Precise prediction of MSSM Higgs boson masses combining fixed-order and effective field theory calculations

Henning Bahl

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Technische Universität München
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

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Abstract

The Minimal Supersymmetric Standard Model (MSSM) is one of the most common models of physics beyond the Standard Model. One of its distinct features is the possibility to predict the mass of the lightest $\mathcal{CP}$-even Higgs boson in terms of only a few relevant model parameters. These calculations fall into two categories: calculations at a fixed-order in perturbation theory and effective field theories (EFT) calculations. Fixed-order calculations capture all effects at a given order in perturbation theory. In case of a large hierarchy between the electroweak scale and the scale of the non-standard particles they, however, become unreliable because large logarithms spoil the perturbative expansion. These logarithms can be resummed with the help of EFT methods. EFTs are therefore reliable for largely separated scales. If the scales are, however, close to each other, terms which would be suppressed in case of a large separation are missed. These are included in fixed-order calculations.

To profit from the advantages of both approaches, we combine both methods building upon the existing fixed-order calculation implemented into the publicly available program `FeynHiggs`. In this process, aspects like double-counting and the involved renormalization schemes have to be taken into account. Extending previous work, we use full two-loop renormalization group equations (RGEs) and full one-loop threshold corrections to include electroweak contributions at the leading and next-to-leading logarithmic level of accuracy into this hybrid approach. Furthermore, we implement the resummation of next-to-next-to-leading logarithms in the approximation of vanishing electroweak gauge couplings employing three-loop RGEs and two-loop threshold corrections. In most cases, these improvements lead to a down shift of about 1 GeV for the standard-like Higgs boson mass. In addition, we investigate the effect of separate thresholds for the superpartners of the gauge and Higgs bosons. These thresholds only yield sizeable contributions for very large scale separations.

After these improvements, which bring the hybrid approach to the same logarithmic level of accuracy as state-of-the-art pure EFT calculations, we perform a detailed comparison of the hybrid approach to these pure EFT calculations. In the course of this comparison, we reveal several differences between both approaches: Firstly, we show how a conversion of the renormalization scheme can spoil the resummation of large logarithms. Secondly, we address terms induced by the Higgs pole mass determination in the fixed order approach. Taken these effects into account, we find good agreement between both approaches for high scales. We are able to explain the remaining numerical differences for large SUSY scales of about 1 GeV by the different parametrization of non-logarithmic terms.

These studies are conducted mainly with a high-scale or split scenario in mind. If the mass of non-standard Higgs bosons is, however, comparable to the electroweak scale, the corresponding EFT is better described by a Two-Higgs-Doublet-Model. To handle such cases accurately, we implement such a low-energy Two-Higgs-Doublet-Model into our hybrid framework. For the combination of the EFT calculation with the fixed-order calculation, we find the normalization of the Higgs doublets to play a crucial role. In our numerical investigation, we find large effects of up to 8 GeV in scenarios with light non-standard Higgs bosons.

All of these results have been implemented into the `Fortran` code `FeynHiggs`.

Zusammenfassung

Das minimale supersymmetrische Standard Modell (MSSM) ist eines der weitverbreitesten Modelle für Physik jenseits des Standard Modells. Die Möglichkeit, die Masse des leichtesten $\mathcal{CP}$-geraden Higgs-Bosons in Abhängigkeit einiger weniger relevanten Parametern vorherzusagen, ist ein charakteristisches Merkmal des MSSM. Es gibt zwei Arten von Rechnungen: Rechnungen in fester Ordnung der Störungstheorie und Rechnungen basierend auf effektiven Feldtheorien (EFT). Rechnungen in fester Ordnung erfassen alle Effekte bis zu einer gewissen Ordnung der Störungstheorie. Wenn die elektroschwache Skala und die Skala der Nichtstandardteilchen weit separiert sind, werden sie aber aufgrund von großen logarithmischen Beiträgen, die die Konvergenz der Störungsreihe verschlechtern, unzuverlässig. Diese Logarithmen können allerdings mithilfe von effektiven Feldtheorien resummiert werden. Diese sind deswegen präzise für weit separierte Skalen. Wenn die Skalen allerdings von vergleichbarer Größenordnung sind, können vernachlässigte Terme, die für weit separierte Skalen unterdrückt sind, numerisch relevant werden. Diese sind wiederum in Rechnungen in fester Ordnung enthalten.

Um von den Vorteilen beider Zugänge zu profitieren, werden beide Methoden auf der Basis der bereits existierenden Rechnung in fester Ordnung, die im öffentlich verfügbaren Programm `FeynHiggs` implementiert ist, kombiniert. Zunächst wird dieser hybride Zugang mit den elektroschwachen Beiträgen auf führendem und nächst-führendem logarithmischen Niveau durch Berücksichtigung der vollen Zwei-Schleifen-Renormierungsgruppengleichungen und der vollen Ein-Schleifen-Schwellenkorrekturen erweitert. Außerdem wird die Resummierung von übernächst-führenden Logarithmen unter der Vernachlässigung von elektroschwachen Eichkopplungen unter Benutzung von Drei-Schleifen-Renormierungsgruppengleichungen und Zwei-Schleifen-Schwellenkorrekturen implementiert. Diese Erweiterungen führen in den meisten Fällen zu einer Absenkung der standardartigen Higgsmasse von etwa 1 GeV. Zudem wird der Einfluss von unabhängigen Schwellen für die Superpartner der Eich- und Higgsbosonen untersucht. Diese Schwellen liefern nur für sehr große Separationen der relevanten Massen einen nicht vernachlässigbaren Beitrag.

Durch diese Erweiterungen ist die Hybridmethode von derselben logarithmischen Präzision wie die besten verfügbaren reinen EFT Rechnungen. Durch einen Vergleich dieser beiden Methoden werden mehrere Unterschiede herausgearbeitet: Zunächst wird gezeigt, wie eine Umrechnung zwischen verschiedenen Renomierungsschemata die Resummierung großer Logarithmen stört, und zweitens detailliert untersucht, wie durch die Bestimmung der Higgspolmassen Terme höhere Ordnung erzeugt werden. Unter Berücksichtigung dieser Effekte sind beide Methode für große Skalen in sehr guter Übereinstimmungen. Die verbleibenden Unterschiede von unter 1 GeV können durch verschiedene Parametrisierungen der nicht-logarithmischen Terme erklärt werden.

Bei den obigen Untersuchungen werden hauptsächlich Szenarien mit nur einer hohen Skala oder Split-Szenarien betrachtet. Wenn die Massen der nicht standardartigen Higgsbosonen allerdings in der Nähe der elektroschwachen Skala liegt, lässt sich die entsprechende EFT besser durch ein Two-Higgs-Doublet-Model beschreiben. Um solche Fälle korrekt berücksichtigen zu können, wird ein solches Two-Higgs-Doublet-Model in den oben beschriebenen hybriden Ansatz implementiert. Bei der Kombination von EFT-Rechnung und der Rechnung in fester Ordnung spielt die Normierung der Higgs-Dubletts eine wichtige Rolle. Numerisch verursacht diese Implementierung große Effekte von bis zu 8 GeV im Falle von leichten Nichtstandard-Higgsbosonen. Alle Resultate dieser Arbeit wurden in den `Fortran` Code `FeynHiggs` implementiert.

Contents

Chapter 1

Introduction

By now, nearly all sectors of the Standard Model (SM) of particle physics have been tested extensively against experimental results. The Higgs sector, however, has become experimentally accessible only a few years ago. In this respect, the landmark discovery of a Higgs boson by ATLAS [1] and CMS [2] at the Large Hadron Collider (LHC) at the European Organization for Nuclear Research (CERN) is of foremost interest. Its mass was measured to be [3]

$$M_h^{\text{exp}} = 125.08 \pm 0.21(\text{stat.}) \pm 0.11(\text{sys.})\ \text{GeV}. \tag{1.1}$$

This measurement fixes the last free parameter of the SM. All measurements related to the Higgs sector are in agreement with SM predictions so far [4]. There is, however, still ample room left for beyond the SM (BSM) physics.

This is especially relevant in view of the fact that despite the success of the SM in making physical predictions, it is clear that BSM physics is needed. Several observations are not explainable within the SM framework: neutrino oscillations, the baryon asymmetry of the universe, dark matter and dark energy. From a theoretical point of view, the SM lacks an explanation for the smallness of the Higgs mass and for the $\mathcal{CP}$-violating θ-term of QCD.

Several concepts have been proposed to face these problems. One of the most common ones is Supersymmetry (SUSY). SUSY explains the smallness of the Higgs mass and provides a dark matter candidate in many phenomenological realizations. In addition, it might help to explain the baryon asymmetry of the universe. Its fundamental idea is to extend the Lorentz spacetime symmetry in the only possible non-trivial way by relating fermions to bosons. The simplest phenomenological model incorporating this concept is the Minimal Supersymmetric Standard Model (MSSM), which extends in the SM in a minimal way to allow for SUSY. As an immediate consequence of this requirement each SM degree of freedom is accompanied by a superpartner, whose spin is shifted by one half with respect to the spin of the corresponding SM particle.

The requirement of SUSY, however, not only enforces the presence of superpartners but also implies the need to introduce a second complex Higgs doublet in order to avoid the appearance of gauge anomalies and to ensure the holomorphicity of the superpotential. This leads to five physical Higgs bosons. In the case of only real parameters (real MSSM), these are the neutral $\mathcal{CP}$-even h and H bosons, the $\mathcal{CP}$-odd A boson and the charged $H^{\pm}$ bosons. One of the $\mathcal{CP}$-even Higgs bosons has to play the role of the SM-like scalar boson discovered at the LHC. As a consequence of SUSY, the Higgs sector depends at the tree-level only on the electroweak gauge couplings and two additional parameters, often chosen to be the mass of the A boson M_A and the ratio of the vacuum expectation values (vevs) of the two doublets, $\tan\beta = v_2/v_1$.

So far no direct experimental evidence for supersymmetric particles has been found. This sets limit on their masses. In addition to these direct searches, the impact of supersymmetric particles on precision observables can be used to constrain the parameter space indirectly. Classical precision observables are for example the mass of the W boson, the effective electroweak mixing angle or the width of the Z boson. As mentioned already, it is a distinct feature of the MSSM that the mass of the SM-like boson discovered at the LHC is not a free parameter, as in the SM, but can be predicted in terms of a few relevant parameters. Therefore, it can be used as an additional precision observable complementing the classical ones.

The tree-level Higgs boson masses are heavily affected by quantum effects. Therefore, to fully exploit the precision reached in the experimental measurement much work has been dedicated to the calculation of higher-order contributions. So far, the full one-loop corrections [5–8], dominant two-loop corrections [9–32] and partial three-loop corrections [33–35] for the light MSSM Higgs boson mass have been calculated by the method of Feynman diagrams. These diagrammatic calculations have the advantage of capturing all terms at a given order in perturbation theory and are therefore expected to be precise for low SUSY scales where logarithmic contributions involving the SUSY scale, contained in the result, are small.

In light of the increasing direct bounds on supersymmetric particles a complementary approach has become increasingly popular: Effective field theory (EFT) calculations allow to resum the logarithmic contributions by means of renormalization group equations (RGEs). These logarithms become large in case of a large mass hierarchy between the electroweak and the SUSY scale [36–40] and can spoil the convergence of the perturbative expansion of fixed-order calculations. EFT calculations, however, are less accurate for relatively low SUSY mass scales owing to the omission of terms which are suppressed only for high SUSY scales. These terms would be generated by higher-dimensional operators in the EFT framework. In most EFT calculations the effect of higher-dimensional operators has been neglected (see [41] for work in this direction).

In order to profit from the advantages of both methods – high accuracy for low SUSY scales in the case of the fixed-order approach versus high accuracy for high SUSY scales in the case of the EFT approach – both approaches have been combined [42] (see [43, 44] for hybrid approaches using a different method). For this combination several subtleties have to be taken into account: The double-counting of terms obtained in both approaches has to be avoided and a conversion between the different employed renormalization schemes has to be performed. This hybrid approach has been implemented into the publicly available code `FeynHiggs` [8, 14, 42, 45–49] supplementing the existing fixed-order calculation with higher-order resummed logarithmic contributions. The advanced development of the hybrid approach is the main content of this thesis.

The original implementation, presented in [42], was restricted to the resummation of leading (LL) and next-to-leading (NLL) logarithms in the limit of vanishing electroweak gauge couplings. In pure EFT calculations however, already a higher level of logarithmic accuracy is available. Therefore, as the first main achievement of this thesis, we advance the hybrid approach introduced in [42] and implemented in `FeynHiggs` the same logarithmic accuracy: We include the missing electroweak contributions at the LL and NLL level as well as next-to-next-to-leading logarithm (NNLL) resummation in the limit of vanishing electroweak gauge couplings. In addition, separate thresholds for the supersymmetric partners of the gauge and Higgs bosons – called gauginos and Higgsinos – are implemented. This improves the result for scenarios in which these gauginos and Higgsinos are lighter than the other SUSY particles.

Having the same logarithmic accuracy, we would expect pure EFT calculations and our hybrid approach to yield very similar results for high SUSY scales. Comparisons between `FeynHiggs` and pure EFT codes in the literature [40, 43, 44] have, however, revealed non-negligible differences between the predicted values for M_h for large SUSY scales. Therefore, we compare our hybrid approach in detail to the pure EFT approach finding three main sources for the observed discrepancy: In a first step, we investigate how the use of different renormalization schemes affects the comparison. `FeynHiggs` by default employs a mixed $\overline{\mathrm{DR}}$/OS scheme, whereas the EFT codes employ a pure $\overline{\mathrm{DR}}$ scheme for the input parameters. We show that the in such cases usually used renormalization scheme conversion of input parameters is not suitable for the comparison of results containing a series of higher-order logarithms. Such a scheme conversion can lead to large shifts corresponding to formally uncontrolled higher-order terms. Secondly, we analytically identify specific terms arising through the determination of the Higgs propagator pole which cancel with subloop renormalization contributions in the irreducible self-energies of the diagrammatic approach for a large SUSY scale. We develop an improved treatment where unwanted effects from incomplete cancellations are avoided. Thirdly, we show how different parametrizations of non-logarithmic terms can explain remaining differences between the results of `FeynHiggs` and pure EFT codes for high scales. This reconciliation of our hybrid approach with the pure EFT approach is the second main achievement of this thesis.

The discussion is so far mainly restricted to scenarios in which the EFT below the SUSY scale is well described by an effective SM or a SM with added gauginos and Higgsinos. This assumes that the non-SM Higgs boson scale is close to the SUSY scale. In the presence of light non-SM Higgs bosons, the EFT below the SUSY scale is better described by an effective Two-Higgs-Doublet-Model (THDM) or a THDM with added gauginos and Higgsinos. These scenarios are especially interesting in the light of the increasingly tight constrains on colored SUSY particles from experimental searches. A previous pure EFT study [39] found large effects originating in the resummation of logarithms of the SUSY scale over the non-SM Higgs scale M_A. As third main achievement, we clarify this situation and implement such an effective THDM into our hybrid approach describing in detail the steps needed to combine the EFT calculation with the fixed-order calculation in this scenario.

This thesis is structured as follows. In the beginning, introductory chapters give a short address to the SM (Chapter 2), Supersymmetry (Chapter 3) and the MSSM (Chapter 4) with a particular focus on the MSSM Higgs sector. Afterwards, we briefly review in Chapter 5 how higher-order corrections are calculated, how these are renormalized and how EFT methods can be used to resum large logarithms. We apply the renormalization procedure to the MSSM Higgs sector in Chapter 6. Afterwards, we come to the main subject of the thesis: In Chapter 7, we describe how the Higgs masses are calculated in the fixed-order and the EFT approach and subsequently explain in detail our hybrid approach. After having clarified all methods used in this thesis, we turn to the actually achieved advances of the hybrid approach. We present the various improvements of the involved EFT calculation in Chapter 8. All issues related to the use of different renormalization schemes are disscussed in Chapter 9. Based upon these achievements, we compare our hybrid approach to the pure EFT approach in Chapter 10 focusing on how the physical mass is obtained as pole of the Higgs propagator. In Chapter 11, we extend our hybrid framework to scenarios with light non-SM Higgs bosons. After comparing `FeynHiggs` to other public codes for the calculation of the MSSM Higgs masses (Chapter 12), we present numerical results in Chapter 13. In this Chapter, we also briefly discuss the remaining theoretical uncertainties. Afterwards, conclusion are presented in Chapter 14. In the Appendix, we provide additional material related to the shifts induced by a renormalization scheme conversion of the input parameters (App. A), logarithms arising through the determination of the Higgs propagator poles (App. B), the used matching conditions between the various EFTs (App. C), the normalization of the Higgs doublets (App. D), the conversion of relevant parameters between the $\overline{\text{DR}}$ and the OS scheme (App. E), as well as the used, before only partly known, renormalization group equations (App. F).

Chapter 2

The Standard Model

2.1 General structure and particle content

The Standard Model (SM) of particle physics is a gauge theory based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ subgroup leads to the strong interaction [50–53], whereas the $SU(2)_L \times U(1)_Y$ subgroup is responsible for the electroweak interactions [54–56]. The corresponding gauge bosons are

- the gluons G^a_μ for $SU(3)_C$ $(a = 1, ..., 8)$ with the gauge coupling g_3,
- the W bosons W^a_μ for $SU(2)_L$ $(a = 1, 2, 3)$ with the gauge coupling g, and
- the B boson B_μ for $U(1)_Y$ with the gauge coupling g'.

In this thesis, $\alpha_s \equiv g_3^2/4\pi$ is sometimes used instead of g_3. The W- and B-Bosons are related to the physical mass eigenstates $W^\pm_\mu$, Z_μ and the photon A_μ, which can be obtained through rotation of the original states with the electroweak mixing angle.

The matter fields of the SM are grouped into three generations. In addition to this overlaying structure, the fermion fields can be categorized according to their behaviour under the SM gauge-group transformations:

left-handed quark doublets	:	$(\mathbf{3}, \mathbf{2})_{1/1}$
right-handed u-type quarks	:	$(\mathbf{3}, \mathbf{1})_{4/3}$
right-handed d-type quarks	:	$(\mathbf{3}, \mathbf{1})_{-2/3}$
left-handed lepton doublets	:	$(\mathbf{1}, \mathbf{2})_{-1}$
right-handed lepton singlets	:	$(\mathbf{1}, \mathbf{1})_{-2}$

Table 2.1: List of SM fermions and their behaviour under gauge-group transformations.

The first number in the round brackets corresponds to $SU(3)_C$ (triplet or singlet), the second one to $SU(2)_L$ (doublet or singlet) and the subscript to the $U(1)_Y$ quantum number (hypercharge).

In addition to the particles listed above, also so-called Faddeev-Popov ghosts are present in the theory. They are unphysical and appear only as virtual particles in Feynman amplitudes. Their introduction is necessary to cancel the effects of the unphysical timelike and longitudinal polarizations of the gauge bosons in a non-Abelian theory and thus to conserve unitarity.

Furthermore, a scalar sector, the Higgs sector, is needed for the generation of gauge boson and fermion masses.

2.2 Higgs sector

Most of the SM particles are massive. This is an apparent problem for the theory, since a mass term in the SM-Lagrangian violates the gauge symmetry. The Higgs mechanism [57–59] solves this problem by breaking the $SU(2)_L \times U(1)_Y$ symmetry of the SM spontaneously down to the $U(1)_{\text{em}}$ symmetry of quantum electodynamics (QED) with the elementary electromagnetic charge e as gauge coupling. Spontaneous symmetry breaking occurs if the vacuum state of the theory is less symmetric than the Lagrangian.

The Higgs mechanism is incorporated by introducing a scalar isospin-doublet Φ with a hypercharge of 1. The Higgs field is coupled to the gauge bosons through the covariant derivative

$$\mathcal{L}_{\text{Higgs,kin.}} = \frac{1}{2}(D_\mu\Phi)^\dagger D^\mu\Phi, \tag{2.1}$$

$$\text{with} \qquad D_\mu = \partial_\mu - igI_aW^a_\mu + i\frac{g'}{2}B_\mu \tag{2.2}$$

with I_a being the weak isospin of the field on which the covariant derivative acts. The Higgs scalar potential can be parametrized as follows,

$$\mathcal{L}_{\text{Higgs,pot.}} = -\mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2. \tag{2.3}$$

To ensure spontaneous symmetry breaking, both μ^2 and λ must be positive. The corresponding non-zero vacuum expectation value (vev) of Φ can be written as follows,

$$\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.4}$$

$$\text{with} \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \approx 174\ \text{GeV [60]}. \tag{2.5}$$

The Higgs field has to be expanded around the vev to examine the physical content of the theory in the phase of the spontaneously broken symmetry,

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ v + \frac{1}{\sqrt{2}}(H(x) + i\chi(x)) \end{pmatrix}. \tag{2.6}$$

The newly introduced fields H, χ and ϕ^+ are defined to have a zero vev. The fields χ and ϕ^+ can be eliminated by choosing a specific gauge, signaling that they are unphysical. The corresponding degrees of freedom are absorbed into the now massive vector boson fields W and Z as longitudinal polarization modes.

For higher-order calculations it is, however, advantageous to work in general gauge. In this case the additional degrees of freedom enter the calculation in the form of non-physical fields named “would-be Goldstone bosons”.

In either case, the W and Z boson masses are given by

$$M_W^2 = \frac{v^2}{2}g^2, \tag{2.7}$$

$$M_Z^2 = \frac{v^2}{2}(g^2 + g'^2). \tag{2.8}$$

The associated weak mixing angle θ_W is related to the masses via

$$\cos\theta_W = M_W/M_Z \tag{2.9}$$

and to the elementary electromagnetic charge via

$$e = g'\cos\theta_W = g\sin\theta_W. \tag{2.10}$$

Also the physical Higgs field itself describes particles with mass

$$M_h^2 = 2\lambda v^2. \tag{2.11}$$

2.3 Fermion masses

Fermion masses are obtained by introducing Yukawa-interaction terms in the Lagrangian which couple the Higgs field to the fermions, i.e.

$$\mathcal{L}^{\text{SM}}_{\text{Yuk}} = -(\mathbf{y}_l)_{ij}\bar{l}_{i,R}\Phi^\dagger L_{j,L} - (\mathbf{y}_d)_{ij}\bar{d}_{i,R}\Phi^\dagger Q_{j,L} - (\mathbf{y}_u)_{ij}\bar{u}_{i,R}(-i\Phi^T\sigma_2)Q_{j,L} + h.c.\,. \tag{2.12}$$

where σ_2 is the second Pauli matrix. L_L is the $SU(2)_L$ lepton doublet; Q_L, the corresponding $SU(2)_L$ quark doublet. The associated right-handed fields, $SU(2)_L$ singlets, are denoted by l_R, d_R and u_R. The indices i and j run over the three fermion generations. $\mathbf{y}_l$, $\mathbf{y}_d$ and $\mathbf{y}_u$ are the Yukawa-coupling matrices for the leptons, down-type quarks and up-type quarks, respectively. They are in general not diagonal leading to fermion mixing, but can be diagonalized by bi-unitary transformations of the fermion fields.

In this thesis we neglect effects from flavour mixing and assume diagonal Yukawa-coupling matrices,

$$\mathbf{y}_l = \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad \mathbf{y}_d = \begin{pmatrix} y_d & 0 & 0 \\ 0 & y_s & 0 \\ 0 & 0 & y_b \end{pmatrix}, \quad \mathbf{y}_u = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \tag{2.13}$$

we obtain the fermion masses by expanding the Higgs field Φ around its vev (see Eq. (2.6)),

$$m_f = y_f v. \tag{2.14}$$

Instead of the Yukawa coupling y_f, we also make use of $\alpha_f \equiv y_f^2/4\pi$.

Chapter 3

Supersymmetry

As mentioned in the introduction, supersymmetry can provide solutions for some of the SM's issues. By relating fermions to bosons quadratic divergences in the corrections to the Higgs boson mass are cancelled between SM-particles and their associated superpartners in a systematic way. In this way the electroweak scale is stabilized. Moreover, supersymmetric theories provide a candidate for dark matter in the form of the lightest supersymmetric particle (at least if the theory is R-symmetric, see Section 3.2). Another benefit of supersymmetry is related to Grand Unified Theories (GUTs). In GUTs, all gauge groups of the SM are unified into one single bigger gauge group containing the SM groups as subgroups. For this to happen, all gauge couplings have to unify at a single scale above which the single gauge group becomes valid. In the SM this is not achievable, whereas in supersymmetric theories the renormalization group equations are altered such that the couplings unify.

3.1 General structure

Following the No-Go-theorem of Mandula [61], the Poincaré-symmetry can only be extended truly – meaning that the extension does not factorize out as a subgroup in form of a gauge group – by a fermionic symmetry. Here, the term "fermionic" indicates that the corresponding symmetry generators obey anticommutation relations. The authors of [62] showed that the maximal extension is given by a set of fermionic operators Q^i_α $(0 \leq i < N)$ with the algebra (here for $N = 1$)

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \tag{3.1a}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \tag{3.1b}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0, \tag{3.1c}$$

where σ^μ are the Pauli-matrices and P_μ is the Poincaré-generator of translations. $\alpha, \beta \in \{1, 2\}$, the dotted components transform as right-handed Weyl-spinors, the undotted ones as left-handed Weyl-spinors. Eq. (3.1a) expresses that the application of SUSY-operators can lead to a translation in normal spacetime. This shows the interweavement of SUSY and normal spacetime-symmetry.

The action of the supersymmetry generators Q can be interpreted geometrically by introducing the concept of superspace parametrized by the normal spacetime coordinates x_μ as well as the additional variables θ_α and $\bar{\theta}_{\dot{\alpha}}$. The action of Q_α, $\bar{Q}_{\dot{\alpha}}$ corresponds to a translation in the θ, $\bar{\theta}$ directions in superspace. It follows from Eq. (3.1) that θ_α and $\bar{\theta}_{\dot{\alpha}}$ have to be anticommuting Grassmann-numbers.

Correspondingly, superfields can be defined, which are an extension of the normal fields in Minkowski space to the superspace. They depend not only on the spacetime coordinates x_μ but also on the supervariables θ_α, $\bar{\theta}_{\dot{\alpha}}$. The anticommuting character of the θ_α, $\bar{\theta}_{\dot{\alpha}}$ allows to write a general superfield $S(x_\mu, \theta_\alpha, \theta_{\dot{\alpha}})$ in the following form (sums over spinor indices are suppressed in the notation),

$$S(x_\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x)$$

$$+ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x). \tag{3.2}$$

Using a generalized covariant derivative

$$D_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{3.3}$$

$$\bar{D}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} - i\bar{\sigma}^{\dot{\alpha}\beta}_\mu\theta_\beta\partial^\mu, \tag{3.4}$$

commuting with Q_α, $\bar{Q}_{\dot{\alpha}}$, special types $\Phi, \Phi^\dagger, V$ of superfields can be defined,

$$\text{left chiral superfield} \;\rightarrow \bar{D}^{\dot{\alpha}}\Phi \quad = 0, \tag{3.5}$$

$$\text{right chiral superfield} \;\rightarrow D_\alpha\Phi^\dagger \quad = 0, \tag{3.6}$$

$$\text{vector superfield} \;\rightarrow V^\dagger - V = 0. \tag{3.7}$$

From the vector superfields, super-field strengths with components

$$W^a_\alpha \equiv -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V^a \tag{3.8}$$

can be built.

For the construction of a general supersymmetric Lagrangian, the following observation is crucial: Only the terms proportional to the maximal possible number of Grassmann variables (e.g. $D(x)$ in Eq. (3.2)) transform under a global supersymmetric transformation such that the corresponding action remains unchanged (see [63], Section 4.6). An integration over the superspace variables θ_α can be used to project on these components due to the integration rules for Grassmann-numbers (see [63], Section 4.1).

So the most general, supersymmetric and renormalizable Lagrangian containing chiral superfields Φ_i and vector superfields V^a with a gauge symmetry (generators T^a) is given by

$$\mathcal{L}_{SUSY} = \left[\int d^2\theta \left(\frac{1}{4}W^{a,\alpha}W^a_\alpha + \mathcal{W}(\Phi_i)\right) + h.c.\right] + \int d^4\theta\Phi_i^\dagger e^{2g_a T^a V^a}\Phi_i \tag{3.9}$$

with the holomorphic superpotential

$$\mathcal{W}(\Phi_i) = c_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{6}y_{ijk}\Phi_i\Phi_j\Phi_k. \tag{3.10}$$

3.2 R-parity

In principle, SUSY-theories allow for baryon- and lepton-number violation. Experimental constraints, e.g. the lifetime of the proton, require the respective couplings to be very small. Introducing a discrete $\mathbb{Z}_2$-symmetry called R-parity forbids the respective terms. It is defined as

$$R = (-1)^{3(B-L)+2s}, \tag{3.11}$$

where B is the baryon number, L is the lepton number and s is the spin quantum number. For SM particles, $R = 1$; for SUSY particles, $R = -1$. As an immediate consequence, a single sparticle can not decay exclusively into SM particles. Therefore, the lightest supersymmetric particle (LSP) is stable. If the LSP is not charged under $SU(3)_C$ and $U(1)_{\text{em}}$, it provides a suitable dark matter candidate.

3.3 Breaking of supersymmetry

If nature realized SUSY as an exact symmetry, the superpartners and their corresponding SM particles would have the same mass. Since no superpartners has been discovered so far, SUSY has to be broken. The breaking mechanism is not known so far. Many models of spontaneous SUSY breaking have been proposed, e.g. gravity-mediated breaking [64] or gauge-mediated breaking [65].

For phenomenological studies it is convenient to simply parametrize our ignorance of SUSY breaking by introducing terms into the Lagrangian which explicitly break SUSY. These terms are required to have positive mass dimension in order not to give rise to quadratically divergent corrections to the Higgs boson mass and to maintain renormalizability. Due to this property, they are called soft-breaking terms. They are thought of to be generated by an unknown breaking mechanism.

The possible soft-breaking terms in the Lagrangian of a general supersymmetric theory respecting gauge invariance as well as renormalizability are [66]

$$\mathcal{L}_{\text{soft-breaking}} = -\left(\frac{1}{2}M_a\lambda^a\lambda^a + \frac{1}{6}a_{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b_{ij}\phi_i\phi_j + c_i\phi_i + \text{h.c.}\right) - m_{ij}^2\phi_i^*\phi_j, \tag{3.12}$$

with ϕ_i being scalars and λ^a being left-chiral Weyl spinors (see for example [63]).

An immediate consequence of the soft-breaking terms is that the sparticles have a higher mass than their SM-partners.

Chapter 4

The Minimal Supersymmetric Standard Model

From the ingredients described in the previous Chapter phenomenological supersymmetric models can be built. The simplest physically viable model is a $N = 1$ supersymmetric extension of the SM called Minimal Supersymmetric Standard Model (MSSM) [64, 67], in which the SM fermions are described by Weyl spinors contained in chiral superfields and SM gauge bosons by vector fields contained in vector superfields. The additional components of the superfields describe superpartners and ensure that the number of fermionic and bosonic degrees of freedom are equal.

4.1 Particle content

In the MSSM each particles of the SM gets a superpartner (a tilde is used to denote the superpartner $\tilde{a}$ of a SM particle a). Particle and superpartner are grouped in superfields.

superfield	**particle**	**superparticle**	**gauge group**
V_Y	B_μ	$\tilde{B}$	$U(1)_Y$
V_w	W_μ	$\tilde{W}_\mu$	$SU(2)_L$
V_c	G_μ	$\tilde{G}$	$SU(3)_C$

Table 4.1: Gauge sector of the MSSM.

The gauge fields of the SM (spin 1) get fermionic superpartners called gauginos (spin 1/2) (see Table 4.1). The specific gauginos are named bino, wino and gluino.

Similary, each chiral SM fermion $f_{L,R}$ ($f = e, \mu, \tau, u, d, c, s, t, b$) gets a scalar superpartner $\tilde{f}_{L,R}$ (spin 0) called like the SM-fermion with a 's' in front. E.g., the superpartner of a top-quark is a stop.

The Higgs sector of the MSSM differs from the SM Higgs sector. It consists of two Higgs doublets ($\mathcal{H}_1$ with $Y = -1$ and $\mathcal{H}_2$ with $Y = +1$). Two doublets are needed to implement Yukawa couplings for up- ($\mathcal{H}_2$) and down-type quarks ($\mathcal{H}_1$) into the holomorpic superpotential (the holomorphicity of the superpotential ensures that the action is invariant under SUSY-transformations). In addition, in the MSSM gauge anomalies $\propto \mathrm{Tr}\{Y^3\}$ exists (see Fig. 4.1).

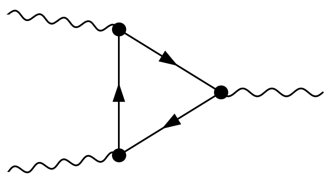


Figure 4.1: Gauge anomaly in the MSSM $\propto \mathrm{Tr}\{Y^3\}$.

Without the introduction of a second Higgs doublet with opposite hypercharge these anomalies would not cancel. The two Higgs doublets are accompanied by two fermionic $SU(2)_L$ doublets, the Higgsinos $\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2$ (with the neutral components $\widetilde{\mathcal{H}}^0_{1,2}$ and the charged components $\widetilde{\mathcal{H}}^\pm_{1,2}$). Higgs and Higgsino doublets are grouped together into the superfields H_1 and H_2. The matter content of the MSSM is summarized in Table 4.2 (see text below Table 2.1 for a explanation of the quantum number notation; the index i denotes the generation):

superfield	**components**	**quantum numbers**
Q_i	$q_{i,L}, \tilde{q}_{i,L}$	$(\mathbf{3},\mathbf{2})_{1/3}$
U_i	$u_{i,R}, \tilde{u}_{i,R}$	$(\mathbf{3},\mathbf{1})_{4/3}$
D_i	$d_{i,R}, \tilde{d}_{i,R}$	$(\mathbf{3},\mathbf{1})_{-2/3}$
L_i	$l_{i,L}, \tilde{l}_{i,L}$	$(\mathbf{1},\mathbf{2})_{-1}$
E_i	$e_{i,R}, \tilde{e}_{i,R}$	$(\mathbf{1},\mathbf{1})_{-2}$
H_1	$\mathcal{H}_1, \widetilde{\mathcal{H}}_1$	$(\mathbf{1},\mathbf{2})_{-1}$
H_2	$\mathcal{H}_2, \widetilde{\mathcal{H}}_2$	$(\mathbf{1},\mathbf{2})_{1}$

Table 4.2: Matter content of the MSSM.

The superfield content and the gauge group determine most of the structure of the MSSM. Missing pieces like the Yukawa coupling are incorporated in the superpotential. Suppressing generation indices, it is given by

$$\mathcal{W}_{\mathrm{MSSM}} = \mu(\mathrm{H}_1 \cdot \mathrm{H}_2) - \mathbf{h}_l(\mathrm{H}_1 \cdot L)E^C - \mathbf{h}_d(\mathrm{H}_1 \cdot Q)D^C - \mathbf{h}_u(Q \cdot \mathrm{H}_2)U^C, \tag{4.1}$$

where $\mathbf{h}_{l,d,u}$ are the Yukawa-coupling matrices, which, as in the SM, can be diagonalized in terms of bi-unitary transformations of the fermion superfields. μ is the Higgsino mass parameter (since the corresponding term in the superpotential is necessary to give mass to the Higgsinos). The product of two superfield doublets is defined by (ϵ is the Levi-Civita-tensor with $\epsilon_{12} = -1$)

$$\Phi_1 \cdot \Phi_2 = \epsilon_{ij}\Phi_1^i\Phi_2^j. \tag{4.2}$$

The superscript C denotes the superfield with charge-conjugated scalar and spinor components.

Specifying the gauge group, the superfield content as well as the superpotential, the Lagrangian is fixed. Most of the mass matrices appearing in the non-SM part of the Lagrangian are not diagonal. Diagonalizing them transforms the gauge eigenstates into mass eigenstates. The corresponding symbols used for the mass eigenstates are listed in Table 4.3.

name	**spin**	**gauge eigenstate**	**mass eigenstate**
Higgs bosons	0	$\mathcal{H}_{1,2}$	$h,\ H,\ A,\ H^\pm$
Goldstone bosons	0	$\mathcal{H}_{1,2}$	$G,\ G^\pm$
squarks	0	$\tilde{q}_{L,R}$	$\tilde{q}_{1,2}$
sleptons	0	$\tilde{l}_{L,R}, \tilde{\nu}_{l_{L,R}}$	$\tilde{l}_{1,2}, \tilde{\nu}_{l_{1,2}}$
neutralinos	1/2	$\widetilde{B},\ \widetilde{\mathcal{H}}^0_{1,2},\ \widetilde{W}^0$	$\tilde{\chi}^0_{1,2,3,4}$
charginos	1/2	$\widetilde{\mathcal{H}}^\pm_{1,2},\ \widetilde{W}^\pm$	$\tilde{\chi}^\pm_{1,2}$
gluino	1/2	$\tilde{g}$	$\tilde{g}$

Table 4.3: Gauge and mass eigenstates of the MSSM.

4.2 Soft-breaking in the MSSM

As discussed in Section 3.3, soft-breaking terms are introduced with general coefficients to parametrize the ignorance of the SUSY breaking mechanism. Specifying Eq. (3.12) for the MSSM yields (i, j are generation indices)

$$\begin{aligned}\mathcal{L}^{\mathrm{MSSM}}_{\text{soft-breaking}} = &-\frac{1}{2}\Big(M_3\tilde{g}\tilde{g} + M_2\tilde{W}\tilde{W} + M_1\tilde{B}\tilde{B} + \mathrm{h.c.}\Big)\\ &-\tilde{m}_1^2\mathcal{H}_1^\dagger\mathcal{H}_1 - \tilde{m}_2^2\mathcal{H}_2^\dagger\mathcal{H}_2 - (b_{\mathcal{H}_1\mathcal{H}_2}\mathcal{H}_1\cdot\mathcal{H}_2 + \mathrm{h.c.})\\ &-\Big[(\mathbf{h}_u\mathbf{A}_u)_{ij}(\tilde{q}_{L,i}\cdot\mathcal{H}_2)\tilde{u}^*_{R,j} + (\mathbf{h}_d\mathbf{A}_d)_{ij}(\mathcal{H}_1\cdot\tilde{q}_{L,i})\tilde{d}^*_{R,j}\end{aligned}$$

$$
\begin{aligned}
&+(\mathbf{h}_l\mathbf{A}_l)_{ij}(\mathcal{H}_1 \cdot \tilde{l}_{L,i})\tilde{l}^*_{R,j} + \text{h.c.}\Big] \\
&-(m^2_{\tilde{q}})_{ij}\tilde{q}^*_{L,i}\tilde{q}_{L,j} - (m^2_{\tilde{u}})_{ij}\tilde{u}^*_{R,i}\tilde{u}_{R,j} - (m^2_{\tilde{d}})_{ij}\tilde{d}^*_{R,i}\tilde{d}_{R,j} \\
&-(m^2_{\tilde{l}_L})_{ij}\tilde{l}^*_{L,i}\tilde{l}_{L,j} - (m^2_{\tilde{l}_R})_{ij}\tilde{l}^*_{R,i}\tilde{l}_{R,j}.
\end{aligned}
\tag{4.3}
$$

$\mathbf{A}_{l,d,u}$ and $m^2_{\tilde{q},\tilde{d},\tilde{u},\tilde{l}_{L,R}}$ are matrices in generation space leading in general to generation mixing. In this thesis we, however, assume them to be diagonal in order to avoid introducing flavour violating terms.

After specifying the soft-breaking terms in the MSSM, the particular sectors are examined more closely following the notations of [8].

4.3 Squark/slepton sector

For each SM fermion f two sfermions $\tilde{f}_L$, $\tilde{f}_R$ exist. Their mass matrix is given by

$$
\mathbf{M}^2_{\tilde{q}} = \begin{pmatrix} m^2_{\tilde{f}_L} + m^2_f + M^2_Z \cos 2\beta (I^f_3 - Q_f s^2_W) & m_f X_f \\ m_f X_f & m^2_{\tilde{f}_R} + m^2_f + M^2_Z \cos 2\beta Q_f s^2_W \end{pmatrix}, \tag{4.4}
$$

where I^f_3 is the isospin, Q_f the electric charge and m_f the mass of the corresponding quark. Note that due to the $SU(2)_L$ gauge symmetry, the soft-breaking masses for the left handed sfermions are equal within one generation of squarks or sleptons (e.g. $m_{\tilde{t}_L} = m_{\tilde{b}_L}$).

The off-diagonal mixing parameter X_f is given in terms of the soft-breaking trilinear coupling parameter A_f, the Higgsino mass parameter μ and the ratio of the vacuum expectation values of the two Higgs doublets $\tan\beta = v_2/v_1$ (see Section 4.6) by

$$
X_f \equiv A_f - \mu\{\cot\beta, \tan\beta\}, \tag{4.5}
$$

where $\cot\beta$ applies for up-type squarks and $\tan\beta$ for down-type squarks and sleptons, respectively. For stop squarks this matrix reads explicitly

$$
\mathbf{M}^2_{\tilde{t}} = \begin{pmatrix} m^2_{\tilde{t}_L} + m^2_t + \cos 2\beta(\frac{1}{2} - \frac{2}{3}s^2_W)M^2_Z & m_t X_t \\ m_t X_t & m^2_{\tilde{t}_R} + m^2_t + \frac{2}{3}\cos 2\beta s^2_W M^2_Z \end{pmatrix} \tag{4.6}
$$

with $X_t \equiv A_t - \mu\cot\beta$.

These mass matrices are diagonalized by unitary matrices $\mathbf{U}_{\tilde{f}}$ ($\mathbf{U}_{\tilde{f}}\mathbf{U}^\dagger_{\tilde{f}} = \mathbb{1}$). The eigenvalues are the squared masses of the sfermion mass eigenstates $\tilde{f}_1, \tilde{f}_2$. They are given by

$$
m^2_{\tilde{f}_{1,2}} = m^2_f + \frac{1}{2}\Big[m^2_{\tilde{f}_L} + m^2_{\tilde{f}_R} + I^f_3 M^2_Z \cos 2\beta \tag{4.7}
$$

$$
\pm\sqrt{[m^2_{\tilde{f}_L} - m^2_{\tilde{f}_R} + M^2_Z\cos 2\beta(I^f_3 - 2Q_f s^2_W)]^2 + 4m^2_f X^2_f}\Big]. \tag{4.8}
$$

4.4 Chargino sector

The charginos $\chi^\pm_{1,2}$ are the mass eigenstates resulting from the charged Higgsinos and electroweak gauginos. Their masses can be obtained from of the mass matrix

$$
\mathbf{X} = \begin{pmatrix} M_2 & \sqrt{2}\sin\beta M_W \\ \sqrt{2}\cos\beta M_W & \mu \end{pmatrix}. \tag{4.9}
$$

The mass eigenstates are determined by diagonalizing the matrix $\mathbf{X}$ using two unitary 2×2 matrices $\mathbf{U}$ and $\mathbf{V}$. The matrices $\mathbf{U}$ and $\mathbf{V}$ rotate the original wino and Higgsino states to the mass eigenstates

$$
\begin{pmatrix} \tilde{\chi}^+_1 \\ \tilde{\chi}^+_2 \end{pmatrix} = \mathbf{V}\begin{pmatrix} \widetilde{W}^+ \\ \widetilde{\mathcal{H}}^+_2 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\chi}^-_1 \\ \tilde{\chi}^-_2 \end{pmatrix} = \mathbf{U}\begin{pmatrix} \widetilde{W}^- \\ \widetilde{\mathcal{H}}^-_1 \end{pmatrix}. \tag{4.10}
$$

The rotation is chosen in a way such that the resulting mass matrix

$$\begin{pmatrix} m_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m_{\tilde{\chi}_2^\pm} \end{pmatrix} = \mathbf{U}^*\mathbf{X}\mathbf{V}^\dagger \tag{4.11}$$

is diagonal. Mathematically, this corresponds to a singular-value decomposition with the masses being the singular values.

4.5 Neutralino sector

The neutralino sector is similar to the chargino sector accommodating the neutral bino, winos and Higgsinos. In the neutralino sector the matrix

$$\mathbf{Y} = \begin{pmatrix} M_1 & 0 & -M_Z s_W c_\beta & M_Z s_W s_\beta \\ 0 & M_2 & M_Z c_W c_\beta & -M_Z c_W s_\beta \\ -M_Z s_W c_\beta & M_Z c_W c_\beta & 0 & -\mu \\ M_Z s_W s_\beta & -M_Z c_W s_\beta & -\mu & 0 \end{pmatrix} \tag{4.12}$$

has to be diagonalized to obtain the mass eigenstates. The abbreviations

$$s_\gamma \equiv \sin\gamma, \qquad c_\gamma \equiv \cos\gamma, \qquad t_\gamma \equiv \tan\gamma \tag{4.13}$$

are introduced for a generic angle γ to keep the expressions short. The electroweak mixing angle θ_W is abbreviated with the subscript W, i.e.

$$s_W \equiv \sin\theta_W, \qquad c_W \equiv \cos\theta_W. \tag{4.14}$$

To obtain positive eigenvalues, the diagonalization is performed by using a Takagi transformation [68]. We denote the resulting mass matrix of the rotated states

$$\begin{pmatrix} \tilde{\chi}_1^0 \\ \tilde{\chi}_2^0 \\ \tilde{\chi}_3^0 \\ \tilde{\chi}_4^0 \end{pmatrix} = \mathbf{N} \begin{pmatrix} \widetilde{B} \\ \widetilde{W}^0 \\ \widetilde{\mathcal{H}}_1^0 \\ \widetilde{\mathcal{H}}_2^0 \end{pmatrix} \tag{4.15}$$

by

$$\begin{pmatrix} m_{\tilde{\chi}_1^0} & 0 & 0 & 0 \\ 0 & m_{\tilde{\chi}_2^0} & 0 & 0 \\ 0 & 0 & m_{\tilde{\chi}_3^0} & 0 \\ 0 & 0 & 0 & m_{\tilde{\chi}_4^0} \end{pmatrix} = \mathbf{N}^*\mathbf{Y}\mathbf{N}^\dagger. \tag{4.16}$$

4.6 Higgs sector

This Section follows in large parts the discussion in [8].

Since the superpotential must be holomorphic, two Higgs doublets are needed. Conventionally they are decomposed as follows,

$$\mathcal{H}_1 = \begin{pmatrix} v_1 + \frac{1}{\sqrt{2}}(\phi_1 - i\chi_1) \\ -\phi_1^- \end{pmatrix}, \qquad \mathcal{H}_2 = \begin{pmatrix} \phi_2^+ \\ v_2 + \frac{1}{\sqrt{2}}(\phi_2 + i\chi_2) \end{pmatrix}, \tag{4.17}$$

where ϕ_i, χ_i and $\phi^\pm$ are real scalar fields and v_1, v_2 are the vacuum expectation values of the doublets. The ratio v_2/v_1 is called $\tan\beta$ ($\tan\beta \equiv v_2/v_1$). Note that in the literature (and also in later parts of this thesis) sometimes also a different notation for the Higgs doublets (and the associated Higgsinos) is used based upon their coupling to up- and down-type quarks (see Eq. (4.1)): $\mathcal{H}_1$ is denoted as $\mathcal{H}_d$, $\mathcal{H}_2$ as $\mathcal{H}_u$ (analogously for the Higgsinos).

In its general form the Higgs potential V_H is given as follows

$$\begin{aligned} V_H =& m_1^2\mathcal{H}_1^\dagger\mathcal{H}_1 + m_2^2\mathcal{H}_2^\dagger\mathcal{H}_2 + m_{12}^2(\mathcal{H}_1\cdot\mathcal{H}_2 + \text{h.c.}) + \\ &+ \frac{1}{8}(g^2 + g'^2)(\mathcal{H}_1^\dagger\mathcal{H}_1 - \mathcal{H}_2^\dagger\mathcal{H}_2)^2 + \frac{1}{2}g^2|\mathcal{H}_1^\dagger\mathcal{H}_2|^2, \end{aligned} \tag{4.18}$$

where $m_{1,2}^2 = \tilde{m}_{1,2}^2 + |\mu|^2$ and $m_{12}^2 = b_{\mathcal{H}_1\mathcal{H}_2}$ (see Eq. (4.3)). The soft breaking terms $\tilde{m}_1^2$, $\tilde{m}_2^2$, $\tilde{m}_{12}^2$ as well as the Higgsino mass parameter μ are assumed to be real in this thesis.

Plugging in the expressions for $\mathcal{H}_1$ and $\mathcal{H}_2$ yields the Higgs potential in terms of ϕ_i, χ_i and $\phi^\pm$,

$$\begin{aligned} V_H =& \text{const} - T_{\phi_1}\phi_1 - T_{\phi_2}\phi_2 + \frac{1}{2}\left(\phi_1, \phi_2, \chi_1, \chi_2\right) \mathbf{M}_{\phi\phi\chi\chi} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix} \\ &+ \left(\phi_1^-, \phi_2^+\right) \mathbf{M}_{\phi^\pm\phi^\pm} \begin{pmatrix} \phi_1^+ \\ \phi_2^- \end{pmatrix} + \text{coupling terms.} \end{aligned} \tag{4.19}$$

The coefficients T_{ϕ_1}, T_{ϕ_2}, also called tadpoles, are

$$T_{\phi_1} = -\sqrt{2}\left(m_1^2 v_1 - m_{12}^2 v_2 + \frac{1}{4}(g^2 + g'^2)(v_1^2 - v_2^2)v_1\right), \tag{4.20}$$

$$T_{\phi_2} = -\sqrt{2}\left(m_2^2 v_2 - m_{12}^2 v_1 - \frac{1}{4}(g^2 + g'^2)(v_1^2 - v_2^2)v_2\right). \tag{4.21}$$

The mass matrices $\mathbf{M}_{\phi\phi\chi\chi}$ and $\mathbf{M}_{\phi^\pm\phi^\pm}$ are given by

$$\mathbf{M}_{\phi\phi\chi\chi} = \begin{pmatrix} \mathbf{M}_\phi & 0 \\ 0 & \mathbf{M}_{\chi\chi} \end{pmatrix}, \tag{4.22}$$

$$\mathbf{M}_\phi = \begin{pmatrix} m_1^2 + \frac{1}{4}(g^2 + g'^2)(3v_1^2 - v_2^2) & -m_{12}^2 - \frac{1}{2}(g^2 + g'^2)v_1 v_2 \\ -m_{12}^2 - \frac{1}{2}(g^2 + g'^2)v_1 v_2 & m_2^2 + \frac{1}{4}(g^2 + g'^2)(3v_2^2 - v_1^2) \end{pmatrix}, \tag{4.23}$$

$$\mathbf{M}_\chi = \begin{pmatrix} m_1^2 + \frac{1}{4}(g^2 + g'^2)(v_1^2 - v_2^2) & -m_{12}^2 \\ -m_{12}^2 & m_2^2 + \frac{1}{4}(g^2 + g'^2)(v_2^2 - v_1^2) \end{pmatrix}, \tag{4.24}$$

$$\mathbf{M}_{\phi^\pm\phi^\pm} = \begin{pmatrix} m_1^2 + \frac{1}{4}g'^2(v_1^2 - v_2^2) + \frac{1}{4}g^2(v_1^2 + v_2^2) & -m_{12}^2 - \frac{1}{2}g^2 v_1 v_2 \\ -m_{12}^2 - \frac{1}{2}g^2 v_1 v_2 & m_2^2 + \frac{1}{4}g'^2(v_2^2 - v_1^2) + \frac{1}{4}g^2(v_1^2 + v_2^2) \end{pmatrix}. \tag{4.25}$$

The mass eigenstates are obtained by a unitary transformation of the ϕ, χ basis,

$$\begin{pmatrix} h \\ H \\ A \\ G \end{pmatrix} = \mathbf{U}_n \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}, \qquad \begin{pmatrix} H^\pm \\ G^\pm \end{pmatrix} = \mathbf{U}_c \begin{pmatrix} \phi_1^\pm \\ \phi_2^\pm \end{pmatrix}. \tag{4.26}$$

The unitary matrices $\mathbf{U}_n$ and $\mathbf{U}_c$ can be parametrized using the angles α, β_n and β_c, i.e.

$$\mathbf{U}_n = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 & 0 \\ \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & -\sin\beta_n & \cos\beta_n \\ 0 & 0 & \cos\beta_n & \sin\beta_n \end{pmatrix}, \qquad \mathbf{U}_c = \begin{pmatrix} -\sin\beta_c & \cos\beta_c \\ \cos\beta_c & \sin\beta_c \end{pmatrix}. \tag{4.27}$$

In this new basis the Higgs potential reads

$$\begin{aligned} V_H =& \text{const.} - T_h \cdot h - T_H \cdot H \\ &+ \frac{1}{2}\left(h, H, A, G\right) \cdot \begin{pmatrix} m_h^2 & m_{hH}^2 & 0 & 0 \\ m_{hH}^2 & m_H^2 & 0 & 0 \\ 0 & 0 & m_A^2 & m_{AG}^2 \\ 0 & 0 & m_{AG}^2 & m_G^2 \end{pmatrix} \cdot \begin{pmatrix} h \\ H \\ A \\ G \end{pmatrix} + \\ &+ \left(H^-, G^-\right) \cdot \begin{pmatrix} m_{H^\pm}^2 & m_{H^\pm G^\pm}^2 \\ m_{H^\pm G^\pm}^2 & m_{G^\pm}^2 \end{pmatrix} \cdot \begin{pmatrix} H^+ \\ G^+ \end{pmatrix} + \\ &+ \text{coupling terms.} \end{aligned} \tag{4.28}$$

Using the modified mass formulas for the massive gauge bosons ($v^2 \to v_1^2 + v_2^2$ in the MSSM),

$$M_Z^2 = \frac{1}{2}(g^2 + g'^2)(v_1^2 + v_2^2), \tag{4.29}$$

$$M_W^2 = \frac{1}{2} g^2 (v_1^2 + v_2^2), \tag{4.30}$$

the Higgs tree-level masses can be expressed in terms of e, s_W, c_W, M_Z, M_W, $\tan\beta$, the tadpoles $T_{\phi_{1,2}}$, the angles α, β_c, β_n and either m_A or $m_{H^\pm}$ (here m_A is chosen). The entries for the $\mathcal{CP}$-even Higgs bosons read

$$\begin{aligned} m_h^2 =& M_Z^2 \sin^2(\alpha+\beta) + m_A^2 \frac{\cos^2(\alpha-\beta)}{\cos^2(\beta-\beta_n)} \\ &+ \frac{e}{2M_Z s_W c_W} T_H \frac{\cos(\alpha-\beta)\sin^2(\alpha-\beta_n)}{\cos^2(\beta-\beta_n)} \\ &+ \frac{e}{2M_Z s_W c_W} T_h \frac{\sin(\alpha-\beta_n)}{2\cos^2(\beta-\beta_n)} (\cos(2\alpha-\beta-\beta_n) + 3\cos(\beta-\beta_n)), \end{aligned} \tag{4.31a}$$

$$\begin{aligned} m_{hH}^2 =& - M_Z^2 \sin(\alpha+\beta)\cos(\alpha+\beta) + m_A^2 \frac{\sin(\alpha-\beta)\cos(\alpha-\beta)}{\cos^2(\beta-\beta_n)} \\ &+ \frac{e}{2M_Z s_W c_W} T_H \frac{\sin(\alpha-\beta)\sin^2(\alpha-\beta_n)}{\cos^2(\beta-\beta_n)} \\ &- \frac{e}{2M_Z s_W c_W} T_h \frac{\cos(\alpha-\beta)\cos^2(\alpha-\beta_n)}{\cos^2(\beta-\beta_n)}, \end{aligned} \tag{4.31b}$$

$$\begin{aligned} m_H^2 =& M_Z^2 \cos^2(\alpha+\beta) + m_A^2 \frac{\sin^2(\alpha-\beta)}{\cos^2(\beta-\beta_n)} \\ &+ \frac{e}{2M_Z s_W c_W} T_H \frac{\cos(\alpha-\beta_n)}{2\cos^2(\beta-\beta_n)} (\cos(2\alpha-\beta-\beta_n) - 3\cos(\beta-\beta_n)) \\ &+ \frac{e}{2M_Z s_W c_W} T_h \frac{\sin(\alpha-\beta)\cos^2(\alpha-\beta_n)}{\cos^2(\beta-\beta_n)}. \end{aligned} \tag{4.31c}$$

The requirement that v_1 and v_2 are indeed the vacuum expectation values of the Higgs potential implies that the tadpoles have to vanish.

With vanishing tadpoles, the remaining entries of the mass matrices in Eq. (4.28) read

$$m_{AG}^2 = -m_A^2 \tan(\beta-\beta_n), \tag{4.32a}$$
$$m_G^2 = m_A^2 \tan^2(\beta-\beta_n), \tag{4.32b}$$
$$m_{H^\pm}^2 = m_A^2 + M_W^2, \tag{4.32c}$$
$$m_{H^\pm G^\pm}^2 = -(m_A^2 + M_W^2)\tan(\beta-\beta_c), \tag{4.32d}$$
$$m_{G^\pm}^2 = (m_A^2 + M_W^2)\tan^2(\beta-\beta_c). \tag{4.32e}$$

To obtain diagonal mass matrices (in Eq. (4.28)), the off-diagonal elements, m_{AG}^2 and $m_{H^\pm G^\mp}$, have to be zero. This is achieved for

$$\beta_c = \beta_n = \beta. \tag{4.33}$$

The mixing angle α can be calculated by demanding that m_{hH}^2 should be zero. Alternatively, one can diagonalize the matrix $\mathbf{M}_\phi$ (see Eq. (4.23)) directly. Both ways result in the tree-level prediction

$$m_h^2 = \frac{1}{2}\left(m_A^2 + M_Z^2 - \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2 2\beta}\right), \tag{4.34}$$

$$m_H^2 = \frac{1}{2}\left(m_A^2 + M_Z^2 + \sqrt{(m_A^2+M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2 2\beta}\right), \tag{4.35}$$

$$m_{H^\pm}^2 = m_A^2 + M_W^2 \tag{4.36}$$

for the masses of the Higgs bosons. The mixing angle α reads

$$\alpha = \arctan\left[-\frac{(m_A^2+M_Z^2)\sin\beta\cos\beta}{M_Z^2\cos^2\beta + m_A^2\sin^2\beta - m_h^2}\right]. \tag{4.37}$$

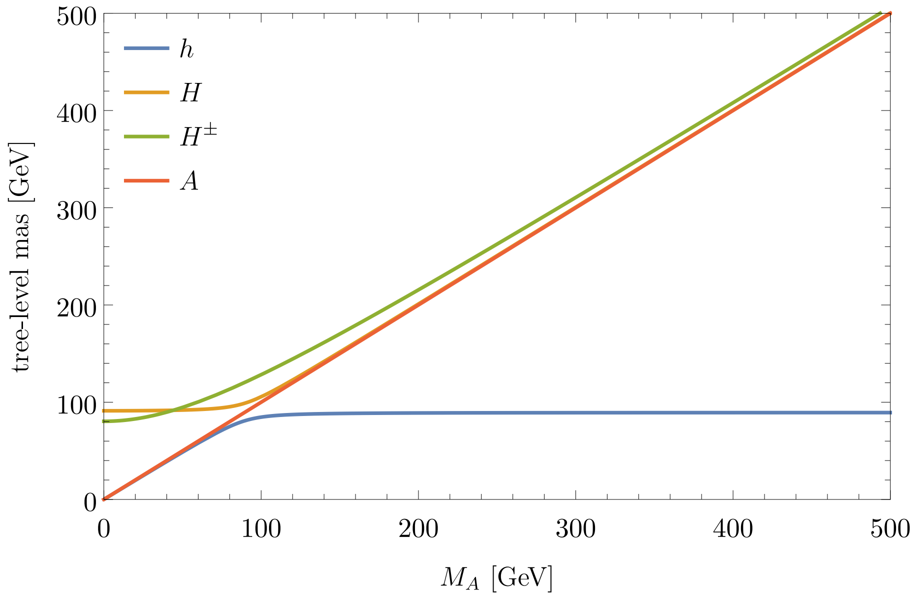


Figure 4.2: Tree-level Higgs boson masses in dependence of M_A for $\tan\beta = 10$.

Conventionally, $-\frac{\pi}{2} < \alpha < 0$ is chosen. Alternatively, we can write

$$\frac{\tan 2\alpha}{\tan 2\beta} = \frac{m_A^2 + M_Z^2}{m_A^2 - M_Z^2}. \tag{4.38}$$

In contrast to the tree-level Higgs boson masses, a Higgs boson mass is denoted with a capitalized M, whenever higher-order corrections are included. In this thesis, $m_A = M_A$, since the tree-level mass m_A does not receive any higher-order corrections in the used on-shell scheme (see Chapter 6).

Fig. 4.2 shows the tree-level masses in dependence of the mass of the A boson. For rising M_A the masses of the H- and $H^\pm$ bosons rise too. In particular the mass of the H boson converges to M_A. In contrast the tree-level mass of the lightest $\mathcal{CP}$-even Higgs m_h remains constant ($m_h \leq M_Z$). In addition, the couplings of the h boson become SM-like, since $\alpha \to \beta - \pi/2$, if $M_A \gg M_Z$. This limit is denoted as the decoupling limit.

As in the SM (see Section 2.3), the fermion masses are generated via Yukawa-interaction terms arising from the superpotential given in Eq. (4.1),

$$\begin{aligned}\mathcal{L}_{\text{Yuk}}^{\text{MSSM}} =& -(\mathbf{h}_l)_{ij}\bar{l}_{i,R}(i\mathcal{H}_1^T\sigma_2)L_{j,L} - (\mathbf{h}_d)_{ij}\bar{d}_{i,R}(i\mathcal{H}_1^T\sigma_2)Q_{j,L} \\ &- (\mathbf{h}_u)_{ij}\bar{u}_{i,R}(-i\mathcal{H}_2^T\sigma_2)Q_{j,L} + h.c.\ ,\end{aligned} \tag{4.39}$$

Here, the above mentioned notation of $\mathcal{H}_{d,u}$ instead of $\mathcal{H}_{1,2}$ becomes obvious.

Inserting the expansion of the Higgs doublets around the corresponding vevs (see Eq. (4.17)) yields

$$m_{e,\mu,\tau} = h_{e,\mu,\tau}v_1 = h_{e,\mu,\tau}c_\beta v, \tag{4.40a}$$
$$m_{d,s,b} = h_{d,s,b}v_1 = h_{d,s,b}c_\beta v, \tag{4.40b}$$
$$m_{u,c,t} = h_{u,c,t}v_2 = h_{u,c,t}s_\beta v. \tag{4.40c}$$

for the masses of the fermions.

Chapter 5

Methods for higher-order calculations

In this Chapter, we summarize basic concepts needed for the calculations presented in this thesis. First, we briefly explain how UV divergencies appearing in higher-order calculations are handled. Second, we discuss the use of effective field theories.

5.1 Regularization and renormalization

In quantum field theories, the classical relations between the different observables (tree-level relations) are modified by quantum corrections. If all couplings are small enough, which is assumed in this thesis, these corrections can be obtained by an expansion in the couplings. The coefficients at each order can be calculated by evaluating the corresponding Feynman diagrams.

Higher-order Feynman diagrams involve internal loops. These depict integrals over the momentum of the internally propagating particle. E.g., integrals of the form

$$\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^2} \frac{1}{k^2 - m^2} \tag{5.1}$$

appear (with m being the mass of the internal particle). Those integrals diverge if $k^2 \to \infty$ (UV divergent). To render the result for physical observables finite, these divergencies have to be absorbed into the bare quantities of the Lagrangian. This procedure is called renormalization.

5.1.1 Regularization

The first step of the renormalization procedure is the regularization of the divergent intergrals. This is achieved by introducing a regularization parameter such that the integral becomes finite. The original integral has to be recovered if the regularization parameter approaches a certain limiting value. Widely used methods are dimensional regularization (DREG) [69–72] and dimensional reduction (DRED) [73, 74].

In dimensional regularization the dimension of the loop integrals is shifted from 4 to D dimensions ($D = 4 - 2\epsilon$ with $\epsilon > 0$),

$$\int \frac{d^4k}{(2\pi)^4} \to \mu_R^{4-D} \int \frac{d^Dk}{(2\pi)^D}. \tag{5.2}$$

The at first hand arbitrary renormalization scale μ_R (mass dimension 1) has to be introduced to preserve the overall mass dimension of the integral. After shifting the integral to dimension D, it is free of divergences. The result can be expanded in ϵ; terms proportional to an inverse power of ϵ reflect the original divergence. In this way, the divergences are parametrized in an analytic form.

The dimension shift is not only applied to loop integrals but also to all other four-dimensional objects. This implies that the number of bosonic and fermionic degrees of freedom are changed asymmetrically. In other words, DREG breaks supersymmetry. Therefore, in supersymmetric theories DRED is used for the regularization of loop integrals. DRED resembles DREG in the way that it shifts the dimension of momenta and loop integral measures to D-dimensions. But, all other four-dimensional objects, like the gauge boson fields, are left untouched. Thus, supersymmetry is conserved.

5.1.2 Renormalization

In principle, renormalization can be carried out in various ways. The basic requirement is that after renormalization all divergences appearing in physical observables are cancelled. In renormalizable theories this cancellation can be reached by the procedure of multiplicative renormalization. In multiplicative renormalization, the original parameters of the Lagrangian are transformed by

$$m \to m_0 = Z_m m = (1+\delta Z_m)m = m + \delta m, \tag{5.3a}$$

$$g \to g_0 \;\; = Z_g g \;\;\; = (1+\delta Z_g)g \;\;\; = g \;\; + \delta g \tag{5.3b}$$

with m being a generic mass parameter and g being a generic coupling parameter. The 'bare' coupling g_0 and mass m_0 are split up into a renormalized finite quantity g (or m) and a counterterm δg (or δm).

If in addition to physical amplitudes, also general Greens functions are required to be finite, also the fields have to be renormalized by introducing field renormalization constants via

$$\phi_0 = \sqrt{Z_\phi}\phi = \sqrt{1+\delta Z_\phi}\phi. \tag{5.4}$$

This procedure can be summarized by

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{counterterms}}, \tag{5.5}$$

where $\mathcal{L}_{\text{ren}}$ is equal to the bare Langrangian $\mathcal{L}_0$ but all bare quantities are replaced by the corresponding renormalized quantities. The counterterm Langrangian $\mathcal{L}_{\text{counterterms}}$ contains all counterterm contributions. The divergent parts of these counterterms are chosen such that all divergencies originating from loop integrals are cancelled. In consequence, all n-point vertex functions are UV finite.

5.1.3 Renormalization schemes

The finite parts of the counterterms are, however, not fixed by this condition and can in principle be chosen freely. In the simplest scheme, the minimal subtraction (MS) scheme, which uses DREG for regularization, the counterterms are chosen such that only the divergent terms are cancelled, i.e., the finite parts of the counterterms are chosen to be zero. In the slightly modified $\overline{\text{MS}}$ scheme, also the additionally terms $\propto \ln 4\pi - \gamma_E$ ($\gamma_E = -\Gamma'(1)$) are absorbed. These terms appear in the calculation of every loop integral. In consequence, all terms proportional to

$$\Delta = \frac{1}{\epsilon} + \ln 4\pi - \gamma_E \tag{5.6}$$

are removed by choosing the counterterms in $\overline{\text{MS}}$ scheme. If DRED is used instead of DREG for regularization, the schemes are called DR (instead of MS) and $\overline{\text{DR}}$ (instead of $\overline{\text{MS}}$).

Another scheme, particularly well-suited for calculating physical observables, is the on-shell scheme. Basically, it is defined such that the pole of a loop-corrected propagator corresponds to the physical mass of the propagating particle. This corresponds to the condition that the renormalized one-particle two-point vertex function (the hat marks a renormalized and therefore finite quantity)

$$\hat{\Gamma}(p^2) = i(p^2 - M^2) + i\hat{\Sigma}(p^2), \tag{5.7}$$

is zero for $p^2 = M^2$, where M is the renormalized mass. The quantity Σ, denoted as self-energy, is the sum of one-particle irreducible loop diagrams with two external legs. The associated renormalized quantity including the counterterm contributions is labelled by $\hat{\Sigma}$,

$$\hat{\Sigma}(p^2) = \Sigma(p^2) + \text{ counterterms}. \tag{5.8}$$

The condition $\hat{\Gamma}(p^2 = M^2) = 0$ implies immediately that

$$\text{Re}\ \hat{\Sigma}(M^2) \stackrel{!}{=} 0. \tag{5.9}$$

has to be fulfilled. This condition has only to be fulfilled for the real part, whereas Im $\hat{\Sigma} \neq 0$ in general (i.e. above particle thresholds). This is the on-shell renormalization condition.

In the OS scheme, the field renormalization constants are typically fixed by demanding that the residues of all propagators are equal to unity, yielding

$$\left.\frac{\partial \hat{\Sigma}(p^2)}{\partial p^2}\right|_{p^2=M^2} = 0. \tag{5.10}$$

The counterterms of the coupling constants can be fixed by certain scattering process. E.g., the OS counterterm of the electromagnetic coupling is fixed by demanding that the quantum corrections to Thompson scattering vanish.

Generally, the determination of the counterterms by relating them to physical observables eliminates the explicit dependence on the renormalization scale μ_R.

5.1.4 Renormalization group

In other schemes, like $\overline{\text{MS}}$, the results for physical observables depend in general explicitly on μ_R. This explicit dependence is cancelled by a intrinsic dependence of the renormalized Lagrangian parameters on μ_R at each level of the perturbative expansion.

This intrinsic dependence of the renormalized parameters can be derived by exploiting the fact that the bare parameters cannot depend on μ_R, as in the case of a coupling constant,

$$0 \stackrel{!}{=} \frac{d}{dt} g_0 = \frac{d}{dt} Z_g g = Z_g \frac{d}{dt} g + g \frac{d}{dt} Z_g, \tag{5.11}$$

where

$$t \equiv \ln \mu_R^2. \tag{5.12}$$

Introducing the abbreviation

$$\beta_g = \frac{d}{dt} g, \tag{5.13}$$

called the beta-function of g, we obtain

$$\begin{aligned} 0 \stackrel{!}{=} \frac{d}{dt} g_0 &= \frac{d}{dt}\left(Z_g g\right) = Z_g \frac{d}{dt} g + g \frac{d}{dt} Z_g = \\ &= Z_g \beta_g + g \left[\frac{\partial}{\partial t} + \sum_i \left(\frac{d}{dt} g_i\right) \frac{\partial}{\partial g_i}\right] Z_g = \\ &= Z_g \beta_g + g \left[\frac{\partial}{\partial t} + \sum_i \beta_{g_i} \frac{\partial}{\partial g_i}\right] Z_g, \end{aligned} \tag{5.14}$$

where the sum runs over all couplings of the theory. This equation allows to derive all beta functions order by order by calculating the renormalization factors Z_g. An analogous equation can be derived for the mass parameters of the theory.

5.2 Effective field theories

If widely separated scales appear in a calculation, for example the electroweak scale and a multi TeV SUSY scale, large logarithmic contributions appear in the calculation. These large logarithms typically exacerbate the convergence of the perturbative expansion rendering diagrammatic fixed-order calculations unreliable. In such situations effective field theory (EFT) are a useful method to resum these logarithms.

The main idea of EFTs is that the physics at low energies depends only marginally on the physics at high energies. This fact can be used by decoupling or integrating out the 'heavy'

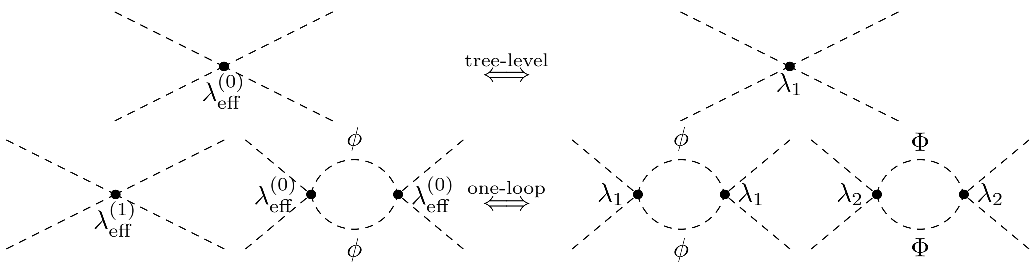


Figure 5.1: Matching the effective theory to the full theory at the tree-level and at the one-loop level.

degrees of freedom leading to an effective Lagrangian containing only 'light' fields suitable to describe the low energy physics.

For illustration, we consider the toy model

$$\mathcal{L}_{\text{toy}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - m^2\phi^2 - M^2\Phi^2 - V(\phi, \Phi), \tag{5.15}$$

$$V(\phi, \Phi) = \frac{\lambda_1}{4!}\phi^4 + \frac{\lambda_2}{4}\phi^2\Phi^2 + \frac{\lambda_3}{4!}\Phi^4 \tag{5.16}$$

with $m \ll M$.

For energies $Q^2 \ll M^2$, we remove the the heavy field Φ from the theory and obtain the effective Lagrangian by writing down all allowed terms involving only ϕ,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2 - \frac{g_{\text{eff}}}{3!}\phi^3 - \frac{\lambda_{\text{eff}}}{4!}\phi^4. \tag{5.17}$$

At this point, the question arises how it is ensured that the effective field theory gives the right result meaning the same result as in the full theory. We achieve this by matching the effective field theory to the full theory at the scale $Q = M$.

Consider e.g. the four-point function of ϕ. At the tree-level, it corresponds to the upper left diagram in Fig. 5.1 in the effective field theory, in the full theory to the upper right one. If the calculation of the four-point function in both, the effective and full field theory, are required to yield the same result, it follows immediately that

$$\lambda_{\text{eff}}^{(0)}(Q = M) \equiv \lambda_{\text{eff}}^{\text{tree-level}}(Q = M) = \lambda_1(Q = M). \tag{5.18}$$

If the result should be identical also at the one-loop level, the results have to be matched accordingly. This is depicted in the bottom row of Fig. 5.1. The one-loop diagrams of the full theory contain loops involving not only the light ϕ but also the heavy field Φ. The contribution of this diagram has to be calculated in the limit $m/M \to 0$ (terms suppressed by the heavy scale M enter in the EFT via higher-dimensional operators as discussed below). In the EFT, this contribution involving Φ has to be compensated by adjusting λ_{eff} at the one-loop level,

$$\lambda_{\text{eff}}(Q = M) = \lambda_{\text{eff}}^{(0)}(Q = M) + \lambda_{\text{eff}}^{(1)}(Q = M). \tag{5.19}$$

This one-loop correction enters through the tree-level diagram (lower left diagram in Fig. 5.1) and is denoted as threshold correction. The procedure can easily be extended to higher loop orders. Applying it to the three-point function of ϕ shows that

$$g_{\text{eff}} = 0. \tag{5.20}$$

So far Φ has only entered through loop corrections which are compensated by adjusting the effective coupling. But Φ can also be responsible that a certain process is allowed in the first place. Consider the same toy-model as above (see Eq. (5.15)) but with the changed potential

$$V(\phi, \Phi) = g\phi^3\Phi. \tag{5.21}$$

Naively, one could think that the effective Lagrangian is again given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2 - \frac{g_{\text{eff}}}{3!}\phi^3 - \frac{\lambda_{\text{eff}}}{4!}\phi^4. \tag{5.22}$$

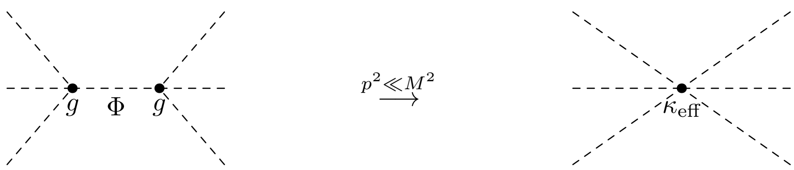


Figure 5.2: Matching the effective theory to the full theory, higher-dimensional terms.

The decoupled field Φ, however, can also mediate interactions of six ϕ fields (see left diagram in Fig. 5.2). To reproduce this effect in the EFT, higher-dimensional terms have to be included into the effective Lagrangian, i.e.

$$\mathcal{L}_{\text{eff}} = \ldots - \frac{\kappa_{\text{eff}}}{6!}\phi^6. \tag{5.23}$$

Clearly, κ_{eff} must have mass dimension -2. Therefore, the interaction term with the coupling κ_{eff} is not renormalizable. This is, however, not an issue, because the EFT is replaced by the full renormalizable theory at $Q = M$. The observation that the internal propagator involved in the Φ-exchange diagram behaves like

$$\frac{1}{p^2 - M^2} \overset{p^2 \ll M^2}{\longrightarrow} -\frac{1}{M^2} \tag{5.24}$$

in the limit $p^2 \ll M^2$ shows that

$$\kappa_{\text{eff}} \propto g^2/M^2. \tag{5.25}$$

In other words, the effects of high-energy physics are suppressed by the scale of these high-energy physics. In this thesis, all such suppressed operators are omitted. This is a good approximation if the scale of calculation is much smaller than the matching scale. If both are comparable, however, numerical important terms might be missed in the EFT calculation without including higher-dimensional operators.

It remains to clarify how this procedure of integrating out heavy particles helps to resum large logarithmic contributions. The main missing ingredient is the use of the renormalization group equations (RGEs) introduced in Section 5.1.4. They are used to run the couplings of the EFTs, fixed by matching to the full theory at the scale of the heavy particles, down to the scale of the light particles. At this scale, we can now calculate all physical observables of interest. Since all heavy particles are integrated out, no large logarithms appear explicitly anymore. These and also all other relevant information about the full high-energy model are implicitly contained in the effective couplings: By looking at the structure of the RGEs – i.e., the derivative with respect to $\ln \mu_R^2$ – it becomes clear that the running down to the low scale resums all appearing large logarithms. Non-logarithmic terms induced by the heavy particles enter through the matching conditions at the high scale.

Chapter 6

Renormalization of the Higgs sector

Before we discuss the calculation of the lightest $\mathcal{CP}$-even Higgs mass in the various approaches, it is indispensable to explain how the Higgs sector of the MSSM is renormalized using the methods presented in Chapter 5. First, we describe in detail the complete one-loop renormalization. After that, we discuss the two-loop renormalization in the limit of vanishing electroweak gauge coupling, which will be needed in Chapter 11. We also explain the renormalization of other sectors relevant for the two-loop renormalization of the Higgs sector.

6.1 One-loop renormalization

Several different schemes for the renormalization of the MSSM sector can be used. Here, we describe the scheme used in `FeynHiggs`, namely a mixed OS/$\overline{\text{DR}}$ scheme. This scheme was in detail described in [8].

6.1.1 Counterterms

According to the presentation in Section 4.6, counterterms have to be introduced for the parameters,

$$
\begin{aligned}
M_Z^2 &\to M_Z^2 \quad + \delta^{(1)} M_Z^2, &\text{(6.1a)}\\
M_W^2 &\to M_W^2 \quad + \delta^{(1)} M_W^2, &\text{(6.1b)}\\
T_h &\to T_h \quad + \delta^{(1)} T_h, &\text{(6.1c)}\\
T_H &\to T_H \quad + \delta^{(1)} T_H, &\text{(6.1d)}\\
T_A &\to T_A \quad + \delta^{(1)} T_A, &\text{(6.1e)}\\
\tan\beta &\to \tan\beta + \delta^{(1)} \tan\beta. &\text{(6.1f)}
\end{aligned}
$$

This implies that the mass matrices get counterterm contributions,

$$
\mathbf{M}_{hHAG} \to \mathbf{M}_{hHAG} + \delta^{(1)}\mathbf{M}_{hHAG}, \tag{6.2}
$$

$$
\mathbf{M}_{H^\pm G^\pm} \to \mathbf{M}_{H^\pm G^\pm} + \delta^{(1)}\mathbf{M}_{H^\pm G^\pm} \tag{6.3}
$$

with

$$
\delta^{(1)}\mathbf{M}_{hHAG} = \begin{pmatrix} \delta^{(1)} m_h^2 & \delta^{(1)} m_{hH}^2 & 0 & 0 \\ \delta^{(1)} m_{hH}^2 & \delta^{(1)} m_H^2 & 0 & 0 \\ 0 & 0 & \delta^{(1)} m_A^2 & \delta^{(1)} m_{AG}^2 \\ 0 & 0 & \delta^{(1)} m_{AG}^2 & \delta^{(1)} m_G^2 \end{pmatrix}, \tag{6.4}
$$

$$
\delta^{(1)}\mathbf{M}_{H^\pm G^\pm} = \begin{pmatrix} \delta^{(1)} m_{H^\pm}^2 & \delta^{(1)} m_{H^\pm G^\pm}^2 \\ \delta^{(1)} m_{H^\pm G^\pm}^2 & \delta^{(1)} m_{G^\pm}^2 \end{pmatrix}. \tag{6.5}
$$

Choosing m_A as independent input parameter, all entries of Eqs. (6.4) and (6.5), can be expressed in terms of $\delta^{(1)}m_A^2$ and the counterterms given in Eq. (6.1). The mass counterterms for the $\mathcal{CP}$-even Higgs bosons read

$$\begin{aligned}\delta^{(1)}m_h^2 &= \delta^{(1)}m_A^2\cos^2(\alpha-\beta) + \delta^{(1)}M_Z^2\sin^2(\alpha+\beta)\\ &+ \frac{e}{2M_Z s_W c_W}\Big(\delta^{(1)}T_H\cos(\alpha-\beta)\sin^2(\alpha-\beta) + \delta^{(1)}T_h\sin(\alpha-\beta)(1+\cos^2(\alpha-\beta))\Big)\\ &+ \delta^{(1)}\tan\beta\cos^2\beta\left(m_A^2\sin 2(\alpha-\beta) + M_Z^2\sin 2(\alpha+\beta)\right), \end{aligned}\tag{6.6a}$$

$$\begin{aligned}\delta^{(1)}m_{hH}^2 &= \frac{1}{2}\Big(\delta^{(1)}m_A^2\sin 2(\alpha-\beta) - \delta^{(1)}M_Z^2\sin 2(\alpha+\beta)\Big)\\ &+ \frac{e}{2M_Z s_W c_W}\Big(\delta^{(1)}T_H\sin^3(\alpha-\beta) - \delta^{(1)}T_h\cos^3(\alpha-\beta)\Big)\\ &- \delta^{(1)}\tan\beta\cos^2\beta\left(m_A^2\cos 2(\alpha-\beta) + M_Z^2\cos 2(\alpha+\beta)\right), \end{aligned}\tag{6.6b}$$

$$\begin{aligned}\delta^{(1)}m_H^2 &= \delta^{(1)}m_A^2\sin^2(\alpha-\beta) + \delta^{(1)}M_Z^2\cos^2(\alpha+\beta)\\ &- \frac{e}{2M_Z s_W c_W}\Big(\delta^{(1)}T_H\cos(\alpha-\beta)(1+\sin^2(\alpha-\beta)) + \delta^{(1)}T_h\sin(\alpha-\beta)\cos^2(\alpha-\beta)\Big)\\ &- \delta^{(1)}\tan\beta\cos^2\beta\left(m_A^2\sin 2(\alpha-\beta) + M_Z^2\sin 2(\alpha+\beta)\right). \end{aligned}\tag{6.6c}$$

In addition to parameter renormalization, also the fields have to be renormalized to guarantee the finiteness of all Green's functions with Higgs fields,

$$\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\end{pmatrix} \to \begin{pmatrix}1+\frac{1}{2}\delta^{(1)}Z_{11} & \frac{1}{2}\delta^{(1)}Z_{12}\\ \frac{1}{2}\delta^{(1)}Z_{12} & 1+\frac{1}{2}\delta^{(1)}Z_{22}\end{pmatrix}\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\end{pmatrix}. \tag{6.7}$$

Here, we introduced also – in contrast to [8] – an off-diagonal field renormalization. This off-diagonal counterterm is not needed to render all appearing Green's function finite but will be helpful in Chapter 11.

Field renormalization constants for the neutral mass eigenstates are introduced via

$$\begin{pmatrix}h\\ H\\ A\\ G\end{pmatrix} \to \begin{pmatrix}1+\frac{1}{2}\delta^{(1)}Z_{hh} & \frac{1}{2}\delta^{(1)}Z_{hH} & 0 & 0\\ \frac{1}{2}\delta^{(1)}Z_{hH} & 1+\frac{1}{2}\delta^{(1)}Z_{HH} & 0 & 0\\ 0 & 0 & 1+\frac{1}{2}\delta^{(1)}Z_{AA} & \frac{1}{2}\delta^{(1)}Z_{AG}\\ 0 & 0 & \frac{1}{2}\delta^{(1)}Z_{AG} & 1+\frac{1}{2}\delta^{(1)}Z_{GG}\end{pmatrix}\begin{pmatrix}h\\ H\\ A\\ G\end{pmatrix}. \tag{6.8}$$

For the charged Higgs this reads

$$\begin{pmatrix}H^\pm\\ G^\pm\end{pmatrix} \to \begin{pmatrix}1+\frac{1}{2}\delta^{(1)}Z_{H^\pm H^\pm} & \frac{1}{2}\delta^{(1)}Z_{H^\pm G^\pm}\\ \frac{1}{2}\delta^{(1)}Z_{H^\pm G^\pm} & 1+\frac{1}{2}\delta^{(1)}Z_{G^\pm G^\pm}\end{pmatrix}\begin{pmatrix}H^\pm\\ G^\pm\end{pmatrix}. \tag{6.9}$$

These counterterms are related to the field renormalization constants of the original doublets by the rotation to the mass eigenstate basis (see Eq. (4.26))),

$$\delta^{(1)}Z_{hh} = s_\alpha^2\delta^{(1)}Z_{11} - s_{2\alpha}\delta^{(1)}Z_{12} + c_\alpha^2\delta^{(1)}Z_{22}, \tag{6.10a}$$

$$\delta^{(1)}Z_{hH} = -s_\alpha c_\alpha\Big(\delta^{(1)}Z_{11} - \delta^{(1)}Z_{22}\Big) + c_{2\alpha}\delta^{(1)}Z_{12}, \tag{6.10b}$$

$$\delta^{(1)}Z_{HH} = c_\alpha^2\delta^{(1)}Z_{11} + s_{2\alpha}\delta^{(1)}Z_{12} + s_\alpha^2\delta^{(1)}Z_{22}, \tag{6.10c}$$

$$\delta^{(1)}Z_{AA} = s_\beta^2\delta^{(1)}Z_{11} - s_{2\beta}\delta^{(1)}Z_{12} + c_\beta^2\delta^{(1)}Z_{22}, \tag{6.10d}$$

$$\delta^{(1)}Z_{AG} = -s_\beta c_\beta\Big(\delta^{(1)}Z_{11} - \delta^{(1)}Z_{22}\Big) + c_{2\beta}\delta^{(1)}Z_{12}, \tag{6.10e}$$

$$\delta^{(1)}Z_{GG} = c_\beta^2\delta^{(1)}Z_{11} + s_{2\beta}\delta^{(1)}Z_{12} + s_\beta^2\delta^{(1)}Z_{22}, \tag{6.10f}$$

$$\delta^{(1)}Z_{H^\pm H^\pm} = s_\beta^2\delta^{(1)}Z_{11} - s_{2\beta}\delta^{(1)}Z_{12} + c_\beta^2\delta^{(1)}Z_{22}, \tag{6.10g}$$

$$\delta^{(1)}Z_{H^\pm G^\pm} = -s_\beta c_\beta\Big(\delta^{(1)}Z_{11} - \delta^{(1)}Z_{22}\Big) + c_{2\beta}\delta^{(1)}Z_{12}, \tag{6.10h}$$

$$\delta^{(1)}Z_{G^\pm G^\pm} = c_\beta^2\delta^{(1)}Z_{11} + s_{2\beta}\delta^{(1)}Z_{12} + s_\beta^2\delta^{(1)}Z_{22}. \tag{6.10i}$$

6.1.2 Renormalization conditions

In order to determine the counterterms, renormalization conditions have to be imposed. As mentioned above, we follow the prescription of [8] and correspondingly employ the OS scheme apart from the $\overline{\rm DR}$ renormalization of the Higgs fields.

For the massive gauge bosons Z and W, the on-shell condition reads

$$\text{Re }\hat{\Sigma}_{ZZ}(M_Z^2) = 0, \qquad \text{Re }\hat{\Sigma}_{WW}(M_W^2) = 0, \tag{6.11}$$

where $\hat{\Sigma}_{ZZ}$ is the renormalized Z self-energy and $\hat{\Sigma}_{WW}$ is the renormalized W self-energy. This implies for the corresponding counterterms that

$$\delta^{(1)}M_Z^2 = \text{Re }\Sigma_{ZZ}^{(1)}(M_Z^2), \qquad \delta^{(1)}M_W^2 = \text{Re }\Sigma_{WW}^{(1)}(M_W^2). \tag{6.12}$$

Also the A boson is renormalized on-shell,

$$\text{Re }\hat{\Sigma}_{AA}(m_A^2) = 0, \tag{6.13}$$

with $\hat{\Sigma}_{AA}$ being the renormalized A boson self-energy. This implies

$$\delta^{(1)}m_A^2 = \text{Re }\Sigma_{AA}(M_A^2) \tag{6.14}$$

and that the A bosons's tree-level mass m_A is equal to its physical mass M_A at the one-loop level. We will therefore use the label M_A instead of m_A in the following.

