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On the algorithmic inversion of the discrete Radon transform ☆

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Abstract

The present paper deals with the computational complexity of the discrete inverse problem of reconstructing finite point sets and more general functionals with finite support that are accessible only through some of the values of their discrete Radon transform. It turns out that this task behaves quite differently from its well-studied companion problem involving 1-dimensional X-rays. Concentrating on the case of coordinate hyperplanes in $\mathbb{R}^d$ and on functionals $\psi : \mathbb{Z}^d \to D$ with $D \in \{\{0, 1, \dots, r\}, \mathbb{N}_0\}$ for some arbitrary but fixed r, we show in particular that the problem can be solved in polynomial time if information is available for m such hyperplanes when $m \leqslant d-1$ but is $\mathbb{NP}$-hard for $m = d$ and $D = \{0, 1, \dots, r\}$. However, for $D = \mathbb{N}_0$, a case that is relevant in the context of contingency tables, the problem is still in $\mathbb{P}$. Similar results are given for the task of determining the uniqueness of a given solution and for a related counting problem.

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1. Introduction

The problem of reconstructing finite point sets from the values of their discrete X-ray transform in a few directions has attracted much attention; see the surveys [10], [11] and the monograph [12]. In particular, the computational complexity of the basic

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underlying tasks has been determined completely, showing that the problem can be solved in polynomial time when m, the number of lines parallel to which the images are taken, is 2 but is $\mathbb{NP}$-complete for any set of at least 3 lines in any dimension d at least 2, [8]. There is much less known about the 'companion' problem that involves the discrete Radon transform. Some uniqueness theorems are due to [5] and [19], some tractability results for contingency tables are given in [13] and [20], and some first intractability results for general k-dimensional X-rays are discussed in [6]. The starting point for the present paper is a recent result of [1] that solves a problem raised by the first named author in 1996 showing that checking the consistency of data that encode the values of the Radon transform for the 3 coordinate planes in $\mathbb{R}^3$ is $\mathbb{NP}$-hard. The present paper extends this result to arbitrary dimensions, determines the computational complexity of related uniqueness and counting problems, and also treats the tasks for the more general objects of weighted sets or, equivalently, for functionals $\psi : \mathbb{Z}^d \to D$ of finite support. The most relevant domains D for the weights are $\{0,1\}$, the standard case of finite point sets, and, more generally, $\{0,1,\dots,r\}$ for some fixed positive integer r, $\mathbb{N}_0$, the case corresponding to contingency tables, and $[0,1]$, $[0,r]$ and $[0,\infty[$, the associated linear programming relaxations. (Of course, in order to deal with these problems in the binary Turing machine model we will actually restrict all input and output data to rationals.)

Our results show that the case of Radon transforms is much more involved than that of the standard X-ray problem in that

- It is not only the parameters d and m that determine the computational complexity but also the intersection pattern of the hyperplanes.
- Even in $\mathbb{R}^4$ there are hyperplanes S_1, S_2, S_3 for which the consistency and uniqueness problems are $\mathbb{NP}$-hard over $\{0,1,\dots,r\}$ and $\mathbb{N}_0$, but there are other hyperplanes S_1, S_2, S_3 for which the problems can be solved in polynomial time over these domains.
- The problems can be solved in polynomial time for $m \leqslant d-1$ coordinate hyperplanes over all relevant domains D, are $\mathbb{NP}$-hard for $m = d$ over $\{0,1,\dots,r\}$ but polynomial-time solvable over $\mathbb{N}_0$, $[0,r]$ and $[0,\infty[$.

Section 2 gives all relevant definitions, a brief overview of the known complexity results, and statements of our main results. Section 3 contains proofs for the new intractability results while some tractability results are proved in Section 4.

2. Definitions, preliminaries and main results

Typically, in discrete tomography the objects that have to be reconstructed are finite lattice sets $F \subseteq \mathbb{Z}^d$, see [12]. This is appropriate for most of the applications, particularly for those involving crystalline structures in semiconductor physics that have to be reconstructed from few of their images under high resolution transmission electron microscopy; see [11]. Of course, crystal lattices are more general than $\mathbb{Z}^3$ but due to the affine invariance of the basic tasks the concentration on the standard lattice is no real restriction of generality. Another line of research focuses on the reconstruction of contingency tables from some of their marginal sums, see [13] and [20]. Here the entries of the tables are nonnegative integers. Also linear programming relaxations to 'fuzzy sets' with weights in $[0,1]$ have been studied in the literature; see [4].

Here we utilize a unified approach by considering functions $\psi : \mathbb{Z}^d \to D$, with finite support $\text{supp}(\psi) = \{x \in \mathbb{Z}^d : \ \psi(x) \neq 0\}$, where D is some specific set of nonnegative numbers. The sets D that will be considered are $\{0,1\}$, or more generally $R = \{0,1,2,\dots,r\}$ for some fixed $r \in \mathbb{N}$, $\mathbb{N}_0$ and the corresponding relaxations $[0,1]$, $[0,1] \cap \mathbb{Q}$, $[0,r]$, $[0,r] \cap \mathbb{Q}$, $\mathbb{R}_+$ and $\mathbb{Q}_+$, where $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the nonnegative rationals and reals, respectively. (We will sometimes express results explicitly for $D = \{0,1\}$ and $D = R$ even though the latter case contains the former since it is only required that r is an arbitrary but fixed positive integer allowing the choice $r = 1$. This is done to emphasize that the corresponding result holds for the 'classical case' of finite subsets of $\mathbb{Z}^d$.) The family of these sets D will be denoted by $\mathscr{D}$.

For any such set D let $\mathscr{F}_D^d$ denote the class of all functions $\psi : \mathbb{Z}^d \to D$ with finite support. In order to emphasize the interpretation of a function $\psi \in \mathscr{F}_{\{0,1\}}^d$ as the finite set $\text{supp}(\psi)$ we will sometimes abbreviate $\mathscr{F}_{\{0,1\}}^d$ by $\mathscr{F}^d$, the notation introduced in [8]. The elements of $\mathscr{F}^d$ are then called *lattice sets*. The values $\psi(x)$ of a function $\psi \in \mathscr{F}_D^d$ will in the following be denoted by ψ_x.

