Geometric Clustering for the Consolidation of Farmland and Woodland

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In many agricultural areas, farmers cultivate a large number of small lots that are scattered across an extended region. In a typical farming area in Bavaria, Germany, about 7 – 20 farmers cultivate between 300 and 1000 lots; Figure 1 provides a visual impression of a typical distribution.

In such a situation, the farmers face serious disadvantages. Because the individual lots are scattered across a large region, there is considerable overhead driving, resulting in an excess of personal and transportation cost. Calculations of the *Bavarian State Institute for Agriculture* show that these additional costs often add up to more than 30% of the part of the farmers' net income coming from their agricultural production. (EU- and other subsidies that constitute a substantial additional part of the income are, of course, typically independent of such aspects of cost-efficient production.) Also, because the single separate lots are rather small, modern heavy machinery cannot be used profitably. Hence, the cost of cultivation is much higher than it would be for fewer, larger lots of the same total size.

In its classical form, land consolidation consists of a complete restructuring of the agricultural area, discarding the current and creating a new lot structure. This process involves extended surveying and new legal assignments of property, and is hence costly, lengthy, and inflexible. After the decision is made, farmers are forced to participate in this process. A typical classical land consolidation process lasts more than a decade and costs about 2500 Euro per hectare. Of course, the land distribution is less rigid in agricultural areas where farmers other than the lot owners cultivate the majority of the lots, through lend–lease agreements. (This is partly caused by inheritance regulations and partly due to the tough economic situation of small farmers.) So, even districts that underwent a classical form of land consolidation in the recent past may look like rag rugs. This is a common situation in Northern Bavaria.

On the other hand, a farmer who rents a lot for cultivation is generally less tied to the lot, and is hence more willing to "trade" it to improve the overall cost structure for his operations. This opens the possibility for conceptually simple lend–lease agreements based on the existing lot structure, i.e., without the nullification of the property structure.

For an optimal redistribution there are some main aspects to be considered. Because large connected pieces of land are desirable for each farmer, while the lot structure, i.e., the dissection of the region into individual lots, is not changed, one aims to assign adjacent lots to the same farmer. Naturally, certain *balancing constraints* need to be satisfied. For instance, the total size of each farmer's land should not change too much in the course of redistribution, neither should its quality of soil, the EU-subsidies attached to his lots, or other possibly relevant parameters. Also, ecological constraints play a role.

The quality of soil, in particular, is typically different in different parts of the region. This means that, in practice, the assigned lots of each farmer will form certain connected patches, which, in turn, should be as close to each other as possible. The lend–lease agreements are completely voluntary. In particular, farmers are allowed to fix some of their lots

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and make only a subset available for redistribution. Then, naturally, the redistributed lots should be adjacent to the fixed ones.

The number of possible reassignments is typically very large. In fact, for k farmers with m lots, it is k^m. So, even for moderate sizes of seven farmers and 419 lots, as in Figure 1, the number of possibilities exceeds 10^{354} and excludes "trial–and–error approaches." For this reason, the lend–lease initiative was first regarded as impractical by the farming community and organizations.

Mathematically, even simple instances of the problem are $\mathbb{NP}$–hard or even harder. Further, standard graph-theoretical methods of redistribution have problems with the balancing constraints. Also, proper visualization and evaluation tools are needed in practice.

The Basic Model

Naturally, there are various ways to model the lend–lease task (isoperimetric models, graph k-partitioning, etc.) Because all existing models had deficits for this particular application, we developed in [12] the model of *geometric clustering*. We will now introduce the model; the two subsequent sections will then justify it by proving that it *captures the intuition* behind "good clusterings" and is also *computationally tractable*.

In a first step, we abstract from the lot geometry and replace each lot by its center. Our task then becomes that of partitioning a finite weighted point set in some Minkowski space $(\mathbb{R}^d, \|\cdot\|)$, under certain balancing constraints, so as to optimize a suitable distance-based function. Here are the "ingredients" of our problem (the interpretation in terms of the consolidation of farmland is given in parentheses):

d is the *dimension of the space of objects* (two coordinates for the lots), m is the *number of objects* (the number of lots), V is the given *point set in* $(\mathbb{R}^d, \|\cdot\|)$ (the set of centers of the lots); of course, $|V| = m$. Further, k denotes the *number of clusters* (the number of farmers); typically, $k \ll m$. Also, s is the number of *features* of the points (size of the lot, quality of soil, EU-subsidies tied to the lot, etc.), the function