Furthermore, we demand that tadpole diagrams vanish to ensure that $v_{1,2}$ are still the true vacua when considering higher-order corrections to the Higgs potential. Accordingly, the tadpole counterterms have to be chosen as follows,

$$\delta T_h = -T_h^{(1)}, \qquad \delta T_H = -T_H^{(1)}, \tag{6.15}$$

where $T_h^{(1)}$ and $T_H^{(1)}$ are the sum of the one-loop h/H tadpole diagrams.

In the on-shell scheme the field renormalization constants are normally chosen such that the residua of the propagators are equal to one. It is in principle possible to use this prescription here for the determination of the field renormalization of the A boson (see e.g. [6]). However, it was shown that this procedure yields numerical unstable results. A better working alternative is to renormalize the fields using $\overline{\rm DR}$ renormalization conditions (for a discussion of this issue, see [75]). The $\overline{\rm DR}$ renormalization conditions for the field renormalization constants read

$$\delta^{(1)}Z_{11} = \delta^{(1)}Z_{11}^{\overline{\rm DR}} = -\left[\text{Re }\Sigma_{11}'(p^2)\right]^{\rm div}, \tag{6.16}$$

$$\delta^{(1)}Z_{22} = \delta^{(1)}Z_{22}^{\overline{\rm DR}} = -\left[\text{Re }\Sigma_{22}'(p^2)\right]^{\rm div}, \tag{6.17}$$

where Σ_{11}' and Σ_{22}' are the derivatives of the self-energies in the original gauge basis with respect to p^2. As said before, the off-diagonal field renormalization constant is not needed to cancel divergences. Therefore, we fix it to be[1]

$$\delta^{(1)}Z_{12} = \delta^{(1)}Z_{12}^{\overline{\rm DR}} = 0. \tag{6.18}$$

Fixing the field counterterms, determines also the counterterm of $\tan\beta$,[2]

$$\delta\tan\beta = \frac{1}{2}t_\beta(\delta^{(1)}Z_{11} - \delta^{(1)}Z_{22}) + \frac{1}{2}(1-t_\beta^2)\delta^{(1)}Z_{12}. \tag{6.19}$$

By this fixing the field renormalization counterterms as specified, a renormalization scale $\mu^{\overline{\rm DR}}$ is introduced. In principle, it can be fixed freely. In this thesis, the default value of `FeynHiggs` is adopted, which is $\mu^{\overline{\rm DR}} = M_t$. This means especially that the input $\overline{\rm DR}$ parameter $\tan\beta$ is defined at the scale M_t.

[1] In Chapter 11, a non-zero finite part will be added to the off-diagonal field renormalization constant.

[2] Actually, the counterterm of $\tan\beta$ also contains a contribution from the vev counterterms, $\delta^{(1)}v_2/v_2 - \delta^{(1)}v_1/v_1$. In the $\overline{\rm DR}$ scheme, this term is, however, equal to zero.

6.1.3 Renormalized one-loop self-energies

Using the counterterms derived above, the renormalized self-energies of the h and H bosons can be written in terms of the unrenormalized self-energies. The field renormalization counterterms are necessary to cancel momentum dependent divergences.

Using the terminology of Eq. (5.8), the renormalized self-energies for the $\mathcal{CP}$-even part of the Higgs sector read as follows,

$$\hat{\Sigma}_{hh}(p^2) = \Sigma_{hh}(p^2) + \delta Z_{hh}(p^2 - m_h^2) - \delta m_h^2, \tag{6.20a}$$

$$\hat{\Sigma}_{hH}(p^2) = \Sigma_{hH}(p^2) + \delta Z_{hH}\left(p^2 - (m_h^2 + m_H^2)/2\right) - \delta m_{hH}^2, \tag{6.20b}$$

$$\hat{\Sigma}_{HH}(p^2) = \Sigma_{HH}(p^2) + \delta Z_{HH}(p^2 - m_H^2) - \delta m_H^2, \tag{6.20c}$$

with the counterterms from Eqs. (6.6) and (6.10).

6.2 Two-loop renormalization

All two-loop corrections implemented in `FeynHiggs` are derived in the limit of vanishing external momentum (for a study of momentum dependent effects at $\mathcal{O}(\alpha_s\alpha_t)$ see [76, 77]) and vanishing electroweak gauge couplings (often also denoted as gaugeless limit) which implies that

$$\alpha \to \beta - \frac{\pi}{2}. \tag{6.21}$$

As mentioned in Section 4.6, the same relation holds in the decoupling limit, $M_Z/M_A \to 0$.

Correspondingly, all counterterms derived in this Section are only valid in the limit of these approximations. The notation follows closely the one of [31], where also more details about the renormalization as well as the applied approximations can be found.

6.2.1 Counterterms

All two-loop counterterms are introduced analogously to the one-loop counterterms as specified in Eqs. (6.1) and (6.2). First, we give the two-loop mass counterterms of the $\mathcal{CP}$-even sector,

$$\begin{aligned}\delta^{(2)} m_h^2 =& M_A^2 c_\beta^4 \left(\delta^{(1)} t_\beta\right)^2 \\ &- \frac{e}{2M_W s_W}\left[\delta^{(2)} T_h + \delta^{(1)} T_h \delta^{(1)} Z_W\right],\end{aligned} \tag{6.22a}$$

$$\begin{aligned}\delta^{(2)} m_{hH}^2 =& M_A^2 c_\beta^2 \delta^{(2)} t_\beta + c_\beta^2 \delta^{(1)} M_A^2 \delta^{(1)} t_\beta - M_A^2 c_\beta^3 s_\beta \left(\delta^{(1)} t_\beta\right)^2 \\ &- \frac{e}{2M_W s_W}\left[\delta^{(2)} T_H + \delta^{(1)} T_H \delta^{(1)} Z_W\right],\end{aligned} \tag{6.22b}$$

$$\delta^{(2)} m_H^2 = \delta^{(2)} m_A^2, \tag{6.22c}$$

where $\delta^{(1)} Z_W$ combines the renormalization constants of the electromagnetic charge, the W boson mass and the electroweak mixing angle. We will define it exactly in Section 6.3.

The two-loop field renormalization is introduced by extending Eq. (6.7),

$$\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\end{pmatrix} \to \begin{pmatrix}1 + \frac{1}{2}\delta^{(1)} Z_{11} + \frac{1}{2}\Delta^{(2)} Z_{11} & \frac{1}{2}\delta^{(1)} Z_{12} + \frac{1}{2}\Delta^{(2)} Z_{12} \\ \frac{1}{2}\delta^{(1)} Z_{12} + \frac{1}{2}\Delta^{(2)} Z_{12} & 1 + \frac{1}{2}\delta^{(1)} Z_{22} + \frac{1}{2}\Delta^{(2)} Z_{22}\end{pmatrix}\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\end{pmatrix}, \tag{6.23}$$

where

$$\Delta^{(2)} Z_{ij} = \delta^{(2)} Z_{ij} - \frac{1}{4}\left(\delta^{(1)} Z_{ij}\right)^2. \tag{6.24}$$

Similarly, we introduce two-loop field renormalization constants in the mass eigenstate basis,

$$\begin{pmatrix}h\\ H\end{pmatrix} \to \begin{pmatrix}1 + \frac{1}{2}\delta^{(1)} Z_{hh} + \frac{1}{2}\delta^{(2)} Z_{hh} & \frac{1}{2}\delta^{(1)} Z_{hH} + \frac{1}{2}\delta^{(2)} Z_{hH} \\ \frac{1}{2}\delta^{(1)} Z_{hH} + \frac{1}{2}\delta^{(2)} Z_{hH} & 1 + \frac{1}{2}\delta^{(1)} Z_{HH} + \frac{1}{2}\delta^{(2)} Z_{HH}\end{pmatrix}\begin{pmatrix}h\\ H\end{pmatrix}, \tag{6.25}$$

$$\begin{pmatrix}A\\ G\end{pmatrix} \to \begin{pmatrix}1 + \frac{1}{2}\delta^{(1)} Z_{AA} + \frac{1}{2}\delta^{(2)} Z_{AA} & \frac{1}{2}\delta^{(1)} Z_{AG} + \frac{1}{2}\delta^{(2)} Z_{AG} \\ \frac{1}{2}\delta^{(1)} Z_{AG} + \frac{1}{2}\delta^{(2)} Z_{AG} & 1 + \frac{1}{2}\delta^{(1)} Z_{GG} + \frac{1}{2}\delta^{(2)} Z_{GG}\end{pmatrix}\begin{pmatrix}A\\ G\end{pmatrix}, \tag{6.26}$$

$$\begin{pmatrix} H^\pm \\ G^\pm \end{pmatrix} \to \begin{pmatrix} 1+\frac{1}{2}\delta^{(1)}Z_{H^\pm H^\pm}+\frac{1}{2}\delta^{(2)}Z_{H^\pm H^\pm} & \frac{1}{2}\delta^{(1)}Z_{H^\pm G^\pm}+\frac{1}{2}\delta^{(2)}Z_{H^\pm G^\pm} \\ \frac{1}{2}\delta^{(1)}Z_{H^\pm G^\pm}+\frac{1}{2}\delta^{(2)}Z_{H^\pm G^\pm} & 1+\frac{1}{2}\delta^{(1)}Z_{G^\pm G^\pm}+\frac{1}{2}\delta^{(2)}Z_{G^\pm G^\pm} \end{pmatrix} \begin{pmatrix} H^\pm \\ G^\pm \end{pmatrix}. \tag{6.27}$$

The two-loop field renormalization constants are related to the ones in the gauge basis via

$$\delta^{(2)}Z_{hh} = c_\beta^2\Delta^{(2)}Z_{11} + s_{2\beta}\Delta^{(2)}Z_{12} + s_\beta^2\Delta^{(2)}Z_{22}, \tag{6.28a}$$

$$\delta^{(2)}Z_{hH} = s_\beta c_\beta\left(\Delta^{(2)}Z_{11} - \Delta^{(2)}Z_{22}\right) - c_{2\beta}\Delta^{(2)}Z_{12}, \tag{6.28b}$$

$$\delta^{(2)}Z_{HH} = s_\beta^2\Delta^{(2)}Z_{11} - s_{2\beta}\Delta^{(2)}Z_{12} + c_\beta^2\Delta^{(2)}Z_{22}, \tag{6.28c}$$

$$\delta^{(2)}Z_{AA} = s_\beta^2\Delta^{(2)}Z_{11} - s_{2\beta}\Delta^{(2)}Z_{12} + c_\beta^2\Delta^{(2)}Z_{22}, \tag{6.28d}$$

$$\delta^{(2)}Z_{AG} = -s_\beta c_\beta\left(\Delta^{(2)}Z_{11} - \Delta^{(2)}Z_{22}\right) + c_{2\beta}\Delta^{(2)}Z_{12}, \tag{6.28e}$$

$$\delta^{(2)}Z_{GG} = c_\beta^2\Delta^{(2)}Z_{11} + s_{2\beta}\delta^{(2)}Z_{12} + s_\beta^2\Delta^{(2)}Z_{22}, \tag{6.28f}$$

$$\delta^{(2)}Z_{H^\pm H^\pm} = s_\beta^2\Delta^{(2)}Z_{11} - s_{2\beta}\Delta^{(2)}Z_{12} + c_\beta^2\Delta^{(2)}Z_{22}, \tag{6.28g}$$

$$\delta^{(2)}Z_{H^\pm G^\pm} = -s_\beta c_\beta\left(\Delta^{(2)}Z_{11} - \Delta^{(2)}Z_{22}\right) + c_{2\beta}\Delta^{(2)}Z_{12}, \tag{6.28h}$$

$$\delta^{(2)}Z_{G^\pm G^\pm} = c_\beta^2\Delta^{(2)}Z_{11} + s_{2\beta}\Delta^{(2)}Z_{12} + s_\beta^2\Delta^{(2)}Z_{22}. \tag{6.28i}$$

The two-loop counterterm for $\tan\beta$ is given by[3]

$$\begin{aligned}\delta^{(2)}t_\beta = {} & \frac{1}{2}t_\beta\left(\delta^{(2)}Z_{22} - \delta^{(2)}Z_{11}\right) + \frac{1}{2}\left(1-t_\beta^2\right)\delta^{(2)}Z_{12} \\ & + \frac{1}{8}t_\beta\left[3\left(\delta^{(1)}Z_{11}\right)^2 - \left(\delta^{(1)}Z_{22}\right)^2\right] - \frac{1}{8}\left(1+2t_\beta - t_\beta^2 - 2t_\beta^3\right)\left(\delta^{(1)}Z_{12}\right)^2 \\ & - \frac{1}{4}t_\beta\delta^{(1)}Z_{11}\delta^{(1)}Z_{22} - \frac{1}{4}\left(1-2t_\beta^2\right)\delta^{(1)}Z_{11}\delta^{(1)}Z_{12} - \frac{1}{4}t_\beta^2\delta^{(1)}Z_{12}\delta^{(1)}Z_{22}.\end{aligned} \tag{6.29}$$

Setting $\delta Z_{12} = 0$, we recover the relation given in [31].

6.2.2 Renormalization conditions

As in the one-loop case, we fix the A boson mass on-shell by the condition,

$$\begin{aligned}\delta^{(2)}m_A^2 = {} & \Sigma_{AA}^{(2)}(0) - M_A^2\left[\delta^{(2)}Z_{AA} + \frac{1}{4}\left(\delta^{(1)}Z_{AA}\right)^2\right] \\ & - \delta^{(1)}Z_{AA}\delta^{(1)}M_A^2 - \delta^{(1)}Z_{AG}\delta^{(1)}m_{AG}^2.\end{aligned} \tag{6.30}$$

Here, we set the external momentum to zero due to the zero-momentum approximation in which we work at the two-loop level. Consequently, field renormalization constants appear in the definition of the counterterm.

The two-loop tadpole counterterms are fixed by demanding that the tadpoles vanish at the two-loop level,

$$T_i^{(2)} + \delta^{(2)}T_i^{\mathbf{Z}} = 0, \tag{6.31}$$

where $T_i^{(2)}$ are the two-loop tadpoles of the i boson ($i = h, H$). The appearing two-loop tadpole counterterms including field renormalization, labelled by '$\mathbf{Z}$', are given by

$$\delta^{(2)}T_h^{\mathbf{Z}} = \frac{1}{2}\left(\delta^{(1)}Z_{hh}\delta^{(1)}T_h + \delta^{(1)}Z_{hH}\delta^{(1)}T_H\right) + \delta^{(2)}T_h, \tag{6.32a}$$

$$\delta^{(2)}T_H^{\mathbf{Z}} = \frac{1}{2}\left(\delta^{(1)}Z_{HH}\delta^{(1)}T_H + \delta^{(1)}Z_{hH}\delta^{(1)}T_h\right) + \delta^{(2)}T_H. \tag{6.32b}$$

These conditions allow to fix $\delta^{(j)}T_i$ appearing in Eq. (6.22).

[3] In the gaugeless limit, the divergent parts of the two vev counterterms again cancel each other as in the case of the one-loop counterterms.

6.2.3 Renormalized two-loop self-energies

The renormalized two-loop self-energies are given by[4]

$$\hat{\Sigma}^{(2)}_{hh}(0) = \Sigma^{(2)}_{hh}(0) - \delta^{(2)} m_h^{\mathbf{Z}}, \tag{6.33a}$$
$$\hat{\Sigma}^{(2)}_{hH}(0) = \Sigma^{(2)}_{hH}(0) - \delta^{(2)} m_{hH}^{\mathbf{Z}}, \tag{6.33b}$$
$$\hat{\Sigma}^{(2)}_{HH}(0) = \Sigma^{(2)}_{HH}(0) - \delta^{(2)} m_H^{\mathbf{Z}}, \tag{6.33c}$$

with the counterterms

$$\delta^{(2)} m_h^{\mathbf{Z}} = \frac{1}{4} M_A^2 \left(\delta^{(1)} Z_{hH}\right)^2 + \delta^{(1)} Z_{hh} \delta^{(1)} m_h^2 + \delta^{(1)} Z_{hH} \delta^{(1)} m_{hH}^2 + \delta^{(2)} m_h^2 \tag{6.34a}$$

$$\begin{aligned}\delta^{(2)} m_{hH}^{\mathbf{Z}} =& \frac{1}{2} \left[\left(\delta^{(1)} Z_{hh} + \delta^{(1)} Z_{HH}\right) \delta^{(1)} m_{hH}^2 + \delta^{(1)} Z_{hH} \left(\delta^{(1)} m_h^2 + \delta^{(1)} m_H^2\right) \right] \\ &+ \frac{1}{4} M_A^2 \delta^{(1)} Z_{HH} \delta^{(1)} Z_{hH} + \frac{1}{2} M_A^2 \delta^{(2)} Z_{hH} + \delta^{(2)} m_{hH}^2,\end{aligned} \tag{6.34b}$$

$$\begin{aligned}\delta^{(2)} m_H^{\mathbf{Z}} =& M_A^2 \left[\delta^{(2)} Z_{HH} + \frac{1}{4} \left(\delta^{(1)} Z_{HH}\right)^2 \right], \\ &+ \delta^{(1)} Z_{HH} \delta^{(1)} m_H^2 + \delta^{(1)} Z_{hH} \delta^{(1)} m_{hH}^2 + \delta^{(2)} m_H^2.\end{aligned} \tag{6.34c}$$

The appearing one-loop counterterms have to be derived in the same approximation as all other two-loop quantities. Applying the limit of vanishing electroweak gauge couplings and vanishing external momentum to the relations in Section 6.1, we obtain

$$\delta^{(1)} m_h^2 = - \frac{e}{2 M_W s_W} \delta^{(1)} T_h, \tag{6.35a}$$
$$\delta^{(1)} m_{hH}^2 = M_A^2 c_\beta^2 \delta^{(1)} t_\beta - \frac{e}{2 M_W s_W} \delta^{(1)} T_H, \tag{6.35b}$$
$$\delta^{(1)} m_H^2 = \delta^{(1)} M_A^2, \tag{6.35c}$$
$$\delta^{(1)} m_{AG}^2 = - M_A^2 c_\beta^2 \delta^{(1)} t_\beta + \frac{e}{2 M_W s_W} \delta^{(1)} T_H, \tag{6.35d}$$
$$\delta^{(1)} m_A^2 = \Sigma^{(1)}_{AA}(0) - M_A^2 \delta^{(1)} Z_{AA}. \tag{6.35e}$$

We observe that the two-loop field renormalization constants appear exclusively in the Z-dependent counterterms of Eq. (6.33), either directly or through the two-loop mass, tadpole and $\tan\beta$ counterterms. In the combinations of Eq. (6.34) they completely drop out and hence are not needed for the renormalized self-energies given in Eq. (6.33). This was already noted for the diagonal field counterterms of $\mathcal{O}(\alpha_t^2)$ in [31].

6.3 Renormalization of other sectors

The two-loop renormalization of the Higgs sector also depends on other sectors, such as e.g. the renormalization of the top and stop sector, which enters via one-loop subrenormalization.

The top-quark mass is fixed on-shell,

$$\delta^{(1)} m_t^2 = m_t^2 \mathrm{Re} \left[\Sigma_t^{(1),L}(m_t^2) + \Sigma_t^{(1),R}(m_t^2) + 2 \Sigma_t^{(1),S}(m_t^2) \right], \tag{6.36}$$

where the top self-energy is decomposed according to its Lorentz structure,

$$\Sigma_t(p) = \not{p} \omega_- \Sigma_t^L(p^2) + \not{p} \omega_+ \Sigma_t^R(p^2) + m_t \gamma_5 \Sigma_t^S(p^2) \tag{6.37}$$

with the chirality projectors

$$\omega_\pm = \frac{1}{2}(1 \pm \gamma_5). \tag{6.38}$$

[4] As mentioned above, we set the external momentum p^2 to zero, since the two-loop corrections used by default in `FeynHiggs` are derived in the approximation of zero external momentum.

To resum QCD corrections, it is advantageous to additionally reparametrize the OS top mass in terms of the SM $\overline{\text{MS}}$ top mass [78],

$$\overline{m}_t(M_t) = M_t \left(1 + \delta_{\text{QCD}}^{\text{SM}} + \delta_{\text{ew}}^{\text{SM}}\right), \tag{6.39}$$

with the SM QCD corrections $\delta_{\text{QCD}}^{\text{SM}}$ and the SM electroweak corrections $\delta_{\text{ew}}^{\text{SM}}$ (for explicit formulas, see e.g. [79]). This reparametrization induces additional finite two-loop terms.

The counterterms of the stop sector are derived from the stop mass matrix (see Eq. (4.6)),

$$\mathbf{M}_{\tilde{t}}^2 \to \mathbf{M}_{\tilde{t}}^2 + \delta^{(1)}\mathbf{M}_{\tilde{t}}^2 \tag{6.40}$$

with

$$\delta^{(1)}\mathbf{M}_{\tilde{t}}^2 = \begin{pmatrix} \delta^{(1)}m_{\tilde{t}_L}^2 + \delta^{(1)}m_t^2 & m_t\delta^{(1)}X_t + X_t\delta^{(1)}m_t \\ m_t\delta^{(1)}X_t + X_t\delta^{(1)}m_t & \delta^{(1)}m_{\tilde{t}_R}^2 + \delta^{(1)}m_t^2 \end{pmatrix}. \tag{6.41}$$

The counterterm of X_t is related to the counterterm of A_t via

$$\delta^{(1)}A_t = \delta^{(1)}X_t + \frac{\mu}{t_\beta}\frac{\delta^{(1)}\mu}{\mu} - \frac{\mu}{t_\beta}\frac{\delta^{(1)}t_\beta}{t_\beta}. \tag{6.42}$$

$\delta^{(1)}\mu$ is the counterterm of the Higgsino mass parameter μ, which we fix in the $\overline{\text{DR}}$ scheme.

To fix the remaining counterterms $\delta^{(1)}m_{\tilde{t}_L}^2$, $\delta^{(1)}m_{\tilde{t}_R}^2$ and $\delta^{(1)}X_t$ we rotate the counterterm matrix to the mass eigenstate basis (using the rotation matrix defined in Section 4.3),

$$\mathbf{U}_{\tilde{t}}\delta^{(1)}\mathbf{M}_{\tilde{t}}\mathbf{U}_{\tilde{t}}^\dagger = \begin{pmatrix} \delta^{(1)}m_{\tilde{t}_1}^2 & \delta^{(1)}m_{\tilde{t}_1\tilde{t}_2}^2 \\ \delta^{(1)}m_{\tilde{t}_1\tilde{t}_2}^2 & \delta^{(1)}m_{\tilde{t}_2}^2 \end{pmatrix} \tag{6.43}$$

and employ the OS scheme by imposing on-shell conditions for the stop masses $M_{\tilde{t}_1}$ and $M_{\tilde{t}_2}$,

$$\delta^{(1)}m_{\tilde{t}_1}^2 = \text{Re}\;\Sigma_{\tilde{t}_1\tilde{t}_1}^{(1)}(M_{\tilde{t}_1}^2), \tag{6.44}$$

$$\delta^{(1)}m_{\tilde{t}_2}^2 = \text{Re}\;\Sigma_{\tilde{t}_2\tilde{t}_2}^{(1)}(M_{\tilde{t}_1}^2). \tag{6.45}$$

A third renormalization condition fixes the mixing of the stops,

$$\delta^{(1)}m_{\tilde{t}_1\tilde{t}_2}^2 = \frac{1}{2}\text{Re}\left[\Sigma_{\tilde{t}_1\tilde{t}_2}^{(1)}(M_{\tilde{t}_1}^2) + \Sigma_{\tilde{t}_1\tilde{t}_2}^{(1)}(M_{\tilde{t}_2}^2)\right], \tag{6.46}$$

leading to

$$\begin{aligned} \delta^{(1)}X_t = \frac{1}{m_t}\Big[&(\delta m_{\tilde{t}_1}^2 - \delta^{(1)}m_{\tilde{t}_2}^2)\mathbf{U}_{\tilde{t},11}\mathbf{U}_{\tilde{t},12} \\ &+ \delta^{(1)}m_{\tilde{t}_1\tilde{t}_2}^2(\mathbf{U}_{\tilde{t},21}\mathbf{U}_{\tilde{t},12} + \mathbf{U}_{\tilde{t},11}\mathbf{U}_{\tilde{t},22}) - X_t\delta^{(1)}m_t^2\Big]. \end{aligned} \tag{6.47}$$

In later parts of this thesis, we will also employ the $\overline{\text{DR}}$ scheme for the renormalization of the stop sector. In this case, the counterterms are still given by the equations written down above, but the UV finite part has to be omitted.

Also the renormalization of the sbottom sector enters via one-loop subrenormalization. The sbottom counterterms are introduced analogously to the stop counterterms. Different renormalization conditions have, however, to be chosen. Owing to the $SU(2)_L$ gauge symmetry, the following relation holds,

$$m_{\tilde{b}_L}^2 = m_{\tilde{t}_L}^2. \tag{6.48}$$

Therefore, $\delta^{(1)}m_{\tilde{b}_L}^2$ is not an independent counterterm but fixed by

$$\begin{aligned} \delta^{(1)}m_{\tilde{b}_L}^2 = \delta^{(1)}m_{\tilde{t}_L}^2 =& \mathbf{U}_{\tilde{t},11}^2\delta^{(1)}m_{\tilde{t}_1}^2 + \mathbf{U}_{\tilde{t},12}^2\delta^{(1)}m_{\tilde{t}_2}^2 \\ &- 2\mathbf{U}_{\tilde{t},12}\mathbf{U}_{\tilde{t},22}\delta^{(1)}m_{\tilde{t}_1\tilde{t}_2}^2 - 2m_t\delta^{(1)}m_t. \end{aligned} \tag{6.49}$$

For the second sbottom mass, we impose the on-shell conditions by setting

$$\delta^{(1)} m^2_{\tilde{b}_2} = \mathrm{Re}\, \Sigma_{\tilde{b}_2\tilde{b}_2}(M^2_{\tilde{b}_2}), \tag{6.50}$$

The bottom-quark mass and the sbottom trilinear coupling A_b are fixed in the $\overline{\mathrm{DR}}$ scheme.

Not only third generation squarks enter at the two-loop level but also neutralinos and charginos. As described in Section 4.4 and Section 4.5, they are mixtures of the bino, the winos and the Higgsinos. The couplings of the bino and wino components vanish in the limit of vanishing electroweak gauge couplings; only the Higgsinos have non-vanishing couplings proportional to the Yukawa couplings.

Therefore, it is sufficient for the two-loop renormalization in the limit of vanishing electroweak gauge couplings to renormalize only the Higgsino mass parameter μ. As already mentioned above, we fix its counterterm in the $\overline{\mathrm{DR}}$ scheme. Alternatively, also an on-shell condition can be applied (see e.g. [31]).

Another counterterm relevant for the two-loop renormalization is $\delta^{(1)} Z_W$ (see e.g. Eq. (6.22)). It is given in terms of the renormalization constants of the electric charge, the W boson mass and the sine of the electroweak mixing angle by

$$\delta^{(1)} Z_W = \frac{\delta^{(1)} e}{e} - \frac{\delta^{(1)} M_W}{M_W} - \frac{\delta^{(1)} s_W}{s_W} \tag{6.51}$$

with (following from Eq. (2.9))

$$\frac{\delta^{(1)} s_W}{s_W} = \frac{c_W^2}{s_W^2}\left(\frac{\delta^{(1)} M_Z}{M_Z} - \frac{\delta^{(1)} M_W}{M_W}\right). \tag{6.52}$$

The W and Z boson mass counterterms as well as the counterterm of the electric charge are fixed using the OS scheme (see Section 6.1.2).

$\delta^{(1)} Z_W$ appears not only in the two-loop Higgs mass counterterms (see Eq. (6.22)) but also enters via renormalization of the top- and bottom-Yukawa couplings. Due to (see Eq. (4.40))

$$h_t = \frac{m_t}{v_2} = \frac{e m_t}{\sqrt{2} s_\beta s_W M_W} \qquad \text{and} \qquad h_b = \frac{m_b}{v_1} = \frac{e m_b}{\sqrt{2} c_\beta s_W M_W}, \tag{6.53}$$

the corresponding counterterms read

$$\delta^{(1)} h_t = h_t \left(\frac{\delta^{(1)} m_t}{m_t} - c_\beta^2 \frac{\delta^{(1)} t_\beta}{t_\beta} + \delta^{(1)} Z_W\right), \tag{6.54}$$

$$\delta^{(1)} h_b = h_b \left(\frac{\delta^{(1)} m_t}{m_t} + s_\beta^2 \frac{\delta^{(1)} t_\beta}{t_\beta} + \delta^{(1)} Z_W\right). \tag{6.55}$$

Instead, the Yukawa couplings can be reparametrized in terms of the Fermi constant G_F, as performed in `FeynHiggs`. In that case, the relation

$$\sqrt{2} G_F = \frac{e^2}{4 s_W^2 M_W^2}(1 + \Delta r) \tag{6.56}$$

has to employed. MSSM predictions for the higher-order contribution Δr can be found in [80–83]. The effect of this reparametrization in the one-loop self-energies is formally of two-loop order.

Chapter 7

Calculation of Higgs-boson masses

The most straightforward way to calculate loop corrections to the MSSM Higgs boson masses is the Feynman diagrammatic (FD) approach (see Section 7.1). In practice, this means that the Higgs self-energies are directly calculated by evaluating the corresponding Feynman diagrams. Despite the conceptional simplicity, this method has some shortcomings. I.e. for non-SM particles higher-order contributions, which are not feasible in the Feynman diagrammatic approach, become important. Such scenarios are easier to handle within an effective field theory (EFT) framework. Renormalization group equations (RGEs) allow to resum the potentially large higher-order corrections effectively (see Section 5.2). However, in this EFT framework terms which are suppressed in case of a high SUSY scale are missed if no higher-dimensional operators are included. They might be especially relevant for light SUSY spectra. To obtain a precise prediction for low as well as high SUSY scales, we therefore combine both approaches. This hybrid method is described in Section 7.3.

7.1 Feynman diagrammatic approach

In the Feynman diagrammatic approach, the prediction of the $\mathcal{CP}$-even neutral Higgs boson is based on the calculation of Higgs self-energies involving contributions from SM particles, extra Higgs bosons, as well as their corresponding superpartners. In this approach the contributions from all sectors of the model and of all particles in the loop can be incorporated at a given order.

In the MSSM with real parameters, after calculating the renormalized Higgs boson self-energies ($\hat{\Sigma}_{hh}$, $\hat{\Sigma}_{hH}$ and $\hat{\Sigma}_{HH}$), the physical masses of the $\mathcal{CP}$-even Higgs bosons h, H can be obtained by finding the poles of their propagator matrix, whose inverse is given by

$$\Delta_{hH}^{-1} = i \begin{pmatrix} p^2 - m_h^2 + \hat{\Sigma}_{hh}^{\text{MSSM}}(p^2) & \hat{\Sigma}_{hH}^{\text{MSSM}}(p^2) \\ \hat{\Sigma}_{hH}^{\text{MSSM}}(p^2) & p^2 - m_H^2 + \hat{\Sigma}_{HH}^{\text{MSSM}}(p^2) \end{pmatrix}. \tag{7.1}$$

We introduced the label "MSSM" to indicate that the corresponding self-energy contains SM-type contributions as well as non-SM contributions.

In `FeynHiggs` the full one-loop corrections to the Higgs self-energies as well as two-loop corrections of $\mathcal{O}(\alpha_t\alpha_s, \alpha_b\alpha_s, \alpha_t^2, \alpha_t\alpha_b, \alpha_b^2)$ are implemented [8, 14, 19, 20, 22, 24, 27, 30, 31, 45–47] ($\alpha_{t,b} = y_{t,b}^2/(4\pi)$ and $\alpha_s = g_3^2/(4\pi)$). While those two-loop corrections in the gaugeless limit have been obtained for vanishing external momentum, there is furthermore an option to incorporate the momentum dependence of the corrections at $\mathcal{O}(\alpha_t\alpha_s)$ [76, 77] (see also [84]).

Finding the (complex) poles corresponds to solving the equation

$$\left(p^2 - m_h^2 + \hat{\Sigma}_{hh}^{\text{MSSM}}(p^2)\right)\left(p^2 - m_H^2 + \hat{\Sigma}_{HH}^{\text{MSSM}}(p^2)\right) - \left(\hat{\Sigma}_{hH}^{\text{MSSM}}(p^2)\right)^2 = 0. \tag{7.2}$$

In the decoupling limit, $M_A \gg M_Z$, the physical mass of the lightest Higgs boson, which becomes SM like, can be obtained as solution of the simpler equation

$$p^2 - m_h^2 + \hat{\Sigma}_{hh}^{\text{MSSM}}(p^2) = 0 \tag{7.3}$$

up to corrections from the hH and HH self-energies, which are suppressed by powers of M_A. In the following discussion we will for simplicity use Eq. (7.3) for determining the pole of the propagator and we will furthermore neglect the imaginary parts of $\hat{\Sigma}_{hh}$.[1] In FeynHiggs the complex poles of the propagator are obtained from the full propagator matrix, taking into account the real and imaginary parts of the Higgs boson self-energies.

Solving Eq. (7.3) iteratively for the case where imaginary parts are neglected yields an expression for the Higgs pole mass,

$$(M_h^2)_{\rm FD} = m_h^2 - \hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2) + \hat{\Sigma}_{hh}^{\rm MSSM\prime}(m_h^2)\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2) + \dots\,, \tag{7.4}$$

where the prime denotes the derivative of the self-energy with respect to the momentum squared. We introduced the label "FD" to indicate that this formula represents the Higgs mass as calculated in the Feynman diagrammatic approach. The ellipsis stands for terms involving higher-order derivatives and products of differentiated self-energies. In App. B, we provide a formula from which these terms can be derived recursively. At a given order, the Higgs pole mass is obtained by expanding Eq. (7.4) to this order.

It is an advantage of the Feynman diagrammatic approach that it allows to take the mass effects of all particles in the loops into account for any pattern of the mass spectrum. If there is, however, a large splitting between the relevant scales, in particular a large mass hierarchy between the electroweak scale and the scale of some or all of the SUSY particles, the fixed-order result will contain numerically large logarithms that can spoil the convergence of the perturbative expansion.

7.2 EFT calculation

Another approach to calculate the mass of the SM-like Higgs boson in the MSSM is using effective field theory (EFT) methods. These allow the resummation of large logarithmic contributions so that contributions beyond the order of fixed-order diagrammatic calculations can be incorporated. Without including higher-dimensional operators in the effective Lagrangian, contributions suppressed by a heavy scale are, however, not captured.

In the simplest EFT framework, all SUSY particles are integrated out from the full theory at a common mass scale $M_{\rm SUSY}$ which is defined by the geometric mean of the stop soft-breaking masses, $M_{\rm SUSY} = \sqrt{m_{\tilde{t}_L} m_{\tilde{t}_R}}$ (see Section 4.3). Below $M_{\rm SUSY}$ the SM remains as the low-energy EFT.[2] The couplings of the EFT are fixed at two different scales.

The Yukawa couplings and the gauge couplings are obtained from physical observables at the low-energy scale, typically chosen to be the OS top mass M_t (or M_Z). E.g., for the SM top-Yukawa coupling and the $SU(2)_L$ gauge coupling this would read

$$y_t(M_t) = \frac{M_t}{v}\left(1 + \Delta y_t\right), \tag{7.5}$$

$$g(M_t) = \frac{\sqrt{2}M_W}{v}\left(1 + \Delta g\right), \tag{7.6}$$

where the Δy_t and Δg represent higher-order corrections (full expressions are listed in [79]) and v is the SM vev.

The remaining free coupling, the Higgs self-coupling λ, is determined at the high-energy scale $M_{\rm SUSY}$ by matching the SM to the MSSM. In the MSSM, the Higgs self-coupling is not a free parameter, but fixed in terms of the gauge couplings,

$$\lambda(M_{\rm SUSY}) = \frac{1}{4}(g^2 + g'^2)\cos^2(2\beta) + \Delta\lambda. \tag{7.7}$$

All couplings on the right side have to be evaluated at the scale $M_{\rm SUSY}$. $\Delta\lambda$ are higher-order corrections. This relation represents the matching condition of λ.

To evolve the couplings between the low- and high-energy scale, renormalization group equations are employed. Solving this system of coupled differential equations with boundary

[1]Since the SM-like Higgs boson mass is close to the electroweak scale, only light SM particles will yield a contribution to the imaginary part. This contribution is negligible.

[2]In case of $M_A \sim M_t$ the effective theory is a Two-Higgs-Doublet model and not the SM, see Chapter 11.

values at the scales M_t and M_{SUSY}, we obtain $\lambda(M_t)$ effectively resumming all large logarithmic contributions (see Section 5.2).

$\lambda(M_t)$ determines the $\overline{\text{MS}}$ mass of the SM Higgs boson at the scale M_t via

$$\left(m_h^{\overline{\text{MS}},\text{SM}}\right)^2 = 2\,\lambda(M_t)\,v_{\overline{\text{MS}}}^2\,, \tag{7.8}$$

with the $\overline{\text{MS}}$ vev (at the scale M_t). The $\overline{\text{MS}}$ vev can be related to the on-shell vev,

$$v_{\text{OS}}^2 = \frac{2s_W^2 M_W^2}{e^2}, \tag{7.9}$$

via the finite part of the corresponding counterterm,

$$v_{\overline{\text{MS}}}^2 = v_{\text{OS}}^2 + \delta^{(1)}v_{\text{OS}}^2\Big|_{\text{fin}}, \tag{7.10}$$

which is given by

$$\frac{\delta^{(1)}v_{\text{OS}}^2}{v_{\text{OS}}^2} = \frac{\delta^{(1)}M_W^2}{M_W^2} + \frac{\delta^{(1)}s_W^2}{s_W^2} - \frac{\delta^{(1)}e^2}{e^2} - \delta^{(1)}Z_{hh}. \tag{7.11}$$

$\delta^{(1)}Z_{hh}$ is the field renormalization counterterm of the SM Higgs field fixed in the $\overline{\text{MS}}$ scheme. It is introduced by

$$\Phi_{\text{SM}} \to \left(1 + \frac{1}{2}\delta^{(1)}Z_{hh}\right)\Phi_{\text{SM}}. \tag{7.12}$$

Therefore, we have (with v_0 being the bare vev),

$$v_0^2 = v_{\text{OS}}^2 + \delta^{(1)}v_{\text{OS}}^2 + \delta^{(1)}Z_{hh} = v_{\text{OS}}^2\left(1 + \frac{\delta^{(1)}M_W^2}{M_W^2} + \frac{\delta^{(1)}s_W^2}{s_W^2} - \frac{\delta^{(1)}e^2}{e^2}\right), \tag{7.13}$$

as expected from Eq. (7.9).

Getting from the running mass (Eq. (7.8)) to the physical Higgs mass one has to solve the pole equation for the Higgs boson propagator,

$$p^2 - \left(m_h^{\overline{\text{MS}},\text{SM}}\right)^2 + \widetilde{\Sigma}_{hh}^{\text{SM}}(p^2) = 0, \tag{7.14}$$

involving the renormalized SM Higgs boson self-energy (denoted by a tilde)

$$\widetilde{\Sigma}_{hh}^{\text{SM}}(p^2) = \Sigma_{hh}^{\text{SM}}(p^2)\Big|_{\text{fin}} - \frac{1}{\sqrt{2}v_{\overline{\text{MS}}}}\,T_h^{\text{SM}}\Big|_{\text{fin}}\,, \tag{7.15}$$

which is renormalized accordingly in the $\overline{\text{MS}}$ scheme at the scale M_t but with the Higgs tadpoles renormalized to zero, i.e. the tadpole counterterm is chosen to cancel the sum of the tadpole diagrams, T_h^{SM}, for the Higgs field,

$$\delta T_h^{\text{SM}} = -T_h^{\text{SM}}\,. \tag{7.16}$$

With all these ingredients, the Higgs pole mass is now obtained as the solution of the equation

$$M_h^2 = 2\lambda(M_t)v_{\overline{\text{MS}}}^2 - \widetilde{\Sigma}_{hh}^{\text{SM}}(M_h^2). \tag{7.17}$$

Expanding the Higgs self-energy perturbatively around the tree-level mass m_h^2 of the MSSM yields

$$\begin{aligned}(M_h^2)_{\text{EFT}} = {} & 2v_{\overline{\text{MS}}}^2\lambda(M_t) - \widetilde{\Sigma}_{hh}^{\text{SM}}(m_h^2) \\ & - \widetilde{\Sigma}_{hh}^{\text{SM}\prime}(m_h^2)\cdot\left[2v_{\overline{\text{MS}}}^2\lambda(M_t) - \widetilde{\Sigma}_{hh}^{\text{SM}}(m_h^2) - m_h^2\right] + \cdots\,,\end{aligned} \tag{7.18}$$

where the ellipsis indicates higher-order terms in the expansion. We use the subscript "EFT" to indicate that this formula is the Higgs mass as calculated in the EFT approach.

We discuss the current status of EFT calculations in Chapter 8.

7.3 Hybrid approach

The fixed-order approach can be combined with the EFT approach in order to supplement the full diagrammatic result with leading higher-order contributions (this is the basic strategy used in `FeynHiggs`). The logarithmic contributions resummed using the EFT approach are incorporated into Eq. (7.3) in the following way,[3]

$$p^2 - m_h^2 + \hat{\Sigma}_{hh}^{\rm MSSM}(p^2) + \Delta_{hh}^{\rm EFT} = 0. \tag{7.19}$$

The quantity $\Delta_{hh}^{\rm EFT}$ contains all logarithmic contributions obtained via the EFT approach as well as subtraction terms compensating the logarithmic terms already present in the diagrammatic fixed-order result for $\hat{\Sigma}_{hh}^{\rm MSSM}$,

$$\Delta_{hh}^{\rm EFT} = -\left[2v_{\overline{\rm MS}}^2\lambda(M_t)\right]_{\rm log} - \left[\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)\right]_{\rm log}. \tag{7.20}$$

The subscript 'log' indicates that only logarithmic contributions have to be taken into account. For the EFT result, which is obtained by numerically solving the system of RGEs, this is achieved in practice by subtracting all non-logarithmic terms contained in the result. These can be extracted by an iterative analytic solution of the RGE system.

The logarithms contained in $\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)$ appear only explicitly when expanding in $v/M_{\rm SUSY}$. They can be derived by either expanding the analytic MSSM result or again by iteratively solving the RGE system up to the two-loop order. Since an expansion for large $M_{\rm SUSY}$ is difficult at the two-loop level from a technical point of view due to the large number of terms in the analytic MSSM result, we use the second alternative. It is important to reparametrize the result of the iterative RGE solution such that all couplings and masses are defined exactly as in the fixed-order result, i.e., G_F and either the OS top mass or the SM $\overline{\rm MS}$ top mass has to be employed for the parameterization of the derived subtraction term (see also Sections 6.3 and 10.2).

Having derived all subtraction terms, we calculate $\Delta_{hh}^{\rm EFT}$ and plug it into Eq. (7.19). We obtain for the physical Higgs mass

$$\begin{aligned}(M_h^2)_{\rm hybrid} &= m_h^2 - \hat{\Sigma}_{hh}^{\rm MSSM}(M_h^2) + \left[2v_{\overline{\rm MS}}^2\lambda(M_t)\right]_{\rm log} + \left[\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)\right]_{\rm log} = \\ &= m_h^2 + \left[2v_{\overline{\rm MS}}^2\lambda(M_t)\right]_{\rm log} - \left[\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)\right]_{\rm nolog} \\ &\quad - \hat{\Sigma}_{hh}^{\rm MSSM\prime}(m_h^2)\left(\left[2v_{\overline{\rm MS}}^2\lambda(M_t)\right]_{\rm log} - \left[\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)\right]_{\rm nolog}\right) + \dots\,.\end{aligned} \tag{7.21}$$

We use the label 'nolog' to indicate that we take only terms without large logarithms into account for the labelled quantity. We again would like to stress that the large logarithms (and thereby the meant non-logarithmic terms) appear only explicitly in $\hat{\Sigma}_{hh}^{\rm MSSM}(m_h^2)$ when expanding in $v/M_{\rm SUSY}$. In this way, we identify the non-logarithmic terms. The subscript "hybrid" is used to indicate that this formula represents the Higgs mass as calculated in the hybrid approach.

As discussed in Chapter 6, a mixed OS/$\overline{\rm DR}$ scheme is employed for renormalization (default choice in `FeynHiggs`). In contrast, in the EFT calculation, i.e. the calculation of $\lambda(M_t)$, all SUSY parameters enter in $\overline{\rm DR}$ renormalized form. Therefore, a conversion of the input parameters to the $\overline{\rm DR}$ scheme becomes necessary. Since we will neglect the bottom-Yukawa coupling in our EFT calculation and all two-loop contributions incorporated in the Feynman diagrammatic calculation are derived in the limit of vanishing electroweak couplings, only the parameters of the stop sector need to be converted. They enter first at the one-loop level. Therefore, a one-loop conversion involving only large logarithmic terms is sufficient to reproduce the logarithms of the Feynman diagrammatic calculation. The conversion of the stop masses does not involve any large logarithms. The conversion of the stop mixing parameter X_t, however, does,

$$X_t^{\overline{\rm DR}} = X_t^{\rm OS}\left\{1 + \left[\frac{\alpha_s}{\pi} - \frac{3\alpha_t}{16\pi}\left(1 - \frac{X_t^2}{M_S^2}\right)\right]\ln\frac{M_S^2}{M_t^2}\right\}. \tag{7.22}$$

[3]For $M_A \sim M_{\rm SUSY}$, the EFT approach allows only to resum logarithms appearing in the calculation of the SM-like Higgs mass. It yields no information about the other non-SM Higgs masses. Motivated by the fact that the top-Yukawa coupling is responsible for the dominant one-loop correction, we in practice include $\Delta_{hh}^{\rm EFT}$ into the $\phi_2\phi_2$ self-energy with a prefactor $1/s_\beta^2$. In this way the shift also affects the hH and HH self-energies.

M_S is the stop mass scale, $M_S^2 = \sqrt{M_{\tilde{t}_1} M_{\tilde{t}_2}}$. Full conversion formulas taking into account also non-logarithmic terms can be found in App. E.

A further issue to be discussed is the treatment of $\tan\beta$. In the EFT approach, $\tan\beta$ appears only in the matching condition of λ at the SUSY scale (see Eq. (7.7)). This means that the $\overline{\text{DR}}$ renormalized $\tan\beta(M_{\text{SUSY}})$ is required as an input of the EFT calculation. In the Feynman diagrammatic calculation, $\tan\beta$ is also a $\overline{\text{DR}}$ renormalized quantity. The corresponding renormalization scale, however, is chosen to be M_t and not M_{SUSY} (see Section 6.1.2). In consequence, we need to relate $\tan\beta(M_t)$, which is used as input of the hybrid calculation, to $\tan\beta(M_{\text{SUSY}})$. This presents a problem, since there is no proper way to define $\tan\beta$ in the EFT below M_{SUSY} where the non SM-like Higgs bosons are integrated out. This problem has already been noted in [85]. We find that without a running of $\tan\beta$ the EFT calculation does not reproduce the one-loop result of the Feynmann diagrammatic calculation. This strongly motivates to evolve $\tan\beta$ between M_t and M_{SUSY} despite the lack of a rigorous definition. In practice, we regard $\tan\beta$ as a high-energy parameter with an evolution according to the one-loop RGE of the MSSM [85],

$$\frac{1}{\tan^2\beta}\,\frac{d\tan^2\beta}{d\ln Q^2} = -\frac{3}{16\pi^2}h_t^2, \tag{7.23}$$

which is determined by the anomalous dimensions of the Higgs fields, with contributions only from the top-quark loops. The parameter h_t denotes the MSSM top-Yukawa coupling, which at lowest order is related to the SM top-Yukawa coupling y_t by

$$y_t = h_t \sin\beta. \tag{7.24}$$

Rewriting the RGE in terms of y_t yields

$$\frac{1}{1+\tan^2\beta}\,\frac{d\tan^2\beta}{d\ln Q^2} = -\frac{3}{16\pi^2}y_t^2. \tag{7.25}$$

Since only SM entries contribute to the running [85], the RGE has not to be modified for scales below M_{SUSY}, even if passing an intermediate threshold. This method reproduces correctly the one-loop result of the diagrammatic calculation.

In principle, for a NLL resummation also the two-loop RGE should be employed, which for the MSSM can be found in [86, 87]. It is, however, unclear which contributions of the two-loop RGE are due to SM particles and which are due to their supersymmetric partners. From a practical point of view, numerical checks suggest that the two-loop running is negligible. Therefore, only the one-loop RGE is used in this work.

Chapter 8

Advances in the hybrid approach

The hybrid approach has first been introduced in [42]. Therein, only the resummation of leading logarithms (LL) and next-to-leading logarithms (NLL) in the limit of vanishing electroweak gauge couplings was considered. In this Chapter, we describe several improvements to the EFT calculation implemented in `FeynHiggs`: the inclusion of electroweak contributions, the implementation of separate gaugino/Higgsino thresholds as well as resummation of next-to-next-to-leading logarithms (NNLL) in the limit of vanishing electroweak gauge couplings.

We will neglect the bottom-Yukawa coupling in our EFT calculation. Contributions proportional to the bottom-Yukawa coupling enter, however, via the Feynman diagrammatic calculation at the one- and two-loop level. Apart of this restriction, the improvements mentioned above, bring the EFT part of the calculation to the same level of accuracy as available in pure EFT codes.

8.1 Electroweak contributions

As a first improvement with respect to [42], we include electroweak contributions in the resummation procedure at the LL and NLL level. Correspondingly, we use the full two-loop RGEs of the SM (see App. F), including terms proportional to the electroweak gauge couplings to evolve the SM couplings.

Another source of logarithms proportional to the electroweak gauge couplings at the NLL level are one-loop threshold corrections. Therefore, the threshold correction of the Higgs self-coupling at the SUSY scale has to be extended at the one-loop level by adding the various electroweak one-loop contributions,

$$\lambda_{\text{SM}}(M_{\text{SUSY}}) = \frac{1}{4}(g^2 + g'^2)\cos^2(2\beta) + \Delta_{\text{stop}}\lambda + \Delta_{\text{heavyH}}\lambda + \Delta_{\text{EWino}}\lambda + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda. \tag{8.1}$$

$\Delta_{\text{stop}}\lambda$ is the contribution from the top and stop sector (extended by electroweak contributions in comparison to [42]); $\Delta_{\text{heavyH}}\lambda$, the contribution from the heavy non-SM Higgs bosons; $\Delta_{\text{EWino}}\lambda$ the contribution from charginos and neutralinos. The term $\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda$ accounts for the fact that the tree-level contribution is expressed in terms of $\overline{\text{MS}}$ renormalized gauge couplings of the SM and not in terms of $\overline{\text{DR}}$ renormalized gauge couplings of the MSSM as well as that the one-loop pure SM corrections differ in the $\overline{\text{MS}}$ and the $\overline{\text{DR}}$ scheme. All of these threshold corrections have been derived in previous works [11, 36, 38, 85]. We use the expressions given in [38], which are also partly listed in App. C. Accordingly, also the relations used to extract SM gauge and Yukawa couplings from physical observables at M_t must include electroweak one-loop corrections [79]. This is especially relevant for the $\overline{\text{MS}}$ top-quark mass, respectively the top-Yukawa coupling, as will be discussed later in Chapter 13.

8.2 Gaugino/Higgsino thresholds

The assumption of a common mass scale for all SUSY particles is quite limiting. To allow for electroweakinos (charginos and neutralinos) lighter than $M_{\rm SUSY}$ (but still above the electroweak scale), we introduce an additional electroweakino threshold at the scale M_χ. We assume that all charginos and neutralinos are approximately mass degenerate (having masses similar to M_χ),

$$M_\chi \sim M_1, M_2, \mu \text{ with } M_Z < M_\chi \leq M_{\rm SUSY}, \tag{8.2}$$

where M_1 and M_2 are the electroweak gaugino soft-breaking masses and μ is the Higgsino mass parameter.

This means that at $M_{\rm SUSY}$ all SUSY particles but charginos and neutralinos are integrated out. The corresponding EFT below $M_{\rm SUSY}$, denoted as split model, is the SM with charginos and neutralinos added. The corresponding effective Lagrangian reads [38]

$$\begin{aligned}\mathcal{L}_{\rm split} =& \mathcal{L}_{\rm SM} + (\ldots) - \frac{1}{2}M_\chi \widetilde{W}\widetilde{W} - \frac{1}{2}M_\chi \widetilde{B}\widetilde{B} - M_\chi(i\widetilde{\mathcal{H}}_u^T\sigma_2)\widetilde{\mathcal{H}}_d \\ &- \frac{1}{\sqrt{2}}H^\dagger\left(\tilde{g}_{2u}\sigma^a\widetilde{W}^a + \tilde{g}_{1u}\widetilde{B}\right)\widetilde{\mathcal{H}}_u \\ &- \frac{1}{\sqrt{2}}(-iH^T\sigma_2)\left(\tilde{g}_{2d}\sigma^a\widetilde{W}^a - \tilde{g}_{1d}\widetilde{B}\right)\widetilde{\mathcal{H}}_d + h.c.\,,\end{aligned} \tag{8.3}$$

with the bino field $\widetilde{B}$, the wino fields $\widetilde{W}$ and the Higgsino fields $\widetilde{\mathcal{H}}_{d,u}$. The ellipsis stands for the associated kinetic terms. The effective Higgs–Higgsino–gaugino couplings are labelled $\tilde{g}_{1u,\ldots}$. The number in the subscript refers to the symmetry group $U(1)_Y$ or $SU(2)_L$, the letter to the involved Higgsino. These effective couplings are determined by a one-loop matching of the split model to the full MSSM at the scale $M_{\rm SUSY}$ (explicit expressions have been derived in [36, 38] and are list in App. C). All couplings are evolved between the electroweakino scale and the stop mass scale using two-loop split model RGEs, which have been derived in [36, 38, 88] and are listed in App. F.

At the scale M_χ, all electroweakinos are integrated out, and the remaining EFT below M_χ is the SM. We match the SM to the split model using the threshold corrections derived in [36, 38] and listed in App. C. I.e., the term $\Delta_{\rm EWino}\lambda$ in Eq. (8.1) is now part of the matching condition of λ at M_χ. Also the top-Yukawa coupling receives a threshold correction at the electroweakino scale. Below M_χ the SM RGEs are used for evolving the couplings.

In addition to allowing for light charginos and neutralinos, we also consider the case of a light gluino. This case is implemented by introducing an additional threshold marked by the gluino mass $M_{\tilde{g}}$, below which the gluino is integrated out. The gluino is also assumed to be heavier than the electroweak scale such that eventually the SM is recovered as the EFT close to the electroweak scale. However, no assumption about the ordering of $M_{\tilde{g}}$ and M_χ is made, i.e. $M_{\tilde{g}} \leq M_\chi$ as well as $M_{\tilde{g}} > M_\chi$ is allowed. Since the gluino does not couple directly to the Higgs boson, no additional one-loop matching condition for λ has to be considered. The same argument applies for the electroweak gauge couplings, the Yukawa couplings (in the absence of sfermions) and the effective Higgs–Higgsino–gaugino couplings of the split model. An explicit calculation also shows that the strong gauge coupling does not receive a threshold correction. However, the presence of the gluino in the EFT above $M_{\tilde{g}}$ modifies the RGEs (see App. F).

8.3 NNLL resummation

As a further improvement, we include resummation at the NNLL level. This is restricted to the dominating contributions resulting from the top-Yukawa coupling and the strong gauge coupling. NNLL resummation at this level requires two-loop threshold corrections of $\mathcal{O}(\alpha_s\alpha_t, \alpha_t^2)$. Therefore, we extend Eq. (8.1) by the corresponding two-loop contributions,

$$\begin{aligned}\lambda_{\rm SM}(M_{\rm SUSY}) =& \frac{1}{4}(g^2+g'^2)\cos^2(2\beta) + \Delta_{\rm stop}\lambda + \Delta_{\rm heavyH}\lambda + \Delta_{\rm EWino}\lambda + \Delta_{\overline{\rm DR}\to\overline{\rm MS}}\lambda \\ &+ \Delta_{\alpha_s\alpha_t}\lambda + \Delta_{\alpha_t^2}\lambda.\end{aligned} \tag{8.4}$$

The $\mathcal{O}(\alpha_s, \alpha_t)$ corrections have been derived in degenerate form (meaning that all involved SUSY masses are assumed to be equal) in [37], based on [18], and in non-degenerate form in [38]; the pure top-Yukawa correction $\mathcal{O}(\alpha_t^2)$ have been obtained in degenerate form in [37, 40], based on [18], and in non-degenerate form in [41]. In `FeynHiggs`, both $\mathcal{O}(\alpha_s\alpha_t)$ and $\mathcal{O}(\alpha_t^2)$ threshold-corrections are implemented in degenerate as well as in non-degenerate form. The non-degenerate expressions are listed in App. C.9.

Also the matching conditions for the SM gauge and Yukawa couplings to physical observables at the scale M_t have to be extended to include the $\mathcal{O}(\alpha_s^2, \alpha_s\alpha_t, \alpha_t^2)$ corrections. These are taken from [79]. The matching condition for the top-Yukawa coupling involves the $\overline{\text{MS}}$ top-quark mass which for NNLL resummation is obtained from the pole mass by means of the standard QCD and top-Yukawa corrections at the two-loop level [79].

Furthermore, three-loop RGEs are needed for the coupling constant evolution. Since only NNL logarithms of $\mathcal{O}(\alpha_s, \alpha_t)$ are to be resummed, we neglect the electroweak gauge couplings at the three-loop level of the needed RGEs.

All couplings of electroweakinos, being present below M_{SUSY} for $M_\chi < M_{\text{SUSY}}$, are proportional to the electroweak gauge couplings when their matching conditions at M_{SUSY} are plugged in. In consequence, their presence has no influence on the form of the three-loop RGEs at this level of approximation. Hence for all considered hierarchies at all scales below M_{SUSY}, the needed three-loop RGEs are just the corresponding SM RGEs, which are well known [89–95] and listed in App. F. The same argument implies that the two-loop matching conditions of λ do not have to be modified for M_χ lower than M_{SUSY}.

In the case of a gluino being lighter than the SUSY scale, we also do not have to modify any threshold corrections. The two-loop threshold correction of the Higgs self-coupling derived in [38] is valid for arbitrary gluino masses. Below M_{SUSY}, the gluino couples only via gluon–gluino–gluino vertex since no squarks are present. Therefore, it affects the Higgs self-coupling only from the three-loop level on. The top-Yukawa coupling is influenced from the two-loop level on. These modifications are beyond the order of NNLL resummation. A light gluino, however, does lead to modifications of the RGEs of the three-loop RGEs. These modifications are unknown. The effect of three-loop running was, however, found to have negligible influence on the final result for M_h (see e.g. [39]). Therefore, we expect also the modifications due to the presence of a gluino to be negligible and use the SM three-loop RGEs for all possible hierarchies.

Chapter 9

Scheme conversion of input parameters

As mentioned in Section 6.3, the fixed-order calculation used for our hybrid approach by default employs the OS scheme for the renormalization of the stop sector. Pure EFT calculations, however, are typically done in the $\overline{\text{DR}}$ scheme. This means in particular that the input parameters of our hybrid calculation and the pure EFT calculations have different definitions. Therefore, a parameter conversion from the OS to the $\overline{\text{DR}}$ scheme or vice versa is crucial for a comparison of our hybrid approach to pure EFT calculations (we will perform such a comparison in Chapter 10).

In this Chapter, we discuss issues related to this conversion between parameters of OS and $\overline{\text{DR}}$ renormalization schemes. We will focus on the case where $\overline{\text{DR}}$ parameters are used as main input. These are then converted to the OS parameters which are inserted into our result in the OS scheme. It should, however, be stressed that the related problems are not intrinsic to the OS approach. The same problems would occur if a $\overline{\text{DR}}$ result were used with OS input parameters. Note that the prediction for the mass of the SM-like Higgs boson within the MSSM is particularly sensitive to higher-order effects of this kind through the pronounced dependence on the stop mixing parameter X_t, which receives large corrections when converting from the $\overline{\text{DR}}$ to the OS scheme or vice versa.

In the case where fixed-order results at the n-loop level obtained in two different renormalization schemes are compared with each other, and higher-order logarithms are unknown and not expected to be particularly enhanced, it is well known that the results based on the same type of corrections in two schemes differ by terms that are of $\mathcal{O}(n+1)$. The same is true for different options regarding how to perform the parameter conversion that differ from each other by higher-order contributions. The numerical differences observed in such a comparison can therefore be used as an indication of the possible size of unknown higher-order corrections.

The situation is different, however, in the case that we are considering here, since the comparison is not performed between fixed-order results but between results incorporating a series of (resummed) higher-order logarithms. It is crucial in such a case that the correct form of the higher-order logarithms that can be derived via the EFT method, which in our case arise from the large splitting between the assumed SUSY scale and the weak scale, is maintained in the parameter conversion. We will demonstrate below that the parameter conversion that is usually applied for a comparison of renormalization schemes in fixed-order results does not maintain the correct form of the higher-order logarithms. Since those higher-order logarithms are numerically important, a conversion carried out in the described way leads to very large numerical discrepancies for large values of the SUSY scale.

9.1 Conversion between $\overline{\text{DR}}$ and OS parameters applicable to fixed-order results

The most straightforward method used for the conversion of $\overline{\text{DR}}$ input parameters to OS parameters in fixed-order results is to derive the shift between a parameter p in the two schemes according to $p^{\text{OS}} = p^{\overline{\text{DR}}} + \Delta p$ at the considered loop order, see e.g. [96].

The input parameters of our calculation are the soft-breaking masses of the squark and slepton sector, the soft-breaking trilinear couplings, the soft-breaking masses of the gaugino sector, the mass of the $\mathcal{CP}$-odd Higgs boson, the Higgsino mass parameter μ as well as $\tan\beta$. The soft-breaking masses and trilinear couplings of the slepton sector and of the first and second generation squarks as well as the bino and wino soft-breaking masses, M_1 and M_2, appear only at the one-loop level of our fixed-order calculation.[1] Therefore, their renormalization scheme is not fixed. The same is true for the gluino soft-breaking mass M_3, which appears only at the two-loop level. A_b, μ and $\tan\beta$ are already fixed in the $\overline{\text{DR}}$ scheme (see Chapter 6). The conversion of the input mass M_A has numerically a very small effect and is therefore neglected. For the remaining parameters $m_{\tilde{t}_L}$, $m_{\tilde{t}_R}$, $m_{\tilde{b}_R}$ and A_t a one-loop conversion is employed.

The corresponding full one-loop level conversion formulas, including logarithmic as well as non-logarithmic terms, read

$$\left(m^2_{\tilde{t}_L}\right)^{\text{OS}} = \left(m^2_{\tilde{t}_L}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_L}\Big|_{\text{fin}}, \tag{9.1}$$

$$\left(m^2_{\tilde{t}_R}\right)^{\text{OS}} = \left(m^2_{\tilde{t}_R}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_R}\Big|_{\text{fin}}, \tag{9.2}$$

$$\left(m^2_{\tilde{b}_R}\right)^{\text{OS}} = \left(m^2_{\tilde{b}_R}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_R}\Big|_{\text{fin}}, \tag{9.3}$$

$$A_t^{\text{OS}} = A_t^{\overline{\text{DR}}} + \delta^{(1)} A_t\Big|_{\text{fin}}. \tag{9.4}$$

Here, the shifts between the OS and $\overline{\text{DR}}$ parameters are given by the finite parts of the associated OS counterterms, which are defined in Eqs. (6.41) and (6.42).

Alternatively, we can also rotate to the corresponding mass eigenstate basis (see Eq. (6.43)) and substitute A_t by X_t (see Eq. (4.5)). Then, we need to convert the stop masses and the stop mixing parameter,

$$M^2_{\tilde{t}_1} = \left(m^2_{\tilde{t}_1}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_1}\Big|_{\text{fin}}, \tag{9.5}$$

$$M^2_{\tilde{t}_2} = \left(m^2_{\tilde{t}_2}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_2}\Big|_{\text{fin}}, \tag{9.6}$$

$$M^2_{\tilde{b}_2} = \left(m^2_{\tilde{b}_2}\right)^{\overline{\text{DR}}} + \delta^{(1)} m^2_{\tilde{t}_2}\Big|_{\text{fin}}, \tag{9.7}$$

$$X_t^{\text{OS}} = X_t^{\overline{\text{DR}}} + \delta^{(1)} X_t\Big|_{\text{fin}}. \tag{9.8}$$

The mass counterterms are determined in Eq. (6.44), Eq. (6.45) and Eq. (6.50). The counterterm of X_t is given in Eq. (6.47). Explicit conversion formulas are listed in [19, 20, 22, 78] and in approximated form in App. E.

This conversion of input parameter has to be distinguished from the conversion necessary to combine the fixed-order calculation and the EFT calculation in our hybrid approach. The on-shell parameters obtained as described above are used as input of the overall hybrid calculation containing the OS renormalized fixed-order calculation. For the EFT calculation, this OS stop mixing parameter is then converted back to the $\overline{\text{DR}}$ scheme using a one-loop conversion containing only large logarithms according to Eq. (7.22). This means in particular that the knowledge of the initial $\overline{\text{DR}}$ parameters is not used any further once the conversion to OS parameters has been carried out. While this procedure is suitable for fixed-order results, it leads to problems if results containing a series of higher-order logarithms are meant to be converted.

Indeed, converting $\overline{\text{DR}}$ input parameter to the OS scheme and using the in this way obtained OS parameters as input for an OS calculation incorporating higher-order logarithms generates additional higher-order terms. These cause a deviation in the logarithmic contributions. This can be seen by investigating the Higgs self-energy up to the two-loop level where the parameter X_t^{OS} obtained from the conversion has been inserted,

$$\hat{\Sigma}^{\text{OS}}_{hh}(X_t^{\text{OS}}) = \hat{\Sigma}^{(1),\text{OS}}_{hh}(X_t^{\text{OS}}) + \hat{\Sigma}^{(2),\text{OS}}_{hh}(X_t^{\text{OS}}). \tag{9.9}$$

[1]All two-loop corrections are calculated in the limit of vanishing electroweak gauge couplings. In addition, the first and second generation Yukawa couplings are negelected.

Using instead Eq. (9.8) to write X_t^{OS} in terms of $X_t^{\overline{\mathrm{DR}}}$,

$$\hat{\Sigma}_{hh}^{\mathrm{OS}}(X_t^{\mathrm{OS}}) = \hat{\Sigma}_{hh}^{(1),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}} + \Delta X_t) + \hat{\Sigma}_{hh}^{(2),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}} + \Delta X_t), \tag{9.10}$$

where we replaced $\delta^{(1)} X_t\Big|_{\mathrm{fin}}$ by ΔX_t for an easier notation. Performing an expansion in ΔX_t yields

$$\begin{aligned}\hat{\Sigma}_{hh}^{\mathrm{OS}}(X_t^{\mathrm{OS}}) &= \hat{\Sigma}_{hh}^{(1),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}}) + \left[\frac{\partial}{\partial X_t}\hat{\Sigma}_{hh}^{(1),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}})\right]\Delta X_t + \hat{\Sigma}_{hh}^{(2),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}}) \\ &\quad + \left[\frac{\partial}{\partial X_t}\hat{\Sigma}_{hh}^{(2),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}})\right]\Delta X_t + \mathcal{O}(\Delta X_t^2) = \\ &= \hat{\Sigma}_{hh}^{\overline{\mathrm{DR}}}(X_t^{\overline{\mathrm{DR}}}) + \left[\frac{\partial}{\partial X_t}\hat{\Sigma}_{hh}^{(2),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}})\right]\Delta X_t + \mathcal{O}(\Delta X_t^2). \end{aligned} \tag{9.11}$$

Thus, the obtained expression obviously differs from the original $\overline{\mathrm{DR}}$ result by terms of 3-loop order and beyond.

Here, we only wrote down explicitly the conversion of X_t. To exactly recover the $\overline{\mathrm{DR}}$ renormalized self-energy, also the other OS parameters have to be converted.

9.2 The case of large higher-order logarithms

The higher-order terms in Eq. (9.11) that are not present in the original $\overline{\mathrm{DR}}$ result contain in general logarithmic contributions which for a result containing a series of higher-order logarithms cause a deviation from the logarithmic corrections determined via the RGE. In our numerical discussion in Chapter 13 below, we will demonstrate that those higher-order contributions that are induced by the parameter conversion can be indeed numerically sizeable.

In a hybrid approach, as pursued in `FeynHiggs`, where a fixed-order result in the OS scheme is combined with higher-order logarithmic expressions expressed in the $\overline{\mathrm{DR}}$ scheme, there is a further issue. It concerns the $\overline{\mathrm{DR}}$ value of X_t used in the EFT part of the calculation. Only logarithmic terms are kept in the relation between $X_t^{\overline{\mathrm{DR}}}$, as used in the EFT calculation, and X_t^{OS}, see Eq. (7.22). If instead an input value for $X_t^{\overline{\mathrm{DR}}}$ were converted to X_t^{OS} using the full one-loop contributions according to Eq. (9.8), the stop mixing parameter used in the EFT calculation of `FeynHiggs` would differ from the input parameter.

In order to properly address the case where $\overline{\mathrm{DR}}$ parameters associated with a result containing a series of higher-order logarithms are used as input for our calculation, we follow the strategy to perform the parameter conversion in the fixed-order result rather than in the infinite series of higher-order logarithms. For this purpose we have extended the fixed-order calculation implemented in `FeynHiggs` such that the incorporated fixed-order result is given in terms of the $\overline{\mathrm{DR}}$ parameters $X_t^{\overline{\mathrm{DR}}}$, $m_{\tilde{t}_1}^{\overline{\mathrm{DR}}}$, $m_{\tilde{t}_2}^{\overline{\mathrm{DR}}}$ (the corresponding soft-breaking parameters are used as the actual input parameters). This new result complements the existing result that is given in terms of the on-shell parameters X_t^{OS}, $M_{\tilde{t}_1}$, $M_{\tilde{t}_2}$. The reparametrisation on which the new result is based can be viewed as the parameter conversion described in the example of the previous section, but truncated at the two-loop level,

$$\hat{\Sigma}_{hh}^{\mathrm{OS}}(X_t^{\mathrm{OS}}) \to \hat{\Sigma}_{hh}^{\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}}) + \left[\frac{\partial}{\partial X_t}\hat{\Sigma}_{hh}^{(1),\mathrm{OS}}(X_t^{\overline{\mathrm{DR}}})\right]\Delta X_t = \hat{\Sigma}_{hh}^{\overline{\mathrm{DR}}}(X_t^{\overline{\mathrm{DR}}}). \tag{9.12}$$

We have used the same procedure also for the stop masses. The two-loop terms that are induced by the conversion at the one-loop level have been added to the two-loop result derived in the on-shell scheme in order to arrive at the corresponding expression in the $\overline{\mathrm{DR}}$ scheme. Explicit expressions for these additional terms can be found in App. A. Using the above result given in terms of $\overline{\mathrm{DR}}$ parameters, the value of X_t that is used in the EFT part of the calculation equals the $\overline{\mathrm{DR}}$ input parameter.

Accordingly, depending on the provided input parameters the evaluation of the prediction for the mass of the SM-like Higgs boson proceeds in the following ways:

- For on-shell input parameters the on-shell fixed-order result is combined with the higher-order logarithms obtained in the EFT approach, where $X_t^{\overline{\mathrm{DR}}}$ used in the EFT calculation is related to X_t^{OS} as specified in Eq. (7.22).

- For $\overline{\mathrm{DR}}$ input parameters in a high-scale SUSY scenario where the impact of higher-order logarithms is expected to be large, the $\overline{\mathrm{DR}}$ fixed-order result is combined with the higher-order logarithms obtained in the EFT approach, where the $X_t^{\overline{\mathrm{DR}}}$ used in the EFT calculation is set equal to the corresponding input parameter.
- For $\overline{\mathrm{DR}}$ input parameters in a low-scale SUSY scenario where the impact of higher-order logarithms is expected to be small, both the fixed-order $\overline{\mathrm{DR}}$ result and the fixed-order on-shell result can be employed, where for the latter the parameter conversion described in the previous section is used.

For the input parameter $m_{\tilde{b}_R}$, we still use a one-loop conversion as specified in Eq. (9.3).

Chapter 10

Comparison between different approaches

In the following we will discuss the differences between the various approaches presented in Chapter 7. It is obvious from the discussion of the previous Chapters that the diagrammatic fixed-order result and the pure EFT result differ by higher-order logarithmic terms that are contained in the EFT result but not in the diagrammatic fixed-order result as well as by non-logarithmic terms that are contained in the diagrammatic fixed-order result but not in the pure EFT result. In the hybrid approach the diagrammatic fixed-order result is supplemented by the higher-order logarithmic terms obtained by the EFT approach.

We focus in the following on the comparison between the hybrid approach and the pure EFT result. In the present Chapter we leave aside issues related to the used renormalization schemes, which were addressed in Chapter 9. While the hybrid approach and the pure EFT approach both incorporate the higher-order logarithmic terms obtained by the EFT approach, this does not necessarily imply that all logarithmic terms in the two results are the same. This is due to the fact that the determination of the Higgs boson masses from the poles of the progagators within the hybrid approach is performed in the full model (in the example considered here the MSSM, incorporating loop contributions from all SUSY particles), while in the EFT approach it is determined in the effective low-scale model (in the considered example the SM). We will demonstrate below that the determination of the propagator pole in the hybrid approach generates logarithmic terms beyond the ones contained in the EFT approach at the two-loop level and beyond which actually cancel in the limit of a heavy SUSY scale with contributions from the subloop renormalization. This cancellation is explicitly demonstrated at the two-loop level. We will furthermore discuss the difference in non-logarithmic terms between the results of the hybrid and the EFT approach.

10.1 Higher-order logarithmic terms from the determination of the propagator poles

In the EFT approach where the Higgs boson mass is determined as the pole of the propagator in the SM as the effective low-scale model, while the SUSY particles have been integrated out, the logarithmic terms are given by (see Eq. (7.18))

$$(M_h^2)^{\log}_{\mathrm{EFT}} = \left[2v^2_{\overline{\mathrm{MS}}}\lambda(M_t)\right]_{\log} - \widetilde{\Sigma}^{\mathrm{SM}\prime}_{hh}(m_h^2)\left[2v^2_{\overline{\mathrm{MS}}}\lambda(M_t)\right]_{\log} + \dots\,. \tag{10.1}$$

The logarithmic terms contained in the result of the hybrid approach are given by (see Eq. (7.21))

$$\begin{aligned}(M_h^2)^{\log}_{\mathrm{FH}} = {}& \left[2v^2_{\overline{\mathrm{MS}}}\lambda(M_t)\right]_{\log} + \left[\hat{\Sigma}^{\mathrm{MSSM}\prime}_{hh}(m_h^2)\right]_{\log}\left[\hat{\Sigma}^{\mathrm{MSSM}}_{hh}(m_h^2)\right]_{\mathrm{nolog}} \\ & - \hat{\Sigma}^{\mathrm{MSSM}\prime}_{hh}(m_h^2)\left[2v^2_{\overline{\mathrm{MS}}}\lambda(M_t)\right]_{\log} + \dots\,. \end{aligned} \tag{10.2}$$

In the decoupling limit ($M_A \gg M_t$, where in particular the light $\mathcal{CP}$-even Higgs boson has SM-like couplings), we can split up the MSSM Higgs self-energy $\hat{\Sigma}^{\mathrm{MSSM}}_{hh}$ into a SM part and a non-SM part,

$$\hat{\Sigma}^{\mathrm{MSSM}}_{hh}(m_h^2) = \hat{\Sigma}^{\mathrm{SM}}_{hh}(m_h^2) + \hat{\Sigma}^{\mathrm{nonSM}}_{hh}(m_h^2). \tag{10.3}$$

In the mixed OS/$\overline{\text{DR}}$ scheme of the full diagrammatic calculation, the Higgs field renormalization constants are fixed in the $\overline{\text{DR}}$ scheme. For scalar propagators, there is no difference between the $\overline{\text{DR}}$ and the $\overline{\text{MS}}$ scheme at the one-loop level. Consequently,

$$\hat{\Sigma}^{\text{SM}\prime}_{hh}(m_h^2) = \widetilde{\Sigma}^{\text{SM}\prime}_{hh}(m_h^2) \tag{10.4}$$

holds (as in Chapter 7, we here use the hat to denote quantities renormalized in the mixed OS/$\overline{\text{DR}}$ scheme presented in Chapter 6 and the tilde to denote $\overline{\text{MS}}$ renormalized quantities).

Using this relation, we obtain for the difference between the higher-order logarithmic terms from the determination of the pole of the propagator obtained in the EFT and the hybrid approach

$$\begin{aligned}\Delta^{\log} &\equiv (M_h^2)^{\log}_{\text{FH}} - (M_h^2)^{\log}_{\text{EFT}} = \\ &= \left[\hat{\Sigma}^{\text{nonSM}\prime}_{hh}(m_h^2)\right]_{\log}\left[\hat{\Sigma}^{\text{MSSM}}_{hh}(m_h^2)\right]_{\text{nolog}} - \hat{\Sigma}^{\text{nonSM}\prime}_{hh}(m_h^2)\left[2v^2_{\overline{\text{MS}}}\lambda(M_t)\right]_{\log} + \ldots \\ &=: \Delta^{\log}_{p^2}.\end{aligned} \tag{10.5}$$

Since this difference, which is of two-loop order and beyond, results only from the momentum dependence of the non-SM contributions to the Higgs self-energy, we call it $\Delta^{\log}_{p^2}$ in the following. We give analytic expressions for $\Delta^{\log}_{p^2}$ in App. B.

In Section 10.3 we will demonstrate at the two-loop level that in the limit of a heavy SUSY scale the quantity $\Delta^{\log}_{p^2}$ consisting of "momentum-dependent non-SM contributions" as given in Eq. (10.5) cancels out with contributions of the Higgs self-energy's subloop renormalization. Before we address this issue we first compare the non-logarithmic terms in the two approaches.

10.2 Non-logarithmic terms

In the EFT approach, the non-logarithmic terms are given by (see Eq. (7.18))

$$\begin{aligned}(M_h^2)^{\text{nolog}}_{\text{EFT}} &= \left[2v^2_{\overline{\text{MS}}}\lambda(M_t)\right]_{\text{nolog}} - \widetilde{\Sigma}^{\text{SM}}_{hh}(m_h^2) \\ &\quad - \widetilde{\Sigma}^{\text{SM}\prime}_{hh}(m_h^2)\left(\left[2v^2_{\overline{\text{MS}}}\lambda(M_t)\right]_{\text{nolog}} - \widetilde{\Sigma}^{\text{SM}}_{hh}(m_h^2) - m_h^2\right) + \ldots\,.\end{aligned} \tag{10.6}$$

By construction, all non-logarithmic terms contained in the result of the hybrid approach originate from the fixed-order diagrammatic calculation (see Eq. (7.21)),

$$(M_h^2)^{\text{nolog}}_{\text{FH}} = m_h^2 - \left[\hat{\Sigma}^{\text{MSSM}}_{hh}(m_h^2)\right]_{\text{nolog}} + \left[\hat{\Sigma}^{\text{MSSM}\prime}_{hh}(m_h^2)\right]_{\text{nolog}}\left[\hat{\Sigma}^{\text{MSSM}}_{hh}(m_h^2)\right]_{\text{nolog}} + \ldots\,. \tag{10.7}$$

Deviations Δ^{nolog} between the non-logarithmic terms in the hybrid approach and the EFT approach arise from the following sources,

$$\Delta^{\text{nolog}} \equiv (M_h^2)^{\text{nolog}}_{\text{FH}} - (M_h^2)^{\text{nolog}}_{\text{EFT}} = \Delta^{\text{nolog}}_{v/M_{\text{SUSY}}} + \Delta^{\text{nolog}}_{\text{para}} + \Delta^{\text{nolog}}_{p^2}. \tag{10.8}$$

We explain the different terms below.

Since all non-logarithmic terms in the hybrid approach originate from the fixed-order diagrammatic calculation, one- and two-loop terms that are suppressed in case of a high SUSY scale, $\Delta^{\text{nolog}}_{v/M_{\text{SUSY}}}$, are included in the result of the hybrid approach. Terms of this kind would result from higher-dimensional operators in the EFT approach. Those terms that are included in the hybrid result as implemented in `FeynHiggs` but not in the publicly available pure EFT results constitute an important source of difference between the corresponding results, which is expected to be sizeable if some or all SUSY particles are relatively light (see also [41] for a discussion of contributions of this kind in the EFT approach). It should be noted that in general terms of $\mathcal{O}(v/M_{\text{SUSY}})$ also originate from solving the full pole mass equation, Eq. (7.2), rather than the approximated one, Eq. (7.3).

At zeroth order in v/M_{SUSY}, the non-logarithmic terms of the EFT approach contained in $\lambda(M_t)$ in Eq. (10.6) agree with the non-SM contributions in Eq. (10.7). They result from the threshold corrections at the matching scale M_{SUSY}. These threshold corrections are so far only known fully at the one-loop order. At the two-loop order only the $\mathcal{O}(\alpha_s\alpha_t, \alpha_t^2, \alpha_t\alpha_b, \alpha_b^2)$

corrections are implemented in publicly available codes so far. Thus, those terms in the non-logarithmic product of self-energy times derivative of the self-energy being not of $\mathcal{O}(\alpha_t^2, \alpha_t\alpha_b, \alpha_b^2)$ are not present in $(M_h^2)_{\text{EFT}}$. At higher orders, all terms involving a derivative of $\hat{\Sigma}_{hh}^{\text{nonSM}}$ are affected. Therefore, we can write

$$\begin{aligned}\Delta_{p^2}^{\text{nolog}} := &\left[\hat{\Sigma}_{hh}^{\text{nonSM}\prime}(m_h^2)\right]_{\text{nolog}}\left[\hat{\Sigma}_{hh}^{\text{MSSM}}(m_h^2)\right]_{\text{nolog}} \\ &- \left[\hat{\Sigma}_{hh}^{\text{nonSM}\prime}(m_h^2)\right]_{\text{nolog}}^{\mathcal{O}(\alpha_t,\alpha_b)}\left[\hat{\Sigma}_{hh}^{\text{MSSM}}(m_h^2)\right]_{\text{nolog}}^{\mathcal{O}(\alpha_t,\alpha_b)} \\ &+ \left[\text{higher-order terms involving } (\partial/\partial p^2)^n\hat{\Sigma}_{hh}^{\text{nonSM}},\, n \geq 1\right]. \end{aligned} \tag{10.9}$$

As we will demonstrate in the following section, also the non-logarithmic non-SM contributions arising from the determination of the pole of the propagator cancel out with contributions of the subloop renormalization in the limit of a high SUSY scale.

Apart from these terms and from the non-logarithmic terms of $\mathcal{O}(v/M_{\text{SUSY}})$ discussed above, $\Delta_{v/M_{\text{SUSY}}}^{\text{nolog}}$, a further difference between the hybrid approach and the EFT aproach is due to the parametrization of the non-logarithmic terms, $\Delta_{\text{para}}^{\text{nolog}}$. In the EFT approach all low-scale parameters are $\overline{\text{MS}}$ quantities. The results of our hybrid approach, on the other hand, are expressed in terms of on-shell parameters. For the top-quark mass both the results expressed in terms of the pole mass, M_t, and the running mass at the scale M_t, $\overline{m}_t(M_t)$ (see [78] for details on the involved reparametrization) have been implemented. As explained in Section 6.3, the Higgs vev is a dependent quantity in our renormalization scheme which is expressed in terms of the on-shell quantities M_W, s_W and e according to Eq. (7.9) (and furthermore reparametrized in terms of the Fermi constant G_F, see Eq. (6.56)). Accordingly, the non-logarithmic terms in the EFT approach are parametrized in terms of the $\overline{\text{MS}}$ quantities $\overline{m}_t(M_t)$ and $v_{\overline{\text{MS}}}(M_t)$, while depending on the option chosen for the top-quark mass the non-logarithmic terms in `FeynHiggs` are expressed in terms of either $\overline{m}_t(M_t)$ and v_{G_F} or M_t and v_{G_F}, where we substituted the Fermi constant by v_{G_F},

$$v_{G_F}^2 = \frac{1}{2\sqrt{2}G_F}. \tag{10.10}$$

Those parametrizations differ from each other by higher-order terms. The observed differences are therefore related to the remaining uncertainties of unknown higher-order corrections.

It should be noted that also within the EFT approach there is a certain freedom for choosing different parametrizations. For instance, the threshold corrections at the matching scale can be expressed in terms of the SM $\overline{\text{MS}}$ top-Yukawa coupling or in terms of the MSSM $\overline{\text{DR}}$ top-Yukawa coupling.

10.3 Terms arising from the determination of the propagator poles at the two-loop level

We saw in Section 10.1 and Section 10.2 that the different determination of the propagator pole in the hybrid approach and the EFT approach gives rise to both logarithmic and non-logarithmic contributions in which the expressions given for the two approaches differ from each other. We will now explicitly demonstrate at the two-loop level that those differences in fact cancel out in the limit of a heavy SUSY scale if all the relevant terms at this order are taken into account.

As a first step, we write down the correction to M_h^2, derived by an explicit diagrammatic calculation. At strict two-loop order, we obtain

$$\begin{aligned}(M_h^2)_{\text{FD}} = &\, m_h^2 - \hat{\Sigma}_{hh}^{\text{MSSM},(1)}(m_h^2) - \hat{\Sigma}_{hh}^{\text{MSSM},(2)}(m_h^2) \\ &+ \left(\hat{\Sigma}_{hh}^{\text{nonSM},(1)\prime}(m_h^2) + \hat{\Sigma}_{hh}^{\text{SM},(1)\prime}(m_h^2)\right)\hat{\Sigma}_{hh}^{\text{MSSM},(1)}(m_h^2). \end{aligned} \tag{10.11}$$

The superscripts indicate the loop-order of the corresponding self-energy.[1]

[1] In our discussion here we treat the two-loop self-energy as the full result containing all contributions that appear at this order. The specific approximations that have been made at the two-loop level in `FeynHiggs` will be discussed below.

We obtain the renormalized two-loop self-energy from the unrenormalized one via

$$\hat{\Sigma}_{hh}^{\mathrm{MSSM},(2)}(m_h^2) = \Sigma_{hh}^{\mathrm{MSSM},(2)}(m_h^2) + (\text{two-loop counterterms}) + (\text{subloop-ren.}). \tag{10.12}$$

The subloop-renormalization can be derived from the one-loop self-energy via a counterterm-expansion. Expressing all couplings appearing in the one-loop self-energy through masses divided by v_{G_F} (for the remainder of this section we drop the subscript "G_F", i.e. we use the shorthand $v \equiv v_{G_F}$), we can write

$$\begin{aligned}
&(\text{subloop-ren.}) = \\
&= (\delta v^2)^{\mathrm{MSSM}} \frac{\partial}{\partial v^2} \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) + \sum_i (\delta m_i)^{\mathrm{MSSM}} \frac{\partial}{\partial m_i} \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) + (\text{field ren.}). = \\
&= -\frac{(\delta v^2)^{\mathrm{MSSM}}}{v^2} \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) + \sum_i (\delta m_i)^{\mathrm{MSSM}} \frac{\partial}{\partial m_i} \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) + (\text{field ren.}),
\end{aligned} \tag{10.13}$$

where we used in the last line that $\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)} \propto 1/v^2$ if all couplings are expressed by the respective mass divided by v. The superscript "MSSM" for the counterterms is used to indicate that these are the counterterms of the full MSSM.

We are interested in terms involving the finite parts of the derivative of the Higgs self-energy, i.e. terms which could potentially cancel the term proportional to $\hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)$ in Eq. (10.11). At first sight it would seem that terms of this kind could arise from an on-shell field renormalization of the Higgs field. It is well-known, however, that those field renormalization constants drop out of the prediction of M_h order by order in perturbation theory (as explained in Chapter 6, we employ a $\overline{\mathrm{DR}}$ renormalization for the Higgs fields). Also the mass counterterms as well as the genuine two-loop counterterms do not contribute terms that are proportional to $\hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)$. The only remaining term is the vev counterterm (with the vev parametrized in terms of the Fermi constant). According to Eq. (7.11) and Eq. (6.56) it is given at the one-loop level by, having the same form in the SM and the MSSM,

$$\frac{\delta v^2}{v^2} = \frac{\delta M_W^2}{M_W^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) - \frac{\delta e^2}{e^2} - \Delta r - \delta Z_{hh}. \tag{10.14}$$

The renormalization constant δZ_{hh} represents within the MSSM the $\overline{\mathrm{DR}}$ field renormalization constant of the SM-like Higgs field, while in the SM it is understood to be the $\overline{\mathrm{MS}}$ field renormalization constant of the Higgs field.