For $k, d \in \mathbb{N}$ with $k \leqslant d-1$, let $\mathscr{S}_{k,d}$ be the set of all k-dimensional subspaces in d-dimensional Euclidean space $\mathbb{R}^d$. Let $\mathscr{L}_{k,d}$ denote the subset of $\mathscr{S}_{k,d}$ of those spaces spanned by vectors $v_1, \dots, v_k \in \mathbb{Z}^d \setminus \{0\}$. The elements of $\mathscr{L}_{k,d}$ will be called *lattice spaces*. For $S \in \mathscr{S}_{k,d}$ let $\mathscr{A}(S)$ denote the set of all k-dimensional affine subspaces of $\mathbb{R}^d$ that are parallel to S.

Let $\psi \in \mathscr{F}_D^d$ and $S \in \mathscr{S}_{k,d}$. The (*discrete*) *k-dimensional X-ray of* ψ *parallel to* S is the function $X_S\psi : \mathscr{A}(S) \to [0,\infty[$ defined by

$$X_S\psi(T) = \sum_{x \in T} \psi_x$$

for $T \in \mathscr{A}(S)$.

The mapping that associates with every $S \in \mathscr{S}_{k,d}$ the X-ray $X_S\psi$ is called the discrete k-dimensional *X-ray transform* of ψ. For $k = 1$ it is the standard discrete X-ray transform while for $k = d-1$ it is called discrete *Radon transform* of ψ.

Given a functional $\psi \in \mathscr{F}_D^d$ its ℓ_1-norm

$$\|\psi\|_1 = \sum_{x \in \mathbb{Z}^d} |\psi_x|$$

will be called *total weight*.

The present paper deals with algorithmic issues related to discrete inverse problems associated with the Radon transform. The natural model of computation is the usual *binary Turing machine model*; see [9] and [14]. Of course, all explicitly encoded data must have finite bit length. Hence we will employ rational points only. So, for most purposes we will restrict the domains D to the subset $\mathscr{D}_{\mathbb{Q}}$ of elements of $\mathscr{D}$ that contain only rational points. Also, all spaces S will be restricted to $\bigcup_{k=1}^{d-1} \mathscr{L}_{k,d}$.

Let $S \in \mathscr{L}_{d-1,d}$. In order to define our basic algorithmic problems we need to parameterize $\mathscr{A}(S)$ appropriately. This is most easily done with the aid of a vector $s \in S^{\perp}$

that has minimum Euclidean distance from the origin among all such nonzero vectors t with $(t+S)\cap\mathbb{Z}^d\neq\emptyset$. Then, of course,

$$\mathscr{A}(S)=\{\sigma s+S:\ \sigma\in\mathbb{Z}\}.$$

Hence we can regard $X_S\psi$ as function on $\mathbb{Z}$, and accordingly represent data for the reconstruction task by means of functions $f_S:\mathbb{Z}\to\mathbb{Q}_+$ with finite support, referred to as *data functions*. The graph

$$\{(\sigma,f(\sigma))\in\mathbb{Z}\times\mathbb{Q}_+:\ f(\sigma)\neq 0\}$$

of f_S is called an *X-set*. In the following we will switch freely between the different representations.

Adequate data structures can also be defined for general k-dimensional X-rays. This has been done explicitly in [8] for 1-dimensional X-rays and can be extended analogously to full generality. Since some more technical details are involved and since we want to concentrate here on the case of the Radon transform anyway we refrain from going into details. However, in order to set our results into perspective we introduce the basic algorithmic tasks for general k-dimensional X-rays.

In the following let always $d,k,m\in\mathbb{N}$ with $2\leqslant d,m$ and $k\leqslant d-1$. Further, $D\in\mathscr{D}_{\mathbb{Q}}$, unless stated otherwise. Also let $S_1,\dots,S_m$ be m different elements of $\mathscr{L}_{k,d}$.

Consistency$_{\mathscr{F}_D^d}(S_1,\dots,S_m)$.
Instance: *Data functions* $f_{S_1},\dots,f_{S_m}$.
Question: *Does there exist a* $\psi\in\mathscr{F}_D^d$ such that $X_{S_i}\psi=f_{S_i}$ for $i=1,\dots,m$?

From a practical point of view, it is more relevant to actually reconstruct a solution. Reconstruction$_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ is defined similarly, the input being the same but the question is replaced by the task of constructing a solution if one exists. Clearly, the reconstruction problem cannot be easier than the consistency problem.

Of course, if ψ is a solution for a given instance $\mathscr{I}$ of Consistency$_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ then its total weight $\|\psi\|_1$ equals $\|f_{S_i}\|_1$ for any $i=1,\dots,m$. Hence $\|f_{S_1}\|_1=\cdots=\|f_{S_m}\|_1$ is a necessary condition for consistency, a condition that can be checked efficiently. In the following we may assume that this condition is satisfied; $n=n(\mathscr{I})$ denotes the corresponding cardinality.

Another important algorithmic task involves checking the uniqueness of a solution.

Uniqueness$_{\mathscr{F}_D^d}$ $(S_1,\dots,S_m)$
Instance: *A functional* $\psi\in\mathscr{F}_D^d$.
Question: *Does there exist* $\psi'\in\mathscr{F}_D^d\setminus\{\psi\}$ *such that* $X_{S_i}\psi=X_{S_i}\psi'$ *for* $i=1,\dots,m$?

Note that Uniqueness$_{\mathscr{F}_D^d}$ $(S_1,\dots,S_m)$ checks, in effect, nonuniqueness; but this way of defining the problem puts it into the class $\mathbb{NP}$ for all our relevant domains.

Uniqueness$_{\mathscr{F}_D^d}$ $(S_1,\dots,S_m)$ is a special case of #(Consistency$_{\mathscr{F}_D^d}(S_1,\dots,S_m)$), the counting problem that asks for the number of solutions.

Here is an overview of the results about the computational complexity of the above tasks that are most relevant for our purposes. As pointed out before the best studied case is that of 1-dimensional X-rays. We begin with (some slight extensions) of results of [8].

