{\psi(z)}$ for all $z \in C$.

(ii) Let X be a Lévy subordinator. Assume an explicit expression for the Laplace exponent $\tilde{\psi}_X : \mathbb{R}_- \to [0,1]$ to be given, which is analytic on the set $\mathbb{C}_- := \{z : \operatorname{Re}(z) \leq 0\}$, then the function $\tilde{\psi}_X$ can be continued on $\mathbb{C}_-$ and it holds that $\mathbb{E}[e^{-z\, X_t}] = e^{t\, \tilde{\psi}(-z)}$, for all $z \in \mathbb{C}_+ := \{z : \operatorname{Re}(z) \geq 0\}$.

Proof
(i) See, for example, [Sato, 1999, Theorem 25.17]. They proof an even more general statement for a larger set C. However, in the present thesis it is enough to consider C as in Equation (2.4).

(ii) The second statement then follows from Theorem 2.8(iii). $\qquad\square$

Remark 2.23 (On analytical characteristic exponents)
The expression for the characteristic exponent in Theorem 2.19(ii) is always analytic on C (cf. [Sato, 1999, Theorem 25.17]), and can therefore be continued on C.

The following theorem presents the characteristic exponent of a compound Poisson process, which can easily be computed knowing the characteristic function of the jump size distribution.

Theorem 2.24 (Characteristic exponent of a compound Poisson process)

Let Z be a compound Poisson process with intensity c and jump size distribution D. Then, the characteristic exponent of Z for all $u \in \mathbb{R}^d$ is given by

$$\psi_Z(u) = c\left(\varphi_D(u) - 1\right),$$

where φ_D denotes the characteristic function of a D-distributed random variable. Moreover, if Z is a compound Poisson process with non-negative jump size distribution on $\mathbb{R}_+$, then, the Laplace exponent of Z for all $u \in \mathbb{R}_+$ is given by

$$\tilde{\psi}_Z(-u) = c\left(\tilde{\varphi}_D(u) - 1\right),$$

where $\tilde{\varphi}_D$ denotes the Laplace transform of a D-distributed random variable.

Proof

See for example [Tankov, 2004, Proposition 3.4]. □

The next theorem investigates the resulting process of a subordinated Lévy process, i.e. a time-changed Lévy proces. This theorem plays a prominent role in our thesis, because it is applied to construct dependent compound Poisson processes, which are then used to model dependent jumps in asset price processes.

Theorem 2.25 (Subordination of Lévy processes)

Let Y be a Lévy process with characteristic exponent ψ_Y and let T be a Lévy subordinator with Laplace exponent $\tilde{\psi}_T$. Assume, Y and T to be independent. Then, the process $Z = \{Z_t\}_{t \geq 0} := \{Y_{T_t}\}_{t \geq 0}$ is a Lévy process with characteristic exponent ψ_Z given by

$$\psi_Z(u) = \tilde{\psi}_T\big(\psi_Y(u)\big), \quad u \in \mathbb{R}^d.$$

In particular, if Y is a one-dimensional Lévy subordinator with Laplace exponent $\tilde{\psi}_Y$, then, Z is a Lévy subordinator with Laplace exponent $\tilde{\psi}_Z$ given by

$$\tilde{\psi}_Z(-u) = \tilde{\psi}_T\big(\tilde{\psi}_Y(-u)\big), \quad u \geq 0.$$

Proof

See, for example, [Sato, 1999, Theorem 30.4] and [Sato, 1999, Theorem 30.1]. □

We now present a nice formula on the characteristic function of integrals with respect to Lévy integrators. For a general introduction on stochastic integration we refer to standard textbooks like Applebaum [2004].

Theorem 2.26 (Characteristic exponent of Lévy integrals)
Let $t > 0$, let X be a d-dimensional Lévy process with characteristic exponent ψ_X and let $f : \mathbb{R}_+ \to \mathbb{C}^d$ be a left-continuous function with limits from the right, such that

$$\psi_X(f(s)) \text{ exists with } \mathbb{E}\left[e^{\mathrm{i} f(s)^\top X_1}\right] = e^{\psi_X(f(s))} \text{ for all } 0 \leq s \leq t, \qquad (2.5)$$

$$\text{there exists an } M > 0 \text{ such that } \operatorname{Re}\left(\psi_X(f(s))\right) < M \text{ for all } 0 \leq s \leq t, \qquad (2.6)$$

Then,

$$\mathbb{E}\left[\exp\left(\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s\right)\right] = \exp\left(\int_0^t \psi_X(f(s))\, \mathrm{d}s\right). \qquad (2.7)$$

Proof
The proof we present is a straightforward multivariate adaption of [Eberlein and Raible, 1999, Lemma 3.1]. For any partition $0 = t_0 < \cdots < t_{N+1} = t$ of the interval $[0, t]$ we get by the independence of increments of X and the definition of the characteristic exponent

$$\mathbb{E}\left[\exp\left(\sum_{k=0}^N \mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] = \prod_{k=0}^N \mathbb{E}\left[\exp\left(\mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)\right)\right]$$

$$= \prod_{k=0}^N \exp\left(\psi_X(f(t_k))\left(t_{k+1} - t_k\right)\right)$$

$$= \exp\left(\sum_{k=0}^N \psi_X(f(t_k))\left(t_{k+1} - t_k\right)\right). \qquad (2.8)$$

If the mesh[5] of the partition goes to zero, $\exp\left(\sum_{k=0}^N \psi_X(f(t_k))\left(t_{k+1} - t_k\right)\right)$ converges to $\exp\left(\int_0^t \psi_X(f(s))\, \mathrm{d}s\right)$. Thus, to show Equation (2.7), we have to make sure that

$$\mathbb{E}\left[\exp\left(\sum_{k=0}^N \mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] \quad \text{converges to} \quad \mathbb{E}\left[\exp\left(\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s\right)\right],$$

---

[5]The mesh of a partition is the length of the longest subinterval.

i.e. that

$$\exp\left(\sum_{k=0}^{N} \mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)\right) \quad \text{converges in } L^1 \text{ to} \quad \exp\left(\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s\right).$$

Since f is left continuous and has limits from the right, $\sum_{k=0}^{N} \mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)$ converges in measure[6] to $\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s$ by the construction of stochastic integrals (cf. [Jacod and Shiryaev, 2003, Proposition I.4.44]). Continuous transformations preserve convergence in measure, and hence

$$\exp\left(\sum_{k=0}^{N} \mathrm{i}\, f(t_k)\left(X_{t_{k+1}} - X_{t_k}\right)\right) \text{converges in measure to} \exp\left(\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s\right). \quad (2.9)$$

By Equation (2.8) and Assumption (2.6), the approximating sequence in Equation (2.9) is uniformly integrable.[7] Therefore, convergence in measure implies convergence in L^1, (cf. [Klenke, 2007, Theorem 6.25]) and we get

$$\mathbb{E}\left[\exp\left(\int_0^t \mathrm{i}\, f(s)\, \mathrm{d}X_s\right)\right] = \exp\left(\int_0^t \psi_X(f(s))\, \mathrm{d}s\right). \qquad \square$$

We conclude this preliminary chapter with a brief introduction to copulas. For a general presentation of the copulas theory, we refer to the book by Nelsen [2006].

Definition 2.27 (Copula)
A *d-dimensional copula* C is the distribution function of a random vector $(U_1, \ldots, U_d)$, where $(U_k)_{k=1,\ldots,d}$ are uniformly distributed on $[0,1]$.
If $(X_1, X_2, \ldots, X_d)$ is a random vector with continuous marginal distribution functions $F_k(t) = \mathbb{P}(X_k \leq t)$ for all $= 1, \ldots, d$, then the distribution function of the vector $\left(F_1(X_1), F_2(X_2), \ldots, F_d(X_d)\right)$ is a copula, called the *copula* of $(X_1, X_2, \ldots, X_d)$. For example, the copula of a d-dimensional normal distribution is called *Gaussian copula*.

The fundamental theorem in the copula theory is Sklar's theorem. In particular, it states that every dependence structure can be characterized by a copula.

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[6] A sequence of random variables $X_1, X_2, \ldots$ converges in measure to a random variable X if $\forall \varepsilon > 0$ $\lim_{n \to \infty} \mathbb{P}(|X - X_n| > \varepsilon) = 0$.

[7] A familiy $\mathcal{H}$ of random variables with finite expectation is said to be uniformly integrable if $\lim_{c \to \infty} \sup_{X \in \mathcal{H}} E\left[\mathbb{1}_{\{|X| \geq c\}} |X|\right] = 0$.

Theorem 2.28 (Sklar's Theorem)

Let F be a joint distribution function with continuous margins $F_1, F_2, \ldots, F_d$ of the random vector $(X_1, X_2, \ldots, X_d)$.

Then, there exists a Copula $C : [0,1]^d \to [0,1]$ such that for all $x_1, x_2, \ldots, x_d \in [-\infty, \infty]$,

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)).$$

Proof

See, for example, [McNeil et al., 2005, p.186 (Theorem 5.3)]. $\qquad\square$

Note that Sklar's theorem also holds for discontinuous marginals. A proof can be found, for example, in Schweizer and Sklar [1983]. For continuous margins, however, the copula is even uniquely determined.

3 Compound Poisson processes with exponentially distributed jumps

A compound Poisson process is a stochastic process with piece-wise constant paths, random jump times, and random jump magnitudes. Such a process is build by a sum of independent and identically distributed random variables, where the number of variables to be added up to time t is given by a Poisson process. In the previous chapter on mathematical preliminaries, we already introduced the mathematical definition, and we have seen that compound Poisson processes are Lévy processes. Furthermore, these are the only Lévy processes having a.s. piece-wise constant paths. Compound Poisson processes are therefore commonly used for modeling random arrivals of random amounts. Thus, some typical fields of applications are the modeling of claim sizes and claim arrival times in non-life insurance mathematics, queuing systems (customer arrivals with required service time), or the modeling of jumps in financial market models.[1] In the present thesis, we will concentrate on the latter and have a look at a specific subclass of compound Poisson processes, where the jump magnitudes are exponentially distributed. As defined in the previous chapter, we call such a process CPP_{Exp} (cf. Definition 2.14).

In this chapter we aim at constructing multi-dimensional compound Poisson processes with each component following a CPP_{Exp}. Obviously, a vector of one-dimensional independent CPP_{Exp} is a multi-dimensional compound Poisson process by Remark 2.17. To obtain dependence between the components one needs more sophisticated approaches. In general, dependence between the components can be created by several construction principles. Deelstra and Petkovic [2010] summarize three possibilities to construct multi-dimensional Lévy processes out of one-dimensional independent Lévy processes:

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[1] For a general introduction to these typical usages of compound Poisson processes, we refer to Mikosch [2009] for insurance, to Baccelli and Bremaud [2003] for queuing theory, and Cont and Tankov [2004] for financial modeling.

(i) Linear combination: Let $n, d \in \mathbb{N}$, M be a $d \times n$ matrix and let X be a vector of n independent one-dimensional Lévy processes. Then, $Z_t := M X_t$ for all $t \geq 0$ is a d-dimensional Lévy process by Remark 2.17.

(ii) Joint time change: Let $d \in \mathbb{N}$, let Y be a Lévy subordinator, and let X be a vector of d independent one-dimensional Lévy processes. Assume X and Y to be independent. Then, $Z_t := X_{Y_t}$ for all $t \geq 0$ is a d-dimensional Lévy process by Theorem 2.25.[2]

(iii) Linking the Lévy measures by a Lévy copula.[3]

Obviously, other constructions are possible to create a dependent multi-dimensional Lévy process (as, e.g., a direct construction from a multi-dimensional infinitely divisible law[4]), but the above construction principles provide flexible instruments, where one starts with independent Lévy processes and ends up with dependent ones. These construction principles applied to compound Poisson processes may be possible choices to introduce dependence between $\mathrm{CPP}_{\mathrm{Exp}}$. Let us have a closer look on each approach. A linear combination of independent compound Poisson processes yields a multi-dimensional compound Poisson process. The jump size distributions of the processes $X^{(1)}, \ldots, X^{(n)}$ have to be chosen in a bespoke way to guarantee that each component follows a $\mathrm{CPP}_{\mathrm{Exp}}$. When it comes to the dependence between jump sizes at joint jump times of some components, the construction by linear combination of independent processes does not provide a flexible dependence structure. Indeed, joint jumps of at least two components are triggered by the same process $X^{(i)}$ for some $1 \leq i \leq n$, and jump sizes are therefore only multiples of the jump size of $X^{(i)}$. Thus, one shortcoming of construction by linear combination is that the jump sizes at joint jumps are always comonotone, which is only one reason for us not considering this approach any further. Another reason is, for example, the large number of parameters one needs to introduce dependence (n times d).

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[2]More generally, a d-dimensional Lévy process Y can be used as time-change process. Each component of Y must be a subordinator, and serves as individual time change process for one component of X. For example, Semeraro [2008] introduced dependence via a multivariate time change process.

[3]The concept of Lévy copulas support a linking of Lévy measures instead of probability measures. As Lévy copulas do not play an important role in the present thesis, we omit their mathematical introduction and recommend Tankov [2004] for more details.

[4]A probability distribution D is infinitely divisible if, for every $n \in \mathbb{N}$, there exist n i.i.d. random variables whose sum follows distribution D. There is a vivid 1-to-1 mapping between those distributions and Lévy processes, see for example [Sato, 1999, Ch. 2]

Constructing a multi-dimensional compound Poisson process by a joint time change with desired marginals is not an obvious task. On first sight, it is not clear, how to choose the processes $X^{(1)}, \ldots, X^{(n)}$ and the subordinator Y. Nevertheless, we present a way to construct such a multi-dimensional process via subordination of compound Poisson processes in Section 3.2. This construction will be the main tool in Chapter 4 and Chapter 5. Our approach has some striking features we discuss later on within this chapter, and we point out why we choose the time change construction principle to introduce dependence to various univariate CPP_{Exp}.

Theoretically, linking the Lévy measures of each component by a Lévy copula appears to be the most elegant and flexible way to introduce dependence, because, based on Sklar's theorem for Lévy copulas, cf. [Kallsen and Tankov, 2006, Theorem 3.6], every joint Lévy measure of the multi-dimensional process can be generated by means of that ansatz. However, Lévy copulas can be cumbersome objects when it comes to practical applications. For example, simulation of Lévy copulas might be quite involved. In contrast to general Lévy processes, the dependence structure of multivariate compound Poisson processes can also be described by usual copulas instead of Lévy copulas. This is achieved by linking the jump size distributions in case of joint defaults. More precisely, every single component of the multivariate process can be separated into individual jumps and common jumps with other components. Jump sizes at common jump times are linked by a copula. Each part of that decomposition is again a compound Poisson process. Hence, for compound Poisson processes, it might be more convenient to work with classical copulas instead of Lévy copulas. Moreover, this separation leads to a very intuitive representation of a multi-dimensional compound Poisson process, especially in low dimensions. In Section 3.5 we examine that pragmatic characterization of our time change construction, which is introduced in the sequel.

Before discussing the multivariate process we first study some distributional properties of a CPP_{Exp} in Section 3.1. Section 3.2 presents the time change construction of multi-dimensional compound Poisson processes with CPP_{Exp} in each component. In Section 3.3, we investigate the implied dependence structure between the resulting processes. Afterwards, we characterize the time-change construction as multidimensional compound Poisson process in Section 3.4. The separation into individual and common jump parts, as discussed above, is examined in Section 3.5. The features making this construction a useful tool for tractable multivariate modeling of financial markets are highlighted in Section 3.6, which concludes the chapter by an outlook on

possible applications. This chapter contains some results, which have already been published in Mai et al. [2014], prior to this thesis.

3.1 Fundamentals: The univariate process

Each component of our multivariate construction shall be a CPP_{Exp} to remain with the univariate case. In this section, we calculate the Laplace exponent of a CPP_{Exp}, as well as the first moments. The results will be used throughout the whole chapter. The first subgraph in Figure 3.1 shows typical CPP_{Exp}-paths.

Theorem 3.1 (Laplace exponent of a CPP_{Exp})
The Laplace exponent of $Z \sim \text{CPP}_{\text{Exp}}(c, \eta)$ is given by

$$\tilde{\psi}_Z(-u) = \frac{-c\,u}{\eta + u}, \quad u \geq 0. \tag{3.1}$$

Proof
The claim follows directly by applying Theorem 2.24 and Example 2.9. Since Equation (3.1) is a crucial formula in this chapter and is often used throughout the whole thesis, we provide a detailed calculation in the following. Let $u \geq 0$, then the Laplace transform of Z_1 is given by

$$\begin{aligned}
\tilde{\varphi}_{Z_1}(u) = \mathbb{E}\left[e^{-u\,Z_1}\right] &= \mathbb{E}\left[\exp\left(-u\sum_{i=1}^{N_1} J_i\right)\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\exp\left(-u\sum_{i=1}^{N_1} J_i\right)\bigg| N_1 = n\right]\mathbb{P}(N_1 = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\prod_{i=1}^{n} e^{-u\,J_i}\right]\mathbb{P}(N_1 = n).
\end{aligned}$$

$J_1, J_2, \ldots$ is a sequence of independent random variables. Hence,

$$\tilde{\varphi}_{Z_1}(u) = \sum_{n=0}^{\infty}\prod_{i=1}^{n} \mathbb{E}\left[e^{-u\,J_i}\right]\mathbb{P}(N_1 = n) = \sum_{n=0}^{\infty}\prod_{i=1}^{n} \tilde{\varphi}_{J_i}(u)\,\mathbb{P}(N_1 = n).$$

Using that J_1, J_2, ... are identically $\text{Exp}(\eta)$-distributed and knowing that the Laplace transform of an $\text{Exp}(\eta)$-distributed random variable is given by $\tilde{\varphi}_{J_1}(u) = \frac{\eta}{\eta + u}$ (cf. Example 2.9), we get

$$\tilde{\varphi}_{Z_1}(u) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{\eta}{\eta + u} \, \mathbb{P}(N_1 = n).$$

Note, that $N_1 \sim \text{Poi}(c)$. Therefore,

$$\tilde{\varphi}_{Z_1}(u) = \sum_{n=0}^{\infty} \left(\frac{\eta}{\eta + u} \right)^n e^{-c} \frac{c^n}{n!} = e^{-c} \sum_{n=0}^{\infty} \left(\frac{c\,\eta}{\eta + u} \right)^n \frac{1}{n!},$$

and the power series representation of the exponential function yields

$$\tilde{\varphi}_{Z_1}(u) = e^{-c} e^{\frac{c\,\eta}{\eta + u}} = e^{-\frac{c\,u}{\eta + u}}.$$

Finally, the Laplace exponent of Z is given by

$$\tilde{\psi}_Z(-u) = \log\left(\tilde{\varphi}_{Z_1}(u) \right) = \frac{-c\,u}{\eta + u}. \qquad \square$$

According to Equation (3.1) the Laplace transform of a CPP_{Exp} has a simple representation. In Remark 2.21, we have seen that the Laplace exponent uniquely determines the distribution of a Lévy subordinator, i.e. any Lévy subordinator having a Laplace exponent like in Equation (3.1) is a CPP_{Exp}. By differentiation of the Laplace transform, the first moments are obtained in the next theorem. Here, it is helpful that the Laplace exponent takes such a simple form.

Theorem 3.2 (Moments of a CPP_{Exp})
Let $Z \sim \text{CPP}_{\text{Exp}}(c, \eta)$. Then, it holds that

$$\mathbb{E}[Z_t] = \frac{c\,t}{\eta}, \qquad \mathbb{E}[Z_t^2] = \frac{2\,c\,t + c^2\,t^2}{\eta^2}, \qquad \text{Var}[Z_t] = \frac{2\,c\,t}{\eta^2}.$$

Proof
From Theorem 2.8(iv), we know that

$$\mathbb{E}[Z_t] = -\tilde{\varphi}_{Z_t}^{'}(0) \qquad \text{and} \qquad \mathbb{E}[Z_t^2] = \tilde{\varphi}_{Z_t}^{''}(0), \tag{3.2}$$

where $\tilde{\varphi}_{Z_t}$ denotes the Laplace transform of Z_t. Let $\tilde{\psi}_Z$ be the Laplace exponent of Z and let $u \in \mathbb{R}_+$. Then, Theorem 2.19(i) and Theorem 3.1 yield

$$\tilde{\varphi}_{Z_t}(u) = (\tilde{\varphi}_{Z_1}(u))^t = \left(\exp\left(\tilde{\psi}_Z(-u) \right) \right)^t = \exp\left(\frac{-c\,u\,t}{\eta + u} \right).$$

Hence,

$$\tilde{\varphi}'_{Z_t}(u) = \frac{-c\,t\,\eta}{(\eta + u)^2} \exp\left(\frac{-c\,u\,t}{\eta + u} \right),$$

and

$$\begin{aligned}
\tilde{\varphi}''_{Z_t}(u) &= \frac{2\,c\,t\,\eta}{(\eta + u)^3} \exp\left(\frac{-c\,u\,t}{\eta + u} \right) + \frac{c^2\,t^2\,\eta^2}{(\eta + u)^4} \exp\left(\frac{-c\,u\,t}{\eta + u} \right) \\
&= \frac{2\,c\,t\,\eta\,(\eta + u) + c^2\,t^2\,\eta^2}{(\eta + u)^4} \exp\left(\frac{-c\,u\,t}{\eta + u} \right).
\end{aligned}$$

Finally, Equation (3.2) gives

$$\mathbb{E}[Z_t] = -\tilde{\varphi}'_{Z_t}(0) = \frac{c\,t}{\eta} \qquad \text{and} \qquad \mathbb{E}[Z_t^2] = \tilde{\varphi}''_{Z_t}(0) = \frac{2\,c\,t + c^2\,t^2}{\eta^2}.$$

Moreover,

$$\mathrm{Var}[Z_t] = \mathbb{E}[Z_t^2] - \mathbb{E}[Z_t]^2 = \frac{2\,c\,t}{\eta^2}. \qquad \square$$

The first moments can alternatively be calculated without knowing the Laplace transform by conditioning of the realization on the number of jumps, as it is usually done in textbooks. More precisely, Wald's formula (cf. [Klenke, 2007, Theorem 5.5]) implies that for square integrable i.i.d. random variables $J_1, J_2, \dots$ and an independent square integrable random variable $N \in \mathbb{N}_0$ we have

$$\mathbb{E}\left[\sum_{k=1}^N J_k \right] = \mathbb{E}[N]\,\mathbb{E}[J_1] \quad \text{and} \quad \mathrm{Var}\left[\sum_{k=1}^N J_k \right] = \mathbb{E}[N]\,\mathrm{Var}[J_1] + \mathrm{Var}[N]\,\left(\mathbb{E}[J_1] \right)^2. \tag{3.3}$$

Knowing the moments of the basic distributions (cf. Theorem 2.6), the claim follows directly.

3.2 Construction of a multi-dimensional process

This section presents the idea how one can construct a multi-dimensional compound Poisson process with each component being a CPP_{Exp}. As already mentioned, this is achieved by a bespoke time change and it is based on subordination of a CPP_{Exp} by another independent CPP_{Exp}. The result is again a CPP_{Exp}, as it is shown in the next theorem.

Theorem 3.3 (Subordination of two independent CPP_{Exp})
Suppose one is given $c_T, \eta_T, c_Y, \eta_Y > 0$. Let $Y = \{Y_t\}_{t \geq 0}$ be a $\text{CPP}_{\text{Exp}}(c_Y, \eta_Y)$ and let $T = \{T_t\}_{t \geq 0}$ be a $\text{CPP}_{\text{Exp}}(c_T, \eta_T)$. Assume that Y and T are independent.
Then, the process $Z = \{Z_t\}_{t \geq 0} := \{Y_{T_t}\}_{t \geq 0}$ is a $\text{CPP}_{\text{Exp}}\left(\frac{c_T\, c_Y}{\eta_T + c_Y}, \frac{\eta_T\, \eta_Y}{\eta_T + c_Y}\right)$.

Proof
The Laplace exponent of a CPP_{Exp} is computed in Theorem 3.1. Hence, for all $u \geq 0$,

$$\tilde{\psi}_Y(-u) = \frac{-c_Y\, u}{\eta_Y + u}, \quad \tilde{\psi}_T(-u) = \frac{-c_T\, u}{\eta_T + u}.$$

By Theorem 2.25, the process Z is again a Lévy subordinator with Laplace exponent

$$\tilde{\psi}_Z(-u) = \tilde{\psi}_T\big(\tilde{\psi}_Y(-u)\big) = \frac{-c_T\, \frac{c_Y\, u}{\eta_Y + u}}{\eta_T + \frac{c_Y\, u}{\eta_Y + u}} = \frac{-c_T\, c_Y\, u}{\eta_T\,(\eta_Y + u) + c_Y\, u} = \frac{-\frac{c_T\, c_Y}{\eta_T + c_Y}\, u}{\frac{\eta_T\, \eta_Y}{\eta_T + c_Y} + u}. \quad (3.4)$$

The Laplace exponent uniquely determines the distribution of a Lévy subordinator (cf. Remark 2.21). Therefore, Theorem 3.1 and Equation (3.4) yield that the process Z must be a $\text{CPP}_{\text{Exp}}\left(\frac{c_T\, c_Y}{\eta_T + c_Y}, \frac{\eta_T\, \eta_Y}{\eta_T + c_Y}\right)$. $\qquad\qquad\square$

Note, that a process generated by subordination of two Lévy subordinator of the same kind is in general not of that kind anymore. To the best of our knowledge, compound Poisson processes with exponentially distributed jump sizes is the only class of Lévy subordinators where this property is indeed fulfilled. The next corollary tells us how one has to choose the parameters c_Y and η_Y to end up with a process $Z \sim \text{CPP}_{\text{Exp}}(c_Z, \eta_Z)$ with bespoke c_Z and η_Z for fixed parameters c_T and η_T. This is the basis for our multivariate construction in Definition 3.5.

Corollary 3.4 (Generating a CPP$_{\text{Exp}}$ by subordination)

Suppose one is given $c_T, \eta_T, \eta_Z > 0$, and $c_Z \in (0, c_T)$. Let $Y = \{Y_t\}_{t \geq 0}$ be a CPP$_{\text{Exp}}\left(\frac{c_Z \eta_T}{c_T - c_Z}, \frac{c_T \eta_Z}{c_T - c_Z}\right)$ and let $T = \{T_t\}_{t \geq 0}$ be a CPP$_{\text{Exp}}(c_T, \eta_T)$. Assume that Y and T are independent.

Then, the process $Z = \{Z_t\}_{t \geq 0} := \{Y_{T_t}\}_{t \geq 0}$ is a CPP$_{\text{Exp}}(c_Z, \eta_Z)$.

Proof

Define $c_Y := \frac{c_Z \eta_T}{c_T - c_Z}$ and $\eta_Y := \frac{c_T \eta_Z}{c_T - c_Z}$.

Then, Theorem 3.3 implies that $Z \sim \text{CPP}_{\text{Exp}}\left(\frac{c_T c_Y}{\eta_T + c_Y}, \frac{\eta_T \eta_Y}{\eta_T + c_Y}\right)$. Hence,

$$\frac{c_T c_Y}{\eta_T + c_Y} = \frac{c_T \frac{c_Z \eta_T}{c_T - c_Z}}{\eta_T + \frac{c_Z \eta_T}{c_T - c_Z}} = \frac{c_T c_Z \eta_T}{\eta_T (c_T - c_Z) + c_Z \eta_T} = c_Z,$$

and

$$\frac{\eta_T \eta_Y}{\eta_T + c_Y} = \frac{\eta_T \frac{c_T \eta_Z}{c_T - c_Z}}{\eta_T + \frac{c_Z \eta_T}{c_T - c_Z}} = \frac{\eta_T c_T \eta_Z}{\eta_T (c_T - c_Z) + c_Z \eta_T} = \eta_Z,$$

concludes the proof. $\qquad\square$

By means of Corollary 3.4 we can introduce dependence to d CPP$_{\text{Exp}}$. Starting with d independent processes, they can be linked by a joint subordination. We call the resulting process a time-changed CPP$_{\text{Exp}}$ and give a mathematical definition.

Definition 3.5 (Time-changed CPP$_{\text{Exp}}$)

Assume, $c_T, \eta_1, \eta_2, \ldots, \eta_d > 0$ and $c_1, c_2, \ldots, c_d \in (0, c_T)$. Let $d + 1$ mutually independent compound Poisson processes be given by $Y^{(i)} \sim \text{CPP}_{\text{Exp}}\left(\frac{c_i}{c_T - c_i}, \frac{c_T \eta_i}{c_T - c_i}\right)$, for all $1 \leq i \leq d$, and $T \sim \text{CPP}_{\text{Exp}}(c_T, 1)$.

Then, we call the d-dimensional processes Z defined by

$$Z = \left\{\left(Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(d)}\right)\right\}_{t \geq 0} := \left\{\left(Y_{T_t}^{(1)}, Y_{T_t}^{(2)}, \ldots, Y_{T_t}^{(d)}\right)\right\}_{t \geq 0}$$

d-dimensional time-changed CPP$_{\text{Exp}}$ with intensities $c_1, c_2, \ldots, c_d$, jump size parameter $\eta_1, \eta_2, \ldots, \eta_d$, and dependence parameter κ, where $\kappa := \frac{1}{c_T} \max_{1 \leq i \leq d} \{c_i\} \in (0, 1)$.
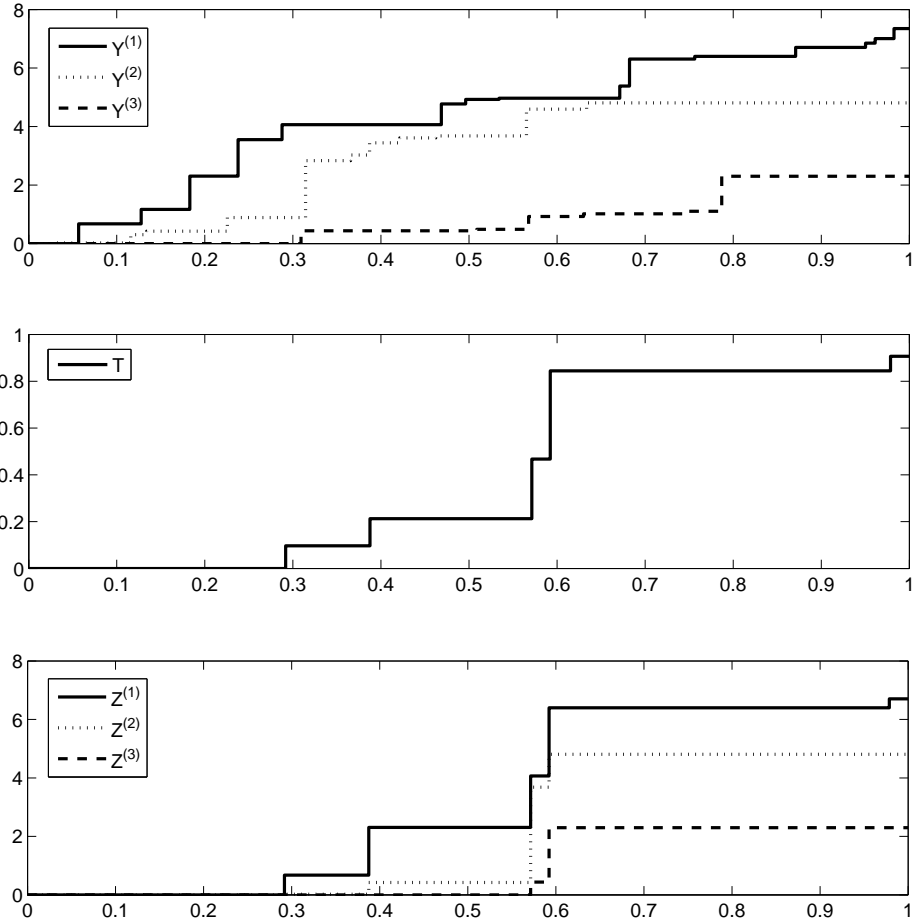
Figure 3.1 The above graph shows paths of three independent compound Poisson processes. The middle graph shows the subordinator T, which serves as joint time transformation. The graph below gives the resulting processes in our construction.

Remark 3.6 (On the construction of time-changed $\mathrm{CPP_{Exp}}$)

(i) A d-dimensional time-changed $\mathrm{CPP_{Exp}}$ Z is a d-dimensional Lévy process, since the subordination of the d-dimensional Lévy process $\left(Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(d)}\right)$ is by Theorem 2.25 again a Lévy process. By Corollary 3.4, every component of Z is a $\mathrm{CPP_{Exp}}$ with $Z^{(i)} \sim \mathrm{CPP_{Exp}}(c_i, \eta_i)$, for all $1 \leq i \leq d$. Hence, a d-

dimensional time-changed $\mathrm{CPP_{Exp}}$ is a d-dimensional compound Poisson process (cf. Remark 2.17(iii)).

(ii) Obviously, the components of a d-dimensional time-changed $\mathrm{CPP_{Exp}}$ are dependent, the dependence is driven by the time-change process T, which is solely determined by the intensity c_T, respectively by the dependence parameter κ. Therefore, we call κ dependence parameter.

(iii) In Corollary 3.4, the time-change process T has $\mathrm{Exp}(\eta_T)$-distributed jumps. In Definition 3.5, however, we fixed the jump size parameter to 1. On first sight, this seems a bit arbitrary. Actually, every other choice for η_T instead of 1 would lead to the same dependence structure between the components of the d-dimensional time-changed $\mathrm{CPP_{Exp}}$. This is proven in Theorem 3.7, where we calculate the characteristic exponent of Z.

Dependence to d compound Poisson processes $Z^{(1)}, \ldots, Z^{(d)}$ with intensities $c_1, \ldots, c_d$ and jump size distributions $\mathrm{Exp}(\eta_1), \ldots, \mathrm{Exp}(\eta_d)$ is introduced by taking the same underlying process T as joint time transformation. We have seen that by choosing the parameters of the original processes $Y^{(1)}, \ldots, Y^{(d)}$ in the right way, the processes $Z^{(1)}, \ldots, Z^{(d)}$ are dependent and have the desired parameters. In the next section, we investigate this construction with specific focus on the dependence structure implied by this construction. Figure 3.1 illustrates the construction of a time-changed $\mathrm{CPP_{Exp}}$ by showing typical sample path of a three-dimensional construction. We observe that they often jump at the same point in time. Moreover, the jump magnitudes at such events are dependent. We see that the dependence structure is quite flexible in the sense that single jumps of just one component, joint jumps of two components, and joint jumps of all components are possible.

3.3 Implied dependence structure

In this section, we further investigate the distributional properties of our multivariate construction of compound Poisson processes with focus on the implied dependence structure. Since we know that a d-dimensional time-changed $\mathrm{CPP_{Exp}}$ is a d-dimensional compound Poisson process, the joint distribution of a time-changed $\mathrm{CPP_{Exp}}$ is determined by its characteristic exponent, which is calculated in the next

theorem. As mentioned in Remark 3.6, we allow the subordinator T to follow a CPP_{Exp} with arbitrarily chosen average jump size. We will see that the characteristic exponent, and therefore the implied dependence structure, does not depend on the actual choice of the average jump size of T. This justifies the choice we made in Definition 3.5 by fixing the average jump size to 1.

Theorem 3.7 (Characteristic exponent of a time-changed CPP_{Exp})

Let a d-dimensional time-changed CPP_{Exp} $Z = \left(Z^{(1)}, Z^{(2)}, \ldots, Z^{(d)}\right)$ be given as defined in Definition 3.5 with the exception that $T \sim \text{CPP}_{\text{Exp}}(c_T, \eta_T)$ rather than $T \sim \text{CPP}_{\text{Exp}}(c_T, 1)$ for some $\eta_T > 0$. Then, the characteristic exponent of Z is given by

$$\psi_Z(u) = \frac{\sum_{k=1}^{d} \frac{\mathrm{i}\, c_T\, c_k\, u_k}{c_T\, \eta_k - \mathrm{i}\, u_k\, (c_T - c_k)}}{1 - \sum_{k=1}^{d} \frac{\mathrm{i}\, c_k\, u_k}{c_T\, \eta_k - \mathrm{i}\, u_k\, (c_T - c_k)}}. \tag{3.5}$$

In particular, for every $t \geq 0$ the joint distribution of $\left(Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(d)}\right)$ is not affected by the choice of the parameter η_T.

Proof

The claim is shown by a multivariate adaption of the proof of Theorem 3.3. Let $u = (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d$. By Theorem 2.25 we know that

$$\psi_Z(u) = \tilde{\psi}_T\big(\psi_Y(u)\big) \tag{3.6}$$

We thus start with the calculation of the characteristic exponent of Y.

$$\psi_Y(u) = \log\left(\mathbb{E}\left[e^{\mathrm{i}\, u^\top Y_1}\right]\right) = \log\left(\mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i}\, u_k\, Y_1^{(k)}}\right]\right).$$

Since $Y^{(1)}, \ldots, Y^{(d)}$ are independent, we get

$$\psi_Y(u) = \log\left(\prod_{k=1}^{d} \mathbb{E}\left[e^{\mathrm{i}\, u_k\, Y_1^{(k)}}\right]\right) = \sum_{k=1}^{d} \log\left(\mathbb{E}\left[e^{\mathrm{i}\, u_k\, Y_1^{(k)}}\right]\right) = \sum_{k=1}^{d} \psi_{Y^{(k)}}(u_k) \tag{3.7}$$

For all $1 \leq k \leq d$, $u \geq 0$, the Laplace exponent of $Y^{(k)}$ is given by

$$\tilde{\psi}_{Y^{(k)}}(-u) = \frac{-\frac{c_k\, \eta_T}{c_T - c_k}\, u}{\frac{c_T\, \eta_k}{c_T - c_k} + u}, \tag{3.8}$$

which was computed in Theorem 3.1. Since the right side in Equation (3.8) is analytic on $\mathbb{C}_-$, Theorem 2.22 yields an expression for the characteristic exponent of $Y^{(k)}$, namely

$$\psi_{Y^{(k)}}(u_k) = \tilde{\psi}_{Y^{(k)}}(\mathrm{i}\,u_k) = \frac{\frac{c_k\,\eta_T}{c_T-c_k}\,\mathrm{i}\,u_k}{\frac{c_T\,\eta_k}{c_T-c_k} - i\,u_k}.$$

Plugging this into Equation (3.7) gives

$$\psi_Y(u) = \sum_{k=1}^d \frac{\frac{c_k\,\eta_T}{c_T-c_k}\,\mathrm{i}\,u_k}{\frac{c_T\,\eta_k}{c_T-c_k} - \mathrm{i}\,u_k} = \sum_{k=1}^d \frac{\mathrm{i}\,c_k\,\eta_T\,u_k}{c_T\,\eta_k - \mathrm{i}\,u_k\,(c_T - c_k)}. \tag{3.9}$$

Again, by using Theorem 3.1, we know that $\tilde{\psi}_T(-u) = \frac{-c_T\,u}{\eta_T+u}$, and by Theorem 2.22 it is allowed to combine Equations (3.6) and (3.9), which yields

$$\psi_Z(u) = \frac{c_T\,\sum_{k=1}^d \frac{\mathrm{i}\,c_k\,\eta_T\,u_k}{c_T\,\eta_k - \mathrm{i}\,u_k\,(c_T - c_k)}}{\eta_T - \sum_{k=1}^d \frac{\mathrm{i}\,c_k\,\eta_T\,u_k}{c_T\,\eta_k - \mathrm{i}\,u_k\,(c_T - c_k)}} = \frac{\sum_{k=1}^d \frac{\mathrm{i}\,c_T\,c_k\,u_k}{c_T\,\eta_k - \mathrm{i}\,u_k\,(c_T - c_k)}}{1 - \sum_{k=1}^d \frac{\mathrm{i}\,c_k\,u_k}{c_T\,\eta_k - \mathrm{i}\,u_k\,(c_T - c_k)}}. \qquad \square$$

The characteristic exponent is indeed independent of the choice of η_T. Hence, any choice leads to the same dependence structure between the components. We fixed it therefore in the first place.

Remark 3.8 (On the characteristic exponent of a time-changed $\mathrm{CPP_{Exp}}$)
Let $z = (z_1, \ldots, z_d) \in C := \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_i) \geq 0, \forall\, 1 \leq i \leq d \right\}$. Since $c_T > c_k > 0$, we get

$$\mathrm{Re}\left(\frac{\mathrm{i}\,c_k\,z_k}{c_T\,\eta_k - \mathrm{i}\,z_k\,(c_T - c_k)} \right) = \frac{-z_k^2\,c_k\,(c_T - c_k) - \mathrm{Im}(z_k)\,c_T\,\eta_k\,c_k}{c_T^2\,\eta_k^2 + z_k^2\,(c_T - c_k)^2} \leq 0, \quad \forall\, 1 \leq k \leq d. \tag{3.10}$$

Moreover, note that for a complex number x with negative real part, it holds that

$$\mathrm{Re}\left(\frac{x}{1-x} \right) = \mathrm{Re}\left(\frac{1}{1 - \mathrm{Re}(x) - \mathrm{i}\,\mathrm{Im}(x)} \right) - 1 = \frac{1 - \mathrm{Re}(x)}{(1 - \mathrm{Re}(x))^2 + (\mathrm{Im}(x))^2} - 1$$

$$\leq \frac{1}{1 - \mathrm{Re}(x)} - 1 \leq 0.$$

Hence, by setting $x := \sum_{k=1}^d \frac{\mathrm{i}\,c_k\,z_k}{c_T\,\eta_k - \mathrm{i}\,z_k\,(c_T - c_k)}$ and using Equations (3.10) and (3.5), we get $\mathrm{Re}\left(\psi_Z(z) \right) < 0$ for all $z \in C$.

Thus, the expression for the characteristic exponent in Equation (3.5) is analytic on C and its real part is negative. Theorem 2.22 therefore ensures that the characteristic exponent of a time-changed $\mathrm{CPP_{Exp}}$ can be continued on C.

Although the characteristic exponent uniquely determines the dependence structure, it does not provide any intuition about the dependence. In the next theorem we therefore calculate the correlation coefficient between two arbitrarily chosen components within our construction to get a better understanding of the implied dependence.

Theorem 3.9 (Correlation between time-changed CPP$_{\mathbf{Exp}}$)

Let a d-dimensional time-changed CPP$_{\text{Exp}}$ $Z = \left(Z^{(1)}, Z^{(2)}, \ldots, Z^{(d)} \right)$ be given as defined in Definition 3.5. For arbitrary $1 \leq i, j \leq d$, $i \neq j$, the correlation of $Z_t^{(i)}$ and $Z_t^{(j)}$ is given by

$$\text{Corr} \left[Z_t^{(i)}, Z_t^{(j)} \right] = \kappa \, \frac{\sqrt{c_i \, c_j}}{c_{max}},$$

where $c_{max} := \max\limits_{1 \leq i \leq d} \{c_i\}$.

Proof

Let us recall the notation of $Z_t^{(k)}$, for all $1 \leq k \leq d$,

$$Z_t^{(k)} = Y_{T_t}^{(k)} = \sum_{l=1}^{N_{T_t}^{(k)}} J_l^{(k)},$$

where $T \sim \text{CPP}_{\text{Exp}}(c_T, 1)$, $N^{(k)}$ is a Poisson process with intensity $\frac{c_k}{c_T - c_k}$ for all $1 \leq k \leq d$, and $J_1^{(k)}, J_2^{(k)}, \ldots$ are $\text{Exp}(\frac{c_T \, \eta_k}{c_T - c_k})$-distributed random variables for all $1 \leq k \leq d$. All quantities are mutually independent.

By conditioning on T_t, the random variables $Z_t^{(i)}$ and $Z_t^{(j)}$ become independent. Hence,

$$\mathbb{E} \left[Z_t^{(i)} \, Z_t^{(j)} \right] = \mathbb{E} \left[\mathbb{E} \left[Z_t^{(i)} \, Z_t^{(j)} \, \Big| \, T_t \right] \right] = \mathbb{E} \left[\mathbb{E} \left[Z_t^{(i)} \, \Big| \, T_t \right] \mathbb{E} \left[Z_t^{(j)} \, \Big| \, T_t \right] \right]. \tag{3.11}$$

By Wald's formula (see Equality (3.3)), we get

$$\mathbb{E} \left[Z_t^{(i)} \, \Big| \, T_t \right] = \mathbb{E} \left[\sum_{l=1}^{N_{T_t}^{(i)}} J_l^{(i)} \, \Bigg| \, T_t \right] = \mathbb{E} \left[N_{T_t}^{(i)} \, \Big| \, T_t \right] \mathbb{E} \left[J_1^{(i)} \, \Big| \, T_t \right]$$

$$= \mathbb{E} \left[N_{T_t}^{(i)} \, \Big| \, T_t \right] \mathbb{E} \left[J_1^{(i)} \, \Big| \, T_t \right] = \frac{T_t \, c_i}{c_T - c_i} \, \frac{c_T - c_i}{c_T \, \eta_i} = \frac{T_t \, c_i}{c_T \, \eta_i}, \tag{3.12}$$

where we used, that the expectation of an $\mathrm{Exp}(\eta)$-distributed random variable is given by $\frac{1}{\eta}$ and the expectation of a $\mathrm{Poi}(c)$-distributed random variable by c. Analogous calculations yield

$$\mathbb{E}\left[Z_t^{(j)} \,\middle|\, T_t \right] = \frac{T_t\, c_j}{c_T\, \eta_j}. \tag{3.13}$$

By combining Equations (3.11), (3.12), and (3.13), we get

$$\mathbb{E}\left[Z_t^{(i)}\, Z_t^{(j)} \right] = \frac{c_i\, c_j}{\eta_i\, \eta_j\, c_T^2} \,\mathbb{E}\left[T_t^2 \right] = \frac{c_i\, c_j}{\eta_i\, \eta_j\, c_T^2}\, \mathbb{E}\left[T_t^2 \right] = \frac{c_i\, c_j}{\eta_i\, \eta_j} \left(\frac{2\, t}{c_T} + t^2 \right), \tag{3.14}$$

where we used, that $\mathbb{E}\left[T_t^2 \right] = 2\, t\, c_T + c_T\, t^2$ as computed in Theorem 3.2. By means of Equation (3.14) and Theorem 3.2 we can calculate the correlation between $Z_t^{(i)}$ and $Z_t^{(j)}$, namely

$$\begin{aligned}
\mathrm{Corr}\left[Z_t^{(i)}, Z_t^{(j)} \right] &= \frac{\mathbb{E}\left[Z_t^{(i)}\, Z_t^{(j)} \right] - \mathbb{E}\left[Z_t^{(i)} \right]\, \mathbb{E}\left[Z_t^{(j)} \right]}{\sqrt{\mathrm{Var}\left[Z_t^{(i)} \right]\, \mathrm{Var}\left[Z_t^{(j)} \right]}} = \frac{\frac{c_i\, c_j}{\eta_i\, \eta_j}\left(\frac{2\, t}{c_T} + t^2 \right) - \frac{c_i\, t}{\eta_i}\, \frac{c_j\, t}{\eta_j}}{\sqrt{\frac{2\, c_i\, t}{\eta_i^2}\, \frac{2\, c_j\, t}{\eta_j^2}}} \\
&= \frac{\sqrt{c_i\, c_j}}{c_T} = \kappa\, \frac{\sqrt{c_i\, c_j}}{c_{max}}, \qquad\qquad \square
\end{aligned}$$

Remark 3.10 (On the correlation between time-changed $\mathbf{CPP_{Exp}}$)

The correlation parameter κ determines the level of correlation and linearly interpolates between the minimal and maximal possible correlation. In particular, correlation coefficients ranging from zero ($\kappa \searrow 0$) to $\frac{\sqrt{c_1 c_2}}{c_{max}}$ ($\kappa \nearrow 1$) are possible, and the correlation is independent of t.

3.4 Characterization as multi-dimensional compound Poisson process

Although we know from Remark 3.6 that a time-changed $\mathrm{CPP_{Exp}}$ is a d-dimensional compound Poisson process, so far, we do not know its explicit representation in accordance with Definition 2.14. In this section we take care of that explicit representation and start with looking at a single component of our construction from a new perspective.

Let $\hat{Z}$ be one component of a d-dimensional time-changed CPP_{Exp}, as defined in Definition 3.5, i.e.

$$\hat{Z}_t = \sum_{k=1}^{N_{T_t}} J_k, \qquad \text{for all } t \geq 0,$$

where $J_1, J_2, \ldots$ is a sequence of independent $\text{Exp}(\frac{c_T \, \eta}{c_T - c})$-distributed random variables, N is a Poisson process with intensity $\frac{c}{c_T - c}$, and $T \sim \text{CPP}_{\text{Exp}}(c_T, 1)$. All quantities are mutually independent. As stated in Remark 3.6, $\hat{Z}$ turns out to be a $\text{CPP}_{\text{Exp}}(c, \eta)$. This representation of $\hat{Z}$ implies that $\hat{Z}$ jumps at time t if and only if N_T jumps at time t. In particular, $\hat{Z}$ jumps at time t only if T jumps at time t. Let τ be a jump time of T. Then, $\hat{Z}$ jumps at τ if and only if $N_{T_\tau} - N_{T_{\tau^-}} \geq 1$, i.e. $\hat{Z}$ jumps at time τ if and only if the jump size of T triggers a jump of N_T. Therefore, $\hat{Z}$ can alternatively be seen as a compound Poisson process with intensity c_T, instead of c, and modified jump size distribution satisfying positive probability for jump magnitudes of size 0. Thus, it makes sense to examine the process N_T in detail, which is done in the next theorem.

Theorem 3.11 (Link to geometric law)
Let N be a Poisson process with intensity $\frac{c}{c_T - c}$ and $T \sim \text{CPP}_{\text{Exp}}(c_T, 1)$. Assume, N and T are independent. Then, the process $N_T = \{N_{T_t}\}_{t \geq 0}$ is a compound Poisson process with intensity c_T and $\text{Geo}\left(1 - \frac{c}{c_T}\right)$-distributed jump sizes.

Proof
Let $u \in \mathbb{R}_+$. From Example 2.9 we know that the Laplace exponent of a $\text{Geo}\left(1 - \frac{c}{c_T}\right)$-distributed random variable X is given by

$$\tilde{\varphi}_X(u) = \frac{1 - \frac{c}{c_T}}{1 - \frac{c}{c_T} \, e^{-u}} = \frac{c_T - c}{c_T - c \, e^{-u}}.$$

Theorem 2.24 yields that the Laplace exponent of a compound Poisson process Z with intensity c_T and $\text{Geo}\left(1 - \frac{c}{c_T}\right)$-distributed jump sizes is given by

$$\tilde{\psi}_Z(-u) = c_T \left(\tilde{\varphi}_X(u) - 1\right) = c_T \left(\frac{c_T - c}{c_T - c \, e^{-u}} - 1\right) = \frac{c_T \, c \, (e^{-u} - 1)}{c_T - c \, e^{-u}}.$$

By Theorem 2.25, the process N_T is a Lévy subordinator. Since the Laplace exponent uniquely determines the distribution of a Lévy subordinator (cf. Remark 2.21), it is

left to show that the Laplace exponent of N_T is given by

$$\tilde{\psi}_{N_T}(-u) = \frac{-c_T\, c\, (1 - e^{-u})}{c_T - c\, e^{-u}}.$$

Using Theorem 2.25, we get

$$\tilde{\psi}_{N_T}(-u) = \tilde{\psi}_T\big(\tilde{\psi}_N(-u)\big) = \tilde{\psi}_T\big(\log\big(\tilde{\varphi}_{N_1}(u)\big)\big).$$

Note that $N_1 \sim \text{Poi}\left(\frac{c}{c_T - c}\right)$. Hence, Example 2.9 and Theorem 3.1 yield

$$\tilde{\psi}_{N_T}(-u) = \tilde{\psi}_T\left(-\frac{c\,(1 - e^{-u})}{c_T - c}\right) = \frac{-c_T\, \frac{c(1 - e^{-u})}{c_T - c}}{1 + \frac{c(1 - e^{-u})}{c_T - c}} = \frac{-c_T\, c\,(1 - e^{-u})}{c_T - c\, e^{-u}}.$$

Therefore, N_T is a compound Poisson process with intensity c_T and $\text{Geo}\left(1 - \frac{c}{c_T}\right)$-distributed jump sizes. $\qquad\square$

Remark 3.12 (On the proof of Theorem 3.11)

The proof of Theorem 3.11 is based on the clever guess that the jump size distribution of N_T is $\text{Geo}\left(1 - \frac{c}{c_T}\right)$. However, this distribution can be calculated in a more elementary way, such that this guess can be derived from these constructive calculations. In the following, it is shown how one can circumvent such a guess.

Proof (Alternative proof of Theorem 3.11)

Instead of guessing the jump size distribution of N_T, it can be calculated. To do so, let $\tau := \inf\left\{t \geq 0 \,\big|\, T_t \neq 0\right\}$, i.e. τ is the first jump time of N_T. Hence, the random variable N_{T_τ} describes the jump size of N_T at time τ and T_τ represents the magnitude of the first jump of T. Using $T_\tau \sim \text{Exp}(1)$ and $N_t \sim \text{Poi}(\frac{ct}{c_T - c})$, $\forall t \in \mathbb{R}_+$, we get for all $k \in \mathbb{N}_0$ that

$$\mathbb{P}\left(N_{T_\tau} = k\right) = \int_0^\infty \mathbb{P}\left(N_{T_\tau} = k \,\big|\, T_\tau = t\right) e^{-t}\, \mathrm{d}t = \int_0^\infty \mathbb{P}\left(N_t = k\right) e^{-t}\, \mathrm{d}t$$

$$= \int_0^\infty \frac{1}{k!}\left(\frac{c\,t}{c_T - c}\right)^k e^{-\frac{c\,t}{c_T - c}}\, e^{-t}\, \mathrm{d}t = \int_0^\infty \frac{1}{k!}\, \frac{c^k}{(c_T - c)^k}\, t^k\, e^{-t\frac{c_T}{c_T - c}}\, \mathrm{d}t. \quad (3.15)$$

The integration by parts formula yields

$$\int_0^\infty \frac{1}{k!}\, \frac{c^k}{(c_T - c)^k}\, t^k\, e^{-t\frac{c_T}{c_T - c}}\, \mathrm{d}t = \left[-\frac{1}{k!}\, \frac{c^k}{(c_T - c)^{k-1}\, c_T}\, t^k\, e^{-t\frac{c_T}{c_T - c}}\right]_{t=0}^{t=\infty}$$

$$+ \int_0^\infty \frac{1}{(k-1)!} \frac{c^k}{(c_T-c)^{k-1} c_T} t^{k-1} e^{-t \frac{c_T}{c_T-c}} \,\mathrm{d}t. \quad (3.16)$$

By multiple application of l'Hôpital's rule we know that

$$\left[-\frac{1}{k!} \frac{c^k}{(c_T-c)^{k-1} c_T} t^k e^{-t \frac{c_T}{c_T-c}} \right]_{t=0}^{t=\infty} = 0,$$

Hence, Equation (3.16) simplifies to

$$\int_0^\infty \frac{1}{k!} \frac{c^k}{(c_T-c)^k} t^k e^{-t \frac{c_T}{c_T-c}} \,\mathrm{d}t = \int_0^\infty \frac{1}{(k-1)!} \frac{c^k}{(c_T-c)^{k-1} c_T} t^{k-1} e^{-t \frac{c_T}{c_T-c}} \,\mathrm{d}t,$$

and by iterative application of the integration by parts formula we get

$$\int_0^\infty \frac{1}{k!} \frac{c^k}{(c_T-c)^k} t^k e^{-t \frac{c_T}{c_T-c}} \,\mathrm{d}t = \cdots = \int_0^\infty \left(\frac{c}{c_T} \right)^k e^{-t \frac{c_T}{c_T-c}} \,\mathrm{d}t$$

$$= \left[-\frac{c_T-c}{c_T} \left(\frac{c}{c_T} \right)^k e^{-t \frac{c_T}{c_T-c}} \right]_{t=0}^{t=\infty} = \frac{c_T-c}{c_T} \left(\frac{c}{c_T} \right)^k.$$

Therefore, Equation (3.15) boils down to

$$\mathbb{P}\left(N_{T_\tau} = k \right) = \left(\frac{c}{c_T} \right)^k \left(1 - \frac{c}{c_T} \right),$$

which shows that $N_{T_\tau} \sim \mathrm{Geo}\left(1 - \frac{c}{c_T} \right)$. □

Theorem 3.11 also motivates an intuitive representation of a d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ $Z = \left(Z^{(1)}, \ldots, Z^{(d)} \right)$ with intensities $c_1, \ldots, c_d$, jump size parameters $\eta_1, \ldots, \eta_d$, and correlation parameter $\kappa \in (0,1)$:

Potential jump times of $Z^{(1)}, \ldots, Z^{(d)}$ are given by a Poisson process with intensity $c_T = \kappa \max_{1 \le i \le d} \{c_i\}$. The jump size of $Z^{(i)}$ at a potential jump time τ is a sum of independent $\mathrm{Exp}\left(\frac{c_T \eta_i}{c_T - c_i} \right)$-distributed random variables. The number of added random variables is given by the jump size of $N_T^{(i)}$ at τ, which is $\mathrm{Geo}\left(1 - \frac{c_i}{c_T} \right)$-distributed. Note that $P\left(N_{T_\tau}^{(i)} = 0 \right) = 1 - \frac{c_i}{c_T} > 0$, and, hence, a *potential jump time* need not necessarily lead to an actual jump of $Z^{(i)}$. Moreover, the jump sizes of $N_T^{(1)}, \ldots, N_T^{(d)}$ are dependent. The joint distribution is investigated in the next theorem.

Theorem 3.13 (Link to a multivariate geometric law)

Let $T \sim \mathrm{CPP}_{\mathrm{Exp}}(c_T, 1)$ and $N^{(1)}, \ldots, N^{(d)}$ be d independent Poisson processes with intensities given by $\frac{c_1}{c_T - c_1}, \ldots, \frac{c_d}{c_T - c_d}$. Assume, T to be independent of $N^{(1)}, \ldots, N^{(d)}$. Let $\tau := \inf \left\{ t \geq 0 \,\middle|\, T_t \neq 0 \right\}$. Then, the joint distribution of the random variables $N_{T_\tau}^{(1)}, \ldots, N_{T_\tau}^{(d)}$ is given by

$$\mathbb{P}\left(N_{T_\tau}^{(1)} = k_1, \ldots, N_{T_\tau}^{(d)} = k_d \right) = \frac{k!}{\prod_{i=1}^d k_i!} \left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i} \right)^{-k-1} \prod_{i=1}^d \left(\frac{c_i}{c_T - c_i} \right)^{k_i},$$

where $k_1, \ldots, k_d \in \mathbb{N}_0$ and $k = \sum_{i=1}^d k_i$.

Moreover, $\left(N_T^{(1)}, \ldots, N_T^{(d)} \right)$ is a d-dimensional compound Poisson process.

Proof

The d-dimensional process $\left(N_T^{(1)}, \ldots, N_T^{(d)} \right)$ is a d-dimensional Lévy process, since the subordination of the d-dimensional Lévy process $\left(N_t^{(1)}, \ldots, N_t^{(d)} \right)$ is again a Lévy process by Theorem 2.25. Using Theorem 3.11, every component of $\left(N_T^{(1)}, \ldots, N_T^{(d)} \right)$ is a compound Poisson process. Hence, $\left(N_T^{(1)}, \ldots, N_T^{(d)} \right)$ is a d-dimensional compound Poisson process (cf. Remark 2.17(iii)). Its jump size distribution can be calculated by a multivariate adaptation of the alternative proof of Theorem 3.11. Using $T_\tau \sim \mathrm{Exp}(1)$ we get

$$\mathbb{P}\left(N_{T_\tau}^{(1)} = k_1, \ldots, N_{T_\tau}^{(d)} = k_d \right) = \int_0^\infty \mathbb{P}\left(N_{T_\tau}^{(1)} = k_1, \ldots, N_{T_\tau}^{(d)} = k_d \,\middle|\, T_\tau = t \right) e^{-t} \, \mathrm{d}t$$

$$= \int_0^\infty \mathbb{P}\left(N_t^{(1)} = k_1, \ldots, N_t^{(d)} = k_d \right) e^{-t} \, \mathrm{d}t$$

The independence between the processes $N^{(1)}, \ldots, N^{(d)}$ and the fact that $N_t^{(i)} \sim \mathrm{Poi}(\frac{c_i \, t}{c_T - c_i})$ $\forall 1 \leq i \leq d$, $\forall t \in \mathbb{R}_+$ yield

$$\mathbb{P}\left(N_{T_\tau}^{(1)} = k_1, \ldots, N_{T_\tau}^{(d)} = k_d \right)$$

$$= \int_0^\infty \prod_{i=1}^d \mathbb{P}\left(N_t^{(i)} = k_i \right) e^{-t} \, \mathrm{d}t$$

$$= \int_0^\infty \prod_{i=1}^d \frac{1}{k_i!} \left(\frac{c_i \, t}{c_T - c_i} \right)^{k_i} e^{-\frac{c_i \, t}{c_T - c_i}} e^{-t} \, \mathrm{d}t$$

$$= \int\limits_0^\infty t^k \exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i}\right)\right) \mathrm{d}t \prod_{i=1}^d \frac{1}{k_i!}\left(\frac{c_i}{c_T - c_i}\right)^{k_i}. \qquad (3.17)$$

The integration by parts formula yields

$$\int\limits_0^\infty t^k \exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i}\right)\right) \qquad\qquad (3.18)$$

$$= \left[-t^k \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}}\right]_{t=0}^{t=\infty} + \int\limits_0^\infty k\, t^{k-1} \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}}\, \mathrm{d}t.$$

By multiple application of l'Hôpital's rule we know that

$$\left[-t^k \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}}\right]_{t=0}^{t=\infty} = 0,$$

Hence, Equation (3.18) simplifies to

$$\int\limits_0^\infty t^k \exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i}\right)\right) = \int\limits_0^\infty k\, t^{k-1} \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}}\, \mathrm{d}t,$$

and by iterative application of the integration by parts formula we get

$$\int\limits_0^\infty t^k \exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i}\right)\right) = \cdots = \int\limits_0^\infty k! \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)^k}\mathrm{d}t,$$

$$= \left[-k! \frac{\exp\left(-t\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)\right)}{\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)^{k+1}}\right]_{t=0}^{t=\infty}$$

$$= \frac{k!}{\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)^{k+1}}.$$

Therefore, Equation (3.17) boils down to

$$\mathbb{P}\left(N_{T_\tau}^{(1)} = k_1, \ldots, N_{T_\tau}^{(d)} = k_d\right) = \frac{k!}{\left(1 + \sum_{i=1}^d \frac{c_i}{c_T-c_i}\right)^{k+1}} \prod_{i=1}^d \frac{1}{k_i!}\left(\frac{c_i}{c_T - c_i}\right)^{k_i}$$

$$= \frac{k!}{\prod_{i=1}^d k_i!}\left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i}\right)^{-k-1} \prod_{i=1}^d \left(\frac{c_i}{c_T - c_i}\right)^{k_i}.$$

And the claim is shown. $\qquad\square$

Remark 3.14 (On the multivariate geometric law in Theorem 3.13)
The distribution in Theorem 3.13 is a multivariate geometric distribution in the sense of Srivastava and Bagchi [1985], which was further characterized by Sreehari and Vasudeva [2012]. In particular, they say, a random variable $X = (X_1, \ldots, X_d)$ taking values in $\mathbb{N}_0^d$ has a multivariate geometric distribution if

$$\mathbb{P}(X = (k_1, \ldots, k_d)) = \frac{k!}{\prod_{i=1}^d k_i!} \, p_0 \prod_{i=1}^d p_i^{k_i} \quad \forall (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \, k := \sum_{i=1}^d k_i,$$

where $0 < p_i < 1$ for all $0 \le i \le d$ and $\sum_{i=0}^d p_i = 1$.

Indeed, it coincides with the law presented in Theorem 3.13 by choosing

$$p_0 = \left(1 + \sum_{i=1}^d \frac{c_i}{c_T - c_i} \right)^{-1}, \quad p_i = \frac{p_0 \, c_i}{c_T - c_i}, \quad \forall 1 \le i \le d.$$

In Remark 3.6 we have seen that a time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ is a d-dimensional compound Poisson process with each component being a $\mathrm{CPP}_{\mathrm{Exp}}$. By means of Theorem 3.13 a characterization of a time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ as d-dimensional compound Poisson process can be established in the following way.

Remark 3.15 (Characterization of a time-changed $\mathrm{CPP_{Exp}}$)
Let $Z = \left(Z^{(1)}, \ldots, Z^{(d)} \right)$ be a d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}$, i.e. for all $1 \le i \le d$,

$$Y_t^{(i)} = \sum_{j=1}^{N_t^{(i)}} J_j^{(i)}, \qquad T_t = \sum_{j=1}^{N_t^T} J_j^T, \qquad Z_t^{(i)} = Y_{T_t}^{(i)},$$

where N^T is a Poisson process with intensity c_T, $N^{(i)}$ is a Poisson process with intensity $\frac{c_i}{c_T - c_i}$ for all $1 \le i \le d$, $J_1^T, J_2^T, \ldots$ are $\mathrm{Exp}(1)$-distributed random variables, and $J_1^{(i)}, J_2^{(i)}, \ldots$ are $\mathrm{Exp}(\frac{c_T \, \eta_i}{c_T - c_i})$-distributed random variables for all $1 \le i \le d$. All quantities are mutually independent. Let $\tau_1, \tau_2, \ldots$ be the arrival times of T and set $\tau_0 = 0$.
Then, Z can be represented by

$$Z_t = \sum_{k=1}^{N_t^T} E_k, \quad \text{for all } t \ge 0, \tag{3.19}$$

where $E_1 = \left(E_1^{(1)}, \ldots, E_1^{(d)} \right)$, $E_2 = \left(E_2^{(1)}, \ldots, E_2^{(d)} \right), \ldots$ is a sequence of independent and identically distributed $\mathbb{R}_+^d$-valued random variables given by

$$E_j^{(i)} := \sum_{k=1+N_{T_{\tau_{j-1}}}^{(i)}}^{N_{T_{\tau_j}}^{(i)}} J_k^{(i)} \quad \text{for all } 1 \leq i \leq d, \text{ and for all } j \geq 1.$$

Equation (3.19) is a characterization of our time change constructed process Z as d-dimensional compound Poisson process in accordance with Definition 2.14. The intensity of Z is given by the intensity of N^T, namely c_T. The jump sizes $E_1, E_2, \ldots$ are d-dimensional random variables having a sum of independent $\text{Exp}(\frac{c_T\, \eta_i}{c_T - c_i})$-distributed random variables in each component. The number of summed variables is given by the jump heights of $N_T^{(1)}, \ldots, N_T^{(d)}$, whose distribution is computed in Theorem 3.13.

3.5 Linear jump decomposition

As mentioned in the introduction of this chapter, there is a vivid way of representing multivariate compound Poisson processes by splitting each component into individual and common parts, and linking the common parts by copulas. This characterization is intuitive, particularly in low dimensions. Following [Cont and Tankov, 2004, Chapter 5.5], the decomposition of a d-dimensional time-changed CPP_{Exp} $Z = \left(Z^{(1)}, \ldots, Z^{(d)} \right)$ can be obtained by splitting into individual and common jump parts. Using the representation of Z in Equation (3.19), i.e.

$$Z_t^{(i)} = \sum_{k=1}^{N_t^T} E_k^{(i)}, \qquad \forall 1 \leq i \leq d,$$

each component of the time-changed CPP_{Exp} Z can be separated in the following way:

$$Z_t^{(i)} = \sum_{k=1}^{N_t^T} E_k^{(i)} \left(\prod_{\substack{j=1 \\ j \neq i}}^{d} \mathbb{1}_{\{E_k^{(j)}=0\}} + \sum_{\{\} \neq A \subseteq \{1,\ldots,d\} \setminus \{i\}} \prod_{j \in A} \mathbb{1}_{\{E_k^{(j)}>0\}} \prod_{\substack{l \notin A \\ l \neq i \\ l \neq i}} \mathbb{1}_{\{E_k^{(l)}=0\}} \right)$$

$$= I_t^{(i)} + \sum_{\{\} \neq A \subseteq \{1,\ldots,d\} \setminus \{i\}} C_t^{(i,A)}, \qquad \forall 1 \leq i \leq d, \forall t \geq 0, \tag{3.20}$$

where

$$I_t^{(i)} := \sum_{k=1}^{N_t^T} \prod_{\substack{j=1 \\ j \neq i}}^{d} \mathbb{1}_{\{E_k^{(j)}=0\}} E_k^{(i)}, \qquad \forall 1 \leq i \leq d, \forall t \geq 0, \tag{3.21}$$

$$C_t^{(i,A)} := \sum_{k=1}^{N_t^T} \prod_{j \in A} \mathbb{1}_{\{E_k^{(j)}>0\}} \prod_{\substack{l \notin A \\ l \neq i}} \mathbb{1}_{\{E_k^{(l)}=0\}} E_k^{(i)}, \qquad \forall 1 \leq i \leq d, \forall t \geq 0. \tag{3.22}$$

Note that the term inside the parentheses in Equation (3.20) boils down to one. The process $I^{(i)}$ consists of all individual jumps of $Z^{(i)}$, i.e. there exists no other component $Z^{(j)}$, $i \neq j$, sharing a jump time with $I^{(i)}$. The process $C^{(i,A)}$ represents all jumps of $Z^{(i)}$, which occur simultaneously with jumps in all components of Z that are given by the set A. Since $\left(E_1^{(1)}, \ldots, E_1^{(d)} \right), \left(E_2^{(1)}, \ldots, E_2^{(d)} \right), \ldots$ is a sequence of independent and identically distributed random variables (cf. Remark (3.19)),

$$\prod_{\substack{j=1 \\ j \neq i}}^{d} \mathbb{1}_{\{E_1^{(j)}=0\}} E_1^{(i)}, \ \prod_{\substack{j=1 \\ j \neq i}}^{d} \mathbb{1}_{\{E_2^{(j)}=0\}} E_2^{(i)}, \ldots \qquad \text{and}$$

$$\prod_{j \in A} \mathbb{1}_{\{E_1^{(j)}>0\}} \prod_{\substack{l \notin A \\ l \neq i}} \mathbb{1}_{\{E_1^{(l)}=0\}} E_1^{(i)}, \ \prod_{j \in A} \mathbb{1}_{\{E_2^{(j)}>0\}} \prod_{\substack{l \notin A \\ l \neq i}} \mathbb{1}_{\{E_2^{(l)}=0\}} E_2^{(i)}, \ldots$$

are also sequences of independent and identically distributed random variables for all $1 \leq i \leq d$. Furthermore, these sequences are independent of the process N^T. Hence, $\forall 1 \leq i \leq d$, $A \subseteq \{1, \ldots, d\} \backslash \{i\}$, $A \neq \{\}$ the processes $I^{(i)}$ and $C^{(i,A)}$ are compound Poisson processes by construction. The intensities and jump size distributions of these compound Poisson processes can be computed by means of Theorem 3.13.

To illustrate the decomposition, let us have a look at the three-dimensional case. For $d = 3$, we get

$$Z^{(1)} = I^{(1)} + C^{(1,\{2\})} + C^{(1,\{3\})} + C^{(1,\{2,3\})},$$
$$Z^{(2)} = I^{(2)} + C^{(2,\{1\})} + C^{(2,\{3\})} + C^{(2,\{1,3\})},$$
$$Z^{(3)} = I^{(3)} + C^{(3,\{1\})} + C^{(3,\{2\})} + C^{(3,\{1,2\})}.$$

As we can see, to characterize a three-dimensional compound Poisson process by means of that representation, one has to determine

- 7 intensities (Note that $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$ share the same intensity, also $C^{(2,\{3\})}$ and $C^{(3,\{2\})}$, $C^{(1,\{3\})}$ and $C^{(3,\{1\})}$, and $C^{(1,\{2,3\})}$, $C^{(2,\{1,3\})}$ and $C^{(3,\{1,2\})}$.),

- 12 jump size distributions,

- 3 two-dimensional copulas to link the jump size distributions of $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$, $C^{(2,\{3\})}$ and $C^{(3,\{2\})}$, and $C^{(1,\{3\})}$ and $C^{(3,\{1\})}$, and

- one three-dimensional copula to link the jump size distributions of $C^{(1,\{2,3\})}$, $C^{(2,\{1,3\})}$ and $C^{(3,\{1,2\})}$.

Obviously, such a representation is way too involved for dimension $d > 2$ to suit as a useful characterization. Note, that in general we need $2^d - 1$ intensities, $d\,2^{(d-1)}$ jump size distributions, and $2^d - d - 1$ copulas. Hence, the idea to characterize the components of a d-dimensional compound Poisson process by sums of dependent compound Poisson processes is reasonable for $d = 2$ only. Therefore, in the sequel we concentrate on the bivariate case.

Let $Z = \left(Z^{(1)}, Z^{(2)}\right)$ be a two-dimensional time-changed $\mathrm{CPP_{Exp}}$ with intensities c_1, c_2, jump size parameter η_1, η_2, and correlation parameter κ. Denote by $c_T = \frac{1}{\kappa} \max\{c_1, c_2\}$ the intensity of the subordinator within the construction of Z. The decomposition is then given by

$$Z^{(1)} = I^{(1)} + C^{(1,\{2\})}, \qquad \text{and} \qquad Z^{(2)} = I^{(2)} + C^{(2,\{1\})}, \tag{3.23}$$

where $I^{(1)}$, $I^{(2)}$, $C^{(1,\{2\})}$, and $C^{(2,\{1\})}$ are compound Poisson processes. $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$ are dependent, all other processes are mutually independent. $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$ have the same intensity, they jump at the same time and have dependent jump sizes. In the following theorem, we investigate the distributions of the compound Poisson processes $I^{(1)}$, $I^{(2)}$, $C^{(1,\{2\})}$, and $C^{(2,\{1\})}$.

Theorem 3.16 (Decomposition of a bivariate time-changed $\mathrm{CPP_{Exp}}$)
Let the processes $I^{(1)}$, $I^{(2)}$, $C^{(1,\{2\})}$, and $C^{(2,\{1\})}$ be defined as in Equation (3.23). Then,

(i) $I^{(1)} \sim \mathrm{CPP_{Exp}} \left(\frac{c_1\,(c_T - c_2)^2}{c_T^2 - c_1\,c_2}, \frac{c_T^2\,\eta_1}{c_T^2 - c_1 c_2} \right)$,

(ii) $I^{(2)} \sim \mathrm{CPP_{Exp}} \left(\frac{c_2\,(c_T - c_1)^2}{c_T^2 - c_1\,c_2}, \frac{c_T^2\,\eta_2}{c_T^2 - c_1 c_2} \right)$,

(iii) $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$ are compound Poisson processes with identical jump times arriving with intensity $\frac{c_1\,c_2\,(2\,c_T - c_1 - c_2)}{c_T^2 - c_1\,c_2}$. The distribution function of the jump

sizes of $C^{(1,\{2\})}$ is given by

$$1 - \frac{c_T^2 - c_1 c_2}{c_2 (2 c_T - c_1 - c_2)} e^{-\eta_1 x} + \frac{(c_T - c_2)^2}{c_2 (2 c_T - c_1 - c_2)} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x}, \quad x \geq 0.$$

The distribution function of the jump sizes of $C^{(2,\{1\})}$ is given by

$$1 - \frac{c_T^2 - c_1 c_2}{c_1 (2 c_T - c_1 - c_2)} e^{-\eta_2 x} + \frac{(c_T - c_1)^2}{c_1 (2 c_T - c_1 - c_2)} e^{-\frac{c_T^2 \eta_2}{c_T^2 - c_1 c_2} x}, \quad x \geq 0.$$

Proof

Let us recall the definition of the compound Poisson processes $I^{(1)}$ and $C^{(1,\{2\})}$ (cf. Equation (3.21) and Equation (3.19)),

$$I_t^{(1)} = \sum_{k=1}^{N_t^T} \mathbb{1}_{\{E_k^{(2)}=0\}} E_k^{(1)}, \qquad C_t^{(1,\{2\})} = \sum_{k=1}^{N_t^T} \mathbb{1}_{\{E_k^{(2)}>0\}} E_k^{(1)}$$

where

$$E_k^{(i)} = \sum_{j=1+N_{T_{\tau_{k-1}}}^{(i)}}^{N_{T_{\tau_k}}^{(i)}} J_j^{(i)} \quad \text{for all } 1 \leq i \leq 2, \text{ and for all } k \geq 1.$$

Here, $N^{(1)}$ is a Poisson process with intensity c_1, $N^{(2)}$ is a Poisson process with intensity c_2, $T \sim \mathrm{CPP}_{\mathrm{Exp}}(c_T, 1)$. We denote by τ_1, τ_2, ... the arrival times of T and N^T is the corresponding Poisson process, $\tau_0 := 0$. $J_1^{(1)}$, $J_2^{(1)}, \ldots$ is a sequence of independent $\mathrm{Exp}(\eta_1)$-distributed random variables, and $J_1^{(2)}$, $J_2^{(2)}, \ldots$ is a sequence of independent $\mathrm{Exp}(\eta_2)$-distributed random variables. All quantities are mutually independent.

We start with proving (i) and derive the jump size distribution of $I_t^{(1)}$ by computing the distribution function of $\mathbb{1}_{\{E_1^{(2)}=0\}} E_1^{(1)}$. Note first that

$$E_1^{(2)} = 0 \Leftrightarrow \sum_{j=1}^{N_{T_{\tau_1}}^{(2)}} J_j^{(2)} = 0 \Leftrightarrow N_{T_{\tau_1}}^{(2)} = 0.$$

Hence, for some $x \in \mathbb{R}_+$

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}=0\}} E_1^{(1)} > x\right) = \mathbb{P}\left(\sum_{j=1}^{N_{T_{\tau_1}}^{(1)}} J_j^{(1)} > x, N_{T_{\tau_1}}^{(2)} = 0\right)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{k} J_j^{(1)} > x, \, N_{T_{\tau_1}}^{(1)} = k, \, N_{T_{\tau_1}}^{(2)} = 0 \right)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{k} J_j^{(1)} > x \right) \mathbb{P}\left(N_{T_{\tau_1}}^{(1)} = k, \, N_{T_{\tau_1}}^{(2)} = 0 \right), \quad (3.24)$$

because the sequence $J_1^{(1)}$, $J_2^{(1)}, \ldots$ is independent of the processes $N^{(1)}$, $N^{(2)}$, and T. Theorem 3.13 yields

$$\mathbb{P}\left(N_{T_{\tau_1}}^{(1)} = k, \, N_{T_{\tau_1}}^{(2)} = 0 \right)$$

$$= \left(\frac{1}{1 + \sum_{i=1}^{2} \frac{c_i}{c_T - c_i}} \right)^{k+1} \left(\frac{c_1}{c_T - c_1} \right)^{k}$$

$$= \left(\frac{(c_T - c_1)(c_T - c_2)}{(c_T - c_1)(c_T - c_2) + c_1(c_T - c_2) + c_2(c_T - c_1)} \right)^{k+1} \left(\frac{c_1}{c_T - c_1} \right)^{k}$$

$$= \left(\frac{(c_T - c_1)(c_T - c_2)}{c_T^2 - c_1 c_2} \right)^{k+1} \left(\frac{c_1}{c_T - c_1} \right)^{k}$$

$$= \frac{(c_T - c_1)(c_T - c_2)}{c_T^2 - c_1 c_2} \left(\frac{c_1(c_T - c_2)}{c_T^2 - c_1 c_2} \right)^{k}. \quad (3.25)$$

A sum of independent and identically exponential distributed random variables follows an Erlang distribution, cf. Definition 2.3. Hence,

$$\mathbb{P}\left(\sum_{j=1}^{k} J_j^{(1)} > x \right) = \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{c_T \, \eta_1}{c_T - c_1} \right)^{n} x^n \, e^{-\frac{c_T \, \eta_1}{c_T - c_1} x}. \quad (3.26)$$

Putting together Equations (3.24), (3.25), and (3.26) yields

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}} E_1^{(1)} > x \right)$$

$$= \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{c_T \, \eta_1}{c_T - c_1} \right)^{n} x^n \, e^{-\frac{c_T \, \eta_1}{c_T - c_1} x} \frac{(c_T - c_1)(c_T - c_2)}{c_T^2 - c_1 c_2} \left(\frac{c_1(c_T - c_2)}{c_T^2 - c_1 c_2} \right)^{k},$$

and rearranging summation gives

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}} E_1^{(1)} > x \right)$$

$$= e^{-\frac{c_T \, \eta_1}{c_T - c_1} x} \frac{(c_T - c_1)(c_T - c_2)}{c_T^2 - c_1 c_2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{c_T \, \eta_1}{c_T - c_1} \right)^{n} x^n \sum_{k=n+1}^{\infty} \left(\frac{c_1(c_T - c_2)}{c_T^2 - c_1 c_2} \right)^{k}. \quad (3.27)$$

Recall the formula for a geometric sum, i.e. $\sum_{k=n+1}^{\infty} \alpha^k = \frac{\alpha^{n+1}}{1-\alpha}$, if $|\alpha| < 1$. Note that $0 < \frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2} < 1$, because $c_T > c_1$ and $c_T > c_2$. Therefore,

$$
\sum_{k=n+1}^{\infty} \left(\frac{c_1\left(c_T-c_2\right)}{c_T^2 - c_1 c_2} \right)^k = \frac{\left(\frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2}\right)^{n+1}}{1 - \frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2}} = \frac{c_1\left(c_T-c_2\right)\left(\frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2}\right)^n}{c_T^2 - c_1 c_2 - c_1\left(c_T-c_2\right)}
$$
$$
= \frac{c_1\left(c_T-c_2\right)}{c_T\left(c_T-c_1\right)} \left(\frac{c_1\left(c_T-c_2\right)}{c_T^2 - c_1 c_2} \right)^n . \tag{3.28}
$$

Combining Equation (3.27) and Equation (3.28) yields

$$
\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}=0\}} E_1^{(1)} > x \right)
$$
$$
= e^{-\frac{c_T \eta_1}{c_T-c_1} x} \frac{c_1\left(c_T-c_2\right)^2}{c_T^3 - c_T c_1 c_2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{c_T \eta_1}{c_T-c_1} \right)^n x^n \left(\frac{c_1\left(c_T-c_2\right)}{c_T^2 - c_1 c_2} \right)^n .
$$

Finally, by the series expansion of the exponential function, we obtain

$$
\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}=0\}} E_1^{(1)} > x \right) = e^{-\frac{c_T \eta_1}{c_T-c_1} x} \frac{c_1\left(c_T-c_2\right)^2}{c_T^3 - c_T c_1 c_2} e^{\frac{c_T \eta_1}{c_T-c_1} x \frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2}}
$$
$$
= \frac{c_1\left(c_T-c_2\right)^2}{c_T^3 - c_T c_1 c_2} e^{\frac{c_T \eta_1}{c_T-c_1} x \left(\frac{c_1(c_T-c_2)}{c_T^2-c_1 c_2} - 1 \right)}
$$
$$
= \frac{c_1\left(c_T-c_2\right)^2}{c_T^3 - c_T c_1 c_2} e^{-\frac{c_T^2 \eta_1}{c_T^2-c_1 c_2} x} .
$$

Overall, we know that $I_t^{(1)}$ is a compound Poisson process with intensity c_T and the jump sizes follow the distribution function

$$
\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}=0\}} E_1^{(1)} \leq x \right) = 1 - \frac{c_1\left(c_T-c_2\right)^2}{c_T^3 - c_T c_1 c_2} e^{-\frac{c_T^2 \eta_1}{c_T^2-c_1 c_2} x} . \tag{3.29}
$$

To see that

$$
I^{(1)} \sim \mathrm{CPP}_{\mathrm{Exp}} \left(\frac{c_1\left(c_T-c_2\right)^2}{c_T^2 - c_1 c_2}, \frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} \right),
$$

one has to erase the zero jumps from the jump size distribution, which we do in the sequel.

Note first, that given a compound Poisson process Z with intensity c and non-negative jump size distribution D with mass on zero, this process can be reformulated without jumps of height zero. More precisely, let F_D be the distribution function of D. Then,

Z is also a compound Poisson process with intensity $c\,(1 - F_D(0))$ and the jump size distribution is given by the distribution function $\frac{F_D(x) - F_D(0)}{1 - F_D(0)}$, $x \geq 0$. This statement follows immediately from the formula for the Laplace exponent $\tilde{\psi}_Z$ of the process Z (cf. Theorem 2.24). Let $u \in \mathbb{R}_+$, then

$$\tilde{\psi}_Z(-u) = c\,(\tilde{\varphi}_D(u) - 1) = c\left(\int\limits_0^\infty e^{-u\,x}\,F_D(\mathrm{d}x) - 1\right).$$

Now, let $\hat{Z}$ be a compound Poisson process with intensity $c\,(1 - F_D(0))$ and jump size distribution function F given by $F(x) = \frac{F_D(x) - F_D(0)}{1 - F_D(0)}$, $x \geq 0$ and $F(x) = 0$, $x < 0$. Then

$$\tilde{\psi}_{\hat{Z}}(-u) = c\,(1 - F_D(0))\left(\int\limits_0^\infty e^{-u\,x}\,F(\mathrm{d}x) - 1\right)$$

$$= c\,(1 - F_D(0))\left(\int\limits_0^\infty \frac{1}{1 - F_D(0)}\,e^{-u\,x}\,F(\mathrm{d}x) - \frac{F_D(0)}{1 - F_D(0)} - 1\right)$$

$$= c\left(\int\limits_0^\infty e^{-u\,x}\,F_D(\mathrm{d}x) - 1\right) = \tilde{\psi}_Z(-u).$$

Since the Laplace exponent uniquely determines the distribution of a Lévy process, both processes, Z and $\hat{Z}$, have the same distribution.

Let us come back to the process $I^{(1)}$. Its jump size distribution (cf. Equation (3.29)) also obtains zero jumps with probability

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}}\,E_1^{(1)} = 0\right) = 1 - \frac{c_1\,(c_T - c_2)^2}{c_T^3 - c_T\,c_1\,c_2}.$$

Hence, $I^{(1)}$ can alternatively be described as a compound Poisson process with intensity

$$c_T\left(1 - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}}\,E_1^{(1)}\right)\right) = \frac{c_1\,(c_T - c_2)^2}{c_T^2 - c_1\,c_2}$$

and the jump sizes follow the distribution function

$$\frac{\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}}\,E_1^{(1)} \leq x\right) - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}}\,E_1^{(1)} = 0\right)}{1 - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}}\,E_1^{(1)} = 0\right)}$$

$$
\begin{aligned}
&= \frac{1 - \frac{c_1(c_T - c_2)^2}{c_T^3 - c_T\, c_1\, c_2}\, e^{-\frac{c_T^2\, \eta_1}{c_T^2 - c_1\, c_2}\, x} - \left(1 - \frac{c_1(c_T - c_2)^2}{c_T^3 - c_T\, c_1\, c_2}\right)}{\frac{c_1(c_T - c_2)^2}{c_T^3 - c_T\, c_1\, c_2}} \\
&= 1 - e^{-\frac{c_T^2\, \eta_1}{c_T^2 - c_1\, c_2}\, x}.
\end{aligned}
$$

Hence,

$$
I^{(1)} \sim \mathrm{CPP}_{\mathrm{Exp}}\left(\frac{c_1\,(c_T - c_2)^2}{c_T^2 - c_1\, c_2}, \frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2}\right),
$$

and (i) is fulfilled. Similar to (i), we get (ii), namely

$$
I^{(2)} \sim \mathrm{CPP}_{\mathrm{Exp}}\left(\frac{c_2\,(c_T - c_1)^2}{c_T^2 - c_1\, c_2}, \frac{c_T^2 \eta_2}{c_T^2 - c_1 c_2}\right).
$$

It is left to prove (iii). Again, we start with computing the jump size distribution of $C_t^{(1,\{2\})}$ by calculating the distribution function of $\mathbb{1}_{\{E_1^{(2)} > 0\}} E_1^{(1)}$. Here, we can use the results from (i). Note first that for some $x \in \mathbb{R}_+$, we get

$$
\begin{aligned}
\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} > 0\}} E_1^{(1)} > x\right) &= \mathbb{P}\left(E_1^{(1)} > x,\, E_1^{(2)} > 0\right) \\
&= \mathbb{P}\left(E_1^{(1)} > x\right) - \mathbb{P}\left(E_1^{(1)} > x,\, E_1^{(2)} = 0\right) \\
&= \mathbb{P}\left(E_1^{(1)} > x\right) - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)} = 0\}} E_1^{(1)} > x\right). \tag{3.30}
\end{aligned}
$$

Equation (3.26) yields

$$
\begin{aligned}
\mathbb{P}\left(E_1^{(1)} > x\right) &= \mathbb{P}\left(\sum_{j=1}^{N_{T_{\tau_1}}^{(1)}} J_j^{(1)} > x\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{k} J_j^{(1)} > x,\, N_{T_{\tau_1}}^{(1)} = k\right) \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{k} J_j^{(1)} > x\right) \mathbb{P}\left(N_{T_{\tau_1}}^{(1)} = k\right). \tag{3.31}
\end{aligned}
$$

By Theorem 3.11, we know that $N_{T_{\tau_1}}^{(1)} \sim \mathrm{Geo}\left(1 - \frac{c_1}{c_T}\right)$. Hence

$$
\mathbb{P}\left(N_{T_{\tau_1}}^{(1)} = k\right) = \left(\frac{c_1}{c_T}\right)^k \left(1 - \frac{c_1}{c_T}\right). \tag{3.32}
$$

Combining Equations (3.26), (3.31), and (3.27), we get

$$
\mathbb{P}\left(E_1^{(1)} > x\right) = \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{c_T\, \eta_1}{c_T - c_1}\right)^n x^n\, e^{-\frac{c_T\, \eta_1}{c_T - c_1}\, x} \left(\frac{c_1}{c_T}\right)^k \left(1 - \frac{c_1}{c_T}\right),
$$

and rearranging summation gives

$$\mathbb{P}\left(E_1^{(1)} > x\right) = e^{-\frac{c_T \eta_1}{c_T - c_1} x} \left(1 - \frac{c_1}{c_T}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{c_T \eta_1}{c_T - c_1}\right)^n x^n \sum_{k=n+1}^{\infty} \left(\frac{c_1}{c_T}\right)^k.$$

Again, the formula for the geometric sum yields

$$
\begin{aligned}
\mathbb{P}\left(E_1^{(1)} > x\right) &= e^{-\frac{c_T \eta_1}{c_T - c_1} x} \left(1 - \frac{c_1}{c_T}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{c_T \eta_1}{c_T - c_1}\right)^n x^n \frac{\left(\frac{c_1}{c_T}\right)^{n+1}}{1 - \frac{c_1}{c_T}} \\
&= e^{-\frac{c_T \eta_1}{c_T - c_1} x} \frac{c_1}{c_T} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{c_1 \eta_1}{c_T - c_1}\right)^n x^n \\
&= \frac{c_1}{c_T} e^{-\eta_1 x}.
\end{aligned}
\tag{3.33}
$$

By combing Equations (3.30), (3.33), and (3.29), we obtain

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} > x\right) = \frac{c_1}{c_T} e^{-\eta_1 x} - \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x}.$$

Overall, we know that $C_t^{(1,\{2\})}$ is a compound Poisson process with intensity c_T and the jump sizes follow the distribution function

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} \le x\right) = 1 - \frac{c_1}{c_T} e^{-\eta_1 x} + \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x}. \tag{3.34}$$

Note, that

$$\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} = 0\right) = 1 - \frac{c_1}{c_T} + \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2}.$$

Therefore, by killing the zero jumps from the jumps size distribution, $C_t^{(1,\{2\})}$ can be seen as a compound Poisson process with intensity

$$c_T \left(1 - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} = 0\right)\right) = c_1 - \frac{c_1 (c_T - c_2)^2}{c_T^2 - c_1 c_2} = \frac{c_1 c_2 (2 c_T - c_1 - c_2)}{c_T^2 - c_1 c_2}$$

and jump size distribution function

$$\frac{\mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} \le x\right) - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} = 0\right)}{1 - \mathbb{P}\left(\mathbb{1}_{\{E_1^{(2)}>0\}} E_1^{(1)} = 0\right)}$$

$$
= \frac{1 - \frac{c_1}{c_T} e^{-\eta_1 x} + \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x} - \left(1 - \frac{c_1}{c_T} + \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2} \right)}{\frac{c_1 c_2 (2 c_T - c_1 - c_2)}{c_T^3 - c_T c_1 c_2}}
$$

$$
= \frac{\frac{c_1}{c_T} \left(1 - e^{-\eta_1 x} \right) - \frac{c_1 (c_T - c_2)^2}{c_T^3 - c_T c_1 c_2} \left(1 - e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x} \right)}{\frac{c_1 c_2 (2 c_T - c_1 - c_2)}{c_T^3 - c_T c_1 c_2}}
$$

$$
= \frac{\left(c_T^2 - c_1 c_2 \right) \left(1 - e^{-\eta_1 x} \right) - (c_T - c_2)^2 \left(1 - e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x} \right)}{c_2 \left(2 c_T - c_1 - c_2 \right)}
$$

$$
= 1 - \frac{c_T^2 - c_1 c_2}{c_2 \left(2 c_T - c_1 - c_2 \right)} e^{-\eta_1 x} + \frac{(c_T - c_2)^2}{c_2 \left(2 c_T - c_1 - c_2 \right)} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x} . \tag{3.35}
$$

Analogously, the jump size distribution function of $C_t^{(2,\{1\})}$ is given by

$$
= 1 - \frac{c_T^2 - c_1 c_2}{c_1 \left(2 c_T - c_1 - c_2 \right)} e^{-\eta_2 x} + \frac{(c_T - c_1)^2}{c_1 \left(2 c_T - c_1 - c_2 \right)} e^{-\frac{c_T^2 \eta_2}{c_T^2 - c_1 c_2} x} .
$$

Hence, (iii) is shown. $\qquad \square$

Remark 3.17 (Jump size distribution of the common parts)
In Theorem 3.16 (iii), the distribution function of the jump magnitudes of $C_t^{(1,\{2\})}$ turned out to be

$$
F(x) := 1 - \frac{c_T^2 - c_1 c_2}{c_2 \left(2 c_T - c_1 - c_2 \right)} e^{-\eta_1 x} + \frac{(c_T - c_2)^2}{c_2 \left(2 c_T - c_1 - c_2 \right)} e^{-\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} x} .
$$

Indeed, this distribution function F is a linear combination of two exponential distribution functions. Let $X \sim \mathrm{Exp}\,(\eta_1)$, and $Y \sim \mathrm{Exp}\left(\frac{c_T^2 \eta_1}{c_T^2 - c_1 c_2} \right)$ and denote by F_X, respectively F_Y, their distribution functions. Then,

$$
F(x) = p\, F_X(x) + (1 - p)\, F_Y(x) \qquad \text{with} \qquad p = \frac{c_T^2 - c_1 c_2}{c_2 \left(2 c_T - c_1 - c_2 \right)} .
$$

On first sight, F looks like a two-dimensional hyperexponential distribution, which is defined by a convex combination of exponential distribution functions. A two-dimensional hyperexponentially distributed random variable can be constructed by an independent Bernoulli-distributed decision variable. Conditioned on the result of the Bernoulli experiment one of the two exponentially distributed random variables

is chosen. Here, however, we do not face a convex combination, since $p > 1$, due to $c_T > c_2$. Hence, the jump sizes of $C_t^{(1,\{2\})}$ are not hyperexponentially distributed. For more information on that distribution we refer to Nelson [1995]. Note, that the alternative representation for the jump size distribution of $C_t^{(1,\{2\})}$ including zero jumps (cf. Equation (3.34)), is a linear combination of these two exponential distributions and the so-called degenerate distribution function, which jumps from zero to one at one.

A construction by means of a Bernoulli distributed decision variable can be obtained for the jump sizes of the overall process $Z^{(1)}$. Every time a jump occurs, a Bernoulli distributed random variable decides whether it is an individual $\mathrm{Exp}\left(\frac{c_T^2\,\eta_1}{c_T^2 - c_1\,c_2}\right)$-distributed jump or a common jump with distribution given by F. The probability for an individual jump is the ratio of the intensities of $I^{(1)}$ and $Z^{(1)}$, namely $\frac{(c_T - c_2)^2}{c_T^2 - c_1\,c_2}$. The convex combination of the distribution function turns out to be an $\mathrm{Exp}(\eta_1)$-distribution function.

All statements in Remark 3.17 with respect to the process $C_t^{(1,\{2\})}$ are also valid for the process $C_t^{(2,\{1\})}$ with slightly changed parameters. Theorem 3.16 tells us how the marginal distributions of the decomposed processes look like. To characterize a two-dimensional $\mathrm{CPP}_{\mathrm{Exp}}$ completely, we have to investigate the copula of the jump size distribution of $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$. Theoretically, the copula can be obtained from the joint distribution function of $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$, which can be calculated by means of Theorem 3.13. Unfortunately, there is not a closed form expression for the joint distribution of the jump sizes. It can only be expressed by an infinite sum, what makes a further investigation of the copula meaningless. Nevertheless, it is possible to compute the correlation of the jump sizes which does not characterize the dependence but provides a useful measure of association.

Theorem 3.18 (Correlation between joint jump sizes)
Let the processes $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$ with the parameters $c_T > c_1, c_2 > 0$ be defined as in Equation (3.23). Define $a := \frac{\min\{c_1, c_2\}}{\max\{c_1, c_2\}}$ and $\kappa := \frac{\max\{c_1, c_2\}}{c_T}$. Then

$$
\mathrm{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right)
$$
$$
= \frac{\kappa^2\,a\left(3 - 2\,\kappa\,(1 + a) + \kappa^2\,a\right)}{\sqrt{\left(f(\kappa, a) + \kappa\,(4 - 3\,\kappa - 2\,a\,\kappa + a\,\kappa^3)\right)\left(f(\kappa, a) + a\,\kappa\,(4 - 3\,a\,\kappa - 2\,\kappa + a^2\,\kappa^3)\right)}},
$$

with $f(\kappa, a) := \left(1 - a\,\kappa^2\right)\left(2 - \kappa - a\,\kappa\right)^2$ and $\pi := \inf\left\{t \geq 0 \,\middle|\, C_t^{(1,\{2\})} \neq 0\right\}$.

Proof

The covariance between $C_t^{(1,\{2\})}$ and $C_t^{(2,\{1\})}$ can be derived from the covariance between the jump size distributions of $C^{(1,\{2\})}$ and $C^{(2,\{1\})}$, for all $t \geq 0$, due to the same jump times. More precisely, let a two-dimensional compound Poisson process $\tilde{Z} = \left(\tilde{Z}^{(1)}, \tilde{Z}^{(2)} \right) = \sum_{i=1}^{\tilde{N}_t} \tilde{J}_i$ be given, where $\tilde{J}_1 = \left(\tilde{J}_1^{(1)}, \tilde{J}_1^{(2)} \right)$, $\tilde{J}_2 = \left(\tilde{J}_2^{(1)}, \tilde{J}_2^{(2)} \right), \dots$ is a sequence of independent and identically distributed random variables. Then, conditioning on $\tilde{N}_t$ yields

$$
\begin{aligned}
\mathbb{E}\left[\tilde{Z}_t^{(1)} \, \tilde{Z}_t^{(2)} \right] &= \sum_{n=0}^{\infty} \mathbb{P}\left(\tilde{N}_t = n \right) \mathbb{E}\left[\sum_{i=1}^{n} \tilde{J}_i^{(1)} \sum_{j=1}^{n} \tilde{J}_j^{(2)} \right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\tilde{N}_t = n \right) \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left[\tilde{J}_i^{(1)} \, \tilde{J}_j^{(2)} \right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\tilde{N}_t = n \right) \left(n \, \mathbb{E}\left[\tilde{J}_1^{(1)} \, \tilde{J}_1^{(2)} \right] + \left(n^2 - n \right) \mathbb{E}\left[\tilde{J}_1^{(1)} \right] \mathbb{E}\left[\tilde{J}_1^{(2)} \right] \right) \\
&= \mathbb{E}\left[\tilde{N}_t \right] \mathbb{E}\left[\tilde{J}_1^{(1)} \, \tilde{J}_1^{(2)} \right] + \mathbb{E}\left[\tilde{N}_t^2 - \tilde{N}_t \right] \mathbb{E}\left[\tilde{J}_1^{(1)} \right] \mathbb{E}\left[\tilde{J}_1^{(2)} \right] \\
&= \mathbb{E}\left[\tilde{N}_t \right] \mathbb{E}\left[\tilde{J}_1^{(1)} \, \tilde{J}_1^{(2)} \right] + \mathbb{E}\left[\tilde{N}_t \right]^2 \mathbb{E}\left[\tilde{J}_1^{(1)} \right] \mathbb{E}\left[\tilde{J}_1^{(2)} \right] \\
&= \mathbb{E}\left[\tilde{N}_t \right] \mathbb{E}\left[\tilde{J}_1^{(1)} \, \tilde{J}_1^{(2)} \right] + \mathbb{E}\left[\tilde{Z}_t^{(1)} \right] \mathbb{E}\left[\tilde{Z}_t^{(2)} \right],
\end{aligned}
$$

where we used that $\mathbb{E}\left[\tilde{N}_t^2 - \tilde{N}_t \right] = \mathbb{E}\left[\tilde{N}_t \right]^2$, which holds for all Poisson processes (cf. Theorem 2.6). Hence, we get

$$
\operatorname{Cov}\left(\tilde{Z}_t^{(1)}, \, \tilde{Z}_t^{(2)} \right) = \mathbb{E}\left[\tilde{N}_t \right] \mathbb{E}\left[\tilde{J}_1^{(1)} \, \tilde{J}_1^{(2)} \right]. \tag{3.36}
$$

In the proof of Theorem 3.16 we have seen two different ways to represent the common part $C^{(1,\{2\})}$: on the one hand, a compound Poisson processes with intensity c_T and jump size distribution given by Equation (3.34), and on the other hand, a compound Poisson processes with intensity $\frac{c_1 c_2 (2 c_T - c_1 - c_2)}{c_T^2 - c_1 c_2}$ and jump size distribution given by Equation (3.35). The same holds true for the process $C^{(2,\{1\})}$ with slightly changed parameters. Hence, Equation (3.36) yields two possible ways to express the covariance of $C_t^{(1,\{2\})}$ and $C_t^{(2,\{1\})}$, namely

$$
\operatorname{Cov}\left(C_t^{(1,\{2\})} \, C_t^{(2,\{1\})} \right) = c_T \, t \, \mathbb{E}\left[\mathbb{1}_{\{E_1^{(2)} > 0\}} E_1^{(1)} \, \mathbb{1}_{\{E_1^{(1)} > 0\}} E_1^{(2)} \right] = c_T \, t \, \mathbb{E}\left[E_1^{(1)} \, E_1^{(2)} \right],
$$

and

$$
\operatorname{Cov}\left(C_t^{(1,\{2\})} \, C_t^{(2,\{1\})} \right) = \frac{c_1 c_2 \left(2 c_T - c_1 - c_2 \right)}{c_T^2 - c_1 c_2} \, t \, \mathbb{E}\left[C_\pi^{(1,\{2\})} \, C_\pi^{(2,\{1\})} \right]
$$

where $\pi := \inf \left\{ t \geq 0 \,\middle|\, C_t^{(1,\{2\})} \neq 0 \right\}$. Therefore, by comparing these two expression, we get

$$\mathbb{E}\left[C_\pi^{(1,\{2\})} \, C_\pi^{(2,\{1\})} \right] = \frac{c_T^3 - c_T \, c_1 \, c_2}{c_1 \, c_2 \, (2 \, c_T - c_1 - c_2)} \, \mathbb{E}\left[E_1^{(1)} \, E_1^{(2)} \right]. \tag{3.37}$$

By definition (cf. Equation (3.19)), $E_1^{(1)} = \sum_{j=1}^{N_{T_{\tau_1}}^{(1)}} J_j^{(1)}$ and $E_1^{(2)} = \sum_{j=1}^{N_{T_{\tau_1}}^{(2)}} J_j^{(2)}$ become independent by conditioning on T_{τ_1}. Wald's formula (see Equation (3.3)) implies

$$\begin{aligned}
\mathbb{E}\left[E_1^{(1)} \, E_1^{(2)} \right] &= \mathbb{E}\left[\mathbb{E}\left[E_1^{(1)} \,\middle|\, T_{\tau_1} \right] \mathbb{E}\left[E_1^{(2)} \,\middle|\, T_{\tau_1} \right] \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[J_1^{(1)} \right] \mathbb{E}\left[N_{T_{\tau_1}}^{(1)} \,\middle|\, T_{\tau_1} \right] \mathbb{E}\left[J_1^{(2)} \right] \mathbb{E}\left[N_{T_{\tau_1}}^{(2)} \,\middle|\, T_{\tau_1} \right] \right].
\end{aligned} \tag{3.38}$$

Note that, $J_1^{(1)} \sim \mathrm{Exp}(\frac{c_T \, \eta_1}{c_T - c_1})$, $J_1^{(2)} \sim \mathrm{Exp}(\frac{c_T \, \eta_2}{c_T - c_2})$, $T_{\tau_1} \sim \mathrm{Exp}(1)$, and $N^{(1)}$ and $N^{(2)}$ are Poisson processes with intensities $\frac{c_1}{c_T - c_1}$, respectively $\frac{c_2}{c_T - c_2}$. All quantities appearing in Equation (3.38) are therefore known and stated in Chapter 1. Hence

$$\begin{aligned}
\mathbb{E}\left[E_1^{(1)} \, E_1^{(2)} \right] &= \mathbb{E}\left[\frac{c_T - c_1}{c_T \, \eta_1} \, \frac{c_1}{c_T - c_1} \, T_{\tau_1} \, \frac{c_T - c_2}{c_T \, \eta_2} \, \frac{c_2}{c_T - c_2} \, T_{\tau_1} \right] \\
&= \frac{c_1 \, c_2}{c_T^2 \, \eta_1 \, \eta_2} \, \mathbb{E}\left[T_{\tau_1}^2 \right] = \frac{2 \, c_1 \, c_2}{c_T^2 \, \eta_1 \, \eta_2}.
\end{aligned} \tag{3.39}$$

Plugging in Equation (3.39) in Equation (3.37) yields

$$\mathbb{E}\left[C_\pi^{(1,\{2\})} \, C_\pi^{(2,\{1\})} \right] = \frac{c_T^3 - c_T \, c_1 \, c_2}{c_1 \, c_2 \, (2 \, c_T - c_1 - c_2)} \, \frac{2 \, c_1 \, c_2}{c_T^2 \, \eta_1 \, \eta_2} = \frac{2(c_T^2 - c_1 \, c_2)}{(2 \, c_T - c_1 - c_2) \, c_T \, \eta_1 \, \eta_2}. \tag{3.40}$$

Due to the fact that the density of $C_\pi^{(1,\{2\})}$ is a linear combination of exponential densities (see Remark 3.17), the expectation and the variance of $C_\pi^{(1,\{2\})}$ is easy to compute. Thus, by defining $p := \frac{c_T^2 - c_1 \, c_2}{c_2(2 \, c_T - c_1 - c_2)}$, we get

$$\mathbb{E}\left[C_\pi^{(1,\{2\})} \right] = p \, \frac{1}{\eta_1} + (1 - p) \frac{c_T^2 - c_1 \, c_2}{c_T^2 \, \eta_1} = \frac{c_T^2 - (1 - p) \, c_1 \, c_2}{c_T^2 \, \eta_1} = \frac{c_T^2 - (1 - p) \, c_1 \, c_2}{c_T^2 \, \eta_1}, \tag{3.41}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left(C_\pi^{(1,\{2\})} \right)^2 \right] &= p \, \frac{2}{\eta_1^2} + (1 - p) \frac{2 \left(c_T^2 - c_1 \, c_2 \right)^2}{c_T^4 \, \eta_1^2} \\
&= \frac{2 \, p \, c_T^4 + 2 \, (1 - p) \left(c_T^4 - 2 \, c_T^2 \, c_1 \, c_2 + c_1^2 \, c_2^2 \right)}{c_T^4 \, \eta_1^2}
\end{aligned}$$

$$= \frac{2\,c_T^4 + 2\,(1-p)\,c_1\,c_2\,\left(-2\,c_T^2 + c_1\,c_2\right)}{c_T^4\,\eta_1^2},$$

and therefore,

$$\mathrm{Var}\left[C_\pi^{(1,\{2\})}\right] = \mathbb{E}\left[\left(C_\pi^{(1,\{2\})}\right)^2\right] - \mathbb{E}\left[C_\pi^{(1,\{2\})}\right]^2$$

$$= \frac{2\,c_T^4 + 2\,(1-p)\,c_1\,c_2\,\left(-2\,c_T^2 + c_1\,c_2\right)}{c_T^4\,\eta_1^2} - \left(\frac{c_T^2 - (1-p)\,c_1\,c_2}{c_T^2\,\eta_1}\right)^2$$

$$= \frac{c_T^4 - 2\,(1-p)\,c_T^2\,c_1\,c_2 + \left(2 - 2p - (1-p)^2\right)c_1^2\,c_2^2}{c_T^4\,\eta_1^2}$$

$$= \frac{c_T^4 - 2\,(1-p)\,c_T^2\,c_1\,c_2 + \left(1 - p^2\right)c_1^2\,c_2^2}{c_T^4\,\eta_1^2}$$

$$= \frac{c_T^4 - 2\,c_T^2\,c_1\,c_2 + c_1^2\,c_2^2 + 2\,p\,c_T^2\,c_1\,c_2 - p^2\,c_1^2\,c_2^2}{c_T^4\,\eta_1^2}$$

$$= \frac{\left(c_T^2 - c_1\,c_2\right)^2}{c_T^4\,\eta_1^2} + \frac{2\,c_T^2\,c_1\,c_2^2\,\left(c_T^2 - c_1\,c_2\right)(2\,c_T - c_1 - c_2) - \left(c_T^2 - c_1\,c_2\right)^2 c_1^2\,c_2^2}{c_T^4\,\eta_1^2\,c_2^2\,(2\,c_T - c_1 - c_2)^2}$$

$$= \frac{\left(c_T^2 - c_1\,c_2\right)^2}{c_T^4\,\eta_1^2} + \frac{c_1\,\left(c_T^2 - c_1\,c_2\right)\left(2\,c_T^2\,(2\,c_T - c_1 - c_2) - \left(c_T^2 - c_1\,c_2\right)c_1\right)}{c_T^4\,\eta_1^2\,(2\,c_T - c_1 - c_2)^2}$$

$$= \frac{\left(c_T^2 - c_1\,c_2\right)^2}{c_T^4\,\eta_1^2} + \frac{c_1\,\left(c_T^2 - c_1\,c_2\right)\left(4\,c_T^3 - 3\,c_T^2\,c_1 - 2\,c_T^2\,c_2 + c_1^2\,c_2\right)}{c_T^4\,\eta_1^2\,(2\,c_T - c_1 - c_2)^2}$$

$$= \frac{\left(c_T^2 - c_1\,c_2\right)A_1}{c_T^4\,\eta_1^2\,(2\,c_T - c_1 - c_2)^2}, \tag{3.42}$$

where $A_1 := \left(c_T^2 - c_1\,c_2\right)(2\,c_T - c_1 - c_2)^2 + c_1\left(4\,c_T^3 - 3\,c_T^2\,c_1 - 2\,c_T^2\,c_2 + c_1^2\,c_2\right)$. Analogously, we get

$$\mathrm{Var}\left[C_\pi^{(2,\{1\})}\right] = \frac{\left(c_T^2 - c_1\,c_2\right)A_2}{c_T^4\,\eta_2^2\,(2\,c_T - c_1 - c_2)^2}, \tag{3.43}$$

where $A_2 := \left(c_T^2 - c_1\,c_2\right)(2\,c_T - c_1 - c_2)^2 + c_2\left(4\,c_T^3 - 3\,c_T^2\,c_2 - 2\,c_T^2\,c_1 + c_1\,c_2^2\right)$. Moreover, by using the definition of p, Equation (3.41) boils down to

$$\mathbb{E}\left[C_\pi^{(1,\{2\})}\right] = \frac{c_T^2 - \left(1 - \frac{c_T^2 - c_1\,c_2}{c_2(2\,c_T - c_1 - c_2)}\right)c_1\,c_2}{c_T^2\,\eta_1}$$

$$= \frac{c_T^2\,c_2\,(2\,c_T - c_1 - c_2) - \left(c_2\,(2\,c_T - c_1 - c_2) - c_T^2 + c_1\,c_2\right)c_1\,c_2}{c_T^2\,c_2\,(2\,c_T - c_1 - c_2)\,\eta_1}$$

$$= \frac{c_T^2\,(2\,c_T - c_1 - c_2) + (c_T - c_2)^2\,c_1}{c_T^2\,(2\,c_T - c_1 - c_2)\,\eta_1}. \tag{3.44}$$

Analogously, we get

$$\mathbb{E}\left[C_\pi^{(2,\{1\})} \right] = \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) + \left(c_T - c_1\right)^2 c_2}{c_T^2 \left(2\,c_T - c_1 - c_2\right) \eta_2}. \tag{3.45}$$

Now, the covariation of $C_\pi^{(1,\{2\})}$ and $C_\pi^{(2,\{1\})}$ can be computed by putting together Equations (3.40), (3.44) and (3.45).

$$
\begin{aligned}
&\mathrm{Cov}\left(C_\pi^{(1,\{2\})},\, C_\pi^{(2,\{1\})} \right) \\
=& \frac{2(c_T^2 - c_1\,c_2)}{(2\,c_T - c_1 - c_2)\,c_T\,\eta_1\,\eta_2} \\
&\quad - \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) + \left(c_T - c_2\right)^2 c_1}{c_T^2 \left(2\,c_T - c_1 - c_2\right) \eta_1}\, \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) + \left(c_T - c_1\right)^2 c_2}{c_T^2 \left(2\,c_T - c_1 - c_2\right) \eta_2} \\
=& \frac{2(c_T^2 - c_1\,c_2)\left(2\,c_T - c_1 - c_2\right) c_T^3 - \left(c_T - c_2\right)^2 \left(c_T - c_1\right)^2 c_1\,c_2}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
&\quad - \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) \left(c_T^2 \left(2\,c_T - c_1 - c_2\right) + \left(c_T - c_2\right)^2 c_1 + \left(c_T - c_1\right)^2 c_2 \right)}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
=& \frac{2(c_T^2 - c_1\,c_2)\left(2\,c_T - c_1 - c_2\right) c_T^3 - \left(c_T - c_2\right)^2 \left(c_T - c_1\right)^2 c_1\,c_2}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
&\quad - \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) \left(2\,c_T^3 - 4\,c_T\,c_1\,c_2 + c_1\,c_2\,(c_1 + c_2) \right)}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
=& \frac{c_T^2 \left(2\,c_T - c_1 - c_2\right) \left(2\,c_T\,c_1\,c_2 - c_1\,c_2\,(c_1 + c_2)\right) - \left(c_T - c_2\right)^2 \left(c_T - c_1\right)^2 c_1\,c_2}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
=& \frac{c_1\,c_2 \left(c_T^2 \left(2\,c_T - c_1 - c_2\right)^2 - \left(c_T - c_2\right)^2 \left(c_T - c_1\right)^2 \right)}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2} \\
=& \frac{c_1\,c_2 \left(c_T^2 - c_1\,c_2\right) \left(3\,c_T^2 - 2\,c_T\,(c_1 + c_2) + c_1\,c_2\right)}{c_T^4 \left(2\,c_T - c_1 - c_2\right)^2 \eta_1\,\eta_2}
\end{aligned}
\tag{3.46}
$$

The correlation can be calculated using Equations (3.42), (3.43), and (3.46), namely

$$
\begin{aligned}
\mathrm{Corr}\left(C_\pi^{(1,\{2\})},\, C_\pi^{(2,\{1\})} \right) &= \frac{\mathrm{Cov}\left(C_\pi^{(1,\{2\})},\, C_\pi^{(2,\{1\})} \right)}{\sqrt{\mathrm{Var}\left[C_\pi^{(1,\{2\})} \right] \mathrm{Var}\left[C_\pi^{(2,\{1\})} \right]}} \\
&= \frac{c_1\,c_2 \left(3\,c_T^2 - 2\,c_T\,(c_1 + c_2) + c_1\,c_2\right)}{\sqrt{A_1\,A_2}}
\end{aligned}
$$

Finally, using the transformations $a = \frac{\min\{c_1, c_2\}}{\max\{c_1, c_2\}}$ and $\kappa = \frac{\max\{c_1, c_2\}}{c_T}$ yields

$$\text{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = \frac{c_1 \, c_2 \left(3 \, c_T^2 - 2 \, c_T \, (c_1 + c_2) + c_1 \, c_2\right)}{\sqrt{A_1 \, A_2}}$$

$$= \frac{c_T^4 \, \kappa^2 \, a \left(3 - 2 \, \kappa \, (1 + a) + \kappa^2 \, a\right)}{\sqrt{A_1 \, A_2}},$$

and by defining $f(\kappa, a) := \left(1 - a \, \kappa^2\right) \left(2 - \kappa - a \, \kappa\right)^2$, we have

$$A_1 \, A_2$$
$$= c_T^8 \left(f(\kappa, a) + \kappa \left(4 - 3 \, \kappa - 2 \, a \, \kappa + a \, \kappa^3\right)\right) \left(f(\kappa, a) + a \, \kappa \left(4 - 3 \, a \, \kappa - 2 \, \kappa + a^2 \, \kappa^3\right)\right).$$

Hence,

$$\text{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right)$$
$$= \frac{\kappa^2 \, a \left(3 - 2 \, \kappa \, (1 + a) + \kappa^2 \, a\right)}{\sqrt{\left(f(\kappa, a) + \kappa \left(4 - 3 \, \kappa - 2 \, a \, \kappa + a \, \kappa^3\right)\right) \left(f(\kappa, a) + a \, \kappa \left(4 - 3 \, a \, \kappa - 2 \, \kappa + a^2 \, \kappa^3\right)\right)}},$$

which concludes the proof. $\qquad\square$

The correlation function in Theorem 3.18 seems a bit cumbersome on first sight. Therefore, we have a look at the limiting cases of the parameters a and κ to get a better intuition about the formula. Note first, that the correlation does not dependent on the actual choice of the jump size parameters η_1 and η_2. Indeed, it only depends on the ratio of the time change intensity and the larger of the two marginal intensity (κ) and the ratio of the two marginal intensities (a).

Remark 3.19 (On the correlation between joint jump sizes)
The limit behavior of the correlation function in Theorem 3.18 is given by the following expressions:

(i) Cases with vanishing correlation,

$$\lim_{\kappa \searrow 0} \text{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = 0, \qquad \lim_{a \searrow 0} \text{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = 0.$$

(ii) Maximal correlation for fixed marginal intensities c_1 and c_2,

$$\lim_{\kappa \nearrow 1} \text{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = \frac{a \, (1 - a)}{\sqrt{\left((1 - a)^3 + (1 - a)\right) \left((1 - a)^3 + a \, (2 - 3 \, a + a^2)\right)}}$$

$$= \frac{a}{\sqrt{\left((1-a)^2+1\right)\left((1-a)^2+a\,(2-a)\right)}}$$

$$= \frac{a}{\sqrt{2-2\,a+a^2}} = \frac{a}{\sqrt{1-(1-a)^2}}.$$

(iii) Correlation, if $c_1 = c_2$,

$$\lim_{a\nearrow 1} \mathrm{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = \frac{\kappa^2\left(3-4\,\kappa+\kappa^2\right)}{(1-\kappa^2)\,(2-2\,\kappa)^2 + \kappa\,(4-5\,\kappa+\kappa^3)}$$

$$= \frac{\kappa^2\,(3-\kappa)}{4\,(1-\kappa^2)\,(1-\kappa) + \kappa\,(4-\kappa-\kappa^2)}$$

$$= \frac{\kappa^2\,(3-\kappa)}{4-5\,\kappa^2+3\,\kappa^3}.$$

(iv) Maximal correlation is obtained by $\displaystyle\lim_{\kappa\nearrow 1}\lim_{a\nearrow 1} \mathrm{Corr}\left(C_\pi^{(1,\{2\})}, C_\pi^{(2,\{1\})}\right) = 1$.

Figure 3.2 illustrates this correlation coefficient. To obtain a significant jump size correlation at common jump times, c_T, c_1, c_2 must be of similar size.
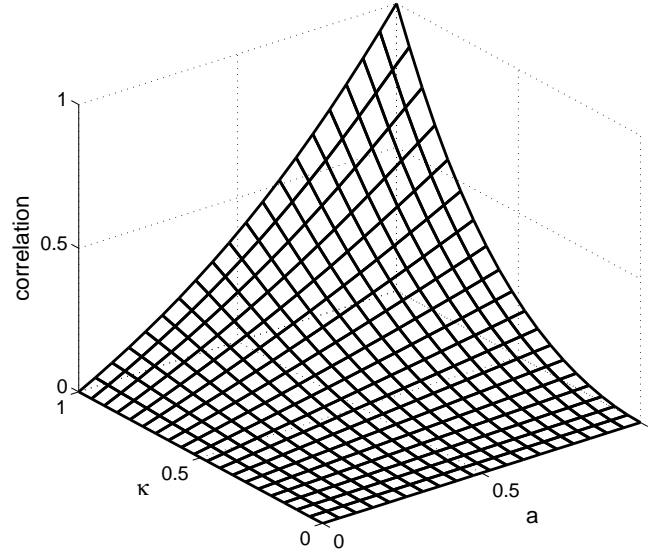

Figure 3.2 Correlation coefficient of the jump sizes at common jump times as a function of a and κ.

3.6 Application: Dependent jump modeling

The time-change construction of dependent CPP_{Exp}, which we presented and thoroughly investigated in this chapter has some striking features making it a useful tool in financial engineering. In this section, we motivate some possible applications in the area of dependent jump modeling in derivative pricing and we give links to the following topics of this thesis. First of all, note that a time-changed CPP_{Exp} can easily be simulated due to its simple construction principle. An efficient algorithm to simulate paths of d-dimensional time-changed CPP_{Exp} is essential, e.g. for derivatives pricing via Monte Carlo simulation. Therefore, the next algorithm shows how such paths can be obtained.

Algorithm 3.20 (Path simulation of time-changed CPP_{Exp})
Suppose the following parameters to be given: parameters for the univariate processes $Z^{(1)}, \ldots, Z^{(d)}$, i.e. $c_1, \ldots, c_d$ and $\eta_1, \ldots, \eta_d$, the dependence parameter κ, and the maturity $t^* > 0$.

(1) Define $c_T := \frac{1}{\kappa} \max_{1 \leq i \leq d} \{c_i\}$ and simulate $N \sim \text{Poi}(c_T \, t^*)$.

(2) Draw N independent and Uniform$[0, t^*]$-distributed random variables and sort them in ascending order, resulting in $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_N \leq t^*$.

(3) Draw N independent and $\text{Exp}(1)$-distributed random variables $E_1, \ldots, E_N$.

(4) For each $1 \leq i \leq d$ do

      For each $1 \leq j \leq N$ do

    (a) Draw a $\text{Poi}\left(E_j \frac{c_i}{c_T - c_i}\right)$-distributed random variables $M_j^{(i)}$.

    (b) Draw an $\text{Erlang}\left(M_j^{(i)}, \frac{c_T \, \eta_i}{c_T - c_i}\right)$-distributed random variable $J_j^{(i)}$.

    (c) Compute $Z_j^{(i)} = \sum_{k=1}^{j} J_k^{(i)}$.

(5) Return $(\tau_1, \ldots, \tau_N)$ and $\left(Z^{(1)}, \ldots, Z^{(d)}\right)$, with $Z^{(i)} := \left(Z_1^{(i)}, \ldots, Z_N^{(i)}\right)$ for all $1 \leq i \leq d$.

Let us have a closer look at the algorithm. The random variable N represents the number of jumps of the subordinator T until time t^*. The points in time $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$ serve as the jump times of T and, moreover, as possible jump times of

$Z^{(1)}, \ldots, Z^{(d)}$. An alternative, and maybe more natural, way to simulate those jump times can be achieved by adding up exponentially distributed random variables until the sum reaches t^* (cf. the construction of Poisson processes in Theorem 2.13). Here, we used that the conditional distribution of the jump times of a Poisson process given the number of jumps N in $[0, t^*]$ coincides with the distribution of the order statistics of N samples obtained from a uniform distribution on $[0, t^*]$.[5] In situations with high expected numbers of jumps of the process T, the classical method becomes faster, because the ordering in Step (2) gets time-consuming, and the classical drawing of jump times is the method of choice. However, for small numbers of N the presented method is more efficient, which will be the case in our practical examples throughout the thesis. The random variable E_j yields the jump size of T at the j-th jump time. In step (4), we use them to determine the parameter of the Erlang distribution, which then gives the actual jump sizes of our resulting processes. Actually, the return variables $Z_1^{(i)}, \ldots, Z_N^{(i)}$ represent the i-th component of the time-change $\mathrm{CPP}_{\mathrm{Exp}}$ at the possible jump times $\tau_1, \ldots, \tau_N$.

Besides the easy-to-simulate setup, another nice property of our construction is that only one parameter, namely κ, suffices to model a quite flexible dependence structure. Note that, for example, individual jumps of only one component, joint jumps of all components, and also joint jumps of an arbitrary subset of marginal processes can be generated by a time-changed $\mathrm{CPP}_{\mathrm{Exp}}$. Furthermore, the jump magnitudes are dependent. Given that we only use one dependence parameter, there are situations where the construction reaches its limits. For example, imagine the following situation. Assume, a time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ be given, whose components can be separated into two sets. One set consisting of one-dimensional processes having huge intensities $c_h > 0$ and the other set having small intensities c_s, $c_h \gg c_s > 0$. Then, by the results of Section 3.3 the components with small intensities behave like independent processes and our construction principle can not explain much dependence between them. In particular, e.g. from Theorem 3.9, we know, that the correlation between two processes with intensity c_h equals κ and can therefore take all values in $(0, 1)$. The correlation between a process with huge intensity c_h and one with a small one c_s is given by $\kappa \sqrt{\frac{c_s}{c_h}}$, which is very small even for $\kappa \nearrow 1$. That is still fine, since these two processes behave very diverse by definition. The problem arises for two processes with small intensity. Then, the correlation vanishes. Similar deductions can be made by using the pragmatic jump decomposition in Section 3.5. Therefore, the construction by subordination as

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[5]see, for example, [Sato, 1999, Proposition 3.4] for a reference

presented in the previous sections might not be suited in some situations, which is not surprising, since the dependence structure is driven by one parameter only, even in high dimensions. To fix the problem, one could come up with two separate subordinations, i.e. in the above mentioned setup, each set of similar processes are subordinated with an individual CPP$_{\text{Exp}}$. The two time-change processes can be dependent as well. On the one hand, such a construction solves the issue, on the other hand, we end up with at least two dependence parameter. Hence, it is a trade-off between flexibility and complexity. All in all, if the intensities are of same order, the dependence structure implied by our construction is quite flexible and we found a handy tool for modeling dependent jump processes. In the following, we give an overview of some possible applications.

A time-changed CPP$_{\text{Exp}}$ can be applied to model dependence for pricing derivatives and serves as a useful tool in all three situations presented in Chapter 1. Actually, the construction is helpful if two or more processes of financial terms have to be modeled dependently, whereas the marginal processes are driven by at least one CPP$_{\text{Exp}}$. Examples for such processes are asset prices, volatilities, interest rates, exchange rates, and (cumulative) hazard rates. *First*, in Chapter 4 we model dependence between stock prices and their volatility processes, ending up with a generalization of the popular Γ-OU-BNS model. Here, both, the log prices and the volatility is driven by a CPP$_{\text{Exp}}$. We will use a two-dimensional time-changed CPP$_{\text{Exp}}$ as stochastic driver. This extends the parameter space by one parameter modeling the dependence between log-price and volatility, which will be the jump dependence parameter κ of the time change construction. A striking advantage of introducing dependence among the jumps in this manner is that the time-changed processes remain in the class of compound Poisson processes with exponential jump heights, which ensures that the marginal processes maintain a tractable structure. A useful property of the Γ-OU-BNS model is its closed form solution of the characteristic function of the log prices, which is essential for a fast model calibration by means of Fourier pricing methods. Our construction by subordination preserves this desirable feature, i.e. there exists also an analytic expression for the characteristic function of the log price in the generalized model. *Secondly*, Chapter 5 presents multivariate jump diffusion models. Time-changed CPP$_{\text{Exp}}$ are used to introduce dependence between two or more processes of the same kind, building tractable multivariate models. We show multi-dimensional versions of the Γ-OU-BNS-model, the Kou-model (cf. Kou [2002]), and a two-sided extension of the BNS-model. The multivariate models are constructed via a bottom-up approach, which means we start

with several one-dimensional models and link them by adding dependence in a second step. This is a striking feature when it comes to calibration. Since the subordination construction ensures that the resulting process stays in the same class of CPP_{Exp}, the univariate log-price processes of the extended models still follow the classical one-dimensional model of that kind and the parameters of the univariate processes may be calibrated separately to univariate derivative prices. The dependence parameters can be calibrated separately afterwards without altering the already fixed marginal distributions. This simplifies the model calibration and is a convenient feature for practical purposes, because it automatically ensures that univariate derivative prices are fitted to the multivariate model. Furthermore, it keeps the number of parameters which cannot be caught from vanilla option price data limited. *Thirdly*, time-changed CPP_{Exp} could be useful as well for modeling wrong way risk in credit valuation adjustment by combining credit risk modeling via hazard rates driven by a CPP_{Exp} and asset price modeling. From a mathematical perspective such an credit-equity ansatz is quite similar to the methods presented in Chapters 4 and 5. Therefore, we attack the problem of calculating wrong way risk from a different angle, namely a model-free ansatz in Chapter 6.

4 Decoupling of volatility jumps and asset jumps in the BNS model

For derivatives valuation, the Black–Scholes model, presented in the seminal papers Samuelson [1965] and Black and Scholes [1973], generated a wave of stochastic models for the description of stock-prices. Since the assumptions of the Black–Scholes model (normally distributed log-returns, independent returns) cannot be observed in neither time series of stock-prices nor option markets (implicitly expressed in terms of the volatility surface), several alternative models have been developed trying to overcome these assumptions. One approach to extend the Black–Scholes model is the incorporation of jumps into the asset price process, inspired by market shocks causing the asset price to jump. These jumps are driven by Lévy processes, e.g. Merton [1976]; Kou [2002] use a compound Poisson process as jump driver, Madan and Senata [1990] a Variance Gamma process, or Barndorff-Nielsen [1997] a normal inverse Gaussian process. Another generalization is achieved by substituting the constant Black–Scholes volatility by a stochastic process, leading to diffusion-style stochastic volatility models as in, e.g., Stein and Stein [1991]; Heston [1993], also enhanced by independent jumps in the asset price process by Bates [1996]; Duffie et al. [2000]. Another approach combining stochastic volatility and jumps in both volatility and asset price process is available with the Barndorff-Nielsen–Shephard (BNS) model class, presented in Barndorff-Nielsen and Shephard [2001] and extended in several papers (e.g. Nicolato and Venardos [2003]).

In this chapter, which is based on our published article Hofmann and Schulz [2016], we present an extension of the BNS model class, mitigating the strong link between asset price jumps and jumps in volatility. Before presenting this generalization in Section 4.2, we give a short introduction to the BNS model class and present one popular member of that class, the so-called Γ-OU-BNS model in Section 4.1. The model is tractable in the sense that the characteristic function of the log-price can be expressed in closed form. This ensures quick and convenient valuation of plain vanilla derivatives (e.g.

for calibration purposes) because Fourier pricing methods like FFT pricing (e.g. Carr and Madan [1999]; Raible [2000]) or the COS method described in Fang and Osterlee [2008] can efficiently be used. The construction of a time-changed CPP_{Exp}, which was discussed in Chapter 3, is applied in Section 4.3 to build a generalization of the Γ-OU-BNS model, which we call weak-link Γ-OU-BNS model. As usual, we conclude the chapter by an application in Section 4.4. Particularly, we investigate the model dynamics of the weak-link Γ-OU-BNS model and employ a calibration exercise.

4.1 Fundamentals: The Barndorff-Nielsen–Shephard model class

In the seminal paper Barndorff-Nielsen and Shephard [2001], a tractable stochastic volatility model class was presented. The variance process $\sigma^2 = \{\sigma_t^2\}_{t\geq 0}$ in the BNS model class is given by a non-Gaussian Ornstein–Uhlenbeck (OU) process, driven by a Lévy subordinator. Furthermore, the same Lévy subordinator adds jumps to the asset price process $S = \{S_t\}_{t\geq 0}$, linking jumps in volatility and jumps in the asset price. Indeed, the dynamics of the log-price $X = \{X_t\}_{t\geq 0} := \{\log(S_t)\}_{t\geq 0}$ and the volatility are governed by the SDEs

$$\mathrm{d}X_t = \left(\mu + \beta\,\sigma_t^2\right)\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + \rho\,\mathrm{d}Z_t, \tag{4.1}$$

$$\mathrm{d}\sigma_t^2 = -\lambda\,\sigma_t^2\,\mathrm{d}t + \mathrm{d}Z_t, \tag{4.2}$$

where $W = \{W_t\}_{t\geq 0}$ is a Brownian motion, $Z = \{Z_t\}_{t\geq 0}$ a Lévy subordinator (independent of W), $\rho \leq 0$, $\sigma_0^2, \lambda > 0$ and the drift coefficients μ, β are given by the martingale drift condition.[1] Since the so-called *leverage parameter* ρ is negative, upward jumps in the volatility are always accompanied by downward jumps in the asset price process, which accounts for modeling the leverage effect, a well documented statistical observation. A solution to Equation (4.2) is given by the OU-process

$$\sigma_t^2 = e^{-\lambda\,t}\,\sigma_0^2 + e^{-\lambda t}\int_0^t e^{\lambda s}\,\mathrm{d}Z_s, \tag{4.3}$$

---

[1]In many formulations of BNS-type models, an additional time change $t \mapsto \lambda t$ is employed to the process $(Z_t)_{t\geq 0}$, which is mainly for mathematical reasons. From a modeling point of view, the formulation without time change is equivalent.

which can be shown by using the product rule of Itô calculus applied to $\mathrm{d}(e^{\lambda t}\sigma_t^2)$, plugging in the SDE (4.2), and integrating from 0 to t. Particularly,

$$\mathrm{d}\left(e^{\lambda t}\sigma_t^2\right) = e^{\lambda t}\,\mathrm{d}\sigma_t^2 + \lambda\,e^{\lambda t}\,\sigma_t^2\,\mathrm{d}t = e^{\lambda t}\,\mathrm{d}Z_t$$

$$\Leftrightarrow e^{\lambda t}\sigma_t^2 - \sigma_0^2 = \int\limits_0^t e^{\lambda s}\,\mathrm{d}Z_s \Leftrightarrow \sigma_t^2 = e^{-\lambda t}\,\sigma_0^2 + e^{-\lambda t}\int_0^t e^{\lambda s}\,\mathrm{d}Z_s$$

Thus, the squared volatility process σ^2 increases by jumps and declines exponentially between any two consecutive jumps. The rate of decay is set by the slow-down parameter $\lambda > 0$. Figure 4.1 illustrates the joint behavior of the asset price process and the volatility process.

One of the most popular choices for the Lévy driver is a $\mathrm{CPP}_{\mathrm{Exp}}$, resulting in the Γ-OU-BNS model.

Definition 4.1 (Γ-OU-BNS model)
Assume the price dynamic follows the SDEs in Equations (4.1) and (4.2) with $\{Z_t\}_{t\geq 0}$ being a $\mathrm{CPP}_{\mathrm{Exp}}(c, \eta)$. Due to the exponential jump sizes, the stationary distribution of σ^2 follows a Gamma law (cf. Barndorff-Nielsen and Shephard [2001]). Thus, we call the variance process $(\sigma_t^2)_{t\geq 0}$ a Γ-Ornstein–Uhlenbeck process and the BNS model with Lévy driver Z a Γ-OU-BNS model.

Under a risk-neutral martingale measure, the drift components of the Γ-OU-BNS model have to satisfy

$$\mu = r - \frac{c\rho}{\eta - \rho}, \qquad \beta = -\frac{1}{2}, \tag{4.4}$$

where r denotes the constant risk-free interest rate. For more details on the choice of the risk neutral measure[2] within this model setup, we refer to Nicolato and Venardos [2003]. A striking feature of this model is the existence of a closed-form expression for the characteristic function of the log-price process X, as stated in the next theorem. Besides the Γ-OU-BNS model there is only one other model specification known with

---

[2]Heuristically, the risk neutral measure is a probability measure such that the asset price coincides with the discounted expectation of the future asset price under this measure. A rigorous mathematical definition can be found in standard textbooks on financial derivative pricing, like Shreve [2004].
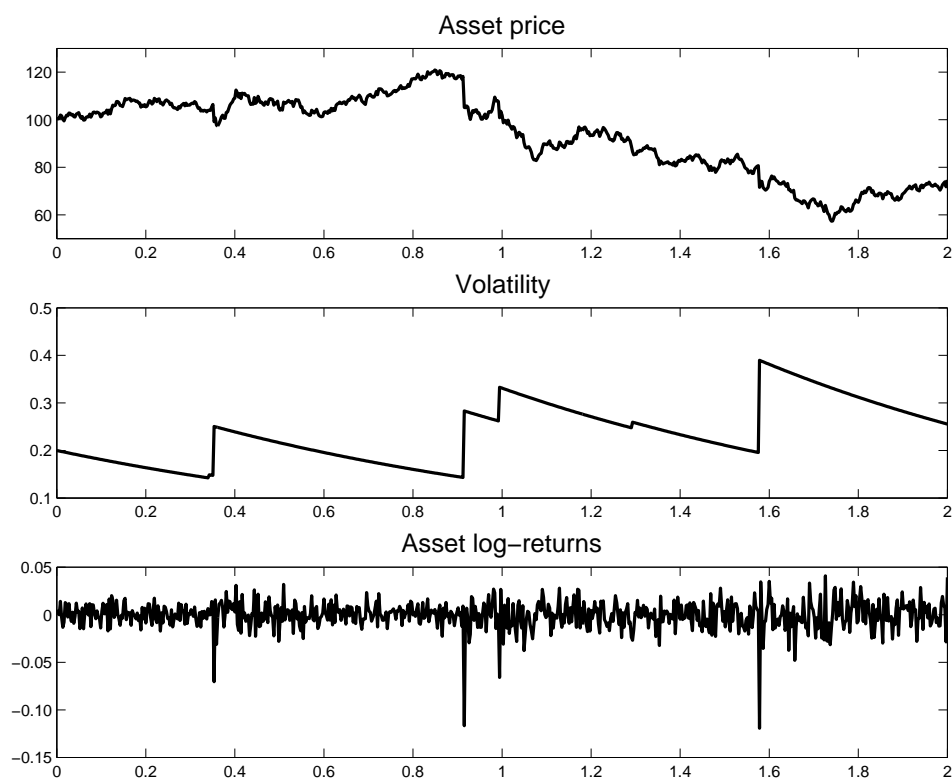
Figure 4.1 The above graph shows the asset price process S, the middle graph shows the volatility process σ, and the graph below the daily log-returns ΔX. A negative jump in the asset value process appears with a simultaneous increase in the volatility, which is a quite realistic stylized fact and can be seen as a market shock. Right after such a shock there is a great nervousness and thus a high volatility. As time goes by the volatility calms down until the next jump occurs. This volatility clustering can be detected in the last graph.

the property of a closed-form characteristic function, namely the so-called IG-OU-BNS model, where the stationary distribution of the variance process is inverse Gaussian (cf. Barndorff-Nielsen et al. [2002]).

Theorem 4.2 (Characteristic function in the Γ-OU-BNS model)

Let $S = \{S_t\}_{t \geq 0}$ follow a Γ-OU-BNS model, then the characteristic function of $X_t = \log(S_t)$ is given by

$$\mathbb{E}\left[e^{\mathrm{i}\, u\, X_t}\right] = \exp\left(\mathrm{i}\, u\,(X_0 + \mu\, t) - g\, h\, \sigma_0^2 + \frac{c}{\eta - f_2}\left(\frac{\eta}{\lambda}\, \log\frac{\eta - f_1}{\eta - \mathrm{i}\, u\, \rho} + f_2\, t\right)\right),$$

with

$$g := \frac{u^2 - 2\,\beta\,\mathrm{i}\, u}{2}, \quad h := \frac{1 - \exp(-\lambda\, t)}{\lambda}, \quad f_1 := \mathrm{i}\, u\, \rho - g\, h, \quad f_2 := \mathrm{i}\, u\, \rho - \frac{g}{\lambda}.$$

Proof

See Nicolato and Venardos [2003]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2 The BNS model with decoupled jumps

The BNS model, as described in the previous section, incorporates the leverage effect in a rather restrictive manner: *Every* jump in the volatility process is accompanied by a jump in the stock price and vice versa. Obviously, this strong link can seriously be doubted. On the one hand, a sudden jump in the stock price may trigger several limit and stop orders. Hence, from an economic perspective, it makes sense that rising volatility can be a side effect of asset prices jumps. On the other hand, suddenly changing volatility can have manifold reasons; some reasons are presented in, e.g., Shiller [1988]. Furthermore, performing a statistical analysis, Jacod and Todorov [2010] scrutinize the jump behavior of the S&P 500 and its volatility process and obtain strong evidence for the existence of separate and joint jumps in both the asset price and its volatility. Thus, a model establishing different levels of dependence between asset price jumps and volatility jumps could provide a more realistic behavior of the stylized facts of asset price dynamics than the classical BNS model. In this section, we mitigate the strong link between asset price jumps and jumps in the volatility and present the BNS model with decoupled jumps, which incorporates a stochastic variance process following a Lévy-subordinator-driven Ornstein-Uhlenbeck process, Lévy jumps in the log-price process, and flexible dependence between log-price process jumps and volatility process jumps.

Definition 4.3 (BNS model with decoupled jumps)
We say that a positive asset price process $S = (S_t)_{t \geq 0}$ follows a *BNS model with decoupled jumps*, if the log-price process $X = \{X_t\}_{t \geq 0} = \{\log S_t\}_{t \geq 0}$ has the dynamics

$$dX_t = (\mu + \beta \, \sigma_t^2) \, dt + \sigma_t \, dW_t + dY_t, \tag{4.5}$$

$$d\sigma_t^2 = -\lambda \, \sigma_t^2 \, dt + dZ_t, \tag{4.6}$$

where $W = \{W_t\}_{t \geq 0}$ is a Brownian motion, $(Y, Z) = \{(Y_t, Z_t)\}_{t \geq 0}$ is a 2-dimensional pure jump Lévy process, i.e. the process has no Brownian component. Furthermore, Z is a subordinator, $\mu, \beta \in \mathbb{R}$, and $\sigma_0^2, \lambda > 0$, and we require that Y and Z are mutually independent of the Brownian motion W.

The main difference between the classical BNS model and the BNS model with decoupled jumps lies, as the name says, in the (weaker, and more flexible) relationship between volatility and asset jumps: In the classical BNS model, every (upward) volatility jump is accompanied by a downward jump in the asset price process, while the parameter ρ steers the magnitude of the asset price process jump. Conversely, in the decoupled model, this one-to-one relationship does not hold anymore: Similar to the development of the Cox–Ingersoll–Ross-type stochastic volatility models from Heston [1993] over Bates [1996] to Duffie et al. [2000], the dependence of volatility and asset prices becomes more sophisticated, since we only assume to have some dependence structure preserving the two-dimensional Lévy structure of Y and Z. Some suggestions how to construct dependent Lévy processes are elaborated in Deelstra and Petkovic [2010], which we already discussed in Chapter 3 for the case of compound Poisson processes. We start with stating one special subclass, which is constructed by linear dependence.

Example 4.4 (General linear dependence BNS model)
Let $\mathbf{Z} = (Z_t^1, \ldots, Z_t^n)_{t \geq 0}$ be an n-dimensional Lévy subordinator with independent coordinate processes and let $\boldsymbol{\rho} \in \mathbb{R}^n$. Furthermore, let $\boldsymbol{\xi} \in \{0, 1\}^n$ with at least one $\xi_j = 1$, $j = 1, \ldots, n$. Then the model following the dynamics

$$dX_t = (\mu + \beta \sigma_t^2) \, dt + \sigma_t \, dW_t + \boldsymbol{\rho}' d\mathbf{Z}_t,$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 \, dt + \xi' d\mathbf{Z}_t,$$

is called the *general linear dependence BNS model*.

Choosing $n = 1$, $\boldsymbol{\xi} = 1$, and $\boldsymbol{\rho} = (\rho)$ with $\rho \leq 0$, the linear dependence model boils down to the "classical" BNS model. Choosing $n = 2$, $\boldsymbol{\xi} = (1,1)'$, and $\boldsymbol{\rho} = (\rho_-, \rho_+)$ with $\rho_- \leq 0 \leq \rho_+$, the linear dependence model reduces to the two-sided BNS model of Bannör and Scherer [2013], which we introduce in more detail in Section 5.1. Furthermore, the construction principle in the model described in Definition 4.3 is very flexible and easily extends several models existing in the literature as, e.g., OU-stochastic volatility versions of jump-diffusion models described in Merton [1976] or Kou [2002].

Besides the linear construction, dependent Lévy processes can also be constructed by a joint time change. When introducing dependence by joint time change between independent Lévy processes, the link between the jumps in the volatility and asset price process becomes weaker and more blurry. Joint time change of two independent Lévy processes causes the probability of joint jumps to rise due to "common clocking", but does not necessarily imply simultaneous jumps (as discussed above). In this setup, time change is not employed to model some kind of *business time* (as, e.g., in Luciano and Schoutens [2006]), but the time change construction is solely used as a technical aid to establish a weak type of dependence between the Lévy processes, causing both joint and separate jumps to occur in a stochastic manner.

As described in Deelstra and Petkovic [2010], a third possibility to construct dependent Lévy processes can be obtained by linking the respective Lévy measures by Lévy copulas, which was promoted by Tankov [2004] and Kallsen and Tankov [2006]. Analogously to linking marginal distributions by a copula (as described in Nelsen [2006]), one may link univariate, independent Lévy measures by a Lévy copula. Lévy copulas are functions fulfilling some regularity conditions linking the tail integrals w.r.t. the Lévy measures. Sklar's theorem for Lévy copulas (cf. [Kallsen and Tankov, 2006, Theorem 3.6]) states that this construction principle is a universal one, i.e. *every* dependence structure in multidimensional Lévy processes can be constructed from independent Lévy processes, linked by some suitable Lévy copula. From a purely mathematics point of view, the universal concept of Lévy copulas makes the above mentioned constructions redundant. But for pricing purposes, a closed-form characteristic function of (integrated) variance and asset price process is typically helpful. Furthermore, a tractable simulation scheme for Monte Carlo simulation is essential. With linear combination and joint time change of independent Lévy processes, the characteristic function of the factors can be calculated at least in a semi-closed form and a simulation scheme is immediately provided, while linking independent Lévy processes with Lévy

copulas typically exhibits difficulties concerning these issues. Therefore, our focus is on linear combination and on time change constructions.

The next theorem derives the finite-dimensional distribution of the log-price process in the BNS model with decoupled jumps, which is done by calculating the joint characteristic function of the log-price process at finitely many points in time. As an immediate corollary we get a semi-closed form solution for the characteristic function of log-prices. In some cases, the characteristic function boils down further admitting a closed form solution, which is essential for fast option pricing via Fourier methods. One specification with closed form characteristic function is the so-called weak-link Γ-OU-BNS model, which we define in the next section. Theorem 4.5 is also used to deduce convergence results for that model.

Theorem 4.5 (Finite-dimensional distribution of the log-price process)
Let the logarithmic price process $\{X_t\}_{t \geq 0}$ follow Equation (4.5) and denote by $\psi_{(Y,Z)}$ the characteristic exponent of the two-dimensional process (Y, Z). Assume, that $\psi_{(Y,Z)}$ is analytic on $C := \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_i) \geq 0, \forall 1 \leq i \leq d\}$. Set $n \in \mathbb{N}$, $0 = t_0 \leq t_1 < \cdots < t_n$, and $u_1, \ldots, u_n \in \mathbb{R}$. Define for all $1 \leq j \leq n$,

$$\tilde{u}_j := \sum_{k=j}^{n} u_k, \qquad \varepsilon(s,t) := 1 - e^{\lambda(s-t)}, \qquad f(u) := \frac{1}{\lambda}\left(\mathrm{i}\, u\, \beta - \frac{u^2}{2}\right)$$

$$a_j(t) := -\mathrm{i}\, f(\tilde{u}_j)\, \varepsilon(t, t_j) - \mathrm{i} \sum_{k=j+1}^{n} f(\tilde{u}_k)\, \varepsilon(t_{k-1}, t_k)\, e^{\lambda(t - t_{k-1})}.$$

Assume, $\forall 1 \leq j \leq n$, there exists an $M > 0$, such that $\operatorname{Re}\left(\psi_{(Y,Z)}\left(\tilde{u}_j, a_j(t)\right)\right) < M$, $\forall t_{j-1} \leq t \leq t_j$. Then,

$$\mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \mathrm{i}\, u_j\, X_{t_j}\right)\right] = \exp\left(\mathrm{i}\, \tilde{u}_1\, X_{t_0} + \sum_{j=1}^{n}\left(\int_{t_{j-1}}^{t_j} \psi_{(Y,Z)}\left(\tilde{u}_j, a_j(s)\right) \mathrm{d}s + A_j\right)\right),$$

where $A_j := f\left(\tilde{u}_j\right) \varepsilon\left(t_{j-1}, t_j\right) e^{-\lambda t_{j-1}} \sigma_0^2 + \mathrm{i}\, \mu\, \tilde{u}_j\left(t_j - t_{j-1}\right)$.

Proof
For the proof we adapt the calculation of the characteristic function of the classical BNS model, cf. Nicolato and Venardos [2003]. Using Equation (4.5) yields

$$\mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \mathrm{i}\, u_j\, X_{t_j}\right)\right]$$

$$
= \mathbb{E}\left[\exp\left(\mathrm{i}\,\tilde{u}_1\,X_{t_0} + \sum_{j=1}^{n} \mathrm{i}\,\tilde{u}_j \left(X_{t_j} - X_{t_{j-1}} \right) \right) \right]
$$

$$
= \mathbb{E}\left[\exp\left(\mathrm{i}\,\tilde{u}_1\,X_{t_0} + \sum_{j=1}^{n} \mathrm{i}\,\tilde{u}_j \left(\int_{t_{j-1}}^{t_j} \left(\mu + \beta\,\sigma_t^2 \right) \mathrm{d}t + \int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t + Y_{t_j} - Y_{t_{j-1}} \right) \right) \right]
$$

$$
= \mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \mathrm{i}\,\tilde{u}_j \left(\int_{t_{j-1}}^{t_j} \beta\,\sigma_t^2\,\mathrm{d}t + \int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t + Y_{t_j} - Y_{t_{j-1}} \right) \right) \right] \tag{4.7}
$$

$$
\times \exp\left(\mathrm{i}\,\tilde{u}_1\,X_{t_0} + \mathrm{i}\,\mu \sum_{j=1}^{n} \tilde{u}_j \left(t_j - t_{j-1} \right) \right).
$$

By conditioning on the trajectory of Z, the squared volatility σ^2 becomes deterministic and $\int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t$ becomes independent of the process Y. Moreover, we know from the theory of stochastic integration with respect to Brownian motion, that $\int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t$ follows a normal distribution with zero mean and variance $\int_{t_{j-1}}^{t_j} \sigma_t^2\,\mathrm{d}t$. Note that the characteristic function of a $\mathcal{N}(0,\sigma^2)$-distributed random variable is given by $\varphi_{\mathcal{N}(0,\sigma^2)}(u) = \exp\left(\frac{-\sigma^2 u^2}{2} \right)$, cf. Example 2.9. Hence,

$$
\mathbb{E}\left[\exp\left(\mathrm{i}\,\tilde{u}_j \int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t \right) \middle| Z \right] = \exp\left(-\frac{1}{2}\,\tilde{u}_j^2 \int_{t_{j-1}}^{t_j} \sigma_t^2\,\mathrm{d}t \right). \tag{4.8}
$$

Therefore, by conditioning on the trajectories of Z, we get

$$
\mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \mathrm{i}\,\tilde{u}_j \left(\int_{t_{j-1}}^{t_j} \beta\,\sigma_t^2\,\mathrm{d}t + \int_{t_{j-1}}^{t_j} \sigma_t\,\mathrm{d}W_t + Y_{t_j} - Y_{t_{j-1}} \right) \right) \right]
$$

$$
= \mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \left(\mathrm{i}\,\beta\,\tilde{u}_j - \frac{1}{2}\,\tilde{u}_j^2 \right) \int_{t_{j-1}}^{t_j} \sigma_t^2\,\mathrm{d}t + \sum_{j=1}^{n} \mathrm{i}\tilde{u}_j \left(Y_{t_j} - Y_{t_{j-1}} \right) \right) \right]. \tag{4.9}
$$

Note, that by the definition of the squared volatility process in Equation (4.6), we get for all $1 \le j \le n$,

$$
\mathrm{d}\sigma_t^2 = -\lambda\,\sigma_t^2\,\mathrm{d}t + \mathrm{d}Z_t
$$

$$
\Leftrightarrow \quad \sigma_{t_j}^2 - \sigma_{t_{j-1}}^2 = -\int_{t_j}^{t_{j-1}} \lambda\,\sigma_t^2\,\mathrm{d}t + Z_{t_j} - Z_{t_{j-1}}
$$

$$\Leftrightarrow \quad \int_{t_j}^{t_{j-1}} \sigma_t^2 \, \mathrm{d}t = \frac{1}{\lambda} \left(-\sigma_{t_j}^2 + \sigma_{t_{j-1}}^2 + Z_{t_j} - Z_{t_{j-1}} \right)$$

$$\Leftrightarrow \quad \int_{t_{j-1}}^{t_j} \sigma_t^2 \, \mathrm{d}t = \frac{e^{-\lambda t_{j-1}} - e^{-\lambda t_j}}{\lambda} \left(\sigma_0^2 + \int_0^{t_{j-1}} e^{\lambda s} \, \mathrm{d}Z_s \right) - \frac{e^{-\lambda t_j}}{\lambda} \int_{t_{j-1}}^{t_j} e^{\lambda s} \, \mathrm{d}Z_s$$

$$+ \frac{1}{\lambda} \left(Z_{t_j} - Z_{t_{j-1}} \right)$$

$$= \frac{e^{-\lambda t_{j-1}} \left(1 - e^{\lambda(t_{j-1}-t_j)} \right)}{\lambda} \left(\sigma_0^2 + \int_0^{t_{j-1}} e^{\lambda s} \, \mathrm{d}Z_s \right)$$

$$+ \int_{t_{j-1}}^{t_j} \frac{1 - e^{\lambda(s-t_j)}}{\lambda} \, \mathrm{d}Z_s$$

$$= \frac{e^{-\lambda t_{j-1}} \, \varepsilon\left(t_{j-1}, \, t_j \right)}{\lambda} \left(\sigma_0^2 + \int_0^{t_{j-1}} e^{\lambda s} \, \mathrm{d}Z_s \right) + \int_{t_{j-1}}^{t_j} \frac{\varepsilon\left(s, \, t_j \right)}{\lambda} \, \mathrm{d}Z_s,$$

where we used that $\sigma_t^2 = e^{-\lambda t} \sigma_0^2 + e^{-\lambda t} \int_0^t e^{\lambda s} \, \mathrm{d}Z_s$, cf. Equation (4.3). Hence,

$$\sum_{j=1}^n \left(\mathrm{i}\,\beta\,\tilde{u}_j - \frac{1}{2}\,\tilde{u}_j^2 \right) \int_{t_{j-1}}^{t_j} \sigma_t^2 \, \mathrm{d}t$$

$$= \sum_{j=1}^n f\left(\tilde{u}_j \right) \varepsilon\left(t_{j-1}, \, t_j \right) e^{-\lambda t_{j-1}} \sigma_0^2 + \sum_{j=1}^n \int_0^{t_{j-1}} f\left(\tilde{u}_j \right) \varepsilon\left(t_{j-1}, \, t_j \right) e^{\lambda(s-t_{j-1})} \, \mathrm{d}Z_s \quad (4.10)$$

$$+ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} f\left(\tilde{u}_j \right) \varepsilon\left(s, \, t_j \right) \mathrm{d}Z_s \,.$$

Rearranging of summands yields

$$\sum_{j=1}^n \int_0^{t_{j-1}} f(\tilde{u}_j)\, \varepsilon(t_{j-1}, \, t_j)\, e^{\lambda(t-t_{j-1})} \, \mathrm{d}Z_t$$

$$= \sum_{j=1}^n \sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_k} f(\tilde{u}_j)\, \varepsilon(t_{j-1}, \, t_j)\, e^{\lambda(t-t_{j-1})} \, \mathrm{d}Z_t$$

$$= \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \sum_{j=k+1}^n f(\tilde{u}_k)\, \varepsilon(t_{j-1}, \, t_j)\, e^{\lambda(t-t_{j-1})} \, \mathrm{d}Z_t \,. \quad (4.11)$$

By combining Equations (4.9), (4.10), and (4.11), we obtain

$$
\mathbb{E}\left[\exp\left(\sum_{j=1}^{n}\mathrm{i}\,\tilde{u}_j\left(\int_{t_{j-1}}^{t_j}\beta\,\sigma_t^2\,\mathrm{d}t+\int_{t_{j-1}}^{t_j}\sigma_t\,\mathrm{d}W_t+Y_{t_j}-Y_{t_{j-1}}\right)\right)\right]
$$

$$
=\mathbb{E}\left[\exp\left(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\sum_{j=k+1}^{n}f(\tilde{u}_k)\,\varepsilon(t_{j-1},t_j)\,e^{\lambda\,(t-t_{j-1})}\,\mathrm{d}Z_t+\sum_{j=1}^{n}\mathrm{i}\tilde{u}_j\left(Y_{t_j}-Y_{t_{j-1}}\right)\right.\right.
$$

$$
\left.\left.+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}f\left(\tilde{u}_j\right)\varepsilon\left(s,t_j\right)\mathrm{d}Z_s\right)\right]\exp\left(\sum_{j=1}^{n}f\left(\tilde{u}_j\right)\varepsilon\left(t_{j-1},t_j\right)e^{-\lambda\,t_{j-1}}\,\sigma_0^2\right)
$$

$$
=\mathbb{E}\left[\exp\left(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathrm{i}\,a_j(s)\,\mathrm{d}Z_t+\sum_{j=1}^{n}\mathrm{i}\tilde{u}_j\left(Y_{t_j}-Y_{t_{j-1}}\right)\right)\right]
$$

$$
\times\exp\left(\sum_{j=1}^{n}f\left(\tilde{u}_j\right)\varepsilon\left(t_{j-1},t_j\right)e^{-\lambda\,t_{j-1}}\,\sigma_0^2\right).
$$

$$
=\prod_{j=1}^{n}\mathbb{E}\left[\exp\left(\int_{t_{j-1}}^{t_j}\mathrm{i}\,a_j(s)\,\mathrm{d}Z_t+\mathrm{i}\tilde{u}_j\left(Y_{t_j}-Y_{t_{j-1}}\right)\right)\right]
$$

$$
\times\exp\left(\sum_{j=1}^{n}f\left(\tilde{u}_j\right)\varepsilon\left(t_{j-1},t_j\right)e^{-\lambda\,t_{j-1}}\,\sigma_0^2\right),
$$

where the latter step follows from the fact that the 2-dimensional Lévy process (Y,Z) has independent increments. Note, that the imaginary part of $a_j(t)$ is positive for all $1\leq j\leq d$ and all $t\in\mathbb{R}_+$. Therefore, $a_j(t)\in C$ and the characteristic exponent $\psi_{(Y,Z)}$ can by continued by Theorem 2.22. Hence, we can apply Theorem 2.26, which gives

$$
\mathbb{E}\left[\exp\left(\sum_{j=1}^{n}\mathrm{i}\,\tilde{u}_j\left(\int_{t_{j-1}}^{t_j}\beta\,\sigma_t^2\,\mathrm{d}t+\int_{t_{j-1}}^{t_j}\sigma_t\,\mathrm{d}W_t+Y_{t_j}-Y_{t_{j-1}}\right)\right)\right]
$$

$$
=\prod_{j=1}^{n}\exp\left(\int_{t_{j-1}}^{t_j}\psi_{(Y,Z)}\left(\tilde{u}_j,a_j(s)\right)\mathrm{d}s+f\left(\tilde{u}_j\right)\varepsilon\left(t_{j-1},t_j\right)e^{-\lambda\,t_{j-1}}\,\sigma_0^2\right). \tag{4.12}
$$

Combining Equations (4.7) and (4.12), we get the final result,

$$
\mathbb{E}\left[\exp\left(\sum_{j=1}^{n} \mathrm{i}\, u_j\, X_{t_j}\right)\right] = \exp\left(\mathrm{i}\, \tilde{u}_1\, X_{t_0} + \sum_{j=1}^{n}\left(\int_{t_{j-1}}^{t_j} \psi_{(Y,Z)}\left(\tilde{u}_j,\, a_j(s)\right)\mathrm{d}s + A_j\right)\right),
$$

where $A_j := f\left(\tilde{u}_j\right)\varepsilon\left(t_{j-1},\, t_j\right)e^{-\lambda\, t_{j-1}}\sigma_0^2 + \mathrm{i}\,\mu\,\tilde{u}_j\left(t_j - t_{j-1}\right).$ $\qquad\square$

Note that, if the characteristic exponent $\psi_{(Y,Z)}$ is given by the expression in Theorem 2.19(ii), then $\psi_{(Y,Z)}$ is always analytic and one of the two assumptions in Theorem 4.5 is fulfilled, cf. Remark 2.23. As an immediate corollary of Theorem 4.5, we obtain a semi-closed form for the characteristic function of the logarithmic price.

Remark 4.6 (Characteristic function of the logarithmic price process)
Let the logarithmic price process $(X_t)_{t\geq 0}$ follow Equation (4.5). With the assumptions and abbreviations defined in Theorem 4.5, we have

$$
\varphi_{X_t}(u) = \exp\left(\mathrm{i}\,u\,X_0 + \mathrm{i}\,u\,\mu\,t + f(u)\,\varepsilon(0,t)\,\sigma_0^2 + \int_0^t \psi_{(Y,Z)}\left(u,\, -\mathrm{i}\,f(u)\,\varepsilon(s,t)\right)\mathrm{d}s\right),
$$

$$(4.13)$$

denoting by $\psi_{(Y,Z)}$ the characteristic exponent of the two-dimensional Lévy process (Y,Z).

It is crucial for practical purposes that the integral appearing in Equation (4.13) is solvable. This depends on the choice of the two-dimensional Lévy process (Y,Z). In case (Y,Z) is constructed by jointly time-changing two independent Lévy processes, the joint characteristic exponent can easily be expressed as a function of the marginal characteristic exponents. In particular, if the two-dimensional Lévy process $(Y,Z) = (Y_t, Z_t)_{t\geq 0}$ is constructed by jointly time-changing two independent Lévy processes $U = (U_t)_{t\geq 0}$, $V = (V_t)_{t\geq 0}$, i.e. it exists a Lévy subordinator $T = (T_t)_{t\geq 0}$ such that $Y_t = U_{T_t}$ and $Z_t = V_{T_t}$ a.s. for all $t > 0$, then the joint characteristic function of (Y_t, Z_t) can be calculated by conditioning on T_t and

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\mathrm{i}\left(u\,U_{T_t} + v\,V_{T_t}\right)\right)\right] &= \mathbb{E}\left[\exp\left(\left(\psi_U(u) + \psi_V(v)\right)T_t\right)\right] \\
&= \exp\left(t\,\tilde{\psi}_T\left(\psi_U(u) + \psi_V(v)\right)\right),
\end{aligned}
$$

where ψ_U, ψ_V are the corresponding characteristic exponents of U, V, and $\tilde{\psi}_T$ is the Laplace exponent of T. A nice example, which falls into that construction principle is the two-dimensional time-changed CPP_{Exp}. It even supports a closed-form solution of that integral appearing in Equation (4.13). By using this process, the model boils down to the weak-link Γ-OU-BNS model, which we introduce in the next section.

4.3 The weak-link Γ-OU-BNS model

A new model fitting in the model class is the weak-link Γ-OU-BNS model, which employs the time change dependence structure between CPP_{Exp} presented in Chapter 3. For this model, we will see that the characteristic function of the log-prices can be calculated in closed-form.

Example 4.7 (Weak-link Γ-OU-BNS model)
Let $X = (X_t)_{t \geq 0}$ follow the dynamics

$$\mathrm{d}X_t = (\mu + \beta\sigma_t^2)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t - \mathrm{d}Y_t,$$
$$\mathrm{d}\sigma_t^2 = -\lambda\sigma_t^2\,\mathrm{d}t + \mathrm{d}Z_t,$$

with $\mu, \beta \in \mathbb{R}$, $\lambda > 0$, $W = \{W_t\}_{t \geq 0}$ being a Brownian motion, and $(Y, Z) = \{(Y_t, Z_t)\}_{t \geq 0}$ being a time-changed CPP_{Exp} with intensities c_Y, c_Z, jump size parameters η_Y, η_Z, and dependence parameter κ, i.e. there exist independent compound Poisson processes $T = \{T_t\}_{t \geq 0}$, $U = \{U_t\}_{t \geq 0}$, $V = \{V_t\}_{t \geq 0}$ with respective intensities c_T, $\frac{c_Y}{c_T - c_Y}$, $\frac{c_Z}{c_T - c_Z}$ fulfilling $c_Y, c_Z > 0$, $c_T = \frac{1}{\kappa}\max\{c_Y, c_Z\}$ and respective jump size distributions $\text{Exp}(1), \text{Exp}(\frac{c_T\,\eta_Y}{c_T - c_Y}), \text{Exp}(\frac{c_T\,\eta_Z}{c_T - c_Z})$, $\eta_Y, \eta_Z > 0$, such that Y and Z can be represented as the T-time-change of the processes U and V, i.e. $Y_t := U_{T_t}$ and $Z_t := V_{T_t}$ a.s. for all $t \geq 0$.

Figure 4.2 shows simulated paths for the classical Γ-OU BNS model and the weak-link Γ-OU-BNS model. The graphs in the upper row show typical asset price paths of the two models. Corresponding to these paths, the graphs beneath exhibit the volatility process and the daily log-returns. On the left side, the jump correlation parameter of the weak-link Γ-OU-BNS model is set to be 80%, on the right side, this parameter is 20%. Thus, we have a strong dependence between the asset price jumps and jumps in the volatility on the left side and a weak dependence on the right side. For the sake

of comparability, the Brownian motions and the asset jump processes of both models coincide within one dependence configuration. Therefore, the difference between the two models is determined solely by the jumps in the volatility process. Moreover, the compound Poisson processes driving the volatility are identically distributed. One can easily see that the volatility jumps are uncoupled from the asset price jumps in the weak-link Γ-OU-BNS model, i.e. there exist asset price jumps without simultaneous volatility jumps and, on the other hand, there are volatility rises without negative asset price jumps. The higher the jump dependence correlation parameter in the weak-link Γ-OU-BNS model, the higher seems the resemblance to the classical BNS model. This impression is confirmed by a mathematical proof in Theorem 4.11.

For the weak-link Γ-OU-BNS model, we immediately obtain an explicit expression for the joint Laplace exponent. Since the joint Laplace exponent of a two-dimensional Lévy process (which appears in the expressions in Theorem 4.5 and Remark 4.6) may be a cumbersome object, we calculate it for the special case of dependence arising from joint time change. The corresponding calculations for dependence arising from linear dependence are straightforward, therefore, we omit them here.

Remark 4.8 (Characteristic exponent for the weak-link Γ-OU-BNS model)
Let $(-Y, Z)$ be the two dimensional jump process in a weak-link Γ-OU-BNS model, i.e. (Y, Z) is a a two-dimensional time-changed CPP_{Exp} with intensities c_Y, c_Z, jump size parameter η_Y, η_Z, and correlation parameter κ. Thus, the time change intensity is given by $c_T = \max\{c_Y, c_Z\}$. Then, the characteristic exponent of $(-Y, Z)$ can be expressed by the characteristic exponent of (Y, Z), which was calculated in Theorem 3.7,

$$\psi_{(-Y, Z)}(u_Y, u_Z) = \psi_{(Y, Z)}(-u_Y, u_Z) = \frac{-\frac{\mathrm{i}\, c_T\, c_Y\, u_Y}{c_T\, \eta_Y + \mathrm{i}\, u_Y\, (c_T - c_Y)} + \frac{\mathrm{i}\, c_T\, c_Z\, u_Z}{c_T\, \eta_Z - \mathrm{i}\, u_Z\, (c_T - c_Z)}}{1 + \frac{\mathrm{i}\, c_Y\, u_Y}{c_T\, \eta_Y + \mathrm{i}\, u_Y\, (c_T - c_Y)} - \frac{\mathrm{i}\, c_Z\, u_Z}{c_T\, \eta_Z - \mathrm{i}\, u_Z\, (c_T - c_Z)}}.$$

As we have seen in Remark 3.8, this expression is continuable on the set $C := \{(u_Y, u_Z) \in \mathbb{C}^d : \text{Im}(u_Y) \leq 0, \text{Im}(u_Z) \geq 0\}$ and its real part is negative.

Alternatively to the weak-link Γ-OU-BNS model, one could employ a weak-link also by a one-sided time-change construction.
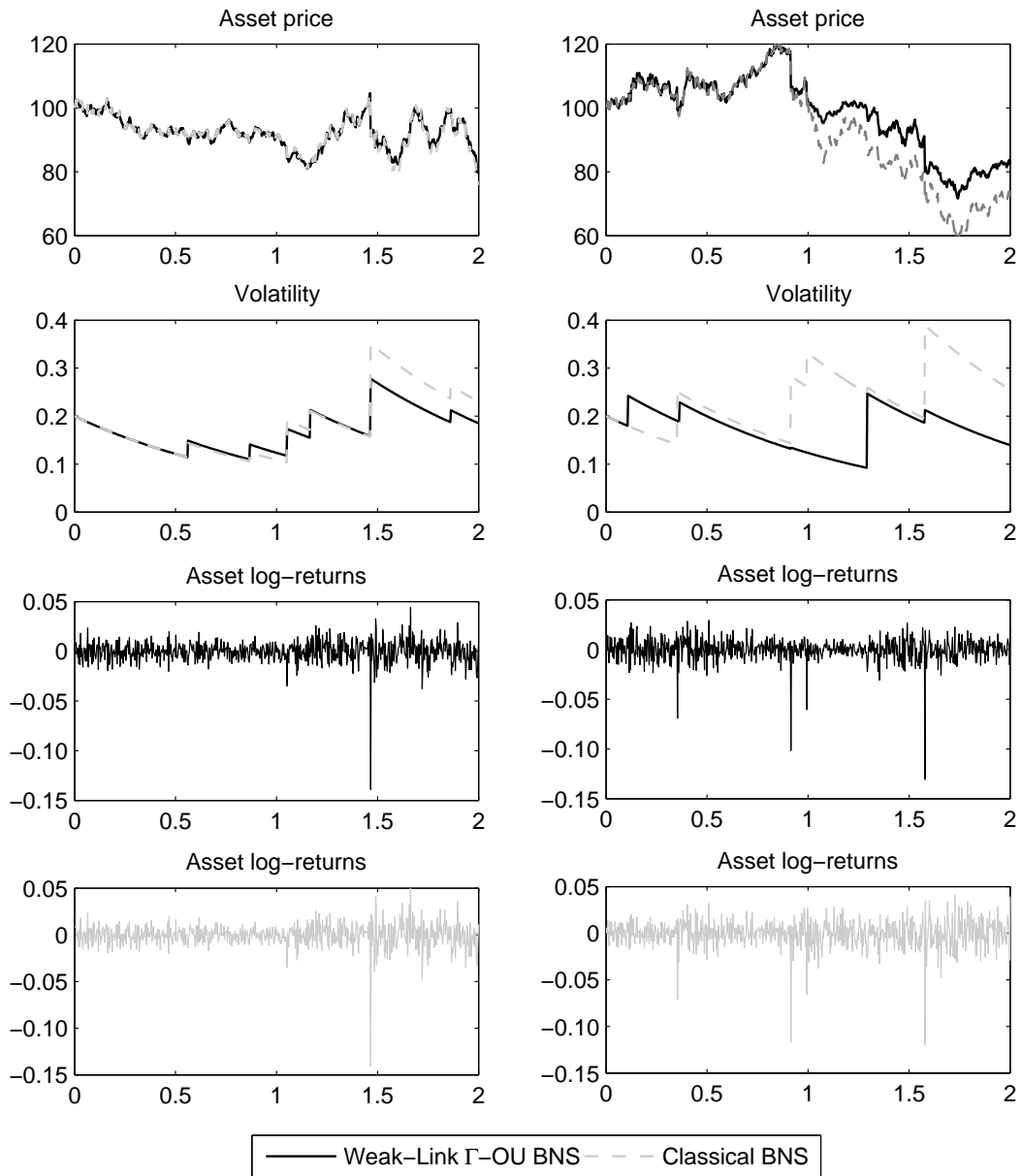
Figure 4.2 Sample paths of the asset price process, the volatility process, and the daily log-returns for the classical BNS model and the weak-link Γ-OU-BNS model. Left: $\kappa = 80\%$, right: $\kappa = 20\%$.

Remark 4.9 (Char. exponent of a one-sided time-change construction)
If the two-dimensional Lévy process $(Y, Z) = (Y_t, U_{Y_t})_{t \geq 0}$ is constructed by two independent processes Y and U with $Y \sim \mathrm{CPP}_{\mathrm{Exp}}(c_Y, \eta_Y)$ and $U \sim \mathrm{CPP}_{\mathrm{Exp}}(c_U, \eta_U)$, then their joint Laplace exponent is given by

$$
\begin{aligned}
\psi_{(Y,Z)}(u_Y, u_Z) &= \log\left(\mathbb{E}\left[e^{\mathrm{i}\,u_Y\,Y_1 + \mathrm{i}\,u_Z\,U_{Y_1}}\right]\right) = \log\left(\mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\,u_Y\,Y_1}\right]\mathbb{E}\left[e^{\mathrm{i}\,u_Z\,U_{Y_1}}\,\big|\,Y_1\right]\right]\right) \\
&= \tilde{\psi}_Y\left(\mathrm{i}\,u_Y + \tilde{\psi}_U(\mathrm{i}\,u_Z)\right) = \tilde{\psi}_Y\left(\mathrm{i}\,u_Y + \frac{\mathrm{i}\,c_U\,u_Z}{\eta_U - \mathrm{i}\,u_Z}\right) \\
&= \frac{c_Y\left(\mathrm{i}\,u_Y + \frac{\mathrm{i}\,c_U\,u_Z}{\eta_U - \mathrm{i}\,u_Z}\right)}{\eta_Y - \mathrm{i}\,u_Y - \frac{\mathrm{i}\,c_U\,u_Z}{\eta_U - \mathrm{i}\,u_Z}} = c_Y\,\frac{\mathrm{i}\,u_Y\,(\eta_U - \mathrm{i}\,u_Z) + \mathrm{i}\,c_U\,u_Z}{(\eta_Y - \mathrm{i}\,u_Y)\,(\eta_U - \mathrm{i}\,u_Z) - \mathrm{i}\,c_U\,u_Z}.
\end{aligned}
$$

This construction is slightly simpler, but less flexible than the weak-link construction. In particular, a one-sided time-change construction only allows for separate jumps in one component, while the jumps in the other component always occur jointly. Later, we show that a model resulting from such a time change construction can be obtained as limit of the weak-link Γ-OU-BNS model (cf. Theorem 4.11).

Theorem 4.10 (Characteristic function of the weak-link Γ-OU-BNS model)
Let $X = \{X_t\}_{t \geq 0}$ be the log-price process following a weak-link Γ-OU-BNS model (cf. Example 4.7) and let $u \in \mathbb{R}_+$. Then the characteristic function of X can be calculated in closed form and is given by

$$
\log \varphi_{X_t}(u) = \mathrm{i}\,u\,(X_0 + \mu\,t) + f(u)\,\varepsilon(0,t)\,\sigma_0^2 - \frac{c_T}{\lambda}\,\alpha(u)\,\log(\gamma(u)) + c_T\,\delta(u)\,t\,,
$$

with the following abbreviations

$$
f(u) := \frac{1}{\lambda}\left(\mathrm{i}u\beta - \frac{u^2}{2}\right),
$$

$$
\varepsilon(s,t) := 1 - e^{\lambda\,(s-t)},
$$

$$
\alpha(u) := \frac{c_T\,c_Z\,\eta_Z\,(c_T\,\eta_Y + \mathrm{i}\,u\,(c_T - c_Y))^2}{f(u)\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right)^2 - c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u)\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right)},
$$

$$
\delta(u) := \frac{f(u)\,(c_T\,c_Z\,\eta_Y + \mathrm{i}\,u\,c_Y\,(c_T - c_Z) + \mathrm{i}\,u\,c_Z\,(c_T - c_Y)) - \mathrm{i}\,u\,c_T\,c_Y\,\eta_Z}{-f(u)\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right) + c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u)},
$$

$$
\gamma(u) := \frac{c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u)}{c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u) - f(u)\,\varepsilon(0,t)\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right)}.
$$

Proof

Since

$$\mathrm{Im}(-\mathrm{i}\,f(u)\,\varepsilon(s,t)) = \frac{u^2}{2\,\lambda}\,\varepsilon(s,t) > 0,$$

Remark 4.8 yields that $\psi_{(-Y,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t))$ exists for all $u \in \mathbb{R}$ and all $0 \leq s \leq t$, and is given by

$$\psi_{(-Y,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t)) = \frac{-\dfrac{\mathrm{i}\,c_T\,c_Y\,u}{c_T\,\eta_Y + \mathrm{i}\,u\,(c_T - c_Y)} + \dfrac{c_T\,c_Z\,f(u)\,\varepsilon(s,t)}{c_T\,\eta_Z - f(u)\,\varepsilon(s,t)\,(c_T - c_Z)}}{1 + \dfrac{\mathrm{i}\,c_Y\,u}{c_T\,\eta_Y + \mathrm{i}\,u\,(c_T - c_Y)} - \dfrac{c_Z\,f(u)\,\varepsilon(s,t)}{c_T\,\eta_Z - f(u)\,\varepsilon(s,t)\,(c_T - c_Z)}}. \quad (4.14)$$

Furthermore, we know from Remark 4.8, that

$$\mathrm{Re}\left(\psi_{(-Y,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t))\right) < 0, \quad \forall\,u \in \mathbb{R},\ \forall\,0 \leq s \leq t.$$

The characteristic function of the log-price process can therefore be expressed by (cf. Remark 4.6)

$$\varphi_{X_t}(u) = \exp\left(\mathrm{i}\,u\,X_0 + \mathrm{i}\,u\,\mu\,t + f(u)\,\varepsilon(0,t)\,\sigma_0^2 + \int_0^t \psi_{(-Y,Z)}\left(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t)\right)\mathrm{d}s\right).$$

The only thing left to show is that

$$\int_0^t \tilde{\psi}_{(-Y,Z)}\left(\mathrm{i}u,\,f(u)\,\varepsilon(s,t)\right)\mathrm{d}s = -\frac{c_T}{\lambda}\,\alpha(u)\,\log(\gamma(u)) + c_T\,\delta(u)\,t. \quad (4.15)$$

Using Equation (4.14)) and the abbreviations $A_1 := c_T\,\eta_Y + \mathrm{i}\,u\,(c_T - c_Y)$ and $A_2 := c_T\,\eta_Z - f(u)\,\varepsilon(s,t)\,(c_T - c_Z)$, we obtain

$$\psi_{(-Y,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t)) = \frac{-\mathrm{i}\,c_T\,c_Y\,u\,A_2 + c_T\,c_Z\,f(u)\,\varepsilon(s,t)\,A_1}{A_1\,A_2 + \mathrm{i}\,c_Y\,u\,A_2 - c_Z\,f(u)\,\varepsilon(s,t)\,A_1}. \quad (4.16)$$

Note that

$$-\mathrm{i}\,c_T\,c_Y\,u\,A_2 + c_T\,c_Z\,f(u)\,\varepsilon(s,t)\,A_1$$
$$= f(u)\,\varepsilon(s,t)\,(c_T\,c_Z\,A_1 + \mathrm{i}\,c_T\,c_Y\,u\,(c_T - c_Z)) - \mathrm{i}\,c_T^2\,c_Y\,u\,\eta_Z$$
$$= f(u)\,\varepsilon(s,t)\,(c_T\,c_Z\,\eta_Y + \mathrm{i}\,u\,c_Y\,(c_T - c_Z) + \mathrm{i}\,u\,c_Z\,(c_T - c_Y))\,c_T - \mathrm{i}\,c_T^2\,c_Y\,u\,\eta_Z,$$

and

$$A_1\,A_2 + \mathrm{i}\,c_Y\,u\,A_2 - c_Z\,f(u)\,\varepsilon(s,t)\,A_1$$

$$
\begin{aligned}
={}& -f(u)\,\varepsilon(s,t)\,(A_1\,(c_T - c_Z) + \mathrm{i}\,c_Y\,u\,(c_T - c_Z) + c_Z\,A_1) + A_1\,c_T\,\eta_Z + \mathrm{i}\,c_Y\,u\,c_T\,\eta_Z \\
={}& -f(u)\,\varepsilon(s,t)\,(A_1\,c_T + \mathrm{i}\,c_Y\,u\,(c_T - c_Z)) + c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u) \\
={}& -f(u)\,\varepsilon(s,t)\,\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right) + c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u).
\end{aligned}
$$

Therefore, Equation (4.16) boils down to

$$
\begin{aligned}
&\psi_{(-Y,\,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t)) \\
={}&c_T\,\frac{f(u)\,\varepsilon(s,t)\,(c_T\,c_Z\,\eta_Y + \mathrm{i}\,u\,c_Y\,(c_T - c_Z) + \mathrm{i}\,u\,c_Z\,(c_T - c_Y)) - \mathrm{i}\,c_T\,c_Y\,u\,\eta_Z}{-f(u)\,\varepsilon(s,t)\,\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right) + c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u)} \\
={}&c_T\,\frac{x\,e^{\lambda s} + y}{z\,e^{\lambda s} + w},
\end{aligned}
$$

with

$$
\begin{aligned}
x &:= -f(u)\,(c_T\,c_Z\,\eta_Y + \mathrm{i}\,u\,c_Y\,(c_T - c_Z) + \mathrm{i}\,u\,c_Z\,(c_T - c_Y))\,e^{-\lambda t}, \\
y &:= f(u)\,(c_T\,c_Z\,\eta_Y + \mathrm{i}\,u\,c_Y\,(c_T - c_Z) + \mathrm{i}\,u\,c_Z\,(c_T - c_Y)) - \mathrm{i}\,u\,c_T\,c_Y\,\eta_Z, \\
z &:= f(u)\,\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right)\,e^{-\lambda t}, \\
w &:= -f(u)\,\left(c_T^2\,\eta_Y + \mathrm{i}\,u\,(c_T^2 - c_Y\,c_Z)\right) + c_T^2\,\eta_Z\,(\eta_Y + \mathrm{i}\,u).
\end{aligned}
$$

To solve the following integral,

$$
\int_0^t \psi_{(-Y,\,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t))\,\mathrm{d}s = \int_0^t c_T\,\frac{x\,e^{\lambda s} + y}{z\,e^{\lambda s} + w}\,\mathrm{d}s, \tag{4.17}
$$

we remark that for arbitrary $x, y, z, w \in \mathbb{C}$ with $z\exp(\lambda s) + w, z, w \neq 0$ for all $s \in [0, t]$, the derivative of the function

$$
\zeta(s) := \frac{1}{\lambda}\left(\frac{x}{z} - \frac{y}{w}\right)\log\left(z\,e^{\lambda s} + w\right) + \frac{y}{w}\,s, \quad s \in [0, t],
$$

turns out to be

$$
\begin{aligned}
\zeta'(s) &= \frac{1}{\lambda}\left(\frac{x}{z} - \frac{y}{w}\right)\frac{z\,\lambda\,e^{\lambda s}}{z\,e^{\lambda s} + w} + \frac{y}{w} = \frac{(w\,x - y\,z)\,e^{\lambda s} + y\,\left(z\,e^{\lambda s} + w\right)}{w\,z\,e^{\lambda s} + w^2} \\
&= \frac{x\,e^{\lambda s} + y}{z\,e^{\lambda s} + w}
\end{aligned}
$$

for all $s \in [0, t]$. Hence, Equation (4.17) boils down to

$$
\int_0^t \psi_{(-Y,\,Z)}(u,\,-\mathrm{i}\,f(u)\,\varepsilon(s,t))\,\mathrm{d}s = c_T\,(\zeta(t) - \zeta(0))
$$

$$= \frac{c_T}{\lambda} \left(\frac{x}{z} - \frac{y}{w} \right) \log \left(\frac{z\, e^{\lambda t} + w}{z + w} \right) + \frac{c_T\, y\, t}{w}. \quad (4.18)$$

Note that

$$\frac{x}{z} = -\frac{c_T\, c_Z\, \eta_Y + \mathrm{i}\, u\, c_Y\, (c_T - c_Z) + \mathrm{i}\, u\, c_Z\, (c_T - c_Y)}{c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z)}, \quad (4.19)$$

$$\frac{y}{w} = \frac{f(u)\, (c_T\, c_Z\, \eta_Y + \mathrm{i}\, u\, c_Y\, (c_T - c_Z) + \mathrm{i}\, u\, c_Z\, (c_T - c_Y)) - \mathrm{i}\, u\, c_T\, c_Y\, \eta_Z}{-f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z)) + c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)} = \delta(u), \quad (4.20)$$

$$\frac{z\, e^{\lambda t} + w}{z + w} = \frac{c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)}{c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u) - f(u)\, \varepsilon(0, t)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))} = \gamma(u), \quad (4.21)$$

and

$$\frac{x}{z} - \frac{y}{w}$$

$$= -\frac{(c_T\, c_Z\, \eta_Y + \mathrm{i}\, u\, c_Y\, (c_T - c_Z) + \mathrm{i}\, u\, c_Z\, (c_T - c_Y))\, c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$+ \frac{\mathrm{i}\, u\, c_T\, c_Y\, \eta_Z\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$= -\frac{(c_T\, c_Z\, \eta_Y^2 + \mathrm{i}\, u\, \eta_Y\, (c_T\, c_Y - 2\, c_Y\, c_Z + 2\, c_T\, c_Z) - u^2\, (c_T\, c_Y - 2\, c_Y\, c_Z + c_T\, c_Z))\, c_T^2\, \eta_Z}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$+ \frac{c_T\, c_Y\, \eta_Z\, (\mathrm{i}\, u\, c_T^2\, \eta_Y - u^2\, (c_T^2 - c_Y\, c_Z))}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$= -\frac{c_T\, c_Z\, \eta_Z\, (c_T^2\, \eta_Y^2 + 2\, \mathrm{i}\, u\, \eta_Y\, c_T\, (c_T - c_Y) - u^2\, (c_T^2\, c_Y - 2\, c_Y\, c_T + c_Y^2))}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$= -\frac{c_T\, c_Z\, \eta_Z\, (c_T\, \eta_Y + \mathrm{i}\, u\, (c_T - c_Y))^2}{f(u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))^2 - c_T^2\, \eta_Z\, (\eta_Y + \mathrm{i}\, u)\, (c_T^2\, \eta_Y + \mathrm{i}\, u\, (c_T^2 - c_Y\, c_Z))}$$

$$= -\alpha(u). \quad (4.22)$$

Combining Equations (4.18), (4.20), (4.21), and (4.22) yields

$$\int_0^t \tilde{\psi}_{(-Y,Z)}\, (\mathrm{i}u,\, f(u)\, \varepsilon(s, t))\, \mathrm{d}s = -\frac{c_T}{\lambda}\, \alpha(u)\, \log(\gamma(u)) + c_T\, \delta(u)\, t\,,$$

which is the desired expression from Equation (4.15), and concludes the proof of the theorem. $\qquad \square$

Obviously, the time-change construction in the weak-link Γ-OU-BNS model always establishes nonlinear dependence between the asset price and the squared volatility process. Therefore, the weak-link Γ-OU-BNS model is not a true extension of the classical Γ-OU-BNS model. But we can show that the classical Γ-OU-BNS model occurs as a limiting case as motivated in Figure 4.2. Thus, the weak-link Γ-OU-BNS model can be considered as an extension where the Γ-OU-BNS model occurs as a limiting case. Dependent on the setting, a dependence structure resulting from a one-sided time change construction (cf. Remark 4.9) occurs as limiting case. Theorem 4.11 investigates the limit behavior of the weak-link Γ-OU-BNS model.

Theorem 4.11 (Limit of the weak-link Γ-OU-BNS model)
Let the logarithmic price process X^κ of a weak-link Γ-OU-BNS model, with κ being the respective jump dependence parameter, c_Y, c_Z the intensities, and η_Y, η_Z the jump size parameters of the driving two-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ (Y, Z). Then, the finite dimensional distribution of X^κ converges in law to the finite dimensional distribution of the limit process X for $\kappa \nearrow 1$, i.e. $(X^\kappa_{t_1}, X^\kappa_{t_2}, \ldots, X^\kappa_{t_n})$ converges in law to $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ for all $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$.[3] The structure of the limiting process X depends on the intensities c_Z and c_Y in the following way:

- $c_Y > c_Z$:
  X can be represented by a construction as described in Remark 4.9, i.e. by the two-dimensional Lévy process $(-\tilde{Y}_t, \tilde{Z}_t)_{t \geq 0} = (-\tilde{Y}_t, \tilde{U}_{\tilde{Y}_t})_{t \geq 0}$, where $\tilde{Y} \sim \mathrm{CPP}_{\mathrm{Exp}}(c_Y, \eta_Y)$ and $\tilde{U} \sim \mathrm{CPP}_{\mathrm{Exp}}(\frac{c_Z\,\eta_Y}{c_Y - c_Z}, \frac{c_Y\,\eta_Z}{c_Y - c_Z})$ are independent.

- $c_Z > c_Y$:
  Again, X is given by the construction as described in Remark 4.9, i.e. by the two-dimensional Lévy process $(-\tilde{Y}_t, \tilde{Z}_t)_{t \geq 0} = (-\tilde{U}_{\tilde{Z}_t}, \tilde{Z}_t)_{t \geq 0}$, where $\tilde{Z} \sim \mathrm{CPP}_{\mathrm{Exp}}(c_Z, \eta_Z)$ and $\tilde{U} \sim \mathrm{CPP}_{\mathrm{Exp}}(\frac{c_Y\,\eta_Z}{c_Z - c_Y}, \frac{c_Z\,\eta_Y}{c_Z - c_Y})$ are independent.

- $c_Y = c_Z$:
  X is given by a classical Γ-OU-BNS model, i.e. by the two-dimensional Lévy process $(\rho\tilde{Z}_t, \tilde{Z}_t)_{t \geq 0}$, where $\rho = -\frac{\eta_Y}{\eta_Z}$ and $\tilde{Z}$ is a $\mathrm{CPP}_{\mathrm{Exp}}(c_Z, \eta_Z)$.

---

[3]This type of convergence is called *finite-dimensional convergence* along $\mathbb{R}_+$, cf. [Jacod and Shiryaev, 2003, IV.3.13].

Proof

Using Theorem 4.5, the problem boils down to showing that the characteristic exponent of (Y^κ, Z^κ) converges pointwise to the characteristic exponent of $(\tilde{Y}, \tilde{Z})$.

By Remark 3.7, we know

$$\psi_{(Y^\kappa, Z^\kappa)}(u_Y, u_Z) = \frac{\frac{\mathrm{i}\, c_T\, c_Y\, u_Y}{c_T\, \eta_Y - \mathrm{i}\, u_Y\, (c_T - c_Y)} + \frac{\mathrm{i}\, c_T\, c_Z\, u_Z}{c_T\, \eta_Z - \mathrm{i}\, u_Z\, (c_T - c_Z)}}{1 - \frac{\mathrm{i}\, c_T\, c_Y\, u_Y}{c_T\, \eta_Y - \mathrm{i}\, u_Y\, (c_T - c_Y)} - \frac{\mathrm{i}\, c_T\, c_Z\, u_Z}{c_T\, \eta_Z - \mathrm{i}\, u_Z\, (c_T - c_Z)}}.$$

Consider $c_Y > c_Z$, then

$$
\begin{aligned}
\lim_{\kappa \nearrow 1} \psi_{(Y^\kappa, Z^\kappa)}(u_Y, u_Z) &= \lim_{c_T \searrow c_Y} \psi_{(Y^\kappa, Z^\kappa)}(u_Y, u_Z) \\
&= c_Y \frac{\frac{\mathrm{i}\, u_Y}{\eta_Y} + \frac{\mathrm{i}\, c_Z\, u_Z}{c_Y\, \eta_Z - \mathrm{i}\, u_Z\, (c_Y - c_Z)}}{1 - \frac{\mathrm{i}\, u_Y}{\eta_Y} - \frac{\mathrm{i}\, c_Z\, u_Z}{c_Y\, \eta_Z - \mathrm{i}\, u_Z\, (c_Y - c_Z)}} \\
&= c_Y \frac{\frac{\mathrm{i}\, u_Y}{\eta_Y}\left(c_Y\, \eta_Z - \mathrm{i}\, u_Z\, (c_Y - c_Z)\right) + \mathrm{i}\, c_Z\, u_Z}{\left(1 - \frac{\mathrm{i}\, u_Y}{\eta_Y}\right)\left(c_Y\, \eta_Z - \mathrm{i}\, u_Z\, (c_Y - c_Z)\right) - \mathrm{i}\, c_Z\, u_Z} \\
&= c_Y \frac{\mathrm{i}\, u_Y \left(\frac{c_Y\, \eta_Z}{c_Y - c_Z} - \mathrm{i}\, u_Z\right) + \mathrm{i}\, \frac{c_Z\, \eta_Y}{c_Y - c_Z}\, u_Z}{(\eta_Y - \mathrm{i}\, u_Y)\left(\frac{c_Y\, \eta_Z}{c_Y - c_Z} - \mathrm{i}\, u_Z\right) - \mathrm{i}\, \frac{c_Z\, \eta_Y}{c_Y - c_Z}\, u_Z},
\end{aligned}
$$

which coincides with the characteristic exponent of a one-sided construction with claimed parameters, cf. Remark 4.9.

In case of $c_Y < c_Z$, we get the result analogously.

Now assume $c_Y = c_Z$, then

$$\lim_{\kappa \nearrow 1} \psi_{(Y^\kappa, Z^\kappa)}(u_Y, u_Z) = \lim_{c_T \searrow c_Y} \psi_{(Y^\kappa, Z^\kappa)}(u_Y, u_Z) = c_Z \frac{\mathrm{i}\, u_Y + \mathrm{i}\, u_Z \frac{\eta_Y}{\eta_Z}}{\eta_Y - \mathrm{i}\, u_Y - \mathrm{i}\, u_Z \frac{\eta_Y}{\eta_Z}},$$

which coincides with the joint characteristic function of $\tilde{Z}$ and $\rho\tilde{Z}$. $\qquad\square$

4.4 Application: The weak-link Γ-OU-BNS model in action

In the previous section, we have shown that the weak-link Γ-OU-BNS model class truly generalizes the classical BNS model and still preserves tractability concerning numerical treatment in the sense of rapid calibration via Fourier pricing methods. Hence, we think that the model is an interesting candidate to serve for derivatives

pricing. In this section, we have a closer look on the calibration to market prices of plain vanilla options and scrutinize the sensitivity of the model-implied volatility skew to changes in the parameters. Furthermore, we point out an example (down-and-in call options) where the weak-link property leads to tremendously different valuation results compared to the classical BNS model.

We begin with a calibration exercise and benchmark the calibration results of the weak-link Γ-OU-BNS model with the results from a classical BNS model. We calibrate both models to intraday quotes of put and call options on the Dow Jones Industrial index as of 2015-Feb-12 provided by Thomson Reuters. The maturities range in between one week and three years with strikes from 6000 to 25000. The spot price is 17895. We performed the calibration exercise with 363 European call and put options.[4] US treasury rates serve as risk-less interest rate approximation. Options are priced via the risk-neutral approach, i.e. we assume the dynamics presented in the previous chapters are given with respect to a martingale measure. Then, the risk-neutral drift parameters in the classical Γ-OU-BNS model have to fulfill (cf. Equation (4.4))

$$\mu = r - \frac{c_Y \rho}{\eta_Z - \rho} = r - \frac{c_Y}{\eta_Y - 1}, \qquad \beta = -\frac{1}{2},$$

where r is the risk-less interest rate. Note, that the parameter ρ of the classical Γ-OU-BNS model can be expressed consistently with the notation of the weak-link model by $\rho = \frac{\eta_Z}{\eta_Y}$. The risk-neutral drift conditions for μ and β in the weak-link Γ-OU-BNS are the same as in the classical one. As objective function for the optimization procedure in the calibration, we employ the mean absolute error (MAE) on the implied volatilities of the options.

The results from the calibration exercise are shown in Table 4.1. We can see that, as one would expect, the weak-link Γ-OU-BNS model observes a slightly better calibration performance than the classical Γ-OU-BNS model with strictly coupled jumps. As one can read from the calibrated parameters, this primarily stems from the model feature to allow for *more* volatility jumps than asset price jumps. Moreover, a two-sided decoupling property of the model (i.e. a jump correlation parameter $\kappa < 1$) cannot be observed in the plain vanilla option prices. In our calibrated model, every downward asset price jump is accompanied by a sudden volatility jump, but not vice versa, i.e. there exist sudden rises in volatility without any immediate downward asset price. This

---

[4]If there are put and call options with the same strike and same maturity, we select the respective option, which was more liquidly traded.

Model	MAE	σ_0	c_Y	c_Z	η_Y	η_Z	λ	κ
"Classical" Γ-OU-BNS	0.94	0.114	0.59	0.59	9.47	6.17	2.37	–
Weak-link Γ-OU-BNS	0.81	0.108	0.84	0.35	10.86	3.60	2.42	0.999

Table 4.1 The calibration performance of the weak-link Γ-OU-BNS model compared to the classical one. The calibration error (MAE) denotes the mean absolute error per option in volatility points.

is actually in line with the arguments of Shiller [1988], where multiple reasons for rising volatility are described. On the other hand, the volatility hikes being accompanied by sudden asset price deterioration may be interpreted as materialization of the leverage effect, where deteriorating asset prices cause higher volatility due to higher leverage on the balance sheet of the respective companies.

Sensitivity of the implied volatility skew

To obtain better intuition for the weak-link model, we observe the impact of the newly introduced parameters on the implied volatility skew. Since the classical Γ-OU-BNS model is solely enhanced by the separate volatility jump intensity as well as the jump correlation parameter, we focus our analysis on the sensitivity towards these parameters.

We start analyzing the impact of the jump correlation parameter κ: As we can see in Figure 4.3, throughout the different maturities, the jump correlation parameter somehow (similar to the asset jump height) controls for the skewness of the plain vanilla prices: Complete independence of volatility and asset price jumps results into a more symmetric volatility skew, while higher jump correlation parameter result in relatively higher put prices and lower call prices. This phenomenon can be explained by the ATM peak of an option's vega. Hence, OTM put prices value joint downward jumps in the asset price and upward jumps in the volatility higher than OTM call prices (the "vanna", i.e. the second order cross derivative w.r.t. implied volatility and the spot price, is negative for OTM puts, while it is positive for OTM calls). But the impact of the jump correlation parameter is far weaker than the impact of the asset price jump height, which drives the skewness much more direct via the respective option's delta.
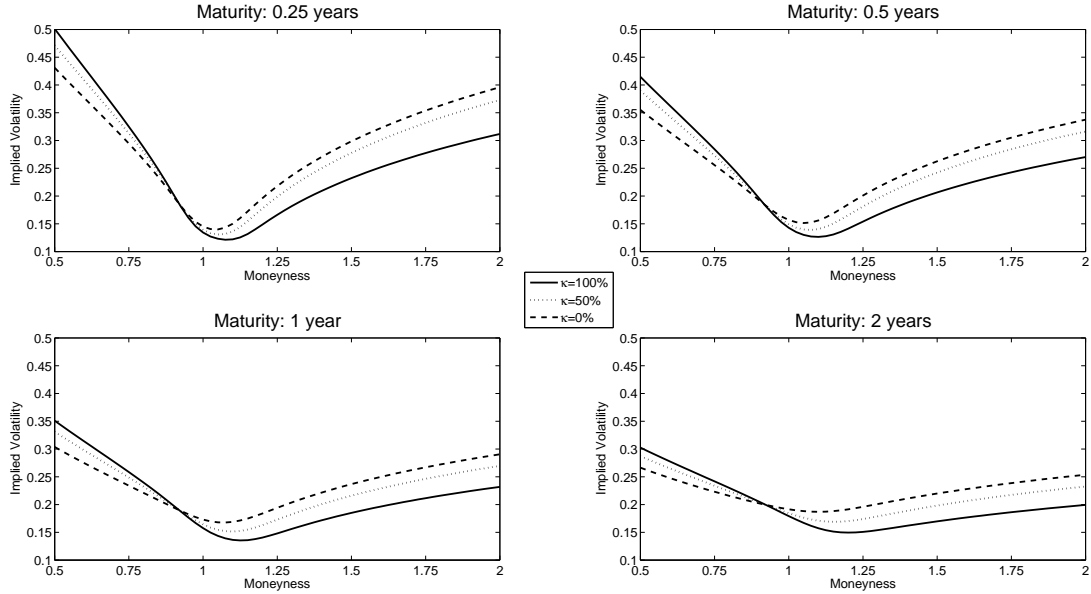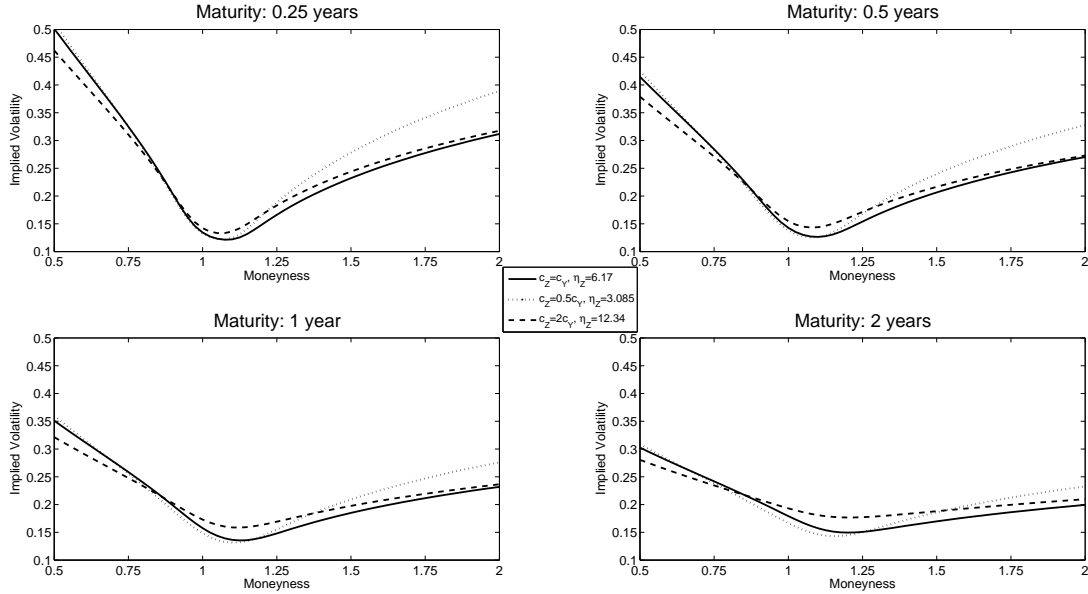
Figure 4.3 Volatility smiles in the Weak-link Γ-OU-BNS model with calibrated parameters of the Γ-OU-BNS model ($\sigma_0 = 0.114$, $c_Y = c_Z = 0.59$, $\eta_Z = 6.17$, $\lambda = 2.37$, $\eta_Y = 9.47$) for different levels of dependence (jump correlation parameter κ). By moneyness, we actually mean the call moneyness, i.e. strike/spot.

We continue by analyzing the impact of the volatility jump intensity on the skewness by assuming constant average expected volatility: As one can see in Figure 4.4, the implied volatilities in the wings rise with more occasional, but sharper upward jumps (low jump intensity, accompanied by higher jumps), while the ATM volatilities become flatter. Overall, the parameter c_Y controls for the pronouncedness of the smile (compared to the jump correlation parameter, controlling the skew).

Jump into the money - where the weak-link property matters

At first sight, the weak-link property of the model does not seem to provide tremendous added value: We observed that fitting the model to a surface of European options does not provide a tremendously better fitting of the observed prices compared to the classical Γ-OU-BNS model. Moreover, particularly the jump correlation parameter κ

Figure 4.4 Volatility smiles in the Weak-link Γ-OU-BNS model with calibrated parameters of the Γ-OU-BNS model ($\sigma_0 = 0.114$, $\lambda = 2.37$, $c_Y = 0.59$, $\eta_Y = 9.47$, $\kappa = 1$) for different volatility jump intensities (c_Z). To keep the average level of volatility comparable, we adjust the jump sizes accordingly such that the average jump impact on the expected volatility remains constant.

seems to be difficult to obtain from quoted European options. We suppose that this is due to the payoff structure of a European option: European call options (which suffice to look at due to put-call parity) solely consider the terminal asset price value to be important. Thus, only the average price fluctuation (concerning volatility and jumps) determines the payoff. But when moving to path-dependent options, the coupling degree of jumps and volatility becomes more important.

Consider a down-and-in call option[5] with a payoff $\mathbb{1}_{\{\{\min_{t \in [0,T]} S_t < BS_0\}\}}(S_T - KS_0)^+$, i.e. a relative strike value $K > 0$, maturity $T > 0$, and a relative barrier level $B < 1$ activating the payoff of the option when crossed downwards.

---

[5]Due to the "barrier parity relationship", we could similarly look at a down-and-out call option, which may be slightly more popular in derivatives markets. We stick to the knock-in option, since we think that it provides more intuition about the weak link feature of the model.
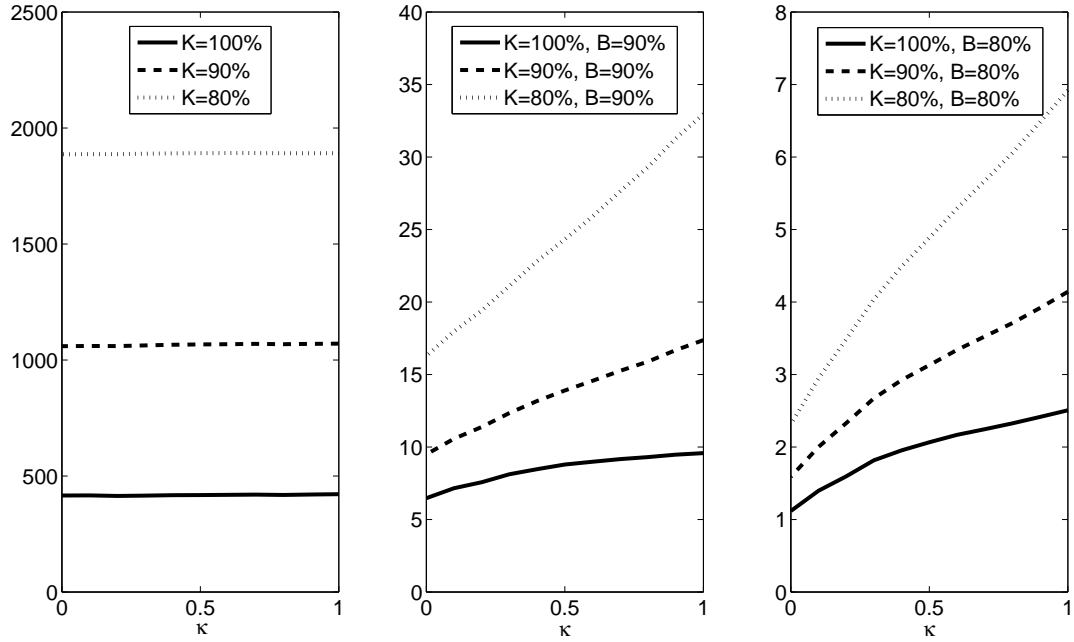
Figure 4.5 Option prices for different levels of jump correlation parameters κ. The maturity of the option is fixed by $T = 0.2$, the spot is $S_0 = 17985$, and the interest rate is assumed to be zero. The left subfigure shows prices of plain vanilla call options, the other two subfigures illustrate prices of down-and-in call options. The sensitivity w.r.t. the jump correlation parameter becomes particularly high for options that are already in-the-money, but yet deactivated.

In comparison to a plain vanilla European call option, the down-and-in call option is much more sensitive to the coupling of asset and volatility jumps: A downward jump causes the option payoff to be activated, while rising volatility enhances the value of the activated European option. Vice versa, rising volatility may enhance the overall value of the barrier option, but in case of a non-activated option, its value is limited in spot regions that are far above the activation barrier. To price these down-and-in call options, we obtained the model parameters from a calibration to market quotes of European options for fixed values of κ. Moreover, we considered a subclass of the weak-link Γ-OU-BNS model class by setting $c_Y = c_Z$. This ensures, that the prices for $\kappa \nearrow 1$ showed in Figure 4.5 correspond to the classical Γ-OU-BNS model, as shown

in Theorem 4.11. As you can see in Figure 4.5, for a set of model choices (by varying the parameter κ) we get, on the one hand, same vanilla prices (by construction), but on the other hand, totally different down-and-in calls. In particular, when the option is in-the-money, but not activated yet, the jump correlation sensitivity is particularly high. The pricing of the down-and-in-call option is done via Monte Carlo simulation and an Euler discretization scheme. We used 100 steps and and 1 000 000 scenarios.

5 Sequential modeling of multivariate financial markets

In this chapter, we present a new methodology to generalize univariate models to the multivariate case. Jump diffusion models, whose jump part is driven by a compound Poisson process with exponentially distributed jumps, are considered. Particularly, time-changed $\mathrm{CPP_{Exp}}$ are used to construct multivariate extensions of three univariate jump diffusion models: the Γ-OU-BNS model – the model which we already introduced in Section 4.1, Kou's model (cf. Kou [2002]) – a model with constant volatility and two-sided exponentially distributed jumps, and the two-sided Γ-OU-BNS model – a combination of the Γ-OU-BNS model and Kou's model. A multivariate extension of the BNS model class employing matrix subordinators is designed in Pigorsch and Stelzer [2008] and pricing in this model is scrutinized in Muhle-Karbe et al. [2012]. Unlike these models, we use a bottom-up approach. That means, we start with d univariate models and merge these to one multivariate model by introducing a certain dependence structure. The most appealing feature of our ansatz is the separability of the marginal distributions from the dependence structure, rendering our multivariate models quite handy. We can divide the model parameters into two sets: the parameters determining the marginal distribution of each one-dimensional model and the parameters determining the dependence structure. This separation feature provides convenient effects in terms of practical issues. For example, a calibration can be carried out in two subsequent steps: first, the univariate models can be calibrated to market quotes of options on single assets, second, one can set the dependence parameters without affecting the already fixed marginal distributions, which we will see in the concluding Section 5.5 on some applications. But first, we shortly introduce the one-dimensional models in this chapter's fundamental Section 5.1 and present the multi-dimensional versions in Sections 5.2, 5.3, and 5.4. The theoretical results within this chapter, as well as the calibration exercises, are also published in Mai et al. [2014] and Bannör et al. [2015].

5.1 Fundamentals: One-dimensional models

This section is dedicated to introduce the univariate models, which are generalized later in this chapter. Since we already investigated the Γ-OU-BNS model in Section 4.1, we only introduce Kou's model and the two-sided Γ-OU-BNS model. All of these models support the striking feature of a closed-form solution for the characteristic function of the log-price process.

Kou's model (cf. Kou [2002]) is an exponential jump diffusion model with constant and deterministic volatility. It supports positive and negative jumps, driven by two independent CPP_{Exp}.

Definition 5.1 (Kou's model)
The asset value process $S = \{S_t\}_{t \geq 0}$ in Kou's model is given by $S_t = S_0 \exp(X_t)$, where

$$X_t = \mu\, t + \sigma\, W_t + Z_t^+ - Z_t^-,$$

with $S_0 > 0$ and $\sigma > 0$. $Z^+ = \{Z_t^+\}_{t \geq 0} \sim \text{CPP}_{\text{Exp}}(c^+, \eta^+), \eta^+ > 1, Z^- = \{Z_t^-\}_{t \geq 0} \sim \text{CPP}_{\text{Exp}}(c^-, \eta^-)$, and $W = \{W_t\}_{t \geq 0}$ is a standard Brownian motion. All processes are mutually independent.

Under an equivalent martingale measure, the drift has to satisfy

$$\mu = r - \frac{\sigma^2}{2} - \frac{c^+}{\eta^+ - 1} + \frac{c^-}{\eta^- + 1},$$

where r denotes the constant risk free interest rate. The drift is similar to the drift in the Γ-OU-BNS model, however, the volatility component is now constant and, additionally, there is a compensation for negative jumps. Relevant for the pricing of options via Fourier inversion methods is the closed-form solution of the characteristic function of X_t, namely

$$\mathbb{E}\left[e^{\mathrm{i}\, u\, X_t}\right] = \exp\left(\mathrm{i}\, u\, X_t + t\left(\mathrm{i}\, u\, \mu - \frac{1}{2}\, \sigma^2\, u^2 + \frac{c^+\, \mathrm{i}\, u}{\eta^+ - \mathrm{i}\, u} - \frac{c^-\, \mathrm{i}\, u}{\eta^- + \mathrm{i}\, u}\right)\right), \quad t \geq 0. \quad (5.1)$$

This formula follows directly from Example 2.9 and the assumption on the independence of W, Z^+, and Z^-.

In contrast to the BNS model, the log-price process X in Kou's model is a Lévy process. Thus, it is not surprising that the characteristic function of X_t is of exponential shape (cf. Theorem 2.19). The BNS model class was extended by Bannör and Scherer [2013] to incorporate two-sided jumps in the asset price process. In the following we define one representative of that class, the two-sided Γ-OU-BNS model, which combines the stochastic volatility from the BNS model with the two-sided jumps from Kou's model.

Definition 5.2 (Two-sided Γ-OU-BNS model)
We say that a stochastic process $\{S_t\}_{t\geq 0}$ follows a *two-sided Γ-OU-BNS model*, if the log-price $X_t := \log S_t$ follows the dynamics of the SDEs

$$
\begin{aligned}
\mathrm{d}X_t &= (\mu + \beta\,\sigma_t^2)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + \rho_+\,\mathrm{d}Z_t^+ + \rho_-\,\mathrm{d}Z_t^-, \\
\mathrm{d}\sigma_t^2 &= -\lambda\,\sigma_t^2\,\mathrm{d}t + \mathrm{d}Z_t^+ + \mathrm{d}Z_t^-,
\end{aligned}
$$

with independent processes $Z^+ = \{Z_t^+\}_{t\geq 0} \sim \mathrm{CPP}_{\mathrm{Exp}}(c^+, \eta^+)$, $Z^- = \{Z_t^-\}_{t\geq 0} \sim \mathrm{CPP}_{\mathrm{Exp}}(c^-, \eta^-)$, and $W = \{W_t\}_{t\geq 0}$ being a Brownian motion independent of Z^+ and Z^-, $\mu \in \mathbb{R}$, $\lambda > 0$, $\rho_+ > 0$, $\rho_- < 0$.[1]

Under a risk neutral martingale measure, the drift has to satisfy

$$
\mu = r - \frac{c^+\,\rho^+}{\eta^+ - \rho^+} + \frac{c^-\,\rho^-}{\eta^- + \rho^-}, \quad \beta = -\frac{1}{2},
$$

where r denotes the risk-neutral drift.

Theorem 5.3 (Characteristic function in a two-sided Γ-OU-BNS model)
Let $S = \{S_t\}_{t\geq 0}$ follow a two-sided Γ-OU-BNS model, then the characteristic function of $X_t = \log(S_t)$ is given by

$$
\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}\,u\,X_t}\right] = \exp\Bigg(&\mathrm{i}\,u\,(X_0 + \mu\,t) - g\,h\,\sigma_0^2 + \frac{c^+}{\eta^+ - f_2^+}\left(\frac{\eta^+}{\lambda}\log\frac{\eta^+ - f_1^+}{\eta^+ - \mathrm{i}\,u\,\rho^+} + f_2^+\,t\right) \\
&+ \frac{c^-}{\eta^- - f_2^-}\left(\frac{\eta^-}{\lambda}\log\frac{\eta^- - f_1^-}{\eta^- - \mathrm{i}\,u\,\rho^-} + f_2^-\,t\right)\Bigg),
\end{aligned}
$$

---

[1] Compared to the original formulation of the model in Bannör and Scherer [2013] and the original BNS model from Nicolato and Venardos [2003], we do not change the clock of the subordinators to $t \mapsto \lambda t$. This formulation is equivalent and more handy in the upcoming multivariate construction.

with

$$g := \frac{u^2 - 2\,\beta\,i\,u}{2}, \quad h := \frac{1 - \exp(-\lambda\,t)}{\lambda},$$

$$f_1^* := i\,u\,\rho^* - g\,h, \quad f_2^* := i\,u\,\rho^* - \frac{g}{\lambda}, \quad * \in \{+, -\}.$$

Proof

See Bannör and Scherer [2013]. □

5.2 A multivariate BNS-Γ-OU model

We model a portfolio of d assets, each represented by a one-dimensional BNS-Γ-OU model. The dependence between the diffusion components is treated as in the standard market models living in a Brownian world. The jump components driving the volatility processes, however, are made dependent via the construction of time-changed CPP$_{\text{Exp}}$ from Chapter 3, making it possible for two or more assets to jump simultaneously, and introducing dependence to the stochastic volatility processes. The induced dependence between the jump components is determined by the dependence parameter κ. Independently of the choice of κ, the marginal distributions remain the same. We are thus able to describe the portfolio model by two separated sets of parameters:

(1) The parameters determining the marginal distributions of the assets: a Γ-OU-process with leverage under an equivalent martingale measure is determined by five parameters: one parameter for the jump intensity, one parameter for the jump sizes, one slow down parameter for the stochastic volatility, one leverage parameter, and one initial value for the volatility process.

(2) One set of parameters for the dependence structure of the assets: a correlation matrix Σ for the Brownian parts and the coefficient $\kappa \in (0, 1)$ for the jump parts.

The construction works as follows. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which we define the following processes.

(a) The process $W = (W^{(1)}, \dots, W^{(d)})$, which is a d-dimensional standard Brownian motion with correlation matrix Σ.

(b) Independently of the process in (a), we define a d-dimensional time-changed $\mathrm{CPP_{Exp}}$, i.e. we have independent Poisson processes $N^{(1)}, \ldots, N^{(d)}$ with intensities $\frac{c_1}{c_T - c_1}, \ldots, \frac{c_d}{c_T - c_d}$. Moreover, for each $i = 1, \ldots, d$ we let $\{J_j^{(i)}\}_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $J_1^{(i)} \sim \mathrm{Exp}\left(\frac{c_T \, \eta_i}{c_T - c_i}\right)$, independently of the previous processes. We suppose the jump size parameters $\eta_1, \ldots, \eta_d > 0$ and the intensities $c_1, \ldots, c_d > 0$ to be given and we set $c_T := \frac{1}{\kappa} \max_{1 \le i \le d}\{c_i\}$.

(c) Independently of the processes in (a) and (b), let $T = \{T_t\}_{t \ge 0} \sim \mathrm{CPP_{Exp}}(c_T, 1)$.

Definition 5.4 (Multivariate BNS-Γ-OU model)
Having defined these processes on our probability space, for each $i = 1, \ldots, d$, we describe asset i in the *multivariate* BNS-Γ-OU *model* by a one-dimensional BNS-Γ-OU model, i.e. $S_t^{(i)} = S_0^{(i)} \exp\left(X_t^{(i)}\right)$, where

$$\mathrm{d}X_t^{(i)} = \left(r - \frac{c_i \, \rho_i}{\eta_i - \rho_i} - \frac{1}{2}\left(\sigma_t^{(i)}\right)^2\right) \mathrm{d}t + \sigma_t^{(i)} \, \mathrm{d}W_t^{(i)} + \rho_i \, \mathrm{d}Z_t^{(i)}, \qquad (5.2)$$

$$\mathrm{d}\left(\sigma_t^{(i)}\right)^2 = -\lambda_i \left(\sigma_t^{(i)}\right)^2 \mathrm{d}t + \mathrm{d}Z_t^{(i)}, \qquad (5.3)$$

with $Z = \left(Z^{(1)}, \ldots, Z^{(d)}\right)$ being a d-dimensional time-changed $\mathrm{CPP_{Exp}}$.

For pricing multi asset options by Fourier methods, a simple expression for the joint characteristic function of $X^{(1)}, \ldots, X^{(d)}$ is useful. In our multivariate BNS-Γ-OU model we can at least calculate it in closed form within the special case of independent Brownian motions. This is done in the next theorem.

Theorem 5.5 (Joint characteristic function in the multivariate BNS model)
Assume that $W^{(1)}, \ldots, W^{(d)}$ are independent. Define for all $0 \le s \le t$, $1 \le k \le d$, and fixed $u = u_1, \ldots, u_d \in \mathbb{C}^d$,

$$f_k(s; u_k) = \frac{u_k - \mathrm{i}\, u_k^2}{2\, \lambda_k}\left(e^{\lambda_k(s-t)} - 1\right) + u_k \, \rho_k.$$

Then, the joint characteristic function φ_{X_t} of $X_t = \left(X_t^{(1)}, \ldots, X_t^{(d)}\right)$ is given by

$$\log\left(\varphi_{X_t}(u)\right) = \sum_{k=1}^{d} \mathrm{i}\, u_k \, t\left(r - \frac{c_k \, \rho_k}{\eta_k - \rho_k}\right) + \frac{\mathrm{i}\, u_k + u_k^2}{2\, \lambda_k}\left(e^{-\lambda_k \, t} - 1\right)\left(\sigma_0^{(k)}\right)^2 - c_T \, t$$

$$+ \int_0^t c_T \left(1 - \sum_{k=1}^d \frac{\mathrm{i}\, c_k\, f_k(s; u_k)}{c_T\, \eta_k - \mathrm{i}\, f_k(s; u_k)\, (c_T - c_k)} \right)^{-1} \mathrm{d}s.$$

Proof
By using the model Equation (5.2), we get

$$\varphi_{X_t}(u) = \mathbb{E} \left[\prod_{k=1}^d e^{\mathrm{i}\, u_k\, X_t^{(k)}} \right]$$

$$= \mathbb{E} \left[\prod_{k=1}^d \exp \left(\mathrm{i}\, u_k \left(r\, t - \frac{c_k\, \rho_k}{\eta_k - \rho_k}\, t - \frac{1}{2} \int_0^t \left(\sigma_s^{(k)} \right)^2 \mathrm{d}s + \int_0^t \sigma_s^{(k)}\, \mathrm{d}W_s^{(k)} + \rho_k\, Z_t^{(k)} \right) \right) \right].$$

$$(5.4)$$

Conditioned on the trajectory of $Z^{(k)}$, the volatility process $\sigma^{(k)}$ is deterministic for all $1 \le k \le d$. Therefore, with a same argument like in Equation (4.8), we get

$$\mathbb{E} \left[\exp \left(\int_0^t \mathrm{i}\, u_k\, \sigma_s^{(k)}\, \mathrm{d}W_s^{(k)} \right) \middle| \left(Z_u^{(k)} \right)_{u \le t} \right] = \exp \left(-\frac{u_k^2}{2} \int_0^t \left(\sigma_s^{(k)} \right)^2 \mathrm{d}s \right).$$

Hence, by conditioning on the trajectories of $Z^{(1)}, \ldots, Z^{(d)}$, Equation (5.4) boils down to

$$\varphi_{X_t}(u) = \mathbb{E} \left[\prod_{k=1}^d \exp \left(-\frac{1}{2} \left(\mathrm{i}\, u_k + u_k^2 \right) \int_0^t \left(\sigma_s^{(k)} \right)^2 \mathrm{d}s + \mathrm{i}\, u_k\, \rho_k\, Z_t^{(k)} \right) \right]$$

$$\exp \left(\sum_{k=1}^d \mathrm{i}\, u_k\, t \left(r - \frac{c_k\, \rho_k}{\eta_k - \rho_k} \right) \right), \qquad (5.5)$$

Using Equation (5.3), the integrated variance process can be simplified for all $1 \le k \le d$, namely

$$\mathrm{d} \left(\sigma_t^{(k)} \right)^2 = -\lambda_k \left(\sigma_t^{(k)} \right)^2 \mathrm{d}t + \mathrm{d}Z_t^{(k)}$$

$$\Leftrightarrow \left(\sigma_t^{(k)} \right)^2 - \left(\sigma_0^{(k)} \right)^2 = -\int_0^t \lambda_k \left(\sigma_s^{(k)} \right)^2 \mathrm{d}s + Z_t^{(k)}$$

$$\Leftrightarrow \int_0^t \left(\sigma_t^{(k)} \right)^2 \mathrm{d}s = \frac{1}{\lambda_k} \left(- \left(\sigma_t^{(k)} \right)^2 + \left(\sigma_0^{(k)} \right)^2 + Z_t^{(k)} \right).$$

Combined with Equation (5.5), we get

$$\varphi_{X_t}(u) = \mathbb{E}\left[\prod_{k=1}^{d}\exp\left(\frac{\mathrm{i}\,u_k + u_k^2}{2\,\lambda_k}\left(\left(\sigma_t^{(k)}\right)^2 - \left(\sigma_0^{(k)}\right)^2 - Z_t^{(k)}\right) + \mathrm{i}\,u_k\,\rho_k\,Z_t^{(k)}\right)\right]$$
$$\exp\left(\sum_{k=1}^{d}\mathrm{i}\,u_k\,t\left(r - \frac{c_k\,\rho_k}{\eta_k - \rho_k}\right)\right),$$

and Equation (4.3) yields

$$\varphi_{X_t}(u) = \mathbb{E}\left[\prod_{k=1}^{d}\exp\left(\int_0^t\left(\frac{\mathrm{i}\,u_k + u_k^2}{2\,\lambda_k}\left(e^{\lambda_k(s-t)} - 1\right) + \mathrm{i}\,u_k\,\rho_k\right)\mathrm{d}Z_s^{(k)}\right)\right]$$
$$\exp\left(\sum_{k=1}^{d}\mathrm{i}\,u_k\,t\left(r - \frac{c_k\,\rho_k}{\eta_k - \rho_k}\right) + \frac{\mathrm{i}\,u_k + u_k^2}{2\,\lambda_k}\left(e^{-\lambda_k\,t} - 1\right)\left(\sigma_0^{(k)}\right)^2\right). \quad (5.6)$$

Define for all $0 \le s \le t$, $1 \le k \le d$,

$$f_k(s; u_k) = \frac{u_k - \mathrm{i}\,u_k^2}{2\,\lambda_k}\left(e^{\lambda_k(s-t)} - 1\right) + u_k\,\rho_k,$$

and note that $f(s) := (f_1(s; u_1), \dots, f_d(s; u_d))$ is continuous. Furthermore, as calculated in Theorem 3.7, the characteristic exponent of the time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ $Z = \left(Z^{(1)}, \dots, Z^{(d)}\right)$ is given by the expression

$$\psi_Z(v) = \frac{\sum_{k=1}^{d}\frac{\mathrm{i}\,c_T\,c_k\,v_k}{c_T\,\eta_k - \mathrm{i}\,v_k\,(c_T - c_k)}}{1 - \sum_{k=1}^{d}\frac{\mathrm{i}\,c_k\,v_k}{c_T\,\eta_k - \mathrm{i}\,v_k\,(c_T - c_k)}}, \quad \forall v \in \mathbb{R}^d. \quad (5.7)$$

Remark 3.8 states that this expression is analytic and can be continued on $C :=$ $\left\{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_i) \ge 0, \forall 1 \le i \le d\right\}$ with $\mathbb{E}\left[e^{\mathrm{i}\,z^\top Z}\right] = e^{\psi_Z(z)}$ for all $z \in$ C. We intend to apply Theorem 2.26 on Equation (5.6). Thus, we must ensure that $f(s) \in C$ and $\mathrm{Re}(\psi_Z(f(s)) \le 0$ for all $0 \le s \le t$. By the definition of f,

$$\mathrm{Im}(f_k(s; u_k)) = \frac{u_k^2}{2\,\lambda_k}\left(1 - e^{\lambda_k(s-t)}\right) > 0, \quad \forall 0 \le s \le t, 1 \le k \le d,$$

and hence, $f(s) \in C$ for all $0 \le s \le t$. Moreover, by Remark 3.8, we know that $\psi_Z(z) \le 0$ for all $z \in C$. Thus $\mathrm{Re}(\psi_Z(f(s)) \le 0$ for all $0 \le s \le t$. By using Theorem 2.26 and Equation (5.7), Equation (5.6) boils down to

$$\varphi_{X_t}(u) = \exp\left(\int_0^t c_T\left(1 - \sum_{k=1}^{d}\frac{\mathrm{i}\,c_k\,f_k(s; u_k)}{c_T\,\eta_k - \mathrm{i}\,f_k(s; u_k)\,(c_T - c_k)}\right)^{-1}\mathrm{d}s\right)\exp\left(-c_T\,t\right)$$

$$\exp\left(\sum_{k=1}^{d} \mathrm{i}\, u_k\, t \left(r - \frac{c_k\, \rho_k}{\eta_k - \rho_k}\right) + \frac{\mathrm{i}\, u_k + u_k^2}{2\,\lambda_k}\left(e^{-\lambda_k\, t} - 1\right)\left(\sigma_0^{(k)}\right)^2\right).$$

And the claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To price path-dependent multi-asset derivatives, it is important to have a fast simulation scheme for the price process. This can be achieved due to the efficient simulation of a time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ (cf. Algorithm 3.20). In the following, we give a simplistic scheme how to simulate path in the multivariate BNS-Γ-OU model.

Algorithm 5.6 (Paths of the asset values in the multivariate BNS model)
Suppose the following parameters to be given: the initial values and initial volatilities for each asset, i.e. $S_0^{(1)}, \ldots, S_0^{(d)}$ and $(\sigma_0^{(1)})^2, \ldots, (\sigma_0^{(d)})^2$, jump parameters for the processes $Z^{(1)}, \ldots, Z^{(d)}$, i.e. $c_1, \ldots, c_d$ and $\eta_1, \ldots, \eta_d$, the slow-down parameters of the volatility processes $\lambda_1, \ldots, \lambda_d$, the leverage parameters $\rho_1, \ldots, \rho_d$, the maturity t^*, the correlation matrix Σ of the d-dimensional Brownian motion $(W^{(1)}, \ldots, W^{(d)})$, the dependence parameter κ for the jump parts, and the constant interest rate r.

(1) Perform Algorithm 3.20 and receive a d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ $Z = \left(Z^{(1)}, \ldots, Z^{(d)}\right)$

(2) Define a partition $0 = t_0 < t_1 < \cdots < t_M = t^*$ on $[0, t^*]$.

(3) For each $1 \leq i \leq M$ do

   (a) Draw a d-dimensional normally distributed vector $B = \left(B^{(1)}, \ldots, B^{(d)}\right)$ with mean 0 and covariance matrix given by Σ.

   (b) For each $1 \leq j \leq d$ do

$$\begin{aligned}
X_{t_i}^{(j)} =& X_{t_{i-1}}^{(j)} + \left(r - \frac{c_j\, \rho_j}{\eta_j - \rho_j} - \frac{1}{2}\left(\sigma_{t_{i-1}}^{(j)}\right)^2\right)(t_i - t_{i-1}) \\
& + \sigma_{t_{i-1}}^{(j)}\, \sqrt{t_i - t_{i-1}}\, B^{(j)} + \rho_j \left(Z_{t_i}^{(j)} - Z_{t_{i-1}}^{(j)}\right) \\
\left(\sigma_{t_i}^{(j)}\right)^2 =& \left(\sigma_{t_{i-1}}^{(j)}\right)^2 \left(1 - \lambda_j\, (t_i - t_{i-1})\right) + Z_{t_i}^{(j)} - Z_{t_{i-1}}^{(j)}
\end{aligned}$$

(4) Return $S_0^{(j)} \exp(X_{t_i}^{(j)})$ for all $1 \leq j \leq d$, $0 \leq i \leq M$.

This algorithm reuses Algorithm 3.20 to draw the dependent jump parts and performs a simple Euler discretization of the SDEs (5.2) and (5.3). A more accurate simulation might be achieved by more sophisticated discretization schemes. For example, one could merge the partition, which is generated in step (2), with the jump times to get a more realistic handling of the jump process. Or, discretizing Equation (4.3) rather than Equation (5.3) gives more accuracy in the simulation of the volatility process and thus also of the log-price process. If the asset value at one point in time is of interest only, the simulation scheme can be further improved, which we investigate in the following. Algorithm 5.8 does not simulate the whole path. The idea is to use the jump times of $T = \{T_t\}_{t \geq 0}$ as grid points for the simulation. Then, the volatility process becomes deterministic between any two consecutive jump times. To account for the change in the asset value process between two consecutive jump times τ_1 and τ_2 of $Z^{(i)}$, one has to add a the random variable R_i given by

$$R_i = \left(r - \frac{c_i \, \rho_i}{\eta_i - \rho_i} \right) (\tau_2 - \tau_1) - \frac{1}{2} \int\limits_{\tau_1}^{\tau_2} \left(\sigma_t^{(i)} \right)^2 \, dt + \int\limits_{\tau_1}^{\tau_2} \sigma_t^{(i)} \, dW_t.$$

Equation 4.3 implies for all $t \in [\tau_1, \tau_2)$

$$\left(\sigma_t^{(i)} \right)^2 = e^{-\lambda_i \, (t - \tau_1)} \left(\sigma_{\tau_1}^{(i)} \right)^2,$$

and therefore, we get

$$\int\limits_{\tau_1}^{\tau_2} \left(\sigma_t^{(i)} \right)^2 \, \mathrm{d}t = \int\limits_{\tau_1}^{\tau_2} e^{-\lambda_i \, (t - \tau_1)} \left(\sigma_{\tau_1}^{(i)} \right)^2 \, \mathrm{d}t = \frac{1}{\lambda_i} \left(\sigma_{\tau_1}^{(i)} \right)^2 \left(1 - e^{-\lambda_i \, (\tau_2 - \tau_1)} \right).$$

Hence,

$$R_i \sim \mathcal{N} \left(\left(r - \frac{c_i \, \rho_i}{\eta_i - \rho_i} \right) (\tau_2 - \tau_1) - \frac{1}{2} \, \nu_{\tau_1, \tau_2}^{(i)} \, , \, \nu_{\tau_1, \tau_2}^{(i)} \right), \tag{5.8}$$

$$\text{where} \quad \nu_{\tau_1, \tau_2}^{(i)} = \frac{1}{\lambda_i} \left(\sigma_{\tau_1}^{(i)} \right)^2 \left(1 - e^{-\lambda_i \, (\tau_2 - \tau_1)} \right).$$

Left to determine is the correlation of $(R_i)_{1 \leq i \leq d}$, which is done in Lemma 5.7. All combined, the asset prices at time τ_2 are given as

$$X_{\tau_2}^{(i)} = X_{\tau_1}^{(i)} + R_i + \rho \left(Z_{\tau_2}^{(i)} - Z_{\tau_1}^{(i)} \right).$$

Lemma 5.7

For all $1 \leq i \leq d$, let R_i be defined as in (5.8) and assume that $\operatorname{Corr}\left(W_t^{(i)}, W_t^{(j)}\right) = \Sigma_{ij}$, then

$$\operatorname{Corr}(R_i\,,\,R_j) = 2\,\Sigma_{ij}\,\frac{\sqrt{\lambda_i\,\lambda_j}}{\lambda_i + \lambda_j}\,\frac{1 - e^{-\frac{1}{2}\,(\lambda_i + \lambda_j)\,(\tau_2 - \tau_1)}}{\sqrt{\left(1 - e^{-\lambda_i\,(\tau_2 - \tau_1)}\right)\left(1 - e^{-\lambda_j\,(\tau_2 - \tau_1)}\right)}},$$

for all $1 \leq i,\,j \leq d$, $i \neq j$.

Proof

$$\operatorname{Corr}(R_i\,,\,R_j) = \operatorname{Corr}\left(\int_{\tau_1}^{\tau_2} \sigma_t^{(i)}\,dW_t^{(i)}, \int_{\tau_1}^{\tau_2} \sigma_t^{(j)}\,dW_t^{(j)}\right)$$

$$= \operatorname{Corr}\left(\int_{\tau_1}^{\tau_2} e^{-\lambda_i\,(t-\tau_1)}\left(\sigma_{\tau_1}^{(i)}\right)^2\,dW_t^{(i)}, \int_{\tau_1}^{\tau_2} e^{-\lambda_j\,(t-\tau_1)}\left(\sigma_{\tau_1}^{(j)}\right)^2\,dW_t^{(j)}\right)$$

$$= \Sigma_{ij}\,\frac{\int_0^{\tau_2 - \tau_1} \sigma_{\tau_1}^{(i)}\,\sigma_{\tau_1}^{(j)}\,e^{-\frac{1}{2}(\lambda_i + \lambda_j)t}\,dt}{\sqrt{\nu_{\tau_1,\tau_2}^{(i)}\,\nu_{\tau_1,\tau_2}^{(j)}}}$$

$$= 2\,\Sigma_{ij}\,\frac{\sqrt{\lambda_i\,\lambda_j}}{\lambda_i + \lambda_j}\,\frac{1 - e^{-\frac{1}{2}\,(\lambda_i + \lambda_j)\,(\tau_2 - \tau_1)}}{\sqrt{\left(1 - e^{-\lambda_i\,(\tau_2 - \tau_1)}\right)\left(1 - e^{-\lambda_j\,(\tau_2 - \tau_1)}\right)}}. \qquad \square$$

Now, we are ready to present the algorithm for computing the final values in the multivariate BNS-Γ-OU model, which is useful, e.g. for pricing multi-asset European options. Note that in contrast to Algorithm 5.8, the simulation of the final value is unbiased.

Algorithm 5.8 (Final asset values in the multivariate BNS model)

Suppose the same parameters to be given as in Algorithm 5.6.

(1) Perform Algorithm 3.20 and get dependent processes $Z^{(1)}, \ldots, Z^{(d)}$ and their jump times $0 =: \tau_0 < \tau_1 < \cdots < \tau_N < \tau_{N+1} := t^*$.

(2) For each $1 \leq i \leq N+1$ do

   (a) Draw a d-dimensional normally distributed vector $B = \left(B^{(1)}, \ldots, B^{(d)}\right)$ with mean 0 and covariance matrix given by the correlation matrix given by Lemma 5.7.

(b) For each $1 \leq j \leq d$ do

$$\hat{\sigma}^2 = \frac{1}{\lambda_j} \left(\sigma_{\tau_{i-1}}^{(j)} \right)^2 \left(1 - e^{-\lambda_j (\tau_i - \tau_{i-1})} \right)$$

$$X_{\tau_i}^{(j)} = X_{\tau_{i-1}}^{(j)} + \left(r - \frac{c_j \, \rho_j}{\eta_j - \rho_j} \right) (\tau_i - \tau_{i-1}) - \frac{1}{2} \hat{\sigma}^2 + \hat{\sigma} \, B^{(j)}$$

$$+ \rho_j \left(Z_{\tau_i}^{(j)} - Z_{\tau_{i-1}}^{(j)} \right)$$

$$\left(\sigma_{\tau_i}^{(j)} \right)^2 = \left(\sigma_{\tau_{i-1}}^{(j)} \right)^2 \left(1 - \lambda_j (\tau_i - \tau_{i-1}) \right) + Z_{\tau_i}^{(j)} - Z_{\tau_{i-1}}^{(j)}$$

(3) Return $S_0^{(j)} \exp(X_{\tau_{N+1}}^{(j)})$ for all $1 \leq j \leq d$.

5.3 A multivariate Kou Model

Like in the multivariate BNS model, we model a portfolio of d assets. But now, each component follows a one-dimensional Kou model. In contrast to the BNS framework, the volatility is constant, but the asset price jump magnitudes are drawn from a double-exponential distribution. The dependence between the diffusion components is treated as before, i.e. we consider d correlated Brownian motions $W^{(1)}, \ldots, W^{(d)}$. The jumps are subdivided into positive and negative components, modeled by two independent $\mathrm{CPP}_{\mathrm{Exp}}$. Jumps of different assets are made dependent via the construction of time-changed $\mathrm{CPP}_{\mathrm{Exp}}$ from Section 3, allowing two or more assets to jump simultaneously. The induced dependence between the positive jump components is determined by the parameter $\kappa^+ \in (0,1)$, and the negative ones by $\kappa^- \in (0,1)$. Independently of the choice of those dependence parameters, the marginal distributions, which are equivalent to the ones in the univariate Kou model, remain the same. We are thus able to describe the portfolio model by two separated sets of parameters:

(1) The parameters determining the marginal distributions of the assets: one parameter for the diffusion volatility, as well as two parameters for the intensities of the jumps and two parameters determining the average jump sizes.

(2) One set of parameters for the dependence structure of the assets: a correlation matrix Σ for the diffusion parts and the coefficients κ^+ and κ^- for the jump parts.

The construction works as follows. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which we define the following processes.

(a) For $i = 1, \ldots, d$, the processes $\mu_i t + \sigma_i W_t^{(i)}$, where $W = (W^{(1)}, \ldots, W^{(d)})$ is a d-dimensional standard Brownian motion with correlation matrix Σ, μ_i denotes the risk neutral drift, and σ_i represents the volatility of the diffusion part.

(b) Independently of the processes in (a), we define two independent time-changed CPP_{Exp}, i.e we define independent Poisson processes $N^{(1)}, \ldots, N^{(d)}$ with intensities $\frac{c_1 \eta_T^+}{c_T^+ - c_1}, \ldots, \frac{c_d \eta_T^+}{c_T^+ - c_d}$, and $N^{(-1)}, \ldots, N^{(-d)}$ with intensities $\frac{c_{-1} \eta_T^-}{c_T^- - c_{-1}}, \ldots, \frac{c_{-d} \eta_T^-}{c_T^- - c_{-d}}$. Moreover, for each $i = 1, \ldots, d$ we let $\{J_j^{(i)}\}_{j \in \mathbb{N}}$ and $\{J_j^{(-i)}\}_{j \in \mathbb{N}}$ be sequences of i.i.d. random variables with $J_1^{(i)} \sim \mathrm{Exp}\left(\frac{c_T^+ \eta_i}{c_T^+ - c_i}\right)$ and $J_1^{(-i)} \sim \mathrm{Exp}\left(\frac{c_T^- \eta_{-i}}{c_T^- - c_{-i}}\right)$, independently of the previous processes. Here, we suppose we have given intensities $c_1, \ldots, c_d > 0$ and $c_{-1}, \ldots, c_{-d} > 0$, and we set $c_T^+ := \max_{1 \le i \le d}\{c_i\}\frac{1}{\kappa^+}$ and $c_T^- := \max_{1 \le i \le d}\{c_{-i}\}\frac{1}{\kappa^-}$.

(c) Independently of the processes in (a) and (b) we let $T^+ = \{T_t^+\}_{t \ge 0}$ and $T^- = \{T_t^-\}_{t \ge 0}$ be compound Poisson processes with intensities c_T^+ and c_T^- and jump size distributions $\mathrm{Exp}\left(\eta_T^+\right)$ and $\mathrm{Exp}\left(\eta_T^-\right)$.

Definition 5.9 (Multivariate Kou model)
Having defined these processes on our probability space, for each $i = 1, \ldots, d$, we describe asset i in the *multivariate Kou model* by a one-dimensional Kou model, i.e. $S_t^{(i)} = S_0^{(i)} \exp\left(X_t^{(i)}\right)$, where

$$X_t^{(i)} = \mu_i\, t + \sigma_i\, W_t^{(i)} + Z_t^{(i)} - Z_t^{(-i)}, \tag{5.9}$$

$$\text{with} \quad Z_t^{(i)} = \sum_{j=1}^{N_{T_t^+}^{(i)}} J_j^{(i)} \quad \text{and} \quad Z_t^{(-i)} = \sum_{j=1}^{N_{T_t^-}^{(-i)}} J_j^{(-i)},$$

i.e. $\left(Z^{(1)}, \ldots, Z^{(d)}\right)$ and $\left(Z^{(-1)}, \ldots, Z^{(-d)}\right)$ are two independent d-dimensional time-changed CPP_{Exp}.

For pricing multi asset options by Fourier methods we need the joint characteristic function of $X^{(1)}, \ldots, X^{(d)}$. The following theorem presents a closed form expression of this joint characteristic function in our multivariate Kou model.

Theorem 5.10 (Joint char. function in the multivariate Kou model)
For all $u = (u_1, \ldots, u_d) \in \mathbb{C}^d$, define

$$
\alpha^+(u) := \sum_{k=1}^{d} \frac{c_k \, \mathrm{i} \, u_k}{c_T^+ \, \eta_k - \mathrm{i} \, u_k \left(c_T^+ - c_k \right)} \quad \text{and} \quad \alpha^-(u) := \sum_{k=1}^{d} \frac{c_{-k} \, \mathrm{i} \, u_k}{c_T^- \, \eta_{-k} + \mathrm{i} \, u_k \left(c_T^- - c_{-k} \right)}.
$$

Then, the joint characteristic function of $X_t = (X_t^{(1)}, \ldots, X_t^{(d)})$ is given by

$$
\varphi_{X_t}(u) = \exp\left(t \left(\mathrm{i} \, u^\top \mu - \frac{1}{2} \, u^\top \widehat{\Sigma} \, u + \frac{c_T^+ \, \alpha^+(u)}{1 - \alpha^+(u)} - \frac{c_T^- \, \alpha^-(u)}{1 + \alpha^-(u)} \right) \right),
$$

where $\widehat{\Sigma}$ denotes the covariance matrix of $(\sigma_1 W_1^{(1)}, \ldots, \sigma_d W_1^{(d)})$, i.e. $\widehat{\Sigma} = \sigma^\top \Sigma \, \sigma$.

Proof

$$
\varphi_{X_t}(u) = \mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, X_t^{(k)}} \right] = \mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \left(\mu_k \, t + \sigma_k \, W_t^{(k)} + Z_t^{(k)} - Z_t^{(-k)} \right)} \right]
$$

$$
= \exp\left(\sum_{k=1}^{d} \mathrm{i} \, u_k \, \mu_k \, t \right) \mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, \sigma_k \, W_t^{(k)}} \right] \mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, Z_t^{(k)}} \right] \mathbb{E}\left[\prod_{k=1}^{d} e^{-\mathrm{i} \, u_k \, Z_t^{(-k)}} \right].
$$

$$
\tag{5.10}
$$

Note that $\mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, \sigma_k \, W_t^{(k)}} \right]$ is the characteristic function of a multivariate normal distributed random variable (cf. Example 2.9), and, hence

$$
\mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, \sigma_k \, W_t^{(k)}} \right] = \exp\left(-\frac{1}{2} \, u^\top \widehat{\Sigma} \, u \, t \right). \tag{5.11}
$$

By conditioning on T_t^+, the processes $Z_t^{(1)}, \ldots, Z_t^{(d)}$ become independent and $Z_t^{(i)} | T_t^+$ is a compound Poisson distributed random variable with expected number of jumps $\frac{T_t^+ \, c_i}{c_T^+ - c_i}$ and jump size distribution $\mathrm{Exp}\left(\frac{c_T^+ \, \eta_i}{c_T^+ - c_i} \right)$ for all $1 \leq i \leq d$. Furthermore, using similar calculations like in the proof of Theorem 3.1, we get for all $1 \leq i \leq d$

$$
\mathbb{E}\left[e^{\mathrm{i} \, u_k \, Z_t^{(k)}} \,\Big|\, T_t^+ \right] = \exp\left(\frac{\frac{T_t^+ \, c_k}{c_T^+ - c_k} \, \mathrm{i} \, u_k}{\frac{c_T^+ \, \eta_k}{c_T^+ - c_k} - \mathrm{i} \, u_k} \right) = \exp\left(\frac{\mathrm{i} \, c_k \, u_k}{c_T^+ \, \eta_k - \mathrm{i} \, u_k \left(c_T^+ - c_k \right)} \, T_t^+ \right).
$$

Hence, using the definition $\alpha^+(u) := \sum_{k=1}^{d} \frac{c_k \, \mathrm{i} \, u_k}{c_T^+ \, \eta_k - \mathrm{i} \, u_k \left(c_T^+ - c_k \right)}$, we get

$$
\mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i} \, u_k \, Z_t^{(k)}} \right] = \mathbb{E}\left[\prod_{k=1}^{d} \mathbb{E}\left[e^{\mathrm{i} \, u_k \, Z_t^{(k)}} \,\Big|\, T_t^+ \right] \right] = \mathbb{E}\left[e^{\alpha^+(u) \, T_t^+} \right]. \tag{5.12}
$$

Furthermore, $\mathrm{Re}(\alpha^+(u)) < 0$, because

$$\mathrm{Re}\left(\alpha^+(u)\right) = \mathrm{Re}\left(\sum_{k=1}^{d} \frac{c_k\,\mathrm{i}\,u_k}{c_T^+\,\eta_k - \mathrm{i}\,u_k\,(c_T^+ - c_k)}\right) = -\sum_{k=1}^{d} \frac{c_k\,u_k^2\,(c_T^+ - c_k)}{(c_T^+)^2\,\eta_k^2 + u_k^2\,(c_T^+ - c_k)^2} < 0.$$

Thus, Theorem 2.22 can be applied on $\mathbb{E}\left[e^{\alpha^+(u)\,T_t^+}\right]$, which yields

$$\mathbb{E}\left[e^{\alpha^+(u)\,T_t^+}\right] = \exp\left(\frac{c_T^+\,\alpha^+(u)}{1 - \alpha^+(u)}\,t\right), \tag{5.13}$$

where we used Theorem 3.1, which states that the Laplace exponent of T^+ is given by $\tilde{\psi}_{T^+}(-u) = \frac{-c_T^+\,u}{1+u}$. Combining Equations (5.12) and (5.13) gives

$$\mathbb{E}\left[\prod_{k=1}^{d} e^{\mathrm{i}\,u_k\,Z_t^{(k)}}\right] = \exp\left(\frac{c_T^+\,\alpha^+(u)}{1 - \alpha^+(u)}\,t\right). \tag{5.14}$$

In a quite similar way we get

$$\mathbb{E}\left[\prod_{k=1}^{d} e^{-\mathrm{i}\,u_k\,Z_t^{(-k)}}\right] = \exp\left(-\frac{c_T^-\,\alpha^-(u)}{1 + \alpha^-(u)}\,t\right). \tag{5.15}$$

Plugging in Equations (5.11), (5.14), and (5.15) into Equation (5.10) yields

$$\varphi_{X_t}(u) = \exp\left(t\left(\mathrm{i}\,u^\top\mu - \frac{1}{2}\,u^\top\widehat{\Sigma}\,u + \frac{c_T^+\,\alpha^+(u)}{1 - \alpha^+(u)} - \frac{c_T^-\,\alpha^-(u)}{1 + \alpha^-(u)}\right)\right),$$

which concludes the proof. $\qquad\square$

Algorithm 5.11 (Paths of the asset values in the multivariate Kou model)
Suppose the following parameters to be given: the initial values $S_0^{(1)}, \ldots, S_0^{(d)}$ for each asset, the constant volatilities $\sigma_1, \ldots, \sigma_d$ for each asset, parameters for the jump processes $Z^{(1)}, \ldots, Z^{(d)}$, i.e. $c_1, \ldots, c_d$ and $\eta_1, \ldots, \eta_d$, parameters for the jump processes $Z^{(-1)}, \ldots, Z^{(-d)}$, i.e. $c_{-1}, \ldots, c_{-d}$ and $\eta_{-1}, \ldots, \eta_{-d}$, the maturity t^*, the correlation matrix Σ of the d-dimensional Brownian motion $(W^{(1)}, \ldots, W^{(d)})$, the dependence parameters κ^+, κ^- for the jump parts, and the constant interest rate r.

(1) Generate a d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}\left(Z^{(1)}, \ldots, Z^{(d)}\right)$ by performing Algorithm 3.20.

(2) Generate a d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}\left(Z^{(-1)}, \ldots, Z^{(-d)}\right)$ by performing Algorithm 3.20.

(3) Define a partition $0 = t_0 < t_1 < \cdots < t_M = t^*$ on $[0, t^*]$.

(4) For each $1 \le i \le M$ do

    (a) Draw a d-dimensional normally distributed vector $B = \left(B^{(1)}, \ldots, B^{(d)}\right)$ with mean 0 and covariance matrix given by Σ.

    (b) For each $1 \le j \le d$ do

$$X_{t_i}^{(j)} = X_{t_{i-1}}^{(j)} + \left(r - \frac{c_j}{\eta_j - 1} + \frac{c_{-j}}{\eta_{-j} - 1} - \frac{\sigma_j^2}{2}\right)(t_i - t_{i-1})$$
$$+ \sigma_j \sqrt{t_i - t_{i-1}}\, B^{(j)} + Z_{t_i}^{(j)} - Z_{t_{i-1}}^{(j)} - Z_{t_i}^{(-j)} + Z_{t_{i-1}}^{(-j)}$$

(5) Return $S_0^{(j)} \exp(X_{t_i}^{(j)})$ for all $1 \le j \le d$, $0 \le i \le M$.

Like for the multivariate BNS-model, we present an efficient and unbiased Algorithm, which simulates the final value of d asset price processes at a fixed time t^*.

Algorithm 5.12 (Final asset values in the multivariate Kou model)
Suppose the same parameters to be given as in Algorithm 5.11.

(1) Perform Algorithm 3.20 and get $\left(Z^{(1)}, \ldots, Z^{(d)}\right)$.

(2) Perform Algorithm 3.20 and get $\left(Z^{(-1)}, \ldots, Z^{(-d)}\right)$.

(3) For each $1 \le i \le d$ draw a random variable

$$R_i \sim \mathcal{N}\left(t^*\left(r - \frac{\sigma_i^2}{2} - \frac{c_i}{\eta_i - 1} + \frac{c_{-i}}{\eta_{-i} + 1}\right), t^* \sigma_i^2\right).$$

    The correlations of $R_1, \ldots, R_d$ are given by the correlation matrix Σ.

(4) Return for each $1 \le i \le n$: $S_0^{(i)} \exp(X_{t^*}^{(i)}) = S_0^{(i)} \exp(R_i + Z_{t^*}^{(i)} - Z_{t^*}^{(-i)})$.

5.4 A multivariate two-sided BNS-Γ-OU model

Now, we mimic the approach to generate multivariate models from the last two sections and apply it to the two-sided BNS-Γ-OU model which combines the BNS-Γ-OU model and the Kou model.

Definition 5.13 (Multivariate two-sided Γ-OU-BNS model)

A d-dimensional stochastic process $\{S_t\}_{t \geq 0}$ with $S_t = (S_t^{(1)}, \ldots, S_t^{(d)})$ follows a *multivariate two-sided Γ-OU-BNS model*, if the dynamics of the log-price vector $X_t = (X_t^{(1)}, \ldots, X_t^{(d)}) = (\log S_t^{(1)}, \ldots, \log S_t^{(d)})$ are governed by the following SDEs:

$$\mathrm{d}X_t^{(j)} = \left(\mu_j + \beta_j \left(\sigma_t^{(j)} \right)^2 \right) \mathrm{d}t + \sigma_t^{(j)} \, \mathrm{d}W_t^{(j)} + \rho_+^{(j)} \, \mathrm{d}Z_t^{+(j)} + \rho_-^{(j)} \, \mathrm{d}Z_t^{-(j)},$$

$$\mathrm{d} \left(\sigma_t^{(j)} \right)^2 = -\lambda_j \left(\sigma_t^{(j)} \right)^2 \mathrm{d}t + \mathrm{d}Z_t^{+(j)} + \mathrm{d}Z_t^{-(j)},$$

with $(W^{(1)}, \ldots, W^{(d)})$ being correlated Brownian motions with correlation matrix Σ and for all $1 \leq j \leq d$, $\mu_j, \beta_j \in \mathbb{R}$, $\rho_+^{(j)} > 0$, $\rho_-^{(j)} < 0$, $\lambda_j > 0$, and $(Z^{+(1)}, Z^{-(1)}), \ldots,$ $(Z^{+(d)}, Z^{-(d)})$ are pairs of $\mathrm{CPP}_{\mathrm{Exp}}$. Furthermore, the $2d$-dimensional Lévy process $(Z^{+(1)}, Z^{-(1)}, \ldots, Z^{+(d)}, Z^{-(d)})$ splits up in two d-dimensional time-changed $\mathrm{CPP}_{\mathrm{Exp}}$.

At first glance, Definition 5.13 looks cumbersome, but it is necessary to capture all combinations of possible dependence. As a simplifying example, one might think about introducing dependence between $(Z^{+(1)}, \ldots, Z^{+(d)})$ on the one hand and between $(Z^{-(1)}, \ldots, Z^{-(d)})$ on the other hand, which coincides with the dependence structure between positive and negative jumps of the multivariate *Kou* model. In this case, positive jumps of the processes are mutually dependent and negative jumps are mutually dependent, but positive jumps occur independently of negative jumps. A closer examination of how to establish the dependence structure between the time-change-dependent compound Poisson processes is made in the following section, since dependence between the jumps has to be introduced in a sound economic manner. In general, positive jumps in one process could occur simultaneously with negative jumps of another process. The multivariate *Kou* model could easily be extended to also support such a generic structure of joint jumps by adopting the notion of the jump components of the two-sided Γ-OU-BNS model. Like in the multivariate Γ-OU-BNS model, in general, the joint characteristic function cannot be expressed in a closed form solution, only in case of uncorrelated Brownian motions. We therefore omit the calculation of the joint characteristic function and conclude this chapter with the following section on some calibration exercises.

5.5 Application: Calibration exercises

A calibration of the presented multivariate models can be carried out in two separate steps. Due to the fact that the marginal distributions can be separated from the dependence structure within our models, it is possible to keep the parameters governing the dependence separated from the parameters governing the marginal distributions. Therefore, in a first step we calibrate independently each univariate model and in a second step the dependence parameters can be set without altering the already fixed marginal distributions. Since there is little market data of multi-asset options, this two step method is very appealing: we can disintegrate one big calibration problem in two smaller ones. The univariate models can be calibrated to prices of plain vanilla options, which can easily be carried out.

In a first example, we consider a two-dimensional model of two equity indexes, the *DAX* and the *ESTX 50 Net Return*. As described above, we employ the two-step calibration approach. Thus, we first fit the univariate models of Kou and BNS type to market quotes. We use market data of European call and put options on the indexes. All market quotes are closing prices of March 30, 2012. Actually, implied volatilities of bid and ask prices of put and call options are given[2]. The expiry dates of these options range from one month to one year. For each maturity, we consider various strikes. Option prices with a wide bid ask spread are withdrawn. If there is a put option and a call option with the same strike and maturity, we use the option having a smaller bid ask spread, which is usually the out-of-the-money option.

After thinning out the implied volatility quotes[3], we calibrate the univariate models to the mid implied volatilities via minimizing the absolute distance of the model implied volatilities to market implied volatilities, with equal weights on every option. Here, option prices in the univariate Kou models and in the univariate BNS models are obtained via Fourier inversion by means of the analytic expression for the characteristic function of the log prices (see Theorem 4.2 and Equation (5.1)).

The calibration of the parameters governing the dependence could be done in a quite similar way. We calibrate the multivariate model with already fixed univariate parameters to market quotes of European multi-asset options, e.g. best-of-two options. Again, prices in the multivariate Kou model can be obtained via Fourier methods. Here we

---

[2]Initially, implied volatilities of the *ESTX 50* (price index) are given. Therefore we transform the strike prices to the *ESTX 50 Net Return* (performance index) and assume the implied volatilities of these indexes to be equal for this calibration exercise.

[3]The cleansing results in 187 mid implied volatilities for ESTX, 328 for DAX.

	S_0	σ_0	c	η	λ	ρ
DAX	6946.8	0.16	1.2426	7.0068	2.8025	-0.5398
ESTX	4210.0	0.1755	0.6506	4.2776	1.7224	-0.4620

Table 5.1 Calibrated parameters in the univariate BNS models.

have to use a multi-dimensional extension of the one-dimensional Fourier method (cf. Eberlein et al. [2010]). Prices in the multivariate BNS model have to be computed via Monte Carlo simulation, because the joint characteristic function of the log prices cannot be expressed in an analytic and fast-to-calculate form. In the bivariate case one has to calibrate only two dependence parameters: the correlation of the two Brownian motions and the parameter κ driving the dependence of the jump parts. Unfortunately, we have not enough market data of multi-asset options to get sensible calibration results. Time series analysis of historical index series may put things right here. This is what practitioners usually do in those situations. Note, that from a theoretical point of view, this yields to a mismatch between the historical measure and the risk-neutral measure. However, due to the lag of data, one could calibrate the marginal models to option data and determine the dependence parameter through a time series analysis. Then, one assumes, that the dependence structure would not change much by a measure change from the historical pricing measure to the risk-neutral measure. In this thesis, we will not elaborate on such historical calibrations. Table 5.1 presents the calibrated parameters of the univariate BNS model and Figure 5.1 shows simulated paths of the bivariate model using the calibrated univariate parameters and fixed dependence parameters.

Analogously, we fit the Kou model to the same set of plain vanilla option quotes. The resulting univariate parameters are presented in Table 5.2. We observe that the intensity for positive jumps is 0, i.e. the option data implies zero probability of upwards jumps. This is not completely surprising in stock price dynamics, which are calibrated to option prices, see for example the empirical studies by Bakshi et al. [1997] and Eraker [2004]). Usually, option prices in the market can be well explained without positive jumps in the model, which underpins once more the tractability of the BNS model. Figure 5.2 shows simulated paths of the bivariate Kou model with calibrated univariate parameters and fixed dependence parameters. In contrast to the BNS model, the volatility stays at the same level. That missing flexibility of volatility clustering
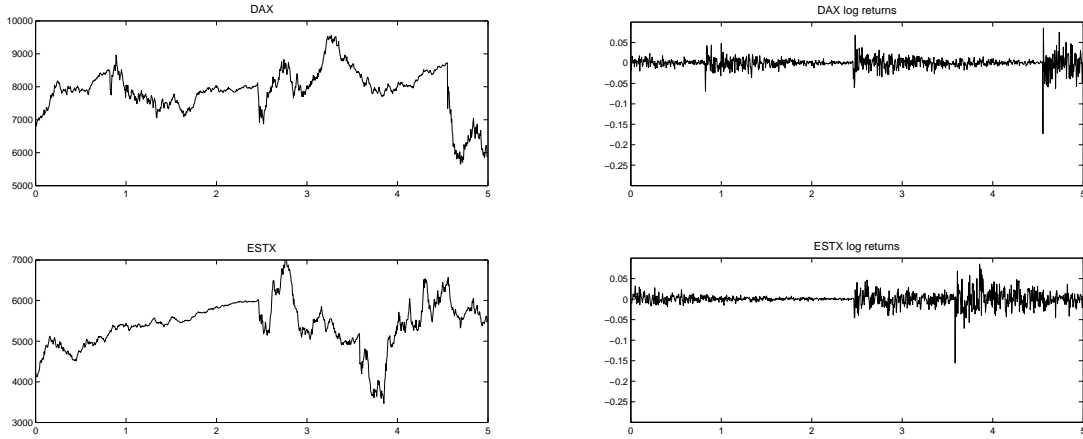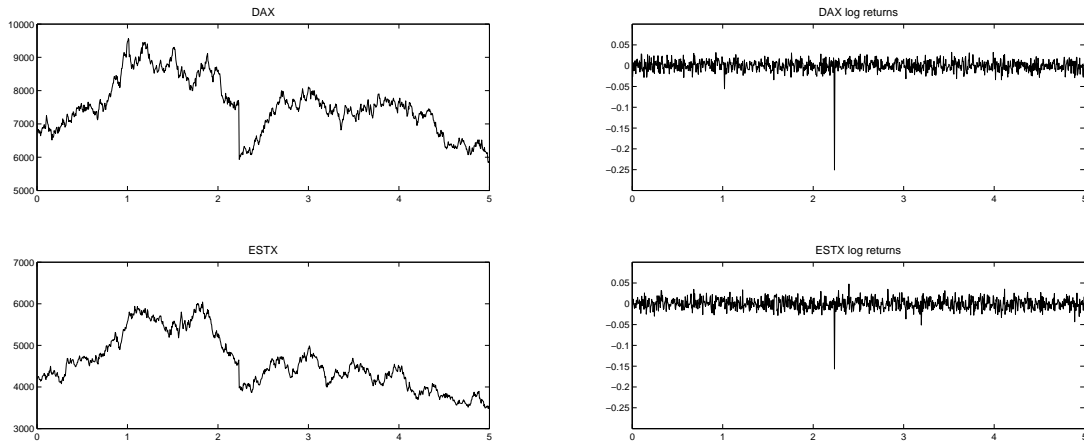
Figure 5.1 The left graphs show simulated paths of the DAX and ESTX with calibrated parameters in the multivariate BNS model. The right graphs show the corresponding daily log returns. We observe one joint jump within this time interval, as well as some individual jumps. Here, the correlation of the Brownian motions is set to 0.5 and $\kappa = 0.7$.

	S_0	σ	c^-	c^+	η^-
DAX	6946.8	0.1673	0.2729	0	3.8953
ESTX	4210.0	0.1816	0.1641	0	2.8379

Table 5.2 Calibrated parameters in the univariate Kou models.

might be the reason for the extreme jump magnitudes in the Kou model.

In the absence of positive jumps in the calibrated bivariate Kou model ($c^+ = 0$), dependence is driven by only two parameters, like in the multivariate BNS model, namely the correlation between the Brownian motions and the dependence parameter for the negative jumps. The calibrated models can then be used to price multi-asset options. As an example, we consider an option with the following payoff at maturity, which is one year:

$$\max\left\{K - \max\left\{e^{X_1^{DAX}}, e^{X_1^{ESTX}}\right\}, 0\right\},$$

i.e. we consider a put option with strike $K > 0$ on the maximum of the two normalized

Figure 5.2 The left graphs show simulated paths of the DAX and ESTX with cali-
brated parameters in the multivariate Kou model with a five years time
horizon. The right graphs show the corresponding daily log returns. We
observe one joint jump within this time interval. Here, the correlation of
the Brownian motions is set to 0.5 and $\kappa = 0.7$.

indexes. This option can be used as an insurance against a global market crash, because
one gets a payoff if the relative performance of both indexes is smaller than K. Here,
X^{DAX} and X^{ESTX} represent the log price processes and we set $K = 0.9$. Figure 5.3
shows model prices of this put option as a function of the dependence parameters.

The valuation of the multi-asset option is done via Monte Carlo pricing. Note that
in the multi-dimensional Kou model it is also possible to use fast Fourier pricing
techniques, because the joint characteristic function of the log-price processes is given
as closed form expression, which was shown in Theorem 5.10. The pricing in the
multi-dimensional BNS model relies on the Monte Carlo method, unless there is no
correlation between the Brownian components, cf. Theorem 5.5.

So far, we just applied two of the three multi-dimensional models we discussed in
this chapter. The remaining one, the two-sided Γ-OU-model, combines the stochastic
volatility of the BNS model and the two-sided jumps of the Kou model. Using the same
market quotes of plain vanilla option prices to fit a bivariate two-sided Γ-OU-model
results in the same model dynamics like in the bivariate (one-sided) Γ-OU-model. That
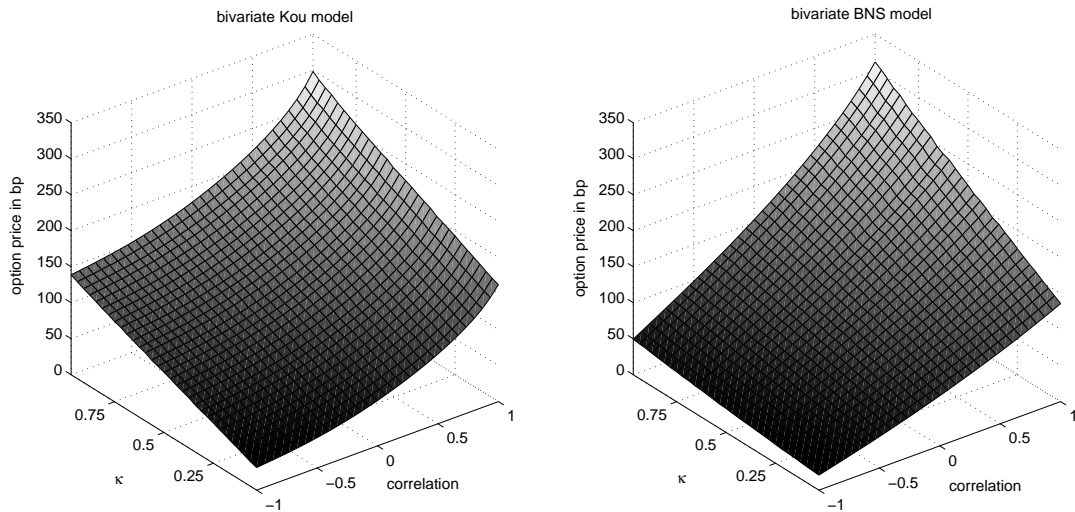
Figure 5.3 The left graph shows option prices in the bivariate Kou model, the right graph shows option prices in the bivariate BNS model. The prices are given in basis points as a function of the two dependence parameters: the correlation of the Brownian motions and the parameter κ for the jump dependence.

is no surprise, since we have seen that the market data does not imply any positive jumps in the Kou model and it is the same in the two-sided Γ-OU-model. In the absence of positive jumps the two-sided model just boils down to a one-sided model. As pointed out above, this is quite common for equity modeling. When modeling other price processes, different from equities, modeling both, negative and positive jumps may be inevitable. In the remainder of this chapter, we therefore present a bivariate exchange rate model based on the two-sided Γ-OU-model presented in the previous section. The symmetry in exchange rates naturally calls for two-sided jumps. By symmetry, we mean the fact that the reciprocal of an exchange rate is again an exchange rate. Jumps in FX rates are mainly driven by unanticipated macroeconomic events (e.g. interest-rate decisions of some central bank) in one of the monetary areas. If we consider a multivariate model with one common currency, e.g. modeling the EUR-USD and the EUR-CHF exchange rates, it is likely that jumps caused by macroeconomic events in the common currency monetary area have an impact on all exchange rates, e.g. the debt crisis of Eurozone countries should affect both the EUR-USD as well as the EUR-

CHF exchange rate. Hence, dependence of the jump processes seem to be a desirable feature of a multivariate model for FX rates with common currency.

To ensure familiarity with the FX markets wording, we recall that an FX rate is the exchange rate between two currencies, expressed as a fraction. The currency in the numerator of the fraction is called (by definition) *domestic currency*, while the currency in the denominator of the fraction is called *foreign currency*.[4] The role each currency plays in an FX rate is defined by market conventions and is often due to historic reasons, so economic interpretations are not necessarily helpful. A more detailed discussion of market conventions of FX rates and derivatives is provided in Reiswich and Wystup [2012], a standard textbook on FX rates modeling is Lipton [2001]. Since we want to model dependence between the jumps in different FX rates, we have to choose the coupling of the compound Poisson drivers carefully and in a way to capture economic intuition. We concentrate on the case of two currency pairs, which illustrates the problems of choosing the jump dependence structure best. When modeling two FX rates, we may want to establish an adequate kind of dependence between the different drivers, accounting separately for positive and negative jumps in the respective FX rate. Depending on which currency is foreign or domestic in the two currency pairs of the FX rates, dependence may be introduced in a different manner to result in sound economic situations. Hence, we can distinguish between the following combinations that may occur for two different FX rates:

1. There are no common currencies, e.g. in the case of EUR-CHF and USD-JPY.

2. In both FX rates the common currency is the foreign currency, e.g. EUR-USD and EUR-CHF.

3. In both FX rates the common currency is the domestic currency, e.g. EUR-CHF and USD-CHF.

4. The common currency is the domestic currency in one FX rate and the foreign currency in the other FX rate, e.g. EUR-USD and USD-CHF.

In each of those situations, one may argue for different kind of dependence structures regarding the compound Poisson drivers. Our suggestions are the following:

---

[4] The wording "foreign" and "domestic" currency does not necessarily reflect whether the currency is foreign or domestic from the point of view of a market participant. The currency EUR, e.g., is *always* foreign currency by market convention. Sometimes, the foreign currency is called *underlying currency*, while the domestic currency is called *accounting* or *base currency*.

1. In case of no common currencies, one may doubt to introduce dependence between jumps due to the missing strong link coming from macroeconomic events. In case that there are no strong ties between the monetary areas (e.g. neither intensive trade relations nor political involvement), this independence assumption may be a realistic choice. In case of strong economic ties like, e.g., between the Eurozone (EUR) and the Czech Republic (CZK), one may employ dependence in the jumps nevertheless, treating the two different currencies similarly as in the common currency case.

2. In case of a common foreign currency, a sudden macroeconomic event strengthening (resp. weakening) the common currency should result in an upward (resp. downward) jump of both FX rates. Hence, it may be a sensible choice to couple the drivers for the positive jumps and to separately couple the drivers for the negative jumps respectively, to ensure the occurrence of joint upward and downward jumps.

3. In case of a common domestic currency, a sudden macroeconomic event strengthening (resp. weakening) the common currency should result in a downward (resp. upward) jump of both FX rates. Hence, coupling (as in 2.) the drivers for positive jumps and separately coupling the drivers for negative jumps yields occurring joint upward and downward jumps.

4. In the last case, a sudden macroeconomic event strengthening (resp. weakening) the common currency should result in an upward (resp. downward) jump of the FX rate where the common currency is the foreign currency, but in a downward (resp. upward) jump of the FX rate where the common currency is the domestic currency. Hence, cross-coupling the driver for positive jumps in the one FX rate with the driver for negative jumps in the other FX rate (and vice versa, respectively) ensures upward jumps in one FX rate being occasionally accompanied by downward jumps in the other FX rate and vice versa.

Summarizing, assuming the currencies CCY1,...,CCY4 to be pairwise different and denoting by $Z^{+(j)}$ (resp. $Z^{-(j)}$) the Lévy driver for the positive (resp. negative) jumps of the jth FX rate, $j = 1, 2$, we suggest to introduce dependence for the jumps along Table 5.3.

Foreign 1	Domestic 1	Foreign 2	Domestic 2	$\tilde{Z}$	$\hat{Z}$
CCY1	CCY2	CCY3	CCY4	no dependence	no dependence
CCY1	CCY2	CCY1	CCY3	$\{Z^{+(1)}, Z^{+(2)}\}$	$\{Z^{-(1)}, Z^{-(2)}\}$
CCY2	CCY1	CCY3	CCY1	$\{Z^{+(1)}, Z^{+(2)}\}$	$\{Z^{-(1)}, Z^{-(2)}\}$
CCY1	CCY2	CCY3	CCY1	$\{Z^{+(1)}, Z^{-(2)}\}$	$\{Z^{-(1)}, Z^{+(2)}\}$

Table 5.3 Suggested dependence structures for different currency combinations.

When two FX rates are modeled and among the two rates there is a common currency, this bivariate model always implicitly defines a model for the missing currency pair which is not modeled directly, e.g. when modeling EUR-USD and EUR-CHF exchange rates simultaneously, the quotient process automatically implies a model for the USD-CHF exchange rate. Similar to the bivariate Garman–Kohlhagen model (cf. Garman and Kohlhagen [1983]), modeling two FX rates directly by a bivariate two-sided BNS model does not necessarily imply a model for the quotient or product process from the same family, but the main structure of a jump-diffusion-type model is maintained. Namely, given two asset-price processes $\{S_t^{(1)}\}_{t\geq 0}$ and $\{S_t^{(2)}\}_{t\geq 0}$ modeled by multivariate two-sided Γ-OU-BNS models, the product and quotient processes $\{S_t^{(1)} S_t^{(2)}\}_{t\geq 0}$ resp. $\{S_t^{(1)}/S_t^{(2)}\}_{t\geq 0}$ are both of jump-diffusion type, which follows directly from $\log(S_t^{(1)} S_t^{(2)}) = X_t^{(1)} + X_t^{(2)}$ and $\log(S_t^{(1)}/S_t^{(2)}) = X_t^{(1)} - X_t^{(2)}$.

The implied model for the third missing FX rate can be used to calibrate the parameters steering the dependence, namely, the correlation between the Brownian motions as well as the jump dependence parameters. Additionally, the calibration performance of the implied model to plain vanilla options yields a plausibility check whether the bivariate model may be useful for the evaluation of true bivariate options, e.g. best-of-two options or spread options.

In the following, we give an example of modeling the FX spot process by the two-sided Γ-OU-BNS dynamics. Alternative approaches model FX forward rates to get a model setup suited for pricing cross-currency derivatives depending on FX forward rates, as for example cross-currency swaps. Multicurrency models built upon FX forward rates (see e.g. Eberlein and Koval [2006]) on the one hand support flexibility to price such derivatives, on the other hand, however, these models do not provide the crucial property of separating the dependence structure from the univariate models, which makes it extremely difficult to calibrate such a multivariate model in a sound manner.

As input data for the second calibration exercise we use option data on exchange rates concerning the three currencies EUR, USD, and SEK. Since the EUR-USD exchange rate can be regarded as an implied exchange rate, i.e.

$$\frac{\text{USD}}{\text{EUR}} = \frac{\frac{\text{SEK}}{\text{EUR}}}{\frac{\text{SEK}}{\text{USD}}},$$

we model the two exchange rates EUR-SEK and USD-SEK directly with two-sided Γ-OU-BNS models. For each currency pair EUR-SEK, USD-SEK, and EUR-USD, we have implied volatilities of 204 different plain vanilla options (different maturities, different moneyness). The option data is as of August 13, 2012, and was provided by Thomson Reuters.

We consider a market with two traded assets, namely $\{\exp(r_{\text{USD}}\,t)\,S_t^{\text{USDSEK}}\}_{t\geq 0}$ and $\{\exp(r_{\text{EUR}}\,t)\,S_t^{\text{EURSEK}}\}_{t\geq 0}$, where S_t^{USDSEK}, S_t^{EURSEK} denote the exchange rates at time t and r_{USD}, r_{EUR}, r_{SEK} denote the risk free interest rates in the corresponding monetary areas. These assets can be seen as the future value of a unit of the respective foreign currency (in this case USD or EUR), valued in the domestic currency (which is SEK). Assume a risk-neutral measure $\mathbb{Q}^{\text{SEK}}$ to be given with numéraire process $\{\exp(r_{\text{SEK}}\,t)\}_{t\geq 0}$, i.e.

$$\{\exp((r_{\text{USD}} - r_{\text{SEK}})\,t)\,S_t^{\text{USDSEK}}\}_{t\geq 0} \quad \text{and} \quad \{\exp((r_{\text{EUR}} - r_{\text{SEK}})\,t)\,S_t^{\text{EURSEK}}\}_{t\geq 0}$$

are martingales with respect to $\mathbb{Q}^{\text{SEK}}$, governed by the SDEs

$$\begin{aligned}
\mathrm{d}X_t^{\star\,\text{SEK}} &= \left(r_{\text{SEK}} - r_\star - \frac{\left(\sigma_t^{\star\,\text{SEK}}\right)^2}{2} - \frac{c_{\star\,\text{SEK}}^+\,\rho_{\star\,\text{SEK}}^+}{\eta_{\star\,\text{SEK}}^+ - \rho_{\star\,\text{SEK}}^+} + \frac{c_{\star\,\text{SEK}}^-\,\rho_{\star\,\text{SEK}}^-}{\eta_{\star\,\text{SEK}}^- + \rho_{\star\,\text{SEK}}^-} \right)\mathrm{d}t \\
&\quad + \sigma_t^{\star\,\text{SEK}}\,\mathrm{d}W_t^{\star\,\text{SEK}} + \rho_{\star\,\text{SEK}}^+\,\mathrm{d}Z_t^{+\star\,\text{SEK}} - \rho_{\star\,\text{SEK}}^-\,\mathrm{d}Z_t^{-\star\,\text{SEK}}, \\
\mathrm{d}\sigma_t^{2\star\,\text{SEK}} &= -\lambda_{\star\,\text{SEK}}\,\sigma_t^{2\star\,\text{SEK}}\,\mathrm{d}t + \mathrm{d}Z_t^{+\star\,\text{SEK}} + \mathrm{d}Z_t^{-\star\,\text{SEK}},
\end{aligned}$$

for $\lambda_{\star\,\text{SEK}}, \rho_{\star\,\text{SEK}}^+, \rho_{\star\,\text{SEK}}^- > 0$, $\star \in \{\text{EUR}, \text{USD}\}$, $\{W_t^{\text{EURSEK}}, W_t^{\text{USDSEK}}\}_{t\geq 0}$ being a two-dimensional Brownian motion with correlation $r \in [-1,1]$, and $\{Z_t^{+\,\text{EURSEK}}, Z_t^{+\,\text{USDSEK}}\}_{t\geq 0}$ and $\{Z_t^{-\,\text{EURSEK}}, Z_t^{-\,\text{USDSEK}}\}_{t\geq 0}$ being (independent) two-dimensional time-changed CPP_{Exp} with parameters

$$\left(c_{\text{EURSEK}}^+, c_{\text{USDSEK}}^+, \eta_{\text{EURSEK}}^+, \eta_{\text{USDSEK}}^+, \kappa^+\right)$$
$$\text{and} \quad \left(c_{\text{EURSEK}}^-, c_{\text{USDSEK}}^-, \eta_{\text{EURSEK}}^-, \eta_{\text{USDSEK}}^-, \kappa^-\right),$$

where κ^+ and κ^- are the jump dependence parameters. Hence, the EUR-SEK, EUR-USD exchange rates follow a bivariate SBNS model. The implied exchange rate process S^{EURUSD} is given by

$$\left\{ S_t^{\mathrm{EURUSD}} \right\}_{t \geq 0} = \left\{ \frac{S_t^{\mathrm{EURSEK}}}{S_t^{\mathrm{USDSEK}}} \right\}_{t \geq 0}.$$

Due to the change-of-numéraire formula for exchange rates (cf. Pelsser [2003]), the process $\{\exp((r_{\mathrm{EUR}} - r_{\mathrm{USD}})\, t)\, S_t^{\mathrm{EURUSD}}\}_{t \geq 0}$ is a martingale with respect to $\mathbb{Q}^{\mathrm{USD}}$, where $\mathbb{Q}^{\mathrm{USD}}$ is determined by the Radon–Nikodým derivative

$$\left. \frac{\mathrm{d}\, \mathbb{Q}^{\mathrm{USD}}}{\mathrm{d}\, \mathbb{Q}^{\mathrm{SEK}}} \right|_t = \frac{S_t^{\mathrm{USDSEK}} \exp(r_{\mathrm{USD}}\, t)}{S_0^{\mathrm{USDSEK}} \exp(r_{\mathrm{SEK}}\, t)}.$$

For calibration purposes, we use the volatility surfaces of the EUR-SEK and USD-SEK exchange rates to fit the univariate parameters. Due to the consistency relationships which have to hold between the exchange rates, we can calibrate the dependence parameters by fitting them to the volatility surface of EUR-USD. Even in presence of other "bivariate options" (e.g. best-of-two options), we argue that European options on the quotient exchange rate currently provide the most liquid and reliable data for a calibration.

The calibration of the presented multivariate model is done in two steps. Again, due to the fact that the marginal distributions can be separated from the dependence structure within our models, it is possible to keep the parameters governing the dependence separated from the parameters governing the marginal distributions. Therefore, in a first step we independently calibrate both univariate models for the EUR-SEK and USD-SEK exchange rates. In a second step we calibrate the parameters driving the dependence structure. In doing so, the fixed univariate parameters are not affected by the second step. Since there is little market data of multi-currency options, this two step method is very appealing: we can disintegrate one big calibration problem in two smaller ones. The univariate models are calibrated to volatility surfaces of the EUR-SEK and USD-SEK exchange rates via minimizing the relative distance of the model implied option prices to market prices, with equal weights on every option. Option prices in the univariate two-sided BNS models are obtained via Fourier inversion (cf. Carr and Madan [1999], Raible [2000]) by means of the characteristic function of the log-prices.

Table 5.4 gives an overview of the calibration results of the two univariate models.

$\star$	$S_0^{\star\,\mathrm{SEK}}$	$\sigma_0^{\star\,\mathrm{SEK}}$	$c^{\star\,\mathrm{SEK}}$	$\eta^{\star\,\mathrm{SEK}}$	$\lambda^{\star\,\mathrm{SEK}}$	$\rho^{\star\,\mathrm{SEK}}$	#options	error
EUR	8.229	0.074	0.71	62.13	3.25	1.66	204	1.08%
USD	6.664	0.078	1.15	40.81	2.19	1.22	204	3.17%

Table 5.4 Calibrated parameters in the two univariate FX models.

To reduce the number of parameters, we use symmetric two-sided Γ-OU-BNS models, i.e. jump parameters of the positive and the negative part coincide. That means, in average the number of upwards jumps equals the number of downwards jumps and the expected absolute jump magnitudes are the same. Furthermore, we assume that the time-change correlation parameters κ^+ and κ^- coincide; maintaining the symmetric structure. The relative error in model prices with respect to market prices of the 204 options can be seen as calibration error. The average relative error in the EUR-SEK-model is about one percent, and in the USD-SEK-model it is around three percent. Hence, the univariate models fit the FX market reasonably well.
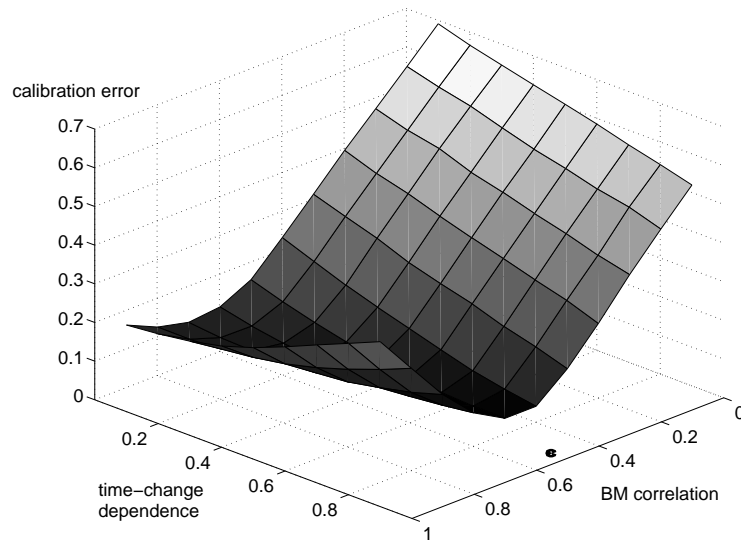
Figure 5.4 The best matching correlation between the two Brownian motions is 0.52 and the optimal time-change dependence parameter is $\kappa = 0.96$.

The calibration of the parameters governing the dependence is done by means of the third implied exchange rate, namely by the volatility surface of EUR-USD. Model prices of EUR-USD-options with payout function f at time t can be obtained by a Monte Carlo simulation of the following expected value:

$$\mathbb{E}_{\mathbb{Q}^{\mathrm{USD}}}\left[f(S_t^{\mathrm{EURUSD}})\exp\left(-r^{\mathrm{USD}}\,t\right)\right] =$$
$$\mathbb{E}_{\mathbb{Q}^{\mathrm{SEK}}}\left[f\left(\frac{S_t^{\mathrm{EURSEK}}}{S_t^{\mathrm{USDSEK}}}\right)\frac{S_t^{\mathrm{USDSEK}}}{S_0^{\mathrm{USDSEK}}}\exp(-r_{\mathrm{SEK}}\,t)\right] \qquad (5.16)$$

Here, we used $100\,000$ simulations to calibrate the dependence parameters. The calibration error of the dependence parameters in terms of average relative error is roughly nine percent, which is still a good result giving consideration to the fact that we try to fit 204 market prices by means of just two parameters in an implicitly specified model. A more complex model, obtained by relaxing the condition that κ^+ and κ^- coincide, leads to even smaller calibration errors. However, we keep the model as simple as possible to maintain tractability. Figure 5.4 illustrates the calibration error of this second step depending on different choices of the dependence parameters. Eventually, the whole model is fixed and could now be used for pricing multi-currency options, for instance a best-of-two call option or spread options.

6 Extremal wrong way risk

In this chapter, we investigate the influence of stochastic dependencies in calculating adjusted derivative prices, recognizing the possibility of default of a contractual party. As motivated in the introductory Chapter 1, we present a model-free approach to compute the bilateral credit valuation adjustment (BCVA) including wrong way risk (WWR) and we derive bounds for extremal cases of wrong way risk, respectively right way risk. This study has already been published in Scherer and Schulz [2016]. In this chapter's fundamental section (Section 6.1) we define the notion of BCVA and WWR and give a broad literature overview of WWR models. In Section 6.2 we present the model-free ansatz, followed by the optimization of the BCVA in Section 6.3. Section 6.4 concludes with a numerical case study. We calculate the model-free BCVA bounds for different portfolio situations and compare our results to the popular Hull–White model (cf. Hull and White [2012]) and to the BCVA optimization results obtained by Helmers et al. [2016].

6.1 Fundamentals: Credit valuation adjustments and wrong way risk

Recognizing counterparty default risk as integral part of the valuation process of financial derivatives has changed the classical view on option pricing. We will first define in rigorous terms, what we mean mathematically when we speak of CVA, DVA, BCVA, and WWR. Pragmatically spoken, the CVA is the expected positive portfolio value in case the counterparty to a derivative transaction defaults first. On the contrary, the DVA is the expected negative portfolio value at someone's own default event, but only if the counterparty is still alive. The BCVA is simply the sum of CVA and DVA. One of the earliest articles proposing price adjustments due to potential default events was published long before the Lehman collapse by Cooper and Mello [1991]. Sorensen and Bollier [1994] were among the first considering a bilateral modification of risk neutral

prices. It is not easy to understand how a realistic dependence structure ultimately should look like, how it is parameterized, and how it is estimated. The BCVA calculation becomes even more difficult if the portfolio value is affected by the credit worthiness of the two parties. In such a case, there is WWR. Classically, one only recognizes WWR, if the dependence structure is positive, i.e. the dependency is in such an extend that high portfolio values imply higher default probabilities of the counterparty. Note that someone's wrong-way risk is the counterparty's right-way risk (negative dependence), and vice versa. Throughout the present thesis, we simply speak of WWR, if the portfolio value and the default times are not stochastically independent. Thus, calculating WWR requires a sound model for the dependence structure between the default times of the two contractual parties and the derivative/portfolio value at the first of the two default times. There exist many proposals, but no market consensus, on how this dependence structure should be modeled to soundly recognize WWR. One of the seminal models taking WWR into account was presented by Duffie and Huang [1996].

In practice, one typically has expertise in using models that explain the marginal laws of the involved default times on the one hand, and stochastic models for the evolution of the underlying to the considered derivative transaction (or portfolios of derivatives) on the other hand. Hence, a variety of WWR models exist generalizing these univariate models. Since the models for the underlying dynamics vary depending on asset classes and type of derivative, it is not surprising that a similar distinction for WWR models exists. Popular WWR models with focus on some specific asset classes are, for example: Interest rate derivatives (Brigo and Pallavicini [2007], Brigo and Pallavicini [2008]), equity derivatives (Brigo and Masetti [2005], Brigo et al. [2011]), products on commodities (Brigo and Bakkar [2009]), or CDS (Brigo and Chourdakis [2008], Leung and Kwok [2005], Hull and While [2001], Walker [2006], Bielecki et al. [2012], Blanchet-Scalliet and Patras [2011], Lipton and Sepp [2009], Brigo et al. [2014]).

In contrast to models dedicated for specific asset classes or bespoke types of derivatives, here are also general WWR models which can be used for arbitrary asset classes. This universality is achieved by introducing the dependence structure in a second step after fixing the marginal models, see, for example, Hull and White [2012], or the so called exposure sampling approaches Sokol [2010], Cespedes et al. [2010], and Rosen and Saunders [2012]. Our methodology follows the very same two-step approach. In most situations market data to estimate/calibrate such univariate models is available, e.g. CDS spreads for the calibration of the default models. With respect to the time

evolution of the underlying processes, or on a higher level the fair portfolio value, it is often an economic scenario generator creating sample paths to be used later on. Much less obvious than specifying the marginal laws, and hence prone to model risk, is the choice of dependence model that is subsequently "pulled upon" the marginal models in use.

Particularly, we aim at maximizing, respectively minimizing, the BCVA across all multivariate distributions whose projections on the marginal laws are consistent with the model of choice for the marginal default times and the portfolio process. The marginal laws are specified in the first place. From a mathematical perspective, we work on a discrete probability space and exploit the machinery of mass-transportation problems. The canonical formulation of this problem is straightforward, see (6.2), but computationally very inefficient. However, it is observed that some information about the dependence structure can be disregarded without changing the optimal solution, which leads to an equivalent reformulation (6.3) of the problem that tremendously reduces the calculation time.

Other studies, like Turnbull [2005] and Cherubini [2013], already investigated upper bounds for valuation adjustments. Yet, the resulting bounds are not necessarily attained by a joint distribution, so they are not tight. Moreover, they only considered the case of unilateral CVA. Closest to our work is Helmers et al. [2016], where the dependence structure between the default times is explained using some fixed copula and it is the dependence to the derivative exposure that is left as a source of model risk. Clearly, this conveniently simplifies the problem from a computational point of view, but it falls short in explaining the full degree of model risk; even though the bivariate Gaussian copula used exemplarily interpolates from counter-monotonicity to co-monotonicity. Glasserman and Yang [2016] use a similar ansatz to derive bounds for the unilateral CVA. Both approaches assume a discrete-time framework and a finite set of portfolio paths coming either from an economic scenario generator or from Monte Carlo samples of a financial market model. There is also active research (see, for example, Puccetti and Rüschendorf [2013], Embrechts et al. [2013]) based on the seminal work by Rüschendorf [1982] and Makarov [1981] in the somewhat related problem of identifying upper and lower bounds for the portfolio value-at-risk (VaR). The extremal portfolio VaR might be derived by the so-called rearrangement algorithm, which optimizes the VaR of a sum of random variables whose marginal laws are fixed, but the dependence structure can freely be chosen.

6.2 Model-free approach

Consistent with a typical IT infrastructure, we assume the time between the valuation date and the considered maturity to be discretized in (not necessarily equidistant) bins. The two considered default times are thus discrete random variables taking values within these bins – their (marginal) probabilities are calibrated to market data (single name CDS, bond spreads, etc.). We assume to work with M trajectories of the (portfolio or individual derivative) value of the considered transaction, produced either by an economic scenario generator or approximating a financial market model by means of a Monte Carlo simulation. Thus, we consider the discrete event space $\Omega = \mathcal{V} \times \mathcal{D} \times \mathcal{D}$ equipped with the power set $\mathcal{P}(\Omega)$ as σ-algebra, where

$$
\begin{aligned}
\mathcal{V} &:= \left\{ v^{(1)}, v^{(2)}, \ldots, v^{(M)} \right\}, \quad v^{(i)} \in \mathbb{R}^K \ \forall i \in \{1, 2, \ldots, M\}, \quad K, M \in \mathbb{N}, \\
\mathcal{D} &:= \{1, 2, \ldots, K+1\}.
\end{aligned}
$$

One state $\omega = (v, d_A, d_B) \in \Omega$ describes a tuple consisting of a trajectory of the portfolio value $v := \{v_j\}_{1 \leq j \leq K}$, a realization of a default time d_A of party A, and a default time d_B of party B. Then, the canonical projections $V : \omega \mapsto v$, $D_A : \omega \mapsto d_A$, $D_B : \omega \mapsto d_B$ represent the three crucial random variables in a BCVA calculation. As each trajectory of the portfolio value represents a result from an economic scenario engine, a discrete-time setup is used. Thus, the set $\{1, 2, \ldots, K\}$ denotes the grid points of the portfolio value simulation. Furthermore, the set $\mathcal{D}$ represents the possible default times of each of the two contractual parties, whereby the state $K+1$ describes a default occurring after the last considered grid point K. The number of grid points K, the simulated time interval, the distribution of the grid points within the simulated time interval, and the number of simulated scenarios M vary depending on the application.[1] We assume that the value path v already incorporates discounting, such that v_j times the respective loss given default ratio can be seen as the discounted value that is lost in case of a default of one party within the time bucket j. Let us denote the loss given default ratios by L^A and L^B. The portfolio value trajectory V and the default times D_A, D_B induce three marginal probability spaces, namely $(\mathcal{V}, \mathcal{P}(\mathcal{V}), \mathbb{P}_V)$, $(\mathcal{D}, \mathcal{P}(\mathcal{D}), \mathbb{P}_A)$, $(\mathcal{D}, \mathcal{P}(\mathcal{D}), \mathbb{P}_B)$. As a standing assumption throughout

---

[1] Typically, banks use 50 to 100 grid points for a CVA calculation. The simulated time horizon is fixed and ranges, depending on the considered application, between one year and 50 years. The location of the grid points is usually non-equidistant, s.t. there are more grid point in the short-term as there are in the long-term. The number of simulations M typically exceeds 1.000.

this work, we assume the marginal distributions $\mathbb{P}_V$, $\mathbb{P}_A$, $\mathbb{P}_B$ to be known, whereas the joint distribution $\mathbb{P}$ on the product space $(\Omega, \mathcal{P}(\Omega))$ is considered unknown.

Remark 6.1 (Valuation of the BCVA)

Define for each $\omega = (v,\, d_A,\, d_B) \in \Omega$ the *state-dependent* BCVA (from the perspective of party A) by

$$
\text{BCVA}_{state}(\omega) := \begin{cases} \max\{0, v_{d_B}\} L^B_{d_B} & \text{if } d_A > d_B, \text{ (loss if party B defaults first)} \\ \min\{0, v_{d_A}\} L^A_{d_A} & \text{if } d_A < d_B, \text{ (loss if party A defaults first)} \\ \xi(v, d_A) & \text{if } d_A = d_B. \text{ (loss in case of a joint default)} \end{cases}
$$
$$(6.1)$$

Note that there are different possible ways on how to deal with the event of a joint default. Therefore, in case of a joint default we express the payoff with the auxiliary function ξ within the above formula to have a flexible setup. Actually, as we are in a discrete-time setup, a joint default in this framework means that both parties default between (the same) two consecutive grid points. In the next section we make clear what this term turns out to be within the prudent best-case/worst-case optimization. Hence, the expected BCVA from the perspective of party A is then given by

$$
\begin{aligned}
\text{BCVA} &= \sum_{\omega \in \Omega} \text{BCVA}_{state}(\omega) \mathbb{P}(\omega) \\
&= \sum_{v \in \mathcal{V}} \sum_{d_A \in \mathcal{D}} \sum_{d_B \in \mathcal{D}} \text{BCVA}_{state}((v,\, d_A,\, d_B)) \mathbb{P}((v,\, d_A,\, d_B)).
\end{aligned}
$$
$$(6.2)$$

As the portfolio value paths result from an economic scenario engine, all paths $v \in \mathcal{V}$ have the identical probability, i.e. $\mathbb{P}_V(v) = 1/M, \forall v \in \mathcal{V}$.[2] Thus, by conditioning on the portfolio value path and using rules for conditional probabilities, Equation (6.2) boils down to

$$
\begin{aligned}
\text{BCVA} &= \frac{1}{M} \sum_{v \in \mathcal{V}} \sum_{(d_A,\, d_B) \in \mathcal{D} \times \mathcal{D}} \text{BCVA}_{state}((v,\, d_A,\, d_B)) \mathbb{P}\left(D_A = d_A, D_B = d_B \,|\, V = v\right) \\
&= \frac{1}{M} \sum_{v \in \mathcal{V}} \sum_{k=1}^{K} \Big(\max\{0, v_k\} \mathbb{P}\left(k = D_B < D_A \,|\, V = v\right) L^B_k \\
&\qquad\qquad + \min\{0, v_k\} \mathbb{P}\left(k = D_A < D_B \,|\, V = v\right) L^A_k
\end{aligned}
$$
$$(6.3)$$

---

[2]If non-identical weights are assigned to the generated paths, this simply corresponds to a different condition (6) in Problem 3.1.

$$+ \xi(v, k)\, \mathbb{P}\left(k = D_A = D_B \,|V = v\right)\Big).$$

In the last step, we plugged in Equation (6.1). The three terms within the big parentheses in Equation (6.3) have a nice interpretation: By summing up the first part over all k and v, we get the loss party A faces in case the counterparty defaults first, the summed up second part represents the gain (negative loss) if party A defaults first, and the third part describes the loss or gain in the case of simultaneous defaults of both parties. Equation (6.3) is the starting point in our optimization procedure in Section 6.3.

This discrete-time ansatz of modeling the random evolution of some underlying, some derived risk-neutral portfolio value, or even a full portfolio containing the netted value of all outstanding positions against the considered counterpart is market practice to overcome the big modeling and implementation challenges financial institutions are imposed to. There is not much academic literature on how such scenarios are ultimately generated. Our considerations, however, are completely generic and thus independent of the choice of a specific scenario engine. Moreover, one can also include the typical math finance view on financial markets by starting out with one's favorite model for the stochastic process describing the underlying of a derivative and drawing a sample of M trajectories from derived value.

6.3 BCVA optimization

In this section, we aim at maximizing, resp. minimizing, the BCVA in Equation (6.3). Using the notation from the previous section, we define $c_{nij} := \mathrm{BCVA}_{state}(v^{(n)}, i, j)$ and $p_{nij} := \mathbb{P}(v^{(n)}, i, j), \forall 1 \leq n \leq M, \forall 1 \leq i, j \leq K + 1$. Note that the values c_{nij} can be precomputed due to Equation (6.1). By Equation (6.2), the BCVA is then given by

$$\mathrm{BCVA} = \sum_{n=1}^{M} \sum_{i=1}^{K+1} \sum_{j=1}^{K+1} c_{nij}\, p_{nij}.$$

In order to optimize this expression over all possible joint distributions $\mathbb{P}$ on Ω preserving the pre-specified marginal laws, one has to find an optimal solution for the atomic probabilities $(p_{nij})_{1 \leq n \leq M, 1 \leq i, j \leq K+1}$ of each state $\omega \in \Omega$. Furthermore, since

the marginal distributions $\mathbb{P}_V$, $\mathbb{P}_A$, and $\mathbb{P}_B$ need to be preserved, one obtains constraints to characterize all feasible distributions. Hence, the following optimization problem maximizes the BCVA.

Optimization problem 6.2 (Maximizing BCVA – brute force)

$$\max \sum_{n=1}^{M} \sum_{i=1}^{K+1} \sum_{j=1}^{K+1} c_{nij}\, p_{nij}$$

subject to the constraints

$$\sum_{n=1}^{M} \sum_{i=1}^{K+1} p_{nij} = \mathbb{P}_B(j), \qquad \forall 1 \leq j \leq K+1, \tag{6.4}$$

$$\sum_{n=1}^{M} \sum_{j=1}^{K+1} p_{nij} = \mathbb{P}_A(i), \qquad \forall 1 \leq i \leq K+1, \tag{6.5}$$

$$\sum_{i=1}^{K+1} \sum_{j=1}^{K+1} p_{nij} = \mathbb{P}_V(n) = \frac{1}{M}, \qquad \forall 1 \leq n \leq M, \tag{6.6}$$

$$0 \leq p_{nij}, \qquad \forall 1 \leq n \leq M, \forall 1 \leq i,j \leq K+1. \tag{6.7}$$

Optimization problem 6.2 is a classic three-dimensional axial transportation problem, which can be solved efficiently, see for example Sharma [1977]. In a framework realistic for the present problem, however, it is almost impossible to solve Optimization problem 6.2 due to the large number of variables. Note that in order to arrive at an optimal solution, one has to find MK^2 atomic probabilities. As mentioned in the previous section, typical values for K and M can easily lead to more than 10 million variables. Due to memory issues and an extremely long computation time, this problem is not tractable anymore.

Helmers et al. [2016] presented within a less general but related model setup a simplification of Optimization problem 6.2 by fixing the dependence structure between the default times of the two parties A and B. Hence, the space of possible joint distributions is reduced by one dimension, corresponding to stricter constraints. On the one hand, this simplifies the optimization problem tremendously. On the other hand, the number of variables is still of order MK^2. Hence, the problem can be solved in less time, as long as no memory issues occur. Nevertheless, the space of possible joint distributions is restricted, such that the BCVA resulting from that simplified problem

is only an optimal solution within a specific subclass of joint probability distributions. In the following we present an optimization problem with order of complexity MK, which maximizes the BCVA over *all* possible joint distributions. The idea is to use the conditional formula in Equation (6.3) instead of Equation (6.2). To do so, we define abbreviations for the ingredients of Equation (6.3). Thus, for all $k \leq K+1$, $n \leq M$,

$$v^+(n,k) := \max\left\{0, v_k^{(n)}\right\} L_k^B, \qquad\qquad v^-(n,k) := \min\left\{0, v_k^{(n)}\right\} L_k^A,$$

$$d_A(n,k) := \mathbb{P}\left(k = D_A \,\middle|\, V = v^{(n)}\right), \qquad d_B(n,k) := \mathbb{P}\left(k = D_B \,\middle|\, V = v^{(n)}\right),$$

$$c_A(n,k) := \mathbb{P}\left(k = D_A < D_B \,\middle|\, V = v^{(n)}\right), \quad c_B(n,k) := \mathbb{P}\left(k = D_B < D_A \,\middle|\, V = v^{(n)}\right),$$

$$c_J(n,k) := \mathbb{P}\left(k = D_A = D_B \,\middle|\, V = v^{(n)}\right).$$

Hence, the BCVA is given by

$$\text{BCVA} = \frac{1}{M}\sum_{n=1}^{M}\sum_{k=1}^{K}\left(\underbrace{v^+(n,k)c_B(n,k)}_{\text{B defaults first, at } k} + \underbrace{v^-(n,k)c_A(n,k)}_{\text{A defaults first, at } k} + \underbrace{\xi(v^{(n)},k)c_J(n,k)}_{\text{A and B default at } k}\right). \tag{6.8}$$
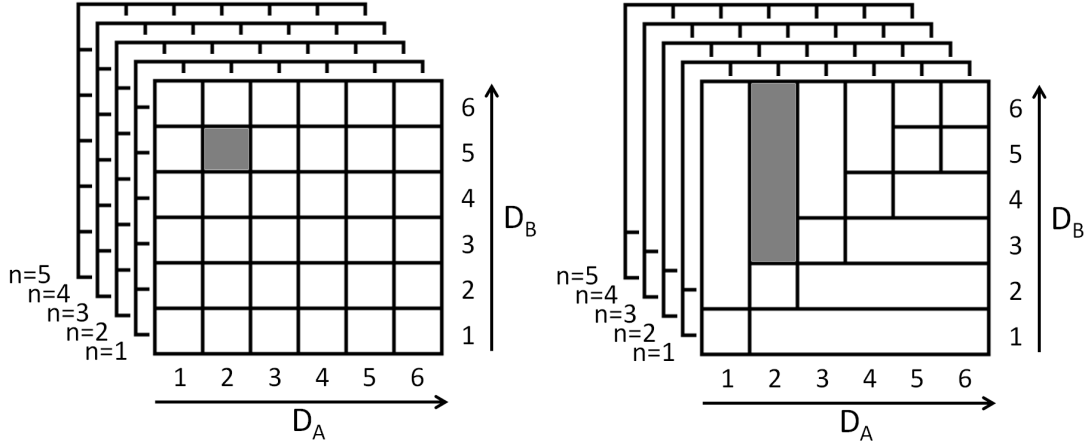

Figure 6.1 Illustration of the variables of the optimization problems for $K = M = 5$. On the left, we visualize Optimization problem 6.2, and on the right, Optimization problem 6.3. For example, the highlighted square on the left corresponds to p_{125}, the highlighted rectangle on the right corresponds to $\sum_{j=3}^{6} p_{12j} = \mathbb{P}(2 = D_A < D_B, V = v^{(1)}) = \mathbb{P}(2 = D_A < D_B | V = v^{(1)})\mathbb{P}(V = v^{(1)}) = c_A(1,2)/5$.

In comparison to Optimization problem 6.2, where MK^2 variables have to be set by the optimizer as illustrated in the left part in Figure 6.1, significantly fewer variables are needed in Equation (6.8). This reduction of complexity is visualized in the right part in Figure 6.1. Hence, Equation (6.8) can be used as objective function for a more advanced optimization to maximize the BCVA, which is done in Optimization problem 6.3. Compared to the naïve optimization problem, the constraints become more evolved. Theorem 6.4 shows that the constraints, as stated in Optimization problem 6.3, are chosen in the right way.

Optimization problem 6.3 (Maximizing BCVA – sophisticated approach)

$$\max \frac{1}{M} \sum_{n=1}^{M} \sum_{k=1}^{K} \left(v^+(n,k)c_B(n,k) + v^-(n,k)c_A(n,k) + \xi(v^{(n)},k)c_J(n,k) \right)$$

subject to the constraints

$$0 \leq c_A(n,k), \quad 0 \leq c_B(n,k), \qquad\qquad \forall 1 \leq n \leq M, \forall 1 \leq k \leq K,$$
$$(6.9)$$

$$0 \leq c_J(n,k), \qquad\qquad \forall 1 \leq n \leq M, \forall 1 \leq k \leq K+1$$
$$(6.10)$$

$$c_A(n,k) + c_J(n,k) \leq d_A(n,k), \qquad\qquad \forall 1 \leq n \leq M, \forall 1 \leq k \leq K,$$
$$(6.11)$$

$$c_B(n,k) + c_J(n,k) \leq d_B(n,k), \qquad\qquad \forall 1 \leq n \leq M, \forall 1 \leq k \leq K,$$
$$(6.12)$$

$$\sum_{k=1}^{K+1} d_A(n,k) = 1, \quad \sum_{k=1}^{K+1} d_B(n,k) = 1, \qquad\qquad \forall 1 \leq n \leq M, \qquad (6.13)$$

$$\sum_{n=1}^{M} d_A(n,k) = M\mathbb{P}_A(k), \quad \sum_{n=1}^{M} d_B(n,k) = M\mathbb{P}_B(k), \quad \forall 1 \leq k \leq K+1, \qquad (6.14)$$

$$\sum_{k=1}^{K} (c_A(n,k) + c_B(n,k)) + \sum_{k=1}^{K+1} c_J(n,k) = 1, \qquad\qquad \forall 1 \leq n \leq M, \qquad (6.15)$$

$$\sum_{i=1}^{k} (d_A(n,i) - c_J(n,i) - c_A(n,i)) \leq \sum_{i=1}^{k-1} c_B(n,i), \qquad \forall 1 \leq n \leq M, \forall 1 \leq k \leq K,$$
$$(6.16)$$

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$$\sum_{i=1}^{k} (d_B(n,i) - c_J(n,i) - c_B(n,i)) \le \sum_{i=1}^{k-1} c_A(n,i), \qquad \forall 1 \le n \le M, \forall 1 \le k \le K.$$

$$(6.17)$$

Theorem 6.4 (The optimal values agree)

A solution of Optimization problem 6.3 maximizes the BCVA. Hence, solving the more tractable Problem 6.3 instead of Problem 6.2 is possible.

Proof

As the objective function coincides with Equation (6.3), there are two steps to prove:

(i) Every joint distribution $\mathbb{P}$ with marginal distributions $\mathbb{P}_V$, $\mathbb{P}_A$, $\mathbb{P}_B$ fulfills the Constraints (6.9)–(6.17).

(ii) For each feasible solution $d_A(n,k)$, $d_B(n,k)$, $c_A(n,k)$, $c_B(n,k)$, $c_J(n,k)$, $\forall 1 \le n \le M, \forall 1 \le k \le K+1$, there is a joint distribution $\mathbb{P} = (p_{nij})_{1 \le n \le M, 1 \le i,j \le K+1}$ with marginal distributions $\mathbb{P}_V$, $\mathbb{P}_A$, $\mathbb{P}_B$.

In order to prove (i), we choose $p_{nij} = \mathbb{P}(v^{(n)}, i, j)$, $\forall 1 \le n \le M, \forall 1 \le i,j \le K+1$ arbitrarily. Then, the variables in Optimization problem 6.3 are given by:

$$d_A(n,k) = M \sum_{i=1}^{K+1} p_{nki}, \qquad\qquad d_B(n,k) = M \sum_{i=1}^{K+1} p_{nik},$$

$$c_A(n,k) = M \sum_{i=k+1}^{K+1} p_{nki}, \qquad\qquad c_B(n,k) = M \sum_{i=k+1}^{K+1} p_{nik},$$

$$c_J(n,k) = M p_{nkk}.$$

Since $p_{nij} \ge 0$, $\forall 1 \le n \le M, \forall 1 \le i,j \le K+1$, Conditions (6.9)–(6.12) are trivial. The Constraints (6.13)–(6.15) are also fulfilled, because $\mathbb{P}$ is the joint distribution with marginals $\mathbb{P}_V$, $\mathbb{P}_A$, $\mathbb{P}_B$, and $\mathbb{P}_V$ is the discrete uniform distribution on $\{1, \ldots, M\}$. To check Condition (6.16), we plug in the definitions and get

$$\sum_{i=1}^{k} (d_A(n,i) - c_J(n,i) - c_A(n,i)) = M \sum_{i=1}^{k} \sum_{j=1}^{i-1} p_{nij} = M \sum_{j=1}^{k-1} \sum_{i=j+1}^{k} p_{nij}$$

$$\le M \sum_{j=1}^{k-1} \sum_{i=j+1}^{K+1} p_{nij} = \sum_{j=1}^{k-1} c_B(n,j).$$

Constraint (6.17) follows analogously.

To prove (ii), let a feasible distribution be given. Me must now construct a joint distribution $p_{nij} = \mathbb{P}(v^{(n)}, i, j), \forall 1 \leq n \leq M, \forall 1 \leq i, j \leq K + 1$, fulfilling the following four conditions (cf. the constraints in Optimization problem 6.2),

(A) $0 \leq p_{nij}, \quad \forall 1 \leq n \leq M, \forall 1 \leq i, j \leq K + 1,$

(B) $\sum_{n=1}^{M} \sum_{i=1}^{K+1} p_{nij} = \mathbb{P}_B(j), \quad \forall 1 \leq j \leq K + 1,$

(C) $\sum_{n=1}^{M} \sum_{j=1}^{K+1} p_{nij} = \mathbb{P}_A(i), \quad \forall 1 \leq i \leq K + 1,$

(D) $\sum_{i=1}^{K+1} \sum_{j=1}^{K+1} p_{nij} = \mathbb{P}_V(n) = \frac{1}{M}, \quad \forall 1 \leq n \leq M.$

We construct the joint distribution $\mathbb{P}$ in a recursive scheme $\forall 1 \leq n \leq M$ by

$$\forall 1 \leq i \leq K + 1, \quad p_{nii} := \frac{c_J(n, i)}{M},$$

$\forall 1 \leq i < j \leq K + 1,$

$$p_{nij} := \frac{\left(\frac{1}{M} c_A(n, i) - \sum_{k=i+1}^{j-1} p_{nik} \right) (d_B(n, j) - c_B(n, j) - c_J(n, j))}{\sum_{k=1}^{j-1} (c_A(n, k) + c_B(n, k) + c_J(n, k) - d_B(n, k))},$$

$\forall 1 \leq j < i \leq K + 1,$

$$p_{nij} := \frac{\left(\frac{1}{M} c_B(n, j) - \sum_{k=j+1}^{i-1} p_{nkj} \right) (d_A(n, i) - c_A(n, i) - c_J(n, i))}{\sum_{k=1}^{i-1} (c_A(n, k) + c_B(n, k) + c_J(n, k) - d_A(n, k))},$$

where, for sake of simplicity, we define $\forall 1 \leq n \leq M, c_A(n, K+1) := 0 =: c_B(n, K+1)$. This recursive construction may seem a little cumbersome on the first sight. The idea is, however, quite simple. The first step is to cut the three dimensions into slices for each $1 \leq n \leq M$, as illustrated in the right part of Figure 6.1. Second, the diagonal entries in each plane are given by c_J. Next, the secondary diagonals are set dependent on the probability mass which is available and not yet fixed in earlier steps. The recursive scheme goes on with the third-rate diagonals, and all the rest of it.

It is left to show that Conditions (A)–(D) are valid for this joint distribution $\mathbb{P}$. Let us consider (A) first. It follows directly from Constraint (6.10) that $0 \leq p_{nii}, \forall 1 \leq n \leq M, \forall 1 \leq i \leq K + 1$. Note, that $\forall 1 \leq n \leq M, \forall 1 \leq j \leq K$ it holds by Constraints (6.17) and (6.12) that

$$\sum_{k=1}^{j-1} (c_A(n, k) + c_B(n, k) + c_J(n, k) - d_B(n, k)) \geq d_B(n, j) - c_B(n, j) - c_J(n, k) \geq 0,$$

and by Constraints (6.13) and (6.15) that

$$\sum_{k=1}^{K} \left(c_A(n,k) + c_B(n,k) + c_J(n,k) - d_B(n,k) \right) = d_B(n, K+1) - c_J(n, K+1).$$

Therefore, we know that $\forall 1 \leq n \leq M$, $\forall 1 \leq i < j \leq K+1$,

$$p_{nij} \leq \frac{1}{M} c_A(n,i) - \sum_{k=i+1}^{j-1} p_{nik}, \tag{6.18}$$

and

$$p_{nij} \geq 0 \Leftrightarrow \frac{1}{M} c_A(n,i) \geq \sum_{k=i+1}^{j-1} p_{nik}. \tag{6.19}$$

Equations (6.18) and (6.19) yield that $p_{nij} \geq 0$, $\forall 1 \leq n \leq M$, $\forall 1 \leq i < j-1 < K+1$. Constraint (6.9) and (6.19) yield that $p_{nij} \geq 0$, $\forall 1 \leq n \leq M$, $\forall 1 \leq i = j-1 < K+1$. Altogether, $p_{nij} \geq 0$, $\forall 1 \leq n \leq M$, $\forall 1 \leq i < j \leq K+1$.

Analogously, by using Constraints (6.11) and (6.16) instead of Constraints (6.12) and (6.17), we get $p_{nij} \geq 0$, $\forall 1 \leq n \leq M$, $\forall 1 \leq j < i \leq K+1$. Hence, Condition (A) is fulfilled.

To check Condition (B), we will use the following auxiliary statements:

(a) $\sum_{i=1}^{j-1} p_{nij} = (d_B(n,j) - c_B(n,j) - c_J(n,j)) \frac{1}{M}$, $\forall 1 \leq n \leq M, \forall 1 < j \leq K+1$,

(b) $\sum_{i=j+1}^{K+1} p_{nij} = c_B(n,j) \frac{1}{M}$, $\forall 1 \leq n \leq M, \forall 1 \leq j < K+1$.

Statement (a) is shown via induction.

Base case ($j=2$): The definition of p_{n12} gives

$$p_{n12} = \frac{\frac{1}{M} c_A(n,1) \left(d_B(n,2) - c_B(n,2) - c_J(n,2) \right)}{c_A(n,1) + c_B(n,1) + c_J(n,1) - d_B(n,1)}.$$

Constraints (6.12) and (6.17) yield that

$$c_B(n,1) + c_J(n,1) - d_B(n,1) = 0. \tag{6.20}$$

Hence, the base case is fulfilled.

Inductive Step $(2, \dots, j-1 \mapsto j)$: By definition of p_{nij}, we get

$$\sum_{i=1}^{j-1} p_{nij} = \sum_{i=1}^{j-1} \frac{\left(\frac{1}{M} c_A(n,i) - \sum_{k=i+1}^{j-1} p_{nik} \right) \left(d_B(n,j) - c_B(n,j) - c_J(n,j) \right)}{\sum_{k=1}^{j-1} \left(c_A(n,k) + c_B(n,k) + c_J(n,k) - d_B(n,k) \right)}. \tag{6.21}$$

Note, that the induction hypothesis yields

$$\sum_{i=1}^{j-1} \sum_{k=i+1}^{j-1} p_{nik} = \sum_{k=1}^{j-1} \sum_{i=1}^{k-1} p_{nik} = \sum_{k=1}^{j-1} \frac{1}{M} \left(d_B(n,k) - c_B(n,k) - c_J(n,k) \right). \qquad (6.22)$$

The combination of Equations (6.21) and (6.22) concludes the inductive step.

To prove Statement (b), we consider the definition of $p_{n(K+1)j}$, namely

$$p_{n(K+1)j} = \frac{\left(\frac{1}{M} c_B(n,j) - \sum_{k=j+1}^{K} p_{nkj} \right) \left(d_A(n,K+1) - c_J(n,K+1) \right)}{\sum_{k=1}^{K} \left(c_A(n,k) + c_B(n,k) + c_J(n,k) - d_A(n,k) \right)}.$$

Combining Constraints (6.13) and (6.15) gives

$$\sum_{k=1}^{K} \left(c_A(n,k) + c_B(n,k) + c_J(n,k) - d_A(n,k) \right) = d_A(n,K+1) - c_J(n,K+1)),$$

and therefore,

$$p_{n(K+1)j} = \frac{1}{M} c_B(n,j) - \sum_{k=j+1}^{K} p_{nkj} \Leftrightarrow \sum_{k=j+1}^{K+1} p_{nkj} = \frac{1}{M} c_B(n,j),$$

and Statement (b) is shown.

Now, we are ready to check Condition (B). By using Statements (a) and (b), we can conclude for all $1 \leq n \leq M$, and $1 < j \leq K+1$ that

$$\sum_{i=1}^{K+1} p_{nij} = \sum_{i=1}^{j-1} p_{nij} + p_{njj} + \sum_{i=j+1}^{K+1} p_{nij} = \frac{1}{M} d_B(n,j). \qquad (6.23)$$

Moreover, Statement (b) and Equation (6.20) yield

$$\sum_{i=1}^{K+1} p_{ni1} = p_{n11} + \sum_{i=2}^{K+1} p_{nij} = \frac{1}{M} c_J(n,1) + \frac{1}{M} c_B(n,1) = \frac{1}{M} d_B(n,1). \qquad (6.24)$$

Combining Constraint (6.14), Equation (6.23), and Equation (6.24) shows Condition (B).

Condition (C) follows analogously to Condition (B) by using Constraints (6.11) and (6.16) instead of Constraints (6.12) and (6.17).

Finally, Condition (D) follows from Constraint (6.13), Equation (6.23), and Equation (6.24). This concludes the proof of Step (ii) and therefore the proof of this theorem. $\qquad \square$

Remark 6.5 (Solutions to Optimization problem 6.3)

As illustrated in Figure 6.1, an optimal solution to Optimization problem 6.3 does not carry all information about the joint distribution $\mathbb{P}$. In fact, only the information needed for the BCVA calculation is available. However, the proof of Theorem 6.4 gives one possible way on how to construct a joint distribution out of this incomplete information.

Remark 6.6 (Joint defaults)

So far, we did not specify the function ξ in Equation (6.1). This function defines how joint defaults of both parties are handled. A joint default means that both parties default within the same interval between two consecutive grid points. In order to optimize the BCVA and search for truly conservative extremes, our choice of this function is as follows: For a BCVA maximization, one treats joint defaults as counterparty defaults, i.e. one assumes that the counterparty defaults first, and vice versa in case of a BCVA minimization. However, other ways of handling joint defaults are imaginable and fit into our setup.

Remark 6.7 (Increase in efficiency)

Using Optimization problem 6.3 has two main advantages over using Optimization problem 6.2, because of the reduced complexity: First, it can be solved in less time. Second, problems with higher values for the number of simulations M and the number of grid points K are solvable.

In the next chapter, we have a look at some examples. We consider a setup with $M = 1.000$ and $K = 84$. In contrast to Optimization problem 6.2, Optimization problem 6.2 is not solvable on a personal computer (with 4 GB RAM) for such high values of M and K, due to memory issues. For simpler problems, where Optimization problem 6.2 is still solvable (e.g. $M = 500$ and $K = 84$), the run time of Problem 6.2 (544 seconds) is roughly ten times higher than the run time of Problem 6.3 (51 seconds).

6.4 Application: Model-free BCVA bounds

In this numerical case study, we perform the BCVA optimization procedure from Section 6.3. In order to calculate the BCVA for a portfolio of financial derivatives, banks usually define a set of risk factors (like interest rates, exchange rates, stock

prices, credit factors, etc.), which are relevant for the portfolio valuation. A scenario generator simulates paths of the risk factors on which the portfolio is repriced resulting in a set of portfolio value paths. The models we presented in the previous sections, for example, could be used to generate those path. Here, we optimize the BCVA for three interest-rate swaps: one in-the-money, one at-the-money, and one out-of-the-money. For each swap we have $M = 1.000$ trajectories of the discounted values on a non-equidistant (84 grid points) discretization of time. These are generated by an economic-scenario engine and visualized in Figure 6.3. The BCVA is calculated from the bank's perspective. Recovery rates are fixed at 40 percent. Besides the portfolio values, we need the default probability distributions of the two parties, which are extracted from a hazard rate model calibrated to CDS spreads and illustrated in Figure 6.2. It is important to mention that in this example the default probability of the counterparty is always higher than the bank's default probability. Since a BCVA-minimization can be seen as a BCVA-maximization from the counterparty's view, we can evaluate the impact of WWR for two basic situation, namely a counterparty with better credit quality than the bank and a counterparty with worse credit rating.
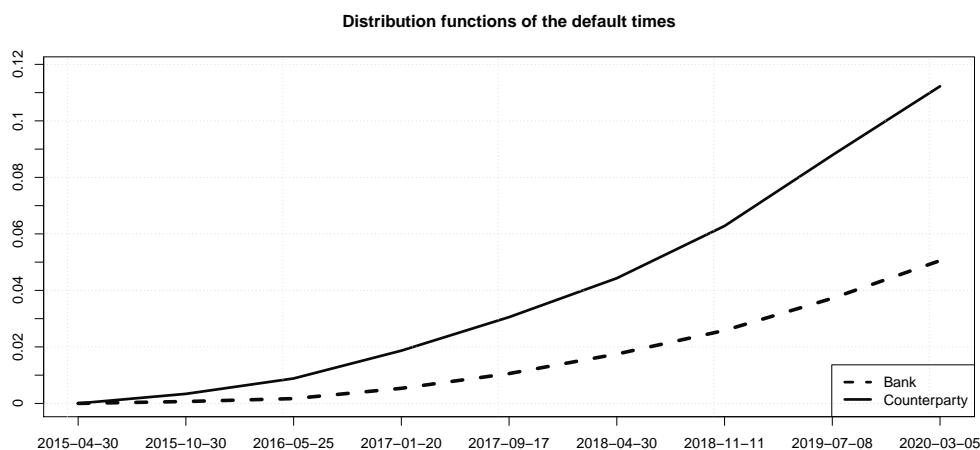

Figure 6.2 Cumulative distribution functions of bank and counterparty.

Note that our setup is flexible enough to optimize a whole portfolio at once. However, for having a better intuition about the results we restrict this numerical example to single products. Obviously, the BCVA can be seen as the sum of CVA (solely driven by the positive part of the portfolio value and the event where the counterparty defaults
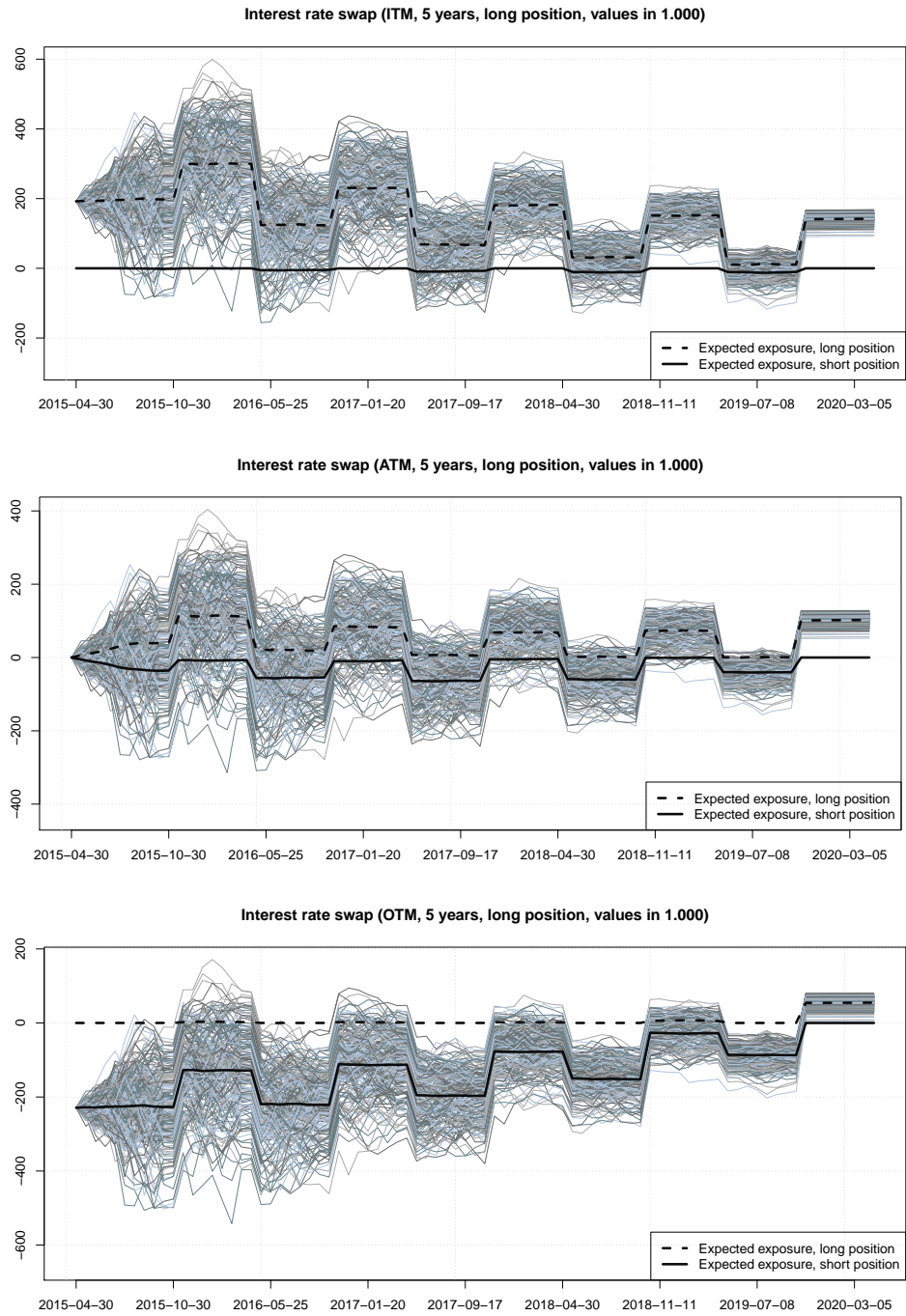
Interest rate swap (ITM, 5 years, long position, values in 1.000)

Interest rate swap (ATM, 5 years, long position, values in 1.000)

Interest rate swap (OTM, 5 years, long position, values in 1.000)

Figure 6.3 The first 250 scenarios of the three products; one in-the-money swap, one at-the-money swap, and one out-of-the-money swap, generated by an economic-scenario engine. For the case study we used 1.000 paths.

first) and DVA (solely driven by the negative part and a first default by the bank). Let us have a look at an extremal case: Assume that every path is positive at each point in time (which is almost true for the considered ITM swap). Then, the DVA vanishes and the BCVA coincides with the CVA. Hence, one can easily construct dependence structures between the default times leading to the maximal BCVA by ensuring that the counterparty defaults before the bank as often as possible. Co-monotonicity or counter-monotonicity between the default times are typical candidates, as we will see within the examples below. Therefore, the overall structure of the portfolio path is very important. We choose three kinds of swaps to cover different situations: (1) the symmetric case (ATM swap) and the two cases where the majority of the paths are either (2) positive or (3) negative (ITM swap and OTM swap). This provides a sound insight into the functioning of the optimizing procedure.

In Figure 6.4 the results of the model-free BCVA bounds are given. For a comparison, we calculated the BCVA in case of no WWR, i.e. under the assumption of independence between the portfolio value and the default times. Furthermore, we implemented the very popular WWR-model by Hull and White [2012]. Within this model, one special dependence structure is induced. By taking the parameters to their extremes, one ends up with an interval of possible Hull–White BCVA values. Here, we adopt the model specifications and the calibration procedure presented in the appendix of Hull and White [2012], which also starts with given trajectories of the portfolio value and a discrete-time setup. The hazard rate directly depends on the portfolio value, and the dependence is driven by one parameter, called b. By letting b go to the extremes, we end up with an interval of Hull–White BCVA values. The maximal and minimal values are also shown in this figure. These values can be seen as the maximal and minimal BCVA coming from an optimization over all dependence structures fitting into the Hull–White WWR-model. Furthermore, by assuming one special joint distribution between the default times of the two parties, one obtains an optimization over a subclass of possible dependence structures which is exactly the setup of Helmers et al. [2016]. Here, we exemplarily chose a Gaussian copula to describe the dependence between the default times. We have a look at different choices of the dependence parameter ρ, namely: $\rho = 1$ (co-monotonicity between the defaults), $\rho = 0$ (independence between the defaults), $\rho = -1$ (counter-monotonicity between the defaults), $\rho = 0.5$ (interpolation between co-monotonicity and independence), and $\rho = -0.5$ (interpolation between counter-monotonicity and independence).
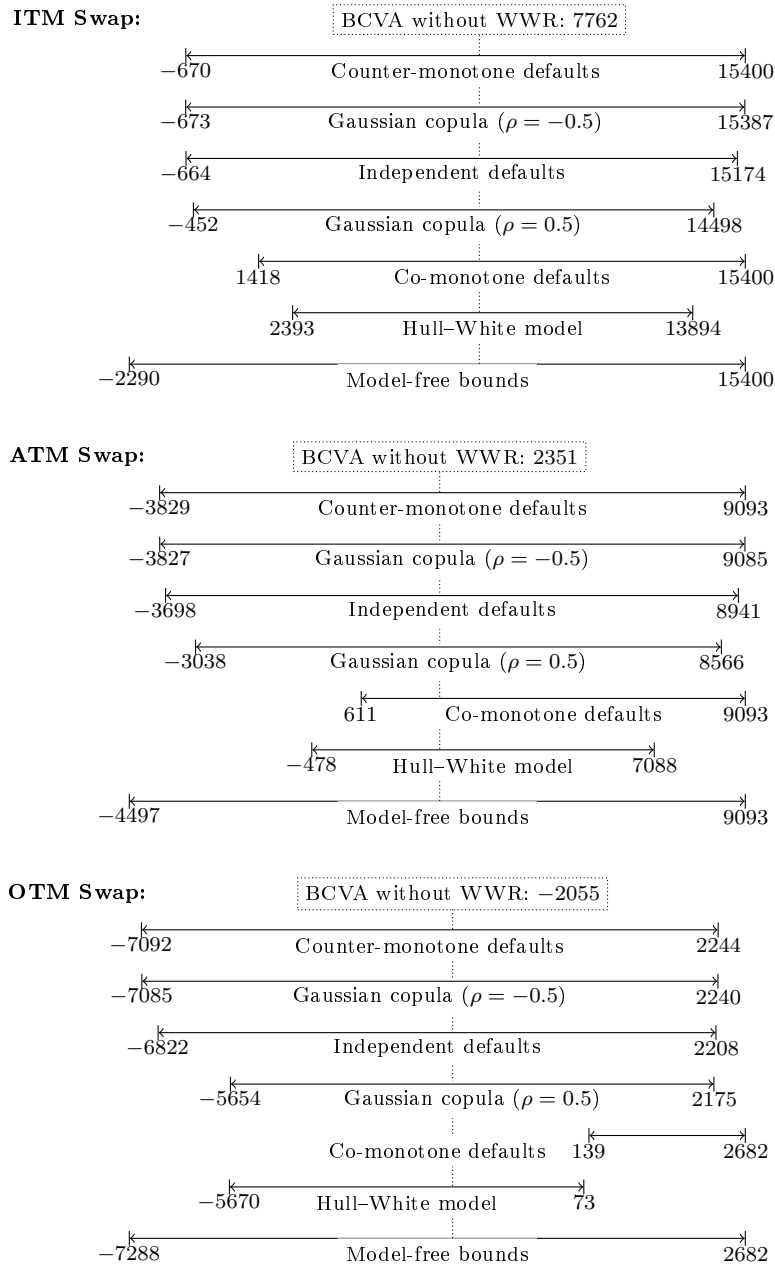
ITM Swap:

BCVA without WWR: 7762

−670   Counter-monotone defaults   15400

−673   Gaussian copula ($\rho = -0.5$)   15387

−664   Independent defaults   15174

−452   Gaussian copula ($\rho = 0.5$)   14498

1418   Co-monotone defaults   15400

2393   Hull–White model   13894

−2290   Model-free bounds   15400

ATM Swap:

BCVA without WWR: 2351

−3829   Counter-monotone defaults   9093

−3827   Gaussian copula ($\rho = -0.5$)   9085

−3698   Independent defaults   8941

−3038   Gaussian copula ($\rho = 0.5$)   8566

611   Co-monotone defaults   9093

−478   Hull–White model   7088

−4497   Model-free bounds   9093

OTM Swap:

BCVA without WWR: −2055

−7092   Counter-monotone defaults   2244

−7085   Gaussian copula ($\rho = -0.5$)   2240

−6822   Independent defaults   2208

−5654   Gaussian copula ($\rho = 0.5$)   2175

Co-monotone defaults   139   2682

−5670   Hull–White model   73

−7288   Model-free bounds   2682

Figure 6.4 BCVA bounds in the respective WWR models. The dashed line in the middle of the illustration corresponds to the BCVA value without WWR. By going to the right side we meet WWR, which is increasing to the maximal possible WWR. Going to the left side results into WWR from the counterparty's perspective. The model-free bounds are the results of the optimization procedure presented in Section 6.3.

Discussion of the optimization results

By studying the results in Figure 6.4 some natural questions arise, which we answer in the following:

- **Can co-monotonicity between the default times always explain maximal BCVA values?**

   In general, no. In many practical situations, however, it is true. It actually depends on the two marginal default distributions. In case the distribution function of the counterparty's default time lies always above the default distribution function of the bank, i.e. $\mathbb{P}(D_A \leq t) < \mathbb{P}(D_B \leq t)$ for all $t \in (0, T]$,[3] co-monotonicity between the default times implies that the counterparty always defaults first. Hence, co-monotonicity can lead to maximal CVA values, whereas the DVA vanishes, because the bank never defaults first. Therefore, the maximal BCVA can be produced by co-monotonicity. In the case where $\mathbb{P}(D_A \leq t) > \mathbb{P}(D_B \leq t)$ for all $t \in (0, T]$, the DVA can be maximized, and the CVA vanishes. For this reason, co-monotonicity leads to the lower BCVA bound. In every other situation, i.e. the two distribution functions have at least one intersection point in $(0, T]$, co-monotonicity is not necessarily able to explain the BCVA bounds. Nevertheless, it is quite realistic in real-world situations that the distribution functions have no intersection points. In our example, the default distribution function of the counterparty lies above the distribution function of the bank's default time, as one can see in Figure 6.2. Hence, the maximal BCVA can be generated by co-monotonicity between the default times due to the special constellation of the default probabilities.

- **Why is the co-monotone-interval for the OTM swap so tiny?**

   As we have seen in the answer to the previous question, co-monotonicity ensures that either the CVA or the DVA vanishes, as long as one default time distribution function is greater than the other within the whole interval $(0, T]$. In our case, the DVA vanishes and therefore, the BCVA is always positive, even in situation where the majority of the portfolio paths is negative. This explains the relatively small interval for the OTM Swap.

- **Why does counter-monotonicity lead to the maximal BCVA for the ITM swap and the ATM swap but not for the OTM swap?**

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[3]T is the point in time corresponding to the last considered grid point K.

Again, this phenomenon can be explained by the default probabilities of the two parties. As long as $\mathbb{P}(D_A \leq T) + \mathbb{P}(D_B \leq T) < 1$, counter-monotonicity guarantees that every default event is a first default, no matter whether it is a default of the bank or the counterparty. The reason is that the counter-monotonicity copula maps early bank defaults to late counterparty defaults, and late bank defaults to early counterparty defaults. Hence, if $\mathbb{P}(D_A \leq T) + \mathbb{P}(D_B \leq T) < 1$, bank defaults occurring before time T are mapped to counterparty defaults after T, and vice versa. In order to maximize the BCVA, the optimizer can map the counterparty default events to the maximal portfolio values and the bank default events can be mapped to portfolio paths with non-negative values. This pragmatic rule leads to the upper BCVA bound as long as there are enough "non-negative" paths, which is the case for the ITM swap and the ATM swap, but not for the OTM swap, as one can see in Figure 6.3. The condition $\mathbb{P}(D_A \leq T) + \mathbb{P}(D_B \leq T) < 1$ is usually fulfilled, as long as T does not take extremely large values or the companies are very risky.

- **Why are the intervals similarly large for totally different dependence structures between the defaults?**
  It is striking that the counter-monotonicity copula and the independence copula lead to similarly large intervals. In the previous question, we already answered why the interval is so large in case of counter-monotonicity. If we assume the default times to be independent, it is very unlikely that a default of one party is not a first default due to the small overall default probability in $(0, T]$ (see Figure 6.2). Hence, in comparison to counter-monotonicity, not all defaults are first defaults, but most of them are. This explains why the independence-interval is slightly smaller than the counter-monotonicity-interval. Furthermore, it is not surprising that the interval of the Gaussian copula with $\rho = -0.5$ is larger than independence, but smaller than counter-monotonicity, as the Gaussian copula interpolates between these extremal dependencies.

- **Independence between the default times seems to be flexible enough to lead to a huge BCVA-interval. Is it recommendable to use?**
  It is right that the independence copula (between the default times) combined with a worst-case dependence to the portfolio values can explain a large interval of possible BCVA values. From this regard, it seems acceptable to use it. In reality, however, it is widely accepted that default times are not independent. Hence, it might be an unrealistic assumption and is therefore not recommended.

Moreover, in order to achieve extreme BCVA values under independent default times the remaining dependence to the underlying must be degenerate.

- **Are the considered dependence structures between the default times realistic?**

  In practice, one usually faces *positive* dependence between the default times. There are different ways how to characterize positive dependence. A positive correlation coefficient between the default times could be one possible choice. Other dependence measures, as the concept of positive quadrant dependence, are also possible. In the special case of a Gaussian copula, one speaks of positive dependence whenever $\rho > 0$. Restricting oneself to positive dependence clearly rules out counter-monotonicity. We implemented it anyway to get a better understanding how the optimization procedure works. The Gaussian copula with positive parameter ρ is, however, just one possible choice and can lead to small intervals, as one can see in the case of an OTM swap. Thus, the model risk is quite high. There might be other *positive* dependence structures leading to much larger intervals, as we will see in the answer to the next question.

- **Are the worst-case dependence structures leading to the BCVA bounds realistic?**

  As we already mentioned, a negative dependence between the default times is quite unrealistic. Thus, it is justifiable to ask to what extent the model-free BCVA bounds are meaningful. One might assume that these bounds can only be produced by extremely unrealistic dependence structures. First, we should mention that the purpose of the present work is not to present realistic WWR-models. The idea is rather to answer the following question: How bad can WWR be? Second, in our perspective, it is hard to decide, if a given dependence structure between the default times and the portfolio paths is realistic. Nevertheless, it is reasonable to force the bivariate dependence between the default times to be non-negative. This can be achieved, for example, by adding the following constraint to Optimization problem 6.3: $P(D_A < t)P(D_B < t) \leq P(D_A < t, D_B < t)$ for all $t \in [0, T]$. This constraint can be seen as a weak PQD[4] condition and can be interpreted as the following: The probability that both parties default before a fixed time t is higher than the probability that both parties default before a fixed time t in case of independence. Hence, dependence structures fulfilling this

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[4]The classical definition for positive quadrant dependence (PQD) is given by $P(D_A < t_A)P(D_B < t_B) \leq P(D_A < t_A, D_B < t_B)$ for all $t_A, t_B \in [0, T]$

condition can be seen as non-negative dependence structures. The results coming from the optimizing procedure including this additional condition are astonishing: the model-free BCVA bounds do not change. This means that by optimizing over a subclass of dependence structures, namely over all dependencies fulfilling the above mentioned constraint, the extremal BCVA coincides with the BCVA bounds. Hence, the BCVA bounds in our examples can even be explained in presence positive dependencies between the default times. To summarize, the extremal BCVA values can stem from reasonable dependence structures among the default times.

7 Conclusion

In this thesis, we presented new approaches for tractable modeling of dependencies between the stochastic drivers in financial models. Particularly, a new stochastic volatility model, which generalized the popular model by Barndorff-Nielsen and Shephard [2001] to allow for decoupled jumps in the volatility process and jumps in the asset price process, has been proposed. We thoroughly investigated the model dynamics and showed some exemplary products, where the dependency between the jump components has a crucial impact. The dependent two-dimensional jump process is constructed by a time-change procedure, which ensures fast pricing and calibrating via Fourier-inversion methods, since the characteristic function of the log-price dynamics of the newly created model can be expressed in closed form.

In the d-dimensional case, the time-change construction of dependent jump processes has also been utilized to construct multi-dimensional versions of well-known univariate models. The easy-to-simulate jump processes fulfill a nice separation property, which allows a tractable sequential calibration procedure. First of all, marginal asset price processes can be calibrated to plain vanilla option prices. Afterwards, the dependence parameters between the assets can be specified in a second step. We emphasized the practicality of this multivariate model in a calibration exercise with real market data.

Last but not least, we presented a model-free approach and formalized an optimization problem that allows to maximize (or minimize) bilateral credit valuation adjustments. We calculated the BCVA considering extremal dependence structures leading to the maximal WWR. These tight BCVA bounds have been compared to the popular WWR model by Hull and White [2012]. Furthermore, we related the results to the BCVA bounds by Helmers et al. [2016], which correspond to subclasses of possible dependence structures between the portfolio value and the default times. In a numerical case study, we have seen that WWR is essential and can lead to exorbitant BCVA values. Furthermore, WWR plays a bigger role than the dependence structure between the default times. The bounds coming from our model-free approach can be used as an

indicator for model risk for existing WWR models in the sense that the flexibility in explaining various BCVA values can be put in relation to the model-free bounds. This sheds some light on the amount of model risk market participants are exposed to, when committing oneself to one specific parametric WWR model.

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