No Dimension Independent Core-Sets for Containment under Homothetics

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ABSTRACT

This paper deals with the containment problem under homothetics, a generalization of the minimal enclosing ball (MEB) problem. We present some new geometric identities and inequalities in the line of Jung's Theorem and show how those effect the hope on fast approximation algorithms using small core-sets as they were developed in recent years for the MEB problem.

Categories and Subject Descriptors

F.2.2 [**Theory of Computation**]: Analysis of Algorithms and Complexity, Nonnumerical Algorithms and Problems, Geometrical Problems and Computations, Pattern Matching

General Terms

Theory, Algorithms

Keywords

Core-Sets, Convex Geometry, Geometric Inequalities, Computational Geometry, Optimal Containment, Approximation Algorithms, Dimension Reduction, k-Center

1. INTRODUCTION

Many well known problems in computational geometry can be classified as some type of optimal containment problem, where the objective is to find an extremal representative C^* of a given class of convex bodies, such that C^* contains a given point set P (or vice versa). These problems arise in many different applications, e.g. facility location, shape fitting and packing problems, clustering, pattern recognition or statistical data reduction. Typical representatives are the minimal enclosing ball (MEB) problem, smallest enclosing cylinders, slabs, boxes, or ellipsoids; see [15] for a survey. Also the well known k-center problem, where P is to be covered by k homothetic copies of a given container C, has to be mentioned in this context.

SCG'11, June 13–15, 2011, Paris, France.

Because of its simple description and the multitude of both theoretical and practical applications there is vast literature concerning the MEB problem. Recently, a main focus has been on so called core-sets, i.e. small subsets S of P requiring (almost) the same dilation factor to be covered as P itself. For the Euclidean MEB problem algorithms constructing core-sets of sizes only depending on the approximation quality but neither on the number of points to be covered nor the dimension have been developed [1, 3, 10, 23], yielding a fully polynomial time approximation scheme (FPTAS) for MEB, but, maybe even more importantly, polynomial time approximation schemes (PTAS) for Euclidean k-center, which also work very well in practice [8]. However, all variants of core-set algorithms for MEB are based on the so called half-space lemma [1, 12] (or equivalent optimality conditions), a property characterizing the Euclidean ball [13], thus not allowing immediate generalization.

Our focus lies on minimal containment under homothetics:

Problem 1.1.
For a given point set $P \subset \mathbb{R}^d$ and a full-dimensional convex body $C \subset \mathbb{R}^d$ (a container) the *minimal containment problem under homothetics* (MCP$_{Hom}$) is to find the least $\rho \geq 0$, such that a translate of ρC contains P. In other words: we are looking for a solution to the following optimization problem:

$$\begin{array}{rl} \min & \rho \\ s.t. & P \subset c + \rho C \\ & c \in \mathbb{R}^d \\ & \rho \geq 0 \end{array} \tag{1}$$

where $c + \rho C := \{c + \rho x : x \in C\}$. For any pair (P, C) as described, we write $R(P, C)$ for the optimal value of (1) and call it the C-*radius* of P. Hence, if C is the Euclidean ball and P is finite this specializes to the MEB problem. If C is 0-symmetric this is the problem of computing the outer d-radius of P with respect to the norm $\|\cdot\|_C$ induced by the gauge body C as considered e.g. in [13, 14]. Choosing $P = -C$, one sees that the well known

$$\textit{Minkowski asymmetry} \qquad s(C) := R(-C, C)$$

is also a special case of containment under homothetics.

Besides direct applications Problem 1 is often the basis for solving much harder containment problems (e.g. k-center or containment under similarities), a good reason for an intensive search for good (approximation) algorithms and, with respect to the k-center results obtained in [3], especially for small core-sets. Whereas there is a rich literature on the Euclidean MEB problem that exhibits many nice properties, only little is known about the general case and how much of the Euclidean properties carry over to this problem (see [7] for an overview of possible solution strategies depending on given container classes and [21] for a new specialized algorithm for polyhedral containers, presented by their defining half-spaces).

Although the known Euclidean core-set algorithms cannot be used to find dimension independent core-sets for other classes of containers, one may think about alternative approaches yielding such core-sets. E.g. if the container has to be a homothetic copy of a given parallelotope, every (with respect to the container) diametral pair of points in P already is a core-set requiring exactly the same dilatation factor to be covered as the whole set P; and such a pair can be found in linear time.

The main goal of this work however is to show that in general there are no dimension independent core-sets for Problem 1.1, not even sublinear ones and this result cannot be improved if the container is restricted to be 0-symmetric, i.e. if it is the unit ball of some normed space. To be more precise we prove the following theorem:

THEOREM 1.2. *(No sublinear core-sets for containment under homothetics)*
For any body $P \subset \mathbb{R}^d$, any container $C \subset \mathbb{R}^d$, and $\varepsilon \geq 0$ there exists an ε-core-set of P of size at most $\left\lceil \frac{d}{1+\varepsilon} \right\rceil + 1$ and for any $\varepsilon < 1$ there exists a body $P \subset \mathbb{R}^d$ and a 0-symmetric container C such that no smaller subset of P suffices.

In order to prove the positive part of Theorem 1.2, we state several new geometric identities and inequalities between radii of convex sets, which are already of quite some value for themselves. A synopsis of the obtained inequalities is presented in Table 1 in the appendix. It is shown how these inequalities can be used to get bounds on the sizes of possible core-sets. Finally, the negative part of the theorem follows by proving that these bounds are best possible.