Theorem 2.1. *Let $S_1,\dots,S_m \in \mathscr{L}_{1,d}$ be m different lattice lines. If $m \geqslant 3$ then the problems* $\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ *and* $\text{Uniqueness}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ *are* $\mathbb{NP}$*-complete while* $\text{Reconstruction}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ *is* $\mathbb{NP}$*-hard in the strong sense for $D=\{0,1\}$, $D=R$ and $D=\mathbb{N}_0$. All three problems are solvable in strongly polynomial time for $D=\mathbb{Q}\cap[0,1]$, $D=\mathbb{Q}\cap[0,r]$ and $D=\mathbb{Q}_+$.* Further, $\#(\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m))$ *is* $\#\mathbb{P}$*-complete for $D=\{0,1\}$, $D=R$ and $D=\mathbb{N}_0$.*

Proof. The results were proved for $D=\{0,1\}$ already in [8, Theorem 3.7] for consistency, in [8, Theorem 4.3] for uniqueness and in [8, Corollary 3.8] for the counting problem. In fact, in [8] it was actually shown, that the statements remain true, even for instances where two data functions f_i take values only in $\{0,1\}$. This observation immediately implies the intractability results for $D=R$ and $D=\mathbb{N}_0$ because such instances permit only solutions ψ with values in $\{0,1\}$.

In the other cases consistency, reconstruction and uniqueness can be reduced to linear programming. In fact, the linear feasibility problem modeling $\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ and $\text{Reconstruction}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ is of the form

$$\begin{aligned}\sum_{x\in T\cap\mathbb{Z}^d} \psi_x &= f_{S_i}(T), & i=1,\dots,m;\ T\in\mathscr{A}(S_i),\\ \psi_x &\in D, & x\in\mathbb{Z}^d.\end{aligned}$$

Note that ψ_x is zero for all but polynomially many $x\in\mathbb{Z}^d$, hence the coefficient matrix is a 0-1-matrix of size bounded by a polynomial in the input.

Given $\psi\in\mathscr{F}_D^d$ specifying an instance of $\text{Uniqueness}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ one can decide for each element x_0 of supp(ψ) whether there exists another solution ψ' with $\psi'_{x_0}\neq\psi_{x_0}$ by solving the linear programs

$$\begin{aligned}&\min\ \psi'_{x_0}\\ \sum_{x\in T\cap\mathbb{Z}^d} \psi'_x &= f_{S_i}(T), & i=1,\dots,m;\ T\in\mathscr{A}(S_i),\\ \psi'_x &\in D, & x\in\mathbb{Z}^d\end{aligned}$$

and

$$\begin{aligned}&\max\ \psi'_{x_0}\\ \sum_{x\in T\cap\mathbb{Z}^d} \psi'_x &= f_{S_i}(T), & i=1,\dots,m;\ T\in\mathscr{A}(S_i),\\ \psi'_x &\in D, & x\in\mathbb{Z}^d.\end{aligned}$$

The tractability assertion follows now with the aid of [18]. □

Of course, the problem $\#(\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m))$ is essentially meaningless for $D\in\{\mathbb{Q}\cap[0,r],\mathbb{Q}_+\}$ because a given instance permits either no, one or infinitely many solutions, since the set of solutions over the closure cl$(\mathscr{F}_D)$ is convex.

The case of higher dimensional X-rays has not been studied as comprehensively as that of 1-dimensional X-rays. Of course, some additional features occur that make the problems more difficult to handle. For instance, not even the case $m=2$ or the membership in $\mathbb{NP}$ extends without further assumptions. For instance, since the intersection of two 2-dimensional lattice subspaces in $\mathbb{R}^3$ is a 1-dimensional lattice subspace it contains an infinite number of lattice points. So, the number of points that belong to a solution need not necessarily be bounded by a polynomial in the bit length of the data functions f_{S_i}. This is no problem for $D=\mathbb{N}_0$ but for bounded D the following general assumption is needed.

Assumption 2.2. Let D be R or $[0,r]\cap\mathbb{Q}$. Then for any instance $\mathscr{I}$ the cardinality $n(\mathscr{I})$ is bounded by a polynomial in the bit length of the data functions.

Remark 2.3. The problem $\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ belongs to $\mathbb{NP}$.

Proof. Checking whether a guessed solution ψ is actually a solution for a given instance can easily be done in time that is polynomial in the input size of $\mathscr{I}$ and the size of ψ. Hence it suffices to show that in case of consistency there exists a solution whose size is polynomial in the size of $\mathscr{I}$. This is clear if $S=\bigcap_{i=1}^m S_i=0$. So, suppose that S contains a line. If $D=\mathbb{N}$ and $D=\mathbb{Q}_+$ then in case of consistency there always exists a solution ψ such that $\text{supp}(\psi)\cap T$ is empty or a singleton for every translate T of S. For $D=R$ and $D=[0,r]\cap\mathbb{Q}$ the existence of a polynomial witness follows from Assumption 2.2, in the latter case with the aid of the fact that in case of consistency the corresponding linear program admits a *basic feasible solution* whose size is bounded by a polynomial in the input length. □

It is not too difficult to see that under the Assumption 2.2 the case $m=2$ is always simple.

Theorem 2.4. *Let $S_1,S_2\in\mathscr{L}_{k,d}$. Then all three problems* $\text{Consistency}_{\mathscr{F}_D^d}(S_1,S_2)$, $\text{Reconstruction}_{\mathscr{F}_D^d}(S_1,S_2)$ *and* $\text{Uniqueness}_{\mathscr{F}_D^d}(S_1,S_2)$ *can be solved in polynomial time for $D\in\mathscr{D}_{\mathbb{Q}}$.*

Proof. Proofs for the planar case can be found in [3], [17], and [16], and the general result for $r=1$ is contained in [7]. For $r>1$ it follows by a simple network-flow argument similar to that of [17]. □

It is also clear that the hardness results of Theorem 2.1 for the case of 1-dimensional X-rays can be used to obtain some hardness results for larger k. The following theorem is taken from [6].