We verified by explicit calculation that in the limit of a large SUSY scale the following relation holds,

$$\frac{(\delta v^2)^{\mathrm{MSSM}}}{v^2} = \frac{(\delta v^2)^{\mathrm{SM}}}{v^2} - \hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2) + \mathcal{O}(v/M_{\mathrm{SUSY}}). \tag{10.15}$$

Using this relation, we can rewrite the two-loop self-energy (omitting terms of $\mathcal{O}(v/M_{\mathrm{SUSY}})$),

$$\hat{\Sigma}_{hh}^{\mathrm{MSSM},(2)}(m_h^2) = \hat{\Sigma}_{hh}^{\mathrm{MSSM},(2)}(m_h^2)\Big|_{(\delta v^2)^{\mathrm{MSSM}} \to (\delta v^2)^{\mathrm{SM}}} + \hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2), \tag{10.16}$$

where the subscript '$(\delta v^2)^{\mathrm{MSSM}} \to (\delta v^2)^{\mathrm{SM}}$' is used to indicate that the MSSM vev counterterm, appearing in the subloop renormalization, is replaced by its SM counterpart.

Plugging this expression back into Eq. (10.13) and Eq. (10.11), we obtain

$$\begin{aligned}
(M_h^2)_{\mathrm{FD}} = {} & m_h^2 - \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) \\
& - \left(\hat{\Sigma}_{hh}^{\mathrm{MSSM},(2)}(m_h^2)\Big|_{(\delta v^2)^{\mathrm{MSSM}} \to (\delta v^2)^{\mathrm{SM}}} + \hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2)\right) \\
& + \left(\hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2) + \hat{\Sigma}_{hh}^{\mathrm{SM},(1)\prime}(m_h^2)\right)\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) = \\
= {} & m_h^2 - \hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2) - \hat{\Sigma}_{hh}^{\mathrm{MSSM},(2)}(m_h^2)\Big|_{(\delta v^2)^{\mathrm{MSSM}} \to (\delta v^2)^{\mathrm{SM}}} \\
& + \hat{\Sigma}_{hh}^{\mathrm{SM},(1)\prime}(m_h^2)\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)}(m_h^2).
\end{aligned} \tag{10.17}$$

We observe that the corresponding subloop renormalization term cancels in Eq. (10.11) the term $\hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)$ involving the non-SM contributions to the Higgs self-energy by which the determination of the propagator pole in the hybrid approach differs from the EFT approach.

The origin of Eq. (10.15) is the different normalization of the SM-like MSSM Higgs doublet Φ_{MSSM} and the SM Higgs doublet Φ_{SM}. Comparing the derivative of the two-point function, appearing in the LSZ factor of amplitudes with external Higgs fields,[2] we obtain in the limit of a heavy SUSY scale,

$$\Phi_{\mathrm{MSSM}}\left(1+\frac{1}{2}\hat{\Sigma}_{hh}^{\mathrm{MSSM},(1)\prime}(m_h^2)\right)=\Phi_{\mathrm{SM}}\left(1+\frac{1}{2}\hat{\Sigma}_{hh}^{\mathrm{SM},(1)\prime}(m_h^2)\right), \tag{10.18}$$

or equivalently

$$\Phi_{\mathrm{MSSM}}=\Phi_{\mathrm{SM}}\left(1-\frac{1}{2}\hat{\Sigma}_{hh}^{\mathrm{nonSM},(1)\prime}(m_h^2)\right). \tag{10.19}$$

Expressed in terms of a relation between the counterterms of the vevs, this implies Eq. (10.15).

While as mentioned above the Higgs field renormalization constant drops out in the Higgs mass prediction order by order, it is nevertheless noteworthy that the introduction of an OS field renormalization constant would lead to

$$\hat{\Sigma}_{hh}^{\mathrm{MSSM}\prime}(m_h^2)\big|_{\delta Z_h^{\mathrm{OS}}}=0 \tag{10.20}$$

and

$$(\delta v^2)^{\mathrm{MSSM}}\big|_{\delta Z_{hh}^{\mathrm{OS}}}=(\delta v^2)^{\mathrm{SM}}\big|_{\delta Z_{hh}^{\mathrm{OS}}}, \tag{10.21}$$

implying that no terms involving $\hat{\Sigma}_{hh}^{\mathrm{nonSM}\prime}$ appear in the subloop renormalization at the two-loop level.

While we have demonstrated this cancellation at the two-loop level, it is to be expected that it would also occur at higher orders. Explicit formulas for higher-order terms of this kind are given in App. B. While the described cancellation occurs at the full two-loop level, only partial cancellations occur between the full one-loop self-energy times its derivative and the two-loop self-energy if for the latter certain approximations are made.

In the fixed-order calculation implemented in `FeynHiggs`, the two-loop self-energies are derived in the gaugeless limit (i.e., two-loop corrections of $\mathcal{O}(\alpha_s\alpha_t,\alpha_s\alpha_b,\alpha_t^2,\alpha_t\alpha_b,\alpha_b^2)$ are incorporated [19, 20, 22, 24, 30, 31]), and by default the external momentum of the two-loop graphs is neglected. Accordingly, all $\mathcal{O}(\alpha_t^2,\alpha_t\alpha_b,\alpha_b^2)$ non-SM terms arising through the determination of the propagator pole at the two-loop level are cancelled in the limit of a large SUSY scale by corresponding subloop renormalization contributions within the diagrammatic calculation (the determination of the propagator pole obviously does not give rise to terms of $\mathcal{O}(\alpha_s\alpha_t,\alpha_s\alpha_b)$). In previous versions of `FeynHiggs`, we have already taken this issue into account and constructed the subtraction terms according to Eq. (7.20) in order to not subtract logarithmic contributions that are needed for the cancellation with the corresponding terms arising from the determination of the propagator poles. For terms arising through the determination of the propagator pole beyond $\mathcal{O}(\alpha_t^2,\alpha_t\alpha_b,\alpha_b^2)$, however, the cancellation in the limit of a large SUSY scale did not occur because the corresponding contributions in the irreducible self-energies at the two-loop level and beyond are not incorporated. In order to avoid unwanted effects from an incomplete cancellation, we have removed the uncompensated terms arising from the determination of the propagator pole in `FeynHiggs`.

[2]It should be noted that such an LSZ factor enters in the EFT approach via the matching condition at the high scale.

Chapter 11

Resummation for low M_A

The discussion in the previous Chapters has focused mainly on single scale scenarios, in which all non-SM particles share a common mass scale (and split scenarios with low mass electroweakinos). In this Chapter, we will discuss another class of phenomenologically interesting scenarios, namely scenarios with light non-SM Higgs bosons. In this case the low energy EFT is not well described by the SM but better by a Two-Higgs-Doublet-Model (THDM). After discussing the EFT calculation incorporating such an effective THDM, we will focus on the implementation of this EFT calculation into the hybrid framework of `FeynHiggs`.

11.1 EFT calculation

With respect to the EFT calculation discussed in Chapter 8, we take into account one additional mass scale: the non-SM Higgs boson scale M_A. M_A marks the scale at which the non-SM Higgs are integrated out. Consequently, we have a set of eight different EFTs: the SM, the SM plus electroweakinos, the SM plus gluino, the SM plus electroweakinos and gluino, the THDM, the THDM plus electroweakinos, the THDM plus gluino as well as the THDM plus electroweakinos and gluino. Leaving aside the gluino threshold, this results into three different low M_A hierarchies (see Fig. 11.1).

In the corresponding EFT calculation, we take into account full one-loop threshold corrections and full two-loop RGEs. This implies full LL and NLL resummation. Additionally, we include $\mathcal{O}(\alpha_s\alpha_t)$ matching conditions for the Higgs self-couplings. $\mathcal{O}(\alpha_t^2)$ threshold corrections for matching the THDM to the MSSM are currently not known. Moreover, three-loop RGEs for the THDM are not yet available. Since the SM three-loop running is negligible, one may believe that this also holds for the three-loop THDM running [97]. Nevertheless, the resummation of NNLL contributions is incomplete.

11.1.1 Relevant EFTs

Here, we give a brief overview of the various EFTs appearing in the various hierarchies. We already described the SM and the SM plus EWinos in Chapter 8. We will not describe EFTs with gluino, since the presence of the gluino does not induce any effective couplings. It, however, does alter the RGEs (see App. F).

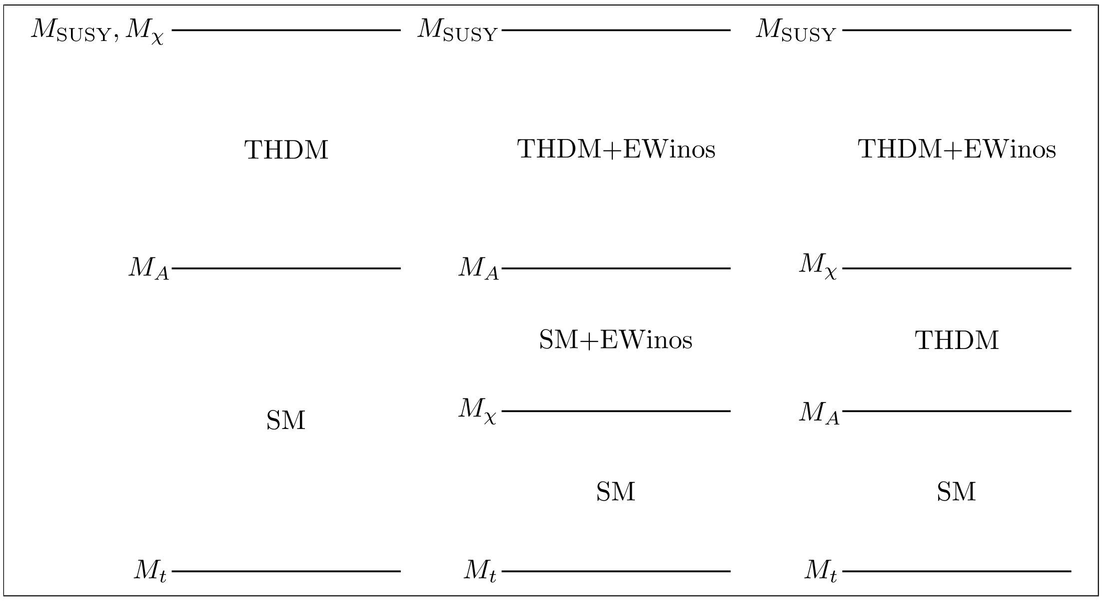


Figure 11.1: EFT towers covered for various hierarchies (gluino threshold not shown).

The Two Higgs Doublet Model

Decoupling all sfermions, gauginos and Higgsinos from the full MSSM leads to a THDM as the remaining effective theory below the SUSY scale. The THDM Higgs potential can be written as follows,

$$\begin{aligned} V_{\text{THDM}}(\Phi_1,\Phi_2) =& m_1^2\Phi_1^\dagger\Phi_1 + m_2^2\Phi_2^\dagger\Phi_2 - m_{12}^2(\Phi_1^\dagger\Phi_2 + \Phi_2^\dagger\Phi_1) + \frac{1}{2}\lambda_1(\Phi_1^\dagger\Phi_1)^2 + \frac{1}{2}\lambda_2(\Phi_2^\dagger\Phi_2)^2 \\ &+ \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{1}{2}\lambda_5\left((\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right) \\ &+ \lambda_6(\Phi_1^\dagger\Phi_1)\left((\Phi_1^\dagger\Phi_2) + (\Phi_2^\dagger\Phi_1)\right) + \lambda_7(\Phi_2^\dagger\Phi_2)\left((\Phi_1^\dagger\Phi_2) + (\Phi_2^\dagger\Phi_1)\right), \end{aligned} \tag{11.1}$$

where $\Phi_{1,2}$ denote the two doublets of scalar fields. Since we consider only the real MSSM, all the coefficients can be chosen as real parameters.

At the minimum of the potential each Higgs field Φ_i acquires a vev,

$$\langle\Phi_i\rangle = \begin{pmatrix} 0 \\ v_i \end{pmatrix}, \quad i = 1, 2. \tag{11.2}$$

Decomposing the Higgs fields into components according to

$$\Phi_i = \begin{pmatrix} \phi_i^+ \\ v_i + \frac{1}{\sqrt{2}}(\phi_i + i\chi_i) \end{pmatrix}. \tag{11.3}$$

and expanding the potential around the minimum yields the mass matrix of the $\mathcal{CP}$-even neutral Higgs bosons,

$$\mathcal{M}_{\phi\phi}^2 = \begin{pmatrix} m_1^2 & -m_{12}^2 \\ -m_{12}^2 & m_2^2 \end{pmatrix} + v^2\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \tag{11.4}$$

with the entries

$$a_{11} = 3\lambda_1 c_\beta^2 + (\lambda_3 + \lambda_4 + \lambda_5)s_\beta^2 + 6\lambda_6 s_\beta c_\beta, \tag{11.5}$$

$$a_{12} = 2(\lambda_3 + \lambda_4 + \lambda_5)s_\beta c_\beta + 6\lambda_6 c_\beta^2 + 6\lambda_7 s_\beta^2, \tag{11.6}$$

$$a_{22} = 3\lambda_2 s_\beta^2 + (\lambda_3 + \lambda_4 + \lambda_5)c_\beta^2 + 6\lambda_7 s_\beta c_\beta. \tag{11.7}$$

With the minimum conditions for the Higgs potential, m_1^2 and m_2^2 can be eliminated; the following relations for the masses of the $\mathcal{CP}$-odd neutral A boson and of the charged $H^\pm$ bosons are obtained,

$$m_A^2 = \frac{m_{12}^2}{s_\beta c_\beta} - v^2(2\lambda_5 + \lambda_6/t_\beta + \lambda_7 t_\beta), \tag{11.8}$$

$$m_{H^\pm} = m_A^2 + v^2(\lambda_5 - \lambda_4), \tag{11.9}$$

and the $\mathcal{CP}$-even mass matrix $\mathcal{M}_{\phi\phi}^2$ can be cast into the following form,

$$\mathcal{M}_{\phi\phi}^2 = m_A^2 \begin{pmatrix} s_\beta^2 & -s_\beta c_\beta \\ -s_\beta c_\beta & c_\beta^2 \end{pmatrix} + 2v^2 \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \tag{11.10}$$

with

$$b_{11} = \lambda_1 c_\beta^2 + 2\lambda_6 s_\beta c_\beta + \lambda_5 s_\beta^2, \tag{11.11}$$

$$b_{12} = (\lambda_3 + \lambda_4) s_\beta c_\beta + \lambda_6 c_\beta^2 + \lambda_7 s_\beta^2, \tag{11.12}$$

$$b_{22} = \lambda_2 s_\beta^2 + 2\lambda_7 s_\beta c_\beta + \lambda_5 c_\beta^2. \tag{11.13}$$

The tree-level mass eigenstates h and H are obtained by a rotation,

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{11.14}$$

with the angle α determined by

$$s_{2\alpha} = \frac{2\mathcal{M}_{\phi_1\phi_2}^2}{\sqrt{\left(\mathcal{M}_{\phi_1\phi_1}^2 - \mathcal{M}_{\phi_2\phi_2}^2\right)^2 + 4\left(\mathcal{M}_{\phi_1\phi_2}^2\right)^2}}, \qquad -\frac{\pi}{2} < \alpha < 0. \tag{11.15}$$

Often, it is useful to work in the Higgs basis instead of the mass eigenstate basis [98]. It is obtained by rotating the original fields $\phi_{1,2}$ in Eq. (11.3) by the angle β,

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{11.16}$$

In this basis, only H_1 acquires a vev, $\langle H_1 \rangle = v \equiv \sqrt{v_1^2 + v_2^2}$ and the mass matrix in Eq. (11.10) is transformed into

$$\mathcal{M}_{HH}^2 = m_A^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2v^2 \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} \tag{11.17}$$

with

$$c_{11} = \lambda_1 c_\beta^4 + \lambda_2 s_\beta^4 + 2(\lambda_3 + \lambda_4 + \lambda_5) s_\beta^2 c_\beta^2 + 4\lambda_6 s_\beta c_\beta^3 + 4\lambda_7 s_\beta^3 c_\beta, \tag{11.18}$$

$$\begin{aligned} c_{12} = &- \lambda_1 s_\beta c_\beta^3 + \lambda_2 s_\beta^3 c_\beta + (\lambda_3 + \lambda_4 + \lambda_5) s_\beta c_\beta c_{2\beta} - \lambda_6 c_\beta^2 (3s_\beta^2 - c_\beta^2) \\ &+ \lambda_7 s_\beta^2 (3c_\beta^2 - s_\beta^2), \end{aligned} \tag{11.19}$$

$$c_{22} = (\lambda_1 + \lambda_2) s_\beta^2 c_\beta^2 - 2(\lambda_3 + \lambda_4) s_\beta^2 c_\beta^2 + \lambda_5 (s_\beta^4 + c_\beta^4) - (\lambda_6 - \lambda_7) s_{2\beta} c_{2\beta}. \tag{11.20}$$

To get from the Higgs basis to the mass eigenstate basis, we have to rotate by the angle $\alpha - \beta$.

We also need the Yukawa part of the effective THDM Lagrangian, which is given for the third generation quarks by

$$\mathcal{L}_{\text{Yuk}}^{\text{THDM}} = -h_t \bar{t}_R (-i\Phi_2^T \sigma_2) Q_L - h_t' \bar{t}_R (-i\Phi_1^T \sigma_2) Q_L + h.c., \tag{11.21}$$

with the third-generation quark doublet Q_L and the Pauli matrix σ_2. h_t and h_t' are the effective top-Yukawa couplings. All other Yukawa couplings are neglected in the EFT calculation; they are, however, fully captured through the diagrammatic calculation at the one-loop level, in case of the bottom-Yukawa coupling also at the two-loop level.

As already noted in [39], the effective THDM with the Yukawa texture as given in Eq. (11.21) is not a type II model where only Φ_2 couples to up-type quarks. Although the tree-level Yukawa sector of the MSSM is that of a THDM of type II (see Eq. (4.39)), loop corrections induce also a coupling of Φ_1 to the top-quark, which enters through the matching procedure in the effective THDM. We take this coupling fully into account in all the affected RGEs and threshold corrections. Hence, we have to deal with 12 coupling constants, consisting of three gauge couplings, seven Higgs self-couplings, and two Yukawa couplings. RGEs for the considered THDM are listed in App. F.

The Two Higgs Doublet Model with Electroweakinos

If in addition to the non-SM Higgs bosons also light electroweak gauginos and Higgsinos (EWinos) are present, the effective Lagrangian below the scale M_{SUSY} is the one of the THDM described above, extended by extra mass and interaction terms

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{\text{THDM}} + (\ldots) - \frac{1}{2}M_\chi \widetilde{W}\widetilde{W} - \frac{1}{2}M_\chi \widetilde{B}\widetilde{B} - M_\chi\,(i\widetilde{\mathcal{H}}_u^T\sigma_2)\widetilde{\mathcal{H}}_d \\ &- \frac{1}{\sqrt{2}}\mathcal{H}_u^\dagger\left(\hat{g}_{2uu}\sigma_a\widetilde{W}^a + \hat{g}_{1uu}\widetilde{B}\right)\widetilde{\mathcal{H}}_u - \frac{1}{\sqrt{2}}\mathcal{H}_d^\dagger\left(\hat{g}_{2dd}\sigma_a\widetilde{W}^a - \hat{g}_{1dd}\widetilde{B}\right)\widetilde{\mathcal{H}}_d \\ &- \frac{1}{\sqrt{2}}(i\mathcal{H}_d^T\sigma_2)\left(\hat{g}_{2du}\sigma_a\widetilde{W}^a + \hat{g}_{1du}\widetilde{B}\right)\widetilde{\mathcal{H}}_u - \frac{1}{\sqrt{2}}(-i\mathcal{H}_u^T\sigma_2)\left(\hat{g}_{2ud}\sigma_a\widetilde{W}^a - \hat{g}_{1ud}\widetilde{B}\right)\widetilde{\mathcal{H}}_d \\ &+ h.c. \end{aligned} \tag{11.22}$$

for the bino field $\widetilde{B}$, the wino fields $\widetilde{W}^a$, and the Higgsino fields $\widetilde{\mathcal{H}}_{u,d}$ (tThe ellipsis denotes their kinetic terms). The associated Higgs fields $\mathcal{H}_{u,d}$ are related to the doublets $\Phi_{1,2}$ in Eq. (11.3) by

$$\mathcal{H}_u = \Phi_2, \tag{11.23}$$

$$\mathcal{H}_d = i\sigma_2\Phi_1^*. \tag{11.24}$$

The coupling constants $\hat{g}_{1uu,1dd,1ud,1du,\ldots}$ are effective Higgs–Higgsino–Gaugino couplings. The numeral in the subscript refers to the attached gauge symmetry (i.e. $U(1)_Y$ or $SU(2)_L$), the first letter to the involved Higgs doublet, and the second letter to the involved Higgsino. Altogether, we now have 20 effective couplings in the game. Corresponding two-loop RGEs for all couplings are listed in App. F.

11.1.2 Matching the EFTs

After having specified the various EFTs, we describe how they are matched to each other. To derive the matching conditions, we have to compare physical amplitudes with external light-particles computed in the EFT valid below the matching scale and the full model (or the more complete EFT) valid above the matching scale. The difference between the physical amplitudes has to be absorbed by adapting the effective couplings in the particular EFT that is to be matched.

Terms contributing to the matching conditions arise from different vertex corrections and from different normalizations of the external fields. The part coming from the vertex corrections is obtained by calculating the vertex functions in the high-energy and the low-energy theory. The difference can then directly be absorbed into the effective coupling of the low-energy theory. At least at the one-loop level, at which we mostly work, this procedure is straightforward. Therefore, we will not go into more details.

If all external fields are non-mixed mass eigenstates, the external leg corrections are just given by the corresponding LSZ factors, the wave-function renormalization. The difference between the LSZ factors in the high-energy and the low-energy theory has again to be absorbed by the low-energy effective coupling.

In case of mixing in the external fields, a more careful discussion is required. Even when the external fields are diagonal at the tree level, loop contributions to the two-point vertices induce mixing between the mass eigenstates at higher orders. This transition has to be included as further external leg corrections, in addition to the LSZ factors. In the MSSM and the THDM, the mixing between the $\mathcal{CP}$-even Higgs bosons h, H is the important issue. It is ascribed to a non-diagonal self-energy Σ_{hH}.

Conveniently, all external leg corrections can be written in form of a single matrix, the **Z**-matrix (see [8] for more details). It gives the relation between the external, asymptotical-free physical states and the tree-level mass eigenstates used for the calculation of the vertex correction. At the one-loop level, the MSSM relation reads

$$\begin{pmatrix} h^{\text{phys}} \\ H^{\text{phys}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\widehat{\Sigma}'_{hh}(m_h^2) & \frac{\widehat{\Sigma}_{hH}(m_h^2)}{m_h^2 - m_H^2} \\ \frac{\widehat{\Sigma}_{hH}(m_H^2)}{m_H^2 - m_h^2} & 1 + \frac{1}{2}\widehat{\Sigma}'_{HH}(m_H^2) \end{pmatrix} \begin{pmatrix} \hat{h} \\ \widehat{H} \end{pmatrix}, \tag{11.25}$$

where we used the symbol $\widehat{\ }$ to mark MSSM quantities. Σ_{hh} and Σ_{HH} are the diagonal self-energies entering the LSZ factors. The prime denotes the derivative with respect to the external momentum. The corresponding relation in the THDM is written as follows,

$$\begin{pmatrix} h^{\text{phys}} \\ H^{\text{phys}} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}\widetilde{\Sigma}'_{hh}(m_h^2) & \frac{\widetilde{\Sigma}_{hH}(m_h^2)}{m_h^2-m_H^2} \\ \frac{\widetilde{\Sigma}_{hH}(m_H^2)}{m_H^2-m_h^2} & 1+\frac{1}{2}\widetilde{\Sigma}'_{HH}(m_H^2) \end{pmatrix} \begin{pmatrix} \widetilde{h} \\ \widetilde{H} \end{pmatrix}. \tag{11.26}$$

where we used the symbol $\widetilde{\ }$ to mark THDM quantities.[1]

Eqs. (11.25) and (11.26) yield the relation between the mass eigenstates of the MSSM and the THDM,

$$\begin{pmatrix} \widetilde{h} \\ \widetilde{H} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{hh}(m_h^2) & \frac{\Delta\Sigma_{hH}(m_h^2)}{m_h^2-m_H^2} \\ \frac{\Delta\Sigma_{hH}(m_H^2)}{m_H^2-m_h^2} & 1+\frac{1}{2}\Delta\Sigma'_{HH}(m_H^2) \end{pmatrix} \begin{pmatrix} \widehat{h} \\ \widehat{H} \end{pmatrix}, \tag{11.27}$$

where the $\Delta\Sigma_{xy}$ summarize the differences between the self-energies, for $x, y \in \{h, H\}$,

$$\Delta\Sigma_{xy}(p^2) \equiv \widehat{\Sigma}_{xy}(p^2) - \widetilde{\Sigma}_{xy}(p^2). \tag{11.28}$$

The mass eigenstates are related to the gauge eigenstates via Eq. (11.14),

$$\begin{pmatrix} \widetilde{h} \\ \widetilde{H} \end{pmatrix} = U_{\widetilde{\alpha}} \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} -s_{\widetilde{\alpha}} & c_{\widetilde{\alpha}} \\ c_{\widetilde{\alpha}} & s_{\widetilde{\alpha}} \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix}, \tag{11.29}$$

$$\begin{pmatrix} \widehat{h} \\ \widehat{H} \end{pmatrix} = U_{\widehat{\alpha}} \begin{pmatrix} \widehat{\Phi}_1 \\ \widehat{\Phi}_2 \end{pmatrix} = \begin{pmatrix} -s_{\widehat{\alpha}} & c_{\widehat{\alpha}} \\ c_{\widehat{\alpha}} & s_{\widehat{\alpha}} \end{pmatrix} \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}. \tag{11.30}$$

With these relations, Eq. (11.27) can be transformed into the gauge eigenstate basis,

$$\begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix} = U_{\widetilde{\alpha}}^T \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{hh}(m_h^2) & \frac{\Delta\Sigma_{hH}(m_h^2)}{m_h^2-m_H^2} \\ \frac{\Delta\Sigma_{hH}(m_H^2)}{m_H^2-m_h^2} & 1+\frac{1}{2}\Delta\Sigma'_{HH}(m_H^2) \end{pmatrix} U_{\widehat{\alpha}} \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}. \tag{11.31}$$

At lowest order, the mixing angles of the THDM and the MSSM are equal and fixed by the condition that they relate the gauge eigenstate basis to the tree-level mass eigenstate basis. At the one-loop level the angle $\widetilde{\alpha}$ of the THDM can be chosen independently (as a free parameter), allowing for a difference

$$\Delta\alpha = \widehat{\alpha} - \widetilde{\alpha}. \tag{11.32}$$

Using this shift to replace $\widetilde{\alpha}$ by $\widehat{\alpha}$ in Eq. (11.31) we obtain, expanded up to the one-loop level,

$$\begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix} = U_{\widehat{\alpha}}^T \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{hh}(m_h^2) & \frac{\Delta\Sigma_{hH}(m_h^2)}{m_h^2-m_H^2} - \Delta\alpha \\ \frac{\Delta\Sigma_{hH}(m_H^2)}{m_H^2-m_h^2} + \Delta\alpha & 1+\frac{1}{2}\Delta\Sigma'_{HH}(m_H^2) \end{pmatrix} U_{\widehat{\alpha}} \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}. \tag{11.33}$$

Next we expand $\Delta\Sigma_{hH}(m_H^2)$ around $p^2 = m_h^2$,

$$\Delta\Sigma_{hH}(m_H^2) = \Delta\Sigma_{hH}(m_h^2) + \Delta\Sigma'_{hH}(m_H^2 - m_h^2) + \mathcal{O}(v/M_{\text{SUSY}}, M_A/M_{\text{SUSY}}). \tag{11.34}$$

All higher-order derivatives of the $\Delta\Sigma_{xy}$ are suppressed by M_{SUSY} and therefore negligible in the EFT calculation. For the same reason, we drop the specification of the external momentum in all derivatives of $\Delta\Sigma_{xy}$ in the following (which is always taken at m_h^2).

Using the expansion (11.34) and rewriting the self-energies partly in the gauge eigenstate basis yields

$$\begin{aligned} \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix} = \Bigg[& \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{11} & \frac{1}{2}\Delta\Sigma'_{12} \\ \frac{1}{2}\Delta\Sigma'_{12} & 1+\frac{1}{2}\Delta\Sigma'_{22} \end{pmatrix} \\ & + \left(\frac{\Delta\Sigma_{hH}(m_h^2)}{m_h^2 - m_H^2} - \frac{1}{2}\Delta\Sigma'_{hH} - \Delta\alpha \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Bigg] \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}. \end{aligned} \tag{11.35}$$

with the notation $\Delta\Sigma_{ij} \equiv \Delta\Sigma_{\phi_i\phi_j}$ for $i, j \in \{1, 2\}$.

[1]In contrast to other Chapters, the symbols "$\widehat{\ }$" and "$\widetilde{\ }$" are not used to label specific renormalization schemes. Here, they are only used to distinguish MSSM and THDM quantities.

The second matrix corresponds to the one-loop part of a unitary matrix and thereby to a basis transformation by a rotation. It can be absorbed by adjusting $\Delta\alpha$ according to

$$\Delta\alpha = \frac{\Delta\Sigma_{hH}(m_h^2)}{m_h^2 - m_H^2} - \frac{1}{2}\Delta\Sigma'_{hH}. \tag{11.36}$$

The first matrix in Eq. (11.35) is not unitary and hence cannot be removed by a basis transformation. Therefore, there is a remaining difference between the normalization of the gauge eigenstates in the MSSM and the THDM, given by the following relation,

$$\begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\Delta\Sigma'_{11} & \frac{1}{2}\Delta\Sigma'_{12} \\ \frac{1}{2}\Delta\Sigma'_{12} & 1 + \frac{1}{2}\Delta\Sigma'_{22} \end{pmatrix} \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix}, \tag{11.37}$$

which corresponds to the one used in [85]. As noted above, it is only valid at the one-loop level.

Replacing the fields $\Phi_{1,2}$ by the corresponding vevs, we immediately obtain

$$\begin{pmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\Delta\Sigma'_{11} & \frac{1}{2}\Delta\Sigma'_{12} \\ \frac{1}{2}\Delta\Sigma'_{12} & 1 + \frac{1}{2}\Delta\Sigma'_{22} \end{pmatrix} \begin{pmatrix} \widehat{v}_1 \\ \widehat{v}_2 \end{pmatrix}. \tag{11.38}$$

This directly implies

$$\widetilde{\beta} = \widehat{\beta} + \frac{1}{2}\Big[(\Delta\Sigma'_{22} - \Delta\Sigma'_{11})\, s_\beta c_\beta + \Delta\Sigma'_{12} c_{2\beta} \Big] = \widehat{\beta} + \frac{1}{2}\Delta\Sigma'_{H_1 H_2} \tag{11.39}$$

or

$$\tan\widetilde{\beta} = \tan\widehat{\beta} + \frac{1}{2c_\beta^2}\Delta\Sigma'_{H_1 H_2}, \tag{11.40}$$

respectively, with $H_{1,2}$ being the fields of the Higgs basis defined in Eq. (11.16).

We have to take care of Eq. (11.37) whenever we match a coupling involving an external Higgs field. The different normalizations of the Higgs fields will introduce derivatives of Higgs self-energies into the matching conditions.

Following this procedure and including vertex corrections, we derived a full set of one-loop threshold corrections for all appearing effective couplings and hierarchies. Below, we list only the tree-level matching conditions and the dominant one-loop corrections, i.e. those proportional to the strong gauge coupling or the top-Yukawa couplings. Full one-loop threshold corrections for all effective couplings including electroweak contributions are listed in App. C.

In addition to the calculation of matching conditions, we will also need Eq. (11.37) for combining the diagrammatic fixed-order calculation and the EFT calculation.

Matching the THDM to the MSSM

The Higgs self-couplings in the THDM scalar potential are fixed at the tree-level in terms of gauge couplings [85],

$$\lambda_{1,\text{tree}}(M_\text{SUSY}) = \lambda_{2,\text{tree}}(M_\text{SUSY}) = \frac{1}{4}(g^2 + g'^2), \tag{11.41}$$

$$\lambda_{3,\text{tree}}(M_\text{SUSY}) = \frac{1}{4}(g^2 - g'^2), \tag{11.42}$$

$$\lambda_{4,\text{tree}}(M_\text{SUSY}) = -\frac{1}{2}g^2, \tag{11.43}$$

$$\lambda_{5,\text{tree}}(M_\text{SUSY}) = \lambda_{6,\text{tree}}(M_\text{SUSY}) = \lambda_{7,\text{tree}}(M_\text{SUSY}) = 0. \tag{11.44}$$

At the one-loop level, these relations receive additional contributions [85],

$$\Delta\lambda_1 = -\frac{1}{2}k h_t^4 \hat{\mu}^4 + \mathcal{O}(g, g'), \tag{11.45}$$

$$\Delta\lambda_2 = 6k h_t^4 \hat{A}_t^2 \left(1 - \frac{1}{12}\hat{A}_t^2\right) + \mathcal{O}(g, g'), \tag{11.46}$$

$$\Delta\lambda_3 = \frac{1}{2}k \hat{\mu}^2 h_t^4 (3 - \hat{A}_t^2) + \mathcal{O}(g, g'), \tag{11.47}$$

$$\Delta\lambda_4 = \frac{1}{2}k\hat{\mu}^2 h_t^4(3-\hat{A}_t^2) + \mathcal{O}(g,g'), \tag{11.48}$$

$$\Delta\lambda_5 = -\frac{1}{2}kh_t^4\hat{\mu}^2\hat{A}_t^2 + \mathcal{O}(g,g'), \tag{11.49}$$

$$\Delta\lambda_6 = \frac{1}{2}kh_t^4\hat{\mu}^3\hat{A}_t + \mathcal{O}(g,g'), \tag{11.50}$$

$$\Delta\lambda_7 = \frac{1}{2}kh_t^4\hat{\mu}\hat{A}_t(\hat{A}_t^2-6) + \mathcal{O}(g,g') \tag{11.51}$$

with $\hat{\mu} = \mu/M_{\text{SUSY}}$ and $\hat{A}_t = A_t/M_{\text{SUSY}}$ (see Section 4.3). The factor $k \equiv (4\pi)^{-2}$ is used to mark the loop-order. A_t is the stop trilinear coupling and h_t the top-Yukawa coupling of the MSSM (see Eq. (4.39)).[2] In addition to these one-loop corrections, we also include $\mathcal{O}(\alpha_s\alpha_t)$ threshold corrections, listed in App. C.9.

For $\hat{\mu} = 1$, the effective top-Yukawa couplings are given by

$$h_t^{\text{THDM}}(M_{\text{SUSY}}) = h_t^{\text{MSSM}}\left\{1 + k\left[\frac{4}{3}g_3^2(1-\hat{A}_t) - \frac{1}{4}h_t^2\hat{A}_t^2\right]\right\} + \mathcal{O}(g,g'), \tag{11.52}$$

$$(h_t')^{\text{THDM}}(M_{\text{SUSY}}) = h_t k\left\{\frac{4}{3}g_3^2\hat{\mu} + \frac{1}{4}h_t^2\hat{A}_t\hat{\mu}\right\} + \mathcal{O}(g,g'). \tag{11.53}$$

The full expressions for $\hat{\mu} \neq 1$ are given in App. C.

The matching condition for $\tan\beta$ is obtained from Eq. (11.40) yielding

$$t_\beta^{\text{THDM}}(M_{\text{SUSY}}) = t_\beta^{\text{MSSM}}(M_{\text{SUSY}})\left[1 + \frac{1}{4}kh_t^2(\hat{A}_t - \hat{\mu}/t_\beta)(\hat{A}_t + \hat{\mu}t_\beta) + \mathcal{O}(g,g')\right]. \tag{11.54}$$

Matching the THDM+EWinos to the MSSM

Neglecting the weak gauge couplings, the relations for matching the THDM to the MSSM are also valid when the THDM+EWinos is matched to the MSSM. The additional effective Higgs–Higgsino–Gaugino couplings of the THDM+EWinos fulfill the tree-level relations

$$\hat{g}_{1uu}(M_{\text{SUSY}}) = \hat{g}_{1dd}(M_{\text{SUSY}}) = g', \tag{11.55}$$

$$\hat{g}_{2uu}(M_{\text{SUSY}}) = \hat{g}_{2dd}(M_{\text{SUSY}}) = g, \tag{11.56}$$

$$\hat{g}_{1ud}(M_{\text{SUSY}}) = \hat{g}_{1du}(M_{\text{SUSY}}) = \hat{g}_{2ud}(M_{\text{SUSY}}) = \hat{g}_{2du}(M_{\text{SUSY}}) = 0. \tag{11.57}$$

Matching the THDM to the THDM+EWinos

Matching the THDM to the THDM+EWinos, the Higgs self-couplings, the gauge couplings, the top-Yukawa couplings and t_β are not modified at the tree-level. If the weak gauge couplings are neglected, there are also no loop corrections. The full one-loop corrections including the weak gauge couplings are listed in App. C.

Matching the SM to the THDM

In this specific case, the characteristic scale for all the couplings below is the mass M_A. In the decoupling limit $M_A \gg M_Z$ ($\alpha \to \beta - \frac{\pi}{2}$), which is assumed when the heavy Higgs bosons are integrated out, the Higgs self-coupling

$$\lambda(M_A) = c_{11} + \Delta\lambda\,, \tag{11.58}$$

with c_{11} from Eq. (11.18), $\beta = \beta^{\text{THDM}}$, and the one-loop correction

$$\Delta\lambda = -3k\left\{(\lambda_6+\lambda_7)c_{2\beta} + (\lambda_6-\lambda_7)c_{4\beta} - \left(\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - (\lambda_3+\lambda_4+\lambda_5)c_{2\beta}\right)s_{2\beta}\right\}^2. \tag{11.59}$$

The SM top-Yukawa coupling y_t is fixed by

$$y_t(M_A) = (h_t^{\text{THDM}}s_\beta + h_t'^{\,\text{THDM}}c_\beta)\left[1 - \frac{3}{8}k\left(h_t^{\text{THDM}}c_\beta - h_t'^{\,\text{THDM}}s_\beta\right)^2\right]. \tag{11.60}$$

[2]For definiteness, we now assign an explicit label for the Yukawa couplings h_t, h_t' introduced in Eq. (11.21) for the THDM.

Matching the SM+EWinos to the THDM+EWinos

Neglecting the weak gauge couplings, the relations for matching the SM to the THDM are also valid if the SM+EWinos is matched to the THDM+EWinos. At tree-level, the effective Higgs–Higgsino–Gaugino couplings of the SM+EWinos and the THDM+EWinos are related by

$$\tilde{g}_{1u}(M_A) = \hat{g}_{1uu}s_\beta + \hat{g}_{1du}c_\beta, \qquad \tilde{g}_{2u} = \hat{g}_{2uu}s_\beta + \hat{g}_{2du}c_\beta, \tag{11.61}$$

$$\tilde{g}_{1d}(M_A) = \hat{g}_{1dd}c_\beta + \hat{g}_{1ud}s_\beta, \qquad \tilde{g}_{2d} = \hat{g}_{2dd}c_\beta + \hat{g}_{2ud}s_\beta. \tag{11.62}$$

One-loop corrections proportional to the electroweak gauge couplings can be found in App. C.

11.1.3 Calculation of pole masses in the EFT approach

The proper way to calculate the physical masses of the $\mathcal{CP}$-even Higgs bosons in the EFT framework depends on the mass hierarchy. For $M_A \gg M_t$, the low-energy theory is the SM (or the SM+EWinos). Therefore, the procedure described in Section 7.2 can be applied. For $M_A \sim M_t$, though, there is no need to integrate out the non-standard Higgs bosons and the low-energy theory is better described by a THDM (or a THDM+EWinos). In this case, the physical masses of the $\mathcal{CP}$-even Higgs bosons are obtained by finding the poles of the propagators, i.e. the zeroes of the determinant of the inverse propagator matrix, depicted here in the Higgs basis as a possible choice,

$$-i\Delta^{-1}_{\widetilde{H}\widetilde{H}}(p^2) = \begin{pmatrix} p^2 - \widetilde{m}^2_{H_1H_1} + \widetilde{\Sigma}_{\widetilde{H}_1\widetilde{H}_1}(p^2) & -\widetilde{m}^2_{H_1H_1} + \widetilde{\Sigma}_{\widetilde{H}_1\widetilde{H}_2}(p^2) \\ -\widetilde{m}^2_{H_1H_1} + \widetilde{\Sigma}_{\widetilde{H}_1\widetilde{H}_2}(p^2) & p^2 - \widetilde{m}^2_{H_2H_2} + \widetilde{\Sigma}_{\widetilde{H}_2\widetilde{H}_2}(p^2) \end{pmatrix}. \tag{11.63}$$

The widetilde "$\widetilde{\ }$" indicates, as in Section 11.1.2, that the corresponding quantities are those of the THDM. The $\widetilde{m}^2_{H_iH_j}$'s are the elements of matrix $\mathcal{M}_{HH}$ defined in Eq. (11.17) and the $\widetilde{\Sigma}$-s are the corresponding self-energies of the THDM (or the THDM+EWinos) renormalized in the $\overline{\text{MS}}$ scheme.

In situations where M_A is larger than M_t, but the separation is also not too large, e.g. $M_A - M_t \sim 100$ GeV, it is difficult to decide if the SM should be used as low-energy theory or the THDM might the better choice. Therefore, a smooth transition between both cases is beneficial. To implement such a transition, we follow a procedure similar to the one introduced in [39]: We include the contribution of the running between M_A and M_t into $\widetilde{m}^2_{H_1H_1}$. The same contribution is in addition added to the off-diagonal entries $\widetilde{m}^2_{H_1H_2}$ with a prefactor $1/t_\beta$ and to $\widetilde{m}^2_{H_2H_2}$ with a prefactor $1/t^2_\beta$.[3] In this way both limits, $M_A \gg M_t$ and $M_A \sim M_t$, are properly recovered.

11.2 Combination of fixed-order and EFT calculation

As in Section 7.3, we want to combine the result of the EFT calculation described in Section 11.1 with the fixed-order calculation implemented in `FeynHiggs`. This combination is done in several steps. First, we have to relate the quantities computed in the EFT approach, namely the entries of the inverse propagator matrix, the two-point vertex function, to those in the fixed-order approach. Second, proper subtraction terms have to be identified and subtracted such that double-counting of terms appearing in both results is avoided. Finally, differences in input parameters resulting from different renormalization schemes have to be considered by proper conversion of the parameters.

We choose to perform the combination in the gauge eigenstate basis. Therefore, we need to know the relation between the two-point vertex function matrix in the full MSSM, denoted by $\Delta^{-1}_{\hat{\phi}\hat{\phi}}$, and in the effective THDM, labelled as $\Delta^{-1}_{\tilde{\phi}\tilde{\phi}}$. Again, as in Section 11.1.2, the symbol $\widehat{\ }$ is used to mark quantities in the full MSSM, and $\widetilde{\ }$ to mark quantities in the effective THDM. Both matrices have to be equal in case of Higgs fields with the same normalization in either

[3] Corresponding to the additional factor $1/t_\beta$ in the top-Yukawa coupling for H_2 for the dominant part of the contribution (see also footnote in Section 7.3)).

of the models. In our case, however, the Higgs field normalization is different, as specified by Eq. (11.37), which leads to the relation

$$\Delta^{-1}_{\widehat{\phi}\widehat{\phi}}(p^2) = \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{11} & \frac{1}{2}\Delta\Sigma'_{12} \\ \frac{1}{2}\Delta\Sigma'_{12} & 1+\frac{1}{2}\Delta\Sigma'_{22} \end{pmatrix} \Delta^{-1}_{\widetilde{\phi}\widetilde{\phi}}(p^2) \begin{pmatrix} 1+\frac{1}{2}\Delta\Sigma'_{11} & \frac{1}{2}\Delta\Sigma'_{12} \\ \frac{1}{2}\Delta\Sigma'_{12} & 1+\frac{1}{2}\Delta\Sigma'_{22} \end{pmatrix}. \tag{11.64}$$

As noted in Section 11.1.2 this formula is valid only in the decoupling limit of $M_{\text{SUSY}} \gg M_t$ and at the one-loop level. Explicit formula for the $\Delta\Sigma'_{ij}$ are listed in App. D.

To takes this relation into account in the combination of the EFT result and the fixed-order result, we now add finite pieces to the one-loop field renormalization constants in order to compensate for the different normalization of the MSSM and THDM Higgs doublets, redefining the one-loop field counterterms fixed in Eqs. (6.16)-(6.18) by

$$\delta^{(1)}Z_{ij} = \delta^{(1)}Z_{ij}\Big|_{\text{div}} + \delta^{(1)}Z_{ij}\Big|_{\text{fin}} \tag{11.65}$$

with the proper choice, according to Eq. (11.64),

$$\delta^{(1)}Z_{11}\Big|_{\text{fin}} = -\Delta\Sigma'_{11}, \quad \delta^{(1)}Z_{22}\Big|_{\text{fin}} = -\Delta\Sigma'_{22}, \quad \delta^{(1)}Z_{12}\Big|_{\text{fin}} = -\Delta\Sigma'_{12}. \tag{11.66}$$

Since Eq. (11.64) is valid only at the one-loop level, it cannot be applied for the two-loop field counterterms $\delta^{(2)}Z_{ij}$. These two-loop terms, however, do not appear in the renormalized two-loop self-energies (see Section 6.2). Based upon the expressions in Section 6.2 it is easy to show that the full dependence of the two-loop self-energies on the one-loop field renormalization constants is given by

$$\hat{\Sigma}^{(2)}_{hh}(0)\Big|_{\delta Z} = \Sigma^{(2)}_{hh}(0)\Big|_{\delta Z} - \frac{e}{2s_W M_W}\left(T^{(2)}_h\Big|_{\delta Z} + \frac{1}{2}s^2_\beta T^{(1)}_h \delta^{(1)}Z_{hh}\right), \tag{11.67}$$

$$\hat{\Sigma}^{(2)}_{hH}(0)\Big|_{\delta Z} = \Sigma^{(2)}_{hH}(0)\Big|_{\delta Z} - \frac{e}{2s_W M_W}\left(T^{(2)}_H\Big|_{\delta Z} + \frac{1}{2}s^2_\beta T^{(1)}_H \delta^{(1)}Z_{hh}\right), \tag{11.68}$$

$$\hat{\Sigma}^{(2)}_{HH}(0)\Big|_{\delta Z} = \Sigma^{(2)}_{HH}(0)\Big|_{\delta Z} - \Sigma^{(2)}_{AA}(0)\Big|_{\delta Z}, \tag{11.69}$$

where we used the subscript δZ to indicate that only contributions proportional to a one-loop field renormalization constant are taken into account.

With the additional finite parts introduced in the field renormalization constants, the inverse propagator matrix of the MSSM becomes equal to that of effective THDM (with restriction to the same perturbative order). Hence, the combination of the fixed-order (MSSM) and the EFT (THDM) approach is straightforward, which means that the MSSM inverse propagator matrix is replaced by

$$\Delta^{-1}_{\widehat{\phi}\widehat{\phi}} \to \Delta^{-1}_{\widehat{\phi}\widehat{\phi}} + \Delta^{\text{EFT}}, \tag{11.70}$$

where Δ^{EFT} contains the resummed logarithms and corresponding subtraction terms,

$$\Delta^{\text{EFT}} = \Delta^{-1}_{\widetilde{\phi}\widetilde{\phi}}\Big|_{\text{logs}} - \Delta^{-1}_{\widehat{\phi}\widehat{\phi}}\Big|_{\text{logs}}. \tag{11.71}$$

We checked numerically that the logarithms of the EFT calculation properly recover the logarithmic behavior of the full fixed-order result when restricted to the same perturbative order.

11.2.1 Redefinition of $\tan\beta$

As mentioned in Chapter 6, for the fixed-order calculation by default the $\overline{\text{DR}}$ scheme is employed for field renormalization of the Higgs doublets and for the renormalization of $\tan\beta$. Thus, there is a renormalization scale entering the diagrammatic calculation. By default, it is chosen to be equal to the pole mass M_t of the top quark. This in particular means that $\tan\beta$ is normally a MSSM $\overline{\text{DR}}$ quantity defined at the scale M_t.

The redefinition of the field renormalization constants by a finite shift, as described above, has an impact on the renormalization and hence the conceptual definition of $\tan\beta$. In presence of an off-diagonal field renormalization constant, the counterterm of $\tan\beta$ is given by (assuming still $\delta^{(i)}v_1/v_1 = \delta^{(i)}v_2/v_2$)

$$\delta^{(1)}t_\beta = \frac{1}{2}t_\beta\left(\delta^{(1)}Z_{22} - \delta^{(1)}Z_{11}\right) + \frac{1}{2}\left(1 - t_\beta^2\right)\delta^{(1)}Z_{12}. \tag{11.72}$$

For the corresponding two-loop counterterm, see Section 6.2. With the finite parts of the field renormalization constants in Eq. (11.66) and switching to the Higgs basis, we find

$$\delta^{(1)}t_\beta\Big|_{\text{fin}} = -\frac{1}{c_\beta^2}\Delta\Sigma'_{H_1H_2}. \tag{11.73}$$

Comparing this result to Eq. (11.40), we realize that $\tan\beta$ by now is not a MSSM quantity anymore, but instead a quantity of the THDM. Furthermore, the scale is changed to M_A, since the THDM part in $\Delta\Sigma'_{H_1H_2}$ is evaluated at the scale M_A. In conclusion, the finite shift in the field normalization constants of the MSSM leads to the conversion

$$t_\beta^{\text{MSSM}}(M_t) \to t_\beta^{\text{THDM}}(M_A). \tag{11.74}$$

Hence, $t_\beta^{\text{THDM}}(M_A)$ is the proper input parameter of the fixed-order calculation.

11.2.2 Conversion of input parameters

The fixed-order calculation employs either the OS or the $\overline{\text{DR}}$ scheme for the renormalization of the stop sector. In case of an OS renormalization, this means in particular that the stop masses and the stop mixing angle are renormalized on-shell. For the EFT calculation, however, respective $\overline{\text{DR}}$ quantities are needed. Therefore, the parameters have to be converted. As argued in Section 7.3, one-loop conversion including only logarithmic terms is sufficient to reproduce the diagrammatic OS expressions from the EFT $\overline{\text{DR}}$ result. Any further terms in the conversion induce higher-order contributions which are presently not under control.

In the case of $M_A = M_{\text{SUSY}}$, only the stop mixing parameter X_t turns out to be affected by large logarithms in the conversion (see Section 7.3). Here, we extend the conversion formulas to the case of $M_A \ll M_{\text{SUSY}}$. As in the case of $M_A = M_{\text{SUSY}}$, we find no large logarithms in the conversion of the stop mass scale M_S. In the conversion formula of the stop mixing parameter, however, additional large logarithms appear in the conversion formula,

$$X_t^{\overline{\text{DR}}}(M_{\text{SUSY}}) = X_t^{\text{OS}}\left\{1 + \left[\frac{\alpha_s}{\pi} - \frac{3\alpha_t}{16\pi}\left(1 - \hat{X}_t^2\right)\right]L - \frac{3}{16\pi}\frac{\alpha_t}{t_\beta^2}\left(1 - \hat{Y}_t^2\right)L_A\right\} \tag{11.75}$$

using the abbreviations

$$L = \ln\left(\frac{M_S^2}{M_t^2}\right), \quad L_A = \ln\left(\frac{M_S^2}{M_A^2}\right), \quad \hat{X}_t = \frac{X_t}{M_S} = \hat{A}_t - \frac{\hat{\mu}}{t_\beta}, \quad \hat{Y}_t = \hat{A}_t + \hat{\mu}t_\beta\,. \tag{11.76}$$

More details and full one-loop expressions for the parameter conversion are given in App. E.

Chapter 12

Comparison to other codes

In this Chapter, we will briefly compare the hybrid calculation of the MSSM Higgs boson masses, presented in this thesis, to other calculations, which are implemented in publicly available codes. This technical comparison will be followed by a numerical comparison in Chapter 13.

Publicly available codes based on diagrammatic fixed-order results or effective potential methods include `CPSuperH` [97, 99, 100], `SoftSUSY` [101], `SPheno` [102, 103] and `SUSPECT` [104]. Publicly available pure EFT calculations are `SUSYHD` [40] and `MhEFT` [37, 39, 105]. `FlexibleSUSY` [106, 107], based on `SARAH` [108–111], includes both a diagrammatic and an EFT result. Furthermore, it also has the option to use a hybrid method different from the one pursued in `FeynHiggs`, called `FlexibleEFTHiggs` [43]. Its basic idea is to include terms suppressed by the SUSY scale into the matching conditions in order to obtain accurate results for both low and high scales. The same approach has been included into `SPheno` [44].

In [112] a comparison of the various diagrammatic codes can be found. A detailed numerical comparison between various diagrammatic and EFT codes has be performed in [43]. Therein, it is also discussed in detail how `FlexibleEFTHiggs` compares to other codes. We therefore focus on a comparison of `FeynHiggs` to pure EFT codes. This comparison will be performed in two simplified scenarios, one single-scale scenario and one scenario with low M_A.

For the single-scale scenario, we will use `SUSYHD` as an exemplary EFT code for comparison. The EFT calculation of `SUSYHD` is quite similar to that in `FeynHiggs`: Both codes implement full leading and next-to-leading resummation and $\mathcal{O}(\alpha_s\alpha_t, \alpha_t^2)$ next-to-next-to-leading resummation of large logarithms. So the levels of accuracy are basically identical. There are, however, several differences which are listed below.

- `SUSYHD` by default uses the top-Yukawa coupling calculated from the OS top mass at the NNNLO level. `FeynHiggs` instead uses the NNLO value by default, which is formally the appropriate setting for the resummation of NNLL contributions. For all numerical results shown in this work, we deactivate the NNNLO corrections to the top-Yukawa coupling in `SUSYHD`.

- `SUSYHD` includes the bottom- and tau-Yukawa couplings in the renormalization group running and also includes corresponding one-loop threshold corrections. In `FeynHiggs`, the bottom and tau Yukawa couplings are set to zero in the EFT calculation. In the part of the fixed-order diagrammatic calculation, however, terms proportional to the bottom-Yukawa coupling are included at the one- and two-loop level (at the one-loop level for the case of the tau-Yukawa coupling).

- `SUSYHD` includes the electroweak gauge couplings in the running up to the three-loop level. `FeynHiggs` takes them into account up to the two-loop level. At the three-loop level, they are set to zero. This difference is numerically completely negligible.

- `FeynHiggs` includes a one-loop running of t_β to relate $t_\beta(M_t)$, which is used as input of `FeynHiggs`, to $t_\beta(M_{\text{SUSY}})$, which enters through the matching at the SUSY scale (see Section 7.3). In contrast, `SUSYHD` uses $t_\beta(M_{\text{SUSY}})$ as input.

- Similarly, FeynHiggs uses a $\overline{\text{DR}}$ renormalized Higgsino mass parameter μ at the scale M_t. The running to the scale M_{SUSY}, at which it enters the EFT calculation via the matching conditions at the SUSY scale, is neglected. SUSYHD uses $\mu(M_{\text{SUSY}})$ as input.

More details on the EFT calculation implemented in SUSYHD are given in [40].

Despite the listed differences including the different treatment of the renormalization scales of t_β and μ, we find excellent agreement between the results of the RGE running of both codes. The numerical difference of the quantity $v^2\lambda(M_t)$ calculated using the two codes is always $\lesssim 50$ GeV^2 for the single scale scenario defined in the beginning of Chapter 13 and $t_\beta \sim \mathcal{O}(10)$. This translates into a difference in M_h of $\lesssim 0.1$ GeV.

For the scenarios with low M_A, there are two other publicly available codes for calculating the Higgs pole masses via a THDM matched to the MSSM: the MhEFT package [105], based on [39], and FlexibleSUSY in the recent version [107], based on [113]. As pointed out in [107], agreement has been found with the MhEFT results. We therefore restrict ourselves to a comparison of FeynHiggs to MhEFT (version 1.1).

The calculation implemented into MhEFT is a pure EFT calculation. Therefore, terms suppressed by heavy scales are missed. Apart of this obvious difference to FeynHiggs, there are some more differences which we list below:

- MhEFT does not employ the $\overline{\text{DR}}$ scheme for renormalization of the SUSY parameters. Instead, $\overline{\text{MS}}$ renormalization is used. Therefore, conversion of the input parameters is needed for the comparison with FeynHiggs. Corresponding conversion formulas can be found in [112].

  Although, as argued in Chapter 9, this conversion will induce unwanted higher-order terms, it is currently the only way to compare both results, since neither FeynHiggs offers the possibility of a $\overline{\text{MS}}$ renormalization nor MhEFT the possibility of a $\overline{\text{DR}}$ renormalization. In practice it is a viable method since the numerical impact of the conversion is almost negligible, owing to the small numerical difference between $\overline{\text{MS}}$ and $\overline{\text{DR}}$ parameters.

- The EFT calculations entering FeynHiggs and MhEFT differ in various aspects. MhEFT assumes a type II THDM as the effective THDM in the evolution equations. Furthermore, EWino contributions to the various threshold corrections are neglected. Also in the RGEs, EWino contributions are neglected at the two-loop level and only taken into account in approximate form at the one-loop level. In addition the one-loop threshold corrections between the SM and the THDM are neglected for the top-Yukawa coupling and approximated for the SM Higgs self-coupling (i.e., the heavy Higgs contribution to the one-loop threshold correction between the SM and the MSSM is used). On the other hand, MhEFT has implemented an approximation for the $\mathcal{O}(\alpha_t^2)$ threshold corrections for the quartic couplings by including the known $\mathcal{O}(\alpha_t^2)$ threshold correction from matching the SM to the MSSM in λ_2, whereas all other self-couplings receive no $\mathcal{O}(\alpha_t^2)$ threshold correction.

- In MhEFT, the THDM self-energies $\widetilde{\Sigma}_{\widetilde{H}_1\widetilde{H}_2}$ and $\widetilde{\Sigma}_{\widetilde{H}_2\widetilde{H}_2}$ (see Eq. (11.63)) are neglected. Thereby, terms of $\mathcal{O}(M_t/M_A)$ are missed.

These differences should be kept in mind, when interpreting the numerical results of the comparison presented in Chapter 13.

Chapter 13

Numerical results

In this Chapter, we present a numerical investigation of the various corrections discussed in the previous chapters. In addition, we will briefly discuss the remaining theoretical uncertainties.

In the course of this investigation, we will restrict ourselves mainly to simplified scenarios. First, we always assume that all slepton and squark soft-breaking masses are equal,

$$m_{\tilde{l}} = m_{\tilde{q}} = M_{\text{SUSY}}. \tag{13.1}$$

Also the gluino mass parameter M_3 is set equal to M_{SUSY}. All trilinear soft-breaking couplings (except of the stop coupling) are set equal zero,

$$A_{e,\mu,\tau,u,d,c,s,b} = 0. \tag{13.2}$$

The stop trilinear coupling A_t is fixed via the stop mixing parameter X_t, which is defined either using the OS scheme or the $\overline{\text{DR}}$ scheme (in this case the associated scale is chosen to be M_{SUSY}). The same scheme is used for the definition of the stop soft-breaking masses.[1]

Second, we set

$$M_1 = M_2 = \mu \equiv M_\chi. \tag{13.3}$$

μ is defined as $\overline{\text{DR}}$ parameter fixed at the renormalization scheme of the fixed-order calculation (by default M_t is used; for scenarios with low M_A, M_{SUSY} will be used).

As a third scale, we take into account the non-SM Higgs boson mass scale marked by the OS mass of the $\mathcal{CP}$-odd Higgs boson M_A.

The ratio of the vevs of the two Higgs doublets, $\tan\beta$, is defined as a MSSM $\overline{\text{DR}}$ parameter at the default renormalization scale M_t. For scenarios in which the effective THDM implementation, described in Chapter 11, is used, however, a different definition will be used. There, the input parameter is $\tan\beta^{\text{THDM}}(M_A)$ (see Section 11.2.1).

For scenarios with $\overline{\text{DR}}$ input parameters, we always use the $\overline{\text{DR}}$ renormalized fixed-order result avoiding parameter conversion if not stated otherwise.

13.1 Advances in the hybrid approach

In this Section, we investigate the numerical effect of the various higher-order contributions discussed in Chapter 8.

First, we examine the contribution of the resummation of logarithms proportional to the electroweak gauge couplings. The left panel of Fig. 13.1 shows M_h as a function of M_{SUSY} for $X_t^{\text{OS}}/M_{\text{SUSY}} = 0$ and $X_t^{\text{OS}}/M_{\text{SUSY}} = 2$. The results with a resummation of logarithms proportional to the electroweak gauge couplings and without such a resummation are compared.

[1] Since no two-loop corrections related to sleptons or first and second generation squarks are implemented, we do not have to specify the scheme of the related soft-breaking parameters. The soft-breaking mass for the right-handed sbotoom is renormalized using either the $\overline{\text{DR}}$ scheme, if for the stop parameters the $\overline{\text{DR}}$ scheme is used, or the OS scheme as defined in Section 6.3, if for the stop parameters the OS scheme is used. The sbottom trilinear coupling is always renormalized in the $\overline{\text{DR}}$ scheme.

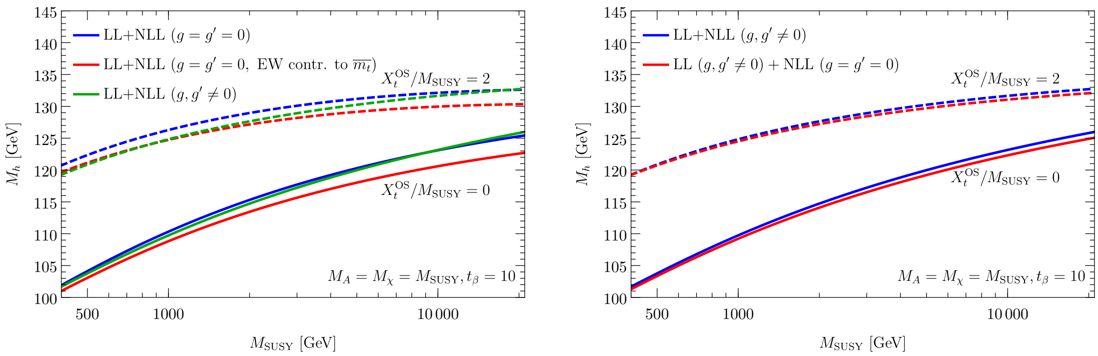


Figure 13.1: Left: M_h as a function of $M_{\rm SUSY}$ for $X_t^{\rm OS}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\rm OS}/M_{\rm SUSY} = 2$ (dashed). The results with (green) and without (blue) resummation of electroweak logarithms (LL+NLL) are compared. Furthermore, the result without resummation of electroweak logarithms but with electroweak NLO corrections to the $\overline{\rm MS}$ top-quark mass (red) is shown. Right: The results with resummation of electroweak logarithms at the LL and NLL level (blue) and at the LL level only (red) are compared.

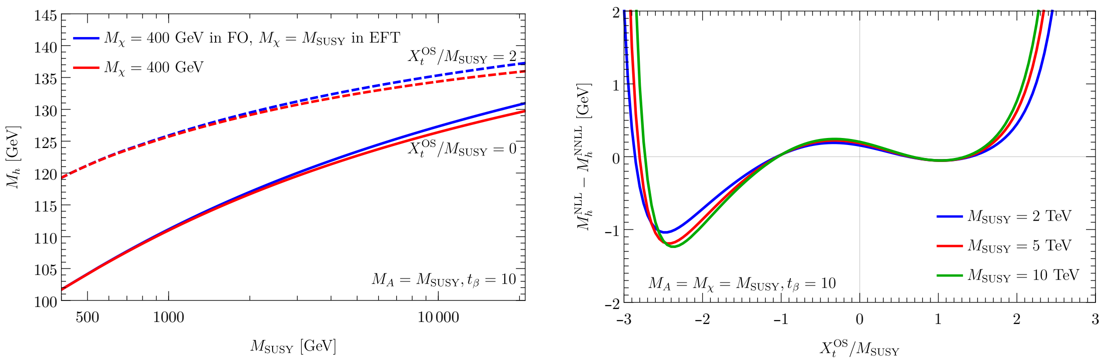


Figure 13.2: Left: M_h as a function of $M_{\rm SUSY}$ for $X_t^{\rm OS}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\rm OS}/M_{\rm SUSY} = 2$ (dashed). The results with (red) and without (blue) electroweakino threshold are compared. Right: The difference between the NLL and the NNLL result as a function of $X_t^{\rm OS}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 2$ TeV (blue), $M_{\rm SUSY} = 5$ TeV (red) and $M_{\rm SUSY} = 10$ TeV (green) is shown.

The latter corresponds, apart from some minor adaptations, to the result presented in [42]. Furthermore, the result without resummation of logarithms proportional to the electroweak gauge couplings but with electroweak NLO corrections to the $\overline{\rm MS}$ top mass is shown. For $X_t^{\rm OS}/M_{\rm SUSY} = 2$, we observe a downwards shift of ~ 1.5 GeV for $M_{\rm SUSY} = 1$ TeV. This shift is almost completely caused by the electroweak NLO corrections to the $\overline{\rm MS}$ top mass yielding a reduction of the $\overline{\rm MS}$ top mass by 1.1 GeV. This translates directly to a downwards shift of M_h [114]. For rising $M_{\rm SUSY}$, the downwards shift caused by the corrections to the $\overline{\rm MS}$ top mass is more and more compensated by the upwards shift caused by the resummed logarithms proportional to the electroweak gauge couplings. For vanishing stop mixing, the behaviour is very similar. For $M_{\rm SUSY} = 1$ TeV, the downwards shift is smaller (~ 1 GeV) owing to the reduced dependence on the $\overline{\rm MS}$ top mass for vanishing stop mixing.

The right panel of Fig. 13.1 shows M_h as a function of $M_{\rm SUSY}$ for $X_t^{\rm OS}/M_{\rm SUSY} = 0$ and $X_t^{\rm OS}/M_{\rm SUSY} = 2$. The results with a resummation of logarithms proportional to the electroweak gauge couplings at the LL and NLL level and with a resummation of logarithms proportional to the electroweak gauge couplings at the LL level and vanishing electroweak gauge couplings at the NLL level are compared. We observe that the effect of a NLL resummation of electroweak logarithms is $\lesssim 0.5$ GeV over the whole $M_{\rm SUSY}$ range for both vanishing and non vanishing mixing. This shows the minor importance of the electroweak NLL resummation in comparison to electroweak LL resummation, which leads to shifts of up to 2.5 GeV for $M_{\rm SUSY} \sim 20$ TeV.

The effect of the electroweakino threshold is investigated in the left panel of Fig. 13.2, which

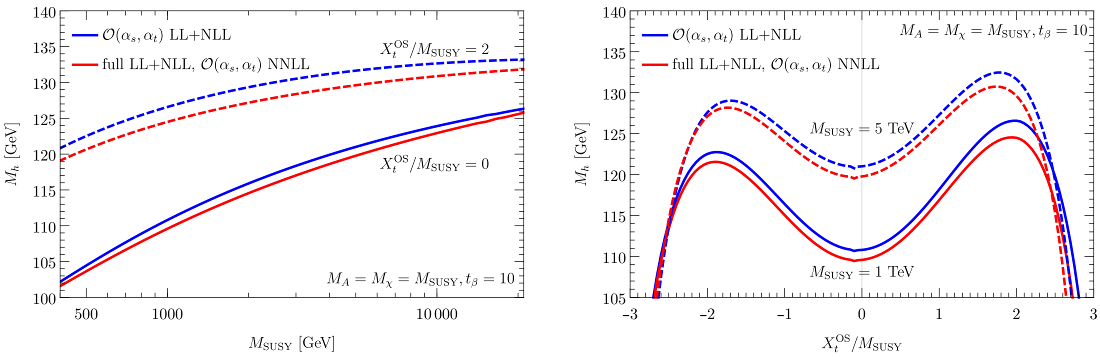


Figure 13.3: Left: M_h as a function of M_{SUSY} for $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 0$ (solid) and $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 2$ (dashed). The results using pure $\mathcal{O}(\alpha_s, \alpha_t)$ LL and NLL resummation (blue) are compared to the results using full LL and NLL as well as $\mathcal{O}(\alpha_s, \alpha_t)$ NNLL resummation. Right: Same as left, but M_h is shown as a function of $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}}$ for $M_{\mathrm{SUSY}} = 1$ TeV (solid) and $M_{\mathrm{SUSY}} = 5$ TeV (dashed).

displays M_h as function of M_{SUSY} for $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 0$ and $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 2$. In contrast to the previous figure, the electroweakino mass scale M_χ is not chosen to be equal to M_{SUSY}, but is fixed to 400 GeV. To disentangle the effect of the electroweakino threshold in the EFT calculation from the fixed-order one-loop corrections due to neutralinos and charginos, we compare the results with a electroweakino threshold to the results without a separate electroweakino threshold. To get the results without a separate electroweakino threshold, we set $M_\chi = M_{\mathrm{SUSY}}$ in the EFT calculation, but keep $M_\chi = 400$ GeV in the Feynman diagrammatic calculation. The plot clearly shows that the implementation of a separate electroweakino threshold becomes only relevant for $M_{\mathrm{SUSY}} \gtrsim 5$ TeV. This behaviour does not depend on the size of the stop mixing.

The effect of a separate gluino threshold is found to be negligible. For M_{SUSY} up to 20 TeV, its inclusion shifts M_h downwards by at most 0.2 GeV for $|X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}}| \leq 2$ and $M_{\tilde{g}} = 1$ TeV. The diagrammatic two-loop corrections capture almost the entire effect of varying $M_{\tilde{g}}$, which can be sizeable (~ 2 GeV) for large stop mixing.

In the right plot of Fig. 13.2, the difference between the results without and with NNLL resummation as a function of $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}}$ is shown for $M_{\mathrm{SUSY}} = 2$ TeV, $M_{\mathrm{SUSY}} = 5$ TeV and $M_{\mathrm{SUSY}} = 10$ TeV. Between $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} \sim -1$ and $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} \sim 1.5$, we observe only small shifts ($\lesssim 0.3$ GeV). For $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} \sim -2$, M_h is shifted upwards by the inclusion of NNLL resummation by up to 1 GeV, whereas M_h is shifted downwards by up to 0.5 GeV for $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 2$. This behaviour is mainly caused by the $\mathcal{O}(\alpha_s\alpha_t)$ matching condition of λ, which exhibits a similar dependence on $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}}$.

Note that the comparison made in the right plot of Fig. 13.2 does not exhibit the effect of the two-loop corrections to the $\overline{\mathrm{MS}}$ top mass, since also for the curve without NNLL resummation the two-loop QCD corrections in the $\overline{\mathrm{MS}}$-mass – pole-mass relation are employed. We have kept them because they constitute the by far dominant part of the two-loop corrections to the $\overline{\mathrm{MS}}$ top mass, shifting the $\overline{\mathrm{MS}}$ top mass down by 1.9 GeV. This downwards shift causes a downwards shift in M_h of about the same size, as discussed before in the context of the electroweak NLO corrections to the $\overline{\mathrm{MS}}$ top mass. Two-loop corrections to the $\overline{\mathrm{MS}}$ top mass are formally not needed in the case of LL and NLL resummation. This means actually that the main effect of going from NLL to NNLL resummation is caused by the higher-order matching condition of the $\overline{\mathrm{MS}}$ top mass, as in the case of including electroweak corrections into the resummation procedure.

To sum up, we analyse the total numerical impact of the improvements discussed in Chapter 8. The comparison is displayed in Fig. 13.3, where the left panel shows M_h as a function of M_{SUSY} for $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 0$ and $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 2$. For $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = 2$, we observe a downwards shift of up to 2 GeV over the whole considered M_{SUSY} range. For vanishing stop mixing, a smaller shift is observed. The right panel in Fig. 13.3 shows M_h as a function of $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}}$ for $M_{\mathrm{SUSY}} = 1$ TeV and $M_{\mathrm{SUSY}} = 5$ TeV. We observe a smaller shift for negative values of X_t; e.g. for $M_{\mathrm{SUSY}} = 1$ TeV the shift is ~ 0.5 GeV for $X_t^{\mathrm{OS}}/M_{\mathrm{SUSY}} = -2$, whereas it amounts to

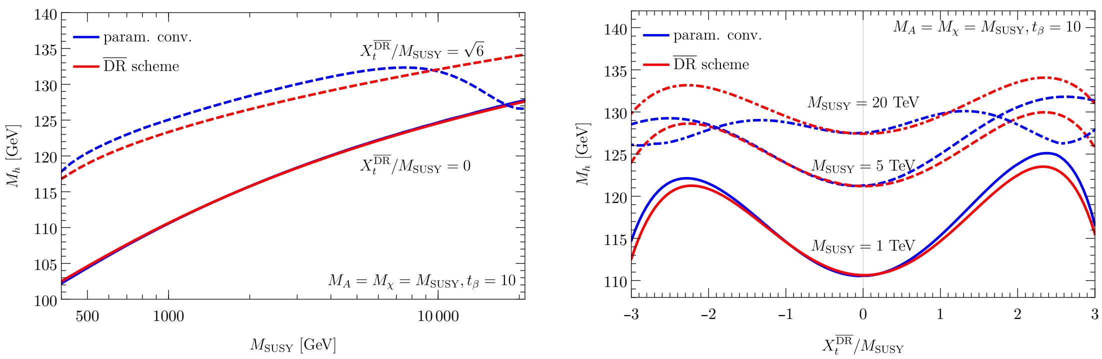


Figure 13.4: Left: M_h as a function of M_{SUSY} for $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = 0$ (solid) and $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = \sqrt{6}$ (dashed). The results using a $\overline{\text{DR}}$ to OS conversion of the input parameters (blue) and a $\overline{\text{DR}}$ renormalization of the fixed-order result (red) are compared. Right: Same as left plot, apart that M_h is shown in dependence of $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}}$ for $M_{\text{SUSY}} = 1$ TeV (solid), $M_{\text{SUSY}} = 5$ TeV (dashed) and $M_{\text{SUSY}} = 20$ TeV (dot-dashed).

~ 2 GeV for $X_t^{\text{OS}}/M_{\text{SUSY}} = 2$. The large positive shift for negative X_t^{OS} by the inclusion of NNLL resummation (see right plot of Fig. 13.2) is compensated by the downwards shift originating from the electroweak NLO correction to the $\overline{\text{MS}}$ top-quark mass. This downwards shift is, however, enhanced by the negative shift induced by NNLL resummation for positive X_t^{OS}. This is the reason for the observed asymmetric behaviour.

13.2 Comparison between pure EFT and hybrid calculation

In this Section, we present a numerical investigation of the effects discussed in Chapter 9 and Chapter 10 and compare the result obtained by `FeynHiggs` to `SUSYHD` as an exemplary pure EFT code.

We first look at the numerical difference between employing a one-loop conversion from $\overline{\text{DR}}$ to OS input parameters ("param. conv.") and using a $\overline{\text{DR}}$ renormalized fixed-order result ("$\overline{\text{DR}}$ scheme"), see the discussion in Chapter 9. The left plot of Fig. 13.4 shows the corresponding results for $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = 0$ and $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = \sqrt{6}$ as a function of M_{SUSY}. One can see that for vanishing stop mixing, the difference between the two methods is negligible small.

In case of $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = \sqrt{6}$, there, however, is a sizeable shift: For $M_{\text{SUSY}} \lesssim 10$ TeV the difference between the two methods leads to an approximately constant shift in the prediction for M_h. The result obtained using a $\overline{\text{DR}}$ fixed-order result is $\sim 1-2$ GeV smaller than the one obtained by the one-loop conversion of the input parameters. The shifts occur not only for scales of a few TeV, but also for very low scales ($M_{\text{SUSY}} \simeq 0.4$ TeV). Therefore, we conclude that at low scales the observed shifts are mainly caused by non-logarithmic higher-order terms by which the $\overline{\text{DR}}$ result and the result involving a parameter conversion differ from each other.

For $M_{\text{SUSY}} \gtrsim 10$ TeV, we observe that the difference between the two results is increasing rapidly to up to 8 GeV in the region up to $M_{\text{SUSY}} \sim 20$ TeV. This behaviour is mainly due to the fact that the parameter conversion that is used for the comparison of fixed-order results induces higher-order logarithmic contributions that are not compatible with the implemented resummation of logarithms to all orders (see the discussion in Chapter 9). For high SUSY scales, where the higher-order logarithmic contributions become numerically large, this mismatch leads to the observed large deviations. To a lesser extent, also the deviation between the input stop mixing parameter and that used in the EFT calculation plays a role in this context (see Chapter 9).

In the right plot of Fig. 13.4 the two results are compared as a function of $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}}$ for $M_{\text{SUSY}} = 1, 5, 20$ TeV. For $M_{\text{SUSY}} = 1$ TeV and $M_{\text{SUSY}} = 5$ TeV the deviations stay relatively small except for the highest values of $|X_t^{\overline{\text{DR}}}/M_{\text{SUSY}}|$. In contrast, for $M_{\text{SUSY}} = 20$ TeV the uncontrolled higher-order contributions induced by the one-loop conversion of the input para-

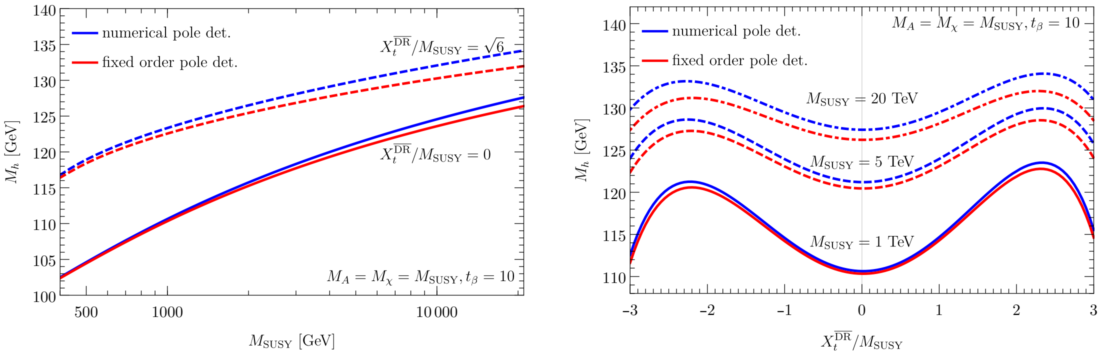


Figure 13.5: Comparison of the M_h predictions using numerical pole determination with the predictions using fixed-order pole determination, such that terms arising from the determination of the propagator pole are omitted that go beyond the level of the corrections implemented in the irreducible self-energies. Left: Prediction for M_h as function of $M_{\rm SUSY}$ for vanishing stop mixing and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 2$. Right: Prediction for M_h as function of of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 1$ TeV (solid), $M_{\rm SUSY} = 5$ TeV (dashed) and $M_{\rm SUSY} = 20$ TeV (dot-dashed).

meters are seen to have a huge effect which even reverts the usual pattern of the dependence on $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$, giving rise to local minima at $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}| \simeq \pm 2.5$. We emphasize again that the same kind of uncontrolled higher-order effects would occur if a one-loop conversion of OS to $\overline{\rm DR}$ parameters would be used as input for a $\overline{\rm DR}$ result containing a series of numerically large higher-order logarithms. Fig. 13.4 shows that numerical instabilities noticed in comparisons of EFT results with `FeynHiggs` carried out in the literature [40, 43, 44] are a consequence of an inappropriate application of the conversion of input parameters between the OS and the $\overline{\rm DR}$ schemes. The higher-order contributions implemented in `FeynHiggs` are seen to be numerically stable up to very high SUSY scales in the considered scenario.

As a next step we investigate the impact of the terms arising from the determination of the propagator pole. As explained in Chapter 10, there occurs a cancellation in the limit of a large SUSY scale between non-SM terms arising through the determination of the propagator pole and contributions from the subloop renormalization of the irreducible self-energy diagrams. This cancellation was incomplete for terms beyond $\mathcal{O}(\alpha_t^2, \alpha_t\alpha_b, \alpha_b^2)$. Therefore, we have modified the determination of the propagator poles such that terms are omitted that would not cancel because their counterpart in the irreducible self-energies is not incorporated at present. In Fig. 13.5, the "old" numerical pole determination is compared to the "new" fixed order pole determination. The difference between the two results corresponds essentially to the terms $\Delta_{p^2}^{\rm logs}$ and $\Delta_{p^2}^{\rm nolog}$ given in Eqs. (10.5) and (10.8).

In the left plot of Fig. 13.5, we show the results as a function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}} = 0$ and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 2$. One observes that the difference grows almost logarithmically with $M_{\rm SUSY}$. This is expected since the largest terms in $\Delta_{p^2}^{\rm logs} + \Delta_{p^2}^{\rm nolog}$ are in fact logarithms of the SUSY scale over M_t. Consequently, for small scales ($M_{\rm SUSY} \lesssim 1$ TeV), these terms induce only a small upwards shift of $\lesssim 0.5$ GeV. For large scales ($M_{\rm SUSY} \gtrsim 5$ TeV), however, this shift grows to up to 1.5 GeV for vanishing stop mixing and 2 GeV for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$. In the right plot of Fig. 13.5, the difference is depicted as a function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 1, 5, 20$ TeV, shown as solid, dashed and dot-dashed lines, respectively. One can see that the difference between the two results is approximately quadratically depependent on $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$. This reflects the $X_t^{\overline{\rm DR}}$ dependence of the derivative of the Higgs boson self-energy (see Eq. (B.17)).

Having investigated the numerical impact of the scheme conversion of the input parameters as well as of the terms arising from the determination of the propagator pole, we now turn to a direct comparison of `FeynHiggs` with `SUSYHD`.[2] The `FeynHiggs` results in this comparison are obtained employing the $\overline{\rm DR}$ renormalization of the stop sector and the fixed-order pole determination.

[2]We use `SUSYHD` with the top-Yukawa coupling evaluated at the NNLO level. Using instead the NNNLO value would shift the results of `SUSYHD` shown here downwards by ~ 0.5 GeV.

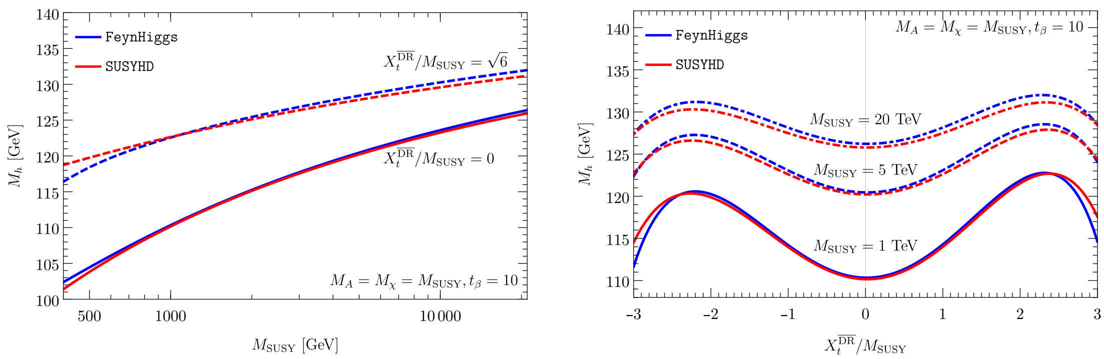


Figure 13.6: Comparison of the M_h predictions of `FeynHiggs` using the $\overline{\rm DR}$ renormalized fixed-order result and the fixed-order pole determination with `SUSYHD`. Left: M_h as function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 2$ (dashed). Right: M_h as function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 1$ TeV (solid), $M_{\rm SUSY} = 5$ TeV (dashed) and $M_{\rm SUSY} = 20$ TeV (dot-dashed).

The left plot of Fig. 13.6 shows M_h as a function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$. For vanishing stop mixing and $M_{\rm SUSY} \gtrsim 1$ TeV, we observe an excellent agreement of the `FeynHiggs` curve with the `SUSYHD` result. Even for very large scales $M_{\rm SUSY} \simeq 20$ TeV, we find agreement within ~ 0.5 GeV in the considered simple numerical scenario, in which all SUSY scales are chosen to be equal to each other. For low scales ($M_{\rm SUSY} \lesssim 1$ GeV), it can be seen that the `FeynHiggs` result is higher by up to ~ 1 GeV compared to the `SUSYHD` result. The origin of this difference are terms suppressed by the SUSY scale, which are included in `FeynHiggs` but not in `SUSYHD`. For $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$, we basically observe the same behaviour as in case of vanishing stop mixing. The overall agreement in the simple numerical scenario is very good (within ~ 1 GeV for $M_{\rm SUSY} \gtrsim 0.5$ TeV). For low scales ($M_{\rm SUSY} \lesssim 0.5$ GeV), the `FeynHiggs` result is lower compared to the `SUSYHD` result by up to ~ 2 GeV. As in the case of vanishing stop mixing, this can be traced back to terms suppressed by the SUSY scale. We will discuss this and investigate the remaining differences in more detail below in Fig. 13.7.

In the right plot of Fig. 13.6 the comparison between the M_h prediction of the new `FeynHiggs` version and `SUSYHD` is shown as a function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 1, 5, 20$ TeV, shown as solid, dashed and dot-dashed lines, respectively. Again one can see an overall very good agreement between both codes for $M_{\rm SUSY} \gtrsim 1$ TeV (within 1 GeV) in the considered simple numerical scenario. The agreement is especially good for small $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$, but the deviations stay below 1 GeV also for increasing mixing in the stop sector except for the highest values of $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$ in the case of $M_{\rm SUSY} = 1$ TeV. The larger deviations of up to ~ 2 GeV for $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}| \gtrsim 2.5$ in the case of $M_{\rm SUSY} = 1$ TeV are due to terms suppressed by $M_{\rm SUSY}$ which become large for increasing $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$.

In Fig. 13.7, we further investigate these remaining differences between `FeynHiggs` and `SUSYHD` observed in Fig. 13.6. In the left plot we show the difference between the results of `FeynHiggs` and `SUSYHD` for M_h^2 (not for M_h). Since in both codes actually M_h^2 is calculated, taking the square root of these results can obscure the true dependences of the difference. As an example, if the difference in M_h^2 is constant when varying $M_{\rm SUSY}$, we would not observe a constant difference when comparing the difference in M_h. We show in the plot the difference in M_h^2 for the case where the fixed-order result of `FeynHiggs` is parametrized in terms of the SM NNLO $\overline{\rm MS}$ top mass. For $M_{\rm SUSY} \lesssim 1$ TeV in the case of vanishing mixing and for $M_{\rm SUSY} \lesssim 3$ TeV in the case of $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ we observe large gradients. For larger scales ($M_{\rm SUSY} \gtrsim 3$ TeV), the difference is only slowly increasing when raising $M_{\rm SUSY}$. For vanishing stop mixing, the difference is growing by ~ 50 GeV^2 when raising $M_{\rm SUSY}$ from 3 TeV to 20 TeV. For $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$, similarly a growth of ~ 50 GeV^2 is recognizable. This behaviour is mostly due to the differences in the EFT calculations implemented in `FeynHiggs` and `SUSYHD` discussed in Chapter 12. In addition, however, we observe an offset relative to the zero axis for $M_{\rm SUSY} \gtrsim 3$ TeV. For vanishing stop mixing, it is small (~ 50 GeV^2), whereas

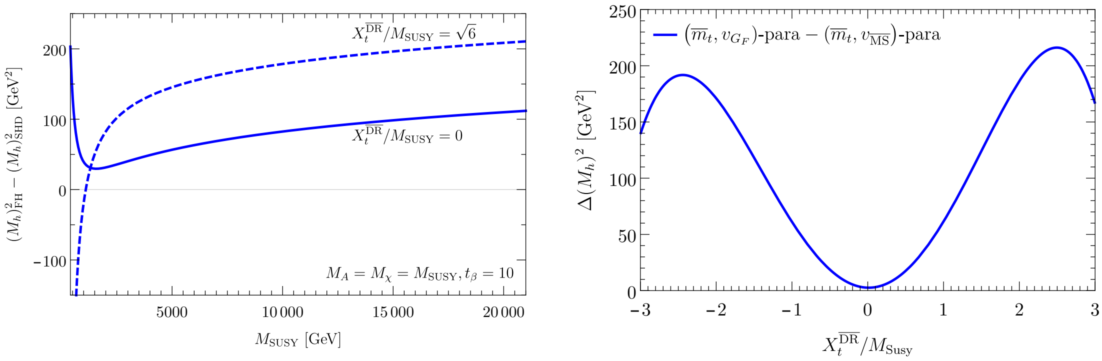


Figure 13.7: Left: Difference of the M_h^2 predictions of FeynHiggs and SUSYHD as a function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed). For the parametrization of the diagrammatic result of FeynHiggs the SM NNLO $\overline{\rm MS}$ top-quark mass is chosen. Right: Differences due to the different parametrization of the top-quark mass and the vev in a fixed-order $\mathcal{O}(\alpha_t\alpha_s, \alpha_t^2)$ calculation, taking into account only non-logarithmic terms, as a function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$. The difference between the result parametrized in terms of the $\overline{\rm MS}$ NNLO top-quark mass and v_{G_F} and the one parametrized in terms of the $\overline{\rm MS}$ NNLO top-quark mass and $v_{\overline{\rm MS}}$ is shown.

for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$, the shift is more significant (~ 150 GeV^2). The nearly constant offset between the two codes can be traced back to the different parametrization of the non-logarithmic terms discussed in Section 10.2.