Notation.
Throughout this paper, we are working in d-dimensional real space and for any $A \subset \mathbb{R}^d$ we write $\mathrm{aff}(A), \mathrm{conv}(A), \mathrm{int}(A)$, and $\mathrm{bd}(A)$ for the affine hull, the convex hull, the interior, and the boundary of A, respectively. For a set $A \subset \mathbb{R}^d$, its dimension is $\dim(A) := \dim(\mathrm{aff}(A))$. Furthermore, for any two sets $A, B \subset \mathbb{R}^d$ and $\rho \in \mathbb{R}$, let $\rho A := \{\rho a : a \in A\}$ and $A + B := \{a + b : a \in A, b \in B\}$ the *ρ-dilatation* of A and the *Minkowski sum* of A and B, respectively. For short, we abbreviate $A + \{c\}$ by $A + c$. Furthermore, $\mathcal{L}_k^d$ and $\mathcal{A}_k^d$ denote the family of all k-dimensional linear and affine subspaces of $\mathbb{R}^d$, respectively, and $A|F$ is used for the orthogonal projection of A onto F for any $F \in \mathcal{A}_k^d$.

We call $C \subset \mathbb{R}^d$ a *body*, if C is convex and compact, and *container* if it is a body with $0 \in \mathrm{int}(C)$. By $\mathcal{C}^d, \mathcal{C}_0^d$ we denote the families of all bodies and all containers, respectively.

We write $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$ for the Euclidean unit ball and $x^T y$ for the standard scalar product of $x, y \in \mathbb{R}^d$. By $H^{\leq}_{(a,\beta)} := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$ we denote the half-space induced by $a \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, bounded by the hyperplane $H^{=}_{(a,\beta)} := \{x \in \mathbb{R}^d : a^T x = \beta\}$.

For $k \in \{1, \dots, d\}$, a k-simplex is the convex hull of $k+1$ affinely independent points. Additionally, let $T^d \in \mathcal{C}^d$ denote some regular d-simplex. Orientation and edge length are not specified as they will be of no interest here.

Finally, for fixed $C \in \mathcal{C}_0^d$ we denote by c_P a possible center for P, i.e. a point such that $P \subset c_P + R(P,C)C$. (Notice that for general C the center c_P may not be unique.)

2. CORE-SETS AND CORE-RADII

As already pointed out in the introduction the concept of ε-core-sets has proved very useful for the special case of the Euclidean MEB problem. Here, we introduce two slightly different definitions for the general MCP_{Hom}: core-sets and center-conform core-sets together with a series of radii closely connected to them. The explicit distinction between the two types of core-sets should help to overcome possible confusion founded in the use of the term core-set for both variants in earlier publications.

DEFINITION 2.1. *(Core-radii and ε-Core-sets)*
For $P \subset \mathbb{R}^d$, $C \in \mathcal{C}_0^d$, and $k = 1, \dots, d$, we call

$$R_k(P,C) := \max\{R(S,C) : S \subset P,\ |S| \leq k+1\}$$

the *k-th core-radius* of P.
For any $\varepsilon \geq 0$, a subset $S \subset P$ such that

(i)
$$R(S,C) \leq R(P,C) \leq (1+\varepsilon)R(S,C)$$

will be called an *ε-core-set* of P, and if

(ii)
$$P \subset c_S + (1+\varepsilon)R(S,C)C$$

a *center-conform ε-core-set* of P,

in each case with respect to C.

Obviously, every center-conform ε-core-set is also an ε-core-set, and it is shown in the appendix that an ε-core-set is a center-conform $(\varepsilon + \sqrt{2\varepsilon + \varepsilon^2})$-core-set if C is the Euclidean ball. Surely, if one is only interested in an approximation of $R(P,C)$ the knowledge of a good core-set suffices. Center-conform core-sets carry the additional information of a suitable center to cover P. However, as the set of centers can itself be quite large, fixing these might not be possible without the knowledge of the whole set.

We present lower bounds on the sizes of core-sets (as these are also lower bounds on the size of center-conform core-sets), and note that most existing positive results (i.e. construction algorithms) already hold for center-conform core-sets. When searching for lower bounds one should notice that there exist ε-core-sets of size at most $k+1$ if and only if the ratio $R(P,C)/R_k(P,C)$ is less or equal to $1+\varepsilon$.

As already observed in [13], the reason for restricting the core-radii to $k \leq d$ follows directly from Helly's Theorem (see e.g. [11]) as the following lemma shows:

LEMMA 2.2. *(0-Core-Sets)*
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $\dim(P) \leq k \leq d$. Then $R_k(P,C) = R(P,C)$, i.e. there exist (center-conform) 0-core-sets of size at most $\dim(P) + 1$ for all P and C.

Furthermore, for $k \leq \dim(P)$, there always exists a simplex $S \subset P$ such that $\dim(S) = k$ and $R(S,C) = R_k(P,C)$.

An explicit proof hereof can be found in the appendix.

Lemma 2.2 can also be seen as a result bounding the combinatorial dimension of Problem 1.1 interpreted as a Generalized Linear Program. As it is not the main focus for our purposes, the reader not familiar with GLPs may be referred to [19, 22]. In order to guaranty the locality condition of GLPs, we simply assume that C is rotund, i.e. C contains no line segments in its boundary. In that case, the optimal solution of the underlying MCP$_{Hom}$ is uniquely determined (as one may easily deduce from the convexity of the problem together with Theorem 2.4).

Remark 2.3. *(Bounding the combinatorial dimension of Problem 1.1)*
Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$ be finite, 2^P its power set, $C \in \mathcal{C}^d$ rotund, and $w : 2^P \to \mathbb{R} \cup \{\pm\infty\}$ such that $w(Q) = R(Q, C)$ for $Q \subset P$. Then (P, w) is a feasible and bounded GLP of combinatorial dimension $\delta \leq d + 1$.

A characterization of optimal solutions for the Euclidean case of the MCP$_{Hom}$ can already be found in [6]. A corollary, known as 'half-space lemma', proved very useful in the construction of fast algorithms for MEB (see, e.g. [1, 3, 12]). However, to our knowledge, literature does not contain any explicit optimality conditions for the general MCP$_{Hom}$.