Theorem 2.5 (Gardner and Gritzmann [6, Theorem 4.5.2]). *Let $S_1,\dots,S_m\in\mathscr{L}_{k,d}$ be m different spaces, and let $D=R$ or $D=\mathbb{N}_0$. Further, let $m\geqslant 3$ and* $\dim(\bigcap_{i=1}^m S_i)=k-1$ *or let $m\geqslant 4$ and $k=2$. Then* $\text{Consistency}_{\mathscr{F}_D^d}(S_1,\dots,S_m)$ *is $\mathbb{NP}$-complete.*

The first more intrinsically 2-dimensional transformation is constructed in [1] to determine the computational complexity of $\textsc{Consistency}_{\mathscr{F}^3}(S_1^3, S_2^3, S_3^3)$ for the standard coordinate planes S_1^3, S_2^3, S_3^3 in $\mathbb{R}^3$.

Theorem 2.6 (Brunetti et al. [1, Theorem 3.3]). $\textsc{Consistency}_{\mathscr{F}^3}$ (S_1^3, S_2^3, S_3^3) *is* $\mathbb{NP}$-*complete*.

This theorem was the starting point for the present paper. We will largely generalize and extend it and give results that show that even though there is an obvious geometric duality between lines and hyperplanes the case of Radon transforms is much more involved than that of the standard X-ray problems in that various phenomena occur for $k = d - 1$ that do not occur for $k = 1$.

To be able to state our results precisely we need some additional notation. In order not to overload the paper with too many technicalities we will state our main results for coordinate hyperplanes and one additional 'diagonal' hyperplane. It should of course be clear that they extend to much more generality.

Let $e_1^d, \ldots, e_d^d$ be the standard unit vectors of $\mathbb{R}^d$, let $e_{d+1}^d = (1/d)\sum_{i=1}^d e_i^d$, and let $\mathbf{1}$ be the 'all-ones' vector de_{d+1}^d. The coordinate hyperplanes are denoted by $S_1^d, \ldots, S_d^d$ i.e.,

$$S_i^d = \{x \in \mathbb{R}^d:\ x^\top e_i^d = 0\}, \quad i = 1, \ldots, d.$$

Further let

$$S_{d+1}^d = \{x \in \mathbb{R}^d:\ x^\top e_{d+1}^d = 0\}.$$

The subfamily of $\mathscr{S}_{d-1,d}$ that is most relevant for the purpose of the present paper is then

$$\mathscr{H}_d = \{S_i^d:\ i = 1, \ldots, d+1\}.$$

Since our problems will from now on be restricted to subsets of $\mathscr{H}_d$ we will use the abbreviation $\textsc{Consistency}_{\mathscr{F}_D^d}(d, m)$ for $\textsc{Consistency}_{\mathscr{F}_D^d}(S_1^d, \ldots, S_m^d)$; the problems $\textsc{Uniqueness}_{\mathscr{F}_D^d}(d, m)$ and $\#(\textsc{Consistency}_{\mathscr{F}_D^d}(d, m))$ are defined accordingly.

Note that for any given instance $\mathscr{I} = (f_1, \ldots, f_m)$ of $\textsc{Consistency}_{\mathscr{F}_D^d}(d, m)$ all solutions are contained in the (generally infinite) lattice set

$$G = G(\mathscr{I}) = \mathbb{Z}^d \cap \bigcap_{i=1}^m (\operatorname{supp}(f_i)e_i^d + S_i),$$

called the *grid* associated with $\mathscr{I}$. Of course, the grid G can be finitely represented and computed in polynomial time from the sets $f_1, \ldots, f_m$ by solving systems of linear equations.

Now we are ready to state our main results. The first is a far reaching intractability result for the algorithmic inversion of the discrete Radon transformation that generalizes Theorem 2.6.

Theorem 2.7. *Let* $d = m \geqslant 3$ *and* $D = \{0,1\}$ *or, more generally,* $D = R$. *Then* $\textsc{Consistency}_{\mathscr{F}_D^d}(d,m)$ *and* $\textsc{Uniqueness}_{\mathscr{F}_D^d}(d,m)$ *are* $\mathbb{NP}$*-complete in the strong sense while* $\#(\textsc{Consistency}_{\mathscr{F}_D^d}(d,m))$ *is* $\#\mathbb{P}$*-complete. For the domains* $D \in \{[0,r] \cap \mathbb{Q}, \mathbb{Q}_+\}$, *the problems* $\textsc{Consistency}_{\mathscr{F}_D^d}(d,m)$ *and* $\textsc{Uniqueness}_{\mathscr{F}_D^d}(d,m)$ *can be solved in strongly polynomial time.*

Let us note in passing that the part of Theorem 2.7 dealing with uniqueness shows in particular that, unless $\mathbb{P} = \mathbb{NP}$, the uniqueness criterion for $d = m = 3$ of [19] cannot be checked efficiently.

On the other hand, Wiegelmann [20] has given the following tractability result that is based on an algebraic study of the underlying toric ideals; it generalizes a result of [13].

Theorem 2.8 (Wiegelmann [20, Section 4.2.2]). *Let* $d = m \geqslant 2$. *Then* $\textsc{Consistency}_{\mathscr{F}_{\mathbb{N}_0}^d}(d,m)$ *and* $\textsc{Uniqueness}_{\mathscr{F}_{\mathbb{N}_0}^d}(d,m)$ *can be solved in polynomial time.*

In Section 4 we will give an elementary proof and simple and fast algorithm for a slight generalization of the former tractability result that shows that whenever $d \geqslant m \geqslant 2$ there is a solution for a given instance $\mathscr{I} = (f_1, \ldots, f_m)$ of $\textsc{Consistency}_{\mathscr{F}_{\mathbb{N}_0}^d}(d,m)$ if and only if $\|f_1\|_1 = \cdots = \|f_m\|_1$. This simple condition is in striking contrast to the more involved characterization of [15] and [16] of the feasibility of instances of $\textsc{Consistency}_{\mathscr{F}_{\{0,1\}}^2}(2,2)$. As another consequence of Theorems 2.7 and 2.8 we see that the complexities for problems over $\{0,1\}$ and $\mathbb{N}_0$ may differ dramatically, another feature that does not occur in the case of 1-dimensional X-rays. On the positive side, we obtain a simple tractability result for $D = R$.