We further analyse the influence of the different ways to parametrize the non-logarithmic terms in the right plot of Fig. 13.7. It shows the difference in M_h^2 obtained from a diagrammatic calculation of $\mathcal{O}(\alpha_t\alpha_s, \alpha_t^2)$ using different parametrizations of the vev for the non-logarithmic one- and two-loop terms (see Section 10.2 for more details). Note that these non-logarithmic terms, apart of $\mathcal{O}(v/M_{\rm SUSY})$ contributions, stay constant when varying $M_{\rm SUSY}$. For $X_t^{\overline{\rm DR}}/M_{\rm SUSY} \sim \sqrt{6}$ the difference between parametrizations in terms of v_{G_F} and $v_{\overline{\rm MS}}$ (both using the SM NNLO $\overline{\rm MS}$ top-quark mass) amounts to ~ 220 GeV^2. Such a shift accounts for the main part of the nearly constant offset observed in the left plot of Fig. 13.7. For $X_t^{\overline{\rm DR}}/M_{\rm SUSY} \sim 0$ the difference between the parametrizations in terms of v_{G_F} and $v_{\overline{\rm MS}}$ is seen to become very small. The nearly constant offset for vanishing stop mixing observed in the left plot of Fig. 13.7 can be explained in a similar way by different parameterization of terms that are not of $\mathcal{O}(\alpha_t\alpha_s, \alpha_t^2)$.

Finally, we briefly comment on the differences between FeynHiggs and other pure EFT codes that have been reported in the literature. In [40] it was claimed that differences between FeynHiggs and SUSYHD of up to ~ 9 GeV would occur for $M_{\rm SUSY} = 2$ TeV and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} \sim \sqrt{6}$. As already noted in [40], this difference was somewhat reduced if the NNLO $\overline{\rm MS}$ top mass was employed in the calculation of FeynHiggs.[3] While at the time of the comparison carried out in [40] the EFT calculation of FeynHiggs was not yet at the same level of accuracy as the one of SUSYHD, the differences claimed in [40] were in fact primarily caused by an inappropriate application of the conversion of input parameters between the $\overline{\rm DR}$ and the OS scheme. The inappropriate parameter conversion, for which the authors of [40] used their own routine, caused a deviation of 3–4 GeV for $M_{\rm SUSY} = 2$ TeV and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} \sim \sqrt{6}$ and was also responsible for the apparent numerical instability at large SUSY scales of the FeynHiggs curve with $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ shown in [40]. The numerical effect of this deviation was larger than the shift caused by employing the NNLO or NNNLO $\overline{\rm MS}$ top-quark mass in FeynHiggs, in contrast to the claim made in [40].

Also the comparison figures between FeynHiggs and FlexibleEFTHiggs as well as SPheno shown in [43, 44] are plagued by deficiencies arising from an inappropriate application of the

[3] In the FeynHiggs version employed in the comparison by default the NLO $\overline{\rm MS}$ top mass was used. This was formally correct for the resummation of the LL and NLL contributions that was implemented in FeynHiggs at that time. As found in Section 13.1, the shift in the top-quark mass from NLO to NNLO generates the main numerical effect when going to NNLL resummation.

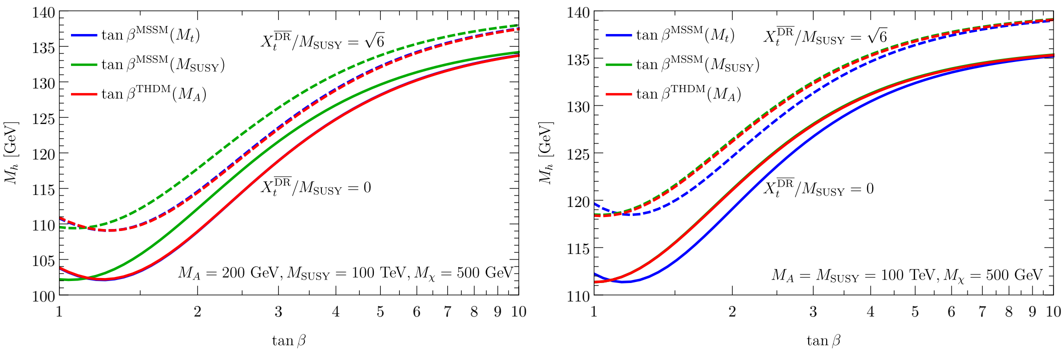


Figure 13.8: Left: M_h as function of $\tan\beta$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed) in a scenario with a low M_A and with different definitions of $\tan\beta$: in the MSSM at the scale M_t (blue) and at the scale $M_{\rm SUSY}$ (red, overlapping with blue), and in the THDM at the scale M_A (green). Right: Same signature, but for $M_A = M_{\rm SUSY}$ (overlapping red and green curves).

parameter conversion between the $\overline{\rm DR}$ and the OS scheme. We stress again that such a parameter conversion would give rise to the same kind of problems when starting from OS parameters and converting to $\overline{\rm DR}$ ones.

13.3 Resummation for low M_A

In this Section, we investigate the numerical impact of the implementation of an effective THDM into `FeynHiggs`. This means in practice that we compare the results using the calculation outlined in Chapter 8 (corresponding to `FeynHiggs2.14.1`, see `www.feynhiggs.de` for a full version history) to those from the calculation presented in Chapter 11, which is implemented in a still private `FeynHiggs` version based on `FeynHiggs2.14.1`. In addition, we show results from `FeynHiggs2.14.0` to point out the impact of the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold corrections [41] , which were implemented as a new feature in `FeynHiggs2.14.1` (see also Section 8.3). The degenerate $\mathcal{O}(\alpha_t^2)$ threshold corrections [40], used in `FeynHiggs2.14.0`, implicitly assume $M_A = M_{\rm SUSY}$. We furthermore compare the results of the calculation presented in this thesis to those of `MhEFT`.

For illustration of the numerical effects, we investigate simplified scenarios as defined in the beginning of this Chapter. We set the gluino mass $M_{\tilde g}$ equal to $M_{\rm SUSY}$[4]. As default values for the figures, we set $M_{\rm SUSY} = 100$ TeV and $M_\chi = 500$ GeV. In combination with low M_A and $\tan\beta$ values, this choice maximizes the numerical impact of the effective THDM.

For the SUSY parameters, we use the $\overline{\rm DR}$ scheme with the corresponding renormalization scale being $M_{\rm SUSY}$. The $\overline{\rm DR}$ scheme is also used for X_t (except in Fig. 13.12, where the OS scheme is used). $\tan\beta$ is defined as $\tan\beta^{\rm THDM}(M_A)$, unless stated otherwise.

Aside from the simplified scenarios, we also study a more complicated situation, the "low-$\tan\beta$-high" scenario proposed by the LHC Higgs Cross Section Working Group in [115].

13.3.1 Shifts from $\tan\beta$ definition

As explained in Section 11.2, we account for the different normalization of the Higgs doublets in the full MSSM and the effective THDM by introducing a finite shift in the field renormalization constants of the fixed-order calculation. This changes the definition of $\tan\beta$: from a MSSM quantity to one of the THDM, along with a change of the renormalization scale from M_t (the default of `FeynHiggs`) to M_A.

We analyze the numerical effect of this redefinition in Fig. 13.8. It shows results of `FeynHiggs` for M_h using different definitions of $\tan\beta$: $\tan\beta^{\rm MSSM}(M_t)$ (default definition in `FeynHiggs`),

[4]Note that our EFT calculation also allows to treat scenarios with $M_{\tilde g}$ as an independent parameter. The numerical effect of the additional threshold, however, is small since the dominant two-loop effect is already captured by the fixed-order calculation (see also Section 13.1)

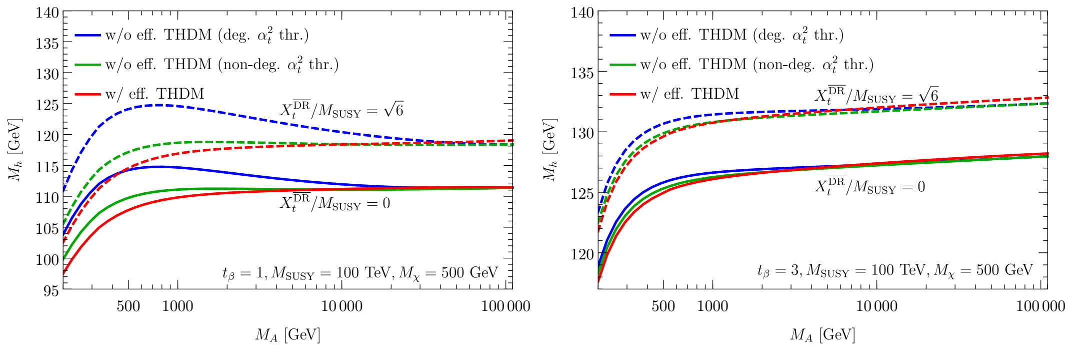


Figure 13.9: M_h as a function of M_A for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed). Left: $\tan\beta = 1$. Right: $\tan\beta = 3$. The results of `FeynHiggs` without effective THDM – using the degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (blue) and using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) – are compared with the results of `FeynHiggs` with effective THDM (red).

$\tan\beta^{\rm THDM}(M_A)$ (default definition in this Section) and, for comparison, $\tan\beta^{\rm MSSM}(M_{\rm SUSY})$ (achieved by shifting the renormalization scale to $M_{\rm SUSY}$). Accordingly, the meaning of the horizontal axis is not the same for the different curves.

The left panel displays a low-M_A scenario. The curves for $\tan\beta^{\rm MSSM}(M_t)$ and $\tan\beta^{\rm THDM}(M_A)$ are very close to each other. This is essentially due to $M_A \sim M_t$, the additional non-logarithmic threshold correction of $\tan\beta$ between the THDM and the MSSM in Eq. (11.54) has only a small numerical impact. In contrast, there is a large hierarchy between M_t (or M_A) and $M_{\rm SUSY}$. Therefore, the third curve for $\tan\beta^{\rm MSSM}(M_{\rm SUSY})$ is shifted upwards for low $\tan\beta$, by up to ~ 2 GeV for $\tan\beta \gtrsim 1.2$. This shift shrinks for rising $\tan\beta$, as a consequence of the decreasing dependence of M_h on $\tan\beta$. For $\tan\beta \lesssim 1.2$ a small downwards shift of up to 2 GeV is visible.

In the right panel, the same set of curves is displayed, but now for M_A equal to $M_{\rm SUSY}$. Therefore, the curves using $\tan\beta^{\rm THDM}(M_A)$ and $\tan\beta^{\rm MSSM}(M_{\rm SUSY})$ are very close; again, the additional non-logarithmic threshold correction of $\tan\beta$ between the THDM and the MSSM turns out to be negligible. Due to the large scale separation between M_t and $M_{\rm SUSY}$ the curve using $\tan\beta^{\rm MSSM}(M_t)$ is shifted downwards by up to 2 GeV between $\tan\beta \sim 1.2$ and $\tan\beta \sim 6$. For $\tan\beta \lesssim 1.2$, a small upwards shift up to 1 GeV is visible.

13.3.2 Impact of the effective THDM

Having investigated the numerical effect of different definitions of $\tan\beta$, we now scrutinize the impact of the implementation of an effective THDM into the hybrid framework of `FeynHiggs`.

In Fig. 13.9, we compare the results of various stages of `FeynHiggs` by showing M_h in dependence of M_A: the previous version without an intermediate effective THDM using degenerate $\mathcal{O}(\alpha_t^2)$ threshold corrections (corresponding to version `2.14.0`) as well as using non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold corrections (corresponding to version `2.14.1`), and the new version with the effective THDM implemented. One observes that the curves of `FeynHiggs` with and without effective THDM converge to each other for rising M_A. This is expected since for $M_A = M_{\rm SUSY}$, the SM+EWinos can be matched directly to the MSSM and no effective THDM is needed. The small remaining deviation of the THDM curve for $M_A = M_{\rm SUSY}$ and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ is caused by the $\mathcal{O}(\alpha_t^2)$ threshold correction, which is part of the current `FeynHiggs` (without effective THDM) but not available for the THDM-modified version. For $M_A \ll M_{\rm SUSY}$ we observe sizeable shifts, in particular in the left panel where $\tan\beta$ is set to 1. The step from degenerate to non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold corrections already induces a downwards shift of up to 5 GeV for vanishing stop mixing and of up to 7 GeV for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$. Implementing now the effective THDM leads to a further shift downwards by up to 2 GeV for vanishing stop mixing and up to 3 GeV for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$.

In the right panel with $\tan\beta = 3$, the curves show the same qualitative behavior, i.e. for low M_A the implementation of an effective THDM shifts M_h downwards, but in comparison to

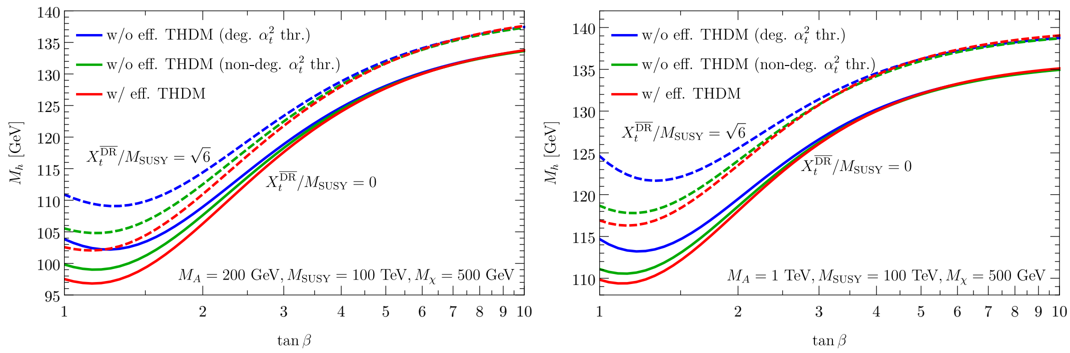


Figure 13.10: M_h as a function of $\tan\beta$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed). Left: $M_A = 200$ GeV. Right: $M_A = 1$ TeV. The results of `FeynHiggs` without effective THDM – using the degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (blue) and using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) – are compared with the results of `FeynHiggs` with effective THDM (red).

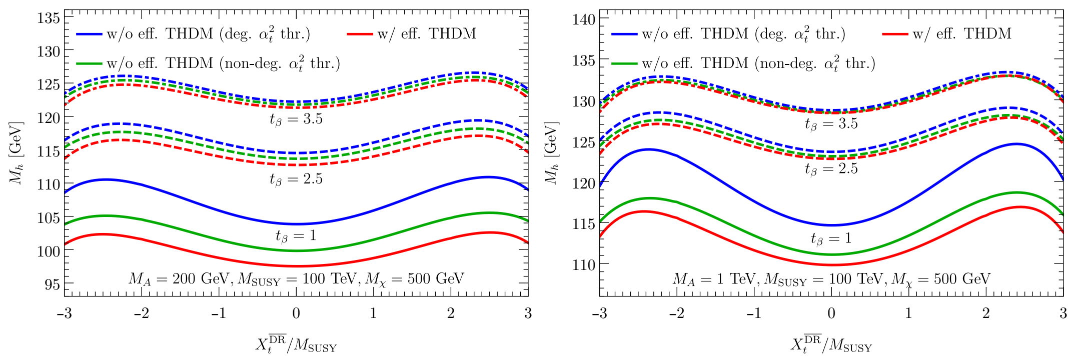


Figure 13.11: M_h as a function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $\tan\beta = 1$, (solid) $\tan\beta = 2.5$ (dashed), and $\tan\beta = 3.5$ (dotdashed). Left: $M_A = 200$ GeV. Right: $M_A = 1$ TeV. The results of `FeynHiggs` without effective THDM – using the degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (blue) and using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) – are compared with the results of `FeynHiggs` with effective THDM (red).

the results with $\tan\beta = 1$, the effects are less pronounced ($\lesssim 1.5$ GeV).

This strong dependence on $\tan\beta$ is visualized more specifically in Fig. 13.10, where M_h is shown versus $\tan\beta$ for the same cases as in Fig. 13.9. In the left panel, the difference between `FeynHiggs` with and without effective THDM is displayed for $M_A = 200$ GeV and in the right panel for a larger value $M_A = 1$ TeV. The effects of the various steps of improvement are most pronounced for low $\tan\beta$ and shrink quickly for increasing values; for $\tan\beta \gtrsim 5$, the shifts are negligible. Again, the use of the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction brings the result without effective THDM closer to that with effective THDM. The curves in the left and right panel behave very similar; the overall M_h values are higher for larger M_A, but the shifts remain of the same size despite the slightly reduced hierarchy between M_A and $M_{\rm SUSY}$.

Next, the dependence on the stop-mixing parameter $X_t^{\overline{\rm DR}}$ is analyzed in Fig. 13.11, presenting M_h versus $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for two different mass scales $M_A = 200$ GeV (left) and $M_A = 1$ TeV (right). As one can see, the difference between M_h predicted by `FeynHiggs` with and without effective THDM is only mildly dependent on $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$. For all values, the effect of including the THDM is a downwards shift of M_h, becoming smaller for increasing $\tan\beta$.

From a phenomenological point of view, shifting the curves according to the various levels of improvement is relevant for the proper determination of the parameter range that predicts M_h compatible with the measurement. We have kept in all the figures the case with degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction in the version without THDM in order to point out the significance

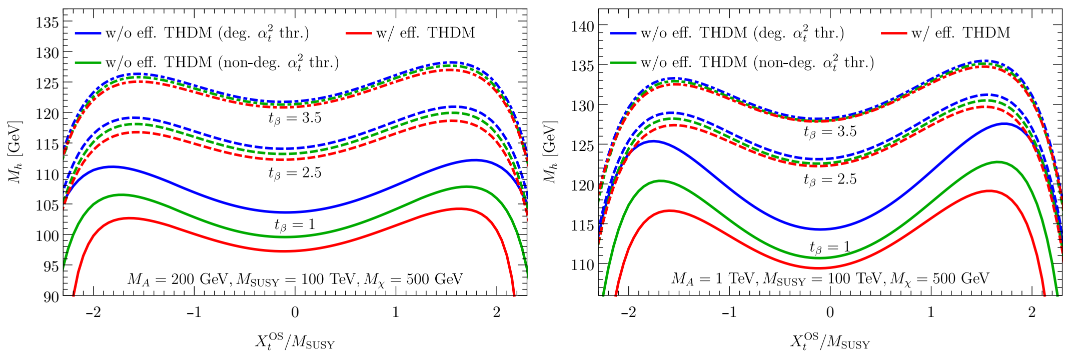


Figure 13.12: M_h as a function of $X_t^{\rm OS}/M_{\rm SUSY}$ for $\tan\beta = 1$ (solid), $\tan\beta = 2.5$ (dashed), and $\tan\beta = 3.5$ (dotdashed). Left: $M_A = 200$ GeV. Right: $M_A = 1$ TeV. The results of `FeynHiggs` without effective THDM – using the degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (blue) and using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) – are compared with the results of `FeynHiggs` with effective THDM (red).

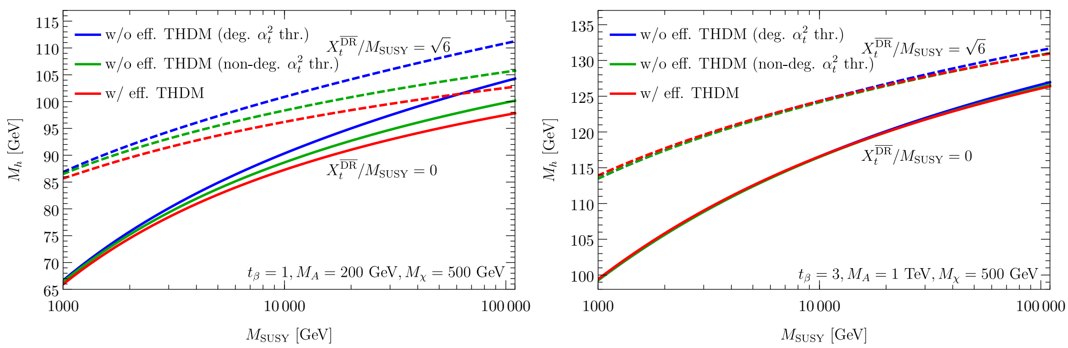


Figure 13.13: M_h as a function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed). Left: $\tan\beta = 1$ and $M_A = 200$ GeV. Right: $\tan\beta = 3$ and $M_A = 1$ TeV. The results of `FeynHiggs` without effective THDM – using the degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (blue) and using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) – are compared with the results of `FeynHiggs` with effective THDM (red).

of going to the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (realized in `FeynHiggs2.14.1`) which already accounts for a substantial part of the shift when turning to the new version with the effective THDM.

So far, all the numerical results refer to the $\overline{\rm DR}$ scheme for the stop-sector renormalization. As a distinct feature of `FeynHiggs`, also the OS scheme can be used for renormalizing the stop input parameters. In order to illustrate the use of OS renormalization, we include Fig. 13.12 as the equivalent of Fig. 13.11, now in the OS scheme, displaying the M_h dependence on $X_t^{\rm OS}/M_{\rm SUSY}$ for $M_A = 200$ GeV (left) and for $M_A = 1$ TeV (right). The overall behavior of the results is similar to the results obtained in the $\overline{\rm DR}$ scheme; also the shifts when turning to the THDM case are similar in size, although slighty more pronounced in the OS scheme.

Here, it is, however, important to note that the shift between `FeynHiggs` with and without effective THDM depends sensitively on the Higgsino mass parameter μ when the OS scheme is used[5]. This is due to the needed conversion of X_t between the $\overline{\rm DR}$ and the OS scheme, according to Eq. (11.75), which involves an extra term that can become large for $M_A \ll M_{\rm SUSY}$, low $\tan\beta$, $\mu \sim M_{\rm SUSY}$ and $X_t^{\rm OS}/M_{\rm SUSY} \sim 2$, inducing large differences between $X_t^{\rm OS}$ and $X_t^{\overline{\rm DR}}$. This signals that in those regions the one-loop conversion is insufficient yielding unreliable results for M_h, and recommends the use of the $\overline{\rm DR}$ scheme.

The $M_{\rm SUSY}$ scale dependence of the effect from implementing the THDM is explicitly shown in Fig. 13.13. In the left panel, we set $\tan\beta = 1$ and $M_A = 200$ GeV to maximize the shift

[5] μ is set to $M_\chi = 500$ GeV in Fig. 13.12

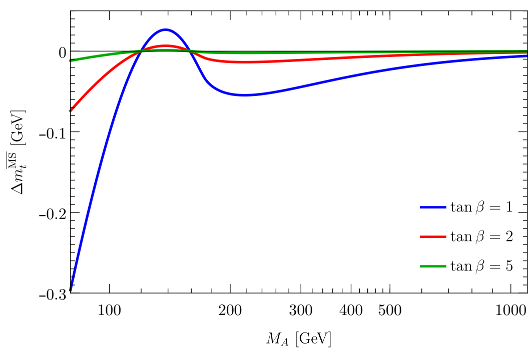


Figure 13.14: Shifts to the SM $\overline{\rm MS}$ top mass induced by non-SM Higgs bosons as a function of M_A for $\tan\beta = 1$ (blue) $\tan\beta = 2$ (red) and $\tan\beta = 5$ (green).

for illustrative purposes. Even for $M_{\rm SUSY} \sim$ few TeV, a sizeable shift occurs between the results with and without effective THDM, despite the small hierarchy between M_A and $M_{\rm SUSY}$. Phenomenologically this observation is, however, of less interest since the Higgs mass values reached are below 115 GeV over the whole considered range of $M_{\rm SUSY}$.

The configuration in the right panel of Fig. 13.13, with $\tan\beta = 3$ and $M_A = 1$ TeV, is more relevant for phenomenology since $M_h \sim 125$ GeV can be reached for $M_{\rm SUSY} \sim 10$ TeV (and $X_t^{\rm DR}/M_{\rm SUSY} = \sqrt{6}$). The difference between the results from FeynHiggs with and without effective THDM, however, is negligible for $M_{\rm SUSY} \lesssim 20$ TeV. We conclude that in the commonly considered scenarios with stop masses around the TeV scale and the h boson playing the role of the SM Higgs boson the additional corrections from an intermediate THDM are negligible.

13.3.3 Results for the heavier Higgs bosons

The role of the SM-like Higgs boson can not only be played by the h boson, also the H boson is a potential candidate (see [116, 117] for recent studies) and deserves a closer inspection. In the following, we investigate the prediction for the mass of H boson within our hybrid approach.

In this class of scenarios M_A is smaller than M_t. In consequence, the proper EFT at the electroweak scale is the THDM and not the SM. In the present study, we take the values of the SM $\overline{\rm MS}$ couplings (y_t, g_1, g_2, g_3) at the scale M_t computed in [79] as boundary values for the EFT calculation. Thus, the EFT at the scale M_t is replaced by the SM, which is then matched to the THDM. This procedure avoids the detailed calculation of the THDM $\overline{\rm MS}$ couplings at the electroweak scale, but neglects THDM-specific terms.

In order to estimate the uncertainty arising from this approximate determination of the boundary values, we investigate the numerical effect of the presence of extra Higgs bosons for the determination of the $\overline{\rm MS}$ top mass, as the parameter with the strongest impact in the Higgs boson mass calculation. As a rule of thumb, a shift of 1 GeV in the top mass implies a shift of the same size in the Higgs masses. As displayed in Fig. 13.14, the shift induced by the presence of extra non-SM Higgs bosons is at most 300 MeV. This value is reached if $M_A = 80$ GeV and $\tan\beta = 1$. For larger M_A and/or larger $\tan\beta$, the shift is quickly diminished below 100 MeV. Accordingly, we estimate the uncertainty induced by neglecting the non-SM Higgs bosons when extracting the $\overline{\rm MS}$ couplings to be below $\mathcal{O}(0.5$ GeV).

In Fig. 13.15, the dependence of M_H on M_A (left) and on $\tan\beta$ (right) is presented. In contrast to the parameters in the previous figures, we set $M_\chi = M_{\rm SUSY} = 10$ TeV to reduce the overall size of M_H. The left panel illustrates the situation for $\tan\beta = 1$, when the differences between the various versions are sizeable. We find an approximately constant shift between the results with and without effective THDM (employing the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction), of about 1 GeV for unmixed top squarks and 4 GeV for $X_t^{\rm DR}/M_{\rm SUSY} = \sqrt{6}$. For the range of input quantities, however, M_H is too large for H playing the role of the SM Higgs boson.

M_H can only be significantly decreased by raising $\tan\beta$. This possibility is analyzed in the right plot of Fig. 13.15, where M_A is set to 80 GeV. The shift between the results with and

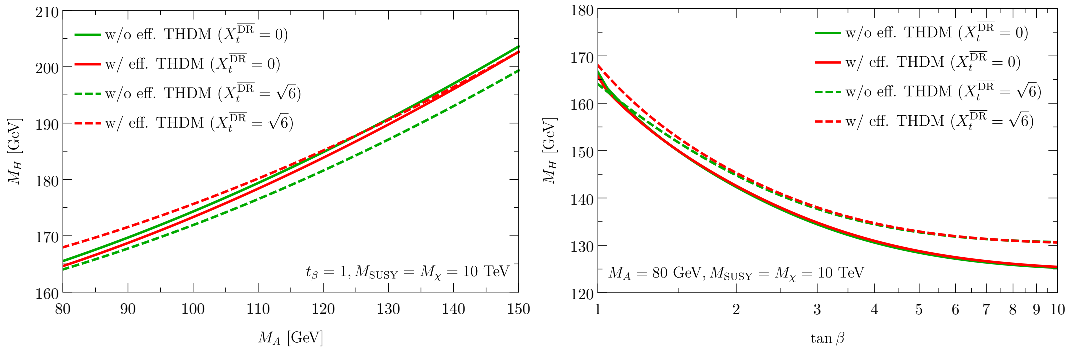


Figure 13.15: Left: M_H as a function of M_A for $\tan\beta = 1$. Right: M_H as a function of $\tan\beta$ for $M_A = 80$ GeV. The results of `FeynHiggs` without effective THDM using the non-degenerate $\mathcal{O}(\alpha_t^2)$ threshold correction (green) and with effective THDM (red) are compared. $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed).

without effective THDM shrinks for rising $\tan\beta$, as it was the case for M_h. To reach the desired value of 125 GeV for M_H, $\tan\beta$ has to be at least > 7. In this region, however, the difference between the results with and without the effective THDM is completely negligible. Also the uncertainty induced by not including contributions from non-SM Higgs bosons in the extraction of the low-energy couplings, estimated above, is totally negligible.

In addition, we also investigated the impact of the effective THDM on the prediction of the charged Higgs mass $M_{H^\pm}$. For the calculation of $M_{H^\pm}$ no resummation of large logarithms was available before. Nevertheless, we only find negligible shifts below 1 GeV in the scenarios considered above.

13.3.4 "Low-$\tan\beta$-high" scenario

In the "low-$\tan\beta$-high" scenario, defined in [115], all soft SUSY-breaking sfermion masses, as well as the gluino mass, are set equal to $M_{\rm SUSY}$. The value of $M_{\rm SUSY}$ is chosen such that the result for M_h is close to the experimentally determined mass and varies between a few TeV (in case of large M_A or $\tan\beta$) and 100 TeV (in case of small M_A or $\tan\beta$). In its original definition, the OS scheme was employed for renormalization, with the OS stop mixing parameter varying with $\tan\beta$ as follows,

$$X_t^{\rm OS}/M_{\rm SUSY} = \begin{cases} 2 & \text{for } \tan\beta \le 2 \\ 0.0375\tan^2\beta - 0.7\tan\beta + 3.25 & \text{for } 2 < \tan\beta \le 8.6 \\ 0 & \text{for } 8.6 < \tan\beta \end{cases} \tag{13.4}$$

Owing to the problems with OS parameters in scenarios with low M_A mentioned in Section 13.3.2, we define all parameters as $\overline{\rm DR}$ quantities[6]. Accordingly, we modify the values for X_t,

$$X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \begin{cases} 0.0375\tan^2\beta - 0.7\tan\beta + 3.25 & \text{for } \tan\beta \le 8.6 \\ 0 & \text{for } 8.6 < \tan\beta \end{cases}. \tag{13.5}$$

In this way, $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ will be close to the value which maximizes M_h when $\tan\beta = 1$ is approached.

The remaining parameters are given by

$$\mu = 1.5 \text{ TeV}, \qquad M_2 = 2 \text{ TeV}, \qquad A_{b,c,s,u,d} = 2 \text{ TeV}. \tag{13.6}$$

M_1 is fixed via the GUT relation $M_1 = \frac{5}{3}\tan^2\theta_W M_2 \approx 0.5 M_2$.

The left panel of Fig. 13.16 contains M_h obtained from the `FeynHiggs` version including the THDM, in dependence of $\tan\beta$ and M_A. One finds that M_h comes close to the experimental

[6]The use of the $\overline{\rm DR}$ scheme will be also be beneficial when comparing with `MhEFT` in the next subsection.

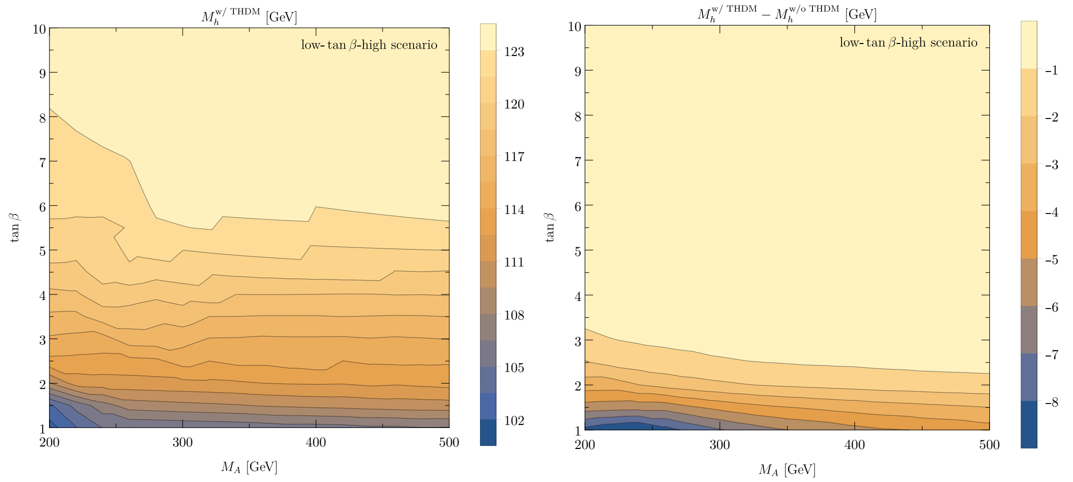


Figure 13.16: Left: M_h computed with **FeynHiggs** including the effective THDM as a function of M_A and $\tan\beta$ in the "low-$\tan\beta$-high" scenario. Right: Same as left plot, but the difference between the result with and without effective THDM is shown.

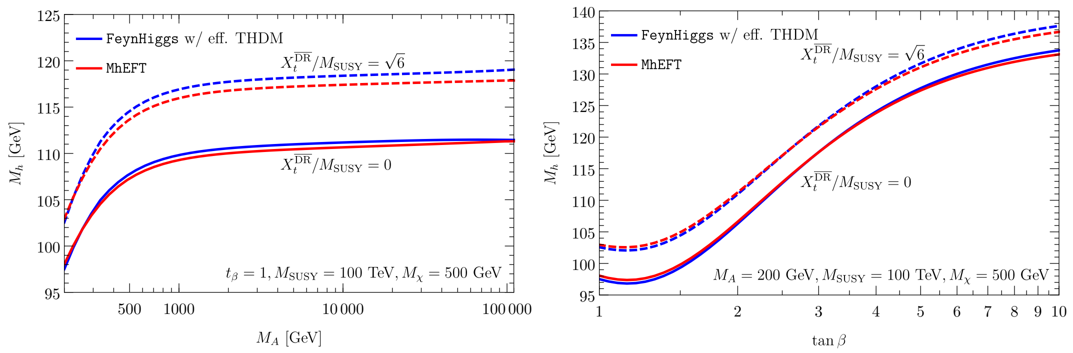


Figure 13.17: Left: M_h as a function of M_A for $\tan\beta = 1$. Right: M_h as a function of $\tan\beta$ for $M_A = 200$ GeV. The results of **FeynHiggs** with effective THDM (blue) and **MhEFT** (red) are compared for $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = 0$ (solid) and $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = \sqrt{6}$ (dashed).

value of 125 GeV only in the upper part of the plot where $\tan\beta \gtrsim 6$. For lower values of $\tan\beta$, M_h drops down to the region around 105 GeV. If additionally M_A is small (~ 200 GeV), M_h is even below 102 GeV. In comparison with the results shown in Fig. 3 of [115], M_h is reduced by several GeV.

The results in [115] were produced using `FeynHiggs2.10.4`. Since then, many additional improvements were implemented in `FeynHiggs` (see discussion above). To point out the effect of the most recent developments since `FeynHiggs2.14.0`, we show the difference between the most topical version of `FeynHiggs` with effective THDM and the non-THDM version `2.14.0` in the right panel of Fig. 13.16. The diagram shows that for the considered scenario the M_h values obtained with an effective THDM are below the values obtained without effective THDM. For $\tan\beta \gtrsim 3$, the downwards shift is small (below 1 GeV). For smaller $\tan\beta$, the shift increases to about 4 GeV for $M_A = 500$ GeV. If in addition also M_A is small (~ 200 GeV), the difference amounts to even more than 8 GeV.

13.3.5 Comparison to `MhEFT`

After investigating the numerical impact of an effective THDM on the hybrid calculation of `FeynHiggs`, we compare our results to `MhEFT` (version `1.1`).

First, we compare the results for M_h in dependence of M_A (see left panel of Fig. 13.17). We choose $\tan\beta = 1$ to maximize the impact of the effective THDM. For vanishing stop mixing, `FeynHiggs` and `MhEFT` are in close agreement. Also for $X_t^{\overline{\text{DR}}}/M_{\text{SUSY}} = \sqrt{6}$, both codes agree

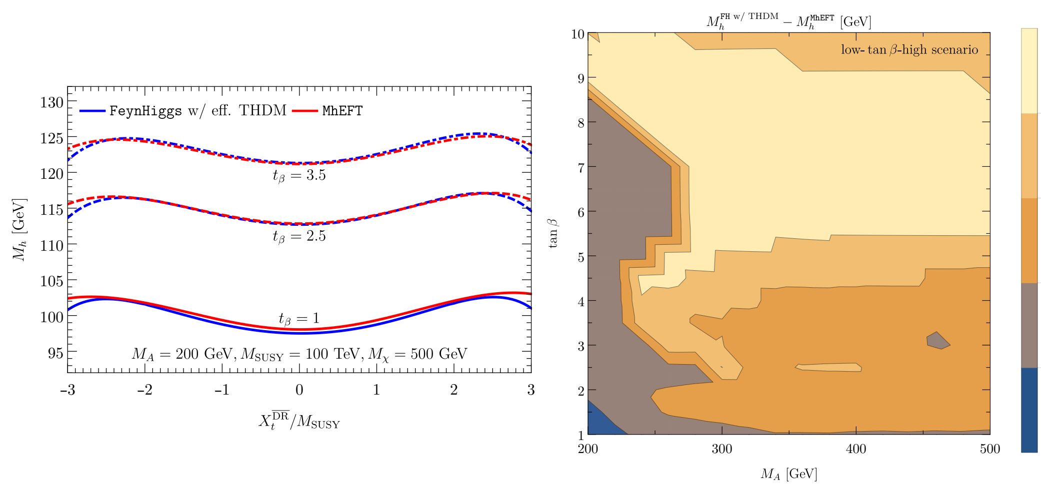


Figure 13.18: Left plot: M_h as a function of X_t for $\tan\beta = 1$ (solid) $\tan\beta = 2.5$ (dashed) and $\tan\beta = 3.5$ (dotdashed). $M_A = 200$ GeV is chosen. The results of `FeynHiggs` with effective THDM (blue) and `MhEFT` (red) are compared. Right plot: M_h in the "low-$\tan\beta$-high" scenario. The difference between `FeynHiggs` with effective THDM and `MhEFT` is displayed.

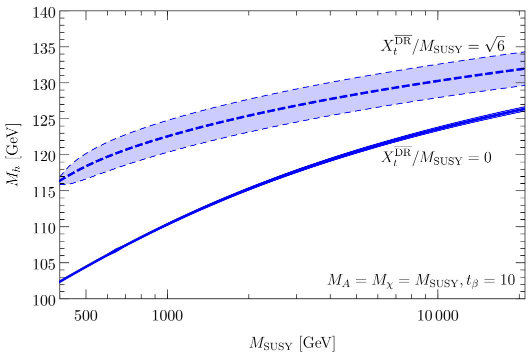
within ~ 1 GeV. The remaining deviation is caused by the different parameterization of non-logarithmic terms (see Section 10.2). For low M_A this constant shift is compensated by terms of $\mathcal{O}(M_t/M_A)$ which are included in `FeynHiggs` but not in `MhEFT`.

In the right panel of Fig. 13.17, the results are compared as a function of $\tan\beta$, setting $M_A = 200$ GeV. The overall good agreement is confirmed. Especially around $\tan\beta \sim 3$ both results are very close to each other, whereas the agreement is slightly worse for smaller or higher values of $\tan\beta$ (but still within 1 GeV). Reasons for the disagreement are again the different parameterization of non-logarithmic terms as well as terms of $\mathcal{O}(M_t/M_A)$.

This behavior is also reflected in the left panel of Fig. 13.18 showing M_h as a function of $X_t^{\overline{\rm DR}}$. For $\tan\beta = 2.5$ and $\tan\beta = 3.5$, `FeynHiggs` and `MhEFT` nearly superpose each other. Only for $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}| > 2.5$, small deviations are visible which originate from the different parameterizations of non-logarithmic terms. These terms become large for large $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$. For $\tan\beta = 1$, a deviation of $\lesssim 1$ GeV is visible for $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}| < 2.5$, which is mainly caused by $\mathcal{O}(M_t/M_A)$ terms.

In the right panel of Fig. 13.18, we have another look at the "low-$\tan\beta$-high" scenario using the $\overline{\rm DR}$ scheme, as defined in Section 13.3.4. In the whole M_A–$\tan\beta$ plane the difference between both codes is smaller than 2 GeV. Especially for low M_A or low $\tan\beta$ both codes agree very well, whereas `FeynHiggs` yields slightly larger results than `MhEFT` in the rest of the parameter plane.

Finally, we comment on the comparison between `FeynHiggs` and `MhEFT` shown in [39] (see Fig. 10 and 11 therein). The authors of [39] compared both codes in the "low-$\tan\beta$-high" scenario and found deviations of up to 15 GeV. According to their claim, this discrepancy was mainly caused by the missing implementation of an effective THDM in `FeynHiggs`. In our Fig. 13.16, right panel, we found, however, the effective THDM to induce shifts of not more than 8 GeV. This raises the question for the origin of the remaining difference of ~ 7 GeV. One reason is certainly the fact that `FeynHiggs` has evolved a lot since version `2.10.2`, which was taken for the comparison in [39]. A second more important reason is the parameter conversion used for the comparison, which was done for the "low-$\tan\beta$-high" scenario defined with OS parameters, Eq. (13.4). Therefore, the OS stop mixing parameter had to be converted to the $\overline{\rm MS}$ scheme which is employed in `MhEFT`. In this conversion, $M_A = M_{\rm SUSY}$ was assumed. Thereby, an important logarithmic contribution was missed (last term in Eq. (11.75)), which is especially large for low $\tan\beta$ and low M_A, thus exactly in the parameter region where the largest deviation between `FeynHiggs` and `MhEFT` was observed.



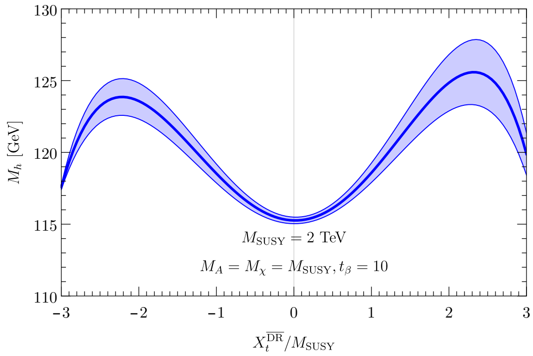


Figure 13.19: M_h predictions of `FeynHiggs` including bands indicating the estimated theoretical uncertainty. Left: M_h as function of $M_{\rm SUSY}$ for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = 0$ (solid) and $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (dashed). Right: M_h as function of $X_t^{\overline{\rm DR}}/M_{\rm SUSY}$ for $M_{\rm SUSY} = 2$ TeV.

13.4 Remaining theoretical uncertainties

In the end, we also want to briefly discuss the remaining theoretical uncertainties. The hybrid calculation of `FeynHiggs` can be improved by either advancing the fixed-order calculation or the EFT calculation. An improvement of the fixed-order calculation would correspond to the inclusion of higher-order non-logarithmic terms, which are partly suppressed in case of a high SUSY scale. The unsuppressed parts of these corrections could also be used to obtain higher-order threshold corrections at the SUSY scale to improve the EFT calculation. The inclusion of these higher-order threshold corrections would then correspond to higher-order unsuppressed non-logarithmic terms as well as logarithms going beyond the current order of resummation. Another way of improving the EFT calculation would be to include higher-order RGEs or higher-order threshold corrections at the electroweak scale. Such improvements would generate logarithms beyond the current order of resummation.

The uncertainty estimate included in `FeynHiggs` focuses so far on the evaluation of the uncertainty of the fixed-order calculation. It consists of three components:

- varying the renormalization scale entering the diagrammatic calculation between $M_t/2$ and $2M_t$ (M_t is the default scale),
- switching between different parametrizations of the top mass (OS top mass and SM $\overline{\rm MS}$ top mass) at the NNLO level,
- deactivating the resummation of the bottom-Yukawa coupling for large $\tan\beta$ (see [118] for more details).

The change in the parametrization of the top mass is performed only for the non-logarithmic terms. The EFT result containing all resummed logarithms is left unchanged.

The resulting estimate is shown in Fig. 13.19. For vanishing stop mixing we observe that the estimated theoretical uncertainty is very small. A change of $M_{\rm SUSY}$ only marginally effects the estimate. For rising $|X_t^{\overline{\rm DR}}/M_{\rm SUSY}|$, the estimate increases to up to ~ 2 GeV for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = \sqrt{6}$ (~ 1.5 GeV for $X_t^{\overline{\rm DR}}/M_{\rm SUSY} = -\sqrt{6}$). Also in this case, the estimate is nearly completely independent of $M_{\rm SUSY}$. Only for $M_{\rm SUSY} \lesssim 500$ GeV a significant decrease is noticeable.

In the considered scenario, the uncertainty estimate originates nearly completely from the change in the parametrization of the top mass. The variation of the renormalization scale and deactivating the resummation of the bottom-Yukawa coupling yield only negligible contributions. This explains the larger uncertainty for large stop mixing, since in this case the dependence on the top mass is stronger.

These results hint at a problem with the estimate of `FeynHiggs`: It does not take into account any uncertainty associated with large logarithms, e.g., in the case of vanishing stop mixing the main correction to the Higgs mass consist of large logarithms. In addition, studies of the uncertainties of pure EFT calculations suggest that the uncertainty in the case of large stop mixing should decrease with rising M_{SUSY} due to a shrinking top-Yukawa coupling at M_{SUSY} [40]. Nevertheless, at least for low M_{SUSY} the results presented here should give a solid estimate of the remaining theoretical uncertainty.

Chapter 14

Conclusions

The discovery of a Higgs boson and the measurement of its mass by the experiments ATLAS [1] and CMS [2] at the LHC enables stringent tests not only of the SM but also of BSM theories. In supersymmetric models, like the MSSM, the Higgs boson mass itself can be predicted in terms of the model parameters and therefore used as a precision observable.

Different methods are used for the calculation of the mass of the lightest $\mathcal{CP}$-even Higgs boson. Perturbative calculations at a fixed order allow to obtain all corrections at a given order and are therefore precise for low SUSY scales. In the light of increasing experimental bounds on SUSY particles from direct searches, however, EFT calculations became increasingly popular. They allow to resum large logarithmic contributions by means of renormalization group equations. These large logarithmic contributions can spoil the convergence of the perturbative expansion in pure fixed-order calculations. They are therefore precise for high SUSY scales. For low scales, though, they become inaccurate since terms which would be suppressed for high SUSY scales are missed, if no higher-dimensional operators are taken into account.

To profit from the advantages of both method – high precision for low scales in the case of the fixed-order calculation and high precision for high scales in the case of the EFT calculation – we described how a hybrid approach allows to combine both techniques. For this combination several subtleties have to be taken into account: Subtraction terms have to be introduced to avoid the double counting of terms contained in both calculations. Moreover, a conversion between the different renormalization schemes used in the fixed-order and the EFT calculation has to be performed.

Originally, this method was restricted to supplementing the existing one- and two-loop fixed-order calculation implemented into the public code `FeynHiggs` by a resummation of leading and next-to-leading logarithms in the limit of vanishing electroweak gauge couplings. In this thesis, we have presented and discussed the inclusion of electroweak contributions. In addition, we implemented separate electroweakino and gluino thresholds, and investigated the effect of the resummation of next-to-next-to-leading logarithms in the limit in the limit of vanishing electroweak gauge couplings. These improvements shift the prediction for M_h, especially pronounced for positive values of the stop-mixing parameter X_t with downwards shifts in M_h of about 2 GeV.

We found that this is mainly caused by the electroweak NLO corrections to the $\overline{\text{MS}}$ top-quark mass. The genuine effect of resumming electroweak contributions shifts the Higgs mass upwards compensating the downwards shift induced by the smaller $\overline{\text{MS}}$ top-quark mass. This effect becomes only relevant for SUSY scales larger than a few TeV. Furthermore, electroweak NLL contributions are found to be much smaller than electroweak LL contributions.

We also investigated the effect of various intermediate thresholds. In our framework, an electroweakino threshold yields significant contributions only for SUSY scales above 5 TeV. We found that a gluino threshold is completely negligible, since the main contributions sensitive to the gluino mass are already captured by the two-loop Feynman diagrammatic result.

Furthermore, we found NNLL resummation of $\mathcal{O}(\alpha_s, \alpha_t)$ to shift the lightest Higgs mass downwards for positive stop mixing, whereas it leads to a larger upwards shift for negative values of X_t.

After discussing these improvements, which brought the EFT part of the calculation in FeynHiggs to the same level of accuracy as dedicated pure EFT codes, we have presented a detailed comparison between various approaches used to predict the mass of the SM-like Higgs boson in the MSSM in a scenario in which all SUSY mass scales are chosen equal to each other. In particular we have compared pure EFT calculations with our hybrid approach. In the literature significant deviations between the results obtained via the two approaches have been reported especially at large SUSY scales. In the course of this investigation, we have identified three sources of the observed differences.

We could show that a large part of the reported discrepancies can be traced back to parameter conversions between different renormalization schemes. In EFT calculations typically the $\overline{\text{DR}}$ scheme is used for the renormalization of SUSY breaking parameters, e.g. the stop mixing parameter. In the diagrammatic calculation of FeynHiggs (in the default case), however, the OS scheme is employed in the scalar top sector. We have demonstrated that the a one-loop scheme conversion of input parameters often used for the comparison of fixed-order results is not suitable for the comparison of results containing a series of higher-order logarithms. This kind of parameter conversion would induce higher-order logarithmic contributions that are not compatible with the implemented resummation of logarithms to all orders. We have shown that the form of the higher-order logarithms obtained in one scheme can manifestly be maintained if the fixed-order part of the calculation is reparametrized to this scheme. In order to enable this approach for $\overline{\text{DR}}$ input parameters, we have extended FeynHiggs such that the results are provided both in terms of the on-shell parameters X_t^{OS}, $M_{\tilde{t}_1} \equiv m_{\tilde{t}_1}^{\text{OS}}$, $M_{\tilde{t}_2} \equiv m_{\tilde{t}_2}^{\text{OS}}$ (as before) and the $\overline{\text{DR}}$ parameters $X_t^{\overline{\text{DR}}}$, $m_{\tilde{t}_1}^{\overline{\text{DR}}}$, $m_{\tilde{t}_2}^{\overline{\text{DR}}}$. In practice, this was achieved by reparametrizing the existing OS fixed-order result. We have demonstrated that many of the apparent discrepancies reported in the literature have mainly been caused by an inappropriate application of the conversion of input parameters between the OS and the $\overline{\text{DR}}$ schemes. This issue is not a problem of the OS renormalization, but analogously appears if OS parameters are used as input for codes employing the $\overline{\text{DR}}$ scheme.

Another difference between pure EFT calculations and the hybrid approach arises from the determination of the poles of the Higgs propagator matrix. We have shown explicitly at the two-loop level that there occurs a cancellation in the limit of a large SUSY scale between non-SM terms arising through the determination of the propagator pole and contributions from the subloop renormalization of the irreducible self-energy diagrams. Since we expect that similar cancellations will happen at higher loops, we have modified the determination of the propagator poles in FeynHiggs such that terms are omitted that would not cancel because their counterpart in the irreducible self-energies is not incorporated at present. Unless otherwise stated, the numerical results presented in this work have been obtained using this new method of pole determination. Numerically, we found that the terms beyond $\mathcal{O}(\alpha_t^2, \alpha_t\alpha_b, \alpha_b^2)$ for which the cancellation was incomplete before are negligible for low scales ($M_{\text{SUSY}} \lesssim 0.5$ TeV). They can be more significant for high scales (~ 1.5 GeV for $M_{\text{SUSY}} \sim 20$ TeV).

Furthermore, we investigated the impact of different parametrizations of the non-logarithmic one- and two-loop terms. In this context, we found the top-quark mass and the vev to be especially relevant. Despite the results being formally identical at the strict two-loop level, using e.g. a SM NNLO $\overline{\text{MS}}$ top-quark mass instead of the OS top-quark mass induces changes in the higher-order non-logarithmic contributions.

In our numerical comparison, we focused on a single scale scenario with a moderate t_β, which is particularly suited for an EFT calculation. We specifically compared the results of FeynHiggs and the EFT code SUSYHD. Using the NNLO value of the $\overline{\text{MS}}$ top-Yukawa coupling in SUSYHD (by default the NNNLO value is used in SUSYHD, which leads to a downward shift by ~ 0.5 GeV in M_h), we find very good agreement between FeynHiggs and SUSYHD for scales $M_{\text{SUSY}} \gtrsim 1$ TeV. Such a good agreement is in fact expected for high SUSY scales since the hybrid approach of FeynHiggs incorporates essentially the same logarithmic contributions as pure EFT calculations. For $M_{\text{SUSY}} \lesssim 1$ TeV we observe significant differences between FeynHiggs and SUSYHD due to terms suppressed by the SUSY scale that are not incorporated in the EFT calculation of SUSYHD. The observed differences stay relatively small for the considered simple scenario with a single SUSY scale, reaching ~ 1 GeV for $M_{\text{SUSY}} \sim 300$ GeV. Larger deviations can be expected in SUSY scenarios with non-negligible mass splittings between the various SUSY particles. Such kind of mass patterns are accounted for in the diagrammatic fixed-order part of the hybrid approach.

Afterwards, we discussed the implementation of an effective THDM into the hybrid framework of `FeynHiggs`. First, we described our EFT calculation. It allows not only to treat the case of light non-SM Higgs bosons but also of light electroweakinos. Furthermore, it includes complete one-loop threshold correction and takes all appearing effective couplings fully into account. In this context, we also discussed how the matching between the different EFTs is performed paying special attention to the different normalization of the Higgs doublets.

This difference in the normalization also plays a crucial role in the combination of the existing fixed-order calculation with the new EFT calculation for low M_A. We accounted for the different normalization by introducing finite field renormalization constants into the fixed-order calculation. We also discussed how this affects the definition of $\tan\beta$ as an input parameter. Moreover, we investigated the effect of low M_A on the scheme conversion of the stop mixing parameter which is necessary if OS input parameters are used.

In our numerical study, we compared the version `2.14.0` and `2.14.1` of `FeynHiggs`, both with the SM as the EFT, to our new computation with an effective THDM. We found the switch to an effective THDM to cause a negative shift in M_h of up to 3 GeV with respect to `FeynHiggs2.14.1`. This maximal value is reached when $\tan\beta \sim 1$ and the hierarchy between the SUSY scale and M_A is large ($M_{\text{SUSY}}/M_A \sim 10^3$). The shift shrinks quickly when $\tan\beta$ is increased. For $\tan\beta \gtrsim 7$, the effects resulting fom the THDM are almost completely negligible. Similarly, the shift decreases when M_A is increased or M_{SUSY} is lowered. Larger shifts, up to 10 GeV, are found when comparing to `FeynHiggs2.14.0`. In that version, the implemented $\mathcal{O}(\alpha_t^2)$ threshold correction implicitly assumed M_A to be equal to M_{SUSY}, leading to an overestimate of M_h in scenarios with $M_A \ll M_{\text{SUSY}}$.

We also investigated predictions for the mass of the second $\mathcal{CP}$-even Higgs boson H. In the phenomenologically most interesting parameter region, where the H boson can play the role of the SM Higgs boson, we found the shift induced by an effective THDM to be negligible. Also the prediction of the charged Higgs boson mass is only marginally affected. In addition, we looked at the "low-tanb-high" benchmark scenario developed by the LHC Higgs Cross Section Working Group. For this scenario, we found corrections of up to -8 GeV in comparison to `FeynHiggs2.14.0` for $\tan\beta \lesssim 3$ with the consequence that the updated M_h prediction is too low for meeting the experimental Higgs boson mass. Finally, we compared our results with those of the code `MhEFT` finding good agreement within 1 GeV throughout the considered parameter space.

In the end, we presented a brief discussion of the remaining theoretical uncertainties. The current uncertainty estimate of `FeynHiggs` focuses on the evaluation of the uncertainty of the fixed-order calculation. Therefore, it is less suitable for high SUSY scales where the uncertainty of large logarithmic contributions becomes relevant. Nevertheless, the results obtained in this thesis provide important input for an improved estimate of the remaining theoretical uncertainties from unknown higher-order corrections. Such an improved estimate will be the subject of future studies, whose final goal will be to fully exploit the high precision reached on the experimental side.

Appendix **A**

Fixed-order conversion: additional two-loop terms

In this Appendix, we describe how the OS renormalized fixed-order calculation can be reparametrized such that the entering stop parameters are defined as $\overline{\text{DR}}$ parameters (see discussion in Chapter 9).

In the limit $M_{\text{SUSY}} \gg M_t$ and degenerate $m_{\tilde{t}_L} = m_{\tilde{t}_R} = M_{\text{SUSY}}$, the one-loop contributions from the stop/top sector to the neutral Higgs self-energies at $\mathcal{O}(\alpha_t)$ are given by (here, we drop the subscript "G_F", i.e. we use the shorthand $v \equiv v_{G_F}$)

$$\hat{\Sigma}_{11} = \frac{1}{16\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\frac{\mu^2 X_t^2}{M_S^4}, \tag{A.1}$$

$$\hat{\Sigma}_{12} = \frac{1}{16\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\frac{\mu X_t}{M_S^2}\left[6 - \frac{X_t^2}{M_S^2} - \frac{1}{t_\beta}\frac{\mu X_t}{M_S^2}\right], \tag{A.2}$$

$$\hat{\Sigma}_{22} = \frac{1}{16\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\left[-12\ln\frac{M_S^2}{m_t^2} - 12\frac{X_t^2}{M_S^2} + \frac{X_t^4}{M_S^4} - \frac{2}{t_\beta}\frac{\mu X_t}{M_S^2}\left(6 - \frac{X_t^2}{M_S^2}\right) + \frac{1}{t_\beta^2}\frac{\mu^2 X_t^2}{M_S^4}\right], \tag{A.3}$$

where $M_S^2 = M_{\tilde{t}_1} M_{\tilde{t}_2}$, and m_t is either the OS top mass or the $\overline{\text{MS}}$ SM top mass.

If we convert the stop masses and the stop mixing parameter from the OS to the $\overline{\text{DR}}$ scheme using the shifts defined in Eqs. (9.1), (9.2) and (9.8), the following two-loop terms are generated (see Eq. (9.12)),

$$\Delta\hat{\Sigma}_{11} = \frac{1}{8\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\left[\frac{\Delta X_t}{M_S}\frac{\mu^2 X_t}{M_S^3} - 2\frac{\Delta M_S}{M_S}\frac{\mu^2 X_t^2}{M_S^4}\right], \tag{A.4}$$

$$\begin{aligned}\Delta\hat{\Sigma}_{12} = &\frac{1}{16\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\left[\frac{\Delta X_t}{M_S}\left(-3\frac{\mu X_t^3}{M_S^3} - \frac{2}{t_\beta}\frac{\mu^2 X_t}{M_S^3} + 6\frac{\mu}{M_S}\right)\right.\\ &\left.+\frac{\Delta M_S}{M_S}\left(4\frac{\mu X_t^3}{M_S^4} + \frac{4}{t_\beta}\frac{\mu^2 X_t^2}{M_S^4} - 12\frac{\mu X_t}{M_S^2}\right)\right],\end{aligned} \tag{A.5}$$

$$\begin{aligned}\Delta\hat{\Sigma}_{22} = &\frac{1}{8\pi^2}\frac{1}{s_\beta^2}\frac{m_t^4}{v^2}\left[\frac{\Delta X_t}{M_S}\left(-2\frac{X_t}{M_S}\left(6 - \frac{X_t^2}{M_S^2}\right) - \frac{3}{t_\beta}\frac{\mu}{M_S}\left(2 - \frac{X_t^2}{M_S^2}\right) + \frac{1}{t_\beta^2}\frac{\mu^2 X_t}{M_S^3}\right)\right.\\ &\left.-2\frac{\Delta M_S}{M_S}\left(6 - 6\frac{X_t^2}{M_S^2} + \frac{X_t^4}{M_S^4} - \frac{2}{t_\beta}\frac{\mu X_t}{M_S^2}\left(3 - \frac{X_t^2}{M_S^2}\right) + \frac{1}{t_\beta^2}\frac{\mu^2 X_t^2}{M_S^4}\right)\right].\end{aligned} \tag{A.6}$$

The quantity ΔM_S is given by

$$\Delta M_S = \frac{1}{4}\left(\frac{\Delta m_{\tilde{t}_1}^2}{M_{\tilde{t}_1}^2} + \frac{\Delta m_{\tilde{t}_2}^2}{M_{\tilde{t}_2}^2}\right) M_S. \tag{A.7}$$

The quantities ΔX_t and $\Delta m_{\tilde{t}_{1,2}}^2$ are given by the finite parts of the associated counterterms (defined in Eqs. (6.44), (6.45) and (6.47)),

$$\Delta m^2_{\tilde{t}_{1,2}} = \delta^{(1)} m^2_{\tilde{t}_{1,2}}\Big|_{\text{fin}}, \tag{A.8}$$

$$\Delta X_t = \delta^{(1)} X_t\Big|_{\text{fin}}. \tag{A.9}$$

Note that for all numerical results presented in this work, we used expressions valid also for low M_{SUSY} ($M_{\text{SUSY}} \sim M_t$) and general SUSY breaking. Note also that the shifts are performed for all self-energies and not only for the hh self-energy as shown exemplary in Chapter 9. Therefore, the procedure remains also valid in non-decoupling scenarios ($M_A \sim M_Z$).

As described in Chapter 9, these two-loop terms are finally added to the respective self-energies, i.e., the $\Delta\hat{\Sigma}$-s are added to the two-loop self-energies obtained from the diagrammatic calculation. Higher-order terms which would be generated by a scheme conversion of the input parameters are omitted. In this way, the renormalization of the stop sector is changed from the OS to the $\overline{\text{DR}}$ scheme.

Appendix **B**

Logarithms arising from the determination of the propagator poles

In this Appendix, we give explicit expressions, valid in the decoupling limit, for the logarithms induced by the momentum dependence of the non-SM contributions to the MSSM Higgs self-energy, i.e. for the quantity $\Delta_{p^2}^{\text{logs}}$ defined in Eq. (10.5).

In order to derive the $(n+1)$th order iterative solution to the Higgs pole mass equation (see Eq. (7.3)) in terms of lower order solutions, Fàa di Bruno's formula (extended chain rule for derivatives) is used,

$$(M_h^2)^{(n+1)} = -\sum_{(a_1,\dots,a_n)\in T_n} \frac{1}{a_1!\cdot\ldots\cdot a_n!}\cdot \cdot\left[\left(\frac{\partial}{\partial p^2}\right)^{(a_1+\ldots+a_n)} \hat{\Sigma}_{hh}^{\text{MSSM}}(p^2)\right]_{p^2=m_h^2}\cdot\prod_{m=1}^{n}(M_h^2)^{(m)}, \tag{B.1}$$

where an n-tuple of non negative integers

$$(a_1,\dots,a_n)\in T_n \text{ if } 1\cdot a_1 + 2\cdot a_2 + \ldots + n\cdot a_n = n. \tag{B.2}$$

The zeroth order correction

$$(M_h^2)^{(0)} = m_h^2 \tag{B.3}$$

serves as starting point of the recursion.

We split $\Delta_{p^2}^{\text{logs}}$ into a leading, a next-to-leading and a next-to-next-to-leading logarithm piece,

$$\Delta_{p^2}^{\text{logs}} = \Delta_{p^2}^{\text{LL}} + \Delta_{p^2}^{\text{NLL}} + \Delta_{p^2}^{\text{NNLL}} + \ldots\ . \tag{B.4}$$

In `FeynHiggs`, the full momentum dependence by default is taken into account only at the one-loop level. At the two-loop level, the external momentum is set to zero (see [76, 77] for a discussion of the momentum dependence at the two-loop level). We can therefore split up the non-SM contributions to the Higgs self-energy into a one- and a two-loop piece,

$$\hat{\Sigma}_{hh}^{\text{nonSM}}(p^2) = \hat{\Sigma}_{hh}^{\text{nonSM},(1)}(p^2) + \hat{\Sigma}_{hh}^{\text{nonSM},(2)}(0). \tag{B.5}$$

To shorten the expressions for the individual contributions, we first introduce abbreviations. We write the non-SM contributions to the Higgs self-energy as

$$\hat{\Sigma}_{hh}^{\text{nonSM},(1)}(m_h^2) = k\left(c_{1,1}^{\chi}L_\chi + c_{1,1}^{A}L_A + c_{1,1}^{\tilde{f}}L_S + c_{1,0}\right), \tag{B.6}$$

$$\hat{\Sigma}_{hh}^{\text{nonSM},(2)}(0) = k^2\left(c_{2,2}L_S^2 + c_{2,1}L_S + c_{2,0}\right), \tag{B.7}$$

where $k \equiv (4\pi)^{-2}$ is used to keep track of the loop order and

$$L_\chi \equiv \ln \frac{M_\chi^2}{m_t^2}, \qquad L_A \equiv \ln \frac{M_A^2}{m_t^2}, \qquad L_S \equiv \ln \frac{M_{\text{SUSY}}^2}{m_t^2}. \tag{B.8}$$

The subscript of a coefficient $c_{a,b}$ indicates that it is the prefactor of the term $k^a L^b$ ($L = L_\chi, L_A, L_S$). The corresponding superscript marks the origin of the respective term (from EWinos χ, from heavy Higgses A or from sfermions $\tilde{f}$). These superscripts are used only at the one-loop level to be able to differentiate between the different types of appearing logarithms (L_χ, L_A and L_S). In the $\overline{\text{DR}}$ scheme, the appearing coefficients up to $\mathcal{O}(v^2/M_{\text{heavy}}^2)$ ($M_{\text{heavy}} = M_\chi, M_A, M_{\text{SUSY}}$) are given by (for the remainder of this section we drop the subscript "G_F", i.e. we use the shorthand $v \equiv v_{G_F}$)

$$c_{1,1}^{\tilde{f}} = -2v^2 \left[6y_t^4 + \frac{3}{2}y_t^2(g^2+g'^2)c_{2\beta} + \frac{1}{2}g^4 + \frac{5}{6}g'^4 + \frac{1}{6}(3g^4+5g'^4)c_{4\beta}\right], \tag{B.9}$$

$$c_{1,1}^{\chi} = -2v^2 \left[\frac{1}{24}g'^4(-11+c_{4\beta}) - \frac{3}{8}g^4(5+c_{4\beta}) - g^2g'^2s_{2\beta}^2\right], \tag{B.10}$$

$$\begin{aligned} c_{1,1}^{A} = -2v^2 \bigg[&\frac{1}{192}g^4(53-28c_{4\beta}-9c_{8\beta}) + \frac{1}{192}g'^4(29-4c_{4\beta}-9c_{8\beta}) \\ &+\frac{1}{8}g^2g'^2(5+3c_{4\beta})s_{2\beta}^2\bigg], \end{aligned} \tag{B.11}$$

$$\begin{aligned} c_{1,0} = -2v^2\bigg\{&6y_t^2\left[\left(y_t^2+\frac{1}{8}(g^2+g'^2)c_{2\beta}\right)\hat{X}_t^2 - \frac{1}{12}y_t^2\hat{X}_t^4\right] \\ &-\frac{1}{4}y_t^2(g^2+g'^2)\hat{X}_t^2c_{2\beta}^2 - \frac{3}{16}(g^2+g'^2)^2s_{4\beta}^2 \\ &-\left[\left(\frac{3}{4}-\frac{1}{6}c_{2\beta}^2\right)g^4 + \frac{1}{2}g^2g'^2 + \frac{1}{4}g'^4\right] \\ &+\frac{1}{24}(s_\beta+c_\beta)^2\cdot \\ &\quad\cdot\Big[-51g^4-24g^2g'^2-13g'^4+\big((g^2+g'^2)c_{4\beta} \\ &\qquad+2(g^2-g'^2)s_{2\beta}\big)(3g^2+g'^2)\Big]\bigg\}, \end{aligned} \tag{B.12}$$

$$c_{2,2} = -2v^2y_t^4\left(-48g_3^2+9y_t^2\right), \tag{B.13}$$

$$c_{2,1} = -2v^2y_t^4\left[8g_3^2\left(4-12\hat{X}_t^2+\hat{X}_t^4\right) - \frac{3}{2}y_t^2\left(20-12\hat{X}_t^2+\hat{X}_t^4\right)\right], \tag{B.14}$$

where all appearing couplings are SM $\overline{\text{MS}}$ couplings evaluated at $Q = M_t$ (g, g' are the electroweak gauge couplings, and $\hat{X}_t \equiv X_t/M_{\text{SUSY}}$). We write the derivative of the non-SM contributions to the Higgs self-energy as

$$\hat{\Sigma}_{hh}^{\text{nonSM},(1)\prime}(m_h^2) = k\left(c'_{1,1}L_\chi + c'_{1,0}\right), \tag{B.15}$$

with the primes denoting that the corresponding coefficient appears in the derivative of the self-energy. We again drop contributions of $\mathcal{O}(v^2/M_{\text{heavy}}^2)$. The coefficient multiplying L_χ originates purely from EWino graphs and reads

$$c'_{1,1} = -\frac{1}{2}(3g^2+g'^2). \tag{B.16}$$

The non-logarithmic coefficient has contributions from EWinos as well as from stops (neglecting all other Yukawa couplings),

$$c'_{1,0} = \underbrace{\frac{1}{2}y_t^2\hat{X}_t^2}_{\text{stop contr.}} \underbrace{-\frac{1}{6}(3g^2+g'^2)(s_\beta+c_\beta)^2}_{\text{EWino contr.}}. \tag{B.17}$$

All higher derivatives of $\hat{\Sigma}_{hh}^{\text{nonSM}}(p^2)$ are suppressed, i.e. of $\mathcal{O}(p^2/M^2_{\text{heavy}})$.

The SM contributions are written in a similar way,

$$\left(\frac{\partial}{\partial p^2}\right)^n \hat{\Sigma}_{hh}^{\text{SM},(1)}(p^2)\bigg|_{p^2=m_h^2} = k\tilde{c}_1^n, \tag{B.18}$$

where the superscript 'n' denotes the nth derivative of $\hat{\Sigma}_{hh}^{\text{SM},(1)}$. Here, we only give explicit expressions for the pure top-Yukawa contributions to the first five derivatives of $\hat{\Sigma}_{hh}^{\text{SM},(1)}$,

$$\tilde{c}_1^{(1)} = -\frac{1}{2}y_t^2 v^0, \tag{B.19}$$

$$\tilde{c}_1^{(2)} = \frac{3}{5}y_t^0 v^{-2}, \tag{B.20}$$

$$\tilde{c}_1^{(3)} = \frac{9}{70}y_t^{-2} v^{-4}, \tag{B.21}$$

$$\tilde{c}_1^{(4)} = \frac{2}{35}y_t^{-4} v^{-6}, \tag{B.22}$$

$$\tilde{c}_1^{(5)} = \frac{4}{77}y_t^{-6} v^{-8}. \tag{B.23}$$

Eq. (B.1) allows now to successively derive all corrections induced by the momentum dependence of the non-SM contributions to the hh self-energy. The generated leading logarithms can be resummed easily, since higher derivatives of $\hat{\Sigma}_{hh}^{\text{nonSM}}$ are always suppressed, as noted before. The resummed expression is given in terms of the c coefficients by

$$\Delta_{p^2}^{\text{LL}} = k^2 \frac{c'_{1,1}L_\chi}{1+kc'_{1,1}L_\chi}\left[c^\chi_{1,1}L_\chi + c^A_{1,1}L_A + c^{\tilde{f}}_{1,1}L_S + kc_{2,2}L_S^2\right]. \tag{B.24}$$

A similar expression can be derived at the NLL level. We obtain

$$\begin{aligned}\Delta_{p^2}^{\text{NLL}} =& k^2\frac{1}{(1+kc'_{1,1}L_\chi)^2}\cdot\\ &\cdot\Big[c^\chi_{1,1}c'_{1,0}L_\chi + c^A_{1,1}c'_{1,0}L_A + c^{\tilde{f}}_{1,1}c'_{1,0}L_S + c_{1,0}c'_{1,1}L_\chi\\ &\quad + k\left(c_{1,0}(c'_{1,1})^2L_\chi^2 + c_{2,1}c'_{1,1}L_\chi L_S + c_{2,2}c'_{1,0}L_S^2\right)\\ &\quad + k^2 c_{2,1}(c'_{1,1})^2L_\chi^2 L_S\Big].\end{aligned} \tag{B.25}$$

At the NLL level, however, additional terms proportional to derivatives of the light self-energy exist. Since these derivatives are not suppressed by a heavy mass, it seems not to be possible to resum the corresponding logarithms. Nevertheless, including terms up to the 7-loop order we find a good convergence behaviour and an induced shift of $\mathcal{O}(\pm 2\ \text{GeV}^2)$ to M_h^2 in the parameter region $M_t < M_{\text{heavy}} \lesssim 20$ TeV. The respective shift in M_h is of $\mathcal{O}(50\ \text{MeV})$. We therefore neglect this contribution completely.

At the NNLL level, we take into account only terms proportional to the strong gauge coupling and the top-Yukawa coupling (terms proportional to electroweak gauge couplings are negligible). We find that at this level all terms include derivatives of the SM self-energy. We also find that this contribution to M_h^2 is not negligible, $\mathcal{O}(20\ \text{GeV}^2)$. Therefore, we include terms up to the 7-loop order, which are given by

$$\begin{aligned}\Delta_{p^2}^{\text{NNLL}} =& k^3L_Sc'_{1,0}\left[c_{2,1} - c^{\tilde{f}}_{1,1}\tilde{c}'_1\right]\\ &- k^4L_S^2c'_{1,0}\left[c_{2,2}c'_{1,0} + c_{2,2}\tilde{c}_1^{(1)} - \frac{1}{2}\left(c^{\tilde{f}}_{1,1}\right)^2\tilde{c}_1^{(1)}\right]\\ &+ k^5L_S^3c'_{1,0}\left[c^{\tilde{f}}_{1,1}c_{2,2}\tilde{c}_1^{(2)} - \frac{1}{6}\left(c^{\tilde{f}}_{1,1}\right)^3\tilde{c}_1^{(3)}\right]\\ &+ \frac{1}{2}k^6L_S^4c'_{1,0}\left[\left(c_{2,2}\right)^2\tilde{c}_1^{(2)} - c_{2,2}\left(c^{\tilde{f}}_{1,1}\right)^2\tilde{c}_1^{(3)} + \frac{1}{12}\left(c^{\tilde{f}}_{1,1}\right)^4\tilde{c}_1^{(4)}\right]\\ &- \frac{1}{2}k^7L_S^5c'_{1,0}\left[\left(c_{2,2}\right)^2c^{\tilde{f}}_{1,1}\tilde{c}_1^{(3)} - \frac{1}{3}c_{2,2}\left(c^{\tilde{f}}_{1,1}\right)^3\tilde{c}_1^{(4)} + \frac{1}{60}\left(c^{\tilde{f}}_{1,1}\right)^5\tilde{c}_1^{(5)}\right]\end{aligned}$$

$$+\,\mathcal{O}(k^8), \tag{B.26}$$

where all terms in the c coefficients proportional to g or g' are set to zero. Correspondingly, the derivatives of the light self-energy only include terms proportional to y_t. These are listed in Eqs. (B.19)-(B.23). This loop expansion quickly converges such that we can safely drop higher-order contributions (8-loop and beyond).

We find the electroweak contributions at the NNLL level and even higher-order logarithms (N^nL with $n > 2$) to be completely negligible. Similar expressions can easily be obtained for the non-logarithmic terms of the same origin (see Eq. (10.9)).

Appendix C

Matching conditions

In this Appendix one-loop formulas for matching the various EFTs to each other are provided. All expressions are derived under the assumption that all particles, which are integrated out, have masses equal to the matching scale. The couplings on the right hand side of all following expressions have to be evaluated at the scale given on the left hand side of the corresponding expressions. Couplings not listed do not receive any one-loop contributions to the matching conditions.

Two-loop threshold corrections for the matching of the SM Higgs self-coupling to the MSSM as well as for the matching of the THDM quartic couplings to the MSSM are given in App. C.9.

We will use of the following abbreviations,

$$\hat{A}_t = \frac{A_t}{M_{\mathrm{SUSY}}}, \tag{C.1}$$

$$\hat{\mu} = \frac{\mu}{M_{\mathrm{SUSY}}}, \tag{C.2}$$

$$\hat{X}_t = \hat{A}_t - \hat{\mu}/t_\beta, \tag{C.3}$$

$$\hat{Y}_t = \hat{A}_t + \hat{\mu} t_\beta. \tag{C.4}$$

C.1 Matching the SM to the MSSM

The threshold correction for matching the SM to the MSSM are well known (see e.g. [38]). Therefore, we list here only the for our purpose most important threshold correction of the Higgs self-coupling,

$$\lambda^{\mathrm{SM}}(M_{\mathrm{SUSY}}) = \frac{1}{4}(g^2+g'^2)\cos^2(2\beta) + \Delta_{\mathrm{stop}}\lambda + \Delta_{\mathrm{heavyH}}\lambda + \Delta_{\mathrm{EWino}}\lambda + \Delta_{\overline{\mathrm{DR}}\to\overline{\mathrm{MS}}}\lambda. \tag{C.5}$$

The individual contributions from stops, non-SM Higgs bosons, electroweakinos and from the $\overline{\mathrm{DR}}$ to $\overline{\mathrm{MS}}$ conversion are given by

$$\begin{aligned}
\Delta_{\mathrm{stop}}\lambda =& 6y_t^2 k\left\{\left[y_t^2 + \frac{1}{8}\left(g^2+g'^2\right)c_{2\beta}\right]\hat{X}_t^2 - \frac{1}{12}y_t^2\hat{X}_t^4\right\} \\
&- \frac{1}{4}k y_t^2\left(g^2+g'^2\right)c_{2\beta}^2\hat{X}_t^2, &\text{(C.6)}\\
\Delta_{\mathrm{heavyH}}\lambda =& -\frac{3}{16}k(g'^2+g^2)^2 s_{4\beta}^2, &\text{(C.7)}\\
\Delta_{\mathrm{EWino}}\lambda =& \frac{1}{24}k(c_\beta+s_\beta)^2\left\{-51g^4 - 24g^2g'^2 - 13g'^4\right. \\
&\left. +(3g^2+g'^2)\left[(g^2+g'^2)c_{4\beta} + 2(g^2-g'^2)s_{2\beta}\right]\right\}, &\text{(C.8)}\\
\Delta_{\overline{\mathrm{DR}}\to\overline{\mathrm{MS}}}\lambda =& -k\left[\left(\frac{3}{4}-\frac{1}{6}c_{2\beta}^2\right)g^4 + \frac{1}{2}g^2g'^2 + \frac{1}{4}g'^4\right]. &\text{(C.9)}
\end{aligned}$$

$$\tag{C.10}$$

C.2 Matching the SM+EWinos to the MSSM

The threshold corrections for matching the SM+EWinos to the MSSM are also known (see e.g. [38]). We extend the known expressions for the effective Higgs–Higgsino–Gaugino couplings $\tilde{g}_{1u,1d,2u,2d}$ by including also terms due to the wave-function renormalization of the external Higgs, which have been neglected in [38]. We split up the matching expressions into four pieces,

$$\tilde{g}_{1u}(M_{\text{SUSY}}) = g' s_\beta + \Delta_{\tilde{f}}\tilde{g}_{1u} + \Delta_H \tilde{g}_{1u} + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{1u}, \tag{C.11a}$$
$$\tilde{g}_{2u}(M_{\text{SUSY}}) = g s_\beta + \Delta_{\tilde{f}}\tilde{g}_{2u} + \Delta_H \tilde{g}_{2u} + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{2u}, \tag{C.11b}$$
$$\tilde{g}_{1d}(M_{\text{SUSY}}) = g' c_\beta + \Delta_{\tilde{f}}\tilde{g}_{1d} + \Delta_H \tilde{g}_{1d} + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{1d}, \tag{C.11c}$$
$$\tilde{g}_{2d}(M_{\text{SUSY}}) = g c_\beta + \Delta_{\tilde{f}}\tilde{g}_{2d} + \Delta_H \tilde{g}_{2d} + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{2d}. \tag{C.11d}$$

The sfermion contributions are given by

$$\Delta_{\tilde{f}}\tilde{g}_{1u} = g' s_\beta k \left(-\frac{5}{2}g'^2 + \frac{1}{4}h_t^2(9 - s_\beta^2 \hat{X}_t^2)\right), \tag{C.12a}$$
$$\Delta_{\tilde{f}}\tilde{g}_{2u} = g s_\beta k \left(-\frac{3}{2}g^2 + \frac{1}{4}h_t^2(9 - s_\beta^2 \hat{X}_t^2\right), \tag{C.12b}$$
$$\Delta_{\tilde{f}}\tilde{g}_{1d} = -g' c_\beta k \left(\frac{5}{2}g'^2 + \frac{1}{4}h_t^2 s_\beta^2 \hat{X}_t^2\right), \tag{C.12c}$$
$$\Delta_{\tilde{f}}\tilde{g}_{2d} = -g c_\beta k \left(\frac{3}{2}g^2 + \frac{1}{4}h_t^2 s_\beta^2 \hat{X}_t^2\right). \tag{C.12d}$$

Note that the new wave-function renormalization contributions proportional to $\hat{X}_t^2$ have been already implemented in `FeynHiggs` from version `2.13.0` on.

Integrating out the heavy Higgs yields

$$\Delta_H \tilde{g}_{1u} = \frac{1}{16} g' s_\beta k \left(21 g^2 c_\beta^2 + g'^2(-2 + 7c_\beta^2)\right), \tag{C.13a}$$
$$\Delta_H \tilde{g}_{2u} = \frac{1}{16} g s_\beta k \left(-g^2(2 + 11c_\beta^2) + 7g'^2 c_\beta^2\right), \tag{C.13b}$$
$$\Delta_H \tilde{g}_{1d} = \frac{1}{16} g' c_\beta k \left(21 g^2 s_\beta^2 + g'^2(-2 + 7s_\beta^2)\right), \tag{C.13c}$$
$$\Delta_H \tilde{g}_{2d} = \frac{1}{16} g c_\beta k \left(-g^2(2 + 11s_\beta^2) + 7g'^2 s_\beta^2\right). \tag{C.13d}$$

Changing the regularization scheme from DRED for $Q > M_{\text{SUSY}}$ to DREG for $Q < M_{\text{SUSY}}$ gives rise to

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{1u} = -\frac{1}{8} g' s_\beta k(3g^2 + g'^2), \tag{C.14a}$$
$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{2u} = \frac{1}{24} g s_\beta k(23g^2 - 3g'^2), \tag{C.14b}$$
$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{1d} = -\frac{1}{8} g' c_\beta k(3g^2 + g'^2), \tag{C.14c}$$
$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\tilde{g}_{2d} = \frac{1}{24} g c_\beta k(23g^2 - 3g'^2). \tag{C.14d}$$

The threshold correction of λ is equivalent to the expression if matching the SM to the MSSM. Only the electroweakino contribution $\Delta_{\text{EWino}}\lambda$ has to be removed. It reappears at the matching scale between SM and SM plus electroweakinos.

C.3 Matching the SM to the SM+EWinos

Integrating out the electroweakinos generates the following one-loop matching condition for the Higgs self-coupling,

$$\begin{aligned}\lambda^{\mathrm{SM}}(M_\chi) =&\lambda^{\mathrm{SM+EWinos}}(M_\chi)\\ &+ k\left\{-\frac{7}{12}(\tilde{g}_{1d}^4+\tilde{g}_{1d}^4)-\frac{9}{4}(\tilde{g}_{2d}^4+\tilde{g}_{1u}^4)-\frac{3}{2}\tilde{g}_{1d}^2g_{1u}^2-\frac{7}{2}g_{2d}^2g_{2u}^2\right.\\ &-\frac{8}{3}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-\frac{7}{6}(\tilde{g}_{1d}^2\tilde{g}_{2d}^2+\tilde{g}_{1u}^2\tilde{g}_{2u}^2)-\frac{1}{6}(\tilde{g}_{1d}^2\tilde{g}_{2u}^2+\tilde{g}_{1u}^2\tilde{g}_{2d}^2)\\ &-\frac{4}{3}(\tilde{g}_{1d}\tilde{g}_{2u}^2+\tilde{g}_{1u}\tilde{g}_{2d})(\tilde{g}_{1d}\tilde{g}_{2d}+\tilde{g}_{1u}\tilde{g}_{2u})\\ &+\frac{2}{3}\tilde{g}_{1d}\tilde{g}_{1u}(\lambda_\chi-2\tilde{g}_{1d}^2-2\tilde{g}_{1u}^2)+2\tilde{g}_{2d}\tilde{g}_{2u}(\lambda_\chi-2\tilde{g}_{2d}^2-2\tilde{g}_{2u}^2)\\ &\left.+\frac{1}{3}\lambda_\chi(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)+\lambda_\chi(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)\right\}. \end{aligned} \tag{C.15}$$

Also the top-Yukawa coupling is affected,

$$\begin{aligned}y_t^{\mathrm{SM}}(M_\chi) = y_t^{\mathrm{SM+EWinos}}(M_\chi)\left\{1-k\left[\frac{1}{6}\tilde{g}_{1u}\tilde{g}_{1d}+\frac{1}{12}(\tilde{g}_{1u}^2+\tilde{g}_{1d}^2)\right.\right.\\ \left.\left.+\ \frac{1}{2}\tilde{g}_{2u}\tilde{g}_{2d}+\frac{1}{4}(\tilde{g}_{2u}^2+\tilde{g}_{2d}^2)\right]\right\}.\end{aligned} \tag{C.16}$$

Assuming that all electroweakinos are mass-degenerate, the matching conditions of the gauge couplings receive no one-loop correction.

C.4 Matching the SM to the THDM

The SM Higgs self-coupling is obtained in terms of the λ_i of the THDM by

$$\lambda^{\mathrm{SM}}(M_A) =\lambda_{\mathrm{tree}}+\Delta\lambda \tag{C.17}$$

with

$$\lambda_{\mathrm{tree}} =\lambda_1 c_\beta^4+\lambda_2 s_\beta^4+2(\lambda_3+\lambda_4+\lambda_5)c_\beta^2 s_\beta^2+4\lambda_6 c_\beta^3 s_\beta+4\lambda_7 c_\beta s_\beta^3, \tag{C.18}$$

$$\begin{aligned}\Delta\lambda =& -3k\left\{(\lambda_6+\lambda_7)c_{2\beta}+(\lambda_6-\lambda_7)c_{4\beta}\right.\\ &\left.-\left(\lambda_1 c_\beta^2-\lambda_2 s_\beta^2-(\lambda_3+\lambda_4+\lambda_5)c_{2\beta}\right)s_{2\beta}\right\}^2.\end{aligned} \tag{C.19}$$

Plugging in the tree-level expressions for the λ_i from the matching of the THDM to the MSSM, we recover the heavy Higgs contribution to the matching condition of the SM Higgs self-coupling to the full MSSM given in Eq. (C.7).

The top-Yukawa coupling of the SM y_t is obtained from the one of the THDM h_t by

$$y_t^{\mathrm{SM}}(M_A) =(h_t s_\beta+h_t' c_\beta)\left[1-\frac{3}{8}k\left(h_t c_\beta-h_t' s_\beta\right)^2\right]. \tag{C.20}$$

This correction corresponds to the heavy Higgs contribution to the threshold of the top-Yukawa coupling when matching the SM to the MSSM given in Eq. (24) of [38].