For brevity, P is said to be optimally contained in C, if $P \subset C$ but there is no $c \in \mathbb{R}^d$ and $\rho < 1$ such that $P \subset c + \rho C$.

Theorem 2.4. *(Optimality condition for Problem 1.1)*
Let $P \in \mathcal{C}^d$ and $C \in \mathcal{C}_0^d$. Then P is optimally contained in C if and only if

(i) $P \subset C$ and

(ii) for some $2 \leq k \leq d+1$, there exist $p_1, \ldots, p_k \in P$ and hyperplanes $H^=_{(a_i,1)}$ supporting P and C in p_i, $i = 1, \ldots, k$ such that $0 \in \operatorname{conv}\{a_1, \ldots, a_k\}$.

The theorem stays valid even if one allows C to be unbounded.

Proof.
Let $C \in \mathcal{C}_0^d$ be given as $C = \bigcap_{a \in N} H^{\leq}_{(a,1)}$ with a suitable set N of outer normals of C.

First, assume (i) and (ii) hold. By (i), $R(P, C) \leq 1$. Now suppose $R(P, C) < 1$. Then there exists $c \in \mathbb{R}^d$ and $0 < \rho < 1$ such that $c + P \subset \rho C$. From (ii) follows $P \cap \operatorname{bd}(C) \neq \emptyset$ and therefore $c \neq 0$. Moreover, as $c + P \subset \rho C$, it follows $\sup_{a \in N} a^T(c + p_i) \leq \rho$ and in particular, $a_i^T(c + p_i) \leq \rho < 1$ for all i. Now, as $0 \in \operatorname{conv}\{a_1, \ldots, a_k\}$, there exist $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ such that $\sum_i \lambda_i a_i = 0$ and $\sum_i \lambda_i a_i^T(c+p_i) < 1$. Using $a_i^T p_i = 1$ one obtains $\sum_i \lambda_i a_i^T c < 0$, an obvious contradiction. Thus, conditions (i) and (ii) imply optimality.

Now, let P be optimally contained in C. As C is compact, we can apply Lemma 2.2 which yields $k \leq d+1$ points $p_i \in P \cap \operatorname{bd}(C)$ for $i = 1, \ldots, k$ such that

$$R(\operatorname{conv}\{p_1, \ldots, p_k\}, C) = 1. \tag{2}$$

Let $A = \{a \in N : \exists i \in \{1, \ldots, k\} \ s.t.\ a^T p_i = 1\}$. Since $P \subset C$, for $a \in A$, we have that $a^T p \leq 1$ for all $p \in P$, and $a^T p_i = 1$ for at least one i by definition of A. We will show that $0 \in \operatorname{conv}(A)$. The statement that there exists a set of at most $d+1$ outer normals with 0 in their convex hull then follows from Caratheodory's Theorem (see [11]). Assume, for a contradiction, that $0 \notin \operatorname{conv}(A)$. Then 0 can be strictly separated from $\operatorname{conv}(A)$, i.e. there exists $y \in \mathbb{R}^d$ with $a^T y \geq 1$ for all $a \in A$. Now, for $A' = \{a \in N : a^T y \leq 0\}$ there exists $\varepsilon > 0$ such that $(A' + \varepsilon \mathbb{B}^d) \cap A = \emptyset$, i.e. $a^T p_i < 1 - \varepsilon$ for all $a \in A'$ and therefore

$$a^T(p_i - \frac{\varepsilon}{\|a\|\|y\|} y) = a^T p_i + \varepsilon \frac{-a^T y}{\|a\|\|y\|} < 1.$$

Moreover, if $a \in N \setminus A'$ then

$$a^T(p_i - \frac{\varepsilon}{\|a\|\|y\|} y) = a^T p_i - \varepsilon \frac{a^T y}{\|a\|\|y\|} < 1.$$

Altogether, $p_i - \frac{\varepsilon}{\|a\|\|y\|} y \in \operatorname{int}(C)$ for all i which contradicts (2).

Finally, the statement about an unbounded container C can easily be obtained from the above by considering the new bounded container $C' = C \cap C''$ where $C'' \in \mathcal{C}_0^d$ such that $P \subset C''$ and $P \cap \operatorname{bd}(C'') = \emptyset$. $\square$

Remark: Besides the direct geometric proof of Theorem 2.4 as stated above, it would be possible to derive the result from the Karush-Kuhn-Tucker conditions for convex optimization.

Corollary 2.5.
Let $P \in \mathcal{C}^d$ and C a polytope in $\mathbb{R}^d$. If $P \subset C$ and P touches every facet of C, then P is optimally contained in C.

Moreover, if C is a polytope with facets $F_i = C \cap H^=_{(a_i,1)}$, $i = 1, \ldots, m$, it is well known [6] that with the choice $\lambda_i = \operatorname{vol}_{d-1}(F_i)$ one has $\sum_{i=1}^m \lambda_i a_i = 0$.

Now, as a further corollary we give a very transparent proof (due to [9]) for the well known fact that the Minkowski asymmetry of a body C is bounded from above by $\dim(C)$:

Corollary 2.6. *(Maximal asymmetry)*
For any $C \in \mathcal{C}^d$, the inequalities $1 \leq s(C) \leq \dim(C)$ hold, with equality, if C is 0-symmetric in the first and if C is a d-simplex in the latter case.

Proof.
Obviously, the Minkowski asymmetry is bounded from below by 1 and $s(C) = 1$ if and only if $C = -C$. For the upper bound we suppose (without loss of generality) that C is full-dimensional. Then Lemma 2.2 yields a d-simplex $S \subset C$ such that $s(C) = R(-C, C) = R(-S, C) \leq R(-S, S) = s(S)$. Thus, it suffices to show $s(S) = \dim(S)$ for any simplex S. Suppose $S = \operatorname{conv}\{x_1, \ldots, x_{d+1}\} \subset \mathbb{R}^d$ is a d-simplex, without loss of generality such that $\sum_{i=1}^{d+1} x_i = 0$. For all $i = 1, \ldots, d+1$, the center of the facet $F_j = d \cdot \operatorname{conv}\{x_i, i \neq j\}$ of dS is $c_j = \sum_{i \neq j} x_i = -x_j$. Hence $-S \subset dS$ and $-S$ touches every facet of dS, showing the optimality of the containment by Corollary 2.5. $\square$

Remark: In [16] also the 'only if' direction for the sharpness of the upper bound for d-simplices in Corollary 2.6 is shown.