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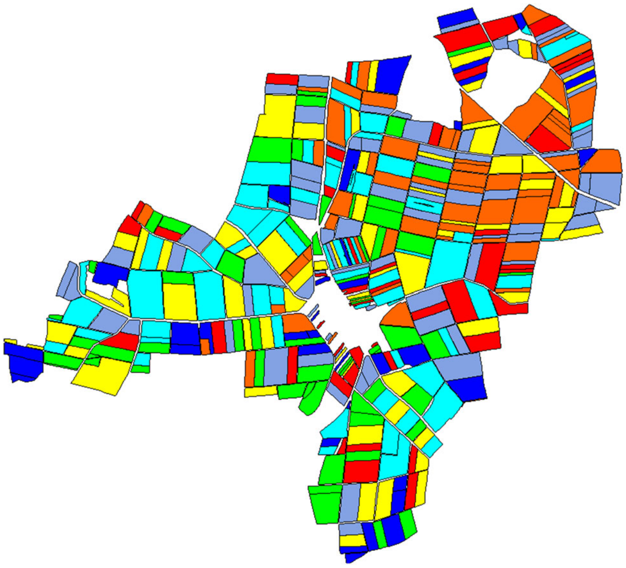

Figure 1. An agricultural region with 7 farmers and 419 lots. Different colors represent different farmers who cultivate the lots.

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$\omega : V \to \mathbb{R}^s$ associates with each lot its *feature vector*, and the vectors $b_1^{\pm}, \ldots, b_k^{\pm} \in \mathbb{R}^s$ specify the *tolerances* (the allowed intervals for each farmer for the total farm size and the other features). Hence, the task is to construct a *partition* $\mathcal{C} = \{V_1, \ldots, V_k\}$ of V satisfying the *balancing constraints*

$$b_i^- \quad \leq \quad \sum_{v \in V_i} \omega(v) \quad \leq \quad b_i^+ \qquad (i = 1, \ldots, k)$$

where, as usual, vector inequalities are to be understood componentwise.

The special case of *prescribed cardinality*, i.e., $s = 1$ and $\omega \equiv 1$, and requiring the clusterings to be *strongly balanced*, i.e., $b_i = b_i^- = b_i^+$ for $i = 1, \ldots, k$ and hence $|V_i| = b_i$, will be referred to as the *combinatorial case*.

The intuition behind the objective function is to move the centers of gravity of the clusters apart. Its construction involves the norm $\|\cdot\|$ on $\mathbb{R}^d$ (recall that for the consolidation of farmland we have $d = 2$) and a second norm $\|\cdot\|_\diamond$ on $\mathbb{R}^{k(k-1)/2}$, where k is again the number of clusters (farmers). $\|\cdot\|_\diamond$ is required to be *monotone*, i.e., $\|x\|_\diamond \leq \|y\|_\diamond$ whenever $x, y \in \mathbb{R}^{k(k-1)/2}$ with $0 \leq x \leq y$. Then the objective function is of the form

$$\max \quad \left\| \left(\|c_1 - c_2\|, \|c_1 - c_3\|, \ldots, \|c_{k-1} - c_k\| \right)^T \right\|_\diamond,$$

where c_i denotes the center of gravity of cluster V_i. Putting things together, we obtain a nonlinear integer maximization problem over a polytope.

Of course, given this kind of formulation it is not obvious, hence important to show, that it models the key features of the problem and that it is computationally efficient enough to handle the problem sizes that are relevant in practice.

As a first hint that the seemingly intractable nonlinear maximization problem might not be so bad after all, note that the nonlinearity depends only on k and not on m. In the example shown in Figure 1, we have $k = 7$ and $m = 419$. Hence the nonlinear part of the problem "lives" only in dimension $kd = 14$.

Capturing the Intuition

In a perfect world, any geometer would like to construct the clustering from a cell-complex dissecting $\mathbb{R}^d$; see Figure 4. This means one would like to find (optimal) clusterings that are "cut out" by cell-decompositions of space; see Figure 5. But is this always possible?

The simple example in dimension one, with three points, and two clusters of Figure 6 exhibits some obstacles to perfection.

Hence, we have to be prepared to accept that certain points are split among clusters. So, it is not enough to consider clusters that are partitions of V, but we need to resort to *fractional* assignments of points, i.e., the clusters C_i must encode for each point of V the portion that belongs to C_i, i.e., must be of the form $C_i = (\xi_{i,1}, \ldots, \xi_{i,m})$ with $\xi_{i,j} \in [0, 1]$.

