# On the Inapproximability of Polynomial-programming, the Geometry of Stable Sets, and the Power of Relaxation

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**Abstract**

The present paper introduces the *geometric rank* as a measure for the quality of relaxations of certain combinatorial optimization problems in the realm of polyhedral combinatorics. In particular, this notion establishes a tight relation between the *maximum stable set* problem from combinatorial optimization, *polynomial programming* from integer non linear programming and *norm maximization*, a basic problem from convex maximization and computational convexity.

As a consequence we obtain very tight inapproximability bounds even for the largely restricted classes of polynomial programming where the polynomial is just a sum of univariate monomials of degree at most $\lceil \log n \rceil$, and it is guaranteed that the maximum is attained at a 0-1-vector. More specifically, unless $\mathbb{NP} = \mathbb{ZPP}$ this problem does not admit a polynomial-time $n^{1-\epsilon}$-approximation for any $\epsilon > 0$, and does not even admit a polynomial-time $n^{1-O(1/\sqrt{\log\log n})}$-approximation, unless $\mathbb{NP} = \mathbb{ZPTIME}(2^{O(\log n (\log\log n)^{3/2})})$. Similar results are also given for norm maximization. In addition we relate the geometric rank of a relaxation of the stable set polytope to the question whether the separation problem for the relaxation can be solved in polynomial time. Again, the results are nearly optimal.

## 1 Introduction and main results

Let $\mathcal{O}$ be any set packing problem, i.e. a combinatorial optimization problem whose instances consist of a collection $\mathcal{V}$ of subsets of a non empty ground set $V$ and whose goal it is to produce a maximum number of disjoint such subsets. Let $P_{\mathcal{O}}(I)$ be the 0-1-polytope spanned by the incidence vectors of the feasible solutions for the instance $I = (V, \mathcal{V})$. Then the set packing problem asks for $\max_{x \in P_{\mathcal{O}}(I)} e^T x$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^{|\mathcal{V}|}$. Since $\mathcal{O}$ is a packing problem, $P_{\mathcal{O}}(I)$ is a monotone polytope, hence contains the standard unit vectors of $\mathbb{R}^{|\mathcal{V}|}$. Now let $P$ be a polytope with

$$P \cap \{0,1\}^{|\mathcal{V}|} \subset P_{\mathcal{O}}(I) \subset P \subset [0,1]^{|\mathcal{V}|}.$$

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Such a polytope is called a *standard relaxation* of $P_{\mathcal{O}}(I)$. The *geometric rank* $g_{\mathcal{O}}(P)$ of the standard relaxation $P$ of $P_{\mathcal{O}}(I)$ is the minimal $p \in [1, \infty]$ such that $\max_{x \in P} \|x\|_p$ is attained at an integral point of $P$. Note that $1 \leq g_{\mathcal{O}}(P) \leq \infty$, and that $\max_{x \in P} \|x\|_p$ is attained at an integral point of $P$ for any $p \geq g_{\mathcal{O}}(P)$. Clearly, $g_{\mathcal{O}}(P) = 1$ means that the optimum of the linear objective function $e^T x$ over $P$ equals that over $P_{\mathcal{O}}(I)$, i.e. $P$ contains a 0-1-point that is optimal.

Various papers have addressed the question of how to measure the quality of relaxations in combinatorial optimization, see e.g. [LMJ94], [Goe95]. The geometric rank is a notion that allows to study the following two questions: What are the algorithmic limitations of strengthening the linear optimization subroutines in polyhedral combinatorics by considering approximative $\ell_p$-norm optimization routines? What are the limitations in terms of polynomial-time separation if we consider relaxations with small geometric rank?

The first question will lead to new inapproximability results while the second will show the limits of polynomial-time separation even for quite coarse approximations.

The following three optimization tasks and their interplay are the central topics of our study. The first is the classical combinatorial optimization problem of finding the cardinality of a maximum stable set in a graph while the other two are particular restrictions of polynomial programming, the last having a specifically strong geometric flavor.

MaxStableSet. Given a graph $G = (V, E)$, find the maximum $\alpha$ so that there is a subset $V^*$ of $V$ with $|V^*| = \alpha$ such that no two vertices of $V^*$ are joined by an edge in $E$.