In the remainder of this section, we show the identity of the (somehow discrete) core-radii to two series of intersection- and cylinder/projection-radii in classical convex geometry. This identity will help us later to use a set of well-known geometric inequalities on these radii to obtain lower bounds on core-set sizes.

Definition 2.7. *(Intersection- and cylinder-radii)* For $P \in \mathcal{C}^d$ and $C \in \mathcal{C}_0^d$ let

$$R_k^{\sigma}(P,C) := \max\{R(P \cap E, C) : E \in \mathcal{A}_k^d\} \quad (3)$$

and

$$R_k^{\pi}(P,C) := \max\{R(P, C+F) : F \in \mathcal{L}_{d-k}^d\} \quad (4)$$

Notice, that, as $C+F$ is unbounded, $R(P, C+F)$ is a slight abuse of notation.

Remark 2.8. *(Cylinder-radii in Euclidean spaces)* When dealing with the Euclidean unit ball $\mathbb{B}^d$, the cylinder-radii can be interpreted as projection-radii, i.e.

$$R_k^{\pi}(P, \mathbb{B}^d) = \max\{R(P|F, \mathbb{B}^d) : F \in \mathcal{L}_k^d\} \ .$$

The following theorem, which is proved in the appendix, states the identity of these three series of radii. To the best of our knowledge, even the equality between the intersection- and projection-radii in the Euclidean case was not shown before.

Theorem 2.9. *(Identity of intersection-, cylinder- and core-radii)*

$$R_k(P,C) = R_k^{\sigma}(P,C) = R_k^{\pi}(P,C)$$

for any $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $k \in \{1, \dots, d\}$.

Remark: The two similar series of radii, where one just replaces the max by min in (3) and (4) differ for certain values of k (e.g. when $k = 2$, consider a regular d-simplex: an optimal projection yields a $(d+1)$-gon, having an outer radius not reachable by any cut through the simplex).

3. GEOMETRIC INEQUALITIES INVOLVING CORE-RADII AND THEIR MEANING FOR CORE-SETS

In this section several geometric inequalities between the core-radii are collected and then used to derive positive or negative results on possible ε-core-set sizes. One should remember that because of Lemma 2.2 we already know the existence of 0-core-sets of size $d+1$, i.e. not depending on the size of P (nor C) and only linearly depending on d.

3.1 General (non-symmetric) containers

Theorem 3.1. *(Inequality relating core-radii)* Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$ and $k, l \in \mathbb{N}$ such that $l \le k \le d$. Then

$$\frac{R_k(P,C)}{R_l(P,C)} \le \frac{k}{l}$$

with equality if $P = -C = T^d$.

Proof.

It suffices to show

$$\frac{R_k(P,C)}{R_{k-1}(P,C)} \le \frac{k}{k-1}. \quad (5)$$

as for $l < k-1$ the claim follows by repeatedly applying (5). Without loss of generality one may assume the existence of a k-simplex $S = \mathrm{conv}\{x_1, \dots, x_{k+1}\} \subset P$ satisfying $R(S,C) = R_k(P,C)$ as (5) is certainly fulfilled if $R_k(P,C) = R_{k-1}(P,C)$. Moreover, it can also be supposed that $\sum_{i=1}^{k+1} x_i = 0$ and $R_{k-1}(S,C) = 1$. Now, let $S_j = \mathrm{conv}\{x_i : i \ne j\}$, $j = 1, \dots, k+1$ denote the facets of S. Since $R_{k-1}(S,C) = 1$, there exist translation vectors $c_j \in \mathbb{R}^d$ such that $S_j + c_j \subset C$ for all $j \in \{1, \dots, k+1\}$. Hence,

$$\sum_{j=1}^{k+1} S_j + \sum_{j=1}^{k+1} c_j \subset (k+1)C. \quad (6)$$

Furthermore, since $\sum_{i=1}^{k+1} x_i = 0$, it follows $-1/k \cdot x_j = 1/k \sum_{i \ne j} x_i \in \mathrm{conv}\{x_i : i \ne j\} = S_j$ for all j and surely $x_j \in S_i$ for all i, j, $i \ne j$. Using (6) this implies

$$(k - \frac{1}{k})x_j \in \sum_{i=1}^{k+1} S_i \subset -\sum_{i=1}^{k+1} c_i + (k+1)C$$

for all j and thus $R(S,C) \le (k+1)/(k - \frac{1}{k})$. However, since $R_{k-1}(S,C) = 1$ we obtain

$$R_k(P,C) = R(S,C) \le \frac{k}{k-1} R_{k-1}(S,C) \le \frac{k}{k-1} R_{k-1}(P,C)$$

proving (5).

The sharpness of the inequality for $-P = C = T^d$ follows directly from showing $R_k(T^d, -T^d) = k$ for $k = 1, \dots, d$:

Since every k-face F of T^d can be covered by the k-face of $-T^d$ parallel to F and since these k-faces are regular k-simplices, one may conclude as in the proof of Corollary 2.6 that F is covered by a copy of $k \cdot (-F)$, showing $R_k(T^d, -T^d) \le k$ for all k.