C.5 Matching the THDM to the MSSM

At tree-level the Higgs self-couplings of the THDM are given by

$$\lambda_{1,\mathrm{tree}}(M_{\mathrm{SUSY}}) = \lambda_{2,\mathrm{tree}}(M_{\mathrm{SUSY}}) =\frac{1}{4}(g^2+g'^2), \tag{C.21a}$$

$$\lambda_{3,\mathrm{tree}}(M_{\mathrm{SUSY}}) =\frac{1}{4}(g^2-g'^2), \tag{C.21b}$$

$$\lambda_{4,\text{tree}}(M_{\text{SUSY}}) = -\frac{1}{2}g^2, \tag{C.21c}$$

$$\lambda_{5,\text{tree}}(M_{\text{SUSY}}) = \lambda_{6,\text{tree}}(M_{\text{SUSY}}) = \lambda_{7,\text{tree}}(M_{\text{SUSY}}) = 0. \tag{C.21d}$$

At one-loop level corrections arise from integrating out the stops, EWinos, as well as from the transition from $\overline{\text{DR}}$ to $\overline{\text{MS}}$. We split up the stop contribution into one part originating from vertex corrections and one part originating from the wave function renormalization (WFR) of the Higgs fields,

$$\lambda_i(M_{\text{SUSY}}) = \lambda_{i,\text{tree}} + \Delta_{\text{Ver.Cor.}}\lambda_i + \Delta_{\text{WFR}}\lambda_i + \Delta_{\text{EWinos}}\lambda_i + \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda_i. \tag{C.22}$$

The stop contributions have originally been calculated by [85] but are listed here for completeness. The vertex corrections from box and triangle diagrams are given by

$$\Delta_{\text{Ver.Cor.}}\lambda_1 = -\frac{1}{2}kh_t^4\hat{\mu}^4 + \frac{3}{4}k(g^2+g'^2)h_t^2\hat{\mu}^2, \tag{C.23a}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_2 = 6kh_t^4\hat{A}_t^2\left(1-\frac{1}{12}\hat{A}_t^2\right) - \frac{3}{4}(g^2+g'^2)h_t^2\hat{A}_t^2, \tag{C.23b}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_3 = \frac{1}{2}k\hat{\mu}^2h_t^4(3-\hat{A}_t^2) - \frac{3}{8}k(g^2-g'^2)h_t^2(\hat{A}_t^2-\hat{\mu}^2), \tag{C.23c}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_4 = \frac{1}{2}k\hat{\mu}^2h_t^4(3-\hat{A}_t^2) + \frac{3}{4}kg^2h_t^2(\hat{A}_t^2-\hat{\mu}^2), \tag{C.23d}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_5 = -\frac{1}{2}h_t^4\hat{\mu}^2\hat{A}_t^2, \tag{C.23e}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_6 = \frac{1}{2}kh_t^4\hat{\mu}^3\hat{A}_t - \frac{3}{8}k(g^2+g'^2)h_t^2\hat{\mu}\hat{A}_t, \tag{C.23f}$$

$$\Delta_{\text{Ver.Cor.}}\lambda_7 = \frac{1}{2}kh_t^4\hat{\mu}\hat{A}_t(\hat{A}_t^2-6) + \frac{3}{8}k(g^2+g'^2)h_t^2\hat{\mu}\hat{A}_t, \tag{C.23g}$$

whereas the WFR corrections read

$$\Delta_{\text{WFR}}\lambda_1 = -2(\hat{\Sigma}'_{11}\lambda_1 + \hat{\Sigma}'_{12}\lambda_6), \tag{C.24a}$$

$$\Delta_{\text{WFR}}\lambda_2 = -2(\hat{\Sigma}'_{22}\lambda_2 + \hat{\Sigma}'_{12}\lambda_7), \tag{C.24b}$$

$$\Delta_{\text{WFR}}\lambda_3 = -(\hat{\Sigma}'_{11} + \hat{\Sigma}'_{22})\lambda_3 - \hat{\Sigma}'_{12}(\lambda_6+\lambda_7), \tag{C.24c}$$

$$\Delta_{\text{WFR}}\lambda_4 = -(\hat{\Sigma}'_{11} + \hat{\Sigma}'_{22})\lambda_4 - \hat{\Sigma}'_{12}(\lambda_6+\lambda_7), \tag{C.24d}$$

$$\Delta_{\text{WFR}}\lambda_5 = -(\hat{\Sigma}'_{11} + \hat{\Sigma}'_{22})\lambda_5 - \hat{\Sigma}'_{12}(\lambda_6+\lambda_7), \tag{C.24e}$$

$$\Delta_{\text{WFR}}\lambda_6 = -\frac{1}{2}(3\hat{\Sigma}'_{11} + \hat{\Sigma}'_{22})\lambda_6 - \frac{1}{2}\hat{\Sigma}'_{12}(\lambda_1+\lambda_3+\lambda_4+\lambda_5), \tag{C.24f}$$

$$\Delta_{\text{WFR}}\lambda_7 = -\frac{1}{2}(\hat{\Sigma}'_{11} + 3\hat{\Sigma}'_{22})\lambda_7 - \frac{1}{2}\hat{\Sigma}'_{12}(\lambda_2+\lambda_3+\lambda_4+\lambda_5), \tag{C.24g}$$

where the $\hat{\Sigma}'_{ij} = \left(\frac{\partial}{\partial p^2}\hat{\Sigma}_{\phi_i\phi_j}\right)|_{p^2=0}$ are given by

$$\hat{\Sigma}'_{11} = \frac{1}{2}kh_t^2\hat{\mu}^2, \tag{C.25a}$$

$$\hat{\Sigma}'_{22} = \frac{1}{2}kh_t^2\hat{A}_t^2, \tag{C.25b}$$

$$\hat{\Sigma}'_{12} = -\frac{1}{2}kh_t^2\hat{A}_t\hat{\mu}. \tag{C.25c}$$

The scheme change from $\overline{\text{DR}}$ to $\overline{\text{MS}}$ yields the additional contributions

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda_{1,2} = -\frac{1}{12}k(7g^4 + 6g^2g'^2 + 3g'^4), \tag{C.26a}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda_3 = -\frac{1}{12}k(7g^4 - 6g^2g'^2 + 3g'^4), \tag{C.26b}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda_4 = -\frac{1}{3}kg^2(g^2+3g'^2), \tag{C.26c}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\lambda_{5,6,7} = 0, \tag{C.26d}$$

which have already been calculated in [119].

The EWino corrections can be obtained by replacing the effective Higgs–Higgsino–Gaugino couplings $\hat{g}_{1uu,1ud,..}$ in the expression for matching the THDM to the THDM+EWinos given below by their tree-level values.

Due to the wave-function renormalization, also β receives a threshold correction,

$$\beta_{\text{THDM}} = \beta_{\text{MSSM}} - \frac{1}{2}\hat{\Sigma}'_{hH}(0). \tag{C.27}$$

$\hat{\Sigma}'_{hH}(0)$ receives corrections from sfermions and EWinos,

$$\hat{\Sigma}'_{hH}(0)_{\tilde{f}} = -\frac{1}{4}kh_t^2 s_{2\beta}(\hat{A}_t - \hat{\mu}/t_\beta)(\hat{A}_t + \hat{\mu}t_\beta), \tag{C.28}$$

$$\hat{\Sigma}'_{hH}(0)_{\text{EWino}} = \frac{1}{6}k(3g^2 + {g'}^2)c_{2\beta}. \tag{C.29}$$

Only when taking into account this threshold, the well known one-loop matching condition of λ (when matching the SM to the MSSM, see Eq. (C.5)) can be recovered from Eq. (C.17) considering the limit $M_A \to M_{\text{SUSY}}$.

The top-Yukawa couplings are obtained at the one-loop level via

$$\begin{aligned} h_t^{\text{THDM}}(M_{\text{SUSY}}) =& h_t\Bigg\{1 + k\Bigg[\frac{4}{3}g_3^2(1-\hat{A}_t) + h_t^2\Big(\mathcal{F}_5(\hat{\mu}) - \frac{1}{4}\hat{A}_t^2\Big) \\ &+ g^2\left(\mathcal{F}_1(\hat{\mu}) - \frac{3}{8}\right) + {g'}^2\left(\mathcal{F}_3(\hat{\mu}) - \frac{1}{72}\right)\Bigg]\Bigg\}, \end{aligned} \tag{C.30}$$

$$(h_t')^{\text{THDM}}(M_{\text{SUSY}}) = h_t k\left\{\frac{4}{3}g_3^2\hat{\mu} + \frac{1}{4}h_t^2\hat{A}_t\hat{\mu} + g^2\mathcal{F}_2(\hat{\mu}) + {g'}^2\mathcal{F}_4(\hat{\mu})\right\}. \tag{C.31}$$

The appearing functions are given by

$$\mathcal{F}_1(\hat{\mu}) = \frac{3}{16(1-\hat{\mu}^2)^2}\Big[7 - 4\hat{\mu}^2 - 3\hat{\mu}^4 + 2\hat{\mu}^2(8-3\hat{\mu}^2)\ln\hat{\mu}^2\Big], \tag{C.32a}$$

$$\mathcal{F}_2(\hat{\mu}) = \frac{3\hat{\mu}^2}{2(1-\hat{\mu}^2)^2}\Big[1 - \hat{\mu}^2 + \ln\hat{\mu}^2\Big], \tag{C.32b}$$

$$\mathcal{F}_3(\hat{\mu}) = \frac{1}{144(1-\hat{\mu}^2)^2}\Big[(55 - 32\hat{A}_t\hat{\mu} + 51\hat{\mu}^2)(1-\hat{\mu}^2) + 2\hat{\mu}^2(72 - 16\hat{A}_t\hat{\mu} - 19\hat{\mu}^2)\ln\hat{\mu}^2\Big], \tag{C.32c}$$

$$\mathcal{F}_4(\hat{\mu}) = \frac{\hat{\mu}^2}{18(1-\hat{\mu}^2)^2}\Big[13(1-\hat{\mu}^2) + (9+4\hat{\mu}^2)\ln\hat{\mu}^2\Big], \tag{C.32d}$$

$$\mathcal{F}_5(\hat{\mu}) = \frac{3}{8(1-\hat{\mu}^2)^2}\Big[-1 + 4\hat{\mu}^2 - 3\hat{\mu}^4 + 2\hat{\mu}^4\ln\hat{\mu}^2\Big], \tag{C.32e}$$

with

$$\mathcal{F}_1(0) = \frac{21}{16}, \qquad \mathcal{F}_1(1) = -\frac{3}{4}, \tag{C.33a}$$

$$\mathcal{F}_2(0) = 0, \qquad \mathcal{F}_2(1) = -\frac{3}{4}, \tag{C.33b}$$

$$\mathcal{F}_3(0) = \frac{55}{144}, \qquad \mathcal{F}_3(1) = -\frac{1}{36}(9+4\hat{A}_t), \tag{C.33c}$$

$$\mathcal{F}_4(0) = 0, \qquad \mathcal{F}_4(1) = -\frac{5}{36}, \tag{C.33d}$$

$$\mathcal{F}_5(0) = -\frac{3}{8}, \qquad \mathcal{F}_5(1) = 0 \tag{C.33e}$$

as limiting values.

C.6 Matching the THDM to the THDM+EWinos

We again split up the matching conditions for the Higgs self-couplings into a piece due to vertex corrections and a piece due to wave-function renormalization,

$$\lambda_i^{\text{THDM}}(M_\chi) = \lambda_i^{\text{THDM+EWinos}} + \Delta_{\text{Ver.Cor.}}\lambda_i + \Delta_{\text{WFR}}\lambda_i. \tag{C.34}$$

The vertex corrections from box and triangle diagrams read

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\lambda_1 = &-\frac{1}{12}k\Big[7\hat{g}_{1dd}^4 + 16\hat{g}_{1dd}^3\hat{g}_{1du} + 2\hat{g}_{1dd}^2(9\hat{g}_{1du}^2 + 7\hat{g}_{2dd}^2 + 8\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2du}^2)\\ &+ 16\hat{g}_{1dd}\hat{g}_{1du}\Big(\hat{g}_{1du}^2 + (\hat{g}_{2dd} + \hat{g}_{2du})^2\Big) + 7\hat{g}_{1du}^4\\ &+ 2\hat{g}_{1du}^2(\hat{g}_{2dd}^2 + 8\hat{g}_{2dd}\hat{g}_{2du} + 7\hat{g}_{2du}^2)\\ &+ 3(\hat{g}_{2dd} + \hat{g}_{2du})^2(9\hat{g}_{2dd}^2 - 2\hat{g}_{2dd}\hat{g}_{2du} + 9\hat{g}_{2du}^2)\Big],\end{aligned} \tag{C.35a}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\lambda_2 = &-\frac{1}{12}k\Big[7\hat{g}_{1ud}^4 + 16\hat{g}_{1ud}^3\hat{g}_{1uu} + 2\hat{g}_{1ud}^2(9\hat{g}_{1uu}^2 + 7\hat{g}_{2ud}^2 + 8\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{2uu}^2)\\ &+ 16\hat{g}_{1ud}\hat{g}_{1uu}\Big(\hat{g}_{1uu}^2 + (\hat{g}_{2ud} + \hat{g}_{2uu})^2\Big) + 7\hat{g}_{1uu}^4\\ &+ 2\hat{g}_{1uu}^2(\hat{g}_{2ud}^2 + 8\hat{g}_{2ud}\hat{g}_{2uu} + 7\hat{g}_{2uu}^2)\\ &+ 3(\hat{g}_{2ud} + \hat{g}_{2uu})^2(9\hat{g}_{2ud}^2 - 2\hat{g}_{2ud}\hat{g}_{2uu} + 9\hat{g}_{2uu}^2)\Big],\end{aligned} \tag{C.35b}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\lambda_3 = &-\frac{1}{12}k\Big[\hat{g}_{1dd}^2(7\hat{g}_{1ud}^2 + 8\hat{g}_{1ud}\hat{g}_{1uu} + 7\hat{g}_{1uu}^2 + 10\hat{g}_{2ud}^2 + 8\hat{g}_{2ud}\hat{g}_{2uu} + 4\hat{g}_{2uu}^2)\\ &+ 2\hat{g}_{1dd}\Big(2\hat{g}_{1du}(2\hat{g}_{1ud}^2 + \hat{g}_{1ud}\hat{g}_{1uu} + 2\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + \hat{g}_{2ud}\hat{g}_{2uu} + 2\hat{g}_{2uu}^2)\\ &- 3(\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} - 3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud})\Big)\\ &+ \hat{g}_{1du}^2(7\hat{g}_{1ud}^2 + 8\hat{g}_{1ud}\hat{g}_{1uu} + 7\hat{g}_{1uu}^2 + 4\hat{g}_{2ud}^2 + 8\hat{g}_{2ud}\hat{g}_{2uu} + 10\hat{g}_{2uu}^2)\\ &- 6\hat{g}_{1du}(-3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu})\\ &+ 10\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 8\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2du} + 4\hat{g}_{1ud}^2\hat{g}_{2du}^2 + 8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}^2\\ &+ 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}^2 + 4\hat{g}_{1uu}^2\hat{g}_{2dd}^2 + 8\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2du}\\ &+ 10\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 27\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 24\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} + 27\hat{g}_{2dd}^2\hat{g}_{2uu}^2\\ &+ 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2 - 12\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2\\ &+ 27\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 24\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu} + 27\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big],\end{aligned} \tag{C.35c}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\lambda_4 = &-\frac{1}{12}k\Big[\hat{g}_{1dd}^2(7\hat{g}_{1ud}^2 + 8\hat{g}_{1ud}\hat{g}_{1uu} + 4\hat{g}_{1uu}^2 - 5\hat{g}_{2ud}^2 - 4\hat{g}_{2ud}\hat{g}_{2uu} - 2\hat{g}_{2uu}^2)\\ &+ 2\hat{g}_{1dd}\Big(\hat{g}_{1du}(4\hat{g}_{1ud}^2 + 5\hat{g}_{1ud}\hat{g}_{1uu} + 4\hat{g}_{1uu}^2 - 2\hat{g}_{2ud}^2 - \hat{g}_{2ud}\hat{g}_{2uu} - 2\hat{g}_{2uu}^2)\\ &+ 3\big(\hat{g}_{1ud}(2\hat{g}_{2dd} + \hat{g}_{2du})(2\hat{g}_{2ud} + \hat{g}_{2uu})\\ &+ \hat{g}_{1uu}(2\hat{g}_{2dd}\hat{g}_{2ud} + 4\hat{g}_{2dd}\hat{g}_{2uu} - \hat{g}_{2du}\hat{g}_{2ud} + 2\hat{g}_{2du}\hat{g}_{2uu})\big)\Big)\\ &+ \hat{g}_{1du}^2(4\hat{g}_{1ud}^2 + 8\hat{g}_{1ud}\hat{g}_{1uu} + 7\hat{g}_{1uu}^2 - 2\hat{g}_{2ud}^2 - 4\hat{g}_{2ud}\hat{g}_{2uu} - 5\hat{g}_{2uu}^2)\\ &+ 6\hat{g}_{1du}\Big(\hat{g}_{1ud}(2\hat{g}_{2dd}\hat{g}_{2ud} - \hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{2du}\hat{g}_{2ud} + 2\hat{g}_{2du}\hat{g}_{2uu})\\ &+ \hat{g}_{1uu}(\hat{g}_{2dd} + 2\hat{g}_{2du})(\hat{g}_{2ud} + 2\hat{g}_{2uu})\Big) - 5\hat{g}_{1ud}^2\hat{g}_{2dd}^2 - 4\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2du}\\ &- 2\hat{g}_{1ud}^2\hat{g}_{2du}^2 - 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}^2 - 2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} - 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}^2\\ &- 2\hat{g}_{1uu}^2\hat{g}_{2dd}^2 - 4\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2du} - 5\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 27\hat{g}_{2dd}^2\hat{g}_{2ud}^2\\ &+ 24\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} + 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2 + 42\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\ &+ 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2 + 24\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu} + 27\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big],\end{aligned} \tag{C.35d}$$

$$\begin{aligned}
\Delta_{\text{Ver.Cor.}}\lambda_5 = &-\frac{1}{12}k\Big[\hat{g}_{1dd}^2(7\hat{g}_{1ud}^2+8\hat{g}_{1ud}\hat{g}_{1uu}-2\hat{g}_{1uu}^2+2\hat{g}_{2ud}^2+4\hat{g}_{2ud}\hat{g}_{2uu}-\hat{g}_{2uu}^2)\\
&+2\hat{g}_{1dd}\Big(\hat{g}_{1du}(4\hat{g}_{1ud}^2+11\hat{g}_{1ud}\hat{g}_{1uu}+4\hat{g}_{1uu}^2+2\hat{g}_{2ud}^2+7\hat{g}_{2ud}\hat{g}_{2uu}+2\hat{g}_{2uu}^2)\\
&+\hat{g}_{1ud}\big(\hat{g}_{2dd}(5\hat{g}_{2ud}+2\hat{g}_{2uu})+2\hat{g}_{2du}(\hat{g}_{2ud}+\hat{g}_{2uu})\big)\\
&+\hat{g}_{1uu}\big(\hat{g}_{2dd}(2\hat{g}_{2ud}-\hat{g}_{2uu})+2\hat{g}_{2du}(\hat{g}_{2ud}+\hat{g}_{2uu})\big)\Big)\\
&+\hat{g}_{1du}^2(-2\hat{g}_{1ud}^2+8\hat{g}_{1ud}\hat{g}_{1uu}+7\hat{g}_{1uu}^2-\hat{g}_{2ud}^2+4\hat{g}_{2ud}\hat{g}_{2uu}+2\hat{g}_{2uu}^2)\\
&+2\hat{g}_{1du}\Big(\hat{g}_{1ud}\big(2\hat{g}_{2dd}(\hat{g}_{2ud}+\hat{g}_{2uu})-\hat{g}_{2du}(\hat{g}_{2ud}-2\hat{g}_{2uu})\big)\\
&+\hat{g}_{1uu}\big(2\hat{g}_{2dd}(\hat{g}_{2ud}+\hat{g}_{2uu})+\hat{g}_{2du}(2\hat{g}_{2ud}+5\hat{g}_{2uu})\big)\Big)+2\hat{g}_{1ud}^2\hat{g}_{2dd}^2\\
&+4\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2du}-\hat{g}_{1ud}^2\hat{g}_{2du}^2+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}^2+14\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}\\
&+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}^2-\hat{g}_{1uu}^2\hat{g}_{2dd}^2+4\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2du}+2\hat{g}_{1uu}^2\hat{g}_{2du}^2\\
&+27\hat{g}_{2dd}^2\hat{g}_{2ud}^2+24\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu}-6\hat{g}_{2dd}^2\hat{g}_{2uu}^2+24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2\\
&+54\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2-6\hat{g}_{2du}^2\hat{g}_{2ud}^2\\
&+24\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}+27\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big],
\end{aligned} \tag{C.35e}$$

$$\begin{aligned}
\Delta_{\text{Ver.Cor.}}\lambda_6 = &-\frac{1}{12}k\Big[\hat{g}_{1dd}^3(7\hat{g}_{1ud}+4\hat{g}_{1uu})+\hat{g}_{1dd}^2(12\hat{g}_{1du}\hat{g}_{1ud}+9\hat{g}_{1du}\hat{g}_{1uu}\\
&+7\hat{g}_{2dd}\hat{g}_{2ud}+4\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{2du}\hat{g}_{2uu})\\
&+\hat{g}_{1dd}\Big(3\hat{g}_{1du}^2(3\hat{g}_{1ud}+4\hat{g}_{1uu})+8\hat{g}_{1du}(\hat{g}_{2dd}+\hat{g}_{2du})(\hat{g}_{2ud}+\hat{g}_{2uu})\\
&+(\hat{g}_{2dd}+\hat{g}_{2du})\big(\hat{g}_{1ud}(7\hat{g}_{2dd}+\hat{g}_{2du})+4\hat{g}_{1uu}(\hat{g}_{2dd}+\hat{g}_{2du})\big)\Big)\\
&+\hat{g}_{1du}^3(4\hat{g}_{1ud}+7\hat{g}_{1uu})+\hat{g}_{1du}^2(\hat{g}_{2dd}\hat{g}_{2ud}+4\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{2du}\hat{g}_{2ud}\\
&+7\hat{g}_{2du}\hat{g}_{2uu})+\hat{g}_{1du}(\hat{g}_{2dd}+\hat{g}_{2du})\Big(4\hat{g}_{1ud}(\hat{g}_{2dd}+\hat{g}_{2du})\\
&+\hat{g}_{1uu}(\hat{g}_{2dd}+7\hat{g}_{2du})\Big)+3(\hat{g}_{2dd}+\hat{g}_{2du})\Big(\hat{g}_{2dd}^2(9\hat{g}_{2ud}+4\hat{g}_{2uu})\\
&+3\hat{g}_{2dd}\hat{g}_{2du}(\hat{g}_{2ud}+\hat{g}_{2uu})+\hat{g}_{2du}^2(4\hat{g}_{2ud}+9\hat{g}_{2uu})\Big)\Big],
\end{aligned} \tag{C.35f}$$

$$\begin{aligned}
\Delta_{\text{Ver.Cor.}}\lambda_7 = &-\frac{1}{12}k\Big[\hat{g}_{1dd}\Big(7\hat{g}_{1ud}^3+12\hat{g}_{1ud}^2\hat{g}_{1uu}+\hat{g}_{1ud}(9\hat{g}_{1uu}^2+7\hat{g}_{2ud}^2+8\hat{g}_{2ud}\hat{g}_{2uu}+\hat{g}_{2uu}^2)\\
&+4\hat{g}_{1uu}\big(\hat{g}_{1uu}^2+(\hat{g}_{2ud}+\hat{g}_{2uu})^2\big)\Big)\\
&+\hat{g}_{1du}\Big(4\hat{g}_{1ud}^3+9\hat{g}_{1ud}^2\hat{g}_{1uu}+4\hat{g}_{1ud}\big(3\hat{g}_{1uu}^2+(\hat{g}_{2ud}+\hat{g}_{2uu})^2\big)\\
&+\hat{g}_{1uu}(7\hat{g}_{1uu}^2+\hat{g}_{2ud}^2+8\hat{g}_{2ud}\hat{g}_{2uu}+7\hat{g}_{2uu}^2)\Big)+7\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}\\
&+4\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{1ud}^2\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1ud}^2\hat{g}_{2du}\hat{g}_{2uu}+8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud}\\
&+8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}+8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}+8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\\
&+\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2ud}+4\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2ud}+7\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu}\\
&+27\hat{g}_{2dd}\hat{g}_{2ud}^3+36\hat{g}_{2dd}\hat{g}_{2ud}^2\hat{g}_{2uu}+21\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2+12\hat{g}_{2dd}\hat{g}_{2uu}^3\\
&+12\hat{g}_{2du}\hat{g}_{2ud}^3+21\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}+36\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}^2+27\hat{g}_{2du}\hat{g}_{2uu}^3\Big].
\end{aligned} \tag{C.35g}$$

The WFR corrections are identical to those listed in Eqs. (C.24a)-(C.24g), but with

$$\hat{\Sigma}'_{11} = -\frac{1}{6}k\Big[(\hat{g}_{1dd}+\hat{g}_{1du})^2+3(\hat{g}_{2dd}+\hat{g}_{2du})^2\Big] \tag{C.36a}$$

$$\hat{\Sigma}'_{22} = -\frac{1}{6}k\Big[(\hat{g}_{1uu}+\hat{g}_{1ud})^2+3(\hat{g}_{2uu}+\hat{g}_{2ud})^2\Big] \tag{C.36b}$$

$$\hat{\Sigma}'_{12} = -\frac{1}{6}k\Big[(\hat{g}_{1uu}+\hat{g}_{1ud})(\hat{g}_{1dd}+\hat{g}_{1du})+3(\hat{g}_{2uu}+\hat{g}_{2ud})(\hat{g}_{2dd}+\hat{g}_{2du})\Big] \tag{C.36c}$$

The matching conditions of the top-Yukawa coupling are purely due to wave-function renormalization,

$$h_t^{\text{THDM}}(M_\chi) = h_t^{\text{THDM+EWinos}} - \frac{1}{2}h_t\hat{\Sigma}'_{22} - \frac{1}{2}h'_t\hat{\Sigma}'_{12}, \tag{C.37a}$$

$$(h'_t)^{\text{THDM}}(M_\chi) = (h'_t)^{\text{THDM+EWinos}} - \frac{1}{2}h'_t\hat{\Sigma}'_{11} - \frac{1}{2}h_t\hat{\Sigma}'_{12}. \tag{C.37b}$$

The threshold correction of β reads

$$\beta_{\text{THDM}}(M_A) = \beta_{\text{THDM+EWinos}} - \frac{1}{2}\hat{\Sigma}'_{hH}(0) \tag{C.38}$$

with

$$\hat{\Sigma}'_{hH}(0) = s_\beta c_\beta\big(\hat{\Sigma}'_{11} - \hat{\Sigma}'_{22}\big) - c_{2\beta}\hat{\Sigma}'_{12} \tag{C.39}$$

In the limit of $M_\chi \to M_{\text{SUSY}}$, we cross-checked the threshold corrections of $\lambda_{1..7}$ against the expressions given in [119] and found agreement.

C.7 Matching the SM+EWinos to the THDM+EWinos

Matching the SM+EWinos to the THDM+EWinos, the thresholds for the SM Higgs self-coupling as well as the top-Yukawa couplings are the same as in the case of matching the SM to the THDM (see Section C.4), since no corresponding unsuppressed diagrams containing heavy Higgs as well as EWinos exist.

We split up the matching condition of the effective Higgs–Higgsino–Gaugino couplings into a part due to vertex corrections and another one due to wave-function renormalization,

$$\tilde{g}_i(M_A) = \tilde{g}_{i,\text{tree}} + \Delta_{\text{Ver.Cor.}}\tilde{g}_i + \Delta_{\text{WFR}}\tilde{g}_i. \tag{C.40}$$

The vertex corrections are given by

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\tilde{g}_{2u} =& \frac{1}{2}(\hat{g}_{2ud}c_\beta - \hat{g}_{2dd}s_\beta)\Big[(\hat{g}_{2dd}\hat{g}_{2uu} - \hat{g}_{1dd}\hat{g}_{1uu})c_\beta^2 \\ &+ (\hat{g}_{1dd}\hat{g}_{1du} - \hat{g}_{1ud}\hat{g}_{1uu} - \hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu})s_\beta c_\beta \\ &+ (\hat{g}_{1du}\hat{g}_{1ud} - \hat{g}_{2du}\hat{g}_{2ud})s_\beta^2\Big],\end{aligned} \tag{C.41a}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\tilde{g}_{2d} =& \frac{1}{2}(\hat{g}_{2uu}c_\beta - \hat{g}_{2du}s_\beta)\Big[(\hat{g}_{2du}\hat{g}_{2ud} - \hat{g}_{1du}\hat{g}_{1ud})c_\beta^2 \\ &+ (\hat{g}_{1dd}\hat{g}_{1du} - \hat{g}_{1ud}\hat{g}_{1uu} - \hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu})s_\beta c_\beta \\ &+ (\hat{g}_{1uu}\hat{g}_{1dd} - \hat{g}_{2dd}\hat{g}_{2uu})s_\beta^2\Big],\end{aligned} \tag{C.41b}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\tilde{g}_{1u} =& \frac{1}{2}(\hat{g}_{1ud}c_\beta - \hat{g}_{1dd}s_\beta)\Big[-(\hat{g}_{1uu}\hat{g}_{1dd} + 3\hat{g}_{2uu}\hat{g}_{2dd})c_\beta^2 \\ &+ (\hat{g}_{1dd}\hat{g}_{1du} - \hat{g}_{1uu}\hat{g}_{1ud} - 3\hat{g}_{2uu}\hat{g}_{2ud} + 3\hat{g}_{2dd}\hat{g}_{2du})s_\beta c_\beta \\ &+ (\hat{g}_{1du}\hat{g}_{1ud} + 3\hat{g}_{2du}\hat{g}_{2ud})s_\beta^2\Big],\end{aligned} \tag{C.41c}$$

$$\begin{aligned}\Delta_{\text{Ver.Cor.}}\tilde{g}_{1d} =& \frac{1}{2}(\hat{g}_{1uu}c_\beta - \hat{g}_{1du}s_\beta)\Big[-(\hat{g}_{1du}\hat{g}_{1ud} + 3\hat{g}_{2du}\hat{g}_{2ud})c_\beta^2 \\ &+ (\hat{g}_{1dd}\hat{g}_{1du} - \hat{g}_{1uu}\hat{g}_{1ud} - 3\hat{g}_{2uu}\hat{g}_{2ud} + 3\hat{g}_{2dd}\hat{g}_{2du})s_\beta c_\beta \\ &+ (\hat{g}_{1uu}\hat{g}_{1dd} + 3\hat{g}_{2uu}\hat{g}_{2dd})s_\beta^2\Big].\end{aligned} \tag{C.41d}$$

The wave-function renormalization contributions read

$$\Delta_{\text{WFR}}\tilde{g}_{2u} = -\frac{1}{16}(\hat{g}_{2uu}s_\beta + \hat{g}_{2du}c_\beta)\Big[(\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2)c_\beta^2$$

$$
\begin{aligned}
&\quad - 2(\hat{g}_{1uu}\hat{g}_{1du} + 2\hat{g}_{2dd}\hat{g}_{2ud} + 5\hat{g}_{2uu}\hat{g}_{2du})s_\beta c_\beta \\
&\quad + (\hat{g}_{1du}^2 + 2\hat{g}_{2dd}^2 + 5\hat{g}_{2du}^2)s_\beta^2\Big], \qquad \text{(C.42a)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\text{WFR}}\tilde{g}_{2d} = &- \frac{1}{16}(\hat{g}_{2dd}c_\beta + \hat{g}_{2ud}s_\beta)\Big[(\hat{g}_{1ud}^2 + 5\hat{g}_{2ud}^2 + 2\hat{g}_{2uu}^2)c_\beta^2 \\
&- 2(\hat{g}_{1dd}\hat{g}_{1ud} + 5\hat{g}_{2dd}\hat{g}_{2ud} + 2\hat{g}_{2uu}\hat{g}_{2du})s_\beta c_\beta \\
&+ (\hat{g}_{1dd}^2 + 5\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2)s_\beta^2\Big], \qquad \text{(C.42b)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\text{WFR}}\tilde{g}_{1u} = &- \frac{1}{16}(\hat{g}_{1uu}s_\beta + \hat{g}_{1du}c_\beta)\Big[(3\hat{g}_{1uu}^2 + 2\hat{g}_{1ud}^2 + 3\hat{g}_{2uu}^2)c_\beta^2 \\
&- 2(2\hat{g}_{1dd}\hat{g}_{1ud} + 3\hat{g}_{1uu}\hat{g}_{1du} + 3\hat{g}_{2uu}\hat{g}_{2du})s_\beta c_\beta \\
&+ (2\hat{g}_{1dd}^2 + 3\hat{g}_{1du}^2 + 3\hat{g}_{2du}^2)s_\beta^2\Big], \qquad \text{(C.42c)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\text{WFR}}\tilde{g}_{1d} = &- \frac{1}{16}(\hat{g}_{1dd}c_\beta + \hat{g}_{1ud}s_\beta)\Big[(2\hat{g}_{1uu}^2 + 3\hat{g}_{1ud}^2 + 3\hat{g}_{2ud}^2)c_\beta^2 \\
&- 2(3\hat{g}_{1dd}\hat{g}_{1ud} + 2\hat{g}_{1uu}\hat{g}_{1du} + 3\hat{g}_{2dd}\hat{g}_{2ud})s_\beta c_\beta \\
&+ (3\hat{g}_{1dd}^2 + 2\hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2)s_\beta^2\Big]. \qquad \text{(C.42d)}
\end{aligned}
$$

C.8 Matching the THDM+EWinos to the MSSM

The threshold corrections for β and λ_i are obtained by taking the respective ones from the matching of the THDM to the MSSM but removing the EWino contributions.

The matching conditions of the effective Higgs–Higgsino–Gaugino couplings, only receive corrections due to sfermions, given by the expressions (at the scale M_{SUSY})

$$\Delta_{\tilde{f}}\hat{g}_{1uu} = g'k\left(-\frac{5}{2}{g'}^2 + \frac{1}{4}h_t^2(9 - \hat{A}_t^2)\right), \qquad \text{(C.43a)}$$

$$\Delta_{\tilde{f}}\hat{g}_{2uu} = gk\left(-\frac{3}{2}g^2 + \frac{1}{4}h_t^2(9 - \hat{A}_t^2)\right), \qquad \text{(C.43b)}$$

$$\Delta_{\tilde{f}}\hat{g}_{1dd} = -g'k\left(\frac{5}{2}{g'}^2 + \frac{1}{4}h_t^2\hat{\mu}^2\right), \qquad \text{(C.43c)}$$

$$\Delta_{\tilde{f}}\hat{g}_{2dd} = -gk\left(\frac{3}{2}g^2 + \frac{1}{4}h_t^2\hat{\mu}^2\right), \qquad \text{(C.43d)}$$

and

$$\Delta_{\tilde{f}}\hat{g}_{1ud} = g' \cdot \frac{1}{4}kh_t^2\hat{A}_t\hat{\mu}, \qquad \text{(C.44a)}$$

$$\Delta_{\tilde{f}}\hat{g}_{2ud} = g \cdot \frac{1}{4}kh_t^2\hat{A}_t\hat{\mu}, \qquad \text{(C.44b)}$$

$$\Delta_{\tilde{f}}\hat{g}_{1du} = g' \cdot \frac{1}{4}kh_t^2\hat{A}_t\hat{\mu}, \qquad \text{(C.44c)}$$

$$\Delta_{\tilde{f}}\hat{g}_{2du} = g \cdot \frac{1}{4}kh_t^2\hat{A}_t\hat{\mu}. \qquad \text{(C.44d)}$$

In the limit $M_A \to M_{\text{SUSY}}$, we recover the corresponding matching conditions of the SM+EWinos to the MSSM, given in Eqs. (C.12a) and (C.12d) only if correctly taking into account the threshold corrections of $\tan\beta$.

The corrections due to the change of the regularization scheme read

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{1uu} = -\frac{1}{8}g'k(3g^2 + {g'}^2), \qquad \text{(C.45a)}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{2uu} = \frac{1}{24}gk(23g^2 - 3{g'}^2), \qquad \text{(C.45b)}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{1dd} = -\frac{1}{8}g'k(3g^2 + {g'}^2), \qquad \text{(C.45c)}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{2dd} = \frac{1}{24}gk(23g^2 - 3g'^2), \tag{C.45d}$$

$$\Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{1du} = \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{1ud} = \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{2du} = \Delta_{\overline{\text{DR}}\to\overline{\text{MS}}}\hat{g}_{2ud} = 0. \tag{C.45e}$$

C.9 Two-loop threshold corrections

C.9.1 Matching the SM to the MSSM

Here, we give the $\mathcal{O}(\alpha_s\alpha_t, \alpha_t^2)$ threshold corrections valid under the assumption of $M_A = M_{\text{SUSY}}$. Note that these expressions also apply to the matching of the SM plus electroweakinos to the MSSM, since the additional electroweakino couplings in the low-energy EFT are zero in the limit of vanishing electroweak gauge couplings. We assume that the one-loop threshold corrections are expressed in terms of the SM top-Yukawa coupling.

The $\mathcal{O}(\alpha_s\alpha_t)$ threshold correction is given by [38]

$$\Delta_{\alpha_s\alpha_t}\lambda = \frac{8}{3}k^2 g_3^2 y_t^4 \left(-12\hat{X}_t - 6\hat{X}_t^2 + 14\hat{X}_t^3 + \frac{1}{2}\hat{X}_t^4 - \hat{X}_t^5\right). \tag{C.46}$$

This expression is valid if $M_{\tilde{g}} = M_{\text{SUSY}}$. If instead $M_{\tilde{g}} \ll M_{\text{SUSY}}$ (e.g. in the case of the SM plus gluino as EFT), the threshold correction reads [38]

$$\Delta_{\alpha_s\alpha_t}^{\text{low } M_{\tilde{g}}}\lambda = -8k^2 g_3^2 y_t^4 \left(3 - 2\hat{X}_t^2 + \frac{1}{6}\hat{X}_t^4\right). \tag{C.47}$$

The $\mathcal{O}(\alpha_t^2)$ threshold correction is given by [40]

$$\begin{aligned}
\Delta_{\alpha_t^2}\lambda = 3k^2\frac{y_t^6}{s_\beta^2}\Big\{ & \frac{1}{2} + 6\hat{\mu}^2 - (8+2\hat{\mu}^2)f_1(\hat{\mu}) + 3\hat{\mu}^2 f_2(\hat{\mu}) + 4f_3(\hat{\mu}) - \frac{1}{2}\hat{X}_t^6 s_\beta^2 \\
& + \hat{X}_t^2\left(-7 - 6\hat{\mu}^2 + 4f_1(\hat{\mu}^2) - 6\hat{\mu}^2 f_1(\hat{\mu}) - 4f_2(\hat{\mu}) - 6\hat{\mu}^2 f_2(\hat{\mu})\right) \\
& + \frac{1}{2}\hat{X}_t^4\left(11 + 2\hat{\mu}^2 - f_1(\hat{\mu}) + 2\hat{\mu}^2 f_1(\hat{\mu}) + f_2(\hat{\mu}) + \hat{\mu}^2 f_2(\hat{\mu})\right) \\
& + c_\beta^2\Big[-\frac{13}{2} + 60K + \pi^2 + \hat{X}_t^2(15-24K) - \frac{25}{4}\hat{X}_t^4 - \hat{X}_t\hat{Y}_t(12+64K) \\
& \qquad + \hat{X}_t^3\hat{Y}_t(4+16K) - \hat{Y}_t^2(3+16K) \\
& \qquad + \hat{X}_t^2\hat{Y}_t^2\left(\frac{14}{3} + 24K\right) - \hat{X}_t^4\hat{Y}_t^2\left(\frac{19}{12} + 8K\right)\Big]\Big\},
\end{aligned} \tag{C.48}$$

with

$$f_1(\hat{\mu}) = \frac{\hat{\mu}^2}{1-\hat{\mu}^2}\ln\hat{\mu}^2, \tag{C.49}$$

$$f_2(\hat{\mu}) = \frac{1}{1-\hat{\mu}^2}\left(1 + \frac{\hat{\mu}^2}{1-\hat{\mu}^2}\ln\hat{\mu}^2\right), \tag{C.50}$$

$$f_3(\hat{\mu}) = \frac{1-2\hat{\mu}^2-2\hat{\mu}^4}{(1-\hat{\mu}^2)^2}\left[\frac{\pi^2}{6} + \hat{\mu}^2\ln\hat{\mu}^2 - \ln\hat{\mu}^2\ln\left(1-\hat{\mu}^2\right) + Li_2(\hat{\mu}^2)\right], \tag{C.51}$$

$$K = -\frac{1}{\sqrt{3}}\int_0^{\pi/6} dx\,\ln(2\cos x) \simeq -0.1953256. \tag{C.52}$$

Li_2 is the dilogarithm function.

C.9.2 Matching the THDM to the MSSM

For deriving the $\mathcal{O}(\alpha_s\alpha_t)$ threshold corrections for the quartic couplings λ_i, we follow the strategy outlined in [39]. As the authors of [39] pointed out, the $\mathcal{O}(\alpha_s\alpha_t)$ threshold corrections do not depend on $\tan\beta$. Therefore, they can be extracted from the threshold correction for the SM quartic coupling λ in the case $M_A \sim M_{\text{SUSY}}$ from matching the SM to the MSSM (see Eq. (C.46)) by selecting the coefficients of the various β-dependent terms according to Eq. (11.58) and Eq. (11.18).

Expressing the one-loop threshold corrections in terms of the MSSM $\overline{\text{DR}}$ renormalized top-Yukawa coupling h_t^{MSSM}, the two-loop $\mathcal{O}(\alpha_s\alpha_t)$ threshold correction for λ at M_{SUSY} given in Eq. (11.58) is modified,

$$\Delta^{h_t}_{\alpha_s\alpha_t}\lambda = -\frac{4}{3}k^2 g_3^2 h_t^4 s_\beta^4 \hat{X}_t\big(24 - 12\hat{X}_t - 4\hat{X}_t^2 + \hat{X}_t^3\big), \tag{C.53}$$

where we used the superscript "h_t" to clarify that this threshold correction is valid, if the corresponding $\mathcal{O}(\alpha_t)$ threshold correction is expressed in terms of the MSSM top-Yukawa coupling h_t.

Inserting $\hat{X}_t$ from Eq. (C.3) and selecting the terms proportional to $(c_\beta^4, s_\beta^4, c_\beta^2 s_\beta^2, c_\beta^3 s_\beta, c_\beta s_\beta^3)$ yields

$$\Delta_{\alpha_s\alpha_t}\lambda_1 = -\frac{4}{3}k^2 g_3^2 h_t^4 \hat{\mu}^4, \tag{C.54a}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_2 = 16k^2 g_3^2 h_t^4\left(-2\hat{A}_t + \hat{A}_t^2 + \frac{1}{3}\hat{A}_t^3 - \frac{1}{12}\hat{A}_t^4\right), \tag{C.54b}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_{345} = 8k^2 g_3^2 h_t^4 \hat{\mu}^2\left(1 + \hat{A}_t - \frac{1}{2}\hat{A}_t^2\right), \tag{C.54c}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_6 = \frac{4}{3}k^2 g_3^2 h_t^4 \hat{\mu}^3\left(-1 + \hat{A}_t\right), \tag{C.54d}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_7 = 4k^2 g_3^2 h_t^4 \hat{\mu}\left(2 - 2\hat{A}_t - \hat{A}_t^2 + \frac{1}{3}\hat{A}_t^3\right), \tag{C.54e}$$

where $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$. These expressions are valid under assumption of $M_{\tilde{g}} = M_{\text{SUSY}}$.

In the case $M_{\tilde{g}} \ll M_{\text{SUSY}}$, the $\mathcal{O}(\alpha_s\alpha_t)$ threshold correction of λ between the SM and the MSSM reads (assuming that the one-loop threshold correction is expressed in terms of the MSSM top-Yukawa coupling),

$$\Delta^{h_t,\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda = -\frac{8}{3}k^2 g_3^2 h_t^4 s_\beta^4\big(9 - 12\hat{X}_t + \hat{X}_t^4\big). \tag{C.55}$$

Selecting again the terms proportional to $(c_\beta^4, s_\beta^4, c_\beta^2 s_\beta^2, c_\beta^3 s_\beta, c_\beta s_\beta^3)$ yields

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_1 = -\frac{8}{3}k^2 g_3^2 h_t^4 \hat{\mu}^4, \tag{C.56a}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_2 = -\frac{8}{3}k^2 g_3^2 h_t^4\left(9 - 12\hat{A}_t^2 + \hat{A}_t^4\right), \tag{C.56b}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_{345} = 8k^2 g_3^2 h_t^4 \hat{\mu}^2\left(2 - \hat{A}_t^2\right), \tag{C.56c}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_6 = \frac{8}{3}k^2 g_3^2 h_t^4 \hat{A}_t \hat{\mu}^3, \tag{C.56d}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_7 = -\frac{8}{3}k^2 g_3^2 h_t^4 \hat{\mu}\left(6 - \hat{A}_t^2\right). \tag{C.56e}$$

Using this method, we get only an information about the sum λ_{345}, leaving thus some some arbitrariness. We follow the arrangement in [39], assigning

$$\Delta_{\alpha_s\alpha_t}\lambda_3 = \frac{1}{2}\Delta_{\alpha_s\alpha_t}\lambda_{345}, \tag{C.57a}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_4 = \frac{1}{2}\Delta_{\alpha_s\alpha_t}\lambda_{345}, \tag{C.57b}$$

$$\Delta_{\alpha_s\alpha_t}\lambda_5 = 0, \tag{C.57c}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_3 = \frac{1}{2}\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_{345}, \tag{C.57d}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_4 = \frac{1}{2}\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_{345}, \tag{C.57e}$$

$$\Delta^{\text{low } M_{\tilde{g}}}_{\alpha_s\alpha_t}\lambda_5 = 0. \tag{C.57f}$$

Other possible distributions yield numerical very similar results.

Appendix D

Difference in field normalization

In this Appendix, we give explicit formulas for the difference of the field normalization between MSSM and THDM fields valid up to $\mathcal{O}(M_t/M_{\rm SUSY})$, $\mathcal{O}(M_A/M_{\rm SUSY})$ and $\mathcal{O}(M_t/M_A)$. They are used in Section 11.2 for fixing the finite part of the field renormalization constants.

The contribution from sfermions is given by

$$\Delta_{\tilde{f}}\Sigma'_{11} = \frac{1}{2}kh_t^2\hat{\mu}^2, \tag{D.1}$$

$$\Delta_{\tilde{f}}\Sigma'_{12} = -\frac{1}{2}kh_t^2\hat{A}_t\hat{\mu}, \tag{D.2}$$

$$\Delta_{\tilde{f}}\Sigma'_{22} = \frac{1}{2}kh_t^2\hat{A}_t^2. \tag{D.3}$$

The contribution from electroweakinos reads

$$\Delta_\chi\Sigma'_{11} = -\frac{1}{6}k\left(3g^2+{g'}^2\right)\left(1+3\ln\frac{M_\chi^2}{\hat{Q}^2}\right), \tag{D.4}$$

$$\Delta_\chi\Sigma'_{12} = -\frac{1}{6}k\left(3g^2+{g'}^2\right), \tag{D.5}$$

$$\Delta_\chi\Sigma'_{22} = -\frac{1}{6}k\left(3g^2+{g'}^2\right)\left(1+3\ln\frac{M_\chi^2}{\hat{Q}^2}\right). \tag{D.6}$$

In addition, also all non SUSY particles, i.e. the particles of the THDM, yield a contribution if the renormalization scales of the THDM and the MSSM are not equal,

$$\Delta_{\rm THDM}\Sigma'_{11} = -\frac{1}{2}k\left(3g^2+{g'}^2\right)\ln\frac{\widehat{Q}^2}{\widetilde{Q}^2}, \tag{D.7}$$

$$\Delta_{\rm THDM}\Sigma'_{12} = 0, \tag{D.8}$$

$$\Delta_{\rm THDM}\Sigma'_{22} = -\frac{1}{2}k\left(3g^2+{g'}^2\right)\ln\frac{\widehat{Q}^2}{\widetilde{Q}^2} + 3kh_t^2\ln\frac{\widehat{Q}^2}{\widetilde{Q}^2}, \tag{D.9}$$

with $\widehat{Q}$ being the renormalization scale of the MSSM and $\widetilde{Q}$ the scale of the THDM.

Appendix E

Scheme conversion of stop parameters: full one-loop formulas

In this Appendix, we list the formulas necessary to convert the parameters of the stop sector from the OS to the $\overline{\mathrm{DR}}$ scheme. Building upon the expressions given in [17, 37], we extend those to the case of $M_A \neq M_S$.

First, we give expression for calculating the $\overline{\mathrm{DR}}$ top-quark mass of the MSSM in terms of the OS top-quark mass,

$$\begin{aligned}\left(m_t^{\overline{\mathrm{DR}}}\right)^2(Q) =M_t^2\Bigg\{1 &- \frac{8}{3}kg_3^2\left[5 + 3\ln\frac{Q^2}{m_t^2} + \ln\frac{M_S^2}{Q^2} - \hat{X}_t\right] \\ &+ \frac{3}{2}k\frac{y_t^2}{s_\beta^2}\left[c_\beta^2\left(\frac{1}{2} - \ln\frac{M_A^2}{Q^2}\right) + s_\beta^2\left(\frac{8}{3} + \ln\frac{Q^2}{m_t^2}\right.\right. \\ &\left.\left. - \ln\frac{M_S^2}{Q^2} + \frac{1}{2} - \hat{\mu}^2 f_2(\hat{\mu})\right]\right\}. \qquad \text{(E.1)}\end{aligned}$$

with M_S being the stop mass scale ($M_S^2 \equiv m_{\tilde{t}_1}m_{\tilde{t}_2}$). For the conversion of this stop mass scale, we obtain

$$\begin{aligned}\left(M_S^{\overline{\mathrm{DR}}}\right)^2(Q) = \left(M_S^{\mathrm{OS}}\right)^2\Bigg\{1 &- \frac{16}{3}kg_3^2\left[2 - \ln\frac{M_S^2}{Q^2}\right] \\ &+ \frac{3}{4}ky_t^2\Bigg[\frac{2}{t_\beta^2}\hat{M}_A^2\ln\frac{M_S^2}{M_A^2} + \frac{2}{t_\beta^2}\hat{M}_A^2\left(1 - \ln\frac{M_S^2}{Q^2}\right) \\ &+ \frac{1}{t_\beta^2}\hat{Y}_t^2\left(\hat{M}_A^2\ln\frac{M_S^2}{M_A^2} + (4 - \hat{M}_A^2)f_A(\hat{M}_A) + 4 - 2\ln\frac{M_S^2}{Q^2}\right) \\ &+ 4\hat{X}_t^2\left(1 - \frac{1}{2}\ln\frac{M_S^2}{Q^2}\right) \\ &+ \frac{2}{s_\beta^2}\Bigg(\hat{\mu}^4\ln\hat{\mu}^2 + (1 - \hat{\mu}^2)\left(3 - 2\ln\frac{M_S^2}{Q^2}\right) \\ &- (1 - \hat{\mu}^2)^2\ln(1 - \hat{\mu}^2)\Bigg)\Bigg]\Bigg\}, \qquad \text{(E.2)}\end{aligned}$$

and for the conversion of the stop mixing parameter,

$$\begin{aligned}X_t^{\overline{\mathrm{DR}}}(Q) =M_S^{\mathrm{OS}}\Bigg\{\hat{X}_t^{\mathrm{OS}} &+ \frac{4}{3}kg_3^2\left[8 + 5\hat{X}_t - \hat{X}_t^2 + 3\hat{X}_t L\right] \\ &+ \frac{1}{4}ky_t^2\Bigg[\frac{6}{t_\beta^2}\hat{Y}_t\left(\hat{M}_A^2\ln\frac{M_S^2}{M_A^2} + (4 - \hat{M}_A^2)f_A(\hat{M}_A) + 2\ln\frac{M_S^2}{Q^2} - 4\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{t_\beta^2}\hat{X}_t \ln\frac{M_S^2}{M_A^2} + \frac{1}{2}\hat{X}_t\bigg(35 - 6\ln\frac{M_S^2}{m_t^2} - 24\ln\frac{M_S^2}{Q^2} + \frac{24}{s_\beta^2}\Big(1 - \ln\frac{M_S^2}{Q^2}\Big)\bigg)\\
&- \frac{6}{s_\beta^2}\hat{X}_t\bigg(1 - \hat{\mu}^2 + \frac{1}{2}f_2(\hat{\mu}) + \hat{\mu}^4\ln\hat{\mu}^2 + (1-\hat{\mu}^4)\ln(1-\hat{\mu}^2)\bigg)\\
&+ \frac{3}{t_\beta^2}\hat{X}_t\hat{Y}_t^2\bigg((1-\hat{M}_A^2)\ln\frac{M_S^2}{M_A^2} - (3-\hat{M}_A^2)f_A(\hat{M}_A) - 2\bigg)\\
&+ \hat{X}_t^3\bigg(3\ln\frac{M_S^2}{m_t^2} - 4\ln 2 - 6\ln|\hat{X}_t|\bigg)\bigg]\bigg\}.
\end{aligned} \tag{E.3}
$$

The appearing loop function f_A depending on $\hat{M}_A \equiv M_A/M_S$ is defined by

$$
f_A(\hat{M}_A) = \frac{\hat{M}_A}{\sqrt{4-\hat{M}_A^2}}\left[\arctan\left(\frac{\hat{M}_A\sqrt{4-\hat{M}_A^2}}{2-\hat{M}_A^2}\right) - \pi\right] \tag{E.4}
$$

with the limiting values

$$
f_A(0) = 0, \qquad f_A(1) = -\frac{2}{3\sqrt{3}}\pi\,. \tag{E.5}
$$

Appendix F

Renormalization group equations

We use the following abbreviations,

$$t \equiv \ln(Q^2), \tag{F.1}$$

$$k \equiv \frac{1}{16\pi^2}. \tag{F.2}$$

k is used to keep track of the loop order. For the convention used for the normalization of λ_i and v_i, see Eqs. (2.3) and (2.5). All RGEs are given at the two-loop order[1] in the form

$$\frac{dg_i}{dt} = \beta_{g_i} = k\beta^{(1)}_{g_i} + k^2\beta^{(2)}_{g_i}, \tag{F.3}$$

with g_i being a generic coupling. The notation $\langle a; b\rangle$ indicates that a is used for $Q < M_{\tilde{g}}$ and b for $Q > M_{\tilde{g}}$.

All RGEs have been derived using the `Mathematica` package `SARAH` [111].

F.1 Standard Model

The beta functions of the SM are given by

$$\beta_{g'} = g'^3 k\left[\frac{41}{12} + \frac{5}{6}k\left(\frac{44}{5}g_3^2 + \frac{27}{10}g^2 + \frac{199}{30}g'^2 - \frac{17}{10}y_t^2\right)\right], \tag{F.4a}$$

$$\beta_g = g^3 k\left[-\frac{19}{12} + \frac{1}{2}k\left(12g_3^2 + \frac{35}{6}g^2 + \frac{3}{2}g'^2 - \frac{3}{2}y_t^2\right)\right], \tag{F.4b}$$

$$\beta_{g_3} = \frac{1}{2}g_3^3 k\left[-\langle 7;5\rangle + k\left(\langle -26;+22\rangle g_3^2 - 2y_t^2 + \frac{9}{2}g^2 + \frac{11}{6}g'^2\right)\right], \tag{F.4c}$$

$$\begin{aligned}\beta_{y_t} = \frac{1}{2}y_t k\Bigg\{&\frac{9}{2}y_t^2 - 8g_3^2 - \frac{9}{4}g^2 - \frac{17}{12}g'^2\\ &+k\left[y_t^2\left(-12y_t^2 - 6\lambda + 36g_3^2 + \frac{225}{16}g^2 + \frac{131}{16}g'^2\right) + \frac{3}{2}\lambda^2 - \langle 108; \frac{284}{3}\rangle g_3^4\right.\\ &\left.-\frac{23}{4}g^4 + \frac{1187}{216}g'^4 + 9g_3^2g^2 + \frac{19}{9}g_3^2g'^2 - \frac{3}{4}g^2g'^2\right]\Bigg\},\end{aligned} \tag{F.4d}$$

$$\begin{aligned}\beta_\lambda = k\Bigg\{&6\left(\lambda^2 + \lambda y_t^2 - y_t^4\right) - \lambda\left(\frac{9}{2}g^2 + \frac{3}{2}g'^2\right) + \frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2g'^2\\ &+k^2\left[\frac{1}{2}\lambda^2\left(-\frac{156}{2}\lambda - 72y_t^2 + 54g^2 + \frac{54}{3}g'^2\right)\right.\\ &+\lambda y_t^2\left(-\frac{3}{2}y_t^2 + 40g_3^2 + \frac{45}{4}g^2 + \frac{85}{12}g'^2\right) + \lambda\left(-\frac{73}{16}g^4 + \frac{629}{48}g'^4 + \frac{39}{8}g^2g'^2\right)\end{aligned}$$

[1] For the SM, we list also the three-loop RGEs in the limit of vanishing electroweak gauge couplings.

$$+2y_t^4\left(15y_t^2-16g_3^2-\frac{4}{3}g'^2\right)+2y_t^2\left(-\frac{9}{8}g^4-\frac{19}{8}g'^4+\frac{21}{4}g^2g'^2\right)$$
$$+\frac{305}{16}g^6-\frac{379}{48}g'^6-\frac{289}{48}g^4g'^2-\frac{559}{48}g^2g'^4\Big]\Big\}. \tag{F.4e}$$

As explained in Section 8.3, in case of $\mathcal{O}(\alpha_s,\alpha_t)$ NNLL resummation also the three-loop RGEs in the limit of vanishing electroweak gauge couplings have to be taken into account. The corresponding three-loop beta-functions are given by

$$\beta_{g_3}^{(3)}=\frac{1}{2}g_3^3\left[y_t^2\left(15y_t^2-40g_3^2\right)+\frac{65}{2}g_3^4\right], \tag{F.5a}$$

$$\beta_{y_t}^{(3)}=\frac{1}{2}y_t\Big[y_t^4\left(58.6028y_t^2+99\lambda-157g_3^2\right)+\frac{1}{2}\lambda y_t^2\left(\frac{15}{8}\lambda+16g_3^2\right)$$
$$+353.765g_3^4y_t^2-\frac{9}{2}\lambda^3-619.35g_3^6\Big], \tag{F.5b}$$

$$\beta_{\lambda}^{(3)}=\frac{1}{4}\lambda^3\left(\frac{6011.35}{2}\lambda+873y_t^2\right)+\frac{1}{2}\lambda^2y_t^2\left(1768.26y_t^2+160.77g_3^2\right)$$
$$-\lambda y_t^4\left(223.382y_t^2+662.866g_3^2\right)+356.968\lambda y_t^2g_3^4$$
$$-2y_t^6\left(243.149y_t^2-250.494g_3^2\right)-100.402y_t^4g_3^4. \tag{F.5c}$$

Note that these three-loop beta-functions are only valid for $Q<M_{\tilde{g}}$.

F.2 Split model

The one and two-loop beta functions of the split model (see effective Lagrangian in Eq. (8.3)) are given by

$$\beta_{g'}=\frac{15}{4}kg'^31+\frac{5}{6}k^2g'^3\Big[-\frac{3}{20}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)+\frac{18}{5}g^2-\frac{9}{20}(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)$$
$$+\frac{44}{5}g_3^2+\frac{104}{15}g'^2-\frac{17}{10}y_t^2\Big], \tag{F.6a}$$

$$\beta_g=-\frac{7}{12}kg^3+\frac{1}{2}k^2g^3\Big[-\frac{1}{4}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)+\frac{106}{3}g^2-\frac{11}{4}(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)$$
$$+12g_3^2+2g'^2-\frac{3}{2}y_t^2\Big], \tag{F.6b}$$

$$\beta_{g_3}=-\frac{1}{2}\langle 7;5\rangle kg_3^3+\frac{1}{2}k^2g_3^3\left[\frac{9}{2}g^2-\langle 26;-22\rangle g_3^2+\frac{11}{6}g'^2-2y_t^2\right], \tag{F.6c}$$

$$\beta_{y_t}=\frac{1}{2}ky_t\left[-\frac{9}{4}g^2-8g_3^2-\frac{17}{12}g'^2+\frac{9}{2}y_t^2+\frac{1}{2}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+3\tilde{g}_{2d}^2+3\tilde{g}_{2u}^2)\right]$$
$$+\frac{1}{2}k^2y_t\Big\{-12y_t^4-\frac{9}{16}\tilde{g}_{1d}^4-\frac{5}{4}\tilde{g}_{1d}^2\tilde{g}_{1u}^2-\frac{9}{16}\tilde{g}_{1u}^4-\frac{17}{4}g^4-\frac{9}{8}\tilde{g}_{1d}^2\tilde{g}_{2d}^2-\frac{45}{16}\tilde{g}_{2d}^4$$
$$-3\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-\frac{9}{8}\tilde{g}_{1u}^2\tilde{g}_{2u}^2-\frac{3}{4}\tilde{g}_{2d}^2\tilde{g}_{2u}^2-\frac{45}{16}\tilde{g}_{2u}^4-\langle 108;\frac{284}{3}\rangle g_3^4$$
$$+g^2\left(\frac{15}{16}\tilde{g}_{1d}^2+\frac{15}{16}\tilde{g}_{1u}^2+\frac{165}{16}\tilde{g}_{2d}^2+\frac{165}{16}\tilde{g}_{2u}^2+9g_3^2\right)$$
$$+\frac{5}{3}g'^2\left(\frac{3}{16}\tilde{g}_{1d}^2+\frac{3}{16}\tilde{g}_{1u}^2-\frac{9}{20}g^2+\frac{9}{16}\tilde{g}_{2d}^2+\frac{9}{16}\tilde{g}_{2u}^2+\frac{19}{15}g_3^2\right)+\frac{1303}{216}g'^4+\frac{3}{2}\lambda^2$$
$$+y_t^2\left(-\frac{9}{8}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)+\frac{225}{16}g^2-\frac{27}{8}(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)+36g_3^2+\frac{131}{16}g'^2-6\lambda\right)\Big\}, \tag{F.6d}$$

$$\beta_\lambda=\frac{1}{2}k\Big[-\tilde{g}_{1d}^4-\tilde{g}_{1u}^4+\frac{9}{4}g^4-5\tilde{g}_{2d}^4-4\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-5\tilde{g}_{2u}^4-2(\tilde{g}_{1u}^2+\tilde{g}_{2d}^2)(\tilde{g}_{1d}^2+\tilde{g}_{2u}^2)$$

$$
\begin{aligned}
&+\frac{3}{2}g^2g'^2+\frac{3}{4}g'^4-12y_t^4+2(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+3\tilde{g}_{2d}^2+3\tilde{g}_{2u}^2)\lambda\\
&-9\left(g^2+\frac{1}{3}g'^2\right)\lambda+12y_t^2\lambda+12\lambda^2\Big]\\
&+\frac{1}{2}k^2\left\{\frac{209}{8}g^6+\frac{1}{2}\left[5(\tilde{g}_{1d}^6+\tilde{g}_{1u}^6)+21(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)\tilde{g}_{2d}^2\tilde{g}_{2u}^2+19\tilde{g}_{1d}^2\tilde{g}_{1u}^2(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)\right.\right.\\
&+7\tilde{g}_{2d}^2\tilde{g}_{2u}^2(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2)+17(\tilde{g}_{1d}^4\tilde{g}_{1u}^2+\tilde{g}_{1d}^2\tilde{g}_{1u}^4+\tilde{g}_{1d}^4\tilde{g}_{2d}^2+\tilde{g}_{1u}^4\tilde{g}_{2u}^2)\\
&+11(\tilde{g}_{1d}^2\tilde{g}_{2d}^4+\tilde{g}_{1u}^2\tilde{g}_{2d}^4)+47(\tilde{g}_{2d}^6+\tilde{g}_{2u}^6)\\
&\left.+\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}(42(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2)+38(\tilde{g}_{2d}^2+\tilde{g}_{2u}^2))\right]\\
&-\frac{137}{8}g'^6-64g_3^2y_t^4-\frac{16}{3}g'^2y_t^4+60y_t^6\\
&+g^4\left[-\frac{3}{4}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+51\tilde{g}_{2d}^2+51\tilde{g}_{2u}^2)-\frac{385}{24}g'^2-\frac{9}{2}y_t^2\right]\\
&+\frac{25}{9}g'^4\left[-\frac{9}{100}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+3\tilde{g}_{2d}^2+3\tilde{g}_{2u}^2)-\frac{171}{50}y_t^2\right]\\
&+g^2\left[-\frac{197}{8}g'^4-4(\tilde{g}_{1d}^2\tilde{g}_{2d}^2+5\tilde{g}_{2d}^4+2\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}+\tilde{g}_{1u}^2\tilde{g}_{2u}^2+2\tilde{g}_{2d}^2\tilde{g}_{2u}^2+5\tilde{g}_{2u}^4)\right.\\
&\left.+\frac{5}{3}g'^2\left(-\frac{3}{10}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2-21\tilde{g}_{2d}^2-21\tilde{g}_{2u}^2)+\frac{63}{5}y_t^2\right)\right]\\
&+\lambda\left[\frac{47}{8}g^4+\frac{223}{8}g'^4+80g_3^2y_t^2-3y_t^4\right.\\
&-\frac{1}{4}(\tilde{g}_{1d}^4+\tilde{g}_{1u}^4+5\tilde{g}_{2d}^4+2\tilde{g}_{1d}^2(\tilde{g}_{2d}^2-6\tilde{g}_{1u}^2)\\
&-80\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}+2\tilde{g}_{1u}^2\tilde{g}_{2u}^2+44\tilde{g}_{2d}^2\tilde{g}_{2u}^2+5\tilde{g}_{2u}^4)\\
&+\frac{5}{3}g'^2\left(\frac{3}{4}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+3\tilde{g}_{2d}^2+3\tilde{g}_{2u}^2)+\frac{17}{2}y_t^2\right)\\
&\left.+g^2\left(\frac{15}{4}(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+11\tilde{g}_{2d}^2+11\tilde{g}_{2u}^2)+\frac{39}{4}g'^2+\frac{45}{2}y_t^2\right)\right]\\
&\left.+\lambda^2\left[54g^2-12(\tilde{g}_{1d}^2+\tilde{g}_{1u}^2+3\tilde{g}_{2d}^2+3\tilde{g}_{2u}^2)+18g'^2-72y_t^2\right]-78\lambda^3\right\},
\end{aligned}\tag{F.6e}
$$

$$
\begin{aligned}
\beta_{\tilde{g}_{1u}}=&\frac{1}{2}k\left[3\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}+\tilde{g}_{1u}\left(\frac{5}{4}\tilde{g}_{1u}^2+2\tilde{g}_{1d}^2+\frac{9}{4}\tilde{g}_{2u}^2+\frac{3}{2}\tilde{g}_{2d}^2-\frac{9}{4}g^2-\frac{3}{4}g'^2+3y_t^2\right)\right]\\
&+\frac{1}{2}k^2\left\{-\frac{3}{4}\tilde{g}_{1u}^5-\frac{9}{4}\tilde{g}_{1d}^3\tilde{g}_{2d}\tilde{g}_{2u}-6\tilde{g}_{1d}\tilde{g}_{1u}^2\tilde{g}_{2d}\tilde{g}_{2u}+\frac{51}{4}\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}g^2-\frac{33}{4}\tilde{g}_{1d}\tilde{g}_{2d}^3\tilde{g}_{2u}\right.\\
&-\frac{9}{2}\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}^3+\frac{3}{4}\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}g'^2-9\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}y_t^2-3\tilde{g}_{1d}\tilde{g}_{2d}\tilde{g}_{2u}\lambda\\
&+\tilde{g}_{1u}^3\left(-\frac{15}{4}\tilde{g}_{1d}^2+\frac{165}{32}g^2-\frac{27}{16}\tilde{g}_{1d}^2-\frac{9}{16}\tilde{g}_{2u}^2+\frac{103}{32}g'^2-\frac{27}{8}y_t^2-3\lambda\right)\\
&+\tilde{g}_{1u}\left[-\frac{9}{4}\tilde{g}_{1d}^4-\frac{17}{4}g^2-\frac{75}{16}\tilde{g}_{1d}^2\tilde{g}_{2d}^2-\frac{45}{16}\tilde{g}_{2d}^4-\frac{75}{16}\tilde{g}_{1d}^2\tilde{g}_{2u}^2-\frac{21}{8}\tilde{g}_{2d}^2\tilde{g}_{2u}^2-\frac{99}{16}\tilde{g}_{2u}^4\right.\\
&+g^2\left(\frac{39}{8}\tilde{g}_{1d}^2+\frac{165}{16}\tilde{g}_{2d}^2+\frac{549}{32}\tilde{g}_{2u}^2\right)+\frac{5}{3}g'^2\left(\frac{3}{40}\tilde{g}_{1d}^2-\frac{27}{20}g^2+\frac{9}{16}\tilde{g}_{2d}^2+\frac{189}{160}\tilde{g}_{2u}^2\right)\\
&+\frac{13}{8}g'^4+y_t^2\left(-\frac{21}{4}\tilde{g}_{1d}^2+\frac{45}{8}g^2-\frac{27}{8}\tilde{g}_{2u}^2+20g_3^2+\frac{85}{24}g'^2\right)\\
&\left.\left.-\frac{27}{4}y_t^4-3\lambda(\tilde{g}_{1d}^2+\tilde{g}_{2u}^2)+\frac{3}{2}\lambda^2\right]\right\},
\end{aligned}\tag{F.6f}
$$

$$
\begin{aligned}
\beta_{\tilde{g}_{1d}}=&\frac{1}{2}k\left[3\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}+\tilde{g}_{1d}\left(\frac{5}{4}\tilde{g}_{1d}^2+2\tilde{g}_{1u}^2+\frac{9}{4}\tilde{g}_{2d}^2+\frac{3}{2}\tilde{g}_{2u}^2-\frac{9}{4}g^2-\frac{3}{4}g'^2+3y_t^2\right)\right]\\
&+\frac{1}{2}k^2\left\{-\frac{3}{4}\tilde{g}_{1d}^5-6\tilde{g}_{1d}^2\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-\frac{9}{4}\tilde{g}_{1u}^3\tilde{g}_{2d}\tilde{g}_{2u}+\frac{51}{4}g^2\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-\frac{9}{2}\tilde{g}_{1u}\tilde{g}_{2d}^3\tilde{g}_{2u}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{33}{4}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}^3+\frac{3}{4}g'^2\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}-9\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}y_t^2-3\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}\lambda\\
&+\tilde{g}_{1d}^3\left(-\frac{15}{4}\tilde{g}_{1u}^2+\frac{165}{32}g^2-\frac{9}{16}\tilde{g}_{2d}^2-\frac{27}{16}\tilde{g}_{2u}^2+\frac{103}{32}g'^2-\frac{27}{8}y_t^2-3\lambda\right)\\
&+\tilde{g}_{1d}\left[-\frac{9}{4}\tilde{g}_{1u}^4-\frac{17}{4}g^2-\frac{75}{16}\tilde{g}_{1u}^2\tilde{g}_{2d}^2-\frac{99}{16}\tilde{g}_{2d}^4-\frac{75}{16}\tilde{g}_{1u}^2\tilde{g}_{2u}^2-\frac{21}{8}\tilde{g}_{2d}^2\tilde{g}_{2u}^2-\frac{45}{16}\tilde{g}_{2u}^4\right.\\
&+g^2\left(\frac{39}{8}\tilde{g}_{1u}^2+\frac{549}{32}\tilde{g}_{2d}^2+\frac{165}{16}\tilde{g}_{2u}^2\right)+\frac{13}{8}g'^4-\frac{27}{4}y_t^4\\
&+\frac{5}{3}g'^2\left(\frac{3}{40}\tilde{g}_{1u}^2-\frac{27}{20}g^2+\frac{189}{160}\tilde{g}_{2d}^2+\frac{9}{16}\tilde{g}_{2u}^2\right)-3(\tilde{g}_{1u}^2+\tilde{g}_{2d}^2)\lambda+\frac{3}{2}\lambda^2\\
&\left.\left.+y_t^2\left(-\frac{21}{4}\tilde{g}_{1u}^2+\frac{45}{8}g^2-\frac{27}{8}\tilde{g}_{2d}^2+20g_3^2+\frac{85}{24}g'^2\right)\right]\right\}, \qquad \text{(F.6g)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\tilde{g}_{2u}}=&\frac{1}{2}k\left[\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}+\tilde{g}_{2u}\left(\frac{1}{2}\tilde{g}_{1d}^2+\frac{3}{4}\tilde{g}_{1u}^2+\tilde{g}_{2d}^2+\frac{11}{4}\tilde{g}_{2u}^2-\frac{33}{4}g^2-\frac{3}{4}g'^2+3y_t^2\right)\right]\\
&+\frac{1}{2}k^2\left\{-\frac{5}{4}\tilde{g}_{1d}^3\tilde{g}_{1u}\tilde{g}_{2d}-\frac{3}{2}\tilde{g}_{1d}\tilde{g}_{1u}^3\tilde{g}_{2d}+\frac{9}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}g^2-\frac{9}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}^3\right.\\
&-\frac{7}{2}\tilde{g}_{2u}^5+\frac{1}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}g'^2-3\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}y_t^2-\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\lambda-4\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}\tilde{g}_{2u}^2\\
&+\tilde{g}_{2u}^3\left(-\frac{15}{16}\tilde{g}_{1d}^2-\frac{59}{16}\tilde{g}_{1u}^2+\frac{875}{32}g^2-\frac{27}{8}\tilde{g}_{2d}^2+\frac{145}{32}g'^2-\frac{45}{8}y_t^2-5\lambda\right)\\
&+\tilde{g}_{2u}\left[-\frac{9}{16}\tilde{g}_{1d}^4-\frac{3}{2}\tilde{g}_{1d}^2\tilde{g}_{1u}^2-\frac{5}{16}\tilde{g}_{1u}^4-\frac{409}{12}g^4-\frac{13}{16}\tilde{g}_{1d}^2\tilde{g}_{2d}^2-\frac{31}{16}\tilde{g}_{1u}^2\tilde{g}_{2d}^2-\frac{11}{8}\tilde{g}_{2d}^4\right.\\
&+g^2\left(\frac{15}{16}\tilde{g}_{1d}^2+\frac{111}{31}\tilde{g}_{1u}^2+\frac{17}{4}\tilde{g}_{2d}^2\right)+\frac{5}{3}g'^2\left(\frac{3}{16}\tilde{g}_{1d}^2+\frac{63}{160}\tilde{g}_{1u}^2+\frac{9}{20}g^2+\frac{3}{20}\tilde{g}_{2d}^2\right)\\
&+\frac{13}{8}g'^4+y_t^2\left(-\frac{9}{8}\tilde{g}_{1u}^2+\frac{45}{8}g^2+\frac{3}{4}\tilde{g}_{2d}^2+20g_3^2+\frac{85}{24}g'^2\right)-\frac{27}{4}y_t^4\\
&\left.\left.-\lambda(\tilde{g}_{1u}^2+\tilde{g}_{2d}^2)+\frac{3}{2}\lambda^2\right]\right\}, \qquad \text{(F.6h)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\tilde{g}_{2d}}=&\frac{1}{2}k\left[\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}+\tilde{g}_{2d}\left(\frac{1}{2}\tilde{g}_{1u}^2+\frac{3}{4}\tilde{g}_{1d}^2+\tilde{g}_{2u}^2+\frac{11}{4}\tilde{g}_{2d}^2-\frac{33}{4}g^2-\frac{3}{4}g'^2+3y_t^2\right)\right]\\
&\frac{1}{2}k^2\left\{-\frac{7}{2}\tilde{g}_{2d}^5-\frac{3}{2}\tilde{g}_{1d}^3\tilde{g}_{1u}\tilde{g}_{2u}-\frac{5}{4}\tilde{g}_{1d}\tilde{g}_{1u}^3\tilde{g}_{2u}+\frac{9}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}g^2-4\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2d}^2\tilde{g}_{2u}\right.\\
&-\frac{9}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}^3+\frac{1}{4}\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}g'^2-3\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}y_t^2-\tilde{g}_{1d}\tilde{g}_{1u}\tilde{g}_{2u}\lambda\\
&+\tilde{g}_{2d}^3\left(-\frac{59}{16}\tilde{g}_{1d}^2-\frac{15}{16}\tilde{g}_{1u}^2+\frac{875}{32}g^2-\frac{27}{8}\tilde{g}_{2u}^2+\frac{145}{32}g'^2-\frac{45}{8}y_t^2-5\lambda\right)\\
&+\tilde{g}_{2d}\left[-\frac{5}{16}\tilde{g}_{1d}^4-\frac{3}{2}\tilde{g}_{1d}^2\tilde{g}_{1u}^2-\frac{9}{16}\tilde{g}_{1u}^4-\frac{409}{12}g^4-\frac{31}{16}\tilde{g}_{1d}^2\tilde{g}_{2u}^2-\frac{13}{16}\tilde{g}_{1u}^2\tilde{g}_{2u}^2-\frac{11}{8}\tilde{g}_{2u}^4\right.\\
&+g^2\left(\frac{111}{32}\tilde{g}_{1d}^2+\frac{15}{16}\tilde{g}_{1u}^2+\frac{17}{4}\tilde{g}_{2u}^2\right)+\frac{5}{3}g'^2\left(\frac{63}{160}\tilde{g}_{1d}^2+\frac{3}{16}\tilde{g}_{1u}^2+\frac{9}{20}g^2+\frac{3}{20}\tilde{g}_{2u}^2\right)\\
&+\frac{13}{8}g'^4+y_t^2\left(-\frac{9}{8}\tilde{g}_{1d}^2+\frac{45}{8}g^2+\frac{3}{4}\tilde{g}_{2u}^2+20g_3^2+\frac{85}{24}g'^2\right)\\
&\left.\left.-\frac{27}{4}y_t^4-\lambda(\tilde{g}_{1u}^2+\tilde{g}_{2d}^2)+\frac{3}{2}\lambda^2\right]\right\}. \qquad \text{(F.6i)}
\end{aligned}
$$

F.3 THDM type III

The Higgs potential of the employed THDM is specified in Eq. (11.1) , the Yukawa couplings in Eq. (11.21).