Of course, MaxStableSet can be regarded as set packing problem. The next problem is a restricted polynomial programming problem that has been studied before by [BR95].

0-1-PolyProg. Given $n, r, s \in \mathbb{N}$, subsets $S_1, \ldots, S_r$ of $\{1, \ldots, n\}$, a vector $b \in \mathbb{Z}^s$ and a matrix $A \in \mathbb{Z}^{s \times n}$, compute the maximum of the multivariate polynomial $f(x) = \sum_{j=1}^r \prod_{i \in S_j} \xi_i$, where $x = (\xi_1, \ldots, \xi_n)^T$, over the polytope $P = \{x \in [0,1]^n \colon Ax \leq b\}$ provided the existence of a 0-1-maximizer is guaranteed.

The third problem depends on a functional $\gamma : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ that is assumed to be evaluable in time that is bounded by a polynomial in $n$.

$\gamma$-NormMax. Given $n, s \in \mathbb{N}$, a rational $s \times n$-matrix $A$ and $b \in \mathbb{Q}^s$, compute

$$N_p(P) = \max_{x \in P} \|x\|_p^p \text{ if } p \neq \infty \quad \text{and} \quad N_\infty(P) = \max_{x \in P} \|x\|_\infty \text{ else,}$$

where $p = \gamma(n)$ and $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$.

As usual, our nonapproximability results hold under the assumption that certain (unlikely) characterizations of $\mathbb{NP}$ do not hold. Therefore recall that $\mathbb{ZPP} = \mathbb{RP} \cap \text{co}\,\mathbb{RP}$ is the class of problems that can be solved in probabilistic polynomial time with zero error. As a generalization, if we replace the polynomial bound on the running time by $O(f(n))$ for some functional $f : \mathbb{N} \to \mathbb{N}$ we obtain the class $\mathbb{ZPTIME}(f(n))$. For some introduction to the theory of computational complexity see e.g. [Jan98].

As shown in [Hås96, Hås99] MaxStableSet does not admit a polynomial-time $|V|^{1-\epsilon}$-approximation for any $\epsilon > 0$, unless $\mathbb{NP} = \mathbb{ZPP}$. Also, by [EH00], MaxStableSet does not even admit a polynomial-time $n^{1-O(1/\sqrt{\log\log n})}$-approximation, unless $\mathbb{NP} = \mathbb{ZPTIME}(2^{O(\log n(\log\log n)^{3/2})})$, i.e., given that $\mathbb{NP}$ does not admit randomized algorithms with slightly super-polynomial expected running time. On the other hand, MaxStableSet can be approximated in polynomial time within $n^{1-O(\log\log n/\log n)}$, [BH92].

Using the former result and a construction of [EHdW84] that relates MaxStableSet to 0-1-PolyProg, [BR95] derived – again under the assumption $\mathbb{NP} \neq \mathbb{ZPP}$ – the inapproximability bound $n^{1/2-\epsilon}$ for 0-1-PolyProg for any $\epsilon > 0$, where the instances are such that $\Omega(\sqrt{n}) = r = O(n)$ and $|S_j| = O(n)$ for $j = 1, \dots, r$.

For constant functions $\gamma \equiv p$ the computational complexity of $p$-NormMax has been studied in detail in [GK93]. Of course, $\infty$-NormMax can be solved in polynomial time, but $p$-NormMax is $\mathbb{NP}$-hard for all other $p$. [BGK00] shows that for $p \in \mathbb{N}$ $p$-NormMax is even $\mathbb{APX}$-hard and a result of [Bri02] indicates that for $p \in \mathbb{N}\backslash\{1\}$ the problem is not even 'likely' to be in $\mathbb{APX}$.

Our main inapproximability result is one for $\gamma$-NormMax.

**Theorem 1.1.** *Let $k \in \mathbb{N}$ and $\lambda : \mathbb{N} \to [1, \infty[$ be any function, $\gamma : \mathbb{N} \to \mathbb{N}$ be a function with $1 + \log(n/k) \leq \gamma(n)$ for all $n \in \mathbb{N}$ that can be evaluated in polynomial time, and assume that there exists a polynomial-time $\lambda$-approximation algorithm for $\gamma$-NormMax. Then there exists a polynomial-time $\lambda$-approximation algorithm for MaxStableSet.*

In conjunction with the inapproximability results of [Hås96, Hås99] and [EH00] for MaxStableSet we obtain the following corollary.

**Corollary 1.2.** *Let $k \in \mathbb{N}$, $\epsilon > 0$, and $\gamma : \mathbb{N} \to \mathbb{N}$ be a function with $1 + \log(n/k) \leq \gamma(n)$ for all $n \in \mathbb{N}$ that can be evaluated in polynomial time. Then there does not exist a polynomial-time $n^{1-\epsilon}$-approximation algorithm for $\gamma$-NormMax, unless $\mathbb{NP} = \mathbb{ZPP}$. In addition there does not exist a polynomial-time $n^{1-O(1/\sqrt{\log\log n})}$-approximation for $\gamma$-NormMax, unless $\mathbb{NP} = \mathbb{ZPTIME}(2^{O(\log n(\log\log n)^{3/2})})$.*

This inapproximability result for $\gamma$-NormMax means that even if $p \to \infty$ (the easy case since $\infty$-NormMax can be solved in polynomial time) norm-maximization over polytopes 'stays pretty intractable on the way'. In the other direction, the geometric study leading to Theorem 1.1 allows to answer the question whether for 0-1-PolyProg the degree bound for the polynomials can be further reduced without weakening the inapproximability result. In fact, as a consequence of Theorem 1.1 we obtain the following inapproximability result for 0-1-PolyProg.

**Corollary 1.3.** *Let $k \in \mathbb{N}$. There is no polynomial-time approximation algorithm for* 0-1-PolyProg *with performance ratio $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $\mathbb{NP} = \mathbb{ZPP}$, even if the instances are restricted to those whose polynomial is a functional, convex on the feasible region, that consists of at most $n$ monomials of degree at most $\lceil 1 + \log(n/k) \rceil$.*

*In addition the same class of instances does not admit a polynomial-time $n^{1-O(1/\sqrt{\log\log n})}$-approximation, unless $\mathbb{NP} = \mathbb{ZPTIME}(2^{O(\log n (\log\log n)^{3/2})})$.*

Note that polynomial-time approximation with error at most $n$ is trivial for the restricted class of instances in Corollary 1.3.

The previous results can be interpreted as showing the limitation of trying to strengthen the linear programming subroutines in polyhedral combinatorics. The following will deal with polynomial-time relaxations of the stable set polytope.

In polyhedral combinatorics the polytopes are $\mathcal{H}$-presented, i.e. given in terms of systems of linear inequalities. So suppose that for each instance $G = (V, E)$ of MaxStableSet, $P(G)$ is a standard $\mathcal{H}$-presented relaxation of the stable set polytope $P_{\mathcal{S}}(G)$ of $G$, i.e.,

$$P(G) \cap \{0,1\}^{|\mathcal{V}|} \subset P_{\mathcal{O}}(I) \subset P(G) \subset [0,1]^{|\mathcal{V}|}.$$

Now, set

$$\mathcal{P} = \{P(G) : G \text{ is a finite graph}\}.$$

and let $\rho : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ be a functional. Then $\mathcal{P}$ is called a *$\rho$-relaxation* of MaxStableSet if $g_{\mathcal{S}}(P(G)) \leq \rho(|V|)$. Furthermore we say that the separation problem for a given *$\rho$-relaxation* $\mathcal{P}$ is solvable in polynomial time if, given $G$ as input, the separation problem for $P(G) \in \mathcal{P}$ is solvable in time bounded by a polynomial in the size of $G$.

Obviously, having an $\infty$-Relaxation is of no particular use since it simply means that the relaxations are contained in the unit-cubes. The other extreme, having a 1-Relaxation, means that linear optimization over the relaxed polytopes solves MaxStableSet. As a trivial example, $g_{\mathcal{S}}(P) = 1$ for $P(G) = [0,1]^n \cap \{x : \sum_{i=1}^n \xi_i \leq \alpha\}$, where $\alpha$ denotes the size of a maximal stable set in the underlying graph $G$ on $n$ vertices. However, we have to 'pay' for the minimal geometric rank $g_{\mathcal{S}}(P) = 1$ with the fact that the separation problem for $P$ is as hard as the original problem.

The following theorem relates the geometric rank of a polyhedral relaxation of the stable set polytope to the question whether the separation problem for the relaxation can be solved in polynomial time.

**Theorem 1.4.** *Let $k \in \mathbb{N}$. Then there exists a* $(1+\log(n/k))$-RELAXATION *for which the separation problem can be solved in polynomial time. Unless $\mathbb{NP} = \mathbb{ZPP}$, this is not the case for any $k$-*RELAXATION*, and, unless $\mathbb{NP} = \mathbb{ZPTIME}(2^{O(\log n (\log\log n)^{3/2})})$, there is no $O(\sqrt{\log\log n})$-*RELAXATION *for which the separation problem can be solved in polynomial time.*

## 2 The geometric rank of relaxations of stable set polytopes

Let $k \in \mathbb{N} \setminus \{1\}$ be fixed. Given a graph $G = (V, E)$ on $n \geq k$ vertices $v_1, \ldots, v_n$, we associate a variable $\xi_i$ with the vertex $v_i$, $i = 1, \ldots, n$, and consider the polytope $P_k(G) \subset \mathbb{R}^n$ that is defined by the following system of linear inequalities.

$$\begin{array}{rcccl} 0 & \leq & \xi_i & \leq & 1 \quad \text{for } i = 1, \ldots, n, \\ & & \xi_i + \xi_j & \leq & 1 \quad \text{for } \{v_i, v_j\} \in E \text{ and} \\ & & \xi_{i_1} + \cdots + \xi_{i_k} & \leq & \alpha_{i_1,\ldots,i_k} \quad \text{for } \{i_1, \ldots, i_k\} \in \mathcal{I}_k \end{array}$$

where $\alpha_{i_1,\ldots,i_k}$ denotes the size of a maximal stable set in the subgraph $G_{i_1,\ldots,i_k}$ of $G$ that is induced by the vertices $v_{i_1}, \ldots, v_{i_k}$ and $\mathcal{I}_k$ denotes the family of all $k$-element subsets of $\{1, \ldots, n\}$.

Note that $P_k(G)$ is a standard relaxation of the stable set polytope $P_{\mathcal{S}}(G)$.

In the following we will write $P_k$ for $P_k(G)$ whenever there is no risk of confusion.

**Lemma 2.1.** *Let $x^S$ be an integral vertex of $P_k$, let $S(x^S)$ denote the stable set associated with it and let $p \in \mathbb{N}$. Then we have $|S(x^S)| = \|x^S\|_p^p$.*

*Proof.* Since $P_k \subset [0,1]^n$, $x^S \in \{0,1\}^n$ and $\|x^S\|_p^p = \|x^S\|_1$. □

**Lemma 2.2.** *Let $1/2 \leq \mu \leq 1$, $q \in ]0, \infty[$, and $l \in [0, 2^q - 1]$. Then*

$$\mu^q + l(1-\mu)^q \leq 1.$$

*Proof.* For $q = 1$ the result is trivial. For $q \neq 1$, first note that

$$\mu^q + l(1-\mu)^q \leq \mu^q + (2^q - 1)(1-\mu)^q =: f_q(\mu).$$

It is easily seen that

$$\mu^* = \left(1 + \left(\frac{1}{2^q - 1}\right)^{1/(q-1)}\right)^{-1}$$

is the unique local extremum of $f_q$ in $[1/2, 1]$, a minimum. Hence $\mu = 1/2, 1$ are the only local maxima of $f_q$ in $[1/2, 1]$ and the assertion follows from

$$f_q(1/2) = \left(\frac{1}{2}\right)^q + (2^q - 1)\left(\frac{1}{2}\right)^q = 2^q 2^{-q} = 1 = f_q(1).$$

□

In the following theorem we derive an upper bound for the geometric rank of the relaxation $P_k(G)$ of the associated stable set polytope $P_{\mathcal{S}}(G)$.