Finally, if F is an arbitrary face of $-T^d$ then $-T^d|F = F$. Thus, if $S_k \subset T^d$ is a k-face of T^d and S_k is optimally contained in a translate of $\rho(-T^d)$ then $-S_k$ is the k-face of $-T^d$ parallel to S_k, and $S_k| - S_k$ is a subset of a translate of $-\rho T^d| - S_k = -\rho S_k$. However, again by Corollary 2.6 it follows $R_k(T^d, -T^d) \ge \rho \ge k$. □

Corollary 3.2. *(No sublinear core-sets for general containers)* For any $P \in \mathcal{C}^d, C \in \mathcal{C}_0^d$ and $\varepsilon \ge 0$ there exists an ε-core-set of P of size at most $\left\lceil \frac{d}{1+\varepsilon} \right\rceil + 1$ and for $P = -C = T^d$ no smaller subset of P will suffice.

Proof.

The case $\varepsilon = 0$ equates to Lemma 2.2. So, let $\varepsilon > 0$ and $k = \left\lceil \frac{d}{1+\varepsilon} \right\rceil$. If $S \subset P$ such that $R(S,C) = R_k(P,C)$ then $|S| \le k+1$ and by Theorem 3.1:

$$R(P,C) \le \frac{d}{k} R(S,C) \ ,$$

with equality if $P = -C = T^d$. Now, by the choice of k it follows $\frac{d}{k} \le 1 + \varepsilon$ with equality if $P = -C = T^d$. □

Remark: Note, that by Lemma 2.2 the minimal size of a 0-core-set depends linearly on d and Corollary 3.2 now shows that allowing $\varepsilon > 0$ does not improve this situation. Thus, Corollary 3.2 already proves Theorem 1.2 for general containers.

On the positive side, one should mention that whenever C is a polytope represented by its facets, i.e. $C = \{x \in \mathbb{R}^d : a_k^T x \le 1, \ k = 1, \dots, r\}$ (therefore also when C is a regular simplex) and $P = \operatorname{conv}\{p_1, \dots, p_n\}$, Problem 1.1 can be rewritten as a Linear Program [7, 14]:

$$\begin{array}{ll} \min & \rho \\ s.t. & \rho + a_k^T c \geq \max\limits_{i=1,\dots,n} a_k^T p_i \\ & c \in \mathbb{R}^d \\ & \rho \geq 0 \end{array}$$

Here the set of points attaining $\max_{i=1,\dots,n} a_k^T p_i$ for each a_k obviously forms a 0-core-set, which can easily be computed and its size is at most the number of facets of C. (Notice that the number of facets of C must be bounded from below by $d+1$ for any full-dimensional container and as the container in the proof of Corollary 3.2 is a polytope, we have the same linear lower bound on the size of ε-core-sets when restricted to polytopal containers.)

It is also worthwhile mentioning that for the choice $P = -C = T^d$ every subset S of d vertices of P yields $R(S, C) = d - 1$, but to cover P by $c_S + \rho C$ we need $\rho \geq \frac{2d}{d-1} R(S, C)$. So, with $C = -T^d$ and $\varepsilon < 1$, we do not have any chance for a center-conform ε-core-set with less than $d+1$ points.

Moreover, the Euclidean distance of the remaining vertex to $c_S + (d-1)C$ is strictly greater than $\frac{1}{\sqrt{2}}$, showing that [20, Theorem 5] cannot be true for $\varepsilon < \frac{1}{\sqrt{2}}$ (as much as we understood it).

However, better results might still be possible by restricting the class of containers.

3.2 Positive results for two special container classes

The most evident (non-trivial) example for a restricted class of containers allowing small core-sets may be parallelotopes. E.g. in [5, §25] the following Proposition is shown:

Proposition 3.3. *(Core-radii for parallelotopes)*
The identity

$$R_1(P, C) = R(P, C)$$

holds true for all $P \in \mathcal{C}^d$ if and only if $C \in \mathcal{C}_0^d$ is a parallelotope.

In terms of core-sets this means that there is a 0-core-set of size two for all $P \in \mathcal{C}^d$, if C is a parallelotope and that these are the only containers with this property.

One should add here that in terms of center-conformity the situation is much worse: since any set S of less than $d+1$ points is subdimensional, the set of possible centers to cover S may be of size up to $R(P, C)$, even in the case that the center of P might be unique.

Surely, the more important restricted class of containers are ellipsoids. In [17] geometric inequalities are derived which relate the radii of Definition 2.7 within each series. Using Theorem 2.9, these inequalities can be presented in a unified way in terms of core-radii:

Proposition 3.4. *(Henk's Inequality)*
Let $P \in \mathcal{C}^d$ and $k, l \in \mathbb{N}$ where $l \leq k \leq d$. Then

$$\frac{R_k(P, \mathbb{B}^d)}{R_l(P, \mathbb{B}^d)} \leq \sqrt{\frac{k(l+1)}{l(k+1)}} \tag{7}$$

with equality if $P = T^d$.

Remark: Because of the affine invariance of (7) one may replace $\mathbb{B}^d$ by any d-dimensional ellipsoid.

Corollary 3.5. *(Dimension independent ε-core-sets for Euclidean containers)*
If $C = \mathbb{B}^d$ then there exits an ε-core-set of P of size at most

$$\left\lceil \frac{1}{2\varepsilon + \varepsilon^2} \right\rceil + 1$$

for any $P \in \mathcal{C}^d$ and any $\varepsilon > 0$. Moreover, this is the best possible d-independent bound.

Proof.
Let $\varepsilon > 0, k = \left\lceil \frac{1}{2\varepsilon+\varepsilon^2} \right\rceil$, and $S \subset P$ such that $R(S, \mathbb{B}^d) = R_k(P, \mathbb{B}^d)$. Then $|S| \leq k+1$ and by Proposition 3.4 and Lemma 2.2:

$$R(P, \mathbb{B}^d) \leq \sqrt{\frac{d(k+1)}{k(d+1)}} \cdot R(S, \mathbb{B}^d)$$

where k is chosen such that $\sqrt{\frac{d(k+1)}{k(d+1)}} \leq 1 + \varepsilon$ independently of $d \in \mathbb{N}$.