Gauge couplings

The one-loop beta functions of the gauge couplings are given by

$$\beta_g^{(1)} = -\frac{3}{2}g^3, \tag{F.7a}$$

$$\beta_{g'}^{(1)} = \frac{7}{2}{g'}^3, \tag{F.7b}$$

$$\beta_{g_3}^{(1)} = -\frac{1}{2}\langle 7;5\rangle g_3^3, \tag{F.7c}$$

the two-loop beta functions by

$$\beta_g^{(2)} = 4g^5 + g^3\left[6g_3^2 - \frac{3}{4}{h_t}^2 - \frac{3}{4}{h_t'}^2 + {g'}^2\right], \tag{F.8a}$$

$$\beta_{g'}^{(2)} = \frac{52}{9}{g'}^5 + {g'}^3\left[\frac{22}{3}g_3^2 - \frac{17}{12}{h_t}^2 - \frac{17}{12}{h_t'}^2 + 3g^2\right], \tag{F.8b}$$

$$\beta_{g_3}^{(2)} = -\langle 13;-11\rangle g_3^5 - g_3^3\left[{h_t}^2 + {h_t'}^2 - \frac{9}{4}g^2 - \frac{11}{12}{g'}^2\right]. \tag{F.8c}$$

Top Yukawa couplings

The one-loop beta functions of the top Yukawa couplings read

$$\beta_{h_t}^{(1)} = \frac{9}{4}{h_t}^3 - h_t\left[4g_3^2 - \frac{9}{4}{h_t'}^2 + \frac{9}{8}g^2 + \frac{17}{24}{g'}^2\right], \tag{F.9a}$$

$$\beta_{h_t'}^{(1)} = \frac{9}{4}{h_t'}^3 - h_t'\left[4g_3^2 - \frac{9}{4}{h_t}^2 + \frac{9}{8}g^2 + \frac{17}{24}{g'}^2\right]. \tag{F.9b}$$

The corresponding two-loop beta functions are given by

$$\begin{aligned}\beta_{h_t}^{(2)} =& -6{h_t}^5 - 6h_t{h_t'}^4 - 9{h_t}^2h_t'\lambda_7 - 3{h_t'}^3\lambda_6\\
&- {h_t}^3\left[3\lambda_2 - 18g_3^2 + 12{h_t'}^2 - \frac{225}{32}g^2 - \frac{131}{32}{g'}^2\right]\\
&- h_t{h_t'}^2\left[3\big(\lambda_3+\lambda_4+\lambda_5\big) - 18g_3^2 - \frac{225}{32}g^2 - \frac{131}{32}{g'}^2\right]\\
&+ h_t\left[\frac{1}{2}\lambda_3\lambda_4 + \frac{3}{4}{\lambda_2}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{3}{4}{\lambda_6}^2 + \frac{9}{4}{\lambda_7}^2\right.\\
&\qquad\left. - \langle 54;\frac{142}{3}\rangle g_3^4 + g_3^2\Big(\frac{9}{2}g^2 + \frac{19}{18}{g'}^2\Big) - \frac{21}{8}g^4 - \frac{3}{8}g^2{g'}^2 + \frac{1267}{432}{g'}^4\right]\\
&+ \frac{3}{4}h_t'\left[\lambda_1\lambda_6 + \lambda_2\lambda_7 + \lambda_3\Big(\lambda_6+\lambda_7\Big) + \lambda_4\Big(\lambda_6+\lambda_7\Big) + \lambda_5\Big(\lambda_6+\lambda_7\Big)\right],\end{aligned} \tag{F.10a}$$

$$\begin{aligned}\beta_{h_t'}^{(2)} =& -6{h_t'}^5 - 6h_t'{h_t}^4 - 9{h_t'}^2h_t\lambda_6 - 3{h_t}^3\lambda_7\\
&- {h_t'}^3\left[3\lambda_1 - 18g_3^2 + 12{h_t}^2 - \frac{225}{32}g^2 - \frac{131}{32}{g'}^2\right]\\
&- h_t'{h_t}^2\left[3\big(\lambda_3+\lambda_4+\lambda_5\big) - 18g_3^2 - \frac{225}{32}g^2 - \frac{131}{32}{g'}^2\right]\\
&+ h_t'\left[\frac{1}{2}\lambda_3\lambda_4 + \frac{3}{4}{\lambda_1}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{9}{4}{\lambda_6}^2 + \frac{3}{4}{\lambda_7}^2\right.\\
&\qquad\left. - \langle 54;\frac{142}{3}\rangle g_3^4 + g_3^2\Big(\frac{9}{2}g^2 + \frac{19}{18}{g'}^2\Big) - \frac{21}{8}g^4 - \frac{3}{8}g^2{g'}^2 + \frac{1267}{432}{g'}^4\right]\\
&+ \frac{3}{4}h_t\left[\lambda_1\lambda_6 + \lambda_2\lambda_7 + \lambda_3\Big(\lambda_6+\lambda_7\Big) + \lambda_4\Big(\lambda_6+\lambda_7\Big) + \lambda_5\Big(\lambda_6+\lambda_7\Big)\right].\end{aligned} \tag{F.10b}$$

Quartic couplings

The one-loop beta functions of the quartic couplings are given by

$$\beta_{\lambda_1}^{(1)} = 6{\lambda_1}^2 + 2{\lambda_3}^2 + 2\lambda_3\lambda_4 + {\lambda_4}^2 + {\lambda_5}^2 + 12{\lambda_6}^2 + \lambda_1\Big(6{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big)$$

$$+ 6\lambda_6 h_t h_t' - 6{h_t'}^4 + \frac{9}{8}g^4 + \frac{3}{4}g^2 {g'}^2 + \frac{3}{8}{g'}^4, \tag{F.11a}$$

$$\beta_{\lambda_2}^{(1)} = 6{\lambda_2}^2 + 2{\lambda_3}^2 + 2\lambda_3\lambda_4 + {\lambda_4}^2 + {\lambda_5}^2 + 12{\lambda_7}^2 + \lambda_2\Big(6{h_t}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big)$$
$$+ 6\lambda_7 h_t h_t' - 6{h_t}^4 + \frac{9}{8}g^4 + \frac{3}{4}g^2 {g'}^2 + \frac{3}{8}{g'}^4, \tag{F.11b}$$

$$\beta_{\lambda_3}^{(1)} = \lambda_1\big(3\lambda_3 + \lambda_4\big) + \lambda_2\big(3\lambda_3 + \lambda_4\big) + 2{\lambda_3}^2 + {\lambda_4}^2 + {\lambda_5}^2 + 2{\lambda_6}^2 + 2{\lambda_7}^2 + 8\lambda_6\lambda_7$$
$$+ \lambda_3\Big(3{h_t}^2 + 3{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big) + 3\lambda_6 h_t h_t' + 3\lambda_7 h_t h_t'$$
$$- 6{h_t}^2{h_t'}^2 + \frac{9}{8}g^4 - \frac{3}{4}g^2{g'}^2 + \frac{3}{8}{g'}^4, \tag{F.11c}$$

$$\beta_{\lambda_4}^{(1)} = \lambda_1\lambda_4 + \lambda_2\lambda_4 + 4\lambda_3\lambda_4 + 2{\lambda_4}^2 + 4{\lambda_5}^2 + 5{\lambda_6}^2 + 5{\lambda_7}^2 + 2\lambda_6\lambda_7$$
$$+ \lambda_4\Big(3{h_t}^2 + 3{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big) + 3\lambda_6 h_t h_t' + 3\lambda_7 h_t h_t'$$
$$- 6{h_t}^2{h_t'}^2 + \frac{3}{2}g^2{g'}^2, \tag{F.11d}$$

$$\beta_{\lambda_5}^{(1)} = \lambda_1\lambda_5 + \lambda_2\lambda_5 + 4\lambda_3\lambda_5 + 6\lambda_4\lambda_5 + 5{\lambda_6}^2 + 2\lambda_6\lambda_7 + 5{\lambda_7}^2$$
$$+ \lambda_5\Big(3{h_t}^2 + 3{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big) + 3\lambda_6 h_t h_t' + 3\lambda_7 h_t h_t' - 6{h_t}^2{h_t'}^2, \tag{F.11e}$$

$$\beta_{\lambda_6}^{(1)} = 6\lambda_1\lambda_6 + 3\lambda_3\big(\lambda_6 + \lambda_7\big) + 2\lambda_4\big(2\lambda_6 + \lambda_7\big) + \lambda_5\big(5\lambda_6 + \lambda_7\big)$$
$$+ \frac{3}{2}\lambda_1 h_t h_t' + \frac{3}{2}\lambda_3 h_t h_t' + \frac{3}{2}\lambda_4 h_t h_t'$$
$$+ \frac{3}{2}\lambda_5 h_t h_t' + \lambda_6\Big(\frac{3}{2}{h_t}^2 + \frac{9}{2}{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big) - 6h_t{h_t'}^3, \tag{F.11f}$$

$$\beta_{\lambda_7}^{(1)} = 6\lambda_2\lambda_7 + 3\lambda_3\big(\lambda_6 + \lambda_7\big) + 2\lambda_4\big(\lambda_6 + 2\lambda_7\big) + \lambda_5\big(\lambda_6 + 5\lambda_7\big)$$
$$+ \frac{3}{2}\lambda_2 h_t h_t' + \frac{3}{2}\lambda_3 h_t h_t' + \frac{3}{2}\lambda_4 h_t h_t'$$
$$+ \frac{3}{2}\lambda_5 h_t h_t' + \lambda_7\Big(\frac{9}{2}{h_t}^2 + \frac{3}{2}{h_t'}^2 - \frac{9}{2}g^2 - \frac{3}{2}{g'}^2\Big) - 6{h_t}^3 h_t'. \tag{F.11g}$$

The corresponding two-loop beta functions are given by

$$\beta_{\lambda_1}^{(2)} = -39{\lambda_1}^3 - \lambda_1\Big(10{\lambda_3}^2 + 10\lambda_3\lambda_4 + 6{\lambda_4}^2 + 7{\lambda_5}^2 + 159{\lambda_6}^2 - 3{\lambda_7}^2\Big) - 8{\lambda_3}^3$$
$$- 12{\lambda_3}^2\lambda_4 - \lambda_3\Big(16{\lambda_4}^2 + 20{\lambda_5}^2 + 66{\lambda_6}^2 + 36\lambda_6\lambda_7 + 18{\lambda_7}^2\Big)$$
$$- 6{\lambda_4}^3 - \lambda_4\Big(22{\lambda_5}^2 + 70{\lambda_6}^2 + 28\lambda_6\lambda_7 + 14{\lambda_7}^2\Big)$$
$$- \lambda_5\Big(74{\lambda_6}^2 + 20\lambda_6\lambda_7 + 10{\lambda_7}^2\Big) - {\lambda_1}^2\Big(36{h_t'}^2 - 27g^2 - 9{g'}^2\Big) - 72\lambda_1\lambda_6 h_t h_t'$$
$$+ {\lambda_3}^2\Big(12g^2 + 4{g'}^2 - 12{h_t}^2\Big) - \lambda_3\lambda_4\Big(12{h_t}^2 - 12g^2 - 4{g'}^2\Big) - 36\lambda_3\lambda_6 h_t h_t'$$
$$+ {\lambda_4}^2\Big(3g^2 + 2{g'}^2 - 6{h_t}^2\Big) - 24\lambda_4\lambda_6 h_t h_t' - {\lambda_5}^2\Big(6{h_t}^2 + {g'}^2\Big) - 12\lambda_5\lambda_6 h_t h_t'$$
$$- {\lambda_6}^2\Big(36{h_t}^2 + 36{h_t'}^2 - 54g^2 - 18{g'}^2\Big)$$
$$+ \lambda_1\Bigg[40g_3^2{h_t'}^2 - \frac{27}{2}{h_t}^2{h_t'}^2 - \frac{3}{2}{h_t'}^4 + {h_t'}^2\Big(\frac{45}{4}g^2 + \frac{85}{12}{g'}^2\Big)$$
$$- \frac{51}{16}g^4 + \frac{39}{8}g^2{g'}^2 + \frac{217}{16}{g'}^4\Bigg]$$
$$+ \lambda_3\Big(\frac{15}{2}g^4 + \frac{5}{2}{g'}^4\Big) + \lambda_4\Big(\frac{15}{4}g^4 + \frac{5}{2}g^2{g'}^2 + \frac{5}{4}{g'}^4\Big) + 12\lambda_5{h_t}^2{h_t'}^2 + 24\lambda_6 h_t{h_t'}^3$$
$$- 32g_3^2{h_t'}^4 + 30{h_t}^2{h_t'}^4 + 30{h_t'}^6 - \frac{8}{3}{h_t'}^4{g'}^2 - {h_t'}^2\Big(\frac{9}{4}g^4 - \frac{21}{2}g^2{g'}^2 + \frac{19}{4}{g'}^4\Big)$$
$$+ \frac{291}{16}g^6 - \frac{101}{16}g^4{g'}^2 - \frac{191}{16}g^2{g'}^4 - \frac{131}{16}{g'}^6, \tag{F.12a}$$

$$\beta_{\lambda_2}^{(2)} = -39{\lambda_2}^3 - \lambda_2\Big(10{\lambda_3}^2 + 10\lambda_3\lambda_4 + 6{\lambda_4}^2 + 7{\lambda_5}^2 - 3{\lambda_6}^2 + 159{\lambda_7}^2\Big) - 8{\lambda_3}^3 - 12{\lambda_3}^2\lambda_4$$

$$
\begin{aligned}
&- \lambda_3\Big(16{\lambda_4}^2 + 20{\lambda_5}^2 + 18{\lambda_6}^2 + 36\lambda_6\lambda_7 + 66{\lambda_7}^2\Big) - 6{\lambda_4}^3 \\
&- \lambda_4\Big(22{\lambda_5}^2 + 14{\lambda_6}^2 + 28\lambda_6\lambda_7 + 70{\lambda_7}^2\Big) \\
&- \lambda_5\Big(10{\lambda_6}^2 + 20\lambda_6\lambda_7 + 74{\lambda_7}^2\Big) - {\lambda_2}^2\Big(36{h_t}^2 - 27g^2 - 9{g'}^2\Big) - 72\lambda_2\lambda_7 h_t h'_t \\
&- {\lambda_3}^2\Big(12{h'_t}^2 - 12g^2 - 4{g'}^2\Big) - 36\lambda_3\lambda_7 h_t h'_t - {\lambda_4}^2\Big(6{h'_t}^2 - 3g^2 - 2{g'}^2\Big) - 24\lambda_4\lambda_7 h_t h'_t \\
&- {\lambda_5}^2\Big(6{h'_t}^2 + {g'}^2\Big) - 12\lambda_5\lambda_7 h_t h'_t - {\lambda_7}^2\Big(36{h_t}^2 + 36{h'_t}^2 - 54g^2 - 18{g'}^2\Big) \\
&+ \lambda_2\Big[40g_3^2{h_t}^2 - \frac{3}{2}{h_t}^4 - \frac{27}{2}{h_t}^2{h'_t}^2 + {h_t}^2\Big(\frac{45}{4}g^2 + \frac{85}{12}{g'}^2\Big) - \frac{51}{16}g^4 + \frac{39}{8}g^2{g'}^2 + \frac{217}{16}{g'}^4\Big] \\
&+ \lambda_3\Big(\frac{15}{2}g^4 + \lambda_4\Big(12g^2 + 4{g'}^2 - 12{h'_t}^2\Big) + \frac{5}{2}{g'}^4\Big) + \lambda_4\Big(\frac{15}{4}g^4 + \frac{5}{2}g^2{g'}^2 + \frac{5}{4}{g'}^4\Big) \\
&+ 12\lambda_5{h_t}^2{h'_t}^2 + {h_t}^4\Big(-32g_3^2 - \frac{8}{3}{g'}^2 + 30{h'_t}^2\Big) - \frac{131}{16}{g'}^6 + 30{h_t}^6 + 24{h_t}^3 h'_t\lambda_7 \\
&- 32g_3^2{h_t}^4 + 30{h_t}^6 + 30{h_t}^4{h'_t}^2 - \frac{8}{3}{h_t}^4{g'}^2 - {h_t}^2\Big(\frac{9}{4}g^4 - \frac{21}{2}g^2{g'}^2 + \frac{19}{4}{g'}^4\Big) \\
&+ \frac{291}{16}g^6 - \frac{101}{16}g^4{g'}^2 - \frac{191}{16}g^2{g'}^4 - \frac{131}{16}{g'}^6,
\end{aligned}
\tag{F.12b}
$$

$$
\begin{aligned}
\beta^{(2)}_{\lambda_3} =&- {\lambda_1}^2\Big(\frac{15}{2}\lambda_3 + 2\lambda_4\Big) - \lambda_1\Big(18{\lambda_3}^2 + 8\lambda_3\lambda_4 + 7{\lambda_4}^2 + 9{\lambda_5}^2 + 31{\lambda_6}^2 + 22\lambda_6\lambda_7 + 11{\lambda_7}^2\Big) \\
&- {\lambda_2}^2\Big(\frac{15}{2}\lambda_3 + 2\lambda_4\Big) - \lambda_2\Big(18{\lambda_3}^2 + 8\lambda_3\lambda_4 + 7{\lambda_4}^2 + 9{\lambda_5}^2 + 11{\lambda_6}^2 + 22\lambda_6\lambda_7 + 31{\lambda_7}^2\Big) \\
&- 6{\lambda_3}^3 - 2{\lambda_3}^2\lambda_4 - \lambda_3\Big(8{\lambda_4}^2 + 9{\lambda_5}^2 + 30{\lambda_6}^2 + 88\lambda_6\lambda_7 + 30{\lambda_7}^2\Big) - 6{\lambda_4}^3 \\
&- \lambda_4\Big(22{\lambda_5}^2 + 34{\lambda_6}^2 + 44\lambda_6\lambda_7 + 34{\lambda_7}^2\Big) - \lambda_5\Big(34{\lambda_6}^2 + 36\lambda_6\lambda_7 + 34{\lambda_7}^2\Big) \\
&- \lambda_1\lambda_3\Big(18{h'_t}^2 - 18g^2 - 6{g'}^2\Big) - \lambda_1\lambda_4\Big(6{h'_t}^2 - 9g^2 - 2{g'}^2\Big) - 24\lambda_1\lambda_7 h_t h'_t \\
&- \lambda_2\lambda_3\Big(18{h_t}^2 - 18g^2 - 6{g'}^2\Big) - \lambda_2\lambda_4\Big(6{h_t}^2 - 9g^2 - 2{g'}^2\Big) - 24\lambda_2\lambda_6 h_t h'_t \\
&- {\lambda_3}^2\Big(6{h_t}^2 + 6{h'_t}^2 - 3g^2 - {g'}^2\Big) - 6\lambda_3\lambda_4 g^2 - 30\lambda_3\lambda_6 h_t h'_t - 30\lambda_3\lambda_7 h_t h'_t \\
&- {\lambda_4}^2\Big(3{h_t}^2 + 3{h'_t}^2 - 3g^2 + {g'}^2\Big) - 12\lambda_4\lambda_6 h_t h'_t - 12\lambda_4\lambda_7 h_t h'_t \\
&- {\lambda_5}^2\Big(3{h_t}^2 + 3{h'_t}^2 - 2{g'}^2\Big) - 6\lambda_5\lambda_6 h_t h'_t - 6\lambda_5\lambda_7 h_t h'_t \\
&- {\lambda_6}^2\Big(12{h'_t}^2 - {g'}^2\Big) - \lambda_6\lambda_7\Big(24{h_t}^2 + 24{h'_t}^2 - 54g^2 - 16{g'}^2\Big) - {\lambda_7}^2\Big(12{h_t}^2 - {g'}^2\Big) \\
&+ \lambda_1\Big(\frac{45}{8}g^4 - \frac{5}{4}g^2{g'}^2 + \frac{15}{8}{g'}^4\Big) + \lambda_2\Big(\frac{45}{8}g^4 - \frac{5}{4}g^2{g'}^2 + \frac{15}{8}{g'}^4\Big) \\
&+ \lambda_3\Big[20g_3^2\Big({h_t}^2 + {h'_t}^2\Big) - \frac{27}{4}{h_t}^4 - \frac{3}{2}{h_t}^2{h'_t}^2 - \frac{27}{4}{h'_t}^4 + {h_t}^2\Big(\frac{45}{8}g^2 + \frac{85}{24}{g'}^2\Big) \\
&\qquad + {h'_t}^2\Big(\frac{45}{8}g^2 + \frac{85}{24}{g'}^2\Big) - \frac{111}{16}g^4 + \frac{11}{8}g^2{g'}^2 + \frac{197}{16}{g'}^4\Big] \\
&+ \lambda_4\Big(12{h_t}^2{h'_t}^2 + \frac{15}{4}g^4 - \frac{3}{2}g^2{g'}^2 + \frac{5}{4}{g'}^4\Big) + 12\lambda_6 h_t{h'_t}^3 + 12\lambda_7{h_t}^3 h'_t \\
&- 32g_3^2{h_t}^2{h'_t}^2 + 30{h_t}^4{h'_t}^2 + 30{h_t}^2{h'_t}^4 - \frac{8}{3}{h_t}^2{h'_t}^2{g'}^2 - {h_t}^2\Big(\frac{9}{8}g^4 + \frac{21}{4}g^2{g'}^2 + \frac{19}{8}{g'}^4\Big) \\
&- {h'_t}^2\Big(\frac{9}{8}g^4 + \frac{21}{4}g^2{g'}^2 + \frac{19}{8}{g'}^4\Big) + \frac{291}{16}g^6 + \frac{11}{16}g^4{g'}^2 + \frac{101}{16}g^2{g'}^4 - \frac{131}{16}{g'}^6,
\end{aligned}
\tag{F.12c}
$$

$$
\begin{aligned}
\beta^{(2)}_{\lambda_4} =&- \frac{7}{2}{\lambda_1}^2\lambda_4 - \lambda_1\Big(20\lambda_3\lambda_4 + 10{\lambda_4}^2 + 12{\lambda_5}^2 + 37{\lambda_6}^2 + 10\lambda_6\lambda_7 + 5{\lambda_7}^2\Big) \\
&- \frac{7}{2}{\lambda_2}^2\lambda_4 - \lambda_2\Big(20\lambda_3\lambda_4 + 10{\lambda_4}^2 + 12{\lambda_5}^2 + 5{\lambda_6}^2 + 10\lambda_6\lambda_7 + 37{\lambda_7}^2\Big) \\
&- 14{\lambda_3}^2\lambda_4 - \lambda_3\Big(14{\lambda_4}^2 + 24{\lambda_5}^2 + 36{\lambda_6}^2 + 40\lambda_6\lambda_7 + 36{\lambda_7}^2\Big) \\
&- \lambda_4\Big(13{\lambda_5}^2 + 34{\lambda_6}^2 + 80\lambda_6\lambda_7 + 34{\lambda_7}^2\Big) - \lambda_5\Big(40{\lambda_6}^2 + 48\lambda_6\lambda_7 + 40{\lambda_7}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda_1\lambda_4\Big(6{h'_t}^2-2{g'}^2\Big)-6\lambda_1\lambda_7h_th'_t-\lambda_2\lambda_4\Big(6{h_t}^2-2{g'}^2\Big)-6\lambda_2\lambda_6h_th'_t\\
&-\lambda_3\lambda_4\Big(12{h_t}^2+12{h'_t}^2-18g^2-2{g'}^2\Big)-12\lambda_3\lambda_6h_th'_t-12\lambda_3\lambda_7h_th'_t\\
&-{\lambda_4}^2\Big(6{h_t}^2+6{h'_t}^2-9g^2-4{g'}^2\Big)-{\lambda_5}^2\Big(12{h_t}^2+12{h'_t}^2-27g^2-8{g'}^2\Big)\\
&-30\lambda_4\lambda_6h_th'_t-30\lambda_4\lambda_7h_th'_t-24\lambda_5\lambda_6h_th'_t-24\lambda_5\lambda_7h_th'_t\\
&-{\lambda_6}^2\Big(30{h'_t}^2-27g^2-7{g'}^2\Big)-\lambda_6\lambda_7\Big(6{h_t}^2+6{h'_t}^2-4{g'}^2\Big)\\
&-{\lambda_7}^2\Big(30{h_t}^2-27g^2-7{g'}^2\Big)+\frac{5}{2}\lambda_1g^2{g'}^2+\frac{5}{2}\lambda_2g^2{g'}^2+\lambda_3\Big(12{h_t}^2{h'_t}^2+g^2{g'}^2\Big)\\
&+\lambda_4\Big[20g_3^2\Big({h_t}^2+{h'_t}^2\Big)-\frac{27}{4}{h_t}^4-\frac{3}{2}{h_t}^2{h'_t}^2-\frac{27}{4}{h'_t}^4+{h_t}^2\Big(\frac{45}{8}g^2+\frac{85}{24}{g'}^2\Big)\\
&\qquad+{h'_t}^2\Big(\frac{45}{8}g^2+\frac{85}{24}{g'}^2\Big)-\frac{231}{16}g^4+\frac{51}{8}g^2{g'}^2+\frac{157}{16}{g'}^4\Big]\\
&+12\lambda_6h_t{h'_t}^3+12\lambda_7{h_t}^3-32g_3^2{h_t}^2{h'_t}^2+30{h_t}^4{h'_t}^2+30{h_t}^2{h'_t}^4\\
&-\frac{8}{3}{h_t}^2{h'_t}^2{g'}^2+\frac{21}{2}{h_t}^2g^2{g'}^2+\frac{21}{2}{h'_t}^2g^2{g'}^2-7g^4{g'}^2-\frac{73}{4}g^2{g'}^4,
\end{aligned}
\tag{F.12d}
$$

$$
\begin{aligned}
\beta^{(2)}_{\lambda_5}=&-\frac{7}{2}{\lambda_1}^2\lambda_5-\lambda_1\Big(20\lambda_3\lambda_5+22\lambda_4\lambda_5+37{\lambda_6}^2+10\lambda_6\lambda_7+5{\lambda_7}^2\Big)\\
&-\frac{7}{2}{\lambda_2}^2\lambda_5-\lambda_2\Big(20\lambda_3\lambda_5+22\lambda_4\lambda_5+5{\lambda_6}^2+10\lambda_6\lambda_7+37{\lambda_7}^2\Big)\\
&-14{\lambda_3}^2\lambda_5-\lambda_3\Big(38\lambda_4\lambda_5+36{\lambda_6}^2+40\lambda_6\lambda_7+36{\lambda_7}^2\Big)-16{\lambda_4}^2\lambda_5\\
&-\lambda_4\Big(38{\lambda_6}^2+44\lambda_6\lambda_7+38{\lambda_7}^2\Big)+3{\lambda_5}^3-\lambda_5\Big(36{\lambda_6}^2+84\lambda_6\lambda_7+36{\lambda_7}^2\Big)\\
&-\lambda_3\lambda_5\Big(12{h_t}^2+12{h'_t}^2-18g^2-8{g'}^2\Big)-12\lambda_3\lambda_6h_th'_t-12\lambda_3\lambda_7h_th'_t\\
&-\lambda_1\lambda_5\Big(6{h'_t}^2+{g'}^2\Big)-6\lambda_1\lambda_7h_th'_t-\lambda_2\lambda_5\Big(6{h_t}^2+{g'}^2\Big)-6\lambda_2\lambda_6h_th'_t\\
&-\lambda_4\lambda_5\Big(18{h_t}^2+18{h'_t}^2-36g^2-12{g'}^2\Big)-18\lambda_4\lambda_6h_th'_t-18\lambda_4\lambda_7h_th'_t\\
&-36\lambda_5\lambda_6h_th'_t-36\lambda_5\lambda_7h_th'_t-{\lambda_6}^2\Big(30{h'_t}^2-27g^2-10{g'}^2\Big)\\
&-\lambda_6\lambda_7\Big(6{h_t}^2+6{h'_t}^2+2{g'}^2\Big)-{\lambda_7}^2\Big(30{h_t}^2-27g^2-10{g'}^2\Big)\\
&+12{h_t}^3h'_t\lambda_6+6{h_t}^2{h'_t}^2\lambda_1+6{h_t}^2{h'_t}^2\lambda_2+12h_t{h'_t}^3\lambda_7\\
&+\lambda_5\Big[20g_3^2\Big({h_t}^2+{h'_t}^2\Big)-\frac{3}{4}{h_t}^4-\frac{27}{2}{h_t}^2{h'_t}^2-\frac{3}{4}{h'_t}^4+{h_t}^2\Big(\frac{45}{8}g^2+\frac{85}{24}{g'}^2\Big)\\
&\qquad+{h'_t}^2\Big(\frac{45}{8}g^2+\frac{85}{24}{g'}^2\Big)-\frac{231}{16}g^4+\frac{19}{8}g^2{g'}^2+\frac{157}{16}{g'}^4\Big]\\
&-32g_3^2{h_t}^2{h'_t}^2+30{h_t}^4{h'_t}^2+30{h_t}^2{h'_t}^4-\frac{8}{3}{h_t}^2{h'_t}^2{g'}^2,
\end{aligned}
\tag{F.12e}
$$

$$
\begin{aligned}
\beta^{(2)}_{\lambda_6}=&-\frac{159}{4}{\lambda_1}^2\lambda_6-\lambda_1\lambda_3\Big(33\lambda_6+9\lambda_7\Big)-\lambda_1\lambda_4\Big(35\lambda_6+7\lambda_7\Big)-\lambda_1\lambda_5\Big(37\lambda_6+5\lambda_7\Big)\\
&+\frac{3}{4}{\lambda_2}^2\lambda_6-\lambda_2\lambda_3\Big(9\lambda_6+9\lambda_7\Big)-\lambda_2\lambda_4\Big(7\lambda_6+7\lambda_7\Big)-\lambda_2\lambda_5\Big(5\lambda_6+5\lambda_7\Big)\\
&-{\lambda_3}^2\Big(16\lambda_6+18\lambda_7\Big)-\lambda_3\lambda_4\Big(34\lambda_6+28\lambda_7\Big)-\lambda_3\lambda_5\Big(36\lambda_6+20\lambda_7\Big)\\
&-{\lambda_4}^2\Big(17\lambda_6+17\lambda_7\Big)-\lambda_4\lambda_5\Big(38\lambda_6+22\lambda_7\Big)-{\lambda_5}^2\Big(18\lambda_6+21\lambda_7\Big)\\
&-\frac{111}{2}{\lambda_6}^3-63{\lambda_6}^2\lambda_7-\frac{33}{2}\lambda_6{\lambda_7}^2-21{\lambda_7}^3\\
&-9\lambda_1\lambda_3h_th'_t-12\lambda_1\lambda_4h_th'_t-15\lambda_1\lambda_5h_th'_t-\lambda_1\lambda_6\Big(36{h'_t}^2-27g^2-9{g'}^2\Big)\\
&-3{\lambda_3}^2h_th'_t-9\lambda_3\lambda_4h_th'_t-12\lambda_3\lambda_5h_th'_t\\
&-\lambda_3\lambda_6\Big(9{h_t}^2+9{h'_t}^2-9g^2-3{g'}^2\Big)-\lambda_3\lambda_7\Big(18{h_t}^2-18g^2-6{g'}^2\Big)-3{\lambda_4}^2h_th'_t\\
&-9\lambda_4\lambda_5h_th'_t-\lambda_4\lambda_6\Big(12{h_t}^2+12{h'_t}^2-18g^2-5{g'}^2\Big)-\lambda_4\lambda_7\Big(12{h_t}^2-9g^2-4{g'}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_5\lambda_6\Big(15{h_t}^2 + 15{h_t'}^2 - 27g^2 - 10{g'}^2\Big) - \lambda_5\lambda_7\Big(6{h_t}^2 + {g'}^2\Big) - 36{\lambda_6}^2 h_t h_t' \\
&- 36\lambda_6\lambda_7 h_t h_t' + 6\lambda_1 h_t {h_t'}^3 + 6\lambda_3 h_t {h_t'}^3 + 6\lambda_4 h_t {h_t'}^3 + 6\lambda_5 {h_t}^3 h_t' \\
&+ \lambda_6\Big[g_3^2\Big(10{h_t}^2 + 30{h_t'}^2\Big) - \frac{27}{8}{h_t}^4 - \frac{3}{2}{h_t}^2{h_t'}^2 - \frac{33}{8}{h_t'}^4 + {h_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2\Big) \\
&\qquad + {h_t'}^2\Big(\frac{135}{16}g^2 + \frac{85}{16}{g'}^2\Big) - \frac{141}{16}g^4 + \frac{29}{8}g^2{g'}^2 + \frac{187}{16}{g'}^4\Big] \\
&+ \lambda_7\Big(6{h_t}^2{h_t'}^2 + \frac{45}{8}g^4 + \frac{5}{4}g^2{g'}^2 + \frac{15}{8}{g'}^4\Big) - 32g_3^2 h_t {h_t'}^3 \\
&+ 30{h_t}^3{h_t'}^3 + 30h_t{h_t'}^5 - \frac{8}{3}h_t{h_t'}^3{g'}^2 - h_t h_t'\Big(\frac{9}{8}g^4 - \frac{21}{4}g^2{g'}^2 + \frac{19}{8}{g'}^4\Big),
\end{aligned}
\tag{F.12f}
$$

$$
\begin{aligned}
\beta_{\lambda_7}^{(2)} =& \frac{3}{4}{\lambda_1}^2\lambda_7 - \lambda_1\lambda_3\Big(9\lambda_6 + 9\lambda_7\Big) - \lambda_1\lambda_4\Big(7\lambda_6 + 7\lambda_7\Big) - \lambda_1\lambda_5\Big(5\lambda_6 + 5\lambda_7\Big) \\
&- \frac{159}{4}{\lambda_2}^2\lambda_7 - \lambda_2\lambda_3\Big(9\lambda_6 + 33\lambda_7\Big) - \lambda_2\lambda_4\Big(7\lambda_6 + 35\lambda_7\Big) - \lambda_2\lambda_5\Big(5\lambda_6 + 37\lambda_7\Big) \\
&- {\lambda_3}^2\Big(18\lambda_6 + 16\lambda_7\Big) - \lambda_3\lambda_4\Big(28\lambda_6 + 34\lambda_7\Big) - \lambda_3\lambda_5\Big(20\lambda_6 + 36\lambda_7\Big) \\
&- {\lambda_4}^2\Big(17\lambda_6 + 17\lambda_7\Big) - \lambda_4\lambda_5\Big(22\lambda_6 + 38\lambda_7\Big) - {\lambda_5}^2\Big(21\lambda_6 + 18\lambda_7\Big) \\
&- 21{\lambda_6}^3 - \frac{33}{2}{\lambda_6}^2\lambda_7 - 63\lambda_6{\lambda_7}^2 - \frac{111}{2}{\lambda_7}^3 \\
&- 9\lambda_2\lambda_3 h_t h_t' - 12\lambda_2\lambda_4 h_t h_t' - 15\lambda_2\lambda_5 h_t h_t' - \lambda_2\lambda_7\Big(36{h_t}^2 - 27g^2 - 9{g'}^2\Big) \\
&- 3{\lambda_3}^2 h_t h_t' - 9\lambda_3\lambda_4 h_t h_t' - 12\lambda_3\lambda_5 h_t h_t' - \lambda_3\lambda_6\Big(18{h_t'}^2 - 18g^2 - 6{g'}^2\Big) \\
&- \lambda_3\lambda_7\Big(9{h_t}^2 + 9{h_t'}^2 - 9g^2 - 3{g'}^2\Big) - 3{\lambda_4}^2 h_t h_t' - 9\lambda_4\lambda_5 h_t h_t' \\
&- \lambda_4\lambda_6\Big(12{h_t'}^2 - 9g^2 - 4{g'}^2\Big) - \lambda_4\lambda_7\Big(12{h_t}^2 + 12{h_t'}^2 - 18g^2 - 5{g'}^2\Big) \\
&- \lambda_5\lambda_6\Big(6{h_t'}^2 + {g'}^2\Big) - \lambda_5\lambda_7\Big(15{h_t}^2 + 15{h_t'}^2 - 27g^2 - 10{g'}^2\Big) - 36\lambda_6\lambda_7 h_t h_t' \\
&- 36{\lambda_7}^2 h_t h_t' + 6\lambda_2{h_t}^3 h_t' + 6\lambda_3{h_t}^3 h_t' + 6\lambda_4{h_t}^3 h_t' + 6\lambda_5 h_t{h_t'}^3 \\
&+ \lambda_6\Big(6{h_t}^2{h_t'}^2 + \frac{45}{8}g^4 + \frac{5}{4}g^2{g'}^2 + \frac{15}{8}{g'}^4\Big) \\
&+ \lambda_7\Big[g_3^2\Big(30{h_t}^2 + 10{h_t'}^2\Big) - \frac{33}{8}{h_t}^4 - \frac{3}{2}{h_t}^2{h_t'}^2 - \frac{27}{8}{h_t'}^4 + {h_t}^2\Big(\frac{135}{16}g^2 + \frac{85}{16}{g'}^2\Big) \\
&\qquad + {h_t'}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2\Big) - \frac{141}{16}g^4 + \frac{29}{8}g^2{g'}^2 + \frac{187}{16}{g'}^4\Big] \\
&- 32g_3^2{h_t}^3 h_t' + 30{h_t}^5 h_t' + 30{h_t}^3{h_t'}^3 - \frac{8}{3}{h_t}^3 h_t'{g'}^2 \\
&- h_t h_t'\Big(\frac{9}{8}g^4 - \frac{21}{4}g^2{g'}^2 + \frac{19}{8}{g'}^4\Big).
\end{aligned}
\tag{F.12g}
$$

F.4 THDM with EWinos

The Lagrangian of the THDM with electroweakinos is specified in Eq. (11.22).

Gauge couplings

The one-loop beta functions of the gauge couplings are given by

$$\beta_g^{(1)} = -\frac{1}{2}g^3, \tag{F.13a}$$

$$\beta_{g'}^{(1)} = \frac{23}{6}{g'}^3, \tag{F.13b}$$

$$\beta_{g_3}^{(1)} = -\frac{1}{2}\langle 7; 5\rangle g_3^3, \tag{F.13c}$$

the two-loop beta functions by

$$\begin{aligned}\beta_g^{(2)} =&\frac{75}{4}g^5 + \frac{1}{8}g^3\Big[48g_3^2 - 6{h_t}^2 - 6{h_t'}^2 + 10{g'}^2 \\ &\qquad - \hat{g}_{1dd}^2 - \hat{g}_{1du}^2 - \hat{g}_{1ud}^2 - \hat{g}_{1uu}^2 - 11\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big],\end{aligned} \tag{F.14a}$$

$$\begin{aligned}\beta_{g'}^{(2)} =&\frac{217}{36}{g'}^5 + \frac{1}{8}{g'}^3\Big[\frac{176}{3}g_3^2 - \frac{34}{3}{h_t}^2 - \frac{34}{3}{h_t'}^2 + 30g^2 \\ &\qquad - \hat{g}_{1dd}^2 - \hat{g}_{1du}^2 - \hat{g}_{1ud}^2 - \hat{g}_{1uu}^2 - 3\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big],\end{aligned} \tag{F.14b}$$

$$\beta_{g_3}^{(2)} = -\langle 13; -11\rangle g_3^5 + g_3^3\Big[-{h_t}^2 - {h_t'}^2 + \frac{9}{4}g^2 + \frac{11}{12}{g'}^2\Big]. \tag{F.14c}$$

Top Yukawa couplings

The one-loop beta functions of the top Yukawa couplings read

$$\begin{aligned}\beta_{h_t}^{(1)} =&\frac{9}{4}{h_t}^3 - \frac{1}{4}h_t\Big[16g_3^2 + \frac{9}{2}g^2 + \frac{17}{6}{g'}^2 - \hat{g}_{1ud}^2 - \hat{g}_{1uu}^2 - 3\hat{g}_{2ud}^2 - 3\hat{g}_{2uu}^2\Big] \\ &+ \frac{9}{4}{h_t}^2{h_t'}^2 + \frac{1}{4}h_t'\Big[\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big],\end{aligned} \tag{F.15a}$$

$$\begin{aligned}\beta_{h_t'}^{(1)} =&\frac{9}{4}{h_t'}^3 - \frac{1}{4}h_t'\Big[16g_3^2 + \frac{9}{2}g^2 + \frac{17}{6}{g'}^2 - \hat{g}_{1dd}^2 - \hat{g}_{1du}^2 - 3\hat{g}_{2dd}^2 - 3\hat{g}_{2du}^2\Big] \\ &+ \frac{9}{4}h_t'{h_t}^2 + \frac{1}{4}h_t\Big[\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big].\end{aligned} \tag{F.15b}$$

The corresponding two-loop beta functions are given by

$$\begin{aligned}\beta_{h_t}^{(2)} =& -6{h_t}^5 + \frac{1}{16}{h_t}^3\Big\{288g_3^2 - 192{h_t'}^2 - 48\lambda_2 + \frac{225}{2}g^2 + \frac{131}{2}{g'}^2 \\ &\qquad - 9\hat{g}_{1ud}^2 - 9\hat{g}_{1uu}^2 - 27\hat{g}_{2ud}^2 - 27\hat{g}_{2uu}^2\Big\} \\ &- \frac{1}{8}{h_t}^2h_t'\Big\{72\lambda_7 + 9\hat{g}_{1dd}\hat{g}_{1ud} + 9\hat{g}_{1du}\hat{g}_{1uu} + 27\hat{g}_{2dd}\hat{g}_{2ud} + 27\hat{g}_{2du}\hat{g}_{2uu}\Big\} \\ &- 6h_t{h_t'}^4 - 3{h_t'}^3\lambda_6 \\ &- h_t{h_t'}^2\Big\{3\big(\lambda_3 + \lambda_4 + \lambda_5\big) - 18g_3^2 \\ &\qquad - \frac{1}{32}\Big(225g^2 + 131{g'}^2 - 18\hat{g}_{1dd}^2 - 18\hat{g}_{1du}^2 - 54\hat{g}_{2dd}^2 - 54\hat{g}_{2du}^2\Big)\Big\} \\ &+ \frac{1}{32}h_t\Big\{24{\lambda_2}^2 + 16{\lambda_3}^2 + 16\lambda_3\lambda_4 + 16{\lambda_4}^2 + 24{\lambda_5}^2 + 24{\lambda_6}^2 + 72{\lambda_7}^2 \\ &\qquad - \langle 1728; \frac{4544}{3}\rangle g_3^4 + g_3^2\Big(144g^2 + \frac{304}{9}{g'}^2\Big) - 60g^4 - 12g^2{g'}^2 + \frac{922}{9}{g'}^4 \\ &\qquad + 5g^2\Big(3\hat{g}_{1ud}^2 + 3\hat{g}_{1uu}^2 + 13\hat{g}_{2ud}^2 + 13\hat{g}_{2uu}^2\Big) \\ &\qquad + 5{g'}^2\Big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\ &\qquad - 9\hat{g}_{1ud}^4 - 9\hat{g}_{1uu}^4 - 45\hat{g}_{2ud}^4 - 45\hat{g}_{2uu}^4 - 12\hat{g}_{2ud}^2\hat{g}_{2uu}^2 \\ &\qquad - 8\hat{g}_{1dd}\Big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu}\Big) - \hat{g}_{1ud}^2\Big(20\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 18\hat{g}_{2ud}^2\Big) \\ &\qquad - 3\hat{g}_{1dd}^2\Big(3\hat{g}_{1ud}^2 + 2\hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2\Big) - 24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} \\ &\qquad - 48\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} - \hat{g}_{2du}^2\Big(18\hat{g}_{2ud}^2 + 45\hat{g}_{2uu}^2\Big)\end{aligned}$$

$$
\begin{aligned}
&- 9\hat{g}_{1uu}^2\Big(\hat{g}_{2du}^2 + 2\hat{g}_{2uu}^2\Big) - \hat{g}_{2dd}^2\Big(45\hat{g}_{2ud}^2 + 18\hat{g}_{2uu}^2\Big)\\
&- 3\hat{g}_{1du}^2\Big(2\hat{g}_{1ud}^2 + 3\hat{g}_{1uu}^2 + 3\hat{g}_{2uu}^2\Big)\Bigg\}\\
+ \frac{1}{32}h_t'\Bigg\{&24\lambda_6\Big(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5\Big) + 24\lambda_7\Big(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5\Big)\\
&+ 5g^2\Big(3\hat{g}_{1dd}\hat{g}_{1ud} + 3\hat{g}_{1du}\hat{g}_{1uu} + 13\hat{g}_{2dd}\hat{g}_{2ud} + 13\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&+ 5{g'}^2\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- 9\hat{g}_{1ud}\hat{g}_{1dd}^3 - \hat{g}_{1dd}^2\Big(10\hat{g}_{1du}\hat{g}_{1uu} + 9\hat{g}_{2dd}\hat{g}_{2ud}\Big)\\
&- \hat{g}_{1dd}\Big[9\hat{g}_{1ud}^3 + 10\hat{g}_{1du}^2\hat{g}_{1ud} + \hat{g}_{1ud}\Big(10\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 9\hat{g}_{2ud}^2\Big)\\
&\qquad + 12\hat{g}_{1du}\Big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}\Big) + 12\hat{g}_{1uu}\Big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\Big)\Big]\\
&- 9\hat{g}_{1du}^3\hat{g}_{1uu} - 45\hat{g}_{2dd}^3\hat{g}_{2ud} - 9\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} - 45\hat{g}_{2du}^3\hat{g}_{2uu} - 9\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu}\\
&- 9\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} - 6\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} - 12\hat{g}_{1ud}\hat{g}_{1uu}\Big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}\Big)\\
&- \hat{g}_{2dd}\Big(45\hat{g}_{2ud}^3 + 6\hat{g}_{2du}^2\hat{g}_{2ud} + 6\hat{g}_{2uu}^2\hat{g}_{2ud}\Big) - \hat{g}_{2du}\Big(45\hat{g}_{2uu}^3 + 6\hat{g}_{2ud}^2\hat{g}_{2uu}\Big)\\
&- \hat{g}_{1du}\Big[9\hat{g}_{1uu}^3 + 10\hat{g}_{1ud}^2\hat{g}_{1uu} + 9\hat{g}_{1uu}\Big(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2\Big)\\
&\qquad + 12\hat{g}_{1ud}\Big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\Big)\Big]\Bigg\},
\end{aligned} \tag{F.16a}
$$

$$
\begin{aligned}
\beta_{h_t'}^{(2)} = &- 6{h_t'}^5 + \frac{1}{16}{h_t'}^3\Bigg\{288g_3^2 - 192{h_t}^2 - 48\lambda_1 + \frac{225}{2}g^2 + \frac{131}{2}{g'}^2\\
&\qquad - 9\hat{g}_{1dd}^2 - 9\hat{g}_{1du}^2 - 27\hat{g}_{2dd}^2 - 27\hat{g}_{2du}^2\Bigg\}\\
&- \frac{1}{8}{h_t'}^2h_t\Bigg\{72\lambda_6 + 9\hat{g}_{1dd}\hat{g}_{1ud} + 9\hat{g}_{1du}\hat{g}_{1uu} + 27\hat{g}_{2dd}\hat{g}_{2ud} + 27\hat{g}_{2du}\hat{g}_{2uu}\Bigg\}\\
&- 6h_t'{h_t}^4 - 3{h_t}^3\lambda_7\\
&- h_t'{h_t}^2\Bigg\{3\big(\lambda_3 + \lambda_4 + \lambda_5\big) - 18g_3^2\\
&\qquad - \frac{1}{32}\Big(225g^2 + 131{g'}^2 - 18\hat{g}_{1ud}^2 - 18\hat{g}_{1uu}^2 - 54\hat{g}_{2ud}^2 - 54\hat{g}_{2uu}^2\Bigg\}\\
&+ \frac{1}{32}h_t'\Bigg\{24{\lambda_1}^2 + 16{\lambda_3}^2 + 16\lambda_3\lambda_4 + 16{\lambda_4}^2 + 24{\lambda_5}^2 + 72{\lambda_6}^2 + 24{\lambda_7}^2\\
&\qquad - \langle 1728; \frac{4544}{3}\rangle g_3^4 + g_3^2\Big(144g^2 + \frac{304}{9}{g'}^2\Big) - 60g^4 - 12g^2{g'}^2 + \frac{922}{9}{g'}^4\\
&\qquad + 5g^2\Big(3\hat{g}_{1dd}^2 + 3\hat{g}_{1du}^2 + 13\hat{g}_{2dd}^2 + 13\hat{g}_{2du}^2\Big)\\
&\qquad + 5{g'}^2\Big(\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&\qquad - 9\hat{g}_{1dd}^4 - 9\hat{g}_{1du}^4 - 45\hat{g}_{2dd}^4 - 45\hat{g}_{2du}^4 - 9\hat{g}_{1ud}^2\hat{g}_{2dd}^2 - 9\hat{g}_{1uu}^2\hat{g}_{2du}^2\\
&\qquad - 8\hat{g}_{1dd}\Big[\hat{g}_{1du}\Big(\hat{g}_{1ud}\hat{g}_{1uu} + 6\hat{g}_{2dd}\hat{g}_{2du}\Big) + 3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\Big]\\
&\qquad - \hat{g}_{1dd}^2\Big(20\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 6\hat{g}_{1uu}^2 + 18\hat{g}_{2dd}^2 + 9\hat{g}_{2ud}^2\Big)\\
&\qquad - 24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} - \hat{g}_{2du}^2\Big(18\hat{g}_{2ud}^2 + 45\hat{g}_{2uu}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \hat{g}_{2dd}^2\Big(12\hat{g}_{2du}^2 + 45\hat{g}_{2ud}^2 + 18\hat{g}_{2uu}^2\Big) \\
&\qquad - \hat{g}_{1du}^2\Big(6\hat{g}_{1ud}^2 + 9\hat{g}_{1uu}^2 + 18\hat{g}_{2du}^2 + 9\hat{g}_{2uu}^2\Big)\Bigg\} \\
&+ \frac{1}{32}h_t\Bigg\{24\lambda_6\Big(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5\Big) + 24\lambda_7\Big(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5\Big) \\
&\qquad + 3g^2\Big(3\hat{g}_{1dd}\hat{g}_{1ud} + 3\hat{g}_{1du}\hat{g}_{1uu} + 13\hat{g}_{2dd}\hat{g}_{2ud} + 13\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&\qquad + 5{g'}^2\Big(\hat{g}_{1dd}\hat{g}_{1ud} + 5\hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&\qquad - 9\hat{g}_{1ud}\hat{g}_{1dd}^3 - \hat{g}_{1dd}^2\Big(10\hat{g}_{1du}\hat{g}_{1uu} + 9\hat{g}_{2dd}\hat{g}_{2ud}\Big) \\
&\qquad - \hat{g}_{1dd}\Big[9\hat{g}_{1ud}^3 + 10\hat{g}_{1du}^2\hat{g}_{1ud} + \hat{g}_{1ud}\Big(10\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 9\hat{g}_{2ud}^2\Big)\Big) \\
&\qquad\qquad + 12\hat{g}_{1du}\Big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu} + 12\hat{g}_{1uu}\Big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\Big)\Big] \\
&\qquad - 9\hat{g}_{1du}^3\hat{g}_{1uu} - 45\hat{g}_{2dd}^3\hat{g}_{2ud} - 9\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} - 45\hat{g}_{2du}^3\hat{g}_{2uu} - 9\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&\qquad - 9\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} - 6\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} - 12\hat{g}_{1ud}\hat{g}_{1uu}\Big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}\Big) \\
&\qquad - \hat{g}_{2dd}\Big(45\hat{g}_{2ud}^3 + 6\hat{g}_{2du}^2\hat{g}_{2ud} + 6\hat{g}_{2uu}^2\hat{g}_{2ud}\Big) - \hat{g}_{2du}\Big(45\hat{g}_{2uu}^3 + 6\hat{g}_{2ud}^2\hat{g}_{2uu}\Big) \\
&\qquad - \hat{g}_{1du}\Big[9\hat{g}_{1uu}^3 + 10\hat{g}_{1ud}^2\hat{g}_{1uu} + 9\hat{g}_{1uu}\Big(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2\Big) \\
&\qquad\qquad - 12\hat{g}_{1ud}\Big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\Big)\Big]\Bigg\}.
\end{aligned}
\tag{F.16b}
$$

Effective Higgs-Higgsino-Gaugino couplings

The one-loop beta functions of the effective Higgs-Higgsino-Gaugino couplings read

$$
\begin{aligned}
\beta_{\hat{g}_{1dd}}^{(1)} &= \frac{5}{8}\hat{g}_{1dd}^3 \\
&+ \frac{1}{8}\hat{g}_{1dd}\Big[12{h'_t}^2 - 9g^2 - 3{g'}^2 + 8\hat{g}_{1du}^2 + 5\hat{g}_{1ud}^2 + 2\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 6\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2\Big] \\
&+ \frac{3}{4}\Big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 2\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu} + 2\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\Big) \\
&+ \frac{3}{2}\hat{g}_{1ud}h_t h'_t,
\end{aligned}
\tag{F.17a}
$$

$$
\begin{aligned}
\beta_{\hat{g}_{1du}}^{(1)} &= \frac{5}{8}\hat{g}_{1du}^3 \\
&+ \frac{1}{8}\hat{g}_{1du}\Big[12{h'_t}^2 - 9g^2 - 3{g'}^2 + 8\hat{g}_{1dd}^2 + 2\hat{g}_{1ud}^2 + 5\hat{g}_{1uu}^2 + 6\hat{g}_{2dd}^2 + 9\hat{g}_{2du}^2 + 3\hat{g}_{2uu}^2\Big] \\
&+ \frac{3}{4}\Big(\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 2\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&+ \frac{3}{2}\hat{g}_{1uu}h_t h'_t,
\end{aligned}
\tag{F.17b}
$$

$$
\begin{aligned}
\beta_{\hat{g}_{1ud}}^{(1)} &= \frac{5}{8}\hat{g}_{1ud}^3 \\
&+ \frac{1}{8}\hat{g}_{1ud}\Big[12{h_t}^2 - 9g^2 - 3{g'}^2 + 5\hat{g}_{1dd}^2 + 2\hat{g}_{1du}^2 + 8\hat{g}_{1uu}^2 + 3\hat{g}_{2dd}^2 + 9\hat{g}_{2ud}^2 + 6\hat{g}_{2uu}^2\Big] \\
&+ \frac{3}{4}\Big(\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) + 2\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud}\big)\Big) \\
&+ \frac{3}{2}\hat{g}_{1dd}h_t h'_t,
\end{aligned}
\tag{F.17c}
$$

$$
\beta_{\hat{g}_{1uu}}^{(1)} = \frac{5}{8}\hat{g}_{1uu}^3
$$

$$
\begin{aligned}
&+\frac{1}{8}\hat{g}_{1uu}\Big[12{h_t}^2-9g^2-3{g'}^2+2\hat{g}_{1dd}^2+5\hat{g}_{1du}^2+8\hat{g}_{1ud}^2+3\hat{g}_{2du}^2+6\hat{g}_{2ud}^2+9\hat{g}_{2uu}^2\Big]\\
&+\frac{3}{4}\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud}+2\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2ud}+\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}+2\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&+\frac{3}{2}\hat{g}_{1du}h_th'_t,
\end{aligned}
\tag{F.17d}
$$

$$
\begin{aligned}
\beta^{(1)}_{\hat{g}_{2dd}}=&\;\frac{11}{8}\hat{g}_{2dd}^3\\
&+\frac{1}{8}\hat{g}_{2dd}\Big[12{h'_t}^2-33g^2-3{g'}^2+3\hat{g}_{1dd}^2+2\hat{g}_{1du}^2+\hat{g}_{1ud}^2+4\hat{g}_{2du}^2+11\hat{g}_{2ud}^2+2\hat{g}_{2uu}^2\Big]\\
&+\frac{1}{4}\Big(2\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du}+\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2ud}+2\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}+\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2ud}+\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&+\frac{3}{2}\hat{g}_{2ud}h_th'_t,
\end{aligned}
\tag{F.17e}
$$

$$
\begin{aligned}
\beta^{(1)}_{\hat{g}_{2du}}=&\;\frac{11}{8}\hat{g}_{2du}^3\\
&+\frac{1}{8}\hat{g}_{2du}\Big[12{h'_t}^2-3{g'}^2-33g^2+2\hat{g}_{1dd}^2+3\hat{g}_{1du}^2+\hat{g}_{1uu}^2+4\hat{g}_{2dd}^2+2\hat{g}_{2ud}^2+11\hat{g}_{2uu}^2\Big]\\
&+\frac{1}{4}\Big(2\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd}+\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2uu}+2\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud}+\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2uu}+\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&+\frac{3}{2}\hat{g}_{2uu}h_th'_t,
\end{aligned}
\tag{F.17f}
$$

$$
\begin{aligned}
\beta^{(1)}_{\hat{g}_{2ud}}=&\;\frac{11}{8}\hat{g}_{2ud}^3\\
&+\frac{1}{8}\hat{g}_{2ud}\Big(12{h_t}^2-3{g'}^2-33g^2+\hat{g}_{1dd}^2+3\hat{g}_{1ud}^2+2\hat{g}_{1uu}^2+11\hat{g}_{2dd}^2+2\hat{g}_{2du}^2+4\hat{g}_{2uu}^2\Big)\\
&+\frac{1}{4}\Big(\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd}+2\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du}+\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2dd}+2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}+\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&+\frac{3}{2}\hat{g}_{2dd}h_th'_t,
\end{aligned}
\tag{F.17g}
$$

$$
\begin{aligned}
\beta^{(1)}_{\hat{g}_{2uu}}=&\;\frac{11}{8}\hat{g}_{2uu}^3\\
&+\frac{1}{8}\hat{g}_{2uu}\Big[12{h_t}^2-3{g'}^2-33g^2+\hat{g}_{1du}^2+2\hat{g}_{1ud}^2+3\hat{g}_{1uu}^2+2\hat{g}_{2dd}^2+11\hat{g}_{2du}^2+4\hat{g}_{2ud}^2\Big]\\
&+\frac{1}{4}\Big(\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2du}+2\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}+2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}+\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\Big)\\
&+\frac{3}{2}\hat{g}_{2du}h_th'_t.
\end{aligned}
\tag{F.17h}
$$

The corresponding two-loop beta functions are given by

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{1dd}}=&-\frac{3}{8}\hat{g}_{1dd}^5\\
&+\hat{g}_{1dd}^3\Big[-\frac{3}{2}\lambda_1-\frac{27}{16}{h'_t}^2+\frac{165}{64}g^2+\frac{103}{64}{g'}^2\\
&\qquad-\frac{1}{32}\Big(60\hat{g}_{1du}^2+24\hat{g}_{1ud}^2+7\hat{g}_{1uu}^2+3\big(3\hat{g}_{2dd}^2+9\hat{g}_{2du}^2+4\hat{g}_{2ud}^2\big)\Big)\Big]\\
&+\hat{g}_{1dd}^2\Big[-\frac{9}{2}\lambda_6\hat{g}_{1ud}-\frac{27}{8}h_th'_t\hat{g}_{1ud}\\
&\qquad-\frac{1}{32}\Big(88\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu}+48\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu}+96\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}\\
&\qquad\qquad+9\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}+54\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}\Big)\Big]\\
&+\hat{g}_{1dd}\Big[10g_3^2{h'_t}^2-\frac{27}{8}{h_t}^2{h'_t}^2-\frac{27}{8}{h'_t}^4-\frac{15}{8}g^4-\frac{9}{8}g^2{g'}^2+\frac{43}{48}{g'}^4\\
&\qquad-\frac{9}{16}{h_t}^2\Big(3\hat{g}_{1ud}^2+2\hat{g}_{1uu}^2+3\hat{g}_{2ud}^2\Big)-\frac{3}{8}h_th'_t\Big(10\hat{g}_{1du}\hat{g}_{1uu}+9\hat{g}_{2dd}\hat{g}_{2ud}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{16} {h'_t}^2 \Big(45 g^2 + \frac{85}{3} {g'}^2 - 3\big(14 \hat{g}_{1du}^2 + 9 \hat{g}_{2dd}^2\big)\Big) \\
&+ \frac{3}{64} g^2 \Big(52 \hat{g}_{1du}^2 + 55 \hat{g}_{1ud}^2 + 34 \hat{g}_{1uu}^2 + 183 \hat{g}_{2dd}^2 + 110 \hat{g}_{2du}^2 + 25 \hat{g}_{2ud}^2\Big) \\
&+ \frac{1}{64} {g'}^2 \Big(4 \hat{g}_{1du}^2 + 103 \hat{g}_{1ud}^2 - 14 \hat{g}_{1uu}^2 + 63 \hat{g}_{2dd}^2 + 30 \hat{g}_{2du}^2 + 33 \hat{g}_{2ud}^2\Big) \\
&+ \frac{3}{4} {\lambda_1}^2 + \frac{1}{2} {\lambda_3}^2 + \frac{1}{2} \lambda_3 \lambda_4 + \frac{1}{2} {\lambda_4}^2 + \frac{3}{4} {\lambda_5}^2 + \frac{9}{4} {\lambda_5}^2 + \frac{3}{4} {\lambda_7}^2 \\
&- \frac{3}{2} \lambda_1 \big(\hat{g}_{1du}^2 + \hat{g}_{2dd}^2\big) - \frac{1}{2} \lambda_3 \big(3 \hat{g}_{1ud}^2 + 2 \hat{g}_{1uu}^2 + 3 \hat{g}_{2ud}^2\big) \\
&- \frac{1}{2} \lambda_4 \big(3 \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\big) - \frac{3}{2} \lambda_5 \hat{g}_{1ud}^2 - 3 \lambda_6 \big(\hat{g}_{1du} \hat{g}_{1uu} + \hat{g}_{2dd} \hat{g}_{2ud}\big) \\
&- \frac{1}{32} \Big(36 \hat{g}_{1du}^4 + \hat{g}_{1du}^2 \big(25 \hat{g}_{1ud}^2 + 43 \hat{g}_{1uu}^2 + 24 \hat{g}_{2uu}^2 + 75 (\hat{g}_{2dd}^2 + \hat{g}_{2du}^2)\big) \\
&\qquad - 6 \hat{g}_{1du} \big(21 \hat{g}_{1uu} \hat{g}_{2dd} \hat{g}_{2ud} + 2 \hat{g}_{1ud} \hat{g}_{2du} \hat{g}_{2ud} \\
&\qquad\qquad + 14 \hat{g}_{1ud} \hat{g}_{2dd} \hat{g}_{2uu} + 10 \hat{g}_{1uu} \hat{g}_{2du} \hat{g}_{2uu}\big) \\
&\qquad - 12 \hat{g}_{1ud}^4 - 7 \hat{g}_{1uu}^4 - 99 \hat{g}_{2dd}^4 - 45 \hat{g}_{2du}^4 - 30 \hat{g}_{2ud}^4 - 12 \hat{g}_{1uu}^2 \hat{g}_{2du}^2 \\
&\qquad - 42 \hat{g}_{2dd}^2 \hat{g}_{2du}^2 - 51 \hat{g}_{1uu}^2 \hat{g}_{2ud}^2 - 129 \hat{g}_{2dd}^2 \hat{g}_{2ud}^2 - 21 \hat{g}_{2du}^2 \hat{g}_{2ud}^2 \\
&\qquad - 3 \hat{g}_{2uu}^2 \big(7 \hat{g}_{1uu}^2 + 7 \hat{g}_{2dd}^2 + 15 \hat{g}_{2du}^2 + 10 \hat{g}_{2ud}^2\big) - 30 \hat{g}_{2dd} \hat{g}_{2du} \hat{g}_{2ud} \hat{g}_{2uu} \\
&\qquad - 36 \hat{g}_{1ud} \hat{g}_{1uu} \big(\hat{g}_{2dd} \hat{g}_{2du} + \hat{g}_{2ud} \hat{g}_{2uu}\big) \\
&\qquad - 3 \hat{g}_{1ud}^2 \big(14 \hat{g}_{1uu}^2 - \hat{g}_{2dd}^2 + 8 \hat{g}_{2ud}^2 + 9 \hat{g}_{2uu}^2\big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ 10 g_3^2 h_t h'_t \hat{g}_{1ud} - \frac{27}{8} {h_t}^3 h'_t \hat{g}_{1ud} - \frac{27}{8} h_t {h'_t}^3 \hat{g}_{1ud} \\
&- \frac{3}{2} {h'_t}^2 \Big(\hat{g}_{1du} \hat{g}_{1ud} \hat{g}_{1uu} + 3 \hat{g}_{2du} \hat{g}_{2ud} \hat{g}_{1uu} + 3 \hat{g}_{1du} \hat{g}_{2dd} \hat{g}_{2du}\Big) \\
&+ h_t h'_t \Big(\frac{45}{16} g^2 \hat{g}_{1ud} + \frac{85}{48} {g'}^2 \hat{g}_{1ud} - \frac{3}{2} \big(\hat{g}_{1ud} \hat{g}_{1uu}^2 + 3 \hat{g}_{2uu} (\hat{g}_{1du} \hat{g}_{2dd} + \hat{g}_{1uu} \hat{g}_{2ud})\big)\Big) \\
&+ \frac{3}{32} g^2 \Big(9 \hat{g}_{1du} \hat{g}_{1ud} \hat{g}_{1uu} + 92 \hat{g}_{2du} \hat{g}_{2ud} \hat{g}_{1uu} - 24 \hat{g}_{2dd} \hat{g}_{2uu} \hat{g}_{1uu} + 68 \hat{g}_{1du} \hat{g}_{2dd} \hat{g}_{2du} \\
&\qquad + 79 \hat{g}_{1ud} \hat{g}_{2dd} \hat{g}_{2ud} + 55 \hat{g}_{1ud} \hat{g}_{2du} \hat{g}_{2uu}\Big) \\
&+ \frac{3}{32} {g'}^2 \Big(3 \hat{g}_{1du} \hat{g}_{1ud} \hat{g}_{1uu} + 4 \hat{g}_{2du} \hat{g}_{2ud} \hat{g}_{1uu} + 4 \hat{g}_{1du} \hat{g}_{2dd} \hat{g}_{2du} \\
&\qquad + 5 \hat{g}_{1ud} \hat{g}_{2dd} \hat{g}_{2ud} + 5 \hat{g}_{1ud} \hat{g}_{2du} \hat{g}_{2uu}\Big) \\
&+ \frac{3}{4} \lambda_1 \lambda_6 \hat{g}_{1ud} + \frac{3}{4} \lambda_2 \lambda_7 \hat{g}_{1ud} + \frac{3}{4} \lambda_3 \lambda_5 \hat{g}_{1ud} + \frac{3}{4} \lambda_3 \lambda_7 \hat{g}_{1ud} \\
&+ \frac{3}{4} \lambda_4 \lambda_6 \hat{g}_{1ud} + \frac{3}{4} \lambda_4 \lambda_7 \hat{g}_{1ud} + \frac{3}{4} \lambda_5 \lambda_6 \hat{g}_{1ud} + \frac{3}{4} \lambda_5 \lambda_7 \hat{g}_{1ud} \\
&- \frac{3}{2} \lambda_1 \hat{g}_{1du} \hat{g}_{2dd} \hat{g}_{2du} - \frac{1}{2} \lambda_3 \hat{g}_{1uu} \Big(\hat{g}_{1du} \hat{g}_{1ud} + 3 \hat{g}_{2du} \hat{g}_{2ud}\Big) \\
&- \frac{1}{2} \lambda_4 \Big(2 \hat{g}_{1du} \hat{g}_{1ud} \hat{g}_{1uu} + 3 \hat{g}_{2dd} \big(\hat{g}_{1ud} \hat{g}_{2ud} + \hat{g}_{1uu} \hat{g}_{2uu}\big)\Big) \\
&- \frac{3}{2} \lambda_5 \Big(\hat{g}_{1du} \hat{g}_{1ud} \hat{g}_{1uu} + \hat{g}_{1ud} \hat{g}_{2dd} \hat{g}_{2ud} + \hat{g}_{1du} \hat{g}_{2ud} \hat{g}_{2uu}\Big) \\
&- \frac{3}{2} \lambda_6 \Big(\hat{g}_{1ud} \hat{g}_{1du}^2 + \hat{g}_{2du} \hat{g}_{2ud} \hat{g}_{1du} + \hat{g}_{2dd} \hat{g}_{2uu} \hat{g}_{1du} + \hat{g}_{1ud} \hat{g}_{2dd}^2 + \hat{g}_{1uu} \hat{g}_{2dd} \hat{g}_{2du}\Big) \\
&- \frac{3}{2} \lambda_7 \Big(\hat{g}_{1ud}^3 + \hat{g}_{1ud} \big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) + \hat{g}_{1uu} \hat{g}_{2ud} \hat{g}_{2uu}\Big) \\
&- \frac{1}{32} \Big[\hat{g}_{1du}^3 \big(29 \hat{g}_{1ud} \hat{g}_{1uu} + 36 \hat{g}_{2dd} \hat{g}_{2du}\big) \\
&\qquad + 3 \hat{g}_{1du}^2 \big(12 \hat{g}_{1uu} \hat{g}_{2du} \hat{g}_{2ud} + 8 \hat{g}_{1uu} \hat{g}_{2dd} \hat{g}_{2uu} + 3 \hat{g}_{1ud} \hat{g}_{2du} \hat{g}_{2uu}\big) \\
&\qquad + 3 \hat{g}_{2uu}^2 \hat{g}_{1du} \big(7 \hat{g}_{1ud} \hat{g}_{1uu} + 44 \hat{g}_{2dd} \hat{g}_{2du}\big) \\
&\qquad + \hat{g}_{1du} \big(18 \hat{g}_{1uu} \hat{g}_{1ud}^3 + 12 \hat{g}_{2dd} \hat{g}_{2du} \hat{g}_{1ud}^2 + \hat{g}_{1ud} \hat{g}_{1uu} (29 \hat{g}_{1uu}^2 + 24 \hat{g}_{2dd}^2 + 45 \hat{g}_{2du}^2)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + 12\hat{g}_{2dd}\hat{g}_{2du}(\hat{g}_{1uu}^2 + 6\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2)\Big) \\
&+ 12\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu}\big(\hat{g}_{1ud}^2 + 2\hat{g}_{1uu}^2 + 4\hat{g}_{2dd}^2\big) \\
&- 3\Big(5\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{1ud}^3 - 4\hat{g}_{1ud}^2\hat{g}_{1uu}\big(\hat{g}_{2du}\hat{g}_{2ud} + 3\hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\quad - \hat{g}_{1ud}\big(15\hat{g}_{2du}\hat{g}_{2uu}^3 + 2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 \\
&\qquad + \hat{g}_{2du}\hat{g}_{2uu}(11\hat{g}_{1uu}^2 + 2\hat{g}_{2dd}^2 + 15\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2) \\
&\qquad + \hat{g}_{2dd}\hat{g}_{2ud}(8\hat{g}_{1uu}^2 + 23\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2 + 23\hat{g}_{2ud}^2)\big) \\
&\quad - 4\hat{g}_{1uu}\hat{g}_{2ud}\big(11\hat{g}_{2du}\hat{g}_{2uu}^2 + 4\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad\quad + \hat{g}_{2du}(\hat{g}_{1uu}^2 + 6\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2)\big)\Big)\Bigg],
\end{aligned}
\tag{F.18a}
$$

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{1du}} = &- \frac{3}{8}\hat{g}_{1du}^5 \\
&+ \hat{g}_{1du}^3\bigg[-\frac{3}{2}\lambda_1 - \frac{27}{16}{h_t'}^2 + \frac{165}{64}g^2 + \frac{103}{64}{g'}^2 \\
&\quad - \frac{1}{32}\Big(60\hat{g}_{1dd}^2 + 7\hat{g}_{1ud}^2 + 3\big(8\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 4\hat{g}_{2uu}^2\big)\Big)\bigg] \\
&+ \hat{g}_{1du}^2\bigg[-\frac{27}{8}h_t h_t'\hat{g}_{1uu} - \frac{9}{2}\lambda_6\hat{g}_{1uu} \\
&\quad - \frac{1}{32}\Big(88\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 54\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{1uu} + 9\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{1uu} \\
&\qquad + 96\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du} + 48\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\Big)\bigg] \\
&+ \hat{g}_{1du}\bigg[10g_3^2{h_t'}^2 - \frac{27}{8}{h_t}^2{h_t'}^2 - \frac{27}{8}{h_t'}^4 - \frac{15}{8}g^4 - \frac{9}{8}g^2{g'}^2 + \frac{43}{48}{g'}^4 \\
&\quad - \frac{9}{16}{h_t}^2\Big(2\hat{g}_{1ud}^2 + 3\big(\hat{g}_{1uu}^2 + \hat{g}_{2uu}^2\big)\Big) - \frac{3}{8}h_t h_t'\Big(10\hat{g}_{1dd}\hat{g}_{1ud} + 9\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&\quad + \frac{1}{16}{h_t'}^2\Big(45g^2 + \frac{85}{3}{g'}^2 - 3\big(14\hat{g}_{1dd}^2 + 9\hat{g}_{2du}^2\big)\Big) \\
&\quad + \frac{3}{64}g^2\Big(52\hat{g}_{1dd}^2 + 34\hat{g}_{1ud}^2 + 55\hat{g}_{1uu}^2 + 110\hat{g}_{2dd}^2 + 183\hat{g}_{2du}^2 + 25\hat{g}_{2uu}^2\Big) \\
&\quad + \frac{1}{64}{g'}^2\Big(4\hat{g}_{1dd}^2 - 14\hat{g}_{1ud}^2 + 103\hat{g}_{1uu}^2 + 30\hat{g}_{2dd}^2 + 63\hat{g}_{2du}^2 + 33\hat{g}_{2uu}^2\Big) \\
&\quad + \frac{3}{4}{\lambda_1}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{9}{4}{\lambda_6}^2 + \frac{3}{4}{\lambda_7}^2 \\
&\quad - \frac{3}{2}\lambda_1\big(\hat{g}_{1dd}^2 + \hat{g}_{2du}^2\big) - \frac{1}{2}\lambda_3\big(2\hat{g}_{1ud}^2 + 3(\hat{g}_{1uu}^2 + \hat{g}_{2uu}^2)\big) - \frac{1}{2}\lambda_4\big(\hat{g}_{1ud}^2 + 3\hat{g}_{1uu}^2\big) \\
&\quad - \frac{3}{2}\lambda_5\hat{g}_{1uu}^2 - 3\lambda_6\big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\quad + \frac{1}{32}\Big(36\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(43\hat{g}_{1ud}^2 + 25\hat{g}_{1uu}^2 + 24\hat{g}_{2ud}^2 + 75(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2)\big) \\
&\qquad + 6\hat{g}_{1dd}\big(10\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 14\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} \\
&\qquad\qquad + 2\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 21\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + 7\hat{g}_{1ud}^4 + 36\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad + 3\hat{g}_{1ud}^2\big(14\hat{g}_{1uu}^2 + 4\hat{g}_{2dd}^2 + 7\hat{g}_{2ud}^2 + 17\hat{g}_{2uu}^2\big) \\
&\qquad + 3\big(4\hat{g}_{1uu}^4 - \hat{g}_{1uu}^2(\hat{g}_{2du}^2 - 9\hat{g}_{2ud}^2 - 8\hat{g}_{2uu}^2) + 15\hat{g}_{2dd}^4 + 33\hat{g}_{2du}^4 \\
&\qquad\quad + 10\hat{g}_{2uu}^4 + 7\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 43\hat{g}_{2du}^2\hat{g}_{2uu}^2 + 10\hat{g}_{2ud}^2\hat{g}_{2uu}^2 \\
&\qquad\quad + 10\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{2dd}^2(14\hat{g}_{2du}^2 + 15\hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big)\Big)\bigg] \\
&+ 10g_3^2 h_t h_t'\hat{g}_{1uu} - \frac{27}{8}{h_t}^3 h_t'\hat{g}_{1uu} - \frac{27}{8}h_t{h_t'}^3\hat{g}_{1uu} \\
&- \frac{3}{2}{h_t'}^2\Big(\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 3\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ h_t h_t'\Big(\frac{45}{16}g^2\hat{g}_{1uu} + \frac{85}{48}g'^2\hat{g}_{1uu} - \frac{3}{2}\big(\hat{g}_{1uu}\hat{g}_{1ud}^2 + 3\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud} + 3\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud}\big)\Big)\\
&+ \frac{3}{32}g^2\Big(9\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 55\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{1uu} + 79\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{1uu} + 68\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du}\\
&\qquad - 24\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} + 92\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\Big)\\
&+ \frac{3}{32}g'^2\Big(3\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 5\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{1uu} + 5\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{1uu}\\
&\qquad + 4\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du} + 4\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\Big)\\
&+ \frac{3}{4}\lambda_1\lambda_6\hat{g}_{1uu} + \frac{3}{4}\lambda_2\lambda_7\hat{g}_{1uu} + \frac{3}{4}\lambda_3\lambda_6\hat{g}_{1uu} + \frac{3}{4}\lambda_3\lambda_7\hat{g}_{1uu}\\
&+ \frac{3}{4}\lambda_4\lambda_6\hat{g}_{1uu} + \frac{3}{4}\lambda_4\lambda_7\hat{g}_{1uu} + \frac{3}{4}\lambda_5\lambda_6\hat{g}_{1uu} + \frac{3}{4}\lambda_5\lambda_7\hat{g}_{1uu}\\
&- \frac{3}{2}\lambda_1\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du} - \frac{1}{2}\lambda_3\hat{g}_{1ud}\Big(\hat{g}_{1dd}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2uu}\Big)\\
&- \frac{1}{2}\lambda_4\Big(2\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + 3\hat{g}_{2du}\big(\hat{g}_{1ud}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2uu}\big)\Big)\\
&- \frac{3}{2}\lambda_5\Big(\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{1uu} + \hat{g}_{1dd}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&- \frac{3}{2}\lambda_6\Big(\hat{g}_{1uu}\hat{g}_{1dd}^2 + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1dd} + \hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{1dd} + \hat{g}_{1uu}\hat{g}_{2du}^2 + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du}\Big)\\
&- \frac{3}{2}\lambda_7\Big(\hat{g}_{1uu}^3 + \hat{g}_{1ud}^2\hat{g}_{1uu} + \hat{g}_{2uu}^2\hat{g}_{1uu} + \hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&- \frac{1}{32}\Big[\hat{g}_{1dd}^3\big(29\hat{g}_{1ud}\hat{g}_{1uu} + 36\hat{g}_{2dd}\hat{g}_{2du}\big)\\
&\qquad + 3\hat{g}_{1dd}^2\big(3\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} + 12\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\big)\\
&\qquad + \hat{g}_{1dd}\Big(29\hat{g}_{1uu}\hat{g}_{1ud}^3 + 12\hat{g}_{1ud}^2\big(\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad\qquad + 3\hat{g}_{1ud}\hat{g}_{1uu}\big(6\hat{g}_{1uu}^2 + 15\hat{g}_{2dd}^2 + 8\hat{g}_{2du}^2 + 7\hat{g}_{2ud}^2\big)\\
&\qquad\qquad + 12\big(2\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2 + \hat{g}_{2ud}\hat{g}_{2uu}(\hat{g}_{1uu}^2 + 4\hat{g}_{2du}^2)\\
&\qquad\qquad\qquad + \hat{g}_{2dd}\hat{g}_{2du}(\hat{g}_{1uu}^2 + 11\hat{g}_{2dd}^2 + 6\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2))\Big)\\
&\qquad + 3\hat{g}_{2uu}^3\big(8\hat{g}_{1ud}\hat{g}_{2dd} + 23\hat{g}_{1uu}\hat{g}_{2du}\big) + 6\hat{g}_{2ud}\hat{g}_{2uu}^2\big(\hat{g}_{1uu}\hat{g}_{2dd} + 8\hat{g}_{1ud}\hat{g}_{2du}\big)\\
&\qquad + 3\hat{g}_{1uu}\hat{g}_{2ud}\Big(15\hat{g}_{2dd}^3 + 11\hat{g}_{1ud}^2\hat{g}_{2dd} + 2\hat{g}_{2du}^2\hat{g}_{2dd} + 15\hat{g}_{2ud}^2\hat{g}_{2dd} + 12\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\Big)\\
&\qquad + 3\hat{g}_{2uu}\Big(4\hat{g}_{2dd}\hat{g}_{1ud}^3 + 8\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{1ud}^2\\
&\qquad\qquad + 4\hat{g}_{2dd}\hat{g}_{1ud}\big(\hat{g}_{1uu}^2 + 11\hat{g}_{2dd}^2 + 6\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2\big)\\
&\qquad\qquad - \hat{g}_{1uu}\hat{g}_{2du}\big(5\hat{g}_{1uu}^2 - 2\hat{g}_{2dd}^2 - 23\hat{g}_{2du}^2 - 2\hat{g}_{2ud}^2\big)\Big)\Big],
\end{aligned}
\tag{F.18b}
$$

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{1ud}} =& -\frac{3}{8}\hat{g}_{1ud}^5\\
&+ \hat{g}_{1ud}^3\Big[-\frac{3}{2}\lambda_2 - \frac{27}{16}{h_t}^2 + \frac{165}{64}g^2 + \frac{103}{64}g'^2\\
&\qquad - \frac{1}{32}\Big(24\hat{g}_{1dd}^2 + 7\hat{g}_{1du}^2 + 3\big(20\hat{g}_{1uu}^2 + 4\hat{g}_{2dd}^2 + 3\hat{g}_{2ud}^2 + 9\hat{g}_{2uu}^2\big)\Big)\Big]\\
&+ \hat{g}_{1ud}^2\Big[-\frac{9}{2}\lambda_7\hat{g}_{1dd} - \frac{27}{8}h_t h_t'\hat{g}_{1dd}\\
&\qquad - \frac{1}{32}\Big(48\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{2dd} + 2\hat{g}_{1uu}\hat{g}_{2ud}\big)\\
&\qquad\qquad + \hat{g}_{1dd}\big(88\hat{g}_{1du}\hat{g}_{1uu} + 9\hat{g}_{2dd}\hat{g}_{2ud} + 54\hat{g}_{2du}\hat{g}_{2uu}\big)\Big)\Big]\\
&+ \hat{g}_{1ud}\Big[10g_3^2{h_t}^2 - \frac{27}{8}{h_t}^4 - \frac{27}{8}{h_t}^2{h_t'}^2 - \frac{15}{8}g^4 - \frac{9}{8}g^2g'^2 + \frac{43}{48}g'^4
\end{aligned}
$$

$$
\begin{aligned}
&+ {h_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{3}{16}\big(14\hat{g}_{1uu}^2 + 9\hat{g}_{2ud}^2\big)\Big) \\
&- \frac{9}{16}{h'_t}^2\Big(3\hat{g}_{1dd}^2 + 2\hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2\Big) - \frac{3}{8}h_t h'_t\Big(10\hat{g}_{1du}\hat{g}_{1uu} + 9\hat{g}_{2dd}\hat{g}_{2ud}\Big) \\
&+ \frac{3}{64}g^2\Big(55\hat{g}_{1dd}^2 + 34\hat{g}_{1du}^2 + 52\hat{g}_{1uu}^2 + 25\hat{g}_{2dd}^2 + 183\hat{g}_{2ud}^2 + 110\hat{g}_{2uu}^2\Big) \\
&+ \frac{1}{64}{g'}^2\Big(103\hat{g}_{1dd}^2 - 14\hat{g}_{1du}^2 + 4\hat{g}_{1uu}^2 + 33\hat{g}_{2dd}^2 + 63\hat{g}_{2ud}^2 + 30\hat{g}_{2uu}^2\Big) \\
&+ \frac{3}{4}{\lambda_2}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{3}{4}{\lambda_6}^2 + \frac{9}{4}{\lambda_7}^2 \\
&- \frac{3}{2}\lambda_2\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) - \frac{1}{2}\lambda_3\big(3\hat{g}_{1dd}^2 + 2\hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2\big) - \frac{1}{2}\lambda_4\big(3\hat{g}_{1dd}^2 + \hat{g}_{1du}^2\big) \\
&- \frac{3}{2}\lambda_5\hat{g}_{1dd}^2 - 3\lambda_7\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&- \frac{1}{32}\Big(12\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(42\hat{g}_{1du}^2 + 25\hat{g}_{1uu}^2 + 24\hat{g}_{2dd}^2 + 27\hat{g}_{2du}^2 - 3\hat{g}_{2ud}^2\big) \\
&\qquad + 12\hat{g}_{1dd}\big(3\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + 7\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2du} \\
&\qquad\qquad + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 3\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad + 7\hat{g}_{1du}^4 + 6\hat{g}_{1du}\hat{g}_{1uu}\big(21\hat{g}_{2dd}\hat{g}_{2ud} + 10\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1du}^2\big(43\hat{g}_{1uu}^2 + 51\hat{g}_{2dd}^2 + 21\hat{g}_{2du}^2 + 12\hat{g}_{2uu}^2\big) \\
&\qquad + 3\big(12\hat{g}_{1uu}^4 + \hat{g}_{1uu}^2(8\hat{g}_{2du}^2 + 25\hat{g}_{2ud}^2 + 25\hat{g}_{2uu}^2) + 10\hat{g}_{2dd}^4 + 33\hat{g}_{2ud}^4 \\
&\qquad\quad + 15\hat{g}_{2uu}^4 + 7\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 15\hat{g}_{2du}^2\hat{g}_{2uu}^2 + 14\hat{g}_{2ud}^2\hat{g}_{2uu}^2 \\
&\qquad\quad + 10\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{2dd}^2(10\hat{g}_{2du}^2 + 43\hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big)\Big)\Big] \\
+ &\, 10g_3^2 h_t h'_t\hat{g}_{1dd} - \frac{27}{8}{h_t}^3 h'_t\hat{g}_{1dd} - \frac{27}{8}h_t{h'_t}^3\hat{g}_{1dd} \\
- &\, \frac{3}{2}{h_t}^2\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud}\big)\Big) \\
+ &\, h_t h'_t\Big(\frac{45}{16}g^2\hat{g}_{1dd} + \frac{85}{48}{g'}^2\hat{g}_{1dd} - \frac{3}{2}\big(\hat{g}_{1dd}\hat{g}_{1du}^2 + 3\hat{g}_{2du}(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud})\big)\Big) \\
- &\, \frac{3}{32}g^2\Big(24\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud} - 68\hat{g}_{1uu}\hat{g}_{2uu}\hat{g}_{2ud} - 92\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} \\
&\qquad - \hat{g}_{1dd}\big(9\hat{g}_{1du}\hat{g}_{1uu} + 79\hat{g}_{2dd}\hat{g}_{2ud} + 55\hat{g}_{2du}\hat{g}_{2uu}\big)\Big) \\
+ &\, \frac{3}{32}{g'}^2\Big(4\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud}\big) + \hat{g}_{1dd}\big(3\hat{g}_{1du}\hat{g}_{1uu} + 5\hat{g}_{2dd}\hat{g}_{2ud} + 5\hat{g}_{2du}\hat{g}_{2uu}\big)\Big) \\
+ &\, \frac{3}{4}\lambda_1\lambda_6\hat{g}_{1dd} + \frac{3}{4}\lambda_2\lambda_7\hat{g}_{1dd} + \frac{3}{4}\lambda_3\lambda_6\hat{g}_{1dd} + \frac{3}{4}\lambda_3\lambda_7\hat{g}_{1dd} \\
+ &\, \frac{3}{4}\lambda_4\lambda_6\hat{g}_{1dd} + \frac{3}{4}\lambda_4\lambda_7\hat{g}_{1dd} + \frac{3}{4}\lambda_5\lambda_6\hat{g}_{1dd} + \frac{3}{4}\lambda_5\lambda_7\hat{g}_{1dd} \\
- &\, \frac{3}{2}\lambda_2\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} - \frac{1}{2}\lambda_3\hat{g}_{1du}\Big(\hat{g}_{1dd}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2uu}\Big) \\
- &\, \lambda_4\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1uu} + \frac{3}{2}\hat{g}_{2ud}\big(\hat{g}_{1dd}\hat{g}_{2dd} + \hat{g}_{1du}\hat{g}_{2du}\big)\Big) \\
- &\, \frac{3}{2}\lambda_5\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1uu} + \hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2ud}\Big) \\
- &\, \frac{3}{2}\lambda_6\Big(\hat{g}_{1dd}^3 + \hat{g}_{1dd}\big(\hat{g}_{1du}^2 + \hat{g}_{2dd}^2\big) + \hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}\Big) \\
- &\, \frac{3}{2}\lambda_7\Big(\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1du}\hat{g}_{2uu}\hat{g}_{2ud} + \hat{g}_{1dd}\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}\Big) \\
+ &\, \frac{1}{32}\Big[3\hat{g}_{1dd}^3\big(5\hat{g}_{2dd}\hat{g}_{2ud} - 6\hat{g}_{1du}\hat{g}_{1uu}\big) \\
&\qquad - 12\hat{g}_{1dd}^2\big(\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2du} + \hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad - \hat{g}_{1dd}\Big(29\hat{g}_{1uu}\hat{g}_{1du}^3 + 3\hat{g}_{1du}^2\big(8\hat{g}_{2dd}\hat{g}_{2ud} + 11\hat{g}_{2du}\hat{g}_{2uu}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \hat{g}_{1du}\hat{g}_{1uu}\big(29\hat{g}_{1uu}^2 + 21\hat{g}_{2du}^2 + 24\hat{g}_{2ud}^2 + 45\hat{g}_{2uu}^2\big) + 45\hat{g}_{2du}\hat{g}_{2uu}^3 \\
&+ 6\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 + 3\hat{g}_{2dd}\hat{g}_{2ud}\big(23\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2 + 23\hat{g}_{2ud}^2\big) \\
&+ 3\hat{g}_{2du}\hat{g}_{2uu}\big(3\hat{g}_{1uu}^2 + 2\hat{g}_{2dd}^2 + 15\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2\big)\Big) \\
&- 12\big(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud}\big)\Big(11\hat{g}_{2uu}^3 \\
&\qquad + \hat{g}_{2uu}\big(\hat{g}_{1du}^2 + 3\hat{g}_{1uu}^2 + 2\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 6\hat{g}_{2ud}^2\big) \\
&\qquad + 2\hat{g}_{2du}\big(\hat{g}_{1du}\hat{g}_{1uu} + 2\hat{g}_{2dd}\hat{g}_{2ud}\big)\Big)\Big],
\end{aligned} \tag{F.18c}
$$

$$
\begin{aligned}
\beta_{\hat{g}_{1uu}}^{(2)} =& -\frac{3}{8}\hat{g}_{1uu}^5 \\
&+ \hat{g}_{1uu}^3\Big[-\frac{3}{2}\lambda_2 - \frac{27}{16}{h_t}^2 + \frac{165}{64}g^2 + \frac{103}{64}{g'}^2 \\
&\qquad - \frac{1}{32}\Big(7\hat{g}_{1dd}^2 + 3\big(8\hat{g}_{1du}^2 + 20\hat{g}_{1ud}^2 + 4\hat{g}_{2du}^2 + 9\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\big)\Big)\Big] \\
&+ \hat{g}_{1uu}^2\Big[-\frac{27}{8}h_t h_t'\hat{g}_{1du} - \frac{9}{2}\lambda_7\hat{g}_{1du} \\
&\qquad - \frac{1}{32}\Big(88\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud} + 96\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud} + 54\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2ud} \\
&\qquad\quad + 48\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud} + 9\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\Big)\Big] \\
&+ \hat{g}_{1uu}\Big[10g_3^2{h_t}^2 - \frac{27}{8}{h_t}^4 - \frac{27}{8}{h_t}^2{h_t'}^2 - \frac{15}{8}g^4 - \frac{9}{8}g^2{g'}^2 + \frac{43}{48}{g'}^4 \\
&\qquad + {h_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{3}{16}\big(14\hat{g}_{1ud}^2 + 9\hat{g}_{2uu}^2\big)\Big) \\
&\qquad - \frac{3}{8}h_t h_t'\Big(10\hat{g}_{1dd}\hat{g}_{1ud} + 9\hat{g}_{2du}\hat{g}_{2uu}\Big) - \frac{9}{16}{h_t'}^2\Big(2\hat{g}_{1dd}^2 + 3\big(\hat{g}_{1du}^2 + \hat{g}_{2du}^2\big)\Big) \\
&\qquad + \frac{3}{64}g^2\Big(34\hat{g}_{1dd}^2 + 55\hat{g}_{1du}^2 + 52\hat{g}_{1ud}^2 + 25\hat{g}_{2du}^2 + 110\hat{g}_{2ud}^2 + 183\hat{g}_{2uu}^2\Big) \\
&\qquad - \frac{1}{64}{g'}^2\Big(14\hat{g}_{1dd}^2 - 103\hat{g}_{1du}^2 - 4\hat{g}_{1ud}^2 - 33\hat{g}_{2du}^2 - 30\hat{g}_{2ud}^2 - 63\hat{g}_{2uu}^2\Big) \\
&\qquad + \frac{3}{4}{\lambda_2}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{3}{4}{\lambda_6}^2 + \frac{9}{4}{\lambda_7}^2 \\
&\qquad - \frac{3}{2}\lambda_2\big(\hat{g}_{1ud}^2 + \hat{g}_{2uu}^2\big) - \frac{1}{2}\lambda_3\big(2\hat{g}_{1dd}^2 + 3(\hat{g}_{1du}^2 + \hat{g}_{2du}^2)\big) - \frac{1}{2}\lambda_4\big(\hat{g}_{1dd}^2 + 3\hat{g}_{1du}^2\big) \\
&\qquad - \frac{3}{2}\lambda_5\hat{g}_{1du}^2 - 3\lambda_7\big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad - \frac{1}{32}\Big(7\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(42\hat{g}_{1du}^2 + 43\hat{g}_{1ud}^2 + 21\hat{g}_{2dd}^2 + 51\hat{g}_{2du}^2 + 12\hat{g}_{2ud}^2\big) \\
&\qquad\quad + 6\hat{g}_{1dd}(6\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + 21\hat{g}_{1ud}\hat{g}_{2uu}\hat{g}_{2du} \\
&\qquad\qquad + 10\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 6\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu}) \\
&\qquad\quad + 12\hat{g}_{1du}^4 + 12\hat{g}_{1du}\hat{g}_{1ud}\big(\hat{g}_{2du}\hat{g}_{2ud} + 7\hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\qquad\quad + \hat{g}_{1du}^2\big(25\hat{g}_{1ud}^2 + 27\hat{g}_{2dd}^2 + 24\hat{g}_{2du}^2 - 3\hat{g}_{2uu}^2\big) \\
&\qquad\quad + 3\big(12\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2(8\hat{g}_{2dd}^2 + 25\hat{g}_{2ud}^2 + 25\hat{g}_{2uu}^2) + 10\hat{g}_{2du}^4 \\
&\qquad\qquad + 15\hat{g}_{2ud}^4 + 33\hat{g}_{2uu}^4 + 7\hat{g}_{2du}^2\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2(43\hat{g}_{2du}^2 + 14\hat{g}_{2ud}^2) \\
&\qquad\qquad + 10\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{2dd}^2(10\hat{g}_{2du}^2 + 15\hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big)\Big)\Big] \\
&+ 10g_3^2 h_t h_t'\hat{g}_{1du} - \frac{27}{8}{h_t}^3 h_t'\hat{g}_{1du} - \frac{27}{8}h_t{h_t'}^3\hat{g}_{1du} \\
&- \frac{3}{2}{h_t}^2\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud} + 3\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud} + 3\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud}\Big) \\
&+ h_t h_t'\Big(\frac{45}{16}g^2\hat{g}_{1du} + \frac{85}{48}{g'}^2\hat{g}_{1du} - \frac{3}{2}\big(\hat{g}_{1du}\hat{g}_{1dd}^2 + 3\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1dd} + 3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{32}g^2\Big(55\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2ud}+68\hat{g}_{1ud}\hat{g}_{2uu}\hat{g}_{2ud}+79\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\\
&\qquad+\hat{g}_{1dd}\big(9\hat{g}_{1du}\hat{g}_{1ud}+92\hat{g}_{2du}\hat{g}_{2ud}-24\hat{g}_{2dd}\hat{g}_{2uu}\big)\Big)\\
&+\frac{3}{32}{g'}^2\Big(3\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud}+4\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud}+5\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2ud}\\
&\qquad+4\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud}+5\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&+\frac{3}{4}\lambda_1\lambda_6\hat{g}_{1du}+\frac{3}{4}\lambda_2\lambda_7\hat{g}_{1du}+\frac{3}{4}\lambda_3\lambda_6\hat{g}_{1du}+\frac{3}{4}\lambda_3\lambda_7\hat{g}_{1du}\\
&+\frac{3}{4}\lambda_4\lambda_6\hat{g}_{1du}+\frac{3}{4}\lambda_4\lambda_7\hat{g}_{1du}+\frac{3}{4}\lambda_5\lambda_6\hat{g}_{1du}+\frac{3}{4}\lambda_5\lambda_7\hat{g}_{1du}\\
&-\frac{3}{2}\lambda_2\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}-\frac{1}{2}\lambda_3\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1ud}+3\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&-\lambda_4\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud}+\frac{3}{2}\hat{g}_{2uu}\big(\hat{g}_{1dd}\hat{g}_{2dd}+\hat{g}_{1du}\hat{g}_{2du}\big)\Big)\\
&-\frac{3}{2}\lambda_5\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{1ud}+\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-\frac{3}{2}\lambda_6\Big(\hat{g}_{1du}^3+\hat{g}_{1dd}^2\hat{g}_{1du}+\hat{g}_{2du}^2\hat{g}_{1du}+\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2du}\Big)\\
&-\frac{3}{2}\lambda_7\Big(\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1dd}\hat{g}_{2uu}\hat{g}_{2ud}+\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{1du}\big(\hat{g}_{1ud}^2+\hat{g}_{2uu}^2\big)\Big)\\
&+\frac{1}{32}\Big[\hat{g}_{1dd}^3\big(29\hat{g}_{1du}\hat{g}_{1ud}+12\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+3\hat{g}_{1dd}^2\big(8\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du}+8\hat{g}_{1du}\hat{g}_{2uu}\hat{g}_{2du}+11\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2ud}+4\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad+\hat{g}_{1dd}\Big(18\hat{g}_{1ud}\hat{g}_{1du}^3+12\hat{g}_{1du}^2\big(\hat{g}_{2du}\hat{g}_{2ud}+3\hat{g}_{2dd}\hat{g}_{2uu}\big)\\
&\qquad\qquad+\hat{g}_{1du}\hat{g}_{1ud}\big(29\hat{g}_{1ud}^2+21\hat{g}_{2dd}^2+45\hat{g}_{2ud}^2+24\hat{g}_{2uu}^2\big)\\
&\qquad\qquad+12\big(6\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}^2+2\hat{g}_{2dd}\hat{g}_{2uu}(\hat{g}_{1ud}^2+2\hat{g}_{2du}^2)\\
&\qquad\qquad\qquad+\hat{g}_{2du}\hat{g}_{2ud}(3\hat{g}_{1ud}^2+11\hat{g}_{2dd}^2+2\hat{g}_{2du}^2+11\hat{g}_{2ud}^2))\Big)\\
&\qquad-15\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{1du}^3-12\hat{g}_{1du}^2\hat{g}_{1ud}\big(\hat{g}_{2dd}\hat{g}_{2du}+\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad-3\hat{g}_{1du}\Big(23\hat{g}_{2du}\hat{g}_{2uu}^3+2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2+\hat{g}_{2du}\hat{g}_{2uu}\big(2\hat{g}_{2dd}^2+23\hat{g}_{2du}^2+2\hat{g}_{2ud}^2\big)\\
&\qquad\qquad+\hat{g}_{2dd}\hat{g}_{2ud}\big(3\hat{g}_{1ud}^2+15\hat{g}_{2dd}^2+2\hat{g}_{2du}^2+15\hat{g}_{2ud}^2\big)\Big)\\
&\qquad-12\hat{g}_{1ud}\hat{g}_{2uu}\big(11\hat{g}_{2ud}^3+3\hat{g}_{1ud}^2\hat{g}_{2ud}+11\hat{g}_{2dd}^2\hat{g}_{2ud}+2\hat{g}_{2du}^2\hat{g}_{2ud}\\
&\qquad\qquad+6\hat{g}_{2uu}^2\hat{g}_{2ud}+4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}\big)\Big],
\end{aligned}
\tag{F.18d}
$$