**Theorem 2.3.** *Let $G = (V, E)$ be a graph on $n$ vertices, and let $n \geq k$. Then $g(P_k) \leq 1 + \log(n/k)$.*

*Proof.* Let $p = 1 + \log(n/k)$, and take any point $x = (\xi_1, \ldots, \xi_n)^T \in P_k$. Without loss of generality we may assume that $1 \geq \xi_1 \geq \cdots \geq \xi_n \geq 0$.

If $\xi_k = 0$ we conclude

$$\|x\|_p^p \leq \sum_{i=1}^{k-1} \xi_i \leq \alpha_{1,\ldots,k-1} \leq \alpha_{1,\ldots,k} \leq \alpha,$$

where $\alpha$ denotes the size of a maximal stable set in $G$.

Otherwise we set $\xi_0 = 1$ and let $l^*$ be the maximal $l$ with $1 \leq l \leq n-k+1$ such that $\xi_{l+k-1} > 1 - \xi_{l-1}$ and $\xi_{l-1} > 1/2$. Obviously, by the definition of $\xi_0$ and since $\xi_k > 0$ this maximum is well-defined.

Note that $\xi_{l^*+k-1} > 1 - \xi_{l^*-1}$ and $\xi_{l^*-1} > 1/2$ imply the following. Since for any edge $\{v_i, v_j\} \in E$ the inequality $\xi_i + \xi_j \leq 1$ is part of the given $\mathcal{H}$-presentation of $P_k$, the set $S^* = \{v_1, \ldots, v_{l^*-1}\}$ is stable. (Of course, if $l^* = 1$, $S^* = \emptyset$.) Further, there is no edge in $G$ connecting a vertex of $S^*$ with a vertex $v_{l^*+m-1}$ for $m = 1, \ldots, k$. Hence, for any stable set $I$ in the subgraph $G_{l^*,\ldots,l^*+k-1}$ the set $I \cup S^*$ is a stable set in $G$. This yields the inequality

$$\alpha_{l^*,\ldots,l^*+k-1} \leq \alpha - |S^*| = \alpha - (l^* - 1).$$

Now, suppose first that $l^* = n - k + 1$; then we have

$$\begin{aligned} \|x\|_p^p &\leq (n-k) + \sum_{i=n-k+1}^{n} \xi_i^p \leq (n-k) + \sum_{i=n-k+1}^{n} \xi_i \\ &\leq (n-k) + \alpha_{n-k+1,\ldots,n} \leq (n-k) + \alpha - (n-k) = \alpha. \end{aligned}$$

Note that $p = 1$ implies $n = k$, whence $l^* = n - k + 1$.

Now, let $l^* \leq n-k$ (and hence $p > 1$). It follows then from the maximality of $l^*$ that $\xi_{l^*} \leq 1/2$ or $\xi_{l^*+k} \leq 1 - \xi_{l^*}$. If $\xi_{l^*} \leq 1/2$ we have

$$\begin{aligned} \|x\|_p^p &\leq (l^*-1) + \frac{n-(l^*-1)}{k} \sum_{i=l^*}^{l^*+k-1} \xi_i^p \\ &\leq (l^*-1) + \frac{n-(l^*-1)}{k} \xi_{l^*}^{p-1} \sum_{i=l^*}^{l^*+k-1} \xi_i \\ &\leq (l^*-1) + \frac{n-(l^*-1)}{k} \left(\frac{1}{2}\right)^{\log(n/k)} \alpha_{l^*,\ldots,l^*+k-1} \\ &\leq (l^*-1) + \frac{n-(l^*-1)}{k} \frac{k}{n} (\alpha - (l^*-1)) \\ &\leq (l^*-1) + \alpha - (l^*-1) = \alpha. \end{aligned}$$

So, let $\xi_{l^*} > 1/2$ but $\xi_{l^*+k} \leq 1 - \xi_{l^*}$. With the aid of Lemma 2.2 we obtain

$$\begin{aligned}
\|x\|_p^p &\leq (l^*-1) + \sum_{i=l^*}^{l^*+k-1} \xi_i^p + \sum_{i=l^*+k}^{n} \xi_i^p \\
&\leq (l^*-1) + \xi_{l^*}^{p-1} \sum_{i=l^*}^{l^*+k-1} \xi_i + \xi_{l^*+k}^{p-1} \sum_{i=l^*+k}^{n} \xi_i \\
&\leq (l^*-1) + \left( \xi_{l^*}^{p-1} + \frac{n-(l^*-1)-k}{k} (1-\xi_{l^*})^{p-1} \right) \sum_{i=l^*}^{l^*+k-1} \xi_i \\
&\leq (l^*-1) + \alpha - (l^*-1) = \alpha.
\end{aligned}$$

All together we have shown $\max_{x \in P_k} \|x\|_p^p \leq \alpha$, and by Lemma 2.1 we have actually equality. This concludes the proof of Theorem 2.3. □

Note that for a given point $x = (\xi_1, \ldots, \xi_n)^T \in P_k$ the previous proof allows to determine a stable set with cardinality at least $\|x\|_p^p$ in polynomial time.

At first sight, the $(1 + \lceil \log(n/k) \rceil)$-relaxation

$$\mathcal{P}_k = \{P_k(G) : G \text{ is a finite graph}\}$$

might seem rather weak. However since we can solve the separation problem for $\mathcal{P}_k$ in polynomial time we see from the second part of Theorem 1.4 that we cannot expect to do much better. Also note, that there are infinitely many pairs $(n, k)$ for which $g(\mathcal{P}_k) \leq 1 + \log(n/k)$ holds with equality. In fact, for the complete graph $K_n$ on $n \geq 2$ vertices, $x_0 = (1/2, \ldots, 1/2)^T \in P_2(K_n)$ and $\|x_0\|_{\log n}^{\log n} = 1 = \alpha(K_n)$. Hence for $K_n$ the bound is sharp for $k = 2$. Also, if $C_n$ is a cycle for some odd $n \in \mathbb{N}$ and $k = n-1$, $x_0 \in P_{n-1}(C_n)$ and $\|x_0\|_{1+\log(n/(n-1))}^{1+\log(n/(n-1))} = (n-1)/2 = \alpha(C_n)$. So, again, the bound is attained. It is, on the other hand, an open problem to determine for which 'interesting classes' of graphs there exist 'significantly' better estimates.

## 3 Reducing MaxStableSet to 0-1-PolyProg

With Theorem 2.3 at hand Theorem 1.1 and the first part of Theorem 1.4 are easy to prove.

*Proof of Theorem 1.1.* Let $k \in \mathbb{N}$, let $\mathcal{A}$ be a polynomial-time $\lambda$-approximation-algorithm for $\gamma$-NormMax and let $G = (V, E)$ be an instance of MaxStableSet. If $|V| < k$ a maximum stable set in $G$ can be computed in polynomial time. Otherwise note that the $\mathcal{H}$-presentation of $P_k$ can be determined in polynomial time and can be given as input to $\mathcal{A}$. $\mathcal{A}$ outputs a $\lambda$-approximation for the $p$th power of the $l_p$-norm-maximum where $p =$

$\gamma(n) < \infty$ that by means of Theorem 2.3 yields also a $\lambda$-approximation for the cardinality of a maximum stable set in G. □

*Proof of Part 1 of Theorem 1.4.* As already observed in the previous proof, $P_k$ is given by an $\mathcal{H}$-presentation of polynomial size. Of course, the separation problem for $\mathcal{H}$-presented polytopes is solvable in polynomial time. Hence the assertion follows from Theorem 2.3. □

Now, observe that while we have monomials of high degree but consisting of just one variable in the norm-maximization problem, the polynomials we are interested in in the context of 0-1-PolyProg are sums of monomials that are multilinear in the occurring variables. But this can be handled by reproducing the original variables suitably often.

*Proof of Corollary 1.3.* Assume we have a polynomial-time approximation algorithm for 0-1-PolyProg of the kind stated in the assertion. Let $I$ be an instance of $(1+\lceil\log(d/k)\rceil)$-NormMax, let $\xi_1,\ldots,\xi_d$ denote its variables and let $d \geq k$. We take $1+\lceil\log(d/k)\rceil$ copies of each and denote them by $\xi_{i_1},\ldots,\xi_{i_{1+\lceil\log(d/k)\rceil}}$, for $i=1,\ldots,d$. Adding the constraints $\xi_{i_1}=\xi_{i_2}=\cdots=\xi_{i_{1+\lceil\log(d/k)\rceil}}$ for $i=1,\ldots,d$ and replacing $\xi_i^{1+\lceil\log(d/k)\rceil}$ by $\Pi_{j=1}^{1+\lceil\log(d/k)\rceil}\xi_{i_j}$ in the associated objective function yields an instance of 0-1-PolyProg in dimension $n=d(1+\lceil\log(d/k)\rceil)$. Now, since $n=d^{1+O(\log\log d/\log d)}$, it follows that for sufficiently large $n$,

$$n^{1-\epsilon} \leq d^{1-\epsilon/2}$$

for each fixed $\epsilon > 0$, and also that

$$\begin{aligned} n^{1-O(1/\sqrt{\log\log n})} &= d^{(1-O(1/\sqrt{\log\log n}))(1+O(\log\log d/\log d))} \\ &< d^{1-O(1/\sqrt{\log\log n})+O(\log\log d/\log d)} \\ &= d^{1-O(1/\sqrt{\log\log n})} = d^{1-O(1/\sqrt{\log\log d})}. \end{aligned}$$

Hence we obtain a polynomial-time $d^{1-\epsilon/2}$-, respectively $d^{1-O(1/\sqrt{\log\log d})}$-approximation algorithm for $(1+\lceil\log(d/k)\rceil)$-NormMax. However, unless $\mathbb{NP}=\mathbb{ZPP}$ or $\mathbb{NP}=\mathbb{ZPTIME}(2^{O(\log d(\log\log d)^{3/2})})$, respectively, such an algorithm does not exist. □

In their reduction of MaxStableSet to 0-1-PolyProg [BR95] assigned to each edge in the graph $G=(V,E)$ two variables and to each of the vertices $v_i$ one subset $S_i$ that consists of the indices of $v_i$'s neighbors. This leads to polynomials whose degree might be of the same order as the number of variables. In contrast, the above reduction that is based on assigning variables to the vertices gives polynomials of degree at most $1+\lceil\log(n/k)\rceil$

## 4 The geometric rank and polynomial time separability

Now we turn to the question whether relaxations for the stable set polytopes can be determined for which the separation problem can be solved in polynomial time and whose geometric rank does not tend to $\infty$.

**Lemma 4.1.** *Let $n, p \in \mathbb{N}$ and*

$$B_n = [0,1]^n \cap \{(\xi_1, \dots, \xi_n)^T : \big(1/n^{1-1/p}\big) \sum_{i=1}^{n} \xi_i \leq 1\}.$$

*Then*

$$\mathbb{B}_p^n \cap [0, \infty[^n \subset B_n \subset (n^{1-1/p})^{1/p} \mathbb{B}_p^n \cap [0, \infty[^n,$$

*where $\mathbb{B}_p^n$ denotes the $l_p$-unit-ball of $\mathbb{R}^n$.*

*Proof.* For the first inclusion take any $x = (\xi_1, \dots, \xi_n)^T \in \mathbb{B}_p^n \cap [0, \infty[^n$. By Hölder's inequality we have

$$\big(1/n^{1-1/p}\big) \sum_{i=1}^{n} \xi_i = \big(1/n^{1-1/p}\big) e^T x \leq 1/n^{1-1/p} \|e\|_{p'} \|x\|_p \leq 1,$$

where $e = (1, \dots, 1)^T \in \mathbb{R}^n$ and $p' = p/(p-1)$.

For the second inclusion we have to show that $\max_{x \in B_n} \|x\|_p^p \leq n^{1-1/p}$. Clearly it suffices to consider vertices of $B_n$ that are contained in the hyperplane $\{x = (\xi_1, \dots, \xi_n)^T : \big(1/n^{1-1/p}\big) \sum_{i=1}^{n} \xi_i = 1\}$. Any such vertex $v$ has precisely $\lfloor n^{1-1/p} \rfloor$ 1-entries, $n - \lceil n^{1-1/p} \rceil$ 0-entries and if $n^{1-1/p} \notin \mathbb{N}$ one additional entry equal to $n^{1-1/p} - \lfloor n^{1-1/p} \rfloor$. In any case,

$$\begin{aligned} \|v\|_p^p &= \lfloor n^{1-1/p} \rfloor + (n^{1-1/p} - \lfloor n^{1-1/p} \rfloor)^p \\ &\leq \lfloor n^{1-1/p} \rfloor + (n^{1-1/p} - \lfloor n^{1-1/p} \rfloor) = n^{1-1/p}. \end{aligned}$$

□

**Lemma 4.2.** *For any $n \in \mathbb{N}$ let $Q_n$ be an $\mathcal{H}$-presented rational polytope with binary size bounded by a polynomial in $n$ such that*

$$\mathbb{B}_p^n \cap [0, \infty[^n \subset Q_n \subset \lambda^{1/p} \mathbb{B}_p^n \cap [0, \infty[^n.$$

*Further, for $n \in \mathbb{N}$ let $P_n$ be a polytope in $[0, \infty[^n$. If the separation problem for $(P_n)_{n \in \mathbb{N}}$ can be solved in polynomial time then $\lambda$-approximations of $\max_{x \in P_n} \|x\|_p^p$ can be computed in polynomial time.*

*Proof.* Let for each $n \in \mathbb{N}$, $m_n \in \mathbb{N}$, $a_{n,1}, \dots, a_{n,m_n} \in \mathbb{Z}^n$, and $\beta_{n,1}, \dots, \beta_{n,m} \in \mathbb{Z}$ such that $Q_n = \bigcap_{i=1}^{m_n} \{x \in \mathbb{R}^n : a_{n,i}^T x \leq \beta_{n,i}\}$. Since the separation problem for $P_n$ can be solved in polynomial time, $\omega_n(a_{n,i}) = \max_{x \in P_n} a_{n,i}^T x$ can be computed in polynomial time; [GLS93]. With $\omega_n = \max_{i=1,\dots,m_n} \omega_n(a_{n,i})$

this yields $P_n \subset \omega_n Q_n$ and $P_n \not\subset \omega Q_n$ for any $\omega$ with $0 \leq \omega < \omega_n$. The first relation implies $P_n \subset \omega_n \lambda^{1/p} \mathbb{B}_p^n$. Further, since $\mathbb{B}_p^n \cap [0,\infty[^n \subset Q_n$ the second yields the existence of a point $x^* \in P_n$ with $\|x^*\|_p \geq \omega_n$. Hence, $\omega_n^p$ provides a $\lambda$-approximation for $\max_{x \in P_n} \|x\|_p^p$. □

Let us mention in passing that in the realm of the algorithmic theory of convex bodies suitable, explicitly constructed polytopes approximating the $l_p$-unit-balls yield asymptotically optimal algorithms for the computation of norm-maxima and other radii of convex bodies [BGK+03]. In the Euclidean case this even gives optimal randomized algorithms that (in a sense surprisingly) have the same performance ratio as deterministic ones [BGK+98].

*Proof of Part 2 of Theorem 1.4.* Assume that for some $p$ we have a standard $\rho$-Relaxation $\mathcal{P} = \{P(G) : G \text{ is finite a graph}\}$ with $\rho(n) \leq p$ for all $n \in \mathbb{N}$ and for which the separation problem is solvable in polynomial time. Then, given a graph $G$ on $n$ vertices as input, we use the polytope $B_n$ of Lemma 4.1 to play the role of $Q_n$ in Lemma 4.2 in order to obtain an $n^{1-1/p}$-approximation for $\max_{x \in P(G)} \|x\|_p^p$. Since $\rho(n) \leq p$, we have $\max_{x \in P(G)} \|x\|_p^p = \alpha(G)$. Hence we obtain a polynomial-time $n^{1-1/p}$-approximation algorithm for MaxStableSet, whence with the setting $\epsilon = 1/p$, a polynomial-time $n^{1-\epsilon}$-approximation algorithm with constant $\epsilon$ or, with the setting $p = O(\sqrt{\log\log n})$, a polynomial-time $n^{1-O(1/\sqrt{\log\log n})}$-approximation. □

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