Now, we show the sharpness of the bound: Let $d \in \mathbb{N}$ such that $\frac{d}{d+1} > (1+\varepsilon)^2 \frac{k}{k+1}$ and choose $P = T^d$. Now, for $k < \frac{1}{2\varepsilon+\varepsilon^2}$ if $S' \subset P$ consists of no more than $k+1$ points then

$$\begin{aligned} R(P, \mathbb{B}^d) &= \sqrt{\frac{d(k+1)}{k(d+1)}} R_k(T^d, \mathbb{B}^d) \\ &> (1+\varepsilon) R_k(T^d, \mathbb{B}^d) \geq (1+\varepsilon) R(S', \mathbb{B}^d). \end{aligned}$$

Hence S' is no ε-core-set of P. □

Remark: Jung's well known inequality [18], relating the diameter and circumradius of P, can be obtained from Proposition 3.4 just by choosing $k = d$ and $l = 1$. As Proposition 3.4, it can be turned into a core-set result saying that in Euclidean spaces of any dimension a diametral pair of points in P is already a $(\sqrt{2} - 1)$-core-set.

A very easy and intuitive algorithm to actually find ε-core-sets of a finite set P was first introduced in [3]. Roughly speaking, it starts with a subset $S \subset P$ of two (good) points and computes (or approximates) the minimum enclosing ball B_S for S. Whenever a dilatation by $(1+\varepsilon)$ of B_S centred at c_S does not cover the whole set P, an uncovered point is added to S and the process is iterated. The analysis in [3] shows that this algorithm produces ε-core-sets of size $O(\frac{1}{\varepsilon^2})$, and it is quite obvious that these are even center-conform.

In [2] the existence of center-conform ε-core-sets of size $\frac{1}{\varepsilon}$ and the sharpness of this bound are shown. Corollary 3.5 shows that the lower bound on the size of the core-sets may only be reduced by a factor of 2, when dropping the center-conformity condition. The latter bound is the somewhat stronger result, as it shows that more freedom in the choice of the core-set does not really improve the situation. On the other hand [2] also presents a construction routine (which is mainly of theoretical value) for their center-conform ε-core-sets of size $\frac{1}{\varepsilon}$.

3.3 Symmetric containers / general normed spaces

As mentioned before, every 0-symmetric container $C \in \mathcal{C}_0^d$ induces a norm $\|\cdot\|_C$ and vice versa. We will always talk about symmetric containers here, but surely one may reformulate all results in terms of general Minkowski spaces.

The positive results in section 3.2 may motivate the hope that symmetry of the container is the key for positive results on dimension-independence.

In [4], Bohnenblust proved an equivalent to Jung's Inequality (see the remark after Corollary 3.5) for general normed spaces. Taking into account the Minkowski asymmetry $s(C)$ of a possibly asymmetric container C, a slightly generalized result and a simplified proof are derived in [8, Lemma 2]; in terms of core-radii it reads as follows:

Proposition 3.6. *(Generalized Bohnenblust)*
Let $P \in \mathcal{C}^d$, $C \in \mathcal{C}_0^d$. Then

$$\frac{R(P,C)}{R_1(P,C)} \leq \frac{(1+s(C))d}{d+1}$$

with equality, if $P = T^d = -C$ or $P = T^d$ and $C = T^d - T^d$.

For completeness a proof formulated in terms of core-radii is given in the appendix.

When looking at the similarity of Jung's and Bohnenblust's inequalities and the way in which Jung's Inequality generalizes to Henk's Inequality (compare Table 1 in the Appendix), one might conjecture

$$\frac{R_k(P,C)}{R_l(P,C)} \overset{?}{\leq} \frac{k(l+1)}{l(k+1)} \quad \text{if } C \in \mathcal{C}_0^d \text{ is 0-symmetric.} \tag{8}$$

This conjecture may even be more encouraged by the fact that one can show that (8) is tight for $P = T^d$ and $C = T^d - T^d$.

If provable, (8) would yield dimension independent ε-core-sets for any $\varepsilon > 0$ and any 0-symmetric container in the same way as shown for Euclidean spaces in Corollary 3.5. However, (8) *is false* for general 0-symmetric $C \in \mathcal{C}_0^d$:

Lemma 3.7.
With $C^d = T^d \cap (-T^d)$,

$$R_k(T^d, C^d) = \begin{cases} \frac{d+1}{2} & \text{if } k \leq \frac{d+1}{2} \\ k & \text{if } k \geq \frac{d+1}{2} \end{cases}.$$

The proof of this lemma gives an explicit center how to cover T^d with a translate of the appropriately scaled copy of C^d and shows via Theorem 2.4 that this inclusion is best possible. The calculations being rather technical, the proof has been moved to the appendix.

Theorem 3.8. *(The correct inequality for general 0-symmetric containers)*
Let $k, l \in \mathbb{N}$ such that $l \leq k \leq d$, $P \in \mathcal{C}^d$ and $C \in \mathcal{C}_0^d$ a 0-symmetric container. Then

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2} \end{cases}.$$

Moreover, let T^k be a k-simplex embedded in the first k coordinates of $\mathbb{R}^d$ and $C^k = (T^k \cap (-T^k)) + (\{0\}^k \times [0,1]^{d-k})$. Then

$$\frac{R_k(T^k,C^k)}{R_l(T^k,C^k)} = \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2} \end{cases}.$$

Proof.
Let $S \subset P$ be a k-simplex such that $R_k(P,C) = R(S,C)$ and assume without loss of generality that $R(S,C) = k$. By Bohnenblust's Inequality, we get that $R_1(S,C) \geq (k+1)/2$ and thus $R_l(P,C) \geq R_1(P,C) \geq (k+1)/2$. Thus $R_k(P,C)/R_l(P,C) \leq 2k/(k+1)$. On the other hand

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k}{l}$$

by Theorem 3.1. Together this yields

$$\frac{R_k(P,C)}{R_l(P,C)} \leq \min\left\{\frac{2k}{k+1}, \frac{k}{l}\right\}$$

which splits into the two cases claimed above. The second statement follows from Lemma 3.7 and the observation that the computation of $R(T^k, C^k)$ is in fact the k-dimensional containment problem of containing T^k in $T^k \cap (-T^k)$. □

With Theorem 3.8 at hand, Theorem 1.2 follows as a simple corollary:

Proof of Theorem 1.2.
For $k = d$ and $l \geq (d+1)/2$ the inequalities in Theorem 3.1 and 3.8 coincide. Hence the proof of Corollary 3.2 can simply be copied up to the additional condition that $\varepsilon < 1$ and the change from $C = T^d$ to $C = T^d \cap (-T^d)$ to show that the bound is best possible. □

On the other hand diametral pairs of points in P are 1-core-sets for any 0-symmetric container C as Bohnenblust's result already shows. Theorem 3.8 then just shows that an arbitrary choice of up to $\lfloor (d-3)/2 \rfloor$ points to add to the core-set may not improve the approximation quality.

Remark: Obviously, Theorem 1.2 also shows the non-existence of sublinear center-conform ε-core-sets for $\varepsilon < 1$. On the other hand we know from Lemma 2.2 that there are linear ones, even if $\varepsilon = 0$.

Acknowledgements. We would like to thank the anonymous referees for their suggestions and for pointing out some additional relevant literature.

APPENDIX

After a short remark on the center-conformity of Euclidean core-sets, this appendix states the proofs of Lemma 2.2 Theorem 2.9, Proposition 3.6 and Lemma 3.7. At the very end, Table 1 summarizes all derived geometric inequalities among the core-radii.

Remark:
If $P \in \mathcal{C}^d$, $\varepsilon > 0$ and $S \subset P$ is an ε-core-set of P with respect to $\mathbb{B}^d$, then S is also a center-conform $(\varepsilon + \sqrt{2\varepsilon + \varepsilon^2})$-core-set of P.

Proof.

Let $p \in P$ such that $\max_{x \in P} \|c_S - x\|_2 = \|c_S - p\|_2$. Further let H be a hyperplane perpendicular to $\mathrm{aff}\{c_S, c_P\}$ passing through c_S. Denote by H^- the halfspace which is bounded by H and does not contain c_P. Then by Theorem 2.4 (or equivalently the half-space lemma cited in the introduction), there is a point $q \in S \cap H^-$ at distance $R(S, \mathbb{B}^d)$ of c_S. Hence

$$\begin{aligned}\|c_S - c_P\|_2^2 &\le \|c_p - q\|_2^2 - \|q - c_S\|_2^2 \\ &\le R(P, \mathbb{B}^d)^2 - R(S, \mathbb{B}^d)^2 \\ &\le (2\varepsilon + \varepsilon^2) R(S, \mathbb{B}^d)^2\end{aligned}$$

and

$$\begin{aligned}\|c_S - p\| &\le \|c_S - c_P\| + \|c_p - p\| \\ &\le \sqrt{2\varepsilon + \varepsilon^2} R(S, \mathbb{B}^d) + R(P, \mathbb{B}^d) \\ &= (1 + \varepsilon + \sqrt{2\varepsilon + \varepsilon^2}) R(S, \mathbb{B}^d).\end{aligned}$$

□

Proof of Lemma 2.2.

Clearly, $R_k(P, C) \le R(P, C)$. To show $R_k(P, C) \ge R(P, C)$ for $k \ge \dim(P)$, observe that by definition of $R_k(P, C)$, any $S \subset P$ with $|S| \le k + 1$ can be covered by a copy of $R_k(P, C)C$. This means $\bigcap_{p \in S}(p - R_k(P, C)C) \ne \emptyset$ for all such S. Now, as the sets $p - R_k(P, C)C$ are compact, Helly's Theorem applied within $\mathrm{aff}(P)$ yields $\bigcap_{p \in P}(p - R_k(P, C)C) \ne \emptyset$. Thus the whole set P can be covered by a single copy of $R_k(P, C)C$.

Moreover, by applying Helly's Theorem within $\mathrm{aff}(S)$ one may always assume that the finite set S with $R(S, C) = R_k(P, C)$ is affinely independent. Hence, if $|S| \le k \le \dim(P)$ one may complete S to the vertex set of a k-dimensional simplex within P. □

Proof of Theorem 2.9.

The inequality $R_k^\sigma(P, C) \le R_k^\pi(P, C)$ is obvious, so it suffices to show $R_k^\pi(P, C) \le R_k(P, C) \le R_k^\sigma(P, C)$.

First, $R_k(P, C) \le R_k^\sigma(P, C)$: By definition of the core-radii there exists $S \subset P$ with $|S| = k + 1$ and $R(S, C) = R_k(P, C)$. Since $\dim(\mathrm{aff}(S)) \le k$, one obtains

$$R_k(P, C) = R(S, C) \le R(P \cap \mathrm{aff}(S), C) \le R_k^\sigma(P, C).$$

Now, $R_k^\pi(P, C) \le R_k(P, C)$: Let $F \in \mathcal{L}_{d-k}^d$ such that $R_k^\pi(P, C) = R(P, C + F)$ and suppose without loss of generality that P is optimally contained in $C + F$ (i.e. the optimal radius and center are $\rho^* = 1$ and $c^* = 0$, respectively). Then it follows from Theorem 2.4 that there exist $m \le d + 1$ points $p_1, \dots, p_m \in P$ and hyperplanes $H^=_{(a_i,1)}$, $i = 1, \dots, m$ such that $H^=_{(a_i,1)}$ supports $C + F$ in p_i and $0 \in \mathrm{conv}\{a_1, \dots, a_m\}$. Since every direction in F is an unbounded direction in $C + F$ one obtains $a_i \in F^\perp$ for all $i = 1, \dots, m$. Now, by Caratheodory's Theorem, there exists a subset $I \subset \{1, \dots, m\}$ with $|I| \le \dim(F^\perp) + 1 = k + 1$ such that $0 \in \mathrm{conv}\{a_i : i \in I\}$. Applying again Theorem 2.4,

$$R_k^\pi(P, C) = R(P, C + F) = R(\mathrm{conv}\{p_i : i \in I\}, C + F) \le$$

$$\le R(\mathrm{conv}\{p_i : i \in I\}, C) \le R_k(P, C).$$

□

Proof of Proposition 3.6.

Without loss of generality, suppose $R_1(P, C) = 1$. This means, for arbitrary $p_1, p_2 \in P$, there is a $c \in \mathbb{R}^d$ such that $p_1, p_2 \in c + C$; explicitly, $p_1 = c + v$ and $p_2 = c + w$ with $v, w \in C$. Hence $p_1 - p_2 = v - w \in C - C$ for all $p_1, p_2 \in P$ and thus $P - P \subset C - C$. Using Corollary 2.6 one obtains

$$\left(1 + \frac{1}{d}\right) P = P + \frac{1}{d} P \subset P - P \subset C - C$$

$$\subset C + s(C)C = (1 + s(C))C$$

and therefore

$$\frac{R(P, C)}{R_1(P, C)} \le \frac{1 + s(C)}{1 + 1/d} = \frac{(1 + s(C))d}{d + 1}$$

Finally, the case $P = T^d = -C$ was shown in 3.1 and one can easily see that by choosing $P = T^d$ and $C = T^d - T^d$ follows $(1 + \frac{1}{d})P$ is optimally contained in $P - P = C$. Hence, $R(P, C) = d/(d + 1)$ and $R_1(P, C) = 1/2$ as the diameter of a body stays equal under central symmetrization. □

Proof of Lemma 3.7.

Let $T^d = \mathrm{conv}\{x_1, \dots, x_{d+1}\} = \bigcap_{i=1}^{d+1} H^{\le}_{(a_i,1)}$ for suitable $a_i \in \mathbb{R}^d$, indexed such that

$$a_j^T x_i = \begin{cases} 1 & \text{if } j \ne i \\ -d & \text{if } j = i \end{cases} \quad \text{for } i, j \in \{1, \dots, d + 1\}$$

Let $k \in \{1, \dots, d + 1\}$ and consider an arbitrary k-face F of T^d, without loss of generality $F = \mathrm{conv}\{x_1, \dots, x_{k+1}\}$.

For $k \le \frac{d+1}{2}$, let $\gamma = -\frac{(k-1)(d+1)}{2(k+1)}$ and $c = \frac{1}{k+1} \sum_{l=1}^{k+1} x_l + \frac{\gamma}{d-k} \sum_{l=k+2}^{d+1} x_l$. Then $-\frac{(k-1)(d+1)}{2(d-k)} \ge -\frac{d+1}{2}$ and for $i \in \{1, \dots, k + 1\}$ and $j \in \{1, \dots, d + 1\}$

$$a_j^T (x_i - c) = \begin{cases} -\frac{(k-1)(d+1)}{2(d-k)} & \text{if } j > k + 1 \\ \frac{d+1}{2} & \text{if } j \le k + 1, j \ne i \\ -\frac{d+1}{2} & \text{if } j = i\ . \end{cases}$$

Hence $F - c \subset \frac{d+1}{2} C^d$. Moreover, the computation shows that $T^d - c$ touches the facets of $\frac{d+1}{2} C^d$ induced by the hyperplanes $H^=_{(a_i,(d+1)/2)}$, $H^=_{(a_i,-(d+1)/2)}$ for $i = 1, \dots, k + 1$ and therefore it follows by Theorem 2.4 that $R_k(T^d, C^d) = \frac{d+1}{2}$.

For $k \ge \frac{d+1}{2}$, let $c = \sum_{i=1}^{k+1} x_i$. Then $1 - k + d \le k$ and for $i \in \{1, \dots, k + 1\}$ and $j \in \{1, \dots, d + 1\}$

$$a_j^T (x_i - c) = \begin{cases} -k & \text{if } j > k + 1 \\ 1 - k + d & \text{if } j \le k + 1, j \ne i \\ -k & \text{if } j = i \end{cases}$$

showing $F - c \subset kC^d$. Here again, the computation shows that $T^d - c$ touches every facet of $-kT^d$ and $R_k(T^d, C^d) = k$ follows by Theorem 2.4. □

$C = \mathbb{B}^d$	$\frac{R(P,\mathbb{B}^d)}{R_1(P,\mathbb{B}^d)} \leq \sqrt{\frac{2d}{d+1}}$ Jung's Inequality	$\frac{R_k(P,\mathbb{B}^d)}{R_l(P,\mathbb{B}^d)} \leq \sqrt{\frac{k(l+1)}{l(k+1)}}$ Henk's Inequality
$C \in \mathcal{C}_0^d$ 0-symm.	$\frac{R(P,C)}{R_1(P,C)} \leq \frac{2d}{d+1}$ Bohnenblust's Inequality	$\frac{R_k(P,C)}{R_l(P,C)} \leq \begin{cases} \frac{2k}{k+1} & \text{for } l \leq \frac{k+1}{2} \\ \frac{k}{l} & \text{for } l \geq \frac{k+1}{2} \end{cases}$ Theorem 3.8
$C \in \mathcal{C}_0^d$	$\frac{R(P,C)}{R_1(P,C)} \leq d$ Theorem 3.1	$\frac{R_k(P,C)}{R_l(P,C)} \leq \frac{k}{l}$ Theorem 3.1

Table 1: Synopsis of the inequalities between core-radii that were collected in this paper.

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