Theorem 2.9. *Let* $d \in \mathbb{N}$ *with* $d > m \geqslant 2$. *Then there is a polynomial time algorithm for* $\textsc{Consistency}_{\mathscr{F}_R^d}(d,m)$ *and* $\textsc{Reconstruction}_{\mathscr{F}_R^d}(d,m)$.

In particular Theorem 2.9 shows that there is no fixed number m_0, independent of d, such that our basic problem $\textsc{Consistency}_{\mathscr{F}^d}(d,m)$ becomes intractable whenever $m \geqslant m_0$. In conjunction with Theorem 2.5 this shows that unlike in the case of 1-dimensional X-rays, the computational complexity of problems involving the Radon transform does not depend on the parameters d and m alone but also on the intersection pattern of the hyperplanes that are involved.

3. Intractability results

By a modification of the approach of [1] for $\textsc{Consistency}_{\mathscr{F}^3}(3,3)$, we will show in this section that for $D = R$ and $d \geqslant 3$ the problems $\textsc{Consistency}_{\mathscr{F}_D^d}(d,d)$ and $\textsc{Uniqueness}_{\mathscr{F}_D^d}(d,d)$ are $\mathbb{NP}$-complete and $\#(\textsc{Consistency}_{\mathscr{F}_D^d}(d,d))$ is $\#\mathbb{P}$-complete. In [1] a reduction is given from a particularly restricted consistency problem, that was shown to be $\mathbb{NP}$-complete in [2]. The construction fails, however, to show the $\mathbb{NP}$-

hardness of the uniqueness problem and the #ℙ-hardness of the counting problem. Here we construct a transformation from the unrestricted 1-dimensional X-ray problem that allows us to invoke Theorem 2.1 and determine the complexity not only in the general case of $\text{CONSISTENCY}_{\mathscr{F}_D^d}(d,d)$ but also for $\text{UNIQUENESS}_{\mathscr{F}_D^d}(d,d)$ and $\#(\text{CONSISTENCY}_{\mathscr{F}_D^d}(d,d))$ at the same time. Together with the results of the next section this will in particular fully characterize the computational complexity for planes orthogonal to the coordinate axes in arbitrary dimension $d \geqslant 3$. The classical counting problem $\#(\text{CONSISTENCY}_{\mathscr{F}^2}(2,2))$ for two 1-dimensional X-rays remains, however, open.

Let $d \geqslant 3$. For $i=1,\dots,d$ let $S_i' = S_i^{d-1}$, and $S_i = S_i^d$. Of course, $S_1,\dots,S_d$ are hyperplanes of $\mathbb{R}^d$ while $S_1',\dots,S_d'$ will be regarded as hyperplanes of $\mathbb{R}^{d-1}$. Whenever useful we will identify $\mathbb{R}^{d-1}$ with the hyperplane of $\mathbb{R}^d$ of those vectors whose last coordinate is 0.

For our complexity results we will use transformations whose validity can be most easily seen with the aid of a geometric argument which relies on the notion of the barycenter c_ψ of ψ. As usual, the *barycenter* of a function $\psi \in \mathscr{F}_{\mathbb{R}_+}^d$ that is not identically 0 is defined by

$$c_\psi = \frac{1}{\|\psi\|_1} \sum_{x \in \operatorname{supp}(\psi)} \psi_x x.$$

Next we see how information about the barycenter of ψ can be derived from its X-rays parallel to a hyperplane S.

Lemma 3.1. *Let S be a hyperplane and let v be a normal to S. If $\psi \in \mathscr{F}_{\mathbb{R}_+}^d$ then*

$$v^\top c_\psi = \frac{\sum_{y \in S^\perp} X_S\psi(y+S)\, v^\top y}{\sum_{T \in \mathscr{A}(S)} X_S\psi(T)}.$$

Proof. Let $U = \operatorname{supp}(\psi)$. First, note that

$$\|\psi\|_1 = \sum_{x \in U} \psi_x = \sum_{T \in \mathscr{A}(S)} \sum_{x \in T \cap U} \psi_x = \sum_{T \in \mathscr{A}(S)} X_S\psi(T).$$

Second,

$$v^\top c_\psi = v^\top \left(\frac{1}{\|\psi\|_1} \sum_{x \in U} \psi_x x \right) = \frac{1}{\|\psi_1\|} \sum_{x \in U} \psi_x v^\top x$$

$$= \frac{1}{\|\psi\|_1} \sum_{T \in \mathscr{A}(S)} \sum_{x \in T \cap U} \psi_x v^\top x = \frac{1}{\|\psi\|_1} \sum_{y \in S^\perp} X_S\, \psi(y+S)\, v^\top y,$$

which concludes the proof. □

Lemma 3.1 implies that the barycenter c_ψ of a function $\psi \in \mathscr{F}_{\mathbb{R}_+}^d$ is uniquely determined by the X-rays parallel to $S_1,\dots,S_d$. In the sequel we only need the weaker statements that $\mathbf{1}^\top c_\psi$ can be computed in polynomial time from $X_{S_i}\psi$, $i=1,\dots,d$ and also from $X_{S_{d+1}}\psi$ alone.

In the following let Z always be a finite subset of $\mathbb{N}_0^d$, let $n\in\mathbb{N}$, let $r\in\mathbb{N}$, and denote by $\mathscr{F}(Z,n,r)$ the set of all $\psi\in\mathscr{F}^d_{[0,r]}$ with $\operatorname{supp}(\psi)\subset Z$ and of total weight n. We will now characterize those elements ψ^* of $\mathscr{F}(Z,n,r)$ which minimize the functional $\gamma:\mathscr{F}(Z,n,r)\to\mathbb{R}$ defined by $\gamma(\psi)=\mathbf{1}^\top c_\psi$.

For $\kappa\in\mathbb{N}$ set

$$P_{Z,\kappa}=\{x\in Z:\ \mathbf{1}^\top x<\kappa\},\qquad H_{Z,\kappa}=\{x\in Z:\ \mathbf{1}^\top x=\kappa\},$$

and

$$\mathscr{F}_\kappa(Z,n,r)=\{\psi\in\mathscr{F}(Z,n,r):\ P_{Z,\kappa}\subset\operatorname{supp}(\psi)\subset P_{Z,\kappa}\cup H_{Z,\kappa}\wedge\psi|_{P_{Z,\kappa}}\equiv r\},$$

where $\psi|_{P_{Z,\kappa}}$ denotes the restriction of ψ to $P_{Z,\kappa}$.

Lemma 3.2. *Let $\kappa,n\in\mathbb{N}$ be such that $\mathscr{F}_\kappa(Z,n,r)\neq\emptyset$. Then a function $\psi\in\mathscr{F}(Z,n,r)$ minimizes the functional γ if and only if $\psi\in\mathscr{F}_\kappa(Z,n,r)$.*

Proof. Let $\psi\in\mathscr{F}(Z,n,r)$, $\psi^*\in\mathscr{F}_\kappa(Z,n,r)$, $U=\operatorname{supp}(\psi)$ and $U^*=\operatorname{supp}(\psi^*)$. Note that

$$\mathbf{1}^\top c_\psi=\mathbf{1}^\top c_{\psi^*}+\frac{1}{\|\psi^*\|_1}\left(\sum_{x\in U}\psi_x\mathbf{1}^\top x-\sum_{x\in U^*}\psi^*_x\mathbf{1}^\top x\right),$$

hence for the ‘if’-assertion it suffices to show that

$$\left(\sum_{x\in U}\psi_x\mathbf{1}^\top x-\sum_{x\in U^*}\psi^*_x\mathbf{1}^\top x\right)\geqslant 0.$$

With the abbreviation P for $P_{Z,\kappa}$ we have

$$\begin{aligned}&\sum_{x\in U}\psi_x\mathbf{1}^\top x-\sum_{x\in U^*}\psi^*_x\mathbf{1}^\top x\\&\quad=\left(\sum_{x\in U\cap P}\psi_x\mathbf{1}^\top x-\sum_{x\in P}r\cdot\mathbf{1}^\top x\right)+\left(\sum_{x\in U\setminus P}\psi_x\mathbf{1}^\top x-\kappa\sum_{x\in U^*\setminus P}\psi^*_x\right)\\&\quad=\left(\sum_{x\in U\cap P}(\psi_x-r)\mathbf{1}^\top x-\sum_{x\in P\setminus U}r\cdot\mathbf{1}^\top x\right)+\left(\sum_{x\in U\setminus P}\psi_x\mathbf{1}^\top x-\kappa\sum_{x\in U^*\setminus P}\psi^*_x\right).\end{aligned}$$

Since $\psi_x-r\leqslant 0$, $\mathbf{1}^\top x\leqslant\kappa-1$ for $x\in P$ and $\mathbf{1}^\top x\geqslant\kappa$ for $x\in U\setminus P$ we have

$$\begin{aligned}&\sum_{x\in U}\psi_x\mathbf{1}^\top x-\sum_{x\in U^*}\psi^*_x\mathbf{1}^\top x\\&\quad\geqslant(\kappa-1)\left(\left(\sum_{x\in U\cap P}\psi_x\right)-r|P|\right)+\kappa\left(\sum_{x\in U\setminus P}\psi_x-\sum_{x\in U^*\setminus P}\psi^*_x\right).\end{aligned}$$

Now, $r|P| + \sum_{x\in U^*\setminus P} \psi^*_x = \|\psi^*\|_1$. Hence

$$\sum_{x\in U} \psi_x \mathbf{1}^\top x - \sum_{x\in U^*} \psi^*_x \mathbf{1}^\top x \geqslant \kappa \left(\sum_{x\in U} \psi_x - \|\psi^*\|_1 \right) = 0,$$

concluding the proof of the 'if'-part of the assertion. As to the 'only if'-part observe that equality holds in the above computation only if

$$r\cdot|P| = \sum_{x\in U\cap P} \psi_x \quad \text{and} \quad \sum_{x\in U\setminus P} \psi_x \mathbf{1}^\top x = \kappa \sum_{x\in U\setminus P} \psi_x,$$

which implies $\psi \in \mathscr{F}_\kappa(Z,n,r)$. □

In the following we describe a lifting procedure that constructs from a feasible instance of $\textsc{Consistency}_{\mathscr{F}^{d-1}_{[0,r]}}(S'_1,S'_2,\ldots,S'_d)$ an equivalent instance of $\textsc{Consistency}_{\mathscr{F}^{d}_{[0,r]}}(S_1,S_2,\ldots,S_d)$.

So let $\mathscr{I}' = (f'_1,\ldots,f'_d)$ be an instance of $\textsc{Consistency}_{\mathscr{F}^{d-1}_{[0,r]}}(S'_1,S'_2,\ldots,S'_d)$. The construction will be described in terms of the corresponding X-sets; let them be of the form

$$X'_i = \{(j,\beta_{i,j}):\ j\in U'_i\} \quad \text{for } i=1,\ldots,d,$$

where $U'_i = \operatorname{supp}(f'_i)$ for $i=1,\ldots,d$. As the algorithmic problems are invariant under translation, we may assume that $0 = \min U'_i$ for $i=1,\ldots,d-1$. This is to say that the grid $G(\mathscr{I}')$ is contained in $\mathbb{N}_0^{d-1}$, and $0\in G(\mathscr{I}')$. Let $\alpha = \max U'_d$, set

$$P = \{x = (\xi_1,\ldots,\xi_d)^\top \in U'_1\times U'_2\times\cdots\times U'_{d-1}\times\mathbb{Z}:\ \mathbf{1}^\top x\in U'_d\setminus\{\alpha\}\}.$$

Now we define an instance $\mathscr{I} = (f_1,\ldots,f_d)$ of $\textsc{Consistency}_{\mathscr{F}^{d}_{[0,r]}}(S_1,S_2,\ldots,S_d)$ by specifying its X-sets as follows:

$$X_i = \{(j,\beta_{i,j} + r\cdot|P\cap(je^d_i + S^d_i)|):\ j\in U'_i\} \quad \text{for } i=1,\ldots,d-1,$$

$$X_d = \{((\alpha-j),\beta_{d,j} + r\cdot|P\cap((\alpha-j)e^d_d + S^d_d)|):\ j\in U'_d\}.$$

Observe that the grids of $\mathscr{I}'$ and $\mathscr{I}$ are related by

$$G(\mathscr{I}) = \{x + je^d_d:\ x\in G(\mathscr{I}')\wedge(\alpha-j)\in U'_d\}.$$

Further, with the notation introduced before Lemma 3.2,

$$P = P_{G(\mathscr{I}),\alpha} \quad \text{and} \quad \Pi_{S_d} H_{G(\mathscr{I}),\alpha} = G(\mathscr{I}'),$$

where Π_{S_d} denotes the orthogonal projection on S_d.

Theorem 3.3. *Let $\mathscr{I}'$ be a feasible instance of* $\textsc{Consistency}_{\mathscr{F}^{d-1}_{[0,r]}}(S'_1,S'_2,\ldots,S'_d)$, *and let $\mathscr{I}$ be the corresponding instance of* $\textsc{Consistency}_{\mathscr{F}^{d}_{[0,r]}}(S_1,S_2,\ldots,S_d)$ *constructed above. Then the solutions in $\mathscr{F}^{d-1}_{[0,r]}$ of $\mathscr{I}'$ are in one-to-one correspondence with the solutions* *in* $\mathscr{F}^d_{[0,r]}$ *of* $\mathscr{I}$*. The bijectivity persists even when the ranges of the functionals are restricted to* $[0,r]\cap\mathbb{Q}$ *or* R*.*

Proof. Let $\psi'\in\mathscr{F}^{d-1}_{[0,r]}$ be a solution for $\mathscr{I}'$. We construct a solution $\psi\in\mathscr{F}^d_{[0,r]}$ for $\mathscr{I}$ as follows. Let

$$F=\{x+(\alpha-\mathbf{1}^\top x)e^d_d:\ x\in\operatorname{supp}(\psi')\}$$

and

$$\psi_x=\begin{cases} r & \text{for } x\in P_{G(\mathscr{I}),\alpha},\\ \psi'_{\Pi_{S_d}(x)} & \text{for } x\in F,\\ 0 & \text{else.}\end{cases}$$

This construction can easily be reversed for solutions

$$\psi\in\mathscr{F}_\alpha(G(\mathscr{I}),\|\psi\|_1,r),$$

simply by taking

$$\psi'_x=\psi_{x+(\alpha-\mathbf{1}^\top x)e^d_d}\quad\text{for all } x\in G(\mathscr{I}').$$

It follows from the explicit construction above that the solutions $\psi'\in\mathscr{F}^{d-1}_{[0,r]}$ of $\mathscr{I}'$ are in one-to-one correspondence with those solutions $\psi\in\mathscr{F}^d_{[0,r]}$ for $\mathscr{I}$ that belong to $\mathscr{F}_\alpha(G(\mathscr{I}),\|\psi\|_1,r)$. Hence all that remains to be shown is that all solutions of $\mathscr{I}$ are of this kind.

So, let $\psi\in\mathscr{F}^d_{[0,r]}$ be a solution for $\mathscr{I}$. By our assumption there exists a solution $\psi'\in\mathscr{F}^{d-1}_{[0,r]}$ for $\mathscr{I}'$. Let $\psi^*\in\mathscr{F}_\alpha(G(\mathscr{I}),\|\psi\|_1,r)$ be the corresponding solution for $\mathscr{I}$, in $\mathscr{F}^d_{[0,r]}$. By Lemma 3.1,

$$\mathbf{1}^\top c_\psi=\mathbf{1}^\top c_{\psi^*},$$

thus Lemma 3.2 implies

$$\psi\in\mathscr{F}_\alpha(G(\mathscr{I}),\|\psi^*\|_1,r),$$

concluding the proof of the first assertion.

The last two claims of the theorem follow by observing, that under the bijection the values of the functions are not changed. Therefore, integers correspond to integers and rationals to rationals. □

The next result is needed for an induction step in the proof of Theorem 2.7.

Remark 3.4. Using the notation introduced before Theorem 3.3, let $\mathscr{I}'$ be a feasible instance of $\text{Consistency}_{\mathscr{F}^{d-1}_{[0,r]}}(S'_1,S'_2,\ldots,S'_d)$ and let $\mathscr{I}=(f_1,\ldots,f_d)$ be the corresponding instance of $\text{Consistency}_{\mathscr{F}^{d}_{[0,r]}}(S_1,S_2,\ldots,S_d)$. Further, we define an additional data function f_{d+1} corresponding to the hyperplane S^d_{d+1} orthogonal to e^d_{d+1} by

$$f_{d+1}(\sigma) = \begin{cases} r \cdot |P \cap (\sigma e^d_{d+1} + S^d_{d+1})| & \text{if } \sigma \neq \alpha, \\ n(\mathscr{I}') & \text{if } \sigma = \alpha. \end{cases}$$

Then $(f_1, \dots, f_d, f_{d+1})$ is an instance $\hat{\mathscr{I}}$ of $\textsc{Consistency}_{\mathscr{F}^d_{[0,r]}}(S_1, S_2, \dots, S_d, S^d_{d+1})$. The sets of solutions of $\mathscr{I}$ and $\hat{\mathscr{I}}$ are identical. This fact persists even when the ranges of the functionals are restricted to $[0,r] \cap \mathbb{Q}$ or R.

Now we can prove Theorem 2.7, our main intractability result. Note that for $D \in \{[0,r] \cap \mathbb{Q}, \mathbb{Q}_+\}$ consistency, reconstruction and uniqueness can be reduced to linear programming with matrices and objective function with values only 0 and 1. From [18] it follows then, that they are solvable in strongly polynomial time.

So let $D = R$ now. Membership of the problems in the relevant complexity classes is trivial.

The hardness assertions for $d = 3$ follow from the corresponding results for (S^2_1, S^2_2, S^2_3) of Theorem 2.1. In fact, since the feasibility of a given instance over $[0,r]$ can be verified in strongly polynomial time the intractability results of Theorems 2.1 persists if we restrict the set of instances to those that are feasible over $[0,r]$. Hence the assertion follows directly from Theorem 3.3.

For $d > 3$ we proceed by induction. Suppose we know already that $\textsc{Consistency}_{\mathscr{F}^{d-1}_R}(d-1, d-1)$ and $\textsc{Uniqueness}_{\mathscr{F}^{d-1}_R}(d-1, d-1)$ are $\mathbb{NP}$-complete in the strong sense, and that $\#(\textsc{Consistency}_{\mathscr{F}^{d-1}_R}(d-1, d-1))$ is $\#\mathbb{P}$-complete. By Remark 3.4 $\textsc{Consistency}_{\mathscr{F}^{d-1}_R}(d-1, d)$ and $\textsc{Uniqueness}_{\mathscr{F}^{d-1}_R}(d-1, d)$ are $\mathbb{NP}$-complete in the strong sense, and the counting problem $\#(\textsc{Consistency}_{\mathscr{F}^{d-1}_R}(d-1, d))$ is $\#\mathbb{P}$-complete. The assertion follows then again with the aid of Theorem 3.3.

4. Tractability results

We begin by proving a lemma characterizing consistency.

Lemma 4.1. *Let $d \geqslant m \geqslant 2$ then there is a solution for a given instance $\mathscr{I} = (f_1, \dots, f_m)$ of* $\textsc{Consistency}_{\mathscr{F}^d_{\mathbb{N}_0}}(d, m)$ *if and only if $\|f_1\|_1 = \cdots = \|f_m\|_1$. If $m \leqslant R$, the assertion persists even for $D = \{0, 1\}$.*

Proof. Let $\mathscr{I} = (f_1, \dots, f_m)$ be an instance of $\textsc{Consistency}_{\mathscr{F}^d_{\mathbb{N}_0}}(d, m)$. All we need to show is that the condition $\|f_1\|_1 = \cdots = \|f_m\|_1$ is sufficient. This can be done by induction over $n = n(\mathscr{I})$. Of course, for $n = 0$ there is nothing to show. So suppose $n \geqslant 1$. For every index $i = 1, \dots, m$ we choose a point $\sigma_i \in \operatorname{supp}(f_i)$ and set

$$f'_i(\sigma) = \begin{cases} f_i(\sigma) - 1 & \text{for } \sigma = \sigma_i, \\ f_i(\sigma) & \text{for } \sigma \neq \sigma_i. \end{cases}$$

Then $\mathscr{I}' = (f_1', \dots, f_m')$ is an instance of $\text{Consistency}_{\mathscr{F}^d_{\mathbb{N}_0}}(d,m)$ with $n(\mathscr{I}') = n-1$. Since $\|f_1'\|_1 = \dots = \|f_m'\|_1$, the instance $\mathscr{I}$ is feasible by the induction hypothesis; let ψ' be a solution. Further, $T = \bigcap_{i=1}^m (\sigma_i e_i^d + S_i^d) \neq \emptyset$, hence we can raise the value of ψ' on one point in T to obtain a solution for $\mathscr{I}$. This implies the first assertion.

Turning to the second, let $\mathscr{I}$ be an instance of $\text{Consistency}_{\mathscr{F}^d_{\{0,1\}}}(S_1, \dots, S_m)$ satisfying the condition. We regard $\mathscr{I}$ as an instance of $\text{Consistency}_{\mathscr{F}^d_{\mathbb{N}_0}}(S_1, \dots, S_m)$, and apply the first part of Lemma 4.1 to obtain a solution ψ. Note that since now $m \leqslant d-1$,

$$S = \bigcap_{i=1}^m S_i$$

is at least 1-dimensional. Let T be any translate of S that meets $\text{supp}(\psi)$. If we replace $\psi|_T$ by an arbitrary functional $\phi_T : T \cap \mathbb{Z}^d \to \{0,1\}$ with support of cardinality $\sum_{x \in T \cap \mathbb{Z}^d} \psi_x$ and define a new functional ψ' by $\psi_x = \phi_T(x)$ for $x \in T$ and $T \in \mathscr{A}(S)$ then we obtain a solution for $\mathscr{I}$ as an instance of $\text{Consistency}_{\mathscr{F}^d_{\{0,1\}}}(S_1, \dots, S_m)$. □

Next, Algorithm 1 is a polynomial-time method proving Theorem 2.8 for the general case of $d \geqslant m \geqslant 2$.

Algorithm 1. *Reconstruction*

Initialize: For $i = 1, \dots, m$, let supp $(f_i) = \{\sigma_i^{(j)} : j = 1, \dots, n_i\}$ and $\tau_i^{(j_i)} \leftarrow f_i(\sigma_i^{(j)})$
while $(\tau_i^{(1)}, \dots, \tau_i^{(n_i)})^\top \neq 0$ for all $i = 1, \dots, m$ **do**
  for $i = 1$ to m **do**
    $j_i \leftarrow \arg\min\{\tau_i^{(j)} : j = 1, \dots, n_i \wedge \tau_i^{(j)} \neq 0\}$
  end for
  $i_0 \leftarrow \arg\min\{\tau_i^{(j_i)} : i = 1, \dots, m\}$
  $y_0 \leftarrow (\sigma_1^{(j_1)}, \sigma_2^{(j_2)}, \dots, \sigma_m^{(j_m)}, 0_{d-m}^\top)^\top$
  $\psi_{y_0} \leftarrow \tau_{i_0}^{(j_0)}$
  for $i = 1$ to m **do**
    $\tau_i^{(j_i)} \leftarrow \tau_i^{(j_i)} - \psi_{y_0}$
  end for
end while
if $\tau_i^{(j)} \neq 0$ for some index i **then**
  report 'infeasible'
else
  output ψ
end if

Note, that each iteration step can be performed in polynomial time. Further, in each step at least one of the $\tau_i^{(j)}$ turns 0, hence the algorithm runs in strongly polynomial time. This implies Theorem 2.8.

Also, Theorem 2.9 is now a consequence of the second part of Lemma 4.1. In fact, due to Assumption 2.2, the construction in its proof can be specified in such a way, so as to run in polynomial time.

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