The most important notion needed now is that of a generalized *Voronoi diagram*, the *power diagram*; see [1], [4] for surveys. Let $s_1, \ldots, s_k \in \mathbb{R}^d$ denote *control points* (also called *sites*) and $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ certain *sizes*, then the corresponding *power diagram* is the cell decomposition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of

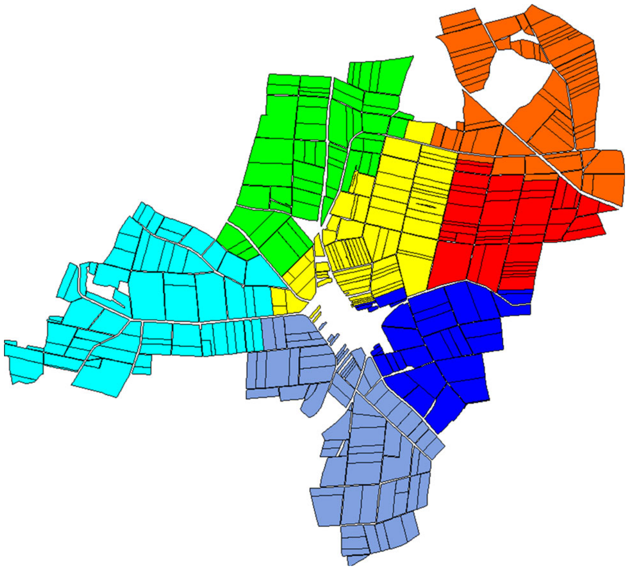

Figure 2. An improved redistribution of lots for the agricultural region of Figure 1.

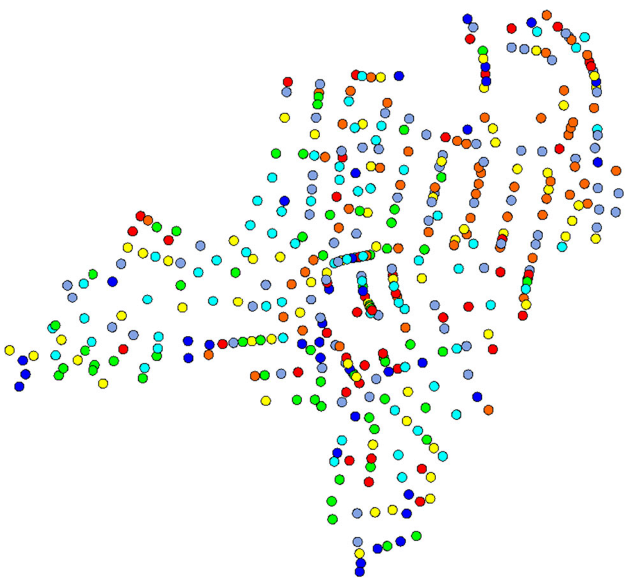

Figure 3. Abstraction from the lot geometry; the lots in Figure 1 are replaced by their centers; the coloring refers to the original coloring.

$\mathbb{R}^d$ defined by

$$P_i = \left\{ x : \|s_i - x\|_{(2)}^2 - \sigma_i \leq \|s_j - x\|_{(2)}^2 - \sigma_j \text{ for all } j \right\}.$$

First note that for $\sigma_1 = \ldots = \sigma_k = 0$, we obtain the classical *Voronoi* or *Dirichlet cells*. Further, adding a real number σ to each of the sizes does not change the cells P_i. Hence we may assume that all sizes are positive. Then the set of points with $\|s_i - x\|_{(2)}^2 = \sigma_i$ is a sphere of radius $\sqrt{\sigma_i}$, and the power diagram can be constructed geometrically; see Figure 7.

Of course, we are mainly interested in power diagrams $\mathcal{P}$ that are *feasible* for our clustering $\mathcal{C}$, i.e., the *support* $supp(C_i) = \{x_j : \xi_{i,j} \neq 0\}$ of C_i is contained in P_i for all i. In fact, we want an even stronger property: $\mathcal{P}$ *supports* $\mathcal{C}$ if $supp(C_i) = V \cap P_i$ for all i.

Further, let us consider the *support multigraph* $G(\mathcal{C})$ of the clustering $\mathcal{C} = (C_1, \ldots, C_k)$. Its vertex set consists of the clusters $C_1, \ldots, C_k$, there is an edge between C_i and C_l precisely for every j for which $x_j \in supp(C_i) \cap supp(C_l)$, and this edge is labeled with x_j. A cycle in $G(\mathcal{C})$ is called *colored* if not all of its labels coincide, and $G(\mathcal{C})$ is called *c-cycle-free* if it does not contain any colored cycle. Then we call the cell-complex $\mathcal{P}$ *strongly feasible* for $\mathcal{C}$, if $\mathcal{P}$ supports $\mathcal{C}$ and $G(\mathcal{C})$ is c-cycle-free.