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{2dd}}=&-\frac{7}{4}\hat{g}_{2dd}^5\\
&+\hat{g}_{2dd}^3\Big[-\frac{5}{2}\lambda_1-\frac{45}{16}{h'_t}^2+\frac{875}{64}g^2+\frac{145}{64}{g'}^2\\
&\qquad-\frac{1}{32}\Big(59\hat{g}_{1dd}^2+15\hat{g}_{1du}^2+21\hat{g}_{2uu}^2+2\big(5\hat{g}_{1ud}^2+27\hat{g}_{2du}^2+56\hat{g}_{2ud}^2\big)\Big)\Big]\\
&+\hat{g}_{2dd}^2\Big[-\frac{15}{2}\lambda_6\hat{g}_{2ud}-\frac{45}{8}h_t h'_t\hat{g}_{2ud}\\
&\qquad-\frac{1}{32}\Big(64\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du}+68\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2du}+81\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2ud}\\
&\qquad\qquad+30\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2ud}+48\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\Big)\Big]\\
&+\hat{g}_{2dd}\Big[10g_3^2{h'_t}^2-\frac{27}{8}{h_t}^2{h'_t}^2-\frac{27}{8}{h'_t}^4-\frac{127}{8}g^4+\frac{3}{8}g^2{g'}^2+\frac{43}{48}{g'}^4\\
&\qquad-\frac{9}{16}{h_t}^2\Big(\hat{g}_{1ud}^2+5\hat{g}_{2ud}^2+2\hat{g}_{2uu}^2\Big)-\frac{3}{8}h_t h'_t\Big(3\hat{g}_{1dd}\hat{g}_{1ud}+2\hat{g}_{2du}\hat{g}_{2uu}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ {h'_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{9}{16}\hat{g}_{1dd}^2 + \frac{3}{8}\hat{g}_{2du}^2\Big) \\
&+ \frac{1}{64}g^2\Big(111\hat{g}_{1dd}^2 + 30\hat{g}_{1du}^2 + 33\hat{g}_{1ud}^2 + 136\hat{g}_{2du}^2 + 875\hat{g}_{2ud}^2 + 86\hat{g}_{2uu}^2\Big) \\
&+ \frac{1}{64}{g'}^2\Big(21\hat{g}_{1dd}^2 + 10\hat{g}_{1du}^2 + 11\hat{g}_{1ud}^2 + 8\hat{g}_{2du}^2 + 145\hat{g}_{2ud}^2 - 14\hat{g}_{2uu}^2\Big) \\
&+ \frac{3}{4}{\lambda_1}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{9}{4}{\lambda_6}^2 + \frac{3}{4}{\lambda_7}^2 \\
&- \frac{1}{2}\lambda_1\big(\hat{g}_{1dd}^2 + \hat{g}_{2du}^2\big) - \frac{1}{2}\lambda_3\big(\hat{g}_{1ud}^2 + 5\hat{g}_{2ud}^2 + 2\hat{g}_{2uu}^2\big) + \frac{1}{2}\lambda_4\big(\hat{g}_{2uu}^2 - 5\hat{g}_{2ud}^2\big) \\
&- \frac{5}{2}\lambda_5\hat{g}_{2ud}^2 - \lambda_6\big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) \\
&- \frac{1}{32}\Big(5\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(24\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 7\hat{g}_{1uu}^2 + 31\hat{g}_{2du}^2 + 27\hat{g}_{2ud}^2\big) \\
&\qquad + 2\hat{g}_{1dd}\big(7\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 26\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu} \\
&\qquad\qquad + 35\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu} + 2\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad + 9\hat{g}_{1du}^4 + 4\hat{g}_{1ud}^4 + 22\hat{g}_{2du}^4 + 56\hat{g}_{2ud}^4 + 21\hat{g}_{2uu}^4 \\
&\qquad + 4\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + 6\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 42\hat{g}_{1ud}^2\hat{g}_{2ud}^2 + 15\hat{g}_{1uu}^2\hat{g}_{2ud}^2 + 19\hat{g}_{2du}^2\hat{g}_{2ud}^2 \\
&\qquad + \hat{g}_{2uu}^2\big(39\hat{g}_{1ud}^2 + 7\hat{g}_{1uu}^2 + 43\hat{g}_{2du}^2 + 56\hat{g}_{2ud}^2\big) + 36\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad + 4\hat{g}_{1du}\hat{g}_{2du}\big(3\hat{g}_{1ud}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^2 + 9\hat{g}_{1uu}^2 + 13\hat{g}_{2du}^2 + 10\hat{g}_{2uu}^2\big)\Big)\Big] \\
&+ 10g_3^2h_th'_t\hat{g}_{2ud} - \frac{27}{8}\hat{g}_{2ud}{h_t}^3h'_t - \frac{27}{8}h_t{h'_t}^3\hat{g}_{2ud} \\
&- \frac{3}{2}{h'_t}^2\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du} - \hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2du} + \hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{32}g^2\Big(36\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du} + 25\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2du} + 39\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2ud} + 15\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2ud} \\
&\qquad + 60\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu} - 24\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{32}{g'}^2\Big(4\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du} + 11\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2du} + 5\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2ud} \\
&\qquad + 5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2ud} + 4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\Big) \\
&+ h'_t\Big(\frac{45}{16}g^2\hat{g}_{2ud} + \frac{85}{48}{g'}^2\hat{g}_{2ud} - \frac{3}{2}\big(\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} + \hat{g}_{2uu}(\hat{g}_{1ud}\hat{g}_{1uu} - \hat{g}_{2ud}\hat{g}_{2uu})\big)\Big) \\
&+ \frac{3}{4}\lambda_1\lambda_6\hat{g}_{2ud} + \frac{3}{4}\lambda_2\lambda_7\hat{g}_{2ud} + \frac{3}{4}\lambda_3\lambda_6\hat{g}_{2ud} + \frac{3}{4}\lambda_3\lambda_7\hat{g}_{2ud} \\
&+ \frac{3}{4}\lambda_4\lambda_6\hat{g}_{2ud} + \frac{3}{4}\lambda_4\lambda_7\hat{g}_{2ud} + \frac{3}{4}\lambda_5\lambda_6\hat{g}_{2ud} + \frac{3}{4}\lambda_5\lambda_7\hat{g}_{2ud} \\
&- \frac{1}{2}\lambda_1\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du} + \frac{1}{2}\lambda_3\Big(\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} - \hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\Big) \\
&- \frac{1}{2}\lambda_4\Big(2\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1dd}\big(\hat{g}_{1ud}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{2}\lambda_5\Big(\hat{g}_{1ud}\big(\hat{g}_{1uu}\hat{g}_{2du} + \hat{g}_{1dd}\hat{g}_{2ud}\big) + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
&- \frac{1}{2}\lambda_6\Big(\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} + \hat{g}_{2ud}\big(\hat{g}_{1dd}^2 + \hat{g}_{2du}^2\big) + \hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{2du} + \hat{g}_{1dd}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{2}\lambda_7\Big(\hat{g}_{2ud}\hat{g}_{1ud}^2 + \hat{g}_{1uu}\hat{g}_{2uu}\hat{g}_{1ud} + \hat{g}_{2ud}\big(5\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{32}\Big[\hat{g}_{1dd}^3\big(24\hat{g}_{1du}\hat{g}_{2du} + \hat{g}_{1ud}\hat{g}_{2ud}\big) \\
&\qquad + 2\hat{g}_{1dd}^2\big(8\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du} - 4\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2du} + 5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2ud} + 12\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1dd}\Big(20\hat{g}_{2du}\hat{g}_{1du}^3 + 10\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{1du}^2 \\
&\qquad\qquad + 4\hat{g}_{1du}\hat{g}_{2du}\big(2\hat{g}_{1ud}^2 + 5\hat{g}_{1uu}^2 + 3(3\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - 8\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}^2 + \hat{g}_{1ud}\hat{g}_{2ud}\big(\hat{g}_{1ud}^2 + 10\hat{g}_{1uu}^2 + 17\hat{g}_{2ud}^2\big) \\
&\qquad + 4\hat{g}_{1uu}\hat{g}_{2uu}\big(4\hat{g}_{1ud}^2 + 6\hat{g}_{2du}^2 + \hat{g}_{2ud}^2\big)\Big) \\
&- 3\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1du}^3\big(9\hat{g}_{1uu}\hat{g}_{2ud} + 20\hat{g}_{1ud}\hat{g}_{2uu}\big) \\
&+ \hat{g}_{2du}\Big(\hat{g}_{2ud}\hat{g}_{2uu}\big(5\hat{g}_{1uu}^2 + \hat{g}_{2du}^2 - 2\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) + 12\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2ud}^2 + 2\hat{g}_{2uu}^2\big)\Big) \\
&+ \hat{g}_{1du}\Big(8\hat{g}_{2uu}\hat{g}_{1ud}^3 + 10\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{1ud}^2 \\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{2uu}\big(5\hat{g}_{1uu}^2 + 3(3\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big) \\
&\qquad + \hat{g}_{1uu}\hat{g}_{2ud}\big(\hat{g}_{2uu}^2 + 9(\hat{g}_{1uu}^2 + \hat{g}_{2du}^2)\big)\Big)\Big],
\end{aligned}
\tag{F.18e}
$$

$$
\begin{aligned}
\beta_{\hat{g}_{2du}}^{(2)} = &- \frac{7}{4}\hat{g}_{2du}^5 \\
&+ \hat{g}_{2du}^3\Big[- \frac{5}{2}\lambda_1 - \frac{45}{16}{h_t'}^2 + \frac{875}{64}g^2 + \frac{145}{64}{g'}^2 \\
&\quad - \frac{1}{32}\Big(15\hat{g}_{1dd}^2 + 59\hat{g}_{1du}^2 + 10\hat{g}_{1uu}^2 + 54\hat{g}_{2dd}^2 + 21\hat{g}_{2ud}^2 + 112\hat{g}_{2uu}^2\Big)\Big] \\
&+ \hat{g}_{2du}^2\Big[- \frac{15}{2}\lambda_6\hat{g}_{2uu} - \frac{45}{8}h_t h_t'\hat{g}_{2uu} - \frac{1}{2}\hat{g}_{1dd}\Big(4\hat{g}_{1du}\hat{g}_{2dd} + 3\hat{g}_{1uu}\hat{g}_{2ud}\Big) \\
&\quad - \frac{1}{32}\hat{g}_{2uu}\Big(30\hat{g}_{1dd}\hat{g}_{1ud} + 81\hat{g}_{1du}\hat{g}_{1uu} + 68\hat{g}_{2dd}\hat{g}_{2ud}\Big)\Big] \\
&+ \hat{g}_{2du}\Big[10g_3^2{h_t'}^2 - \frac{27}{8}{h_t}^2{h_t'}^2 - \frac{27}{8}{h_t'}^4 - \frac{127}{8}g^4 + \frac{3}{8}g^2{g'}^2 + \frac{43}{48}{g'}^4 \\
&\quad - \frac{9}{16}{h_t}^2\Big(\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2\Big) - \frac{3}{8}h_t h_t'\Big(3\hat{g}_{1du}\hat{g}_{1uu} + 2\hat{g}_{2dd}\hat{g}_{2ud}\Big) \\
&\quad + {h_t'}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{9}{16}\hat{g}_{1du}^2 + \frac{3}{8}\hat{g}_{2dd}^2\Big) \\
&\quad + \frac{1}{64}g^2\Big(30\hat{g}_{1dd}^2 + 111\hat{g}_{1du}^2 + 33\hat{g}_{1uu}^2 + 136\hat{g}_{2dd}^2 + 86\hat{g}_{2ud}^2 + 875\hat{g}_{2uu}^2\Big) \\
&\quad + \frac{1}{64}{g'}^2\Big(10\hat{g}_{1dd}^2 + 21\hat{g}_{1du}^2 + 11\hat{g}_{1uu}^2 + 8\hat{g}_{2dd}^2 - 14\hat{g}_{2ud}^2 + 145\hat{g}_{2uu}^2\Big) \\
&\quad + \frac{3}{4}{\lambda_1}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{9}{4}{\lambda_6}^2 + \frac{3}{4}{\lambda_7}^2 \\
&\quad - \frac{1}{2}\lambda_1\big(\hat{g}_{1du}^2 + \hat{g}_{2dd}^2\big) - \frac{1}{2}\lambda_3\big(\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2\big) + \frac{1}{2}\lambda_4\big(\hat{g}_{2ud}^2 - 5\hat{g}_{2uu}^2\big) \\
&\quad - \frac{5}{2}\lambda_5\hat{g}_{2uu}^2 - \lambda_6\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\quad - \frac{1}{32}\Big(9\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(24\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 7\hat{g}_{1uu}^2 + 13\hat{g}_{2dd}^2 + 10\hat{g}_{2ud}^2\big) \\
&\qquad + 2\hat{g}_{1dd}\big(7\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 6\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{1uu} \\
&\qquad\qquad + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 2\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad + 5\hat{g}_{1du}^4 + 4\hat{g}_{1uu}^4 + 22\hat{g}_{2dd}^4 + 21\hat{g}_{2ud}^4 + 56\hat{g}_{2uu}^4 + 4\hat{g}_{1ud}^2\hat{g}_{1uu}^2 \\
&\qquad + 6\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 7\hat{g}_{1ud}^2\hat{g}_{2ud}^2 + 39\hat{g}_{1uu}^2\hat{g}_{2ud}^2 + 43\hat{g}_{2dd}^2\hat{g}_{2ud}^2 \\
&\qquad + \hat{g}_{2uu}^2\big(15\hat{g}_{1ud}^2 + 42\hat{g}_{1uu}^2 + 19\hat{g}_{2dd}^2 + 56\hat{g}_{2ud}^2\big) \\
&\qquad + 36\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 2\hat{g}_{1du}\hat{g}_{2dd}\big(35\hat{g}_{1uu}\hat{g}_{2ud} + 26\hat{g}_{1ud}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^2 + 9\hat{g}_{1uu}^2 + 31\hat{g}_{2dd}^2 + 27\hat{g}_{2uu}^2\big)\Big)\Big] \\
&+ 10g_3^2 h_t h_t'\hat{g}_{2uu} - \frac{27}{8}{h_t}^3 h_t'\hat{g}_{2uu} - \frac{27}{8}h_t'{h_t'}^3\hat{g}_{2uu} \\
&+ h_t h_t'\Big(\frac{45}{16}g^2\hat{g}_{2uu} + \frac{85}{48}{g'}^2\hat{g}_{2uu} - \frac{3}{2}\big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{2ud}(\hat{g}_{1ud}\hat{g}_{1uu} - \hat{g}_{2ud}\hat{g}_{2uu})\big)\Big) \\
&- \frac{3}{2}{h_t'}^2\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd} - \hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd} + \hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{32} g^2 \Big(24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2ud} - 25\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{2ud} - 39\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2uu} \\
&\qquad - 3\hat{g}_{1dd}\big(12\hat{g}_{1du}\hat{g}_{2dd} + 20\hat{g}_{1uu}\hat{g}_{2ud} + 5\hat{g}_{1ud}\hat{g}_{2uu}\big)\Big) \\
&+ \frac{1}{32} {g'}^2 \Big(4\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd} + 11\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd} + 4\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud} \\
&\qquad + 5\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2uu} + 5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2uu}\Big) \\
&+ \frac{3}{4}\lambda_1\lambda_6\hat{g}_{2uu} + \frac{3}{4}\lambda_2\lambda_7\hat{g}_{2uu} + \frac{3}{4}\lambda_3\lambda_6\hat{g}_{2uu} + \frac{3}{4}\lambda_3\lambda_7\hat{g}_{2uu} \\
&+ \frac{3}{4}\lambda_4\lambda_6\hat{g}_{2uu} + \frac{3}{4}\lambda_4\lambda_7\hat{g}_{2uu} + \frac{3}{4}\lambda_5\lambda_6\hat{g}_{2uu} + \frac{3}{4}\lambda_5\lambda_7\hat{g}_{2uu} \\
&- \frac{1}{2}\lambda_1\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd} + \frac{1}{2}\lambda_3\Big(\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu} - \hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud}\Big) \\
&- \frac{1}{2}\lambda_4\Big(2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{2}\lambda_5\Big(\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd} + \hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big)\Big) \\
&- \frac{1}{2}\lambda_6\Big(\hat{g}_{2uu}\hat{g}_{1du}^2 + \hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1dd}\hat{g}_{2ud}\big) + \hat{g}_{2dd}\big(\hat{g}_{1dd}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{2}\lambda_7\Big(\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud} + \hat{g}_{2uu}\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{32}\Big[\hat{g}_{2uu}^3\big(2\hat{g}_{2dd}\hat{g}_{2ud} - 17\hat{g}_{1du}\hat{g}_{1uu}\big) \\
&\qquad - 4\hat{g}_{2uu}^2\big(3\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd} + 3\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd} + \hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2ud} + 3\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud}\big) \\
&\qquad - \hat{g}_{2uu}\Big(9\hat{g}_{1ud}\hat{g}_{1dd}^3 + \hat{g}_{1dd}^2\big(10\hat{g}_{1du}\hat{g}_{1uu} - 3\hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\qquad\qquad + \hat{g}_{1dd}\hat{g}_{1ud}\big(10\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 10\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + \hat{g}_{2ud}^2\big) + \hat{g}_{1du}^3\hat{g}_{1uu} \\
&\qquad\qquad - 8\hat{g}_{1du}^2\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1du}\hat{g}_{1uu}\big(10\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 - 8\hat{g}_{2ud}^2\big) \\
&\qquad\qquad + \hat{g}_{2dd}\hat{g}_{2ud}\big(5\hat{g}_{1ud}^2 + \hat{g}_{2dd}^2 + \hat{g}_{2ud}^2\big)\Big) \\
&\qquad - 4(\hat{g}_{1du}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2ud})\Big(5\hat{g}_{1dd}^3 + \hat{g}_{1dd}\big(6\hat{g}_{1du}^2 + 5\hat{g}_{1ud}^2 + 2\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 3\hat{g}_{2ud}^2\big) \\
&\qquad\qquad + 4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 6\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\Big)\Big],
\end{aligned}
\tag{F.18f}
$$

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{2ud}} = &- \frac{7}{4}\hat{g}_{2ud}^5 \\
&+ \hat{g}_{2ud}^3\Big[- \frac{5}{2}\lambda_2 + \frac{875}{64}g^2 + \frac{145}{64}{g'}^2 - \frac{45}{16}{h_t}^2 \\
&\qquad - \frac{1}{32}\Big(10\hat{g}_{1dd}^2 + 59\hat{g}_{1ud}^2 + 15\hat{g}_{1uu}^2 + 112\hat{g}_{2dd}^2 + 21\hat{g}_{2du}^2 + 54\hat{g}_{2uu}^2\Big)\Big] \\
&+ \hat{g}_{2ud}^2\Big[- \frac{15}{2}\lambda_7\hat{g}_{2dd} - \frac{45}{8}h_t h'_t\hat{g}_{2dd} \\
&\qquad - \frac{1}{32}\Big(81\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd} + 30\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2dd} + 68\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd} \\
&\qquad\qquad + 48\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} + 64\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}\Big)\Big] \\
&+ \hat{g}_{2ud}\Big[10g_3^2{h_t}^2 - \frac{27}{8}{h_t}^4 - \frac{27}{8}{h_t}^2{h'_t}^2 - \frac{127}{8}g^4 + \frac{3}{8}g^2{g'}^2 + \frac{43}{48}{g'}^4 \\
&\qquad + {h_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{9}{16}\hat{g}_{1ud}^2 + \frac{3}{8}\hat{g}_{2uu}^2\Big) \\
&\qquad - \frac{3}{8}h_t h'_t\Big(3\hat{g}_{1dd}\hat{g}_{1ud} + 2\hat{g}_{2du}\hat{g}_{2uu}\Big) - \frac{9}{16}{h'_t}^2\Big(\hat{g}_{1dd}^2 + 5\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2\Big) \\
&\qquad + \frac{1}{64}g^2\Big(33\hat{g}_{1dd}^2 + 111\hat{g}_{1ud}^2 + 30\hat{g}_{1uu}^2 + 875\hat{g}_{2dd}^2 + 86\hat{g}_{2du}^2 + 136\hat{g}_{2uu}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{64} g'^2 \Big(11\hat{g}_{1dd}^2 + 21\hat{g}_{1ud}^2 + 10\hat{g}_{1uu}^2 + 145\hat{g}_{2dd}^2 - 14\hat{g}_{2du}^2 + 8\hat{g}_{2uu}^2 \Big) \\
&+ \frac{3}{4}{\lambda_2}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{3}{4}{\lambda_6}^2 + \frac{9}{4}{\lambda_7}^2 \\
&- \frac{1}{2}\lambda_2\big(\hat{g}_{1ud}^2 + \hat{g}_{2uu}^2\big) - \frac{1}{2}\lambda_3\big(\hat{g}_{1dd}^2 + 5\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2\big) + \frac{1}{2}\lambda_4\big(\hat{g}_{2du}^2 - 5\hat{g}_{2dd}^2\big) \\
&- \frac{5}{2}\lambda_5\hat{g}_{2dd}^2 - \lambda_7\big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) \\
&- \frac{1}{32}\Big(4\hat{g}_{1dd}^4 + \hat{g}_{1dd}^2\big(4\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 7\hat{g}_{1uu}^2 + 42\hat{g}_{2dd}^2 + 39\hat{g}_{2du}^2\big) \\
&\qquad + 2\hat{g}_{1dd}\big(7\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 6\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{1uu} \\
&\qquad\qquad + 18\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + 35\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + 5\hat{g}_{1ud}^4 + 9\hat{g}_{1uu}^4 + 56\hat{g}_{2dd}^4 + 21\hat{g}_{2du}^4 + 22\hat{g}_{2uu}^4 \\
&\qquad + 24\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + 27\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 10\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 56\hat{g}_{2dd}^2\hat{g}_{2du}^2 \\
&\qquad + \hat{g}_{2uu}^2\big(31\hat{g}_{1ud}^2 + 13\hat{g}_{1uu}^2 + 19\hat{g}_{2dd}^2 + 43\hat{g}_{2du}^2\big) \\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 4\hat{g}_{1du}\hat{g}_{2uu}\big(13\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2du}\big) \\
&\qquad + \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^2 + 9\hat{g}_{1uu}^2 + 15\hat{g}_{2dd}^2 + 7\hat{g}_{2du}^2 + 6\hat{g}_{2uu}^2\big)\Big)\Big] \\
&+ 10g_3^2 h_t h_t' \hat{g}_{2dd} - \frac{27}{8}\hat{g}_{2dd}{h_t}^3 h_t' - \frac{27}{8} h_t {h_t'}^3 \hat{g}_{2dd} \\
&- \frac{3}{2}{h_t}^2\Big(\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} - \hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{2du} + \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}\Big) \\
&+ h_t h_t'\Big(\frac{45}{16}g^2\hat{g}_{2dd} + \frac{85}{48}g'^2\hat{g}_{2dd} + \frac{3}{2}\big(\hat{g}_{2dd}\hat{g}_{2du}^2 - \hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2du} - \hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2uu}\big)\Big) \\
&+ \frac{1}{32}g^2\Big(39\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd} + 15\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2dd} + 25\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd} - 24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2du} \\
&\qquad + 60\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} + 36\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{32}g'^2\Big(5\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd} + 5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2dd} + 11\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd} \\
&\qquad + 4\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du} + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}\Big) \\
&+ \frac{3}{4}\lambda_1\lambda_6\hat{g}_{2dd} + \frac{3}{4}\lambda_2\lambda_7\hat{g}_{2dd} + \frac{3}{4}\lambda_3\lambda_6\hat{g}_{2dd} + \frac{3}{4}\lambda_3\lambda_7\hat{g}_{2dd} \\
&+ \frac{3}{4}\lambda_4\lambda_6\hat{g}_{2dd} + \frac{3}{4}\lambda_4\lambda_7\hat{g}_{2dd} + \frac{3}{4}\lambda_5\lambda_6\hat{g}_{2dd} + \frac{3}{4}\lambda_5\lambda_7\hat{g}_{2dd} \\
&- \frac{1}{2}\lambda_2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu} + \frac{1}{2}\lambda_3\Big(\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu} - \hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2du}\Big) \\
&- \frac{1}{2}\lambda_4\hat{g}_{1ud}\Big(\hat{g}_{1dd}\hat{g}_{2dd} + \hat{g}_{1du}\hat{g}_{2du} - \hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- \frac{1}{2}\lambda_5\Big(\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1dd}\big(\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1du}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{2}\lambda_6\Big(\hat{g}_{2dd}\hat{g}_{1dd}^2 + \hat{g}_{1du}\hat{g}_{2du}\hat{g}_{1dd} + \hat{g}_{2dd}\big(5\hat{g}_{2dd}^2 + \hat{g}_{2du}^2\big)\Big) \\
&- \frac{1}{2}\lambda_7\Big(\hat{g}_{2dd}\hat{g}_{1ud}^2 + \hat{g}_{1ud}\big(\hat{g}_{1uu}\hat{g}_{2du} + \hat{g}_{1du}\hat{g}_{2uu}\big) + \hat{g}_{2uu}\big(\hat{g}_{1dd}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{32}\Big[\hat{g}_{1dd}^3\big(\hat{g}_{1ud}\hat{g}_{2dd} + 8\hat{g}_{1uu}\hat{g}_{2du}\big) \\
&\qquad + 2\hat{g}_{1dd}^2\big(5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2dd} + 8\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2du} + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1dd}\Big(\hat{g}_{2dd}\hat{g}_{1ud}^3 + 24\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{1ud}^2 + \hat{g}_{1ud}\hat{g}_{2dd}\big(10\hat{g}_{1uu}^2 + 17\hat{g}_{2dd}^2 - 8\hat{g}_{2du}^2\big) \\
&\qquad\qquad + 10\hat{g}_{1du}^2\big(\hat{g}_{1ud}\hat{g}_{2dd} + 2\hat{g}_{1uu}\hat{g}_{2du}\big) + 4\hat{g}_{1du}\hat{g}_{2uu}\big(4\hat{g}_{1ud}^2 + 3\hat{g}_{2dd}^2 + 6\hat{g}_{2du}^2\big) \\
&\qquad\qquad + 4\hat{g}_{1uu}\hat{g}_{2du}\big(5\hat{g}_{1uu}^2 + 3(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + 3\hat{g}_{2uu}^2)\big)\Big) \\
&\qquad + 9\hat{g}_{1du}^3\hat{g}_{1uu}\hat{g}_{2dd} + 5\hat{g}_{1du}^2\hat{g}_{2uu}\big(4\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2du}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \hat{g}_{1du}\Big(10\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{1ud}^2 + 4\hat{g}_{1ud}\hat{g}_{2du}\big(\hat{g}_{2dd}^2 + 6\hat{g}_{2uu}^2\big) \\
&\qquad + \hat{g}_{1uu}\hat{g}_{2dd}\big(9\hat{g}_{1uu}^2 + \hat{g}_{2du}^2 + 9\hat{g}_{2uu}^2\big)\Big) \\
&+ \hat{g}_{2uu}\Big(24\hat{g}_{1uu}\hat{g}_{1ud}^3 - 8\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1ud}^2 \\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}\big(5\hat{g}_{1uu}^2 + 3(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + 3\hat{g}_{2uu}^2)\big) \\
&\qquad - \hat{g}_{2dd}\hat{g}_{2du}\big(3\hat{g}_{1uu}^2 + 2\hat{g}_{2dd}^2 - \hat{g}_{2du}^2 - \hat{g}_{2uu}^2\big)\Big)\Big],
\end{aligned}
\tag{F.18g}
$$

$$
\begin{aligned}
\beta^{(2)}_{\hat{g}_{2uu}} = &- \frac{7}{4}\hat{g}_{2uu}^5 \\
&+ \hat{g}_{2uu}^3\Big[-\frac{5}{2}\lambda_2 + \frac{875}{64}g^2 + \frac{145}{64}{g'}^2 - \frac{45}{16}{h_t}^2 \\
&\qquad - \frac{1}{32}\Big(10\hat{g}_{1du}^2 + 15\hat{g}_{1ud}^2 + 59\hat{g}_{1uu}^2 + 21\hat{g}_{2dd}^2 + 112\hat{g}_{2du}^2 + 54\hat{g}_{2ud}^2\Big)\Big] \\
&+ \hat{g}_{2uu}^2\Big[-\frac{15}{2}\lambda_7\hat{g}_{2du} - \frac{45}{8}h_t h_t'\hat{g}_{2du} \\
&\qquad - \frac{1}{32}\Big(48\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd} + 68\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd} + 30\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2du} \\
&\qquad\qquad + 81\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du} + 64\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\Big)\Big] \\
&+ \hat{g}_{2uu}\Big[10g_3^2{h_t}^2 - \frac{27}{8}{h_t}^4 - \frac{27}{8}{h_t}^2{h_t'}^2 - \frac{127}{8}g^4 + \frac{3}{8}g^2{g'}^2 + \frac{43}{48}{g'}^4 \\
&\qquad + {h_t}^2\Big(\frac{45}{16}g^2 + \frac{85}{48}{g'}^2 - \frac{9}{16}\hat{g}_{1uu}^2 + \frac{3}{8}\hat{g}_{2ud}^2\Big) \\
&\qquad - \frac{3}{8}h_t h_t'(3\hat{g}_{1du}\hat{g}_{1uu} + 2\hat{g}_{2dd}\hat{g}_{2ud}) - \frac{9}{16}{h_t'}^2\Big(\hat{g}_{1du}^2 + 2\hat{g}_{2dd}^2 + 5\hat{g}_{2du}^2\Big) \\
&\qquad + \frac{1}{64}g^2\Big(33\hat{g}_{1du}^2 + 30\hat{g}_{1ud}^2 + 111\hat{g}_{1uu}^2 + 86\hat{g}_{2dd}^2 + 875\hat{g}_{2du}^2 + 136\hat{g}_{2ud}^2\Big) \\
&\qquad + \frac{1}{64}{g'}^2\Big(11\hat{g}_{1du}^2 + 10\hat{g}_{1ud}^2 + 21\hat{g}_{1uu}^2 - 14\hat{g}_{2dd}^2 + 145\hat{g}_{2du}^2 + 8\hat{g}_{2ud}^2\Big) \\
&\qquad + \frac{3}{4}{\lambda_2}^2 + \frac{1}{2}{\lambda_3}^2 + \frac{1}{2}\lambda_3\lambda_4 + \frac{1}{2}{\lambda_4}^2 + \frac{3}{4}{\lambda_5}^2 + \frac{3}{4}{\lambda_6}^2 + \frac{9}{4}{\lambda_7}^2 \\
&\qquad - \frac{1}{2}\lambda_2\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) - \frac{1}{2}\lambda_3\big(\hat{g}_{1du}^2 + 2\hat{g}_{2dd}^2 + 5\hat{g}_{2du}^2\big) + \frac{1}{2}\lambda_4\big(\hat{g}_{2dd}^2 - 5\hat{g}_{2du}^2\big) \\
&\qquad - \frac{5}{2}\lambda_5\hat{g}_{2du}^2 - \lambda_7\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\qquad - \frac{1}{32}\Big(4\hat{g}_{1du}^4 + \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^2 + 9\hat{g}_{1uu}^2 + 39\hat{g}_{2dd}^2 + 42\hat{g}_{2du}^2\big) \\
&\qquad\qquad + 2\hat{g}_{2ud}\hat{g}_{1du}\big(35\hat{g}_{1uu}\hat{g}_{2dd} + 6\hat{g}_{1ud}\hat{g}_{2du}\big) \\
&\qquad\qquad + 9\hat{g}_{1ud}^4 + 5\hat{g}_{1uu}^4 + 21\hat{g}_{2dd}^4 + 56\hat{g}_{2du}^4 + 22\hat{g}_{2ud}^4 \\
&\qquad\qquad + 24\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + 10\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 27\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 56\hat{g}_{2dd}^2\hat{g}_{2du}^2 \\
&\qquad\qquad + \hat{g}_{2ud}^2\big(13\hat{g}_{1ud}^2 + 31\hat{g}_{1uu}^2 + 43\hat{g}_{2dd}^2 + 19\hat{g}_{2du}^2\big) + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} \\
&\qquad\qquad + 2\hat{g}_{1dd}\big(7\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 26\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu} \\
&\qquad\qquad\qquad + 18\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\qquad\qquad + \hat{g}_{1dd}^2\big(4\hat{g}_{1du}^2 + 9\hat{g}_{1ud}^2 + 7\hat{g}_{1uu}^2 + 7\hat{g}_{2dd}^2 + 15\hat{g}_{2du}^2 + 6\hat{g}_{2ud}^2\big)\Big)\Big] \\
&+ 10g_3^2 h_t h_t'\hat{g}_{2du} - \frac{27}{8}{h_t}^3 h_t'\hat{g}_{2du} - \frac{27}{8}h_t{h_t'}^3\hat{g}_{2du} \\
&- \frac{3}{2}{h_t}^2\Big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd} - \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd} + \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\Big) \\
&+ h_t h_t'\Big(\frac{45}{16}g^2\hat{g}_{2du} + \frac{85}{48}{g'}^2\hat{g}_{2du} + \frac{3}{2}\big(\hat{g}_{2du}\hat{g}_{2dd}^2 - \hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd} - \hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{32}g^2\Big(60\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}-24\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2dd}+25\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}+15\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2du}\\
&\qquad+39\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}+36\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\Big)\\
&+\frac{1}{32}{g'}^2\Big(4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}+11\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}+5\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2du}\\
&\qquad+5\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\Big)\\
&+\frac{3}{4}\lambda_1\lambda_6\hat{g}_{2du}+\frac{3}{4}\lambda_2\lambda_7\hat{g}_{2du}+\frac{3}{4}\lambda_3\lambda_6\hat{g}_{2du}+\frac{3}{4}\lambda_3\lambda_7\hat{g}_{2du}\\
&+\frac{3}{4}\lambda_4\lambda_6\hat{g}_{2du}+\frac{3}{4}\lambda_4\lambda_7\hat{g}_{2du}+\frac{3}{4}\lambda_5\lambda_6\hat{g}_{2du}+\frac{3}{4}\lambda_5\lambda_7\hat{g}_{2du}\\
&-\frac{1}{2}\lambda_2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}+\frac{1}{2}\lambda_3\Big(\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}-\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}\Big)\\
&-\lambda_4\Big(\frac{1}{2}\hat{g}_{1uu}\big(\hat{g}_{1dd}\hat{g}_{2dd}+\hat{g}_{1du}\hat{g}_{2du}\big)+\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\Big)\\
&-\frac{1}{2}\lambda_5\Big(\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1du}\big(\hat{g}_{1uu}\hat{g}_{2du}+\hat{g}_{1dd}\hat{g}_{2ud}\big)\Big)\\
&-\frac{1}{2}\lambda_6\Big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd}+\hat{g}_{2du}\big(\hat{g}_{1du}^2+\hat{g}_{2dd}^2+5\hat{g}_{2du}^2\big)\Big)\\
&-\frac{1}{2}\lambda_7\Big(\hat{g}_{2du}\hat{g}_{2ud}^2+\hat{g}_{2ud}\big(\hat{g}_{1du}\hat{g}_{1ud}+\hat{g}_{1dd}\hat{g}_{1uu}\big)+\hat{g}_{1uu}\big(\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{1uu}\hat{g}_{2du}\big)\Big)\\
&-\frac{1}{32}\Big[9\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{1dd}^3\\
&\quad+5\hat{g}_{1dd}^2\big(4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}+2\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\big)\\
&\quad+\hat{g}_{1dd}\Big(9\hat{g}_{2du}\hat{g}_{1ud}^3+\hat{g}_{1ud}\hat{g}_{2du}\big(10\hat{g}_{1uu}^2+\hat{g}_{2dd}^2+9\hat{g}_{2ud}^2\big)\\
&\qquad+2\hat{g}_{1du}^2\big(8\hat{g}_{1uu}\hat{g}_{2dd}+5\hat{g}_{1ud}\hat{g}_{2du}\big)+4\hat{g}_{1du}\hat{g}_{2ud}\big(4\hat{g}_{1uu}^2+6\hat{g}_{2dd}^2+3\hat{g}_{2du}^2\big)\\
&\qquad+4\hat{g}_{1uu}\hat{g}_{2dd}\big(\hat{g}_{2du}^2+6\hat{g}_{2ud}^2\big)\Big)\\
&\quad+\hat{g}_{1du}^3\big(8\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{1uu}\hat{g}_{2du}\big)+8\hat{g}_{1du}^2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\\
&\quad+\hat{g}_{2ud}\Big(20\hat{g}_{1uu}\hat{g}_{1ud}^3-3\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1ud}^2+12\hat{g}_{1ud}\hat{g}_{1uu}\big(2\hat{g}_{1uu}^2+\hat{g}_{2dd}^2+\hat{g}_{2du}^2+3\hat{g}_{2ud}^2\big)\\
&\qquad-\hat{g}_{2dd}\hat{g}_{2du}\big(8\hat{g}_{1uu}^2-\hat{g}_{2dd}^2+2\hat{g}_{2du}^2-\hat{g}_{2ud}^2\big)\Big)\\
&\quad+\hat{g}_{1du}\Big(20\hat{g}_{2dd}\hat{g}_{1ud}^3+10\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{1ud}^2+12\hat{g}_{1ud}\hat{g}_{2dd}\big(2\hat{g}_{1uu}^2+\hat{g}_{2dd}^2+\hat{g}_{2du}^2+3\hat{g}_{2ud}^2\big)\\
&\qquad+\hat{g}_{1uu}\hat{g}_{2du}\big(\hat{g}_{1uu}^2-8\hat{g}_{2dd}^2+17\hat{g}_{2du}^2\big)\Big)\Big]. \qquad \text{(F.18h)}
\end{aligned}
$$

Quartic couplings

The one-loop beta functions of the quartic couplings are given by

$$
\begin{aligned}
\beta_{\lambda_1}^{(1)}=&6{\lambda_1}^2+2{\lambda_3}^2+2\lambda_3\lambda_4+{\lambda_4}^2+{\lambda_5}^2+12{\lambda_6}^2\\
&+\lambda_1\Big(6{h_t'}^2-\frac{9}{2}g^2-\frac{3}{2}{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2\Big)\\
&+\lambda_6\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-6{h_t'}^4+\frac{9}{8}g^4+\frac{3}{4}g^2{g'}^2+\frac{3}{8}{g'}^4-\frac{1}{2}\hat{g}_{1dd}^4-\hat{g}_{1dd}^2\big(\hat{g}_{1du}^2+\hat{g}_{2dd}^2\big)-2\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}\\
&-\frac{1}{2}\hat{g}_{1du}^4-\hat{g}_{1du}^2\hat{g}_{2du}^2-\frac{5}{2}\hat{g}_{2dd}^4-\hat{g}_{2dd}^2\hat{g}_{2du}^2-\frac{5}{2}\hat{g}_{2du}^4, \qquad \text{(F.19a)}\\
\beta_{\lambda_2}^{(1)}=&6{\lambda_2}^2+2{\lambda_3}^2+2\lambda_3\lambda_4+{\lambda_4}^2+{\lambda_5}^2+12{\lambda_7}^2\\
&+\lambda_2\Big(6{h_t}^2-\frac{9}{2}g^2-\frac{3}{2}{g'}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\Big)\\
&+\lambda_7\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- 6{h_t}^4 + \frac{9}{8}g^4 + \frac{3}{4}g^2{g'}^2 + \frac{3}{8}{g'}^4 - \frac{1}{2}\hat{g}_{1ud}^4 - \hat{g}_{1ud}^2\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) - 2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} \\
&- \frac{1}{2}\hat{g}_{1uu}^4 - \hat{g}_{1uu}^2\hat{g}_{2uu}^2 - \frac{5}{2}\hat{g}_{2ud}^4 - \hat{g}_{2ud}^2\hat{g}_{2uu}^2 - \frac{5}{2}\hat{g}_{2uu}^4, \qquad \text{(F.19b)}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(1)}_{\lambda_3} =& \lambda_1(3\lambda_3 + \lambda_4) + \lambda_2(3\lambda_3 + \lambda_4) + 2{\lambda_3}^2 + {\lambda_4}^2 + {\lambda_5}^2 + 2{\lambda_6}^2 + 8\lambda_6\lambda_7 + 2{\lambda_7}^2 \\
&+ \frac{1}{2}\lambda_3\Big(6{h_t}^2 + 6{h_t'}^2 - 9g^2 - 3{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 \\
&\qquad + 3\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{2}\lambda_6\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{2}\lambda_7\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- 6{h_t}^2{h_t'}^2 + \frac{9}{8}g^4 - \frac{3}{4}g^2{g'}^2 + \frac{3}{8}{g'}^4 - \frac{1}{2}\hat{g}_{1dd}^2\big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2\big) \\
&+ \hat{g}_{1dd}\Big(\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu} - 2\hat{g}_{2du}\hat{g}_{2ud}\big)\Big) - \frac{1}{2}\hat{g}_{1du}^2\big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 2\hat{g}_{2uu}^2\big) \\
&+ \hat{g}_{1du}\Big(\hat{g}_{1ud}\big(\hat{g}_{2du}\hat{g}_{2ud} - 2\hat{g}_{2dd}\hat{g}_{2uu}\big) + \hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\Big) - \hat{g}_{1ud}^2\hat{g}_{2dd}^2 - \hat{g}_{1uu}^2\hat{g}_{2du}^2 \\
&- \frac{5}{2}\hat{g}_{2dd}^2\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) + 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} - \frac{5}{2}\hat{g}_{2du}^2\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big), \qquad \text{(F.19c)}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(1)}_{\lambda_4} =& \lambda_1\lambda_4 + \lambda_2\lambda_4 + 4\lambda_3\lambda_4 + 2{\lambda_4}^2 + 4{\lambda_5}^2 + 5{\lambda_6}^2 + 2\lambda_6\lambda_7 + 5{\lambda_7}^2 \\
&+ \frac{1}{2}\lambda_4\Big(3{h_t}^2 + 3{h_t'}^2 - 9g^2 - 3{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 \\
&\qquad + 3\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{2}\lambda_6\Big(3h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{2}\lambda_7\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- 6{h_t}^2{h_t'}^2 + \frac{3}{2}g^2{g'}^2 + \frac{1}{2}\hat{g}_{1dd}^2\big(\hat{g}_{2ud}^2 - \hat{g}_{1ud}^2\big) \\
&- \hat{g}_{1dd}\Big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} - \hat{g}_{1uu}\big(\hat{g}_{2du}\hat{g}_{2ud} - 2\hat{g}_{2dd}\hat{g}_{2uu}\big)\Big) \\
&+ \frac{1}{2}\hat{g}_{1du}^2\big(\hat{g}_{2uu}^2 - \hat{g}_{1uu}^2\big) + \hat{g}_{1du}\Big(\hat{g}_{1ud}\big(\hat{g}_{2dd}\hat{g}_{2uu} - 2\hat{g}_{2du}\hat{g}_{2ud}\big) - 2\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{2}\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + \frac{1}{2}\hat{g}_{1uu}^2\hat{g}_{2du}^2 + \frac{1}{2}\hat{g}_{2dd}^2\big(4\hat{g}_{2uu}^2 - 5\hat{g}_{2ud}^2\big) - 5\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&+ \frac{1}{2}\hat{g}_{2du}^2\big(2\hat{g}_{2ud}^2 - 5\hat{g}_{2uu}^2\big), \qquad \text{(F.19d)}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(1)}_{\lambda_5} =& \lambda_1\lambda_5 + \lambda_2\lambda_5 + 4\lambda_3\lambda_5 + 6\lambda_4\lambda_5 + 5{\lambda_6}^2 + 2\lambda_6\lambda_7 + 5{\lambda_7}^2 \\
&+ \frac{1}{2}\lambda_5\Big(6{h_t}^2 + 6{h_t'}^2 - 9g^2 - 3{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 \\
&\qquad + 3\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{2}\lambda_6\Big(6h_th_t'\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&+ \frac{1}{2}\lambda_7\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) - 6{h_t}^2{h_t'}^2 \\
&- \frac{1}{2}\hat{g}_{1dd}^2\hat{g}_{1ud}^2 - \hat{g}_{1dd}\Big(\hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{2ud}\hat{g}_{2uu}\big) + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\Big) - \frac{1}{2}\hat{g}_{1du}^2\hat{g}_{1uu}^2 \\
&- \hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu} - \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} - \frac{5}{2}\hat{g}_{2dd}^2\hat{g}_{2ud}^2 - \hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&- \frac{5}{2}\hat{g}_{2du}^2\hat{g}_{2uu}^2, \qquad \text{(F.19e)}
\end{aligned}
$$

$$
\beta^{(1)}_{\lambda_6} = 6\lambda_1\lambda_6 + 3\lambda_3(\lambda_6 + \lambda_7) + 2\lambda_4(2\lambda_6 + \lambda_7) + \lambda_5(5\lambda_6 + \lambda_7)
$$

$$
\begin{aligned}
&+\frac{1}{4}\big(\lambda_1+\lambda_3+\lambda_4+\lambda_5\big)\Big(6h_t h_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&+\frac{1}{4}\lambda_6\Big(6{h_t}^2+18{h_t'}^2-18g^2-6{g'}^2+3\hat g_{1dd}^2+3\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+9\hat g_{2dd}^2+9\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-6h_t{h_t'}^3-\frac{1}{2}\hat g_{1dd}^3\hat g_{1ud}-\frac{1}{2}\hat g_{1dd}^2\big(\hat g_{1du}\hat g_{1uu}+\hat g_{2dd}\hat g_{2ud}\big)\\
&-\frac{1}{2}\hat g_{1dd}\Big(\hat g_{1du}^2\hat g_{1ud}+\hat g_{1du}\big(\hat g_{2dd}\hat g_{2uu}+\hat g_{2du}\hat g_{2ud}\big)+\hat g_{1ud}\hat g_{2dd}^2+\hat g_{1uu}\hat g_{2dd}\hat g_{2du}\Big)\\
&-\frac{1}{2}\hat g_{1du}^3\hat g_{1uu}-\frac{1}{2}\hat g_{1du}^2\hat g_{2du}\hat g_{2uu}-\frac{1}{2}\hat g_{1du}\big(\hat g_{1ud}\hat g_{2dd}\hat g_{2du}+\hat g_{1uu}\hat g_{2du}^2\big)-\frac{5}{2}\hat g_{2dd}^3\hat g_{2ud}\\
&-\frac{1}{2}\hat g_{2dd}^2\hat g_{2du}\hat g_{2uu}-\frac{1}{2}\hat g_{2dd}\hat g_{2du}^2\hat g_{2ud}-\frac{5}{2}\hat g_{2du}^3\hat g_{2uu},
\end{aligned}
\tag{F.19f}
$$

$$
\begin{aligned}
\beta^{(1)}_{\lambda_7}=&6\lambda_2\lambda_7+3\lambda_3(\lambda_6+\lambda_7)+2\lambda_4(\lambda_6+2\lambda_7)+\lambda_5(\lambda_6+5\lambda_7)\\
&+\frac{1}{4}\big(\lambda_2+\lambda_3+\lambda_4+\lambda_5\big)\Big(6h_t h_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&+\frac{1}{4}\lambda_7\Big(18{h_t}^2+6{h_t'}^2-18g^2-6{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2\\
&\qquad+3\hat g_{1ud}^2+3\hat g_{1uu}^2+3\hat g_{2dd}^2+3\hat g_{2du}^2+9\hat g_{2ud}^2+9\hat g_{2uu}^2\Big)\\
&-6{h_t}^3h_t'-\frac{1}{2}\hat g_{1dd}\Big(\hat g_{1ud}^3+\hat g_{1ud}\big(\hat g_{1uu}^2+\hat g_{2ud}^2\big)+\hat g_{1uu}\hat g_{2ud}\hat g_{2uu}\Big)\\
&-\frac{1}{2}\hat g_{1du}\big(\hat g_{1ud}^2\hat g_{1uu}+\hat g_{1ud}\hat g_{2ud}\hat g_{2uu}+\hat g_{1uu}^3+\hat g_{1uu}\hat g_{2uu}^2\big)-\frac{1}{2}\hat g_{1ud}^2\hat g_{2dd}\hat g_{2ud}\\
&-\frac{1}{2}\hat g_{1ud}\hat g_{1uu}\big(\hat g_{2dd}\hat g_{2uu}+\hat g_{2du}\hat g_{2ud}\big)-\frac{1}{2}\hat g_{1uu}^2\hat g_{2du}\hat g_{2uu}\\
&-\frac{1}{2}\hat g_{2dd}\big(5\hat g_{2ud}^3+\hat g_{2ud}\hat g_{2uu}^2\big)-\frac{1}{2}\hat g_{2du}\big(\hat g_{2ud}^2\hat g_{2uu}+5\hat g_{2uu}^3\big).
\end{aligned}
\tag{F.19g}
$$

We split up the two-loop beta functions into four pieces,

$$
\beta^{(2)}_{\lambda_i}=\beta^{(2),\lambda^3}_{\lambda_i}+\beta^{(2),\lambda^2}_{\lambda_i}+\beta^{(2),\lambda^1}_{\lambda_i}+\beta^{(2),\lambda^0}_{\lambda_i}.
\tag{F.20}
$$

The λ^i piece contains all contributions proportional to i quartic couplings.

The λ^3 pieces are given by

$$
\begin{aligned}
\beta^{(2),\lambda^3}_{\lambda_1}=&-39{\lambda_1}^3-10\lambda_1{\lambda_3}^2-10\lambda_1\lambda_3\lambda_4-6\lambda_1{\lambda_4}^2-7\lambda_1{\lambda_5}^2-159\lambda_1{\lambda_6}^2+3\lambda_1{\lambda_7}^2\\
&-8{\lambda_3}^3-12{\lambda_3}^2\lambda_4-16\lambda_3{\lambda_4}^2-20\lambda_3{\lambda_5}^2-66\lambda_3{\lambda_6}^2-36\lambda_3\lambda_6\lambda_7-18\lambda_3{\lambda_7}^2\\
&-6{\lambda_4}^3-22\lambda_4{\lambda_5}^2-70\lambda_4{\lambda_6}^2-28\lambda_4\lambda_6\lambda_7-14\lambda_4{\lambda_7}^2-74\lambda_5{\lambda_6}^2\\
&-20\lambda_5\lambda_6\lambda_7-10\lambda_5{\lambda_7}^2,
\end{aligned}
\tag{F.21a}
$$

$$
\begin{aligned}
\beta^{(2),\lambda^3}_{\lambda_2}=&-39{\lambda_2}^3-10\lambda_2{\lambda_3}^2-10\lambda_2\lambda_3\lambda_4-6\lambda_2{\lambda_4}^2-7\lambda_2{\lambda_5}^2+3\lambda_2{\lambda_6}^2-159\lambda_2{\lambda_7}^2-8{\lambda_3}^3\\
&-12{\lambda_3}^2\lambda_4-16\lambda_3{\lambda_4}^2-20\lambda_3{\lambda_5}^2-18\lambda_3{\lambda_6}^2-36\lambda_3\lambda_6\lambda_7-66\lambda_3{\lambda_7}^2-6{\lambda_4}^3\\
&-22\lambda_4{\lambda_5}^2-14\lambda_4{\lambda_6}^2-28\lambda_4\lambda_6\lambda_7-70\lambda_4{\lambda_7}^2-10\lambda_5{\lambda_6}^2\\
&-20\lambda_5\lambda_6\lambda_7-74\lambda_5{\lambda_7}^2,
\end{aligned}
\tag{F.21b}
$$

$$
\begin{aligned}
\beta^{(2),\lambda^3}_{\lambda_3}=&-\frac{15}{2}{\lambda_1}^2\lambda_3-2{\lambda_1}^2\lambda_4-18\lambda_1{\lambda_3}^2-8\lambda_1\lambda_3\lambda_4-7\lambda_1{\lambda_4}^2-9\lambda_1{\lambda_5}^2-31\lambda_1{\lambda_6}^2\\
&-22\lambda_1\lambda_6\lambda_7-11\lambda_1{\lambda_7}^2-\frac{15}{2}{\lambda_2}^2\lambda_3-2{\lambda_2}^2\lambda_4-18\lambda_2{\lambda_3}^2-8\lambda_2\lambda_3\lambda_4-7\lambda_2{\lambda_4}^2\\
&-9\lambda_2{\lambda_5}^2-11\lambda_2{\lambda_6}^2-22\lambda_2\lambda_6\lambda_7-31\lambda_2{\lambda_7}^2-6{\lambda_3}^3-2{\lambda_3}^2\lambda_4-8\lambda_3{\lambda_4}^2\\
&-30\lambda_3{\lambda_7}^2-6{\lambda_4}^3-22\lambda_4{\lambda_5}^2-34\lambda_4{\lambda_6}^2-44\lambda_4\lambda_6\lambda_7-9\lambda_3{\lambda_5}^2\\
&-30\lambda_3{\lambda_6}^2-88\lambda_3\lambda_6\lambda_7-34\lambda_4{\lambda_7}^2-34\lambda_5{\lambda_6}^2-36\lambda_5\lambda_6\lambda_7-34\lambda_5{\lambda_7}^2,
\end{aligned}
\tag{F.21c}
$$

$$
\begin{aligned}
\beta^{(2),\lambda^3}_{\lambda_4}=&-\frac{7}{2}{\lambda_1}^2\lambda_4-20\lambda_1\lambda_3\lambda_4-10\lambda_1{\lambda_4}^2-12\lambda_1{\lambda_5}^2-37\lambda_1{\lambda_6}^2-10\lambda_1\lambda_6\lambda_7-5\lambda_1{\lambda_7}^2\\
&-\frac{7}{2}{\lambda_2}^2\lambda_4-20\lambda_2\lambda_3\lambda_4-10\lambda_2{\lambda_4}^2-12\lambda_2{\lambda_5}^2-5\lambda_2{\lambda_6}^2-10\lambda_2\lambda_6\lambda_7-37\lambda_2{\lambda_7}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 14\lambda_3{}^2\lambda_4 - 14\lambda_3\lambda_4{}^2 - 24\lambda_3\lambda_5{}^2 - 36\lambda_3\lambda_6{}^2 - 40\lambda_3\lambda_6\lambda_7 - 36\lambda_3\lambda_7{}^2 - 13\lambda_4\lambda_5{}^2 \\
&- 34\lambda_4\lambda_6{}^2 - 80\lambda_4\lambda_6\lambda_7 - 34\lambda_4\lambda_7{}^2 - 40\lambda_5\lambda_6{}^2 - 48\lambda_5\lambda_6\lambda_7 - 40\lambda_5\lambda_7{}^2, \qquad \text{(F.21d)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\lambda_5}^{(2),\lambda^3} =& - \frac{7}{2}\lambda_1{}^2\lambda_5 - 20\lambda_1\lambda_3\lambda_5 - 22\lambda_1\lambda_4\lambda_5 - 37\lambda_1\lambda_6{}^2 - 10\lambda_1\lambda_6\lambda_7 - 5\lambda_1\lambda_7{}^2 - \frac{7}{2}\lambda_2{}^2\lambda_5 \\
&- 20\lambda_2\lambda_3\lambda_5 - 22\lambda_2\lambda_4\lambda_5 - 5\lambda_2\lambda_6{}^2 - 10\lambda_2\lambda_6\lambda_7 - 37\lambda_2\lambda_7{}^2 - 14\lambda_3{}^2\lambda_5 - 38\lambda_3\lambda_4\lambda_5 \\
&- 36\lambda_3\lambda_6{}^2 - 40\lambda_3\lambda_6\lambda_7 - 36\lambda_3\lambda_7{}^2 - 16\lambda_4{}^2\lambda_5 - 38\lambda_4\lambda_6{}^2 - 44\lambda_4\lambda_6\lambda_7 \\
&- 38\lambda_4\lambda_7{}^2 + 3\lambda_5{}^3 - 36\lambda_5\lambda_6{}^2 - 84\lambda_5\lambda_6\lambda_7 - 36\lambda_5\lambda_7{}^2, \qquad \text{(F.21e)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\lambda_6}^{(2),\lambda^3} =& - \frac{159}{4}\lambda_1{}^2\lambda_6 - 33\lambda_1\lambda_3\lambda_6 - 9\lambda_1\lambda_3\lambda_7 - 35\lambda_1\lambda_4\lambda_6 - 7\lambda_1\lambda_4\lambda_7 - 37\lambda_1\lambda_5\lambda_6 - 5\lambda_1\lambda_5\lambda_7 \\
&+ \frac{3}{4}\lambda_2{}^2\lambda_6 - 9\lambda_2\lambda_3\lambda_6 - 9\lambda_2\lambda_3\lambda_7 - 7\lambda_2\lambda_4\lambda_6 - 7\lambda_2\lambda_4\lambda_7 - 5\lambda_2\lambda_5\lambda_6 - 5\lambda_2\lambda_5\lambda_7 \\
&- 16\lambda_3{}^2\lambda_6 - 18\lambda_3{}^2\lambda_7 - 34\lambda_3\lambda_4\lambda_6 - 28\lambda_3\lambda_4\lambda_7 - 36\lambda_3\lambda_5\lambda_6 - 20\lambda_3\lambda_5\lambda_7 \\
&- 17\lambda_4{}^2\lambda_6 - 17\lambda_4{}^2\lambda_7 - 38\lambda_4\lambda_5\lambda_6 - 22\lambda_4\lambda_5\lambda_7 - 18\lambda_5{}^2\lambda_6 \\
&- 21\lambda_5{}^2\lambda_7 - \frac{111}{2}\lambda_6{}^3 - 63\lambda_6{}^2\lambda_7 - \frac{33}{2}\lambda_6\lambda_7{}^2 - 21\lambda_7{}^3, \qquad \text{(F.21f)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\lambda_7}^{(2),\lambda^3} =& \frac{3}{4}\lambda_1{}^2\lambda_7 - 9\lambda_1\lambda_3\lambda_6 - 9\lambda_1\lambda_3\lambda_7 - 7\lambda_1\lambda_4\lambda_6 - 7\lambda_1\lambda_4\lambda_7 - 5\lambda_1\lambda_5\lambda_6 - 5\lambda_1\lambda_5\lambda_7 \\
&- \frac{159}{4}\lambda_2{}^2\lambda_7 - 9\lambda_2\lambda_3\lambda_6 - 33\lambda_2\lambda_3\lambda_7 - 7\lambda_2\lambda_4\lambda_6 - 35\lambda_2\lambda_4\lambda_7 - 5\lambda_2\lambda_5\lambda_6 \\
&- 37\lambda_2\lambda_5\lambda_7 - 18\lambda_3{}^2\lambda_6 - 16\lambda_3{}^2\lambda_7 - 28\lambda_3\lambda_4\lambda_6 - 34\lambda_3\lambda_4\lambda_7 - 20\lambda_3\lambda_5\lambda_6 \\
&- 36\lambda_3\lambda_5\lambda_7 - 17\lambda_4{}^2\lambda_6 - 17\lambda_4{}^2\lambda_7 - 22\lambda_4\lambda_5\lambda_6 - 38\lambda_4\lambda_5\lambda_7 - 21\lambda_5{}^2\lambda_6 \\
&- 18\lambda_5{}^2\lambda_7 - 21\lambda_6{}^3 - \frac{33\lambda_6{}^2\lambda_7}{2} - 63\lambda_6\lambda_7{}^2 - \frac{111}{2}\lambda_7{}^3. \qquad \text{(F.21g)}
\end{aligned}
$$

The λ^2 pieces are given by

$$
\begin{aligned}
\beta_{\lambda_1}^{(2),\lambda^2} =& - 6\lambda_1{}^2\Big(6h_t'^2 - \frac{9}{2}g^2 - \frac{3}{2}g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big) \\
&- 12\lambda_1\lambda_6\Big(12h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- 2\lambda_3\big(\lambda_3 + \lambda_4\big)\Big(6h_t{}^2 - 6g^2 - 2g'^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\
&- 6\lambda_3\lambda_6\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- \lambda_4{}^2\Big(6h_t{}^2 - 3g^2 - 2g'^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\
&- 2\big(2\lambda_4 + \lambda_5\big)\lambda_6\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- \lambda_5{}^2\Big(6h_t{}^2 + g'^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\
&- 6\lambda_6{}^2\Big(6h_t{}^2 + 6h_t'^2 - 9g^2 - 3g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 \\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big), \qquad \text{(F.22a)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\lambda_2}^{(2),\lambda^2} =& - 6\lambda_2{}^2\Big(6h_t{}^2 + \frac{9}{2}g^2 - \frac{3}{2}g'^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\
&- 12\lambda_2\lambda_7\Big(12h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- 2\lambda_3\big(\lambda_3 + \lambda_4\big)\Big(6h_t'^2 - 6g^2 - 2g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big) \\
&- 2\big(3\lambda_3 + 2\lambda_4 + \lambda_5\big)\lambda_7\Big(6h_th_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big) \\
&- \lambda_4{}^2\Big(6h_t'^2 - 3g^2 - 2g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big) \\
&- \lambda_5{}^2\Big(6h_t'^2 + g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big) \\
&- 6\lambda_7{}^2\Big(6h_t{}^2 + 6h_t'^2 - 9g^2 - 3g'^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big), && \text{(F.22b)}\\
\beta_{\lambda_3}^{(2),\lambda^2} =& - 3\lambda_1\lambda_3\Big(6{h_t'}^2 - 6g^2 - 2{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&- \lambda_1\lambda_4\Big(6{h_t'}^2 - 9g^2 - 2{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&- 4\big(\lambda_1\lambda_7 + \lambda_2\lambda_6\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- \lambda_2\big(3\lambda_3 + \lambda_4\big)\Big(6{h_t}^2 - 6g^2 - 2{g'}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- {\lambda_3}^2\Big(6{h_t}^2 + 6{h_t'}^2 - 3g^2 - {g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2dd}^2\\
&\qquad + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- 6\lambda_3\lambda_4 g^2\\
&- 5\lambda_3\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- \frac{1}{2}{\lambda_4}^2\Big(6{h_t}^2 + 6{h_t'}^2 - 6g^2 + 2{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- 2\lambda_4\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- \frac{1}{2}{\lambda_5}^2\Big(6{h_t}^2 + 6{h_t'}^2 - 4{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- \lambda_5\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- 2\big({\lambda_6}^2 + {\lambda_7}^2\big)\Big(6{h_t'}^2 - \frac{1}{2}{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&- 4\lambda_6\lambda_7\Big(6{h_t}^2 + 6{h_t'}^2 - 9g^2 - 4{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big), && \text{(F.22c)}\\
\beta_{\lambda_4}^{(2),\lambda^2} =& - \big(\lambda_1 + \lambda_2\big)\lambda_4\Big(6{h_t'}^2 - 2{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&- \big(\lambda_1\lambda_7 + \lambda_2\lambda_6\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- 2\lambda_3\lambda_4\Big(6{h_t}^2 + 6{h_t'}^2 - 9g^2 - {g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- 2\lambda_3\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- {\lambda_4}^2\Big(6{h_t}^2 + 6{h_t'}^2 - 9g^2 - 4{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- 5\lambda_4\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- 2{\lambda_5}^2\Big(6{h_t}^2 + 6{h_t'}^2 - \frac{27}{2}g^2 - 4{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big)\\
&- 4\lambda_5\big(\lambda_6 + \lambda_7\big)\Big(6h_t h_t' + \hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&- 5\big({\lambda_6}^2 + {\lambda_7}^2\big)\Big(6{h_t'}^2 - \frac{27}{5}g^2 - \frac{7}{5}{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\Big)\\
&- \lambda_6\lambda_7\Big(6{h_t}^2 + 6{h_t'}^2 - 4{g'}^2 + \hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\\
&\qquad + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big), && \text{(F.22d)}
\end{aligned}
$$

$$\begin{aligned}
\beta_{\lambda_5}^{(2),\lambda^2} =& -\big(\lambda_1\lambda_7+\lambda_2\lambda_6\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-\big(\lambda_1+\lambda_2\big)\lambda_5\Big(6{h_t}^2+{g'}^2+\hat g_{1ud}^2+\hat g_{1uu}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-2\lambda_3\lambda_5\Big(6{h_t}^2+6{h_t'}^2-9g^2-4{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-2\lambda_3\big(\lambda_6+\lambda_7\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-3\lambda_4\lambda_5\Big(6{h_t}^2+6{h_t'}^2-12g^2-4{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-3\lambda_4\big(\lambda_6+\lambda_7\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-5\big({\lambda_6}^2+{\lambda_7}^2\big)\Big(6{h_t'}^2-\frac{27}{5}g^2-2{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+3\hat g_{2dd}^2+3\hat g_{2du}^2\Big)\\
&-\lambda_6\lambda_7\Big(6{h_t}^2+6{h_t'}^2+2{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-6\lambda_5\big(\lambda_6+\lambda_7\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big),
\end{aligned}\tag{F.22e}$$

$$\begin{aligned}
\beta_{\lambda_6}^{(2),\lambda^2} =& -\frac{1}{2}\lambda_3\big(3\lambda_1+\lambda_3+3\lambda_4\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-\frac{3}{2}\lambda_3\lambda_6\Big(6{h_t}^2+6{h_t'}^2-6g^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2-2{g'}^2\Big)\\
&-3\lambda_3\lambda_7\Big(6{h_t}^2-6g^2+\hat g_{1ud}^2+\hat g_{1uu}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2-2{g'}^2\Big)\\
&-2\big(\lambda_1\lambda_5+\lambda_3\lambda_5\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-6\lambda_1\lambda_6\Big(6{h_t'}^2-\frac{9}{3}g^2+\hat g_{1dd}^2+\hat g_{1du}^2+3\hat g_{2dd}^2+3\hat g_{2du}^2-\frac{3}{2}{g'}^2\Big)\\
&-2\lambda_1\lambda_4\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-\frac{1}{2}{\lambda_4}^2\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-2\lambda_4\lambda_6\Big(6{h_t}^2+6{h_t'}^2-9g^2-\frac{5}{2}{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-2\lambda_4\lambda_7\Big(6{h_t}^2-\frac{9}{2}g^2-2{g'}^2+\hat g_{1ud}^2+\hat g_{1uu}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&-\frac{3}{2}\lambda_4\lambda_5\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-6\lambda_6\big(\lambda_6+\lambda_7\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-\frac{5}{2}\lambda_5\lambda_6\Big(6{h_t}^2+6{h_t'}^2-\frac{54}{5}g^2-4{g'}^2+\hat g_{1dd}^2+\hat g_{1du}^2+\hat g_{1ud}^2+\hat g_{1uu}^2\\
&\qquad+3\hat g_{2dd}^2+3\hat g_{2du}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)\\
&+\lambda_5\lambda_7\Big(6{h_t}^2+{g'}^2+\hat g_{1ud}^2+\hat g_{1uu}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big),
\end{aligned}\tag{F.22f}$$

$$\begin{aligned}
\beta_{\lambda_7}^{(2),\lambda^2} =& -\frac{3}{2}\lambda_2\big(\lambda_3+\frac{4}{3}\lambda_4\big)\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-\frac{5}{2}\lambda_2\lambda_5\Big(6h_th_t'+\hat g_{1dd}\hat g_{1ud}+\hat g_{1du}\hat g_{1uu}+3\hat g_{2dd}\hat g_{2ud}+3\hat g_{2du}\hat g_{2uu}\Big)\\
&-6\lambda_2\lambda_7\Big(6{h_t}^2-\frac{9}{2}g^2-\frac{3}{2}{g'}^2+\hat g_{1ud}^2+\hat g_{1uu}^2+3\hat g_{2ud}^2+3\hat g_{2uu}^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\lambda_3\big(\lambda_3+3\lambda_4\big)\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-2\lambda_3\lambda_5\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-3\lambda_3\lambda_6\Big(6{h_t'}^2-6g^2-2{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2\Big)\\
&-\frac{3}{2}\lambda_3\lambda_7\Big(6{h_t}^2+6{h_t'}^2-6g^2-2{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2\\
&\qquad\qquad+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\Big)\\
&-\frac{1}{2}{\lambda_4}^2\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-\frac{3}{2}\lambda_4\big(\lambda_5+\frac{2}{3}\lambda_7\big)\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big)\\
&-2\lambda_4\lambda_6\Big(6{h_t'}^2-\frac{9}{2}g^2-2{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2\Big)\\
&-2\lambda_4\lambda_7\Big(6{h_t}^2+6{h_t'}^2-9g^2-\frac{5}{2}{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2\\
&\qquad\qquad+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\Big)\\
&-\frac{5}{2}\lambda_5\lambda_7\Big(6{h_t}^2+6{h_t'}^2+\frac{54}{5}g^2-4{g'}^2+\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2\\
&\qquad\qquad+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\Big)\\
&-6\big(\lambda_6+\lambda_7\big)\lambda_7\Big(6h_th_t'+\hat{g}_{1dd}\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{1uu}+3\hat{g}_{2dd}\hat{g}_{2ud}+3\hat{g}_{2du}\hat{g}_{2uu}\Big).
\end{aligned}
\tag{F.22g}
$$

The λ^1 pieces are given by

$$
\begin{aligned}
\beta_{\lambda_1}^{(2),\lambda^1}=&\lambda_1\Bigg\{40g_3^2{h_t'}^2-\frac{27}{2}{h_t}^2{h_t'}^2-\frac{3}{2}{h_t'}^4+\frac{69}{16}g^4+\frac{39}{8}g^2{g'}^2+\frac{691}{48}{g'}^4+\frac{5}{4}{h_t'}^2\Big[9g^2+\frac{17}{3}{g'}^2\Big]\\
&\quad+\frac{15}{8}g^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+11\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)\Big]\\
&\quad+\frac{5}{8}{g'}^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)\Big]\\
&\quad-\frac{1}{8}\Big[\hat{g}_{1dd}^4-\hat{g}_{1dd}^2\big(12\hat{g}_{1du}^2-9\hat{g}_{1ud}^2-6\hat{g}_{1uu}^2-2\hat{g}_{2dd}^2-9\hat{g}_{2ud}^2\big)\\
&\qquad+8\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu}-10\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}+3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)+\hat{g}_{1du}^4\\
&\qquad+\hat{g}_{1du}^2\big(6\hat{g}_{1ud}^2+9\hat{g}_{1uu}^2+2\hat{g}_{2du}^2+9\hat{g}_{2uu}^2\big)+24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}\\
&\qquad+9\hat{g}_{1ud}^2\hat{g}_{2dd}^2+9\hat{g}_{1uu}^2\hat{g}_{2du}^2+5\hat{g}_{2dd}^4+44\hat{g}_{2dd}^2\hat{g}_{2du}^2+45\hat{g}_{2dd}^2\hat{g}_{2ud}^2\\
&\qquad+18\hat{g}_{2dd}^2\hat{g}_{2uu}^2-24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+5\hat{g}_{2du}^4\\
&\qquad+18\hat{g}_{2du}^2\hat{g}_{2ud}^2+45\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big]\Bigg\}\\
&+\lambda_3\Big[\frac{15}{2}g^4+\frac{5}{2}{g'}^4+2\hat{g}_{1dd}\hat{g}_{1uu}\big(\hat{g}_{1du}\hat{g}_{1ud}+3\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+6\hat{g}_{2dd}\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{1ud}-\hat{g}_{2du}\hat{g}_{2ud}\big)\Big]\\
&+\lambda_4\Big[\frac{15}{4}g^4+\frac{5}{2}g^2{g'}^2+\frac{5}{4}{g'}^4+2\big(\hat{g}_{1dd}\hat{g}_{1uu}+\hat{g}_{2dd}\hat{g}_{2uu}\big)\big(\hat{g}_{1du}\hat{g}_{1ud}+\hat{g}_{2du}\hat{g}_{2ud}\big)\Big]\\
&+\lambda_5\Big[12{h_t}^2{h_t'}^2+\hat{g}_{1dd}^2\hat{g}_{1ud}^2+2\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}+\hat{g}_{1du}^2\hat{g}_{1uu}^2\\
&\qquad+2\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}+5\hat{g}_{2dd}^2\hat{g}_{2ud}^2+5\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big]\\
&+2\lambda_6\Big[12h_t{h_t'}^3+\hat{g}_{1dd}^3\hat{g}_{1ud}+\hat{g}_{1dd}^2(2\hat{g}_{1du}\hat{g}_{1uu}+\hat{g}_{2dd}\hat{g}_{2ud})
\end{aligned}
$$

$$
\begin{aligned}
&+ \hat{g}_{1dd}\Big(2\hat{g}_{1du}^2\hat{g}_{1ud} + 4\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{2dd}\big(\hat{g}_{1ud}\hat{g}_{2dd} + 4\hat{g}_{1uu}\hat{g}_{2du}\big)\Big) \\
&+ \hat{g}_{1du}^3\hat{g}_{1uu} + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1du}\hat{g}_{2du}(4\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2du}) \\
&+ 5\hat{g}_{2dd}^3\hat{g}_{2ud} - 2\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} - 2\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 5\hat{g}_{2du}^3\hat{g}_{2uu}\Big],
\end{aligned}
\tag{F.23a}
$$

$$
\begin{aligned}
\beta_{\lambda_2}^{(2),\lambda^1} =&\lambda_2\Bigg\{40g_3^2{h_t}^2 - \frac{27}{2}{h_t}^2{h_t'}^2 - \frac{3}{2}{h_t}^4 + \frac{69}{16}g^4 + \frac{39}{8}g^2{g'}^2 + \frac{691}{48}{g'}^4 + \frac{5}{4}{h_t}^2\Big[9g^2 + \frac{17}{3}{g'}^2\Big] \\
&+ \frac{15}{8}g^2\Big[\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 11\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big] \\
&+ \frac{5}{8}{g'}^2\Big[\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big] \\
&- \frac{1}{8}\Big[3\hat{g}_{1dd}^2\big(3\hat{g}_{1ud}^2 + 2\hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2\big) + 8\hat{g}_{1dd}\hat{g}_{1uu}\big(\hat{g}_{1du}\hat{g}_{1ud} + 3\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + 3\hat{g}_{1du}^2\Big(2\hat{g}_{1ud}^2 + 3\big(\hat{g}_{1uu}^2 + \hat{g}_{2uu}^2\big)\Big) + 24\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1ud}^4 \\
&\qquad - 12\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + 9\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 2\hat{g}_{1ud}^2\hat{g}_{2ud}^2 - 80\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad + \hat{g}_{1uu}^4 + 9\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 2\hat{g}_{1uu}^2\hat{g}_{2uu}^2 + 45\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 18\hat{g}_{2dd}^2\hat{g}_{2uu}^2 \\
&\qquad - 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 18\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 45\hat{g}_{2du}^2\hat{g}_{2uu}^2 \\
&\qquad + 5\hat{g}_{2ud}^4 + 44\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 5\hat{g}_{2uu}^4\Big]\Bigg\} \\
&+ \lambda_3\Big[\frac{15}{2}g^4 + \frac{5}{2}{g'}^4 + 2\hat{g}_{1dd}\hat{g}_{1uu}\big(\hat{g}_{1du}\hat{g}_{1ud} + 3\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + 6\hat{g}_{2dd}\hat{g}_{2uu}\big(\hat{g}_{1du}\hat{g}_{1ud} - \hat{g}_{2du}\hat{g}_{2ud}\big)\Big] \\
&+ \lambda_4\Big[\frac{15}{4}g^4 + \frac{5}{2}g^2{g'}^2 + \frac{5}{4}{g'}^4 + 2\big(\hat{g}_{1dd}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2uu}\big)\big(\hat{g}_{1du}\hat{g}_{1ud} + \hat{g}_{2du}\hat{g}_{2ud}\big)\Big] \\
&+ \lambda_5\Big[12{h_t}^2{h_t'}^2 + \hat{g}_{1dd}^2\hat{g}_{1ud}^2 + 2\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1du}^2\hat{g}_{1uu}^2 \\
&\qquad + 2\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu} + 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 5\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big] \\
&+ 2\lambda_7\Big[12{h_t}^3h_t' + \hat{g}_{1dd}\Big(\hat{g}_{1ud}^3 + \hat{g}_{1ud}\big(2\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) + 4\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
&\qquad + \hat{g}_{1du}\big(2\hat{g}_{1ud}^2\hat{g}_{1uu} + 4\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big) + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} \\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} + 5\hat{g}_{2dd}\hat{g}_{2ud}^3 \\
&\qquad - 2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 - 2\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2uu}^3\Big],
\end{aligned}
\tag{F.23b}
$$

$$
\begin{aligned}
\beta_{\lambda_3}^{(2),\lambda^1} =&\big(\lambda_1 + \lambda_2\big)\Big[\frac{45}{8}g^4 - \frac{5}{4}g^2{g'}^2 + \frac{15}{8}{g'}^4 + \hat{g}_{1dd}\hat{g}_{1uu}\big(\hat{g}_{1du}\hat{g}_{1ud} + 5\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{2dd}\hat{g}_{2uu}\big(5\hat{g}_{1du}\hat{g}_{1ud} - 7\hat{g}_{2du}\hat{g}_{2ud}\big)\Big] \\
&+ \lambda_3\Bigg\{20g_3^2\big({h_t}^2 + {h_t'}^2\big) - \frac{27}{4}{h_t}^4 - \frac{3}{2}{h_t}^2{h_t'}^2 - \frac{27}{4}{h_t'}^4 + \frac{9}{16}g^4 + \frac{11}{8}g^2{g'}^2 + \frac{631}{48}{g'}^4 \\
&+ \frac{5}{8}\big({h_t}^2 + {h_t'}^2\big)\Big[9g^2 + \frac{17}{3}{g'}^2\Big] \\
&+ \frac{15}{16}g^2\Big[\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 11\big(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{16}g'^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2+3\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2+\hat{g}_{2ud}^2+\hat{g}_{2uu}^2\big)\Big]\\
&-\frac{1}{16}\Big[9\hat{g}_{1dd}^4+2\hat{g}_{1dd}^2\big(2\hat{g}_{1du}^2+\hat{g}_{1ud}^2-2\hat{g}_{1uu}^2+9\hat{g}_{2dd}^2-7\hat{g}_{2ud}^2\big)\\
&\qquad+16\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu}-3\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}+2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\\
&\qquad\qquad+2\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}+3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+9\hat{g}_{1du}^4-2\hat{g}_{1du}^2\big(2\hat{g}_{1ud}^2-\hat{g}_{1uu}^2-9\hat{g}_{2du}^2+7\hat{g}_{2uu}^2\big)\\
&\qquad+16\hat{g}_{1du}\big(3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}+2\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud}+2\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big)\\
&\qquad+9\hat{g}_{1ud}^4-4\hat{g}_{1ud}^2\hat{g}_{1uu}^2-14\hat{g}_{1ud}^2\hat{g}_{2dd}^2+18\hat{g}_{1ud}^2\hat{g}_{2ud}^2\\
&\qquad-48\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}+9\hat{g}_{1uu}^4-14\hat{g}_{1uu}^2\hat{g}_{2du}^2+18\hat{g}_{1uu}^2\hat{g}_{2uu}^2\\
&\qquad+45\hat{g}_{2dd}^4+60\hat{g}_{2dd}^2\hat{g}_{2du}^2+10\hat{g}_{2dd}^2\hat{g}_{2ud}^2-44\hat{g}_{2dd}^2\hat{g}_{2uu}^2\\
&\qquad-48\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+45\hat{g}_{2du}^4-44\hat{g}_{2du}^2\hat{g}_{2ud}^2\\
&\qquad+10\hat{g}_{2du}^2\hat{g}_{2uu}^2+45\hat{g}_{2ud}^4+60\hat{g}_{2ud}^2\hat{g}_{2uu}^2+45\hat{g}_{2uu}^4\Big]\Bigg\}\\
&+\lambda_4\Big[12{h_t}^2{h_t'}^2+\frac{15}{4}g^4-\frac{3}{2}g^2{g'}^2+\frac{5}{4}{g'}^4+\hat{g}_{1dd}^2\big(\hat{g}_{1ud}^2-\hat{g}_{2ud}^2\big)\\
&\quad+4\hat{g}_{1dd}\hat{g}_{2dd}\big(\hat{g}_{1du}\hat{g}_{2du}+\hat{g}_{1ud}\hat{g}_{2ud}\big)+\hat{g}_{1du}^2\big(\hat{g}_{1uu}^2-\hat{g}_{2uu}^2\big)\\
&\quad+4\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}-\hat{g}_{1ud}^2\hat{g}_{2dd}^2+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}-\hat{g}_{1uu}^2\hat{g}_{2du}^2\\
&\quad+\hat{g}_{2dd}^2\big(5\hat{g}_{2ud}^2-4\hat{g}_{2du}^2\big)+5\hat{g}_{2du}^2\hat{g}_{2uu}^2-4\hat{g}_{2ud}^2\hat{g}_{2uu}^2\Big]\\
&-2\lambda_5\Big[\hat{g}_{1dd}\hat{g}_{1du}\big(\hat{g}_{2ud}\hat{g}_{2uu}-\hat{g}_{1ud}\hat{g}_{1uu}\big)+\hat{g}_{2dd}\hat{g}_{2du}\big(\hat{g}_{1ud}\hat{g}_{1uu}-5\hat{g}_{2ud}\hat{g}_{2uu}\big)\Big]\\
&+\lambda_6\Big[12h_t{h_t'}^3+\hat{g}_{1dd}^3\hat{g}_{1ud}+\hat{g}_{1dd}^2(\hat{g}_{1du}\hat{g}_{1uu}+\hat{g}_{2dd}\hat{g}_{2ud})\\
&\quad+\hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud}-\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)\Big)\\
&\quad+\hat{g}_{1dd}\Big(\hat{g}_{1ud}\big(\hat{g}_{1uu}^2+\hat{g}_{2dd}^2\big)-\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}+5\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big)+\hat{g}_{1du}^3\hat{g}_{1uu}\\
&\quad+\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu}+\hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu}-\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du}+5\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}+\hat{g}_{1uu}\hat{g}_{2du}^2\big)\\
&\quad+5\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)+5\hat{g}_{2dd}^3\hat{g}_{2ud}+5\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}\\
&\quad+\hat{g}_{2dd}\hat{g}_{2ud}\big(5\hat{g}_{2du}^2-7\hat{g}_{2uu}^2\big)+5\hat{g}_{2du}^3\hat{g}_{2uu}-7\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Big]\\
&+\lambda_7\Big[12{h_t}^3h_t'+\hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu}+\hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud}+5\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)\Big)\\
&\quad+\hat{g}_{1dd}\Big(\hat{g}_{1ud}^3+\hat{g}_{1ud}\big(\hat{g}_{1uu}^2+\hat{g}_{2ud}^2\big)+\hat{g}_{1uu}\big(5\hat{g}_{2dd}\hat{g}_{2du}-\hat{g}_{2ud}\hat{g}_{2uu}\big)\Big)\\
&\quad+\hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu}+5\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du}-\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}+\hat{g}_{1uu}^3+\hat{g}_{1uu}\hat{g}_{2uu}^2\big)\\
&\quad+\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}-\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)+\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu}\\
&\quad-7\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}+\hat{g}_{2dd}\Big(5\hat{g}_{2ud}\big(\hat{g}_{2ud}^2+\hat{g}_{2uu}^2\big)-7\hat{g}_{2du}^2\hat{g}_{2ud}\Big)\\
&\quad+5\hat{g}_{2du}\hat{g}_{2uu}\big(\hat{g}_{2ud}^2+\hat{g}_{2uu}^2\big)\Big],
\end{aligned}
\tag{F.23c}
$$

$$
\begin{aligned}
\beta_{\lambda_4}^{(2),\lambda^1}=&\big(\lambda_1+\lambda_2\big)\Big[\frac{5}{2}g^2{g'}^2+\hat{g}_{1dd}\hat{g}_{1uu}\big(\hat{g}_{1du}\hat{g}_{1ud}-\hat{g}_{2du}\hat{g}_{2ud}\big)+\hat{g}_{2dd}\hat{g}_{2uu}\big(5\hat{g}_{2du}\hat{g}_{2ud}-\hat{g}_{1du}\hat{g}_{1ud}\big)\Big]\\
&+\lambda_3\Big[12{h_t}^2{h_t'}^2+g^2{g'}^2+\hat{g}_{1dd}^2\big(\hat{g}_{1ud}^2-\hat{g}_{2ud}^2\big)+4\hat{g}_{1dd}\hat{g}_{2dd}\big(\hat{g}_{1ud}\hat{g}_{2ud}+\hat{g}_{1uu}\hat{g}_{2uu}\big)\\
&\quad+\hat{g}_{1du}^2\big(\hat{g}_{1uu}^2-\hat{g}_{2uu}^2\big)+4\hat{g}_{1du}\hat{g}_{2du}\big(\hat{g}_{1ud}\hat{g}_{2ud}+\hat{g}_{1uu}\hat{g}_{2uu}\big)\\
&\quad-\hat{g}_{1ud}^2\hat{g}_{2dd}^2-\hat{g}_{1uu}^2\hat{g}_{2du}^2+\hat{g}_{2dd}^2\big(5\hat{g}_{2ud}^2-4\hat{g}_{2uu}^2\big)+\hat{g}_{2du}^2\big(5\hat{g}_{2uu}^2-4\hat{g}_{2ud}^2\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda_4 \Bigg\{ 20 g_3^2 \big({h_t}^2 + {h'_t}^2\big) - \frac{27}{4}{h_t}^4 - \frac{3}{2}{h_t}^2{h'_t}^2 - \frac{27}{4}{h'_t}^4 - \frac{111}{16}g^4 + \frac{51}{8}g^2{g'}^2 + \frac{511}{48}{g'}^4 \\
&\quad + \frac{5}{8}\big({h_t}^2 + {h'_t}^2\big)\left[9g^2 + \frac{17}{3}{g'}^2\right] \\
&\quad + \frac{15}{16}g^2\left[\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 11\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^2\right] \\
&\quad + \frac{5}{16}{g'}^2\left[\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\right] \\
&\quad - \frac{1}{16}\Bigg[9\hat{g}_{1dd}^4 + 2\hat{g}_{1dd}^2\big(2\hat{g}_{1du}^2 + \hat{g}_{1ud}^2 - 2\hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 - 7\hat{g}_{2ud}^2\big) \\
&\qquad + 16\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 5\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} \\
&\qquad\qquad - 6\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad - 9\hat{g}_{1du}^4 - 2\hat{g}_{1du}^2\big(2\hat{g}_{1ud}^2 - \hat{g}_{1uu}^2 - 9\hat{g}_{2du}^2 + 7\hat{g}_{2uu}^2\big) \\
&\qquad + 16\hat{g}_{1du}\big(3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} - 6\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} + 2\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + 9\hat{g}_{1ud}^4 + 4\hat{g}_{1ud}^2\hat{g}_{1uu}^2 - 14\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 18\hat{g}_{1ud}^2\hat{g}_{2ud}^2 \\
&\qquad + 80\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 9\hat{g}_{1uu}^4 - 14\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 18\hat{g}_{1uu}^2\hat{g}_{2uu}^2 \\
&\qquad - 45\hat{g}_{2dd}^4 - 68\hat{g}_{2dd}^2\hat{g}_{2du}^2 + 10\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 84\hat{g}_{2dd}^2\hat{g}_{2uu}^2 \\
&\qquad - 48\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 45\hat{g}_{2du}^4 + 84\hat{g}_{2du}^2\hat{g}_{2ud}^2 \\
&\qquad + 10\hat{g}_{2du}^2\hat{g}_{2uu}^2 + 45\hat{g}_{2ud}^4 - 68\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 45\hat{g}_{2uu}^4\Bigg]\Bigg\} \\
&+ 2\lambda_5\left[\hat{g}_{1dd}\hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{1uu} + 5\hat{g}_{2ud}\hat{g}_{2uu}\big) + \hat{g}_{2dd}\hat{g}_{2du}\big(5\hat{g}_{1ud}\hat{g}_{1uu} - 7\hat{g}_{2ud}\hat{g}_{2uu}\big)\right] \\
&+ \lambda_6\Bigg[12h_t{h'_t}^3 + \hat{g}_{1dd}^3\hat{g}_{1ud} + \hat{g}_{1dd}^2\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\quad + \hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud} + 5\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{1ud}\big(\hat{g}_{1uu}^2 + \hat{g}_{2dd}^2\big) + \hat{g}_{1uu}\big(5\hat{g}_{2dd}\hat{g}_{2du} - \hat{g}_{2ud}\hat{g}_{2uu}\big)\Big) \\
&\quad + \hat{g}_{1du}^3\hat{g}_{1uu} + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&\quad + \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} + 5\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} - \hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2du}^2\big) \\
&\quad - \hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) + 5\hat{g}_{2dd}^3\hat{g}_{2ud} - 7\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&\quad + \hat{g}_{2dd}\big(5\hat{g}_{2ud}\hat{g}_{2uu}^2 - 7\hat{g}_{2du}^2\hat{g}_{2ud}\big) + 5\hat{g}_{2du}^3\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Bigg] \\
&+ \lambda_7\Bigg[12{h_t}^3h'_t + \hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu} \\
&\quad + \hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud} - \hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) + \hat{g}_{1ud}^3 + \hat{g}_{1ud}\big(\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) \\
&\qquad - \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 5\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
&\quad + \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} - \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + 5\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big) \\
&\quad + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} + 5\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&\quad + 5\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big(5\hat{g}_{2du}^2 + 5\hat{g}_{2ud}^2 - 7\hat{g}_{2uu}^2\big) \\
&\quad + \hat{g}_{2du}\big(5\hat{g}_{2uu}^3 - 7\hat{g}_{2ud}^2\hat{g}_{2uu}\big)\Bigg],
\end{aligned} \tag{F.23d}
$$

$$
\begin{aligned}
\beta_{\lambda_5}^{(2),\lambda^1} = &\big(\lambda_1 + \lambda_2\big)\Bigg[6{h_t}^2{h'_t}^2 + \frac{1}{2}\big(\hat{g}_{1dd}^2\hat{g}_{1ud}^2 + 2\hat{g}_{1dd}\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1du}^2\hat{g}_{1uu}^2 \\
&\qquad + 2\hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu} + 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 5\hat{g}_{2du}^2\hat{g}_{2uu}^2\big)\Bigg] \\
&+ 2\lambda_3\big(\hat{g}_{1dd}\hat{g}_{1du} + \hat{g}_{2dd}\hat{g}_{2du}\big)\big(\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{2ud}\hat{g}_{2uu}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_4\Big[2\hat{g}_{1dd}\hat{g}_{1du}\big(\hat{g}_{1ud}\hat{g}_{1uu}+3\hat{g}_{2ud}\hat{g}_{2uu}\big)+6\hat{g}_{2dd}\hat{g}_{2du}\big(\hat{g}_{1ud}\hat{g}_{1uu}-\hat{g}_{2ud}\hat{g}_{2uu}\big)\Big]\\
&+\lambda_5\Big\{20g_3^2\big({h_t}^2+{h'_t}^2\big)-\frac{3}{4}{h_t}^4-\frac{27}{2}{h_t}^2{h'_t}^2-\frac{3}{4}{h'_t}^4-\frac{111}{16}g^4+\frac{19}{8}g^2{g'}^2\\
&\quad+\frac{511}{48}{g'}^4+\frac{9}{8}\big({h_t}^2+{h'_t}^2\big)\Big[9g^2+\frac{17}{3}{g'}^2\Big]\\
&\quad+\frac{15}{16}g^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2+11\hat{g}_{2dd}^2+11\hat{g}_{2du}^2+11\hat{g}_{2ud}^2+11\hat{g}_{2uu}^2\Big]\\
&\quad+\frac{5}{16}{g'}^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+\hat{g}_{1ud}^2+\hat{g}_{1uu}^2+3\hat{g}_{2dd}^2+3\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\Big]\\
&\quad-\frac{1}{16}\Big[\hat{g}_{1dd}^4+2\hat{g}_{1dd}^2\big(10\hat{g}_{1du}^2+9\hat{g}_{1ud}^2-10\hat{g}_{1uu}^2+\hat{g}_{2dd}^2+9\hat{g}_{2ud}^2\big)\\
&\qquad+16\hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu}+3\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}\\
&\qquad\qquad-8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}+3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+\hat{g}_{1du}^4-2\hat{g}_{1du}^2\big(10\hat{g}_{1ud}^2-9\hat{g}_{1uu}^2-\hat{g}_{2du}^2-9\hat{g}_{2uu}^2\big)\\
&\qquad-16\hat{g}_{1du}\hat{g}_{1ud}\big(8\hat{g}_{2du}\hat{g}_{2ud}-3\hat{g}_{2dd}\hat{g}_{2uu}\big)+\hat{g}_{1ud}^4+20\hat{g}_{1ud}^2\hat{g}_{1uu}^2\\
&\qquad+18\hat{g}_{1ud}^2\hat{g}_{2dd}^2+2\hat{g}_{1ud}^2\hat{g}_{2ud}^2+48\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}+\hat{g}_{1uu}^4\\
&\qquad+18\hat{g}_{1uu}^2\hat{g}_{2du}^2+2\hat{g}_{1uu}^2\hat{g}_{2uu}^2+5\hat{g}_{2dd}^4+12\hat{g}_{2dd}^2\hat{g}_{2du}^2+90\hat{g}_{2dd}^2\hat{g}_{2ud}^2\\
&\qquad+68\hat{g}_{2dd}^2\hat{g}_{2uu}^2-48\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+5\hat{g}_{2du}^4+68\hat{g}_{2du}^2\hat{g}_{2ud}^2\\
&\qquad+90\hat{g}_{2du}^2\hat{g}_{2uu}^2+5\hat{g}_{2ud}^4-12\hat{g}_{2ud}^2\hat{g}_{2uu}^2+5\hat{g}_{2uu}^4\Big]\Big\}\\
&+\lambda_6\Big[12{h_t}^3h'_t+2\hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu}\\
&\quad+\hat{g}_{1dd}\Big(2\hat{g}_{1du}^2\hat{g}_{1ud}+4\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+\hat{g}_{1ud}^3+\hat{g}_{1ud}\hat{g}_{2ud}^2+4\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}\Big)\\
&\quad+\hat{g}_{1du}\big(4\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du}+\hat{g}_{1uu}^3+\hat{g}_{1uu}\hat{g}_{2uu}^2\big)+\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}\\
&\quad+\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu}-2\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}-2\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud}+5\hat{g}_{2dd}\hat{g}_{2ud}^3\\
&\quad+5\hat{g}_{2du}\hat{g}_{2uu}^3\Big]\\
&+\lambda_7\Big[12h_t{h'_t}^3+\hat{g}_{1dd}^3\hat{g}_{1ud}+\hat{g}_{1dd}^2\hat{g}_{2dd}\hat{g}_{2ud}+\hat{g}_{1dd}\Big(\hat{g}_{1ud}\big(2\hat{g}_{1uu}^2+\hat{g}_{2dd}^2\big)+4\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&\quad+\hat{g}_{1du}^3\hat{g}_{1uu}+\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu}+\hat{g}_{1du}\big(2\hat{g}_{1ud}^2\hat{g}_{1uu}+4\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}+\hat{g}_{1uu}\hat{g}_{2du}^2\big)\\
&\quad+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}+5\hat{g}_{2dd}^3\hat{g}_{2ud}-2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2\\
&\quad+5\hat{g}_{2du}^3\hat{g}_{2uu}-2\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Big],
\end{aligned} \tag{F.23e}
$$

$$
\begin{aligned}
\beta_{\lambda_6}^{(2),\lambda^1}=&\frac{1}{2}\lambda_1\Big[12h_t{h'_t}^3+\hat{g}_{1dd}^3\hat{g}_{1ud}+\hat{g}_{1dd}^2\big(2\hat{g}_{1du}\hat{g}_{1uu}+\hat{g}_{2dd}\hat{g}_{2ud}\big)\\
&+\hat{g}_{1dd}\Big(2\hat{g}_{1du}^2\hat{g}_{1ud}+4\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{2du}\hat{g}_{2ud}\big)+\hat{g}_{2dd}\big(\hat{g}_{1ud}\hat{g}_{2dd}+4\hat{g}_{1uu}\hat{g}_{2du}\big)\Big)\\
&+\hat{g}_{1du}^3\hat{g}_{1uu}+\hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu}+\hat{g}_{1du}\hat{g}_{2du}\big(4\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{1uu}\hat{g}_{2du}\big)\\
&+5\hat{g}_{2dd}^3\hat{g}_{2ud}-2\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}-2\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud}+5\hat{g}_{2du}^3\hat{g}_{2uu}\Big]\\
&+\frac{1}{2}\lambda_3\Big[12h_t{h'_t}^3+\hat{g}_{1dd}^3\hat{g}_{1ud}+\hat{g}_{1dd}^2\big(\hat{g}_{1du}\hat{g}_{1uu}+\hat{g}_{2dd}\hat{g}_{2ud}\big)\\
&\quad+\hat{g}_{1dd}\big(\hat{g}_{1du}^2\hat{g}_{1ud}+\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1ud}\hat{g}_{1uu}^2\\
&\qquad+\hat{g}_{1ud}\hat{g}_{2dd}^2+\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}+3\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \hat{g}_{1du}^3\hat{g}_{1uu} + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&+ \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2du}^2\big) \\
&+ 3\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 3\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + 5\hat{g}_{2dd}^3\hat{g}_{2ud} + \hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&+ \hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} - 3\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 + 5\hat{g}_{2du}^3\hat{g}_{2uu} - 3\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Big] \\
+ \frac{1}{2}\lambda_4\Big[&12h_t {h'_t}^3 + \hat{g}_{1dd}^3\hat{g}_{1ud} + \hat{g}_{1dd}^2\big(\hat{g}_{1du}\hat{g}_{1uu} + \hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&+ \hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud} + 3\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{1ud}\hat{g}_{1uu}^2 + \hat{g}_{1ud}\hat{g}_{2dd}^2 + 3\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
&+ \hat{g}_{1du}^3\hat{g}_{1uu} + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&+ \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} + 3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2du}^2\big) \\
&+ \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + 5\hat{g}_{2dd}^3\hat{g}_{2ud} - 3\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&- 3\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 + 5\hat{g}_{2du}^3\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Big] \\
+ \frac{1}{2}\lambda_5\Big[&12{h_t}^3 h'_t + 2\hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu} \\
&+ \hat{g}_{1dd}\Big(2\hat{g}_{1du}^2\hat{g}_{1ud} + 4\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{1ud}^3 + \hat{g}_{1ud}\hat{g}_{2ud}^2 + 4\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}\Big) \\
&+ \hat{g}_{1du}\big(4\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big) + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&- 2\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} - 2\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2ud}^3 + 5\hat{g}_{2du}\hat{g}_{2uu}^3\Big] \\
+ \lambda_6\Big\{&10g_3^2\big({h_t}^2 + 3{h'_t}^2\big) - \frac{27}{8}{h_t}^4 - \frac{3}{2}{h_t}^2{h'_t}^2 - \frac{33}{8}{h'_t}^4 - \frac{21}{16}g^4 + \frac{29}{8}g^2{g'}^2 \\
&+ \frac{601}{48}{g'}^4 + \frac{5}{16}\big({h_t}^2 + 3{h'_t}^2\big)\Big[9g^2 + \frac{17}{3}{g'}^2\Big] \\
&+ \frac{15}{32}g^2\Big[3\hat{g}_{1dd}^2 + 3\hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 33\hat{g}_{2dd}^2 + 33\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^2\Big] \\
&+ \frac{5}{32}{g'}^2\Big[3\hat{g}_{1dd}^2 + 3\hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 9\hat{g}_{2dd}^2 + 9\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big] \\
&- \frac{1}{32}\Big[11\hat{g}_{1dd}^4 - \hat{g}_{1dd}^2\big(4\hat{g}_{1du}^2 - 4\hat{g}_{1ud}^2 + 8\hat{g}_{1uu}^2 - 22\hat{g}_{2dd}^2 - 20\hat{g}_{2ud}^2\big) \\
&\qquad - 16\hat{g}_{1dd}\Big(\hat{g}_{1du}\big(2\hat{g}_{1ud}\hat{g}_{1uu} + 7\hat{g}_{2dd}\hat{g}_{2du} + 8\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad\qquad + 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} - 6\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\Big) \\
&\qquad + 11\hat{g}_{1du}^4 - \hat{g}_{1du}^2\big(8\hat{g}_{1ud}^2 - 4\hat{g}_{1uu}^2 - 22\hat{g}_{2du}^2 - 20\hat{g}_{2uu}^2\big) \\
&\qquad + 32\hat{g}_{1du}\big(3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} - 4\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} - \hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + 9\hat{g}_{1ud}^4 + 20\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + 20\hat{g}_{1ud}^2\hat{g}_{2dd}^2 + 18\hat{g}_{1ud}^2\hat{g}_{2ud}^2 \\
&\qquad - 128\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 48\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 9\hat{g}_{1uu}^4 + 20\hat{g}_{1uu}^2\hat{g}_{2du}^2 \\
&\qquad + 18\hat{g}_{1uu}^2\hat{g}_{2uu}^2 + 55\hat{g}_{2dd}^4 + 100\hat{g}_{2dd}^2\hat{g}_{2du}^2 + 20\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 104\hat{g}_{2dd}^2\hat{g}_{2uu}^2 \\
&\qquad - 32\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 55\hat{g}_{2du}^4 + 104\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 20\hat{g}_{2du}^2\hat{g}_{2uu}^2 \\
&\qquad + 45\hat{g}_{2ud}^4 + 12\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 45\hat{g}_{2uu}^4\Big]\Big\} \\
+ \frac{1}{2}\lambda_7\Big[&12{h_t}^2{h'_t}^2 + \frac{45}{4}g^4 + \frac{5}{2}g^2{g'}^2 + \frac{15}{4}{g'}^4 + \hat{g}_{1dd}^2\hat{g}_{1ud}^2 \\
&+ 2\hat{g}_{1dd}\big(2\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 4\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \hat{g}_{1du}^2\hat{g}_{1uu}^2 + 2\hat{g}_{1du}\hat{g}_{2uu}\big(4\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2du}\big)\\
&+ 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 - 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 5\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big],
\end{aligned}
\tag{F.23f}
$$

$$
\begin{aligned}
\beta_{\lambda_7}^{(2),\lambda^1} =& \frac{1}{2}\lambda_2\Big[12{h_t}^3h_t' + \hat{g}_{1dd}\Big(\hat{g}_{1ud}^3 + \hat{g}_{1ud}\big(2\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2\big) + 4\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&+ \hat{g}_{1du}\big(2\hat{g}_{1ud}^2\hat{g}_{1uu} + 4\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big) + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}\\
&+ 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} + 5\hat{g}_{2dd}\hat{g}_{2ud}^3\\
&- 2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 - 2\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2uu}^3\Big]\\
&+ \frac{1}{2}\lambda_3\Big[12{h_t}^3h_t' + \hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu}\\
&\quad + \hat{g}_{1dd}\Big(\hat{g}_{1du}^2\hat{g}_{1ud} + 3\hat{g}_{1du}\big(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}\big) + \hat{g}_{1ud}^3 + \hat{g}_{1ud}\hat{g}_{1uu}^2\\
&\qquad + \hat{g}_{1ud}\hat{g}_{2ud}^2 + 3\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big)\\
&\quad + \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} + 3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big)\\
&\quad + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\\
&\quad + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} - 3\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} - 3\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2ud}^3\\
&\quad + \hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 + \hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2uu}^3\Big]\\
&+ \frac{1}{2}\lambda_4\Big[12{h_t}^3h_t' + \hat{g}_{1dd}^2\hat{g}_{1du}\hat{g}_{1uu}\\
&\quad + \hat{g}_{1dd}\big(\hat{g}_{1du}^2\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1ud}^3 + \hat{g}_{1ud}\hat{g}_{1uu}^2\\
&\qquad + \hat{g}_{1ud}\hat{g}_{2ud}^2 + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\quad + \hat{g}_{1du}\big(\hat{g}_{1ud}^2\hat{g}_{1uu} + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^3 + \hat{g}_{1uu}\hat{g}_{2uu}^2\big)\\
&\quad + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 3\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\\
&\quad + \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2ud}^3\\
&\quad - 3\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 - 3\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2uu}^3\Big]\\
&+ \frac{1}{2}\lambda_5\Big[12h_t{h_t'}^3 + \hat{g}_{1dd}^3\hat{g}_{1ud} + \hat{g}_{1dd}^2\hat{g}_{2dd}\hat{g}_{2ud}\\
&\quad + \hat{g}_{1dd}\Big(\hat{g}_{1ud}\big(2\hat{g}_{1uu}^2 + \hat{g}_{2dd}^2\big) + 4\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\Big) + \hat{g}_{1du}^3\hat{g}_{1uu}\\
&\quad + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1du}\big(2\hat{g}_{1ud}^2\hat{g}_{1uu} + 4\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2du}^2\big)\\
&\quad + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} + 5\hat{g}_{2dd}^3\hat{g}_{2ud}\\
&\quad - 2\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu}^2 + 5\hat{g}_{2du}^3\hat{g}_{2uu} - 2\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\Big]\\
&+ \frac{1}{2}\lambda_6\Big[12{h_t}^2{h_t'}^2 + \frac{45}{4}g^4 + \frac{5}{2}g^2{g'}^2 + \frac{15}{2}{g'}^4 + \hat{g}_{1dd}^2\hat{g}_{1ud}^2\\
&\quad + 2\hat{g}_{1dd}\big(2\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + \hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 4\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big) + \hat{g}_{1du}^2\hat{g}_{1uu}^2\\
&\quad + 2\hat{g}_{1du}\hat{g}_{2uu}\big(4\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2du}\big) + 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2\\
&\quad - 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 5\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big]\\
&+ \lambda_7\Big\{10g_3^2\big(3{h_t}^2 + {h_t'}^2\big) - \frac{33}{8}{h_t}^4 - \frac{3}{2}{h_t'}^2 - \frac{27}{8}{h_t'}^4 - \frac{21}{16}g^4 + \frac{29}{8}g^2{g'}^2\\
&\quad + \frac{601}{48}{g'}^4 + \frac{5}{16}\big(3{h_t}^2 + {h_t'}^2\big)\Big[9g^2 + \frac{17}{3}{g'}^2\Big]\\
&\quad + \frac{15}{32}g^2\Big[\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + 3\hat{g}_{1ud}^2 + 3\hat{g}_{1uu}^2 + 11\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 33\hat{g}_{2ud}^2 + 33\hat{g}_{2uu}^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{32}g'^2\Big[\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\big(\hat{g}_{1ud}^2+\hat{g}_{1uu}^2+\hat{g}_{2dd}^2+\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\big)\Big]\\
&-\frac{1}{32}\Big[9\hat{g}_{1dd}^4+2\hat{g}_{1dd}^2\big(10\hat{g}_{1du}^2+2\hat{g}_{1ud}^2-4\hat{g}_{1uu}^2+9\hat{g}_{2dd}^2+10\hat{g}_{2ud}^2\big)\\
&\quad-16\hat{g}_{1dd}\Big(\hat{g}_{1du}\big(2\hat{g}_{1ud}\hat{g}_{1uu}-3\hat{g}_{2dd}\hat{g}_{2du}+8\hat{g}_{2ud}\hat{g}_{2uu}\big)+2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\\
&\qquad\qquad+8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}-6\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\Big)\\
&\quad+9\hat{g}_{1du}^4-2\hat{g}_{1du}^2\big(4\hat{g}_{1ud}^2-2\hat{g}_{1uu}^2-9\hat{g}_{2du}^2-10\hat{g}_{2uu}^2\big)\\
&\quad+32\hat{g}_{1du}\big(3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}-4\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud}-\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big)\\
&\quad+11\hat{g}_{1ud}^4-4\hat{g}_{1ud}^2\hat{g}_{1uu}^2+20\hat{g}_{1ud}^2\hat{g}_{2dd}^2+22\hat{g}_{1ud}^2\hat{g}_{2ud}^2\\
&\quad-128\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}-112\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}+11\hat{g}_{1uu}^4\\
&\quad+20\hat{g}_{1uu}^2\hat{g}_{2du}^2+22\hat{g}_{1uu}^2\hat{g}_{2uu}^2+45\hat{g}_{2dd}^4+12\hat{g}_{2dd}^2\hat{g}_{2du}^2+20\hat{g}_{2dd}^2\hat{g}_{2ud}^2\\
&\quad+104\hat{g}_{2dd}^2\hat{g}_{2uu}^2-32\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+45\hat{g}_{2du}^4+104\hat{g}_{2du}^2\hat{g}_{2ud}^2\\
&\quad+20\hat{g}_{2du}^2\hat{g}_{2uu}^2+55\hat{g}_{2ud}^4+100\hat{g}_{2ud}^2\hat{g}_{2uu}^2+55\hat{g}_{2uu}^4\Big]\Bigg\}.
\end{aligned}
\tag{F.23g}
$$

The λ^0 pieces are given by

$$
\begin{aligned}
\beta_{\lambda_1}^{(2),\lambda^0}=&-32g_3^2h_t'^4+30h_t{}^2h_t'^4+30h_t'^6-\frac{8}{3}h_t'^4g'^2-\frac{133}{16}g^2g'^2+\frac{195}{16}g^6-\frac{425}{48}g'^6-\frac{605}{48}g^2g'^4\\
&-\frac{1}{4}h_t'^2\Big(9g^4-42g^2g'^2+19g'^4\Big)-\frac{3}{8}g^4\Big(\hat{g}_{1dd}^2+\hat{g}_{1du}^2+51\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)\Big)\\
&-\frac{1}{4}g^2g'^2\Big(\hat{g}_{1dd}^2+\hat{g}_{1du}^2-21\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)\Big)-\frac{1}{8}g'^4\Big(\hat{g}_{1dd}^2+\hat{g}_{1du}^2+3\big(\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)\Big)\\
&-2g^2\Big(\hat{g}_{1dd}^2\hat{g}_{2dd}^2+2\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}+\hat{g}_{1du}^2\hat{g}_{2du}^2+5\hat{g}_{2dd}^4+2\hat{g}_{2dd}^2\hat{g}_{2du}^2+5\hat{g}_{2du}^4\Big)\\
&+\frac{1}{4}\Big[5\hat{g}_{1dd}^6+\hat{g}_{1dd}^4\big(17\hat{g}_{1du}^2+5\hat{g}_{1ud}^2+2\hat{g}_{1uu}^2+17\hat{g}_{2dd}^2+3\hat{g}_{2ud}^2\big)\\
&\quad+2\hat{g}_{1dd}^3\big(4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu}+21\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2du}+2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}+6\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\quad+\hat{g}_{1dd}^2\Big(17\hat{g}_{1du}^4+\hat{g}_{1du}^2\big(7\hat{g}_{1ud}^2+7\hat{g}_{1uu}^2+19\hat{g}_{2dd}^2+19\hat{g}_{2du}^2+3\hat{g}_{2ud}^2+3\hat{g}_{2uu}^2\big)\\
&\qquad+4\hat{g}_{1du}\big(4\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu}+\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud}+\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud}\big)\\
&\qquad+10\hat{g}_{1ud}^2\hat{g}_{2dd}^2+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}+2\hat{g}_{1uu}^2\hat{g}_{2dd}^2+6\hat{g}_{1uu}^2\hat{g}_{2du}^2\\
&\qquad+11\hat{g}_{2dd}^4+21\hat{g}_{2dd}^2\hat{g}_{2du}^2+10\hat{g}_{2dd}^2\hat{g}_{2ud}^2+2\hat{g}_{2dd}^2\hat{g}_{2uu}^2\\
&\qquad-4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+10\hat{g}_{2du}^2\hat{g}_{2ud}^2\Big)\\
&\quad+2\hat{g}_{1dd}\Big(\hat{g}_{1du}^3\big(4\hat{g}_{1ud}\hat{g}_{1uu}+21\hat{g}_{2dd}\hat{g}_{2du}\big)\\
&\qquad+2\hat{g}_{1du}^2\big(\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}+\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu}+4\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad+\hat{g}_{1du}\big(3\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2du}+2\hat{g}_{1ud}\hat{g}_{1uu}(\hat{g}_{2dd}^2+\hat{g}_{2du}^2)+3\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2du}\\
&\qquad\qquad+19\hat{g}_{2dd}^3\hat{g}_{2du}-2\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu}+19\hat{g}_{2dd}\hat{g}_{2du}^3\\
&\qquad\qquad+5\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2+5\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2-2\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad-2\hat{g}_{1ud}\hat{g}_{2dd}^2\big(\hat{g}_{2dd}\hat{g}_{2ud}-3\hat{g}_{2du}\hat{g}_{2uu}\big)\\
&\qquad+2\hat{g}_{1uu}\hat{g}_{2du}\big(4\hat{g}_{2dd}^2\hat{g}_{2ud}-\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}+5\hat{g}_{2du}^2\hat{g}_{2ud}\big)\Big)\\
&\quad+5\hat{g}_{1du}^6+\hat{g}_{1du}^4\big(2\hat{g}_{1ud}^2+5\hat{g}_{1uu}^2+17\hat{g}_{2du}^2+3\hat{g}_{2uu}^2\big)\\
&\quad+4\hat{g}_{1du}^3\hat{g}_{2uu}\big(3\hat{g}_{1ud}\hat{g}_{2dd}+\hat{g}_{1uu}\hat{g}_{2du}\big)\\
&\quad+\hat{g}_{1du}^2\Big(2\hat{g}_{1ud}^2\big(3\hat{g}_{2dd}^2+\hat{g}_{2du}^2\big)+4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}+10\hat{g}_{1uu}^2\hat{g}_{2du}^2\\
&\qquad+21\hat{g}_{2dd}^2\hat{g}_{2du}^2+10\hat{g}_{2dd}^2\hat{g}_{2uu}^2-4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}+11\hat{g}_{2du}^4\\
&\qquad+2\hat{g}_{2du}^2\hat{g}_{2ud}^2+10\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\hat{g}_{1du}\Big(\hat{g}_{1ud}\hat{g}_{2dd}\big(5\hat{g}_{2dd}^2\hat{g}_{2uu} - \hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud} + 4\hat{g}_{2du}^2\hat{g}_{2uu}\big)\\
&\qquad + \hat{g}_{1uu}\hat{g}_{2du}^2\big(3\hat{g}_{2dd}\hat{g}_{2ud} - \hat{g}_{2du}\hat{g}_{2uu}\big)\Big)\\
&+ \hat{g}_{1ud}^2\hat{g}_{2dd}^2\big(5\hat{g}_{2dd}^2 + \hat{g}_{2du}^2\big) + \hat{g}_{1uu}^2\hat{g}_{2du}^2\big(\hat{g}_{2dd}^2 + 5\hat{g}_{2du}^2\big) + 47\hat{g}_{2dd}^6\\
&+ \hat{g}_{2dd}^4\left(7\hat{g}_{2du}^2 + 47\hat{g}_{2ud}^2 + 10\hat{g}_{2uu}^2\right) - 24\hat{g}_{2dd}^3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\
&+ 7\hat{g}_{2dd}^2\hat{g}_{2du}^2\Big(\hat{g}_{2du}^2 + 3\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) - 24\hat{g}_{2dd}\hat{g}_{2du}^3\hat{g}_{2ud}\hat{g}_{2uu}\\
&+ 47\hat{g}_{2du}^6 + \hat{g}_{2du}^4\left(10\hat{g}_{2ud}^2 + 47\hat{g}_{2uu}^2\right)\Bigg],
\end{aligned}
\tag{F.23h}
$$

$$
\begin{aligned}
\beta_{\lambda_2}^{(2),\lambda^0} =& -32g_3^2{h_t}^4 + 30{h_t}^6 + 30{h_t}^4{h_t'}^2 - \frac{8}{3}{h_t}^4{g'}^2 + \frac{195}{16}g^6 - \frac{133}{16}g^4{g'}^2 - \frac{605}{48}g^2{g'}^4 - \frac{425}{48}{g'}^6\\
&- \frac{1}{4}{h_t}^2\left(9g^4 - 42g^2{g'}^2 + 19{g'}^4\right) - \frac{3}{8}g^4\Big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 51\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big)\\
&- \frac{1}{4}g^2{g'}^2\Big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 - 21\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) - \frac{1}{8}{g'}^4\Big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big)\\
&- 2g^2\Big(\hat{g}_{1ud}^2\hat{g}_{2ud}^2 + 2\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^2\hat{g}_{2uu}^2 + 5\hat{g}_{2ud}^4 + 2\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 5\hat{g}_{2uu}^4\Big)\\
&+ \frac{1}{4}\Bigg[\hat{g}_{1dd}^2\Big(5\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2\big(7\hat{g}_{1uu}^2 + 10\hat{g}_{2ud}^2\big) + 6\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 2\hat{g}_{1uu}^4\\
&\qquad + 2\hat{g}_{1uu}^2\big(3\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) + \hat{g}_{2ud}^2\big(5\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big)\\
&+ 4\hat{g}_{1dd}\Big(\hat{g}_{1du}\big(2\hat{g}_{1ud}^3\hat{g}_{1uu} + \hat{g}_{1ud}^2\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1ud}\hat{g}_{1uu}(2\hat{g}_{1uu}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\\
&\qquad + \hat{g}_{1uu}^2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad + \hat{g}_{1ud}^3\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1ud}^2\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad + \hat{g}_{1ud}\Big(\hat{g}_{2du}\hat{g}_{2uu}\big(\hat{g}_{1uu}^2 + 3\hat{g}_{2ud}^2\big) - \hat{g}_{2dd}\hat{g}_{2ud}^3\Big)\\
&\qquad + \hat{g}_{1uu}\hat{g}_{2ud}\big(3\hat{g}_{1uu}^2\hat{g}_{2du} - \hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu} + 5\hat{g}_{2du}\hat{g}_{2ud}^2 + 4\hat{g}_{2du}\hat{g}_{2uu}^2\big)\Big)\\
&+ \hat{g}_{1du}^2\Big(2\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2\big(7\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + 6\hat{g}_{2uu}^2\big) + 6\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\\
&\qquad + 5\hat{g}_{1uu}^4 + 10\hat{g}_{1uu}^2\hat{g}_{2uu}^2 + \hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 5\hat{g}_{2uu}^4\Big)\\
&+ 4\hat{g}_{1du}\Big(3\hat{g}_{1ud}^3\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1ud}^2\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud}\\
&\qquad + \hat{g}_{1ud}\big(\hat{g}_{1uu}^2(4\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud})\\
&\qquad\qquad + \hat{g}_{2uu}(4\hat{g}_{2dd}\hat{g}_{2ud}^2 + 5\hat{g}_{2dd}\hat{g}_{2uu}^2 - \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu})\big)\\
&\qquad + \hat{g}_{1uu}\hat{g}_{2uu}\left(\hat{g}_{1uu}^2\hat{g}_{2du} + \hat{g}_{2uu}(3\hat{g}_{2dd}\hat{g}_{2ud} - \hat{g}_{2du}\hat{g}_{2uu})\right)\Big)\\
&+ 5\hat{g}_{1ud}^6 + \hat{g}_{1ud}^4\big(17\hat{g}_{1uu}^2 + 3\hat{g}_{2dd}^2 + 17\hat{g}_{2ud}^2\big) + 42\hat{g}_{1ud}^3\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\\
&+ \hat{g}_{1ud}^2\Big(17\hat{g}_{1uu}^4 + \hat{g}_{1uu}^2\big(3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 19(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\\
&\qquad + 10\hat{g}_{2dd}^2\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) - 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\
&\qquad + \hat{g}_{2ud}^2\big(2\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2 + 21\hat{g}_{2uu}^2\big)\Big)\\
&+ 2\hat{g}_{1ud}\hat{g}_{1uu}\Big(\hat{g}_{2ud}\hat{g}_{2uu}\big(21\hat{g}_{1uu}^2 + 5\hat{g}_{2du}^2 + 19(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\\
&\qquad + 5\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} - 2\hat{g}_{2dd}\hat{g}_{2du}\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big)\\
&+ 5\hat{g}_{1uu}^6 + \hat{g}_{1uu}^4\big(3\hat{g}_{2du}^2 + 17\hat{g}_{2uu}^2\big)\\
&+ \hat{g}_{1uu}^2\Big(\hat{g}_{2uu}^2\big(2\hat{g}_{2dd}^2 + 21\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^2\big) - 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\
&\qquad + 10\hat{g}_{2du}^2\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big)\\
&+ \hat{g}_{2dd}^2\big(47\hat{g}_{2ud}^4 + 21\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 10\hat{g}_{2uu}^4\big) - 24\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\\
&+ \hat{g}_{2du}^2\big(10\hat{g}_{2ud}^4 + 21\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 47\hat{g}_{2uu}^4\big)
\end{aligned}
$$

$$
+ 47\hat{g}_{2ud}^6 + 7\hat{g}_{2ud}^4\hat{g}_{2uu}^2 + 7\hat{g}_{2ud}^2\hat{g}_{2uu}^4 + 47\hat{g}_{2uu}^6\Big], \tag{F.23i}
$$

$$
\begin{aligned}
\beta_{\lambda_3}^{(2),\lambda^0} =& -32g_3^2{h_t}^2{h_t'}^2 + 30{h_t}^4{h_t'}^2 + 30{h_t'}^4{h_t}^2 - \frac{8}{3}{h_t}^2{h_t'}^2{g'}^2 + \frac{195g^6}{16} - \frac{425{g'}^6}{48} \\
&+ \frac{43}{16}g^4{g'}^2 + \frac{335}{48}g^2{g'}^4 - \frac{1}{8}\big({h_t}^2 + {h_t'}^2\big)\Big(9g^4 + 42g^2{g'}^2 + 19{g'}^4\Big) \\
&- \frac{3}{16}g^4\Big(\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 51\hat{g}_{2dd}^2 + 51\hat{g}_{2du}^2 + 51\hat{g}_{2ud}^2 + 51\hat{g}_{2uu}^2\Big) \\
&+ \frac{1}{8}g^2{g'}^2\Big(\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 - 21\hat{g}_{2dd}^2 - 21\hat{g}_{2du}^2 - 21\hat{g}_{2ud}^2 - 21\hat{g}_{2uu}^2\Big) \\
&- \frac{1}{16}{g'}^4\Big(\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 + 3\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2\Big) \\
&- 2g^2\Big(\hat{g}_{1dd}^2\hat{g}_{2ud}^2 - \hat{g}_{1ud}\big(\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2ud} - 2\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad - \hat{g}_{1uu}\big(\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2uu} - 2\hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\big) + \hat{g}_{1du}^2\hat{g}_{2uu}^2 \\
&\qquad + \hat{g}_{1ud}^2\hat{g}_{2dd}^2 + \hat{g}_{1uu}^2\hat{g}_{2du}^2 + 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 5\hat{g}_{2dd}^2\hat{g}_{2uu}^2 \\
&\qquad - 8\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 5\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 5\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big) \\
&+ \frac{1}{8}\Big[\hat{g}_{1dd}^4\big(10\hat{g}_{1ud}^2 + 7(\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2)\big) \\
&\quad + 2\hat{g}_{1dd}^3\big(4\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 16\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu} - 5\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{1uu} - 2\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\quad + \hat{g}_{1dd}^2\Big(10\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2\big(19\hat{g}_{1uu}^2 + 24(\hat{g}_{2dd}^2 + \hat{g}_{2ud}^2)\big) \\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2du} + 7\hat{g}_{2ud}\hat{g}_{2uu}\big) + 7\hat{g}_{1uu}^4 + 26\hat{g}_{2ud}^4 \\
&\qquad + 9\hat{g}_{1uu}^2\hat{g}_{2dd}^2 + 15\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 15\hat{g}_{1uu}^2\hat{g}_{2ud}^2 + 32\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 33\hat{g}_{2du}^2\hat{g}_{2ud}^2 \\
&\qquad + 9\hat{g}_{1uu}^2\hat{g}_{2uu}^2 + 7\hat{g}_{2dd}^2\hat{g}_{2uu}^2 - 2\hat{g}_{2ud}^2\hat{g}_{2uu}^2 - 28\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad + 2\hat{g}_{1du}\big(2\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} - 3\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} \\
&\qquad\qquad + 18\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} - 6\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1du}^2\big(19\hat{g}_{1ud}^2 + 19\hat{g}_{1uu}^2 + 18(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\Big) \\
&\quad + 2\hat{g}_{1dd}\Big(4\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{1du}^3 \\
&\qquad - \hat{g}_{1du}^2\big(6\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} - 18\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud} \\
&\qquad\qquad + 3\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} - 2\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + 2\hat{g}_{1du}\big(2\hat{g}_{1uu}\hat{g}_{1ud}^3 + \hat{g}_{1ud}^2(7\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}) \\
&\qquad\qquad + \hat{g}_{1ud}\hat{g}_{1uu}(2\hat{g}_{1uu}^2 + \hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad + 17\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2 + 17\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2 - 9\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad\qquad - 9\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}^2(7\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu})\big) \\
&\qquad - 2\hat{g}_{1ud}^3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{1ud}^2\hat{g}_{1uu}\big(6\hat{g}_{2du}\hat{g}_{2ud} - \hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\qquad - 2\hat{g}_{1ud}\big(9\hat{g}_{2ud}\hat{g}_{2dd}^3 - 11\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd}^2 + 3\hat{g}_{2dd}\hat{g}_{2ud}(\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad - 11\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + \hat{g}_{1uu}^2(3\hat{g}_{2dd}\hat{g}_{2ud} - \hat{g}_{2du}\hat{g}_{2uu})\big) \\
&\qquad - \hat{g}_{1uu}\big(3\hat{g}_{2uu}\hat{g}_{2dd}^3 - 14\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 + 3\hat{g}_{2dd}\hat{g}_{2uu}(7\hat{g}_{2du}^2 + 7\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad - \hat{g}_{1uu}^2(16\hat{g}_{2du}\hat{g}_{2ud} - 5\hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\qquad - 2\hat{g}_{2du}\hat{g}_{2ud}(16\hat{g}_{2du}^2 + 16\hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big)\Big) \\
&\quad + 14\hat{g}_{1ud}^4\hat{g}_{2dd}^2 + 14\hat{g}_{1uu}^4\hat{g}_{2du}^2 - 96\hat{g}_{2dd}^3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\quad - 2\hat{g}_{1du}^3\big(5\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} - 16\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + 2\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\quad - 96\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\big(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) \\
&\quad + \hat{g}_{1du}^4\big(7\hat{g}_{1ud}^2 + 10\hat{g}_{1uu}^2 + 14\hat{g}_{2uu}^2\big) + \hat{g}_{2dd}^4\big(94\hat{g}_{2ud}^2 + 57\hat{g}_{2uu}^2\big) \\
&\quad + \hat{g}_{2du}^4\big(57\hat{g}_{2ud}^2 + 94\hat{g}_{2uu}^2\big) + \hat{g}_{2du}^2\big(57\hat{g}_{2ud}^4 + 53\hat{g}_{2uu}^2\hat{g}_{2ud}^2 + 94\hat{g}_{2uu}^4\big)
\end{aligned}
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\begin{aligned}
&+ 4\hat{g}_{1ud}\hat{g}_{1uu}\big(17\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}^2 - 9\hat{g}_{2dd}\hat{g}_{2du}(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) + 17\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&+ \hat{g}_{2dd}^2\big(94\hat{g}_{2ud}^4 + 53\hat{g}_{2uu}^2\hat{g}_{2ud}^2 + 57\hat{g}_{2uu}^4 + 53\hat{g}_{2du}^2(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\\
&+ \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2(19\hat{g}_{1uu}^2 + 15\hat{g}_{2dd}^2 + 9\hat{g}_{2du}^2 + 9\hat{g}_{2ud}^2 + 15\hat{g}_{2uu}^2)\\
&\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}(\hat{g}_{2dd}\hat{g}_{2du} + 7\hat{g}_{2ud}\hat{g}_{2uu}) + 10\hat{g}_{1uu}^4 + 26\hat{g}_{2uu}^4 + 7\hat{g}_{2du}^2\hat{g}_{2ud}^2\\
&\qquad + 33\hat{g}_{2dd}^2\hat{g}_{2uu}^2 + 32\hat{g}_{2du}^2\hat{g}_{2uu}^2 - 2\hat{g}_{2ud}^2\hat{g}_{2uu}^2 - 28\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\
&\qquad + 24\hat{g}_{1uu}^2(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2)\big)\\
&+ \hat{g}_{1uu}^2\big(\hat{g}_{2dd}^2(7\hat{g}_{2uu}^2 - 2\hat{g}_{2du}^2) - 28\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}\\
&\qquad + \hat{g}_{2du}^2(26\hat{g}_{2du}^2 + 33\hat{g}_{2ud}^2 + 32\hat{g}_{2uu}^2)\big)\\
&+ \hat{g}_{1ud}^2\big(26\hat{g}_{2dd}^4 - \hat{g}_{2dd}^2(2\hat{g}_{2du}^2 - 32\hat{g}_{2ud}^2 - 33\hat{g}_{2uu}^2) - 28\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}\\
&\qquad + 7\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 18\hat{g}_{1uu}^2(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2)\big)\\
&- 2\hat{g}_{1du}\Big(\hat{g}_{1ud}^3\big(5\hat{g}_{2du}\hat{g}_{2ud} - 16\hat{g}_{2dd}\hat{g}_{2uu}\big) + \hat{g}_{1ud}^2\hat{g}_{1uu}\big(6\hat{g}_{2du}\hat{g}_{2uu} - 2\hat{g}_{2dd}\hat{g}_{2ud}\big)\\
&\qquad - \hat{g}_{1ud}\big(32\hat{g}_{2uu}\hat{g}_{2dd}^3 - 21\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2\\
&\qquad\qquad + 2\hat{g}_{2dd}\hat{g}_{2uu}(7\hat{g}_{2du}^2 + 7\hat{g}_{2ud}^2 + 16\hat{g}_{2uu}^2)\\
&\qquad\qquad - 3\hat{g}_{1uu}^2(\hat{g}_{2du}\hat{g}_{2ud} - 6\hat{g}_{2dd}\hat{g}_{2uu})\\
&\qquad\qquad - 3\hat{g}_{2du}\hat{g}_{2ud}(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big)\\
&\qquad + 2\hat{g}_{1uu}\big(3\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd}^2 - 11\hat{g}_{2dd}\hat{g}_{2ud}(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2)\\
&\qquad\qquad + \hat{g}_{2du}\hat{g}_{2uu}(\hat{g}_{1uu}^2 + 9\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 9\hat{g}_{2uu}^2)\big)\Big)\Big],
\end{aligned}
\tag{F.23j}
$$

$$
\begin{aligned}
\beta_{\lambda_4}^{(2),\lambda^0} =& - 32g_3^2{h_t}^2{h_t'}^2 + 30{h_t}^4{h_t'}^2 + 30{h_t}^2{h_t'}^4 - \frac{8}{3}{h_t}^2{h_t'}^2{g'}^2 + \frac{21}{2}{h_t}^2g^2{g'}^2\\
&+ \frac{21}{2}{h_t'}^2g^2{g'}^2 - 11g^4{g'}^2 - \frac{235}{12}g^2{g'}^4\\
&+ \frac{1}{4}g^2{g'}^2\Big(\hat{g}_{1dd}^2 + \hat{g}_{1du}^2 + \hat{g}_{1ud}^2 + \hat{g}_{1uu}^2 - 21\hat{g}_{2dd}^2 - 21\hat{g}_{2du}^2 - 21\hat{g}_{2ud}^2 - 21\hat{g}_{2uu}^2\Big)\\
&+ g^2\Big(\hat{g}_{1dd}^2\hat{g}_{2ud}^2 - \hat{g}_{1ud}\big(4\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2ud} - 2\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} + 4\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad - 2\hat{g}_{1uu}\big(2\hat{g}_{1dd}\hat{g}_{2dd}\hat{g}_{2uu} - \hat{g}_{1dd}\hat{g}_{2du}\hat{g}_{2ud} + 2\hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2uu}\big) + \hat{g}_{1du}^2\hat{g}_{2uu}^2\\
&\qquad + \hat{g}_{1ud}^2\hat{g}_{2dd}^2 + \hat{g}_{1uu}^2\hat{g}_{2du}^2 - 10\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 8\hat{g}_{2dd}^2\hat{g}_{2uu}^2\\
&\qquad - 20\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + 8\hat{g}_{2du}^2\hat{g}_{2ud}^2 - 10\hat{g}_{2du}^2\hat{g}_{2uu}^2\Big)\\
&+ \frac{1}{4}\Big[\hat{g}_{1dd}^4\big(5\hat{g}_{1ud}^2 - 2\hat{g}_{2ud}^2\big)\\
&\qquad + \hat{g}_{1dd}^3\big(11\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} - 5\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{1uu} + 10\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{1uu} + 16\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\big)\\
&\qquad + \hat{g}_{1dd}^2\Big(5\hat{g}_{1ud}^4 + 3\hat{g}_{1ud}^2\big(2\hat{g}_{1uu}^2 + \hat{g}_{2dd}^2 + \hat{g}_{2ud}^2\big) + 8\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad\qquad - 8\hat{g}_{2ud}^4 - 3\hat{g}_{1uu}^2\hat{g}_{2du}^2 - 3\hat{g}_{1uu}^2\hat{g}_{2ud}^2 - 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 - 9\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 2\hat{g}_{2dd}^2\hat{g}_{2uu}^2\\
&\qquad\qquad + 5\hat{g}_{2ud}^2\hat{g}_{2uu}^2 + 19\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\\
&\qquad\qquad + \hat{g}_{1du}\big(8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} + 18\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud}\\
&\qquad\qquad\qquad + 3\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + 12\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big)\\
&\qquad\qquad + \hat{g}_{1du}^2\big(6\hat{g}_{1ud}^2 + 6\hat{g}_{1uu}^2 - 3(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\Big)\\
&\qquad + \hat{g}_{1dd}\Big(11\hat{g}_{1ud}\hat{g}_{1uu}\hat{g}_{1du}^3\\
&\qquad\qquad + \hat{g}_{1du}^2\big(12\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\\
&\qquad\qquad\qquad + 18\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 8\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu}\big)\\
&\qquad\qquad + \hat{g}_{1du}\Big(11\hat{g}_{1uu}\hat{g}_{1ud}^3 + 8(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu})\hat{g}_{1ud}^2\\
&\qquad\qquad\qquad + \hat{g}_{1ud}\hat{g}_{1uu}\big(11\hat{g}_{1uu}^2 + 5(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\\
&\qquad\qquad\qquad + 8\big(\hat{g}_{1uu}^2(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}) + 3\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} + 3\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}
\end{aligned}
$$

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\begin{aligned}
&\qquad\qquad\qquad - \hat{g}_{2dd}\hat{g}_{2du}\big(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\big)\Big) \\
&\qquad + 16\hat{g}_{1ud}^3\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{1ud}^2\hat{g}_{1uu}\big(\hat{g}_{2du}\hat{g}_{2ud} + 6\hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\qquad + 4\hat{g}_{1ud}\big(6\hat{g}_{2ud}\hat{g}_{2dd}^3 - 2\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd}^2 + 3\hat{g}_{2dd}\hat{g}_{2ud}(\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad - 2\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + \hat{g}_{1uu}^2(3\hat{g}_{2dd}\hat{g}_{2ud} + 2\hat{g}_{2du}\hat{g}_{2uu})\big) \\
&\qquad + \hat{g}_{1uu}\big(6\hat{g}_{2uu}\hat{g}_{2dd}^3 + 5\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 + 6\hat{g}_{2dd}\hat{g}_{2uu}(5\hat{g}_{2du}^2 + 5\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad - 5\hat{g}_{1uu}^2(\hat{g}_{2du}\hat{g}_{2ud} - 2\hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\qquad - \hat{g}_{2du}\hat{g}_{2ud}(19\hat{g}_{2du}^2 + 19\hat{g}_{2ud}^2 - 5\hat{g}_{2uu}^2))\big)\Big) \\
&- 2\hat{g}_{1ud}^4\hat{g}_{2dd}^2 - 2\hat{g}_{1uu}^4\hat{g}_{2du}^2 + 45\hat{g}_{2dd}^3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&+ \hat{g}_{1du}^3\big(10\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} - 5\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + 16\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&- \hat{g}_{2du}^4\big(18\hat{g}_{2ud}^2 - 47\hat{g}_{2uu}^2\big) + \hat{g}_{2dd}^4\big(47\hat{g}_{2ud}^2 - 18\hat{g}_{2uu}^2\big) + \hat{g}_{1du}^4\big(5\hat{g}_{1uu}^2 - 2\hat{g}_{2uu}^2\big) \\
&+ 45\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\big(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) \\
&- \hat{g}_{2du}^2\big(18\hat{g}_{2ud}^4 + 20\hat{g}_{2uu}^2\hat{g}_{2ud}^2 - 47\hat{g}_{2uu}^4\big) \\
&- 8\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}^2 - 3\hat{g}_{2du}(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\hat{g}_{2dd} + \hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&+ \hat{g}_{2dd}^2\big(47\hat{g}_{2ud}^4 - 20\hat{g}_{2uu}^2\hat{g}_{2ud}^2 - 18\hat{g}_{2uu}^4 - 20\hat{g}_{2du}^2(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big) \\
&+ \hat{g}_{1uu}^2\big(\hat{g}_{2dd}^2(5\hat{g}_{2du}^2 + 2\hat{g}_{2uu}^2) + 19\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd} \\
&\qquad - \hat{g}_{2du}^2(8\hat{g}_{2du}^2 + 9\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2)\big) \\
&- \hat{g}_{1ud}^2\big(8\hat{g}_{2dd}^4 - \hat{g}_{2dd}^2(5\hat{g}_{2du}^2 - 5\hat{g}_{2ud}^2 - 9\hat{g}_{2uu}^2) - 19\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd} \\
&\qquad - 2\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 3\hat{g}_{1uu}^2(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2)\big) \\
&+ \hat{g}_{1du}^2\Big(5\hat{g}_{1uu}^4 + 3\hat{g}_{1uu}^2\big(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2\big) + 8\hat{g}_{1ud}\hat{g}_{1uu}\big(\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad - 8\hat{g}_{2uu}^4 + 2\hat{g}_{2du}^2\hat{g}_{2ud}^2 - 9\hat{g}_{2dd}^2\hat{g}_{2uu}^2 - 5\hat{g}_{2du}^2\hat{g}_{2uu}^2 + 5\hat{g}_{2ud}^2\hat{g}_{2uu}^2 \\
&\qquad + 19\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1ud}^2\big(6\hat{g}_{1uu}^2 - 3(\hat{g}_{2dd}^2 + \hat{g}_{2uu}^2)\big)\Big) \\
&+ \hat{g}_{1du}\Big(5\hat{g}_{1ud}^3\big(2\hat{g}_{2du}\hat{g}_{2ud} - \hat{g}_{2dd}\hat{g}_{2uu}\big) + 4\hat{g}_{1ud}^2\hat{g}_{1uu}\big(2\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad - \hat{g}_{1ud}\big(19\hat{g}_{2uu}\hat{g}_{2dd}^3 - 30\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 - \hat{g}_{2dd}\hat{g}_{2uu}(5\hat{g}_{2du}^2 + 5\hat{g}_{2ud}^2 - 19\hat{g}_{2uu}^2) \\
&\qquad\qquad - 3\hat{g}_{1uu}^2(6\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\qquad - 6\hat{g}_{2du}\hat{g}_{2ud}(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2)\big) \\
&\qquad + 4\hat{g}_{1uu}\big(3\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd}^2 - 2\hat{g}_{2dd}\hat{g}_{2ud}(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2) \\
&\qquad\qquad + \hat{g}_{2du}\hat{g}_{2uu}(4\hat{g}_{1uu}^2 + 6\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 6\hat{g}_{2uu}^2))\big)\Big)\Big],
\end{aligned} \tag{F.23k}
$$

$$
\begin{aligned}
\beta_{\lambda_5}^{(2),\lambda^0} =& - 32g_3^2{h_t}^2{h_t'}^2 + 30{h_t}^4{h_t'}^2 + 30{h_t}^2{h_t'}^4 - \frac{8}{3}{h_t}^2{h_t'}^2{g'}^2 \\
& - 2g^2\Big(\hat{g}_{2uu}\big(\hat{g}_{1dd}\hat{g}_{1du}\hat{g}_{2ud} + \hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2du} + 5\hat{g}_{2du}^2\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1ud}\hat{g}_{2dd}\big(\hat{g}_{1dd}\hat{g}_{2ud} + \hat{g}_{1uu}\hat{g}_{2du}\big) + 5\hat{g}_{2dd}^2\hat{g}_{2ud}^2 + 2\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
& + \frac{1}{4}\Big[5\hat{g}_{1dd}^4\hat{g}_{1ud}^2 + \hat{g}_{1dd}^3\big(11\hat{g}_{1du}\hat{g}_{1ud}\hat{g}_{1uu} + 5\hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu} + 12\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\qquad + \hat{g}_{1dd}^2\Big(6\hat{g}_{1du}^2\big(\hat{g}_{1ud}^2 + \hat{g}_{1uu}^2\big) \\
&\qquad\qquad + \hat{g}_{1du}\big(8\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2uu} + 8\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2ud} + 8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2ud} + 6\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad\qquad + 5\hat{g}_{1ud}^4 + \hat{g}_{1ud}^2\big(6\hat{g}_{1uu}^2 + 5(\hat{g}_{2dd}^2 + \hat{g}_{2ud}^2)\big) \\
&\qquad\qquad + \hat{g}_{1ud}\hat{g}_{1uu}(13\hat{g}_{2dd}\hat{g}_{2du} + 8\hat{g}_{2ud}\hat{g}_{2uu}) + \hat{g}_{2dd}\hat{g}_{2ud}(3\hat{g}_{2dd}\hat{g}_{2ud} + 7\hat{g}_{2du}\hat{g}_{2uu})\Big) \\
&\qquad + \hat{g}_{1dd}\Big(\hat{g}_{1du}^3(11\hat{g}_{1ud}\hat{g}_{1uu} + 5\hat{g}_{2ud}\hat{g}_{2uu}) \\
&\qquad\qquad + \hat{g}_{1du}^2\big(6\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{1ud}\hat{g}_{2du}\hat{g}_{2uu} + 8\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + 8\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big) \\
&\qquad\qquad + \hat{g}_{1du}\big(11\hat{g}_{1ud}^3\hat{g}_{1uu} + \hat{g}_{1ud}^2(8\hat{g}_{2dd}\hat{g}_{2du} + 13\hat{g}_{2ud}\hat{g}_{2uu})
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \hat{g}_{1ud}\hat{g}_{1uu}(11\hat{g}_{1uu}^2 + 5(\hat{g}_{2dd}^2 + \hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)) \\
&\quad + \hat{g}_{1uu}^2(8\hat{g}_{2dd}\hat{g}_{2du} + 13\hat{g}_{2ud}\hat{g}_{2uu}) + 11\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} \\
&\quad + 8\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2 + 8\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2 + 11\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu} \\
&\quad + 3\hat{g}_{2ud}^3\hat{g}_{2uu} + 3\hat{g}_{2ud}\hat{g}_{2uu}^3) \\
&+ 12\hat{g}_{1ud}^3\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{1ud}^2\hat{g}_{1uu}(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}) \\
&+ \hat{g}_{1ud}\big(\hat{g}_{1uu}^2(6\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{2du}\hat{g}_{2uu}) + 8\hat{g}_{2dd}^3\hat{g}_{2ud} + 8\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} \\
&\quad + \hat{g}_{2dd}(6\hat{g}_{2du}^2\hat{g}_{2ud} + 8\hat{g}_{2ud}^3 + 6\hat{g}_{2ud}\hat{g}_{2uu}^2) + 8\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu}\big) \\
&+ 8\hat{g}_{1uu}\big(\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2) + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}^2)\Big) \\
&+ 5\hat{g}_{1du}^4\hat{g}_{1uu}^2 + 12\hat{g}_{1du}^3\hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2uu} \\
&+ \hat{g}_{1du}^2\big(6\hat{g}_{1ud}^2\hat{g}_{1uu}^2 + \hat{g}_{1ud}\hat{g}_{1uu}(13\hat{g}_{2dd}\hat{g}_{2du} + 8\hat{g}_{2ud}\hat{g}_{2uu}) + 5\hat{g}_{1uu}^4 \\
&\quad + 5\hat{g}_{1uu}^2(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2) + \hat{g}_{2du}\hat{g}_{2uu}(7\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu})\big) \\
&+ 2\hat{g}_{1du}\Big(\hat{g}_{1ud}^2\hat{g}_{1uu}(4\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}) \\
&\quad + 4\hat{g}_{1ud}\big(\hat{g}_{1uu}^2(\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{2du}\hat{g}_{2ud}) + \hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2ud} \\
&\qquad + \hat{g}_{2dd}\hat{g}_{2uu}(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2) + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}^2\big) \\
&\quad + \hat{g}_{1uu}\big(\hat{g}_{2du}\hat{g}_{2uu}(6\hat{g}_{1uu}^2 + 4\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 4\hat{g}_{2uu}^2) \\
&\qquad + 3\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} + 4\hat{g}_{2dd}\hat{g}_{2ud}(\hat{g}_{2du}^2 + \hat{g}_{2uu}^2))\Big) \\
&+ 5\hat{g}_{1ud}^3\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}\big(3\hat{g}_{2dd}\hat{g}_{2ud} + 7\hat{g}_{2du}\hat{g}_{2uu}\big) \\
&+ \hat{g}_{1ud}\hat{g}_{1uu}\Big(5\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2du} + 3\hat{g}_{2dd}^3\hat{g}_{2du} + 8\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} \\
&\quad + \hat{g}_{2dd}\hat{g}_{2du}\big(3\hat{g}_{2du}^2 + 11(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big) + 8\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\Big) \\
&+ \hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu}\big(7\hat{g}_{2dd}\hat{g}_{2ud} + 3\hat{g}_{2du}\hat{g}_{2uu}\big) + 47\hat{g}_{2dd}^4\hat{g}_{2ud}^2 + 9\hat{g}_{2dd}^3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&- \hat{g}_{2dd}^2\big(2\hat{g}_{2du}^2(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) - 47\hat{g}_{2ud}^4 + 2\hat{g}_{2ud}^2\hat{g}_{2uu}^2\big) \\
&+ 9\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\big(\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big) + 47\hat{g}_{2du}^4\hat{g}_{2uu}^2 \\
&- \hat{g}_{2du}^2\big(2\hat{g}_{2ud}^2\hat{g}_{2uu}^2 - 47\hat{g}_{2uu}^4\big)\Big],
\end{aligned}
\tag{F.23l}
$$

$$
\begin{aligned}
\beta_{\lambda_6}^{(2),\lambda^0} =& - 32g_3^2h_th_t'^3 + 30h_t^3h_t'^3 + 30h_th_t'^5 - \frac{8}{3}h_th_t'^3g'^2 \\
&- \frac{1}{8}h_th_t'\Big(9g^4 - 42h_th_t'g^2g'^2 + 19h_th_t'g'^4\Big) \\
&- \frac{3}{16}g^4\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 51\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{8}g^2g'^2\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} - 21\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big) \\
&- \frac{1}{16}g'^4\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big) \\
&- g^2\Big(\hat{g}_{1dd}^2\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{1dd}\big(\hat{g}_{1du}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1du}\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{1ud}\hat{g}_{2dd}^2 + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}\big) \\
&\quad + \hat{g}_{1du}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1du}\hat{g}_{2du}\big(\hat{g}_{1ud}\hat{g}_{2dd} + \hat{g}_{1uu}\hat{g}_{2du}\big) \\
&\quad + 2\big(5\hat{g}_{2dd}^3\hat{g}_{2ud} + \hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 5\hat{g}_{2du}^3\hat{g}_{2uu}\big)\Big) \\
&+ \frac{1}{8}\Big[10\hat{g}_{1ud}\hat{g}_{1dd}^5 + \hat{g}_{1dd}^4\big(11\hat{g}_{1du}\hat{g}_{1uu} + 12\hat{g}_{2dd}\hat{g}_{2ud}\big) \\
&\quad + \hat{g}_{1dd}^3\big(10\hat{g}_{1ud}^3 + 23\hat{g}_{1du}^2\hat{g}_{1ud} + 8\hat{g}_{1uu}^2\hat{g}_{1ud} + 22\hat{g}_{2dd}^2\hat{g}_{1ud} + 8\hat{g}_{2ud}^2\hat{g}_{1ud} \\
&\qquad + 13\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du} + 6\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu} + 13\hat{g}_{1du}(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu})\big) \\
&\quad + \hat{g}_{1dd}^2\Big(23\hat{g}_{1uu}\hat{g}_{1du}^3 + 14\hat{g}_{1du}^2\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big) \\
&\qquad + \hat{g}_{1du}\big(19\hat{g}_{1uu}\hat{g}_{1ud}^2 + 5\hat{g}_{1ud}(9\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{2ud}\hat{g}_{2uu})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \hat{g}_{1uu}(7\hat{g}_{1uu}^2 + 13\hat{g}_{2dd}^2 + 11\hat{g}_{2du}^2 + 5\hat{g}_{2ud}^2 + 3\hat{g}_{2uu}^2))\\
&\qquad + 10\hat{g}_{2dd}\hat{g}_{2ud}^3 + 14\hat{g}_{2dd}^3\hat{g}_{2ud} + 13\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud} + 16\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud}\\
&\qquad + 4\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2ud} + 8\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 6\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} + 15\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}\\
&\qquad + 2\hat{g}_{1ud}\hat{g}_{1uu}(11\hat{g}_{2du}\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2uu})\Big)\\
&+ \hat{g}_{1dd}\Big(11\hat{g}_{1ud}\hat{g}_{1du}^4 + 13\hat{g}_{1du}^3\big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}\big)\\
&\qquad + \hat{g}_{1du}^2\big(7\hat{g}_{1ud}^3 + \hat{g}_{1ud}(19\hat{g}_{1uu}^2 + 11\hat{g}_{2dd}^2 + 13\hat{g}_{2du}^2 + 3\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2)\\
&\qquad\qquad + 5\hat{g}_{1uu}(9\hat{g}_{2dd}\hat{g}_{2du} + 2\hat{g}_{2ud}\hat{g}_{2uu})\big)\\
&\qquad + \hat{g}_{1du}\big(11\hat{g}_{2uu}\hat{g}_{2dd}^3 + 43\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 + 5\hat{g}_{2uu}^3\hat{g}_{2dd} + 43\hat{g}_{2du}^2\hat{g}_{2uu}\hat{g}_{2dd}\\
&\qquad\qquad + \hat{g}_{2ud}^2\hat{g}_{2uu}\hat{g}_{2dd} + 5\hat{g}_{2du}\hat{g}_{2ud}^3 + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}^2 + 11\hat{g}_{2du}^3\hat{g}_{2ud}\\
&\qquad\qquad + \hat{g}_{1uu}^2(19\hat{g}_{2du}\hat{g}_{2ud} + 7\hat{g}_{2dd}\hat{g}_{2uu}) + \hat{g}_{1ud}^2(7\hat{g}_{2du}\hat{g}_{2ud} + 19\hat{g}_{2dd}\hat{g}_{2uu})\\
&\qquad\qquad + 8\hat{g}_{1ud}\hat{g}_{1uu}(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu})\big)\\
&\qquad + 10\hat{g}_{1ud}^3\hat{g}_{2dd}^2 + 7\hat{g}_{1ud}^2\hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2du}\\
&\qquad + \hat{g}_{1uu}\big(11\hat{g}_{2du}\hat{g}_{2dd}^3 + 6\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}^2 + 3\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2dd}\\
&\qquad\qquad + \hat{g}_{2dd}\hat{g}_{2du}(11\hat{g}_{2du}^2 + 21\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2) + 26\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad + 2\hat{g}_{1ud}\big(4\hat{g}_{2dd}^4 + \hat{g}_{2dd}^2(7\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2 + 4\hat{g}_{2uu}^2) + 4\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}\\
&\qquad\qquad + 5\hat{g}_{2du}^2\hat{g}_{2ud}^2 + 2\hat{g}_{1uu}^2(\hat{g}_{2dd}^2 + 2\hat{g}_{2du}^2))\big)\Big)\\
&+ 10\hat{g}_{1du}^5\hat{g}_{1uu} + 94\hat{g}_{2dd}^5\hat{g}_{2ud} + 94\hat{g}_{2du}^5\hat{g}_{2uu} + 12\hat{g}_{1du}^4\hat{g}_{2du}\hat{g}_{2uu} + 9\hat{g}_{2dd}^4\hat{g}_{2du}\hat{g}_{2uu}\\
&+ \hat{g}_{1ud}^2\hat{g}_{2dd}\big(8\hat{g}_{2ud}\hat{g}_{2dd}^2 + 7\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd} + \hat{g}_{2du}^2\hat{g}_{2ud}\big)\\
&+ \hat{g}_{1uu}^2\hat{g}_{2du}\big(\hat{g}_{2uu}\hat{g}_{2dd}^2 + 7\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd} + 8\hat{g}_{2du}^2\hat{g}_{2uu}\big)\\
&+ 2\hat{g}_{1ud}\hat{g}_{1uu}\big(5\hat{g}_{2uu}\hat{g}_{2dd}^3 + 3\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2\hat{g}_{2uu}\hat{g}_{2dd} + 5\hat{g}_{2du}^3\hat{g}_{2ud}\big)\\
&+ 3\hat{g}_{2dd}\hat{g}_{2du}^2\hat{g}_{2ud}\big(3\hat{g}_{2du}^2 + 7\hat{g}_{2ud}^2 - 5\hat{g}_{2uu}^2\big) + \hat{g}_{2dd}^3\hat{g}_{2ud}\big(5\hat{g}_{2du}^2 + 94\hat{g}_{2ud}^2 + 8\hat{g}_{2uu}^2\big)\\
&+ \hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu}\big(5\hat{g}_{2du}^2 - 15\hat{g}_{2ud}^2 + 21\hat{g}_{2uu}^2\big)\\
&+ \hat{g}_{1du}^3\big(10\hat{g}_{1uu}^3 + 8\hat{g}_{1ud}^2\hat{g}_{1uu} + 22\hat{g}_{2du}^2\hat{g}_{1uu} + 8\hat{g}_{2uu}^2\hat{g}_{1uu}\\
&\qquad + 13\hat{g}_{1ud}\hat{g}_{2dd}\hat{g}_{2du} + 6\hat{g}_{1ud}\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&+ 2\hat{g}_{2du}^3\big(47\hat{g}_{2uu}^3 + 4\hat{g}_{2ud}^2\hat{g}_{2uu}\big)\\
&+ \hat{g}_{1du}^2\big(\hat{g}_{1ud}^2(6\hat{g}_{2dd}\hat{g}_{2ud} + 4\hat{g}_{2du}\hat{g}_{2uu}) + 2\hat{g}_{1ud}\hat{g}_{1uu}(5\hat{g}_{2du}\hat{g}_{2ud} + 11\hat{g}_{2dd}\hat{g}_{2uu})\\
&\qquad + 13\hat{g}_{2dd}^2\hat{g}_{2du}\hat{g}_{2uu} + 2\hat{g}_{2du}\hat{g}_{2uu}(8\hat{g}_{1uu}^2 + 7\hat{g}_{2du}^2 + 5\hat{g}_{2uu}^2)\\
&\qquad + \hat{g}_{2dd}\hat{g}_{2ud}(15\hat{g}_{2du}^2 + 8\hat{g}_{2uu}^2)\big)\\
&+ \hat{g}_{1du}\Big(3\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{1ud}^3 + 4\hat{g}_{1ud}^2\hat{g}_{1uu}\big(2\hat{g}_{2dd}^2 + \hat{g}_{2du}^2\big)\\
&\qquad + \hat{g}_{1ud}\big(11\hat{g}_{2du}\hat{g}_{2dd}^3 + 26\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}^2 + 7\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2dd}\\
&\qquad\qquad + \hat{g}_{2dd}\hat{g}_{2du}(11\hat{g}_{2du}^2 + \hat{g}_{2ud}^2 + 21\hat{g}_{2uu}^2) + 6\hat{g}_{2du}^2\hat{g}_{2ud}\hat{g}_{2uu}\big)\\
&\qquad + 2\hat{g}_{1uu}\big(\hat{g}_{2dd}^2(7\hat{g}_{2du}^2 + 5\hat{g}_{2uu}^2) + 4\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd}\\
&\qquad\qquad + 5\hat{g}_{1uu}^2\hat{g}_{2du}^2 + 2\hat{g}_{2du}^2(2\hat{g}_{2du}^2 + 2\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2))\big)\Big)\Big],
\end{aligned}
\tag{F.23m}
$$

$$
\begin{aligned}
\beta_{\lambda_7}^{(2),\lambda^0} =& -32g_3^2{h_t}^3h_t' + 30{h_t}^5h_t' + 30{h_t}^3{h_t'}^3 - \frac{8}{3}{h_t}^3h_t'{g'}^2 - \frac{1}{8}h_th_t'\Big(9g^4 - 42g^2{g'}^2 + 19{g'}^4\Big)\\
&- \frac{3}{16}g^4\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 51\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big)\\
&- \frac{1}{8}g^2{g'}^2\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} - 21\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big)\\
&- \frac{1}{16}{g'}^4\Big(\hat{g}_{1dd}\hat{g}_{1ud} + \hat{g}_{1du}\hat{g}_{1uu} + 3\big(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu}\big)\Big)\\
&- g^2\Big(\hat{g}_{1ud}\big(\hat{g}_{1dd}\hat{g}_{2ud}^2 + \hat{g}_{1du}\hat{g}_{2ud}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2dd}\hat{g}_{2uu} + \hat{g}_{1uu}\hat{g}_{2du}\hat{g}_{2ud}\big)\\
&\qquad + \hat{g}_{2uu}\big(\hat{g}_{1dd}\hat{g}_{1uu}\hat{g}_{2ud} + \hat{g}_{1du}\hat{g}_{1uu}\hat{g}_{2uu} + \hat{g}_{1uu}^2\hat{g}_{2du} + 2\hat{g}_{2du}\hat{g}_{2ud}^2 + 10\hat{g}_{2du}\hat{g}_{2uu}^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} + 2\hat{g}_{2dd}\hat{g}_{2ud}\big(5\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2\big)\Big) \\
&+ \frac{1}{8}\Big[12\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{1ud}^4 + 13\hat{g}_{1ud}^3\hat{g}_{1uu}\big(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\quad + \hat{g}_{1ud}^2\big(10\hat{g}_{2ud}\hat{g}_{2dd}^3 + 8\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2dd}^2 + 14\hat{g}_{2ud}^3\hat{g}_{2dd} + 13\hat{g}_{2ud}\hat{g}_{2uu}^2\hat{g}_{2dd} \\
&\qquad\quad + 15\hat{g}_{2du}\hat{g}_{2ud}^2\hat{g}_{2uu} + 14\hat{g}_{1uu}^2(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu})\big) \\
&\quad + \hat{g}_{1ud}\hat{g}_{1uu}\big(5\hat{g}_{2uu}\hat{g}_{2dd}^3 + \hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2dd}^2 + \hat{g}_{2dd}\hat{g}_{2uu}(\hat{g}_{2du}^2 + 43\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^2) \\
&\qquad\qquad + 13\hat{g}_{1uu}^2(\hat{g}_{2du}\hat{g}_{2ud} + \hat{g}_{2dd}\hat{g}_{2uu}) + \hat{g}_{2du}\hat{g}_{2ud}(5\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2 + 43\hat{g}_{2uu}^2)\big) \\
&\quad + 12\hat{g}_{1uu}^4\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1dd}^3\big(10\hat{g}_{1ud}^3 + 7\hat{g}_{1uu}^2\hat{g}_{1ud} + 10\hat{g}_{2ud}^2\hat{g}_{1ud} + 3\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\quad + \hat{g}_{2dd}^3\big(94\hat{g}_{2ud}^3 + 21\hat{g}_{2uu}^2\hat{g}_{2ud}\big) + \hat{g}_{2dd}^2\hat{g}_{2du}\big(8\hat{g}_{2uu}^3 - 15\hat{g}_{2ud}^2\hat{g}_{2uu}\big) \\
&\quad + \hat{g}_{2du}^3\big(94\hat{g}_{2uu}^3 + 21\hat{g}_{2ud}^2\hat{g}_{2uu}\big) + \hat{g}_{2du}\big(94\hat{g}_{2uu}^5 + 5\hat{g}_{2ud}^2\hat{g}_{2uu}^3 + 9\hat{g}_{2ud}^4\hat{g}_{2uu}\big) \\
&\quad + \hat{g}_{2dd}\hat{g}_{2ud}\big(94\hat{g}_{2ud}^4 + 5\hat{g}_{2uu}^2\hat{g}_{2ud}^2 + 9\hat{g}_{2uu}^4 + \hat{g}_{2du}^2(8\hat{g}_{2ud}^2 - 15\hat{g}_{2uu}^2)\big) \\
&\quad + \hat{g}_{1du}^3\big(7\hat{g}_{1uu}\hat{g}_{1ud}^2 + 3\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud} + 10\hat{g}_{1uu}(\hat{g}_{1uu}^2 + \hat{g}_{2uu}^2)\big) \\
&\quad + \hat{g}_{1uu}^2\big(\hat{g}_{2du}\hat{g}_{2uu}(10\hat{g}_{2du}^2 + 13\hat{g}_{2ud}^2 + 14\hat{g}_{2uu}^2) + \hat{g}_{2dd}\hat{g}_{2ud}(8\hat{g}_{2du}^2 + 15\hat{g}_{2uu}^2)\big) \\
&\quad + \hat{g}_{1du}^2\big(\hat{g}_{1ud}^2(4\hat{g}_{2dd}\hat{g}_{2ud} + 6\hat{g}_{2du}\hat{g}_{2uu}) + \hat{g}_{1ud}\hat{g}_{1uu}(7\hat{g}_{2du}\hat{g}_{2ud} + 19\hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\quad + \hat{g}_{2uu}(16\hat{g}_{2du}\hat{g}_{1uu}^2 + \hat{g}_{2du}\hat{g}_{2ud}^2 + 8\hat{g}_{2du}\hat{g}_{2uu}^2 + 7\hat{g}_{2dd}\hat{g}_{2ud}\hat{g}_{2uu})\big) \\
&\quad + \hat{g}_{1dd}^2\Big(8\hat{g}_{2dd}\hat{g}_{2ud}^3 + 7\hat{g}_{2du}\hat{g}_{2uu}\hat{g}_{2ud}^2 + \hat{g}_{2dd}\hat{g}_{2uu}^2\hat{g}_{2ud} + 16\hat{g}_{1ud}^2\hat{g}_{2dd}\hat{g}_{2ud} \\
&\qquad\quad + 6\hat{g}_{1uu}^2\hat{g}_{2dd}\hat{g}_{2ud} + 4\hat{g}_{1uu}^2\hat{g}_{2du}\hat{g}_{2uu} + \hat{g}_{1ud}\hat{g}_{1uu}\big(19\hat{g}_{2du}\hat{g}_{2ud} + 7\hat{g}_{2dd}\hat{g}_{2uu}\big) \\
&\qquad\quad + \hat{g}_{1du}\big(19\hat{g}_{1uu}\hat{g}_{1ud}^2 + 7\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{1ud} + 4\hat{g}_{1uu}(2\hat{g}_{1uu}^2 + 2\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2)\big)\Big) \\
&\quad + \hat{g}_{1du}\Big(11\hat{g}_{1uu}\hat{g}_{1ud}^4 + \hat{g}_{1ud}^3\big(6\hat{g}_{2dd}\hat{g}_{2du} + 13\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad\quad + \hat{g}_{1ud}^2\hat{g}_{1uu}\big(23\hat{g}_{1uu}^2 + 5\hat{g}_{2dd}^2 + 3\hat{g}_{2du}^2 + 13\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^2\big) \\
&\qquad\quad + \hat{g}_{1ud}\big(5\hat{g}_{1uu}^2(2\hat{g}_{2dd}\hat{g}_{2du} + 9\hat{g}_{2ud}\hat{g}_{2uu}) + 21\hat{g}_{2dd}^2\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad\qquad + \hat{g}_{2dd}\hat{g}_{2du}(6\hat{g}_{2ud}^2 + 26\hat{g}_{2uu}^2) + \hat{g}_{2ud}\hat{g}_{2uu}(\hat{g}_{2du}^2 + 11(\hat{g}_{2ud}^2 + \hat{g}_{2uu}^2))\big) \\
&\qquad\quad + 2\hat{g}_{1uu}\big(5\hat{g}_{1uu}^4 + \hat{g}_{1uu}^2(4\hat{g}_{2du}^2 + 11\hat{g}_{2uu}^2) + 4\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu} \\
&\qquad\qquad\quad + \hat{g}_{2du}^2(5\hat{g}_{2ud}^2 + 2\hat{g}_{2uu}^2) + \hat{g}_{2uu}^2(4\hat{g}_{2dd}^2 + 7\hat{g}_{2ud}^2 + 4\hat{g}_{2uu}^2)\big)\Big) \\
&\quad + \hat{g}_{1dd}\Big(10\hat{g}_{1ud}^5 + \hat{g}_{1ud}^3\big(23\hat{g}_{1uu}^2 + 8\hat{g}_{2dd}^2 + 22\hat{g}_{2ud}^2\big) \\
&\qquad\quad + 5\hat{g}_{1ud}^2\hat{g}_{1uu}\big(2\hat{g}_{2dd}\hat{g}_{2du} + 9\hat{g}_{2ud}\hat{g}_{2uu}\big) \\
&\qquad\quad + \hat{g}_{1ud}\big(11\hat{g}_{1uu}^4 + \hat{g}_{1uu}^2(3\hat{g}_{2dd}^2 + 5\hat{g}_{2du}^2 + 11\hat{g}_{2ud}^2 + 13\hat{g}_{2uu}^2) \\
&\qquad\qquad + 2\hat{g}_{2dd}^2(2\hat{g}_{2ud}^2 + 5\hat{g}_{2uu}^2) + 8\hat{g}_{2du}\hat{g}_{2ud}\hat{g}_{2uu}\hat{g}_{2dd} \\
&\qquad\qquad + 2\hat{g}_{2ud}^2(4\hat{g}_{2du}^2 + 4\hat{g}_{2ud}^2 + 7\hat{g}_{2uu}^2)\big) \\
&\qquad\quad + 2\hat{g}_{1du}\big(5\hat{g}_{2du}\hat{g}_{2ud}^3 + 3\hat{g}_{2dd}\hat{g}_{2uu}\hat{g}_{2ud}^2 + 3\hat{g}_{2du}\hat{g}_{2uu}^2\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2uu}^3 \\
&\qquad\qquad + \hat{g}_{1uu}^2(11\hat{g}_{2du}\hat{g}_{2ud} + 5\hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\qquad + \hat{g}_{1ud}^2(5\hat{g}_{2du}\hat{g}_{2ud} + 11\hat{g}_{2dd}\hat{g}_{2uu}) \\
&\qquad\qquad + 4\hat{g}_{1ud}\hat{g}_{1uu}(\hat{g}_{2dd}\hat{g}_{2ud} + \hat{g}_{2du}\hat{g}_{2uu})\big) \\
&\qquad\quad + \hat{g}_{1uu}\big(11\hat{g}_{2uu}\hat{g}_{2ud}^3 + 26\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2ud}^2 + 11\hat{g}_{2uu}^3\hat{g}_{2ud} + \hat{g}_{2dd}^2\hat{g}_{2uu}\hat{g}_{2ud} \\
&\qquad\qquad + 21\hat{g}_{2du}^2\hat{g}_{2uu}\hat{g}_{2ud} + 6\hat{g}_{2dd}\hat{g}_{2du}\hat{g}_{2uu}^2 \\
&\qquad\qquad + \hat{g}_{1uu}^2(6\hat{g}_{2dd}\hat{g}_{2du} + 13\hat{g}_{2ud}\hat{g}_{2uu})\big) \\
&\qquad\quad + \hat{g}_{1du}^2\big(8\hat{g}_{1ud}^3 + \hat{g}_{1ud}(19\hat{g}_{1uu}^2 + 4\hat{g}_{2ud}^2 + 8\hat{g}_{2uu}^2) \\
&\qquad\qquad + 7\hat{g}_{1uu}\hat{g}_{2ud}\hat{g}_{2uu})\big)\Big)\Big]. \qquad \text{(F.23n)}
\end{aligned}
$$

List of Figures

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Bibliography

[1] G. Aad et al. *„Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC“*. Phys.Lett. B716 (2012), pp. 1–29. DOI: 10.1016/j.physletb.2012.08.020. arXiv: arXiv:1207.7214 [hep-ex].

[2] S. Chatrchyan et al. *„Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC“*. Phys. Lett. B716 (2012), pp. 30–61. DOI: 10.1016/j.physletb.2012.08.021. arXiv: 1207.7235 [hep-ex].

[3] G. Aad et al. *„Combined Measurement of the Higgs Boson Mass in pp Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments“*. Phys. Rev. Lett. 114 (2015), p. 191803. DOI: 10.1103/PhysRevLett.114.191803. arXiv: 1503.07589 [hep-ex].

[4] G. Aad et al. *„Measurements of the Higgs boson production and decay rates and constraints on its couplings from a combined ATLAS and CMS analysis of the LHC pp collision data at $\sqrt{s} = 7$ and 8 TeV“*. JHEP 08 (2016), p. 045. DOI: 10.1007/JHEP08(2016)045. arXiv: 1606.02266 [hep-ex].

[5] P. H. Chankowski, S. Pokorski and J. Rosiek. *„Complete on-shell renormalization scheme for the minimal supersymmetric Higgs sector“*. Nucl.Phys. B423 (1994), pp. 437–496. DOI: 10.1016/0550-3213(94)90141-4. arXiv: hep-ph/9303309 [hep-ph].

[6] A. Dabelstein. *„The One loop renormalization of the MSSM Higgs sector and its application to the neutral scalar Higgs masses“*. Z.Phys. C67 (1995), pp. 495–512. DOI: 10.1007/BF01624592. arXiv: hep-ph/9409375 [hep-ph].

[7] D. M. Pierce et al. *„Precision corrections in the minimal supersymmetric standard model“*. Nucl.Phys. B491 (1997), pp. 3–67. DOI: 10.1016/S0550-3213(96)00683-9. arXiv: hep-ph/9606211 [hep-ph].

[8] M. Frank et al. *„The Higgs Boson Masses and Mixings of the Complex MSSM in the Feynman-Diagrammatic Approach“*. JHEP 0702 (2007), p. 047. DOI: 10.1088/1126-6708/2007/02/047. arXiv: hep-ph/0611326 [hep-ph].

[9] R. Hempfling and A. H. Hoang. *„Two loop radiative corrections to the upper limit of the lightest Higgs boson mass in the minimal supersymmetric model“*. Phys.Lett. B331 (1994), pp. 99–106. DOI: 10.1016/0370-2693(94)90948-2. arXiv: hep-ph/9401219 [hep-ph].

[10] J. A. Casas et al. *„The Lightest Higgs boson mass in the minimal supersymmetric standard model“*. Nucl.Phys. B436 (1995), pp. 3–29. DOI: 10.1016/0550-3213(94)00508-C. arXiv: hep-ph/9407389 [hep-ph].

[11] M. S. Carena, M. Quiros and C. E. M. Wagner. *„Effective potential methods and the Higgs mass spectrum in the MSSM“*. Nucl.Phys. B461 (1996), pp. 407–436. DOI: 10.1016/0550-3213(95)00665-6. arXiv: hep-ph/9508343 [hep-ph].

[12] S. Heinemeyer, W. Hollik and G. Weiglein. *„Precise prediction for the mass of the lightest Higgs boson in the MSSM“*. Phys.Lett. B440 (1998), pp. 296–304. DOI: 10.1016/S0370-2693(98)01116-2. arXiv: hep-ph/9807423 [hep-ph].

[13] S. Heinemeyer, W. Hollik and G. Weiglein. *„QCD corrections to the masses of the neutral CP - even Higgs bosons in the MSSM“*. Phys.Rev. D58 (1998), p. 091701. DOI: 10.1103/PhysRevD.58.091701. arXiv: hep-ph/9803277 [hep-ph].

[14] S. Heinemeyer, W. Hollik and G. Weiglein. „*The Masses of the neutral CP - even Higgs bosons in the MSSM: Accurate analysis at the two loop level*". Eur.Phys.J. C9 (1999), pp. 343–366. DOI: 10.1007/s100529900006. arXiv: hep-ph/9812472 [hep-ph].

[15] R. Zhang. „*Two loop effective potential calculation of the lightest CP even Higgs boson mass in the MSSM*". Phys.Lett. B447 (1999), pp. 89–97. DOI: 10.1016/S0370-2693(98)01575-5. arXiv: hep-ph/9808299 [hep-ph].

[16] S. Heinemeyer, W. Hollik and G. Weiglein. „*The Mass of the lightest MSSM Higgs boson: A Compact analytical expression at the two loop level*". Phys.Lett. B455 (1999), pp. 179–191. DOI: 10.1016/S0370-2693(99)00417-7. arXiv: hep-ph/9903404 [hep-ph].

[17] J. R. Espinosa and R. Zhang. „*MSSM lightest CP even Higgs boson mass to O(alpha(s) alpha(t)): The Effective potential approach*". JHEP 0003 (2000), p. 026. DOI: 10.1088/1126-6708/2000/03/026. arXiv: hep-ph/9912236 [hep-ph].

[18] J. R. Espinosa and R. Zhang. „*Complete two loop dominant corrections to the mass of the lightest CP even Higgs boson in the minimal supersymmetric standard model*". Nucl.Phys. B586 (2000), pp. 3–38. DOI: 10.1016/S0550-3213(00)00421-1. arXiv: hep-ph/0003246 [hep-ph].

[19] G. Degrassi, P. Slavich and F. Zwirner. „*On the neutral Higgs boson masses in the MSSM for arbitrary stop mixing*". Nucl.Phys. B611 (2001), pp. 403–422. DOI: 10.1016/S0550-3213(01)00343-1. arXiv: hep-ph/0105096 [hep-ph].

[20] A. Brignole et al. „*On the O(alpha(t)**2) two loop corrections to the neutral Higgs boson masses in the MSSM*". Nucl.Phys. B631 (2002), pp. 195–218. DOI: 10.1016/S0550-3213(02)00184-0. arXiv: hep-ph/0112177 [hep-ph].

[21] S. P. Martin. „*Two loop effective potential for a general renormalizable theory and softly broken supersymmetry*". Phys. Rev. D65 (2002), p. 116003. DOI: 10.1103/PhysRevD.65.116003. arXiv: hep-ph/0111209 [hep-ph].

[22] A. Brignole et al. „*On the two loop sbottom corrections to the neutral Higgs boson masses in the MSSM*". Nucl.Phys. B643 (2002), pp. 79–92. DOI: 10.1016/S0550-3213(02)00748-4. arXiv: hep-ph/0206101 [hep-ph].

[23] S. P. Martin. „*Two loop effective potential for the minimal supersymmetric standard model*". Phys. Rev. D66 (2002), p. 096001. DOI: 10.1103/PhysRevD.66.096001. arXiv: hep-ph/0206136 [hep-ph].

[24] A. Dedes, G. Degrassi and P. Slavich. „*On the two loop Yukawa corrections to the MSSM Higgs boson masses at large tan beta*". Nucl.Phys. B672 (2003), pp. 144–162. DOI: 10.1016/j.nuclphysb.2003.08.033. arXiv: hep-ph/0305127 [hep-ph].

[25] S. P. Martin. „*Complete two loop effective potential approximation to the lightest Higgs scalar boson mass in supersymmetry*". Phys. Rev. D67 (2003), p. 095012. DOI: 10.1103/PhysRevD.67.095012. arXiv: hep-ph/0211366 [hep-ph].

[26] S. P. Martin. „*Two loop scalar self energies in a general renormalizable theory at leading order in gauge couplings*". Phys. Rev. D70 (2004), p. 016005. DOI: 10.1103/PhysRevD.70.016005. arXiv: hep-ph/0312092 [hep-ph].

[27] S. Heinemeyer et al. „*High-precision predictions for the MSSM Higgs sector at O(alpha(b) alpha(s))*". Eur.Phys.J. C39 (2005), pp. 465–481. DOI: 10.1140/epjc/s2005-02112-6. arXiv: hep-ph/0411114 [hep-ph].

[28] S. P. Martin. „*Strong and Yukawa two-loop contributions to Higgs scalar boson self-energies and pole masses in supersymmetry*". Phys. Rev. D71 (2005), p. 016012. DOI: 10.1103/PhysRevD.71.016012. arXiv: hep-ph/0405022 [hep-ph].

[29] S. P. Martin. „*Two-loop scalar self-energies and pole masses in a general renormalizable theory with massless gauge bosons*". Phys. Rev. D71 (2005), p. 116004. DOI: 10.1103/PhysRevD.71.116004. arXiv: hep-ph/0502168 [hep-ph].

[30] S. Heinemeyer et al. „*The Higgs sector of the complex MSSM at two-loop order: QCD contributions*". Phys.Lett. B652 (2007), pp. 300–309. DOI: 10.1016/j.physletb.2007.07.030. arXiv: arXiv:0705.0746 [hep-ph].

[31] W. Hollik and S. Paßehr. „*Higgs boson masses and mixings in the complex MSSM with two-loop top-Yukawa-coupling corrections*“. JHEP 10 (2014), p. 171. DOI: `10.1007/JHEP10(2014)171`. arXiv: `1409.1687 [hep-ph]`.

[32] S. Paßehr and G. Weiglein. „*Two-loop top and bottom Yukawa corrections to the Higgs-boson masses in the complex MSSM*“. Eur. Phys. J. C78.3 (2018), p. 222. DOI: `10.1140/epjc/s10052-018-5665-8`. arXiv: `1705.07909 [hep-ph]`.

[33] S. P. Martin. „*Three-loop corrections to the lightest Higgs scalar boson mass in supersymmetry*“. Phys. Rev. D75 (2007), p. 055005. DOI: `10.1103/PhysRevD.75.055005`. arXiv: `hep-ph/0701051 [hep-ph]`.

[34] R. V. Harlander et al. „*Higgs boson mass in supersymmetry to three loops*“. Phys. Rev. Lett. 100 (2008). [Phys. Rev. Lett.101,039901(2008)], p. 191602. DOI: `10.1103/PhysRevLett.101.039901,10.1103/PhysRevLett.100.191602`. arXiv: `0803.0672 [hep-ph]`.

[35] P. Kant et al. „*Light MSSM Higgs boson mass to three-loop accuracy*“. JHEP 08 (2010), p. 104. DOI: `10.1007/JHEP08(2010)104`. arXiv: `1005.5709 [hep-ph]`.

[36] G. F. Giudice and A. Strumia. „*Probing High-Scale and Split Supersymmetry with Higgs Mass Measurements*“. Nucl.Phys. B858 (2012), pp. 63–83. DOI: `10.1016/j.nuclphysb.2012.01.001`. arXiv: `arXiv:1108.6077 [hep-ph]`.

[37] P. Draper, G. Lee and C. E. M. Wagner. „*Precise Estimates of the Higgs Mass in Heavy SUSY*“. Phys.Rev. D89.5 (2014), p. 055023. DOI: `10.1103/PhysRevD.89.055023`. arXiv: `arXiv:1312.5743 [hep-ph]`.

[38] E. Bagnaschi et al. „*Higgs Mass and Unnatural Supersymmetry*“. JHEP 1409 (2014), p. 092. DOI: `10.1007/JHEP09(2014)092`. arXiv: `arXiv:1407.4081 [hep-ph]`.

[39] G. Lee and C. E. