Two-sided Moment Matching-Based Reduction for MIMO Quadratic-Bilinear Systems

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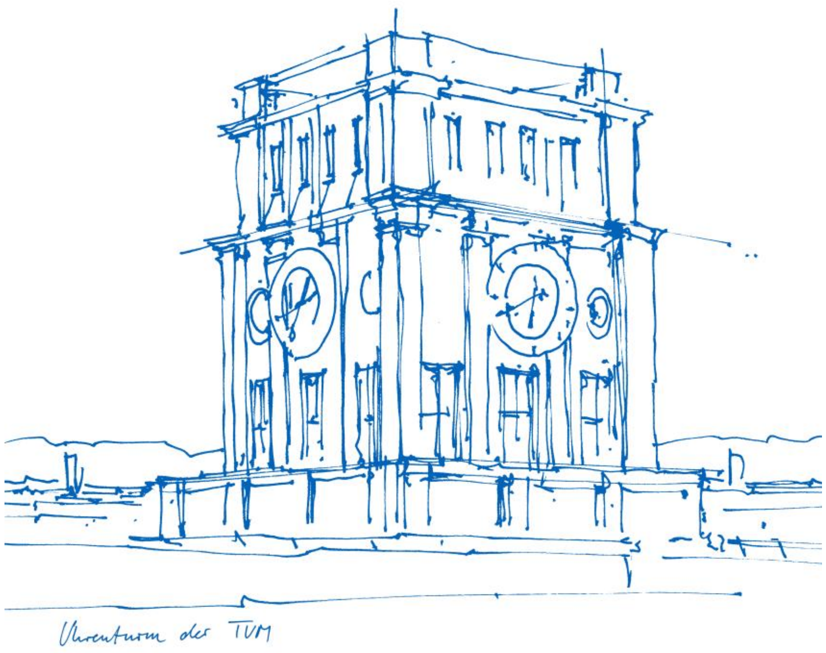

Motivation

Given a large-scale nonlinear control system of the form

$\det(\mathbf{E}) \neq 0$

$$\mathbf{\Sigma} : \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

$\mathbf{x}(t) \in \mathbb{R}^n$

with $\mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{f}(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{q \times n}$

Simulation, design, control and optimization cannot be done efficiently!

MOR

Reduced order model (ROM)

$$\mathbf{\Sigma}_r : \begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{f}_r(\mathbf{x}_r(t)) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t), \quad \mathbf{x}_r(0) = \mathbf{x}_{r,0} \end{cases}$$

$\mathbf{x}_r(t) \in \mathbb{R}^r, \ r \ll n$

with $\mathbf{E}_r \in \mathbb{R}^{r \times r}$, $\mathbf{f}_r(\mathbf{x}_\mathbf{r}(t)) : \mathbb{R}^r \to \mathbb{R}^r$ and $\mathbf{B}_r \in \mathbb{R}^{r \times m}$, $\mathbf{C}_r \in \mathbb{R}^{q \times r}$

Goal:
$\mathbf{y}_r(t) \approx \mathbf{y}(t)$

State-of-the-Art: Overview

Reduction of nonlinear (parametric) systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ Simulation-based:
  - POD, TPWL
  - Reduced Basis, Empirical Gramians

Reduction of bilinear systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}u + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ Carleman bilinearization (approx.)
- ❗ Large increase of dimension: $n + n^2$
- ☑ Generalization of well-known methods:
  - Balanced truncation
  - **Krylov subspace methods**
  - **$\mathcal{H}_2$ (pseudo)-optimal approaches**

Reduction of quadratic-bilinear systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{H}(\mathbf{x} \otimes \mathbf{x}) + \mathbf{N}\mathbf{x}u + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ **Quadratic-bilinearization** (no approx.!)
- ☑ Minor increase of dimension: $2n,\ 3n$
- ☑ Generalization of well-known methods:
  - **Krylov subspace methods**
  - **$\mathcal{H}_2$-optimal approaches**
- ☐ Reduction methods for **MIMO** models

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Quadratic-Bilinearization Process

SISO Quadratic-bilinear system:

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,\mathbf{x}\otimes\mathbf{x} + \mathbf{N}\,\mathbf{x}\,u + \mathbf{b}\,u$$

$$y = \mathbf{c}^T\,\mathbf{x}$$

Quadratic terms of **x**

States-input coupling

$\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}$

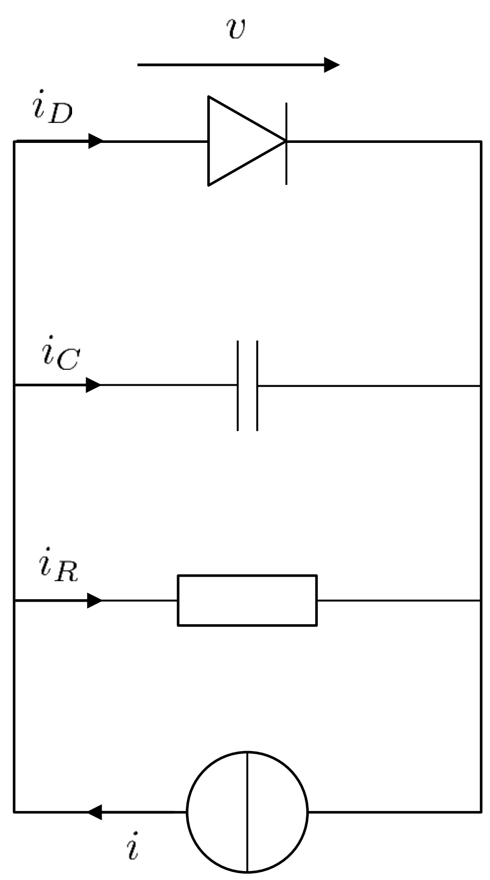
$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{b}, \mathbf{c} \in \mathbb{R}^{n}$

Objective: Bring general nonlinear systems to the quadratic-bilinear (QB) form

1. **Polynomialization:** Convert nonlinear system into an equivalent polynomial system
2. **Quadratic-bilinearization:** Convert the polynomial system into a QBDAE

Quadratic-Bilinearization Process – Example



$$i_C + i_R + i_D = i \qquad \text{with} \quad \begin{cases} i_C = C\dot{v} \\ i_R = \dfrac{v}{R} \\ i_D = e^{\alpha v} - 1 \end{cases}$$

Nonlinear ODE: $\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - e^{\alpha v} + 1 + i\right)$

1 **Polynomialization step:** Introduce new variable and its Lie derivative

$$w = e^{\alpha v} - 1$$

$$\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - w + i\right)$$

$$\begin{aligned} \dot{w} &= (\alpha e^{\alpha v})\dot{v} \\ &= \frac{\alpha}{C}\left(-\frac{vw}{R} - w^2 + wi - \frac{v}{R} - w + i\right) \end{aligned}$$

Quadratic-Bilinearization Process – Example

2 **Quadratic-bilinearization step:** Convert polynomial system into a QBDAE

$$\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - w + i\right)$$

$$\dot{w} = \frac{\alpha}{C}\left(-\frac{vw}{R} - w^2 + wi - \frac{v}{R} - w + i\right)$$

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{\mathbf{E}}\underbrace{\begin{bmatrix}\dot{v}\\\dot{w}\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix}-\frac{1}{RC} & -\frac{1}{C}\\-\frac{\alpha}{RC} & -\frac{\alpha}{C}\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}v\\w\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}0 & 0 & 0 & 0\\0 & -\frac{\alpha}{RC} & 0 & -\frac{\alpha}{C}\end{bmatrix}}_{\mathbf{H}}\underbrace{\begin{bmatrix}v^2\\vw\\vw\\w^2\end{bmatrix}}_{\mathbf{x}\otimes\mathbf{x}} + \underbrace{\begin{bmatrix}0 & 0\\0 & \frac{\alpha}{C}\end{bmatrix}}_{\mathbf{N}}\underbrace{\begin{bmatrix}v\\w\end{bmatrix}}_{\mathbf{x}}\underbrace{i}_{u} + \underbrace{\begin{bmatrix}\frac{1}{C}\\\frac{\alpha}{C}\end{bmatrix}}_{\mathbf{b}}\underbrace{i}_{u}$$

Equivalent **representation**	Dimension slightly **increased**	Transformation **not unique**

The matrix **H** can be seen as a **tensor**

Variational Analysis of Nonlinear Systems

[Rugh '81]

Assumption: Nonlinear system can be broken down into a series of homogeneous subsystems that depend nonlinearly from each other (Volterra theory)

For an input of the form $\alpha u(t)$, we assume that the response should be of the form

$$\mathbf{x}(t) = \alpha \mathbf{x}_1(t) + \alpha^2 \mathbf{x}_2(t) + \alpha^3 \mathbf{x}_3(t) + \dots$$

Inserting the assumed input and response in the QB system and comparing coefficients of α^k, we obtain the variational equations:

$$\begin{aligned}
\mathbf{E}\dot{\mathbf{x}}_1 &= \mathbf{A}\mathbf{x}_1 + \mathbf{b}u \\
\mathbf{E}\dot{\mathbf{x}}_2 &= \mathbf{A}\mathbf{x}_2 + \mathbf{H}\mathbf{x}_1 \otimes \mathbf{x}_1 + \mathbf{N}\mathbf{x}_1 u \\
\mathbf{E}\dot{\mathbf{x}}_3 &= \mathbf{A}\mathbf{x}_3 + \mathbf{H}\left(\mathbf{x}_1 \otimes \mathbf{x}_2 + \mathbf{x}_2 \otimes \mathbf{x}_1\right) + \mathbf{N}\mathbf{x}_2 u \\
&\vdots \\
\mathbf{E}\dot{\mathbf{x}}_k &= \mathbf{A}\mathbf{x}_k + \sum_{i=1}^{k-1} \mathbf{H}\left(\mathbf{x}_i \otimes \mathbf{x}_{k-i}\right) + \mathbf{N}\mathbf{x}_{k-1} u, \quad k = 4, 5, 6, \dots
\end{aligned}$$

Generalized Transfer Functions (SISO)

[Rugh ’81]

Series of generalized transfer functions can be obtained via the growing exponential approach:

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

1st subsystem:

$$G_1(s_1) = -\mathbf{c}^T(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

2nd subsystem:

$$G_2(s_1, s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$\mathbf{H}$ is symmetric

$$\mathbf{H}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{H}(\mathbf{v} \otimes \mathbf{u})$$

$$G_2(s_1, s_2) = -\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}) - \frac{1}{2}\mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$s_1 = s_2 = \sigma$

$$G_2(\sigma, \sigma) = -\mathbf{c}^T\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b}\right]$$

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Moments of QB-Transfer Functions

Taylor coefficients of the transfer function: $G(s) = \underbrace{G(s_0)}_{m_0} + \underbrace{\frac{dG(s_0)}{ds}}_{m_1}(s-s_0) + \underbrace{\frac{1}{2!}\frac{d^2G(s_0)}{ds^2}}_{m_2}(s-s_0)^2 + \dots$

1st subsystem: $G_1(s_1) = -\mathbf{c}^T(\mathbf{A}-s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b}$

$$\mathbf{A}_s = \mathbf{A} - s\mathbf{E}$$

$$\frac{\partial}{\partial s}\mathbf{A}_s^{-1}(s) = -\mathbf{A}_s^{-1}\frac{\partial \mathbf{A}_s}{\partial s}\mathbf{A}_s^{-1} = \mathbf{A}_s^{-1}\mathbf{E}\mathbf{A}_s^{-1}$$

$$\frac{\partial G_1}{\partial s_1} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

2nd subsystem: $G_2(s_1,s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b}+\mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$

$$\begin{aligned}\frac{\partial G_2}{\partial s_1} = &-\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_2}^{-1}\mathbf{b}]\\ &-\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_2}^{-1}\mathbf{b}]\\ &+\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{b}+\mathbf{A}_{s_2}^{-1}\mathbf{b}]\\ &+\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}]\end{aligned}$$

Multimoments approach (SISO)

[Breiten '12]

Algorithm 1 QB Multimoment Matching (SISO)

Input: $\mathbf{E}$, $\mathbf{A}$, $\mathbf{H}$, $\mathbf{N}$, $\mathbf{b}$, $\mathbf{c}$, shift σ, reduced order of first transfer function q_1 and of the second transfer function q_2

Output: Projection matrices $\mathbf{V}$, $\mathbf{W}$

1: $\mathbf{V}_1 = \mathcal{K}_{q_1}(\mathbf{A}_\sigma^{-1}\mathbf{E}, \mathbf{A}_\sigma^{-1}\mathbf{b})$

2: $\mathbf{W}_1 = \mathcal{K}_{q_1}(\mathbf{A}_{2\sigma}^{-T}\mathbf{E}^T, \mathbf{A}_{2\sigma}^{-T}\mathbf{c})$

3: **for** $i = 1 : q_2$ **do**

4: $\quad \mathbf{V}_2^i = \mathcal{K}_{q_2-i+1}(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{N}\mathbf{V}_1(:,i))$

5: $\quad \mathbf{W}_2^i = \mathcal{K}_{q_2-i+1}(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{N}^T\mathbf{W}_1(:,i))$

6: $\quad$ **for** $j = 1 : \min(q_2 - i + 1, i)$ **do**

7: $\quad\quad \mathbf{V}_3^{i,j} = \mathcal{K}_{q_2-i+1}(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{V}_1(:,i) \otimes \mathbf{V}_1(:,j)))$

8: $\quad\quad \mathbf{W}_3^{i,j} = \mathcal{K}_{q_2-i+1}(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{H}^{(2)}(\mathbf{V}_1(:,i) \otimes \mathbf{W}_1(:,j)))$

9: $\quad$ **end for**

10: **end for**

11: $\operatorname{span}(\mathbf{V}) = \operatorname{span}(\mathbf{V}_1) \cup \bigcup_i \operatorname{span}(\mathbf{V}_2^i) \cup \bigcup_{i,j} \operatorname{span}(\mathbf{V}_3^{i,j})$

12: $\operatorname{span}(\mathbf{W}) = \operatorname{span}(\mathbf{W}_1) \cup \bigcup_i \operatorname{span}(\mathbf{W}_2^i) \cup \bigcup_{i,j} \operatorname{span}(\mathbf{W}_3^{i,j})$

linear

bilinear

quadratic

$$\frac{\partial^i G_1}{\partial s_1^i}(\sigma) = \frac{\partial^i G_{1,r}}{\partial s_1^i}(\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^i G_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i G_{1,r}}{\partial s_1^i}(2\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j} G_2(\sigma, \sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j} G_{2,r}(\sigma, \sigma), \qquad i + j \leq 2q_2 - 1$$

$q_1 + q_2^2 + q_2^2$
columns per shift

$$\operatorname{span}(\mathbf{V}) = \operatorname{span}(\mathbf{V}_{\text{lin}}) \cup \operatorname{span}(\mathbf{V}_{\text{b}}) \cup \operatorname{span}(\mathbf{V}_{\text{q}})$$

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Hermite approach (SISO)

Theorem: Two-sided rational interpolation [Breiten '15]

Let $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$ be nonsingular, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V},\ \mathbf{H}_r = \mathbf{W}^T\mathbf{H}(\mathbf{V}\otimes\mathbf{V}),\ \mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V},$ $\mathbf{b}_r = \mathbf{W}^T\mathbf{b},\ \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$ with $\mathbf{V},\ \mathbf{W}\in\mathbb{R}^{n\times r}$ having full rank such that

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma_i}^{-1}[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c},\ \mathbf{A}_{\sigma_i}^{-T}[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c}) - \frac{1}{2}\mathbf{N}^T\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c})]\}$$

with $\sigma_i \notin \{\Lambda(\mathbf{A},\mathbf{E}),\Lambda(\mathbf{A}_r,\mathbf{E}_r\}$.

2 columns per shift

Then:

$$G_1(\sigma_i) = G_{1,r}(\sigma_i)\ \checkmark \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)\ \checkmark$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i)\ \checkmark \qquad \frac{\partial G_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial G_{2,r}}{\partial s_j}(\sigma_i,\sigma_i)\ \checkmark$$

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Krylov subspaces for SISO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Multimoments approach [Gu '11, Breiten '12]:

$$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_{\mathrm{lin}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{b}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{q}})$$

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma}^{-1}\mathbf{b})\right\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{2\sigma}^{-T}\mathbf{N}^T\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{2\sigma}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma}^{-T}\mathbf{c})\right\}$$

$$G_1(\sigma_i) = G_{1,r}(\sigma_i) \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i) \qquad \frac{\partial}{\partial s_j}G_2(\sigma_i,\sigma_i) = \frac{\partial}{\partial s_j}G_{2,r}(\sigma_i,\sigma_i)$$

- Quadratic and bilinear dynamics are treated **separately**
- **Higher-order moments** can be matched
- **3** Krylov directions per shift

Hermite approach [Breiten '15]:

$$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_{\mathrm{lin}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{qb}})$$

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b}\right]\right\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{\sigma}^{-T}\left[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma}^{-T}\mathbf{c}) - \frac{1}{2}\mathbf{N}^T\mathbf{A}_{2\sigma}^{-T}\mathbf{c}\right]\right\}$$

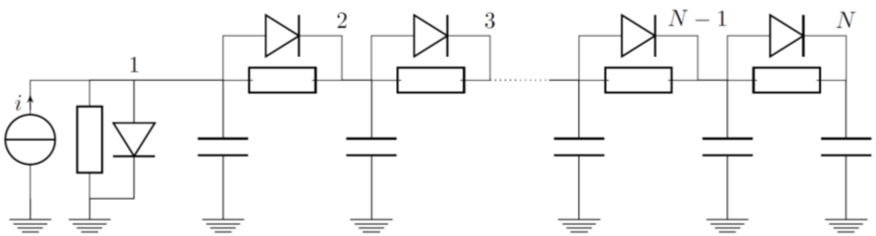
$$G_1(\sigma_i) = G_{1,r}(\sigma_i) \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i) \qquad \frac{\partial}{\partial s_j}G_2(\sigma_i,\sigma_i) = \frac{\partial}{\partial s_j}G_{2,r}(\sigma_i,\sigma_i)$$

- Quadratic and bilinear dynamics are treated **together (as one)**
- **Only 0th and 1st moments** can be matched
- **2** Krylov directions per shift

Numerical Examples: SISO RC-Ladder

SISO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Input/Output: $u(t) = e^{-t}; \quad y(t) = v_1(t)$

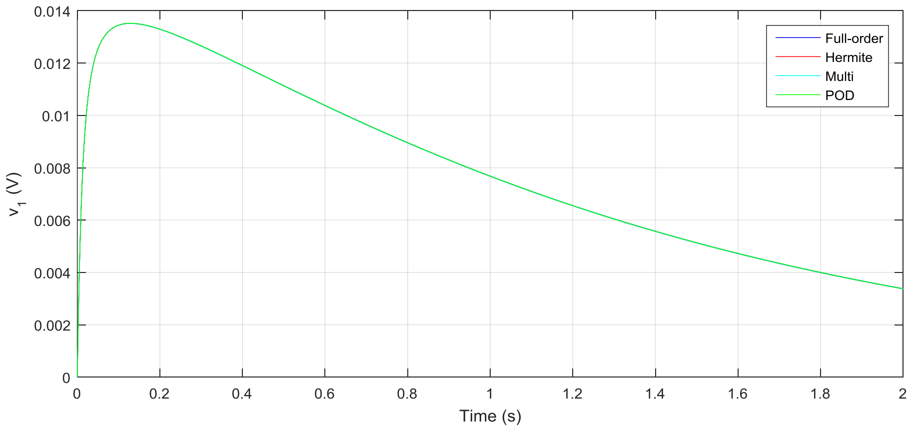
Reduction information:

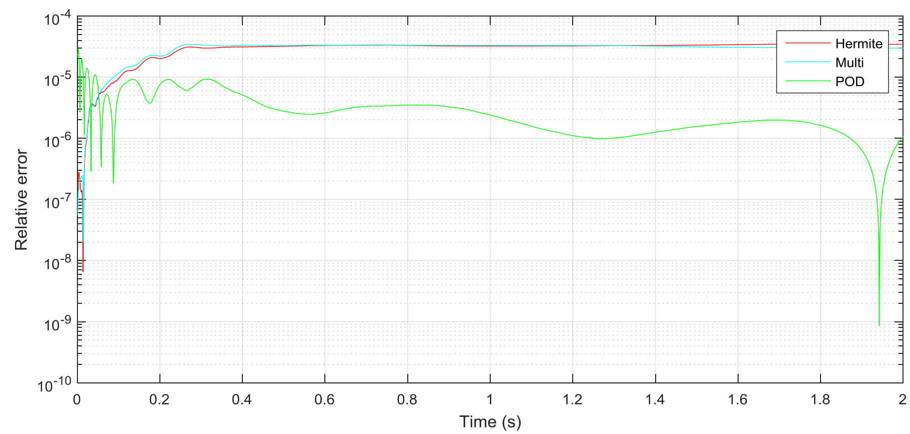
$n = 1000; \quad \text{Shifts } s_0 \text{ gotten from IRKA}$

$t_{\text{sim,orig}} = 17.6 \text{ s}$

$r_{\text{her}} = 12$ $\qquad r_{\text{multi}} = 18$

$t_{\text{sim,her}} = 0.116 \text{ s}$ $\qquad t_{\text{sim,multi}} = 0.122 \text{ s}$





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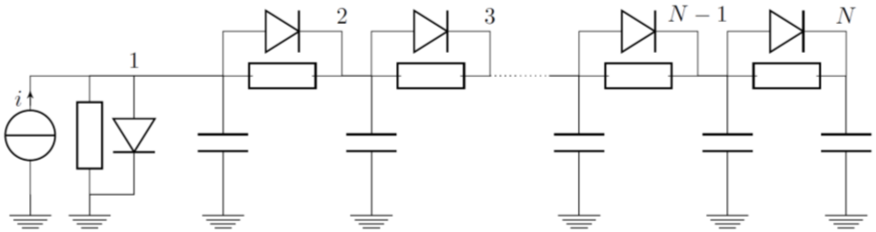
Numerical Examples: SISO RC-Ladder

SISO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Input/Output: $u(t) = {}^{1}\!/_{2} \left[\cos\left(2\pi {}^{t}\!/_{10}\right) + 1\right]$

$y(t) = v_1(t)$

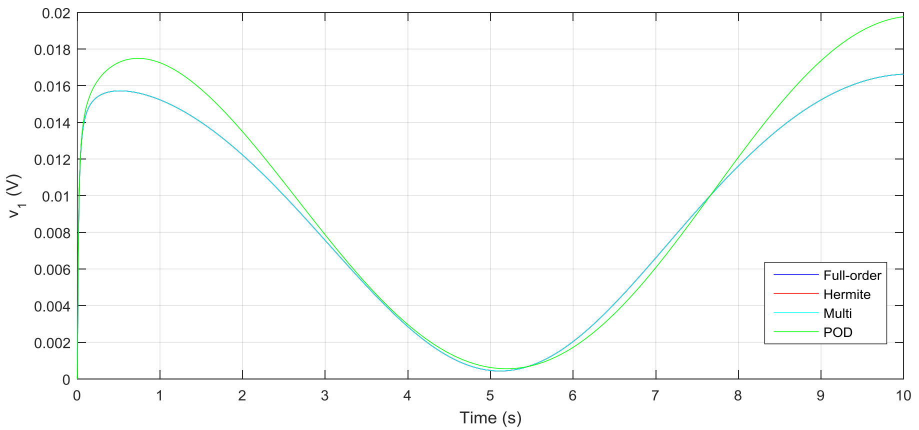
Reduction information:

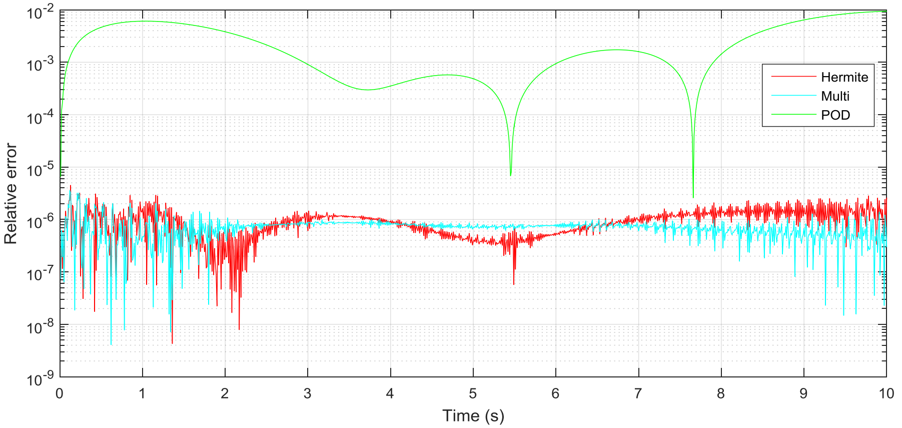
$n = 1000;$ Shifts s_0 gotten from IRKA

$t_{\text{sim,orig}} = 25.5$ s

$r_{\text{her}} = 12$ $\quad$ $r_{\text{multi}} = 18$

$t_{\text{sim,her}} = 0.468$ s $\quad$ $t_{\text{sim,multi}} = 0.788$ s





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MIMO quadratic-bilinear systems

MIMO Quadratic-bilinear system:

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,\mathbf{x}\otimes\mathbf{x} + \sum_{j=1}^{m}\mathbf{N}_j\,\mathbf{x}\,u_j + \mathbf{B}\,\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{x}$$

One bilinear matrix for each input

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n\times n}$

$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{B} \in \mathbb{R}^{n\times m}, \;\; \mathbf{C} \in \mathbb{R}^{p\times n}$

$$\bar{\mathbf{N}} = [\mathbf{N}_1 \;\; \mathbf{N}_2 \;\; \ldots \;\; \mathbf{N}_m] \in \mathbb{R}^{n\times n\cdot m}$$

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,(\mathbf{x}\otimes\mathbf{x}) + \bar{\mathbf{N}}\,(\mathbf{u}\otimes\mathbf{x}) + \mathbf{B}\,\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{x}$$

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Transfer matrices of a MIMO QB system

Generalized transfer matrices can be obtained similarly via the growing exponential approach:

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

1st subsystem:

$$\mathbf{G}_1(s_1) = -\mathbf{C}(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{B} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{B}$$

2nd subsystem:

$$\mathbf{G}_2(s_1, s_2) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m \otimes \left(\mathbf{A}_{s_1}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B})\right)\right]$$

$$s_1 = s_2 = \sigma$$

$$\mathbf{G}_2(\sigma, \sigma) = -\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) - \bar{\mathbf{N}}\left(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}\right)\right]$$

Transfer matrices with

$$\dim(\mathbf{G}_1(s)) = (p, m)$$
$$\dim(\mathbf{G}_2(s_1, s_2)) = (p, m^2)$$

The quadratic term **cannot be simplified**

$$\mathbf{H}(\mathbf{U} \otimes \mathbf{V}) \neq \mathbf{H}(\mathbf{V} \otimes \mathbf{U})$$

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Moments of QB-Transfer Matrices

$\mathbf{A}_s = \mathbf{A} - s\mathbf{E}$

1st subsystem: $\mathbf{G}_1(s_1) = -\mathbf{C}(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{B} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{B}$

$$\frac{\partial}{\partial s}\mathbf{A}_s^{-1}(s) = -\mathbf{A}_s^{-1}\frac{\partial \mathbf{A}_s}{\partial s}\mathbf{A}_s^{-1} = \mathbf{A}_s^{-1}\mathbf{E}\mathbf{A}_s^{-1}$$

$$\frac{\partial \mathbf{G}_1}{\partial s_1} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{B}$$

2nd subsystem: $\mathbf{G}_2(s_1, s_2) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m \otimes (\mathbf{A}_{s_1}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B}))\right]$

$$\begin{aligned}\frac{\partial \mathbf{G}_2}{\partial s_1}(\sigma, \sigma) = & -\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes \mathbf{A}_\sigma^{-1}\mathbf{B}) \\ & -\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_\sigma^{-1}\mathbf{E}\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes \mathbf{A}_\sigma^{-1}\mathbf{B} + \mathbf{A}_\sigma^{-1}\mathbf{B} \otimes \mathbf{A}_\sigma^{-1}\mathbf{E}\mathbf{A}_\sigma^{-1}\mathbf{B}) \\ & +\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_\sigma^{-1}\mathbf{B}) \\ & +\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_\sigma^{-1}\mathbf{E}\mathbf{A}_\sigma^{-1}\mathbf{B})\end{aligned}$$

This term **cannot be simplified**

$\mathbf{H}(\mathbf{U} \otimes \mathbf{V}) \neq \mathbf{H}(\mathbf{V} \otimes \mathbf{U})$

⇒ **Matching of 1st moment of 2nd transfer function much more involved!**

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Block-Multimoments approach (MIMO)

<u>Idea:</u> Straightforward extension of the **multimoments approach** to the MIMO case

Algorithm 1 QB Multimoment Matching (MIMO)

Input: $\mathbf{E}$, $\mathbf{A}$, $\mathbf{H}$, $\bar{\mathbf{N}}$, $\mathbf{B}$, $\mathbf{C}$, shift σ, reduced order of first transfer function q_1 and of the second transfer function q_2

Output: Projection matrices $\mathbf{V}$, $\mathbf{W}$

$\mathbf{V}_1 = \mathcal{K}_{q_1}(\mathbf{A}_\sigma^{-1}\mathbf{E}, \mathbf{A}_\sigma^{-1}\mathbf{B})$

$\mathbf{W}_1 = \mathcal{K}_{q_1}(\mathbf{A}_{2\sigma}^{-T}\mathbf{E}^T, \mathbf{A}_{2\sigma}^{-T}\mathbf{C}^T)$

linear

for $i = 1 : q_2$ **do**

$\mathbf{V}_2^i = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes (\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B})\right)$

$\mathbf{W}_2^i = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{I}_m \otimes (\mathbf{A}_{2\sigma}^{-1}\mathbf{E})^{i-1}\mathbf{A}_{2\sigma}^{-1}\mathbf{B})\right)$

bilinear

for $j = 1 : \min(q_2 - i + 1, i)$ **do**

$\mathbf{V}_3^{i,j} = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{H}((\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes (\mathbf{A}_\sigma^{-1}\mathbf{E})^{j-1}\mathbf{A}_\sigma^{-1}\mathbf{B})\right)$

$\mathbf{W}_3^{i,j} = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{H}^{(2)}((\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes (\mathbf{A}_{2\sigma}^{-1}\mathbf{E})^{i-1}\mathbf{A}_{2\sigma}^{-1}\mathbf{B})\right)$

quadratic

end for

end for

$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_1) \cup \bigcup_i \mathrm{span}(\mathbf{V}_2^i) \cup \bigcup_{i,j} \mathrm{span}(\mathbf{V}_3^{i,j})$

$\mathrm{span}(\mathbf{W}) = \mathrm{span}(\mathbf{W}_1) \cup \bigcup_i \mathrm{span}(\mathbf{W}_2^i) \cup \bigcup_{i,j} \mathrm{span}(\mathbf{W}_3^{i,j})$

$$\frac{\partial^i \mathbf{G}_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \mathbf{G}_{1,r}}{\partial s_1^i}(\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^i \mathbf{G}_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i \mathbf{G}_{1,r}}{\partial s_1^i}(2\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j}\mathbf{G}_2(\sigma,\sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j}\mathbf{G}_{2,r}(\sigma,\sigma), \qquad i + j \leq 2q_2 - 1$$

$m \cdot (q_1 + q_2^2 + q_2^2)$
columns per shift

$$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_{\mathrm{lin}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{b}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{q}})$$

Block-Hermite approach (MIMO)

Aim: Extension of the **hermite** approach to the MIMO case. Is that possible??

Propositions for Block-Hermite approach:

1

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B},\ \mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B})\right]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T,\ \mathbf{A}_{\sigma_i}^{-T}[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T) - \bar{\mathbf{N}}^{(2)}(\mathbf{I}_m\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T)]\}$$

$$\mathbf{H}(\mathbf{U}\otimes\mathbf{V}) \neq \mathbf{H}(\mathbf{V}\otimes\mathbf{U})$$

$$\mathbf{G}_1(\sigma_i) = \mathbf{G}_{1,r}(\sigma_i) \quad ✓$$

$$\mathbf{G}_1(2\sigma_i) = \mathbf{G}_{1,r}(2\sigma_i) \quad ✓$$

$$\mathbf{G}_2(\sigma_i,\sigma_i) = \mathbf{G}_{2,r}(\sigma_i,\sigma_i) \quad ✓$$

$$\frac{\partial\mathbf{G}_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial\mathbf{G}_{2,r}}{\partial s_j}(\sigma_i,\sigma_i) \quad ✗$$

⇓ $\mathbf{W}$ must be adapted!

2

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B},\ \mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B})\right]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T,\ \mathbf{A}_{\sigma_i}^{-T}[(\mathbf{H}+\mathbf{J})^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T) - \bar{\mathbf{N}}^{(2)}(\mathbf{I}_m\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T)]\}$$

$$\mathbf{J} = \mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{E}\otimes\mathbf{E}^{-1}\mathbf{A}_{\sigma_i})$$

m + m² columns per shift

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Krylov subspaces for MIMO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Idea: Combine **multimoments** and **hermite** approaches!

Block tensor-based approach:

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\dots,k}\Big\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \mathbf{A}_{\sigma_i}^{-1}\mathbf{E}\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \dots, (\mathbf{A}_{\sigma_i}^{-1}\mathbf{E})^m\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{B})\right]\Big\}$$

quadratic-bilinear

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\dots,k}\Big\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T, \mathbf{A}_{\sigma_i}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T), \mathbf{A}_{\sigma_i}^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{I}_m\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T)\Big\}$$

quadratic

bilinear

$$\frac{\partial^l\mathbf{G}_1}{\partial s^l}(\sigma_i) = \frac{\partial^l\mathbf{G}_{1,r}}{\partial s^l}(\sigma_i) \quad ✓ \qquad l = 0,\dots,m$$

$$\mathbf{G}_1(2\sigma_i) = \mathbf{G}_{1,r}(2\sigma_i) \quad ✓$$

$$\mathbf{G}_2(\sigma_i,\sigma_i) = \mathbf{G}_{2,r}(\sigma_i,\sigma_i) \quad ✓$$

$$\frac{\partial\mathbf{G}_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial\mathbf{G}_{2,r}}{\partial s_j}(\sigma_i,\sigma_i) \quad ✓ \qquad j = 1,2$$

- **Subsystem interpolation**
- **(m+1) + 4** moments matched
- **(m+1)·m + m² = m + 2m²** columns per shift

Krylov subspaces for MIMO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Idea: Add **tangential directions**!

Tangential tensor-based approach:

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\Big\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i, \mathbf{A}_{\sigma_i}^{-1}\mathbf{E}\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i, \ldots, (\mathbf{A}_{\sigma_i}^{-1}\mathbf{E})^m\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i, \mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i) - \bar{\mathbf{N}}(\mathbf{r}_i \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i)\right]\Big\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\Big\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i, \mathbf{A}_{\sigma_i}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i), \mathbf{A}_{\sigma_i}^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{r}_i \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i)\Big\}$$

$$\left[\frac{\partial^l \mathbf{G}_1}{\partial s^l}(\sigma_i)\right]\mathbf{r}_i = \left[\frac{\partial^l \mathbf{G}_{1,r}}{\partial s^l}(\sigma_i)\right]\mathbf{r}_i \qquad l = 0, \ldots, m$$

$$\mathbf{l}_i^T\left[\mathbf{G}_1(2\sigma_i)\right] = \mathbf{l}_i^T\left[\mathbf{G}_{1,r}(2\sigma_i)\right]$$

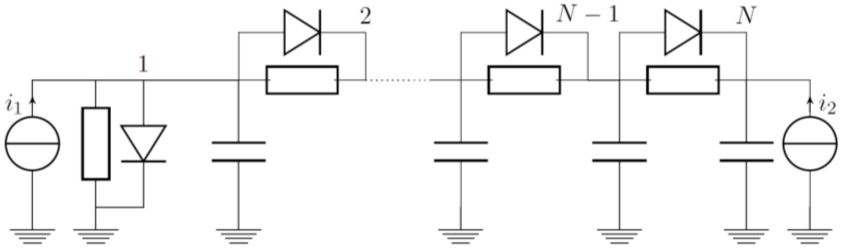
$$\left[\mathbf{G}_2(\sigma_i, \sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) = \left[\mathbf{G}_{2,r}(\sigma_i, \sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i)$$

$$\mathbf{l}_i^T\left[\frac{\partial \mathbf{G}_2}{\partial s_j}(\sigma_i, \sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) = \mathbf{l}_i^T\left[\frac{\partial \mathbf{G}_{2,r}}{\partial s_j}(\sigma_i, \sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) \qquad j = 1, 2$$

- **Tangential sub-system interpolation**
- **(m+1) + 4** moments matched
- **3** columns per shift

Numerical Examples: MIMO RC-Ladder

MIMO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Inputs/Outputs: $\mathbf{u}(t) = \sin(2t) \cdot [1 \;\; 1]^T$

$$\mathbf{y}(t) = [v_1(t) \;\; v_{N-1,N}]^T$$

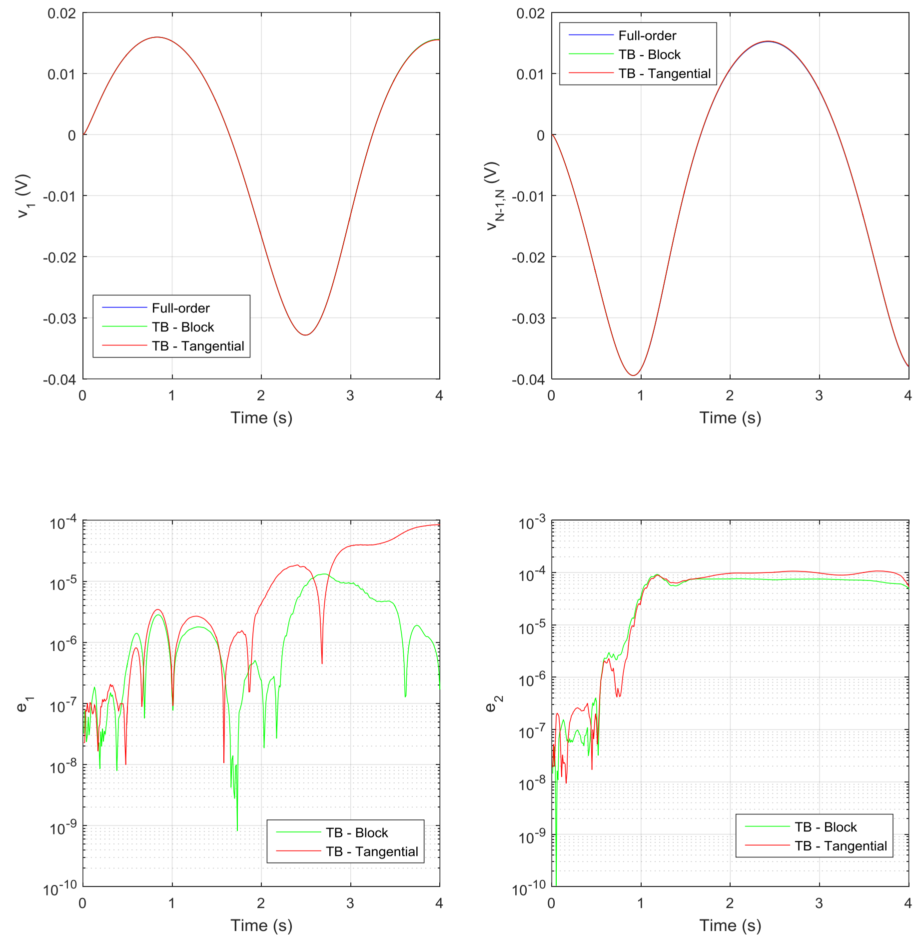
Reduction information:

$n = 800$; Shifts s_0 gotten from IRKA

$t_{\text{sim,orig}} = 17.4$ s

$r_{\text{block}} = 30$ $\qquad$ $r_{\text{tang}} = 21$

$t_{\text{sim,block}} = 0.232$ s $\qquad$ $t_{\text{sim,tang}} = 0.109$ s



Numerical Examples: FitzHugh-Nagumo

$$\epsilon \frac{\partial v}{\partial t}(x,t) = \epsilon^2 \frac{\partial^2 v}{\partial x^2}(x,t) + f(v(x,t)) - w(x,t) + g$$
$$\frac{\partial w}{\partial t}(x,t) = hv(x,t) - \gamma w(x,t) + g$$

Nonlinearity: $f(v) = v(v-0.1)(1-v)$

Inputs: $\mathbf{u}(t) = \begin{bmatrix} 5 \cdot 10^4 t^3 e^{-15t} \\ 1 \end{bmatrix}$

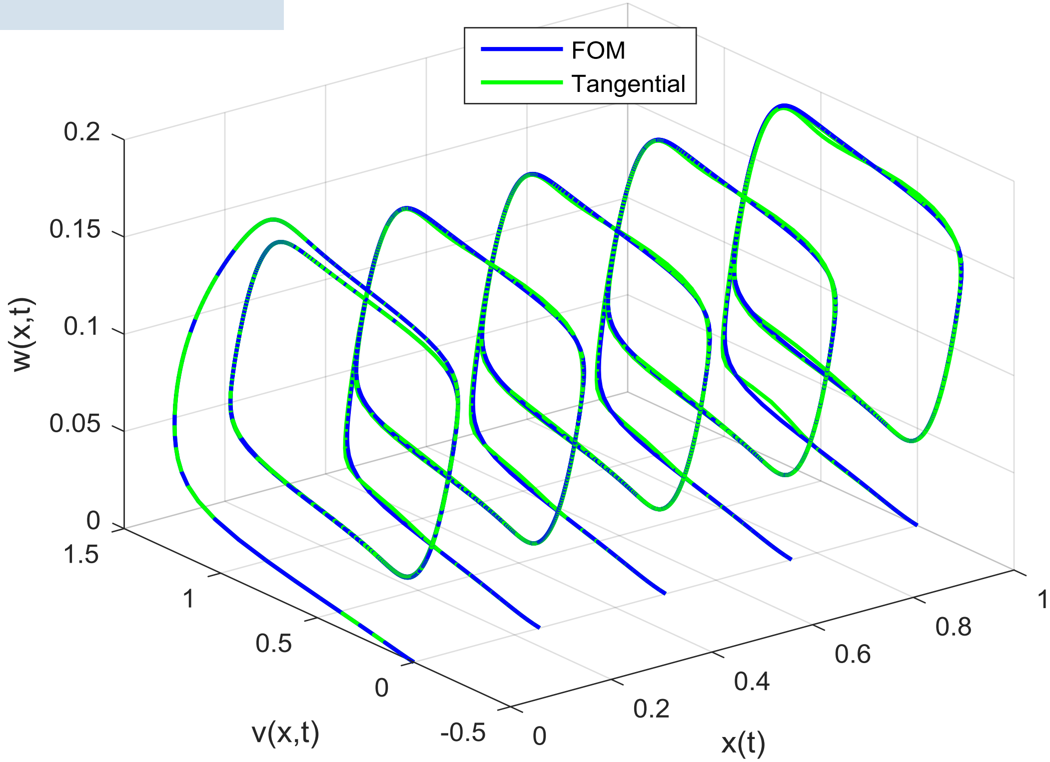
Reduction information:

$n = 1500;$ Shifts s_0 gotten from IRKA

$t_{sim,orig} = 518$ s

$r_{\text{tang}} = 15$

$t_{\text{sim,tang}} = 0.631$ s



Conclusions & Outlook

Summary:

- Many smooth nonlinear systems can be equivalently transformed into QB systems
- Systems theory and Krylov subspaces for SISO QB systems
- Extension of systems theory and Krylov subspaces to **MIMO case**

Conclusions:

- Transfer matrices make Krylov subspace methods more complicated in MIMO case
- **Tangential directions**: good option
- **Choice of shifts and tangential directions** plays an important role

Outlook:

- **Optimal** choice of **shifts** (comparison with T-QB-IRKA)
- **Stability preserving** methods
- Other **benchmark** models

Thank you for your attention!