Interestingly enough, the existence of strongly feasible power diagrams can be most easily accessed via another geometric object that lies in $\mathbb{R}^{kd}$. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a feasible clustering with corresponding centers of gravity $c_1, \ldots, c_k$. The *gravity vector* of $\mathcal{C}$ is then given by $\mathbf{c} := (c_1^T, \ldots, c_k^T)^T$, and the *gravity body* Q is defined by

$$Q := conv\big\{ \mathbf{c} \in \mathbb{R}^{kd} : \mathbf{c} \text{ is the gravity vector of a feasible clustering} \big\}.$$

In the case of strongly balanced clusterings, the gravity bodies are in fact polytopes. As a simple example let us consider the combinatorial case with $d = 1$, $m = k$, and $V = \{1, \ldots, m\}$. Then the corresponding gravity polytope is the well-known *permutahedron*, i.e.,

$$conv\Big\{ \big(\pi(1), \ldots, \pi(m)\big)^T : \pi \text{ is a permutation of } 1, \ldots, m \Big\}.$$

Figure 8 shows the permutahedron for $m = 3$.

As it turns out, the gravity bodies capture the main properties of feasible power diagrams and allow us to grasp all of them simultaneously. This is the key for finding "best-fitting" power diagrams. In particular, we call $\mathcal{C}$ an *extremal clustering* if $\mathcal{C}$'s gravity vector is an extreme point of Q. Recalling that a convex function attains its maximum over a nonempty compact convex set at an extreme point, here is the main justification that our model captures, indeed, the intuition behind good clusterings.

THEOREM 1. [14]

(a) *Each extremal clustering admits a strongly feasible power diagram.*
(b) *At most, $k - 1$ of the (weighted) points are fractionally assigned.*
(c) *In the strongly balanced case, a clustering $\mathcal{C}$ is extremal if and only if $\mathcal{C}$ admits a strongly feasible power diagram.*

Let us point out that Theorem 1 extends and generalizes various previous results, most notably those for the combinatorial case of [6], [2], [3], and [7] (see [14] for further references).

Let us mention that it is also possible to characterize strongly feasible *centroidal* power diagrams (where centers coincide with sites) in terms of the local optima of some ellipsoidal function over the gravity body; [14]. The global optima can also be characterized in terms of the separation properties of the corresponding clusterings; [14]. Further,

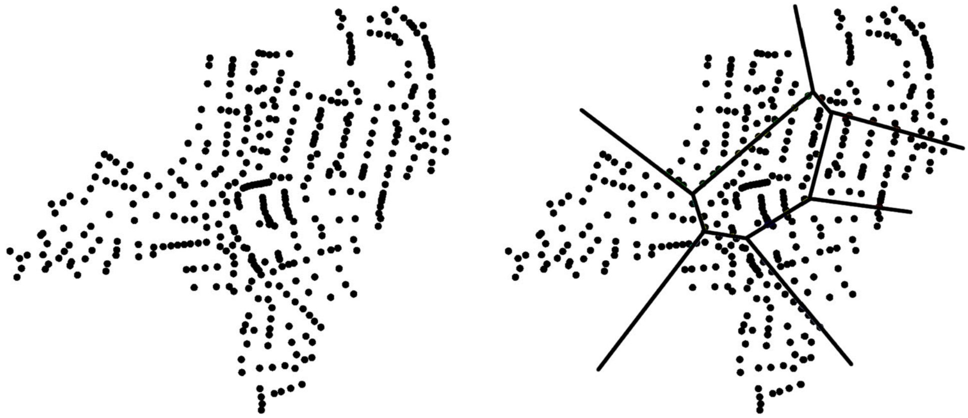

Figure 4. The point set V (left), and a partition of $\mathbb{R}^2$ (right).

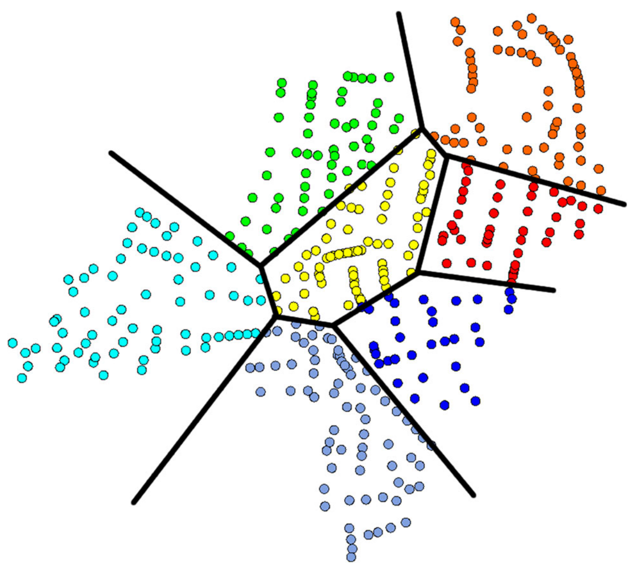

Figure 5. Clustering defined by the membership of points in 2-cells of the cell-decomposition of Figure 4 (right).

there are interpretations and extensions to the realm of machine learning involving soft margins, [9].

Algorithmic Tractability

Of course, because of the potentially exponential number of local maxima, convex maximization is in general $\mathbb{NP}$-hard. Hence it is necessary to resort to approximations. However, [13] gives tight and very favorable worst-case error bounds for these approximations that we will explain now. The main objects are again geometric in nature, the *clustering bodies*

$$C := \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^{kd} : \left\| \begin{pmatrix} \|c_1 - c_2\| \\ \vdots \\ \|c_{k-1} - c_k\| \end{pmatrix} \right\|_\diamond \leq 1 \right\},$$

which are just the level sets of our objective functions. Depending on the chosen norms, these bodies can be polyhedral or smooth or a mixture of both. For instance, if both norms are ℓ_1-norms, we obtain the polar of the Cartesian product of permutahedra; if both norms are Euclidean, we obtain a Euclidean cylinder with

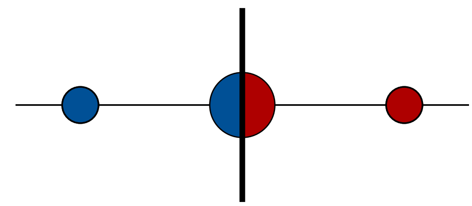

Figure 6. Illustration of an example with parameters $d = 1$, $m = 3$, $k = 2$, $\omega(v_1) = \omega(v_3) = 1$, $\omega(v_2) = 2$, $b_1 = b_2$ for which there is no feasible clustering whose two point sets are strictly separated. The different sizes of the discs indicate the different weights of the corresponding points.

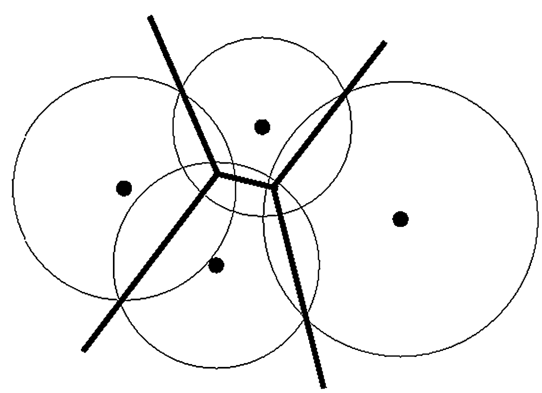

Figure 7. Power diagram defined by four control points s_1, s_2, s_3, s_4 (black dots) and positive sizes $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. The circles are centered at the s_i and have radius $\sqrt{\sigma_i}$.

d-dimensional lineality space; if the inner norm is arbitrary while the outer norm is ℓ_1 or ℓ_∞, the bodies turn out to be the polars of the Minkowski sum or the convex hull of certain diagonally embedded copies of scaled unit balls in $(\mathbb{R}^d, \|\cdot\|)$'s conjugate space.

These and other structural results can be used to obtain tight polyhedral approximations for the corresponding clustering bodies. But how can we use such approximations?

Suppose we have access to an approximation of a clustering body by polyhedra with only polynomially many facets. Then we can devise the following polynomial-time algorithm:

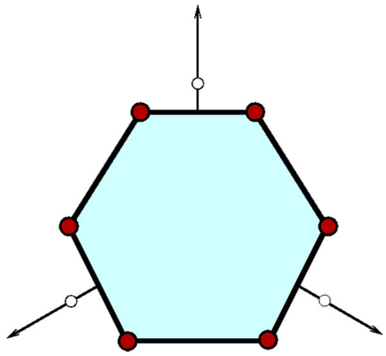

Figure 8. The permutahedron in $\mathbb{R}^3$.

- Solve a *linear program* for each facet (the objective function vector is an outer normal of the facet).
- Take the maximum of the obtained values.

Naturally, the quality of the approximate solution of our convex maximization task depends on the error of the polyhedral approximation of the clustering body.

Here is the "fully Euclidean" case of the much more general approximation results obtained in [13].

Theorem 2. [13] *Let* $\|\cdot\|_\diamond = \|\cdot\| = \|\cdot\|_{(2)}$. *Then the clustering body can be approximated by a polytope with polynomially many facets up to an error of*

$$O\left(\left(\frac{kd}{\log(kd)}\right)^{\frac{1}{2}}\right).$$

Let us point out that this sharpens the results for general ℓ_p-norm-maximization of [5], [17], [15], [16] by a factor of $\sqrt{k}$. Further note that the worst-case upper bound does not depend on m but only on d and k, confirming our heuristic argument after the introduction of the objective function.

In addition to the results provided in detail here, [10] studies an alternative approach for the case that farmers actually prescribe certain lots as nuclei for redistribution whereas [8] contains a detailed in-depth study of the diameter of the relevant partition polytopes in the combinatorial case.

It should be mentioned that the algorithms developed all run on a laptop within a few seconds to half a minute for the practically relevant sizes. The experience described in the next section will make clear that flexibility and short response times are essential for success in practice.

Practical Issues

In cooperation with the *Bavarian Association of Farmers*, we moderated lend–lease actions in some areas in Northern Bavaria. As it turned out, the optimization tool was relevant in different phases of the process. Of course, it was used up front to show the potential that is inherent in the lend–lease agreements in the specific region. Our tools for economic evaluation were used to estimate the financial benefit for each individual farmer; see Figure 9.

In practice, it was necessary to have a tool for manual redistributions available; see Figure 10.

That may at first sound strange because we can produce better redistributions with our optimization tools. However, there are two reasons. Because the participation is voluntary to the degree that the farmers can decide for each lot if it should be subject to redistribution, at the beginning of the process the farmers only entered lots of inferior quality. Then, of course, the potential of the method is limited. Having the chance to "play" with a comfortable, easy to access, and transparent tool (with the results being projected on a screen and hence visible to all participants) increased the confidence in the self-determined and controlled character of the procedure. (Also a kind of "video-game fun effect" took place that was favorable for the atmosphere of the meeting.) After a certain "initializing phase" the farmers were convinced of the potential and fairness of the method and, hence, put most of their lots into the shared pool for redistribution.

A second reason for needing the manual tool was that there were many more restrictions on the redistribution of lots than were ever specified explicitly in detail. For instance, some farmers were willing to participate but were not willing to trade a certain lot with a certain other farmer. Of course, this was not discussed openly and was found out only through the process of "manual post-optimization."

Since certain practically relevant classes of additional constraints (such as which lots are allowed for redistribution, or which farmers do not trade lots with certain others) can be entered into the model very easily, the optimization tool could also be applied at intermediate stages to foster the dynamics of the meeting. Naturally, at the end, solutions were available that were (at least nearly) optimal with respect to all identified additional restrictions.

Ludwig Geis, a farmer who has participated in one of the lend–lease procedures in Northern Bavaria, assesses the impact as follows: *The consequences of the implementation … are enormous, economically but also from an ecological point of view. In particular, in addition to the lower cost of cultivation, there is less need for pesticides, a higher yield, and less trouble among neighbors.*

The method has been applied in various regions. Even more, our tools have already entered the curriculum for farmers' training. In fact, the examples in the Figures 1 and 2 were produced for visualization and training purposes for schooling farmers. (Because of data protection regulations we could not use the real data of the farmers of a region.)

In a separate project with the *Bayerisches Staatsministerium für Landesentwicklung und Umweltfragen* (Bavarian Ministry for State Development and Environmental Matters) our method was further applied with a special focus on ecological issues related to aspects of environmental measures to foster biodiversity.

Additional Challenges in Forestry

In many forest regions in Northern Bavaria, an efficient and sustainable cultivation has become virtually impossible because of inheritance regulations and frequent changes of ownership. In fact, the average sizes of the lots have become less than a hectare. Further, the lots themselves often are of a shape badly suited for cultivation (e.g., long, but very

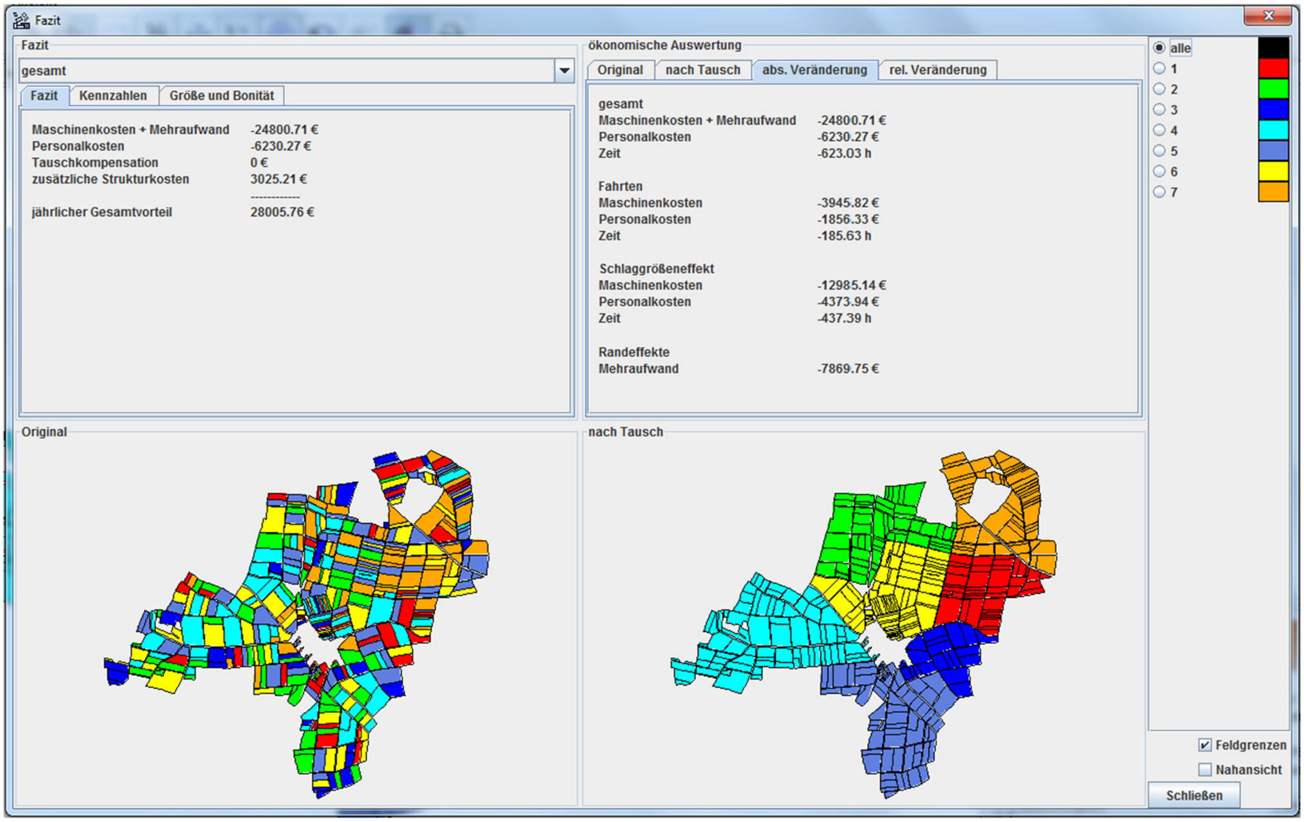

Figure 9. The tools include economic evaluations on various levels – depending on the specific structure of the region. Here stars, spanning trees and traveling salesman tours play a role, but also empirical cost functions that depend on the size and shape of collections of lots are incorporated.

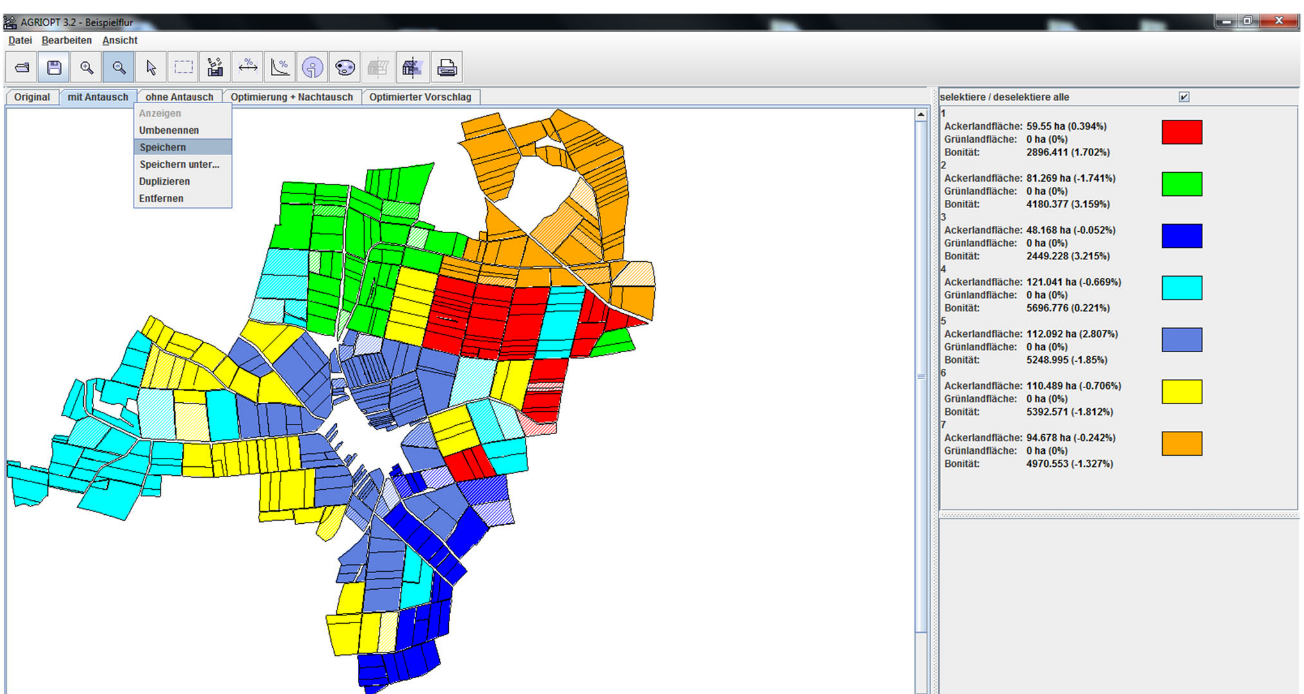

Figure 10. A manual tool for drag and drop redistribution and assessment.

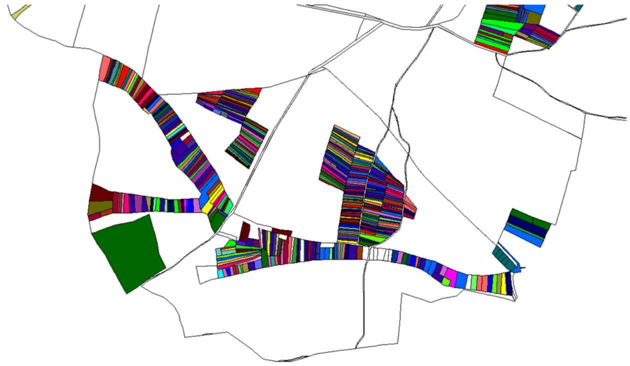

Figure 11. A typical forest region. Different colors represent different owners.

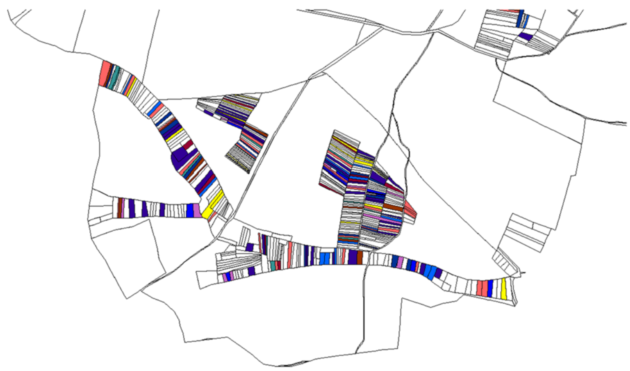

Figure 12. Selection of ten owners with highest "land exchange potential"; their 162 lots.

narrow). Hence, in principle, the problem is the same as that in agriculture. However, there are some new challenges in practice related to the different time frame of production, the different number of owners, and the different relation of the owners to their particular lots. Currently, in a project funded by the *Bayerisches Staatsministerium für Ernährung, Landwirtschaft und Forsten* (Bavarian Ministry for Food, Agriculture, and Forests) the tools are being customized for specific applications in forestry.

To conclude, we mention one additional practical preprocessing problem that is caused by the fact that even in small regions there are often several hundred different owners, many of whom own only some tiny lots. Figure 11 depicts an example; it consists of 460 lots that belong to 127 owners.

With so many different owners, it is very difficult, if not virtually impossible in practice, to enter negotiations with all stakeholders. Thus, the forestry offices want to select some smaller subset of owners that are asked to participate. Naturally, it is desirable to identify owners who provide sufficient and somehow best room for improving the cost structure in the region. After this is done, lend–lease agreements are initiated just as in agriculture. The question of identifying a best set of k owners leads mathematically to a particularly structured weighted dense subgraph problem that, while $\mathbb{NP}$-hard in general, can be tackled surprisingly efficiently; [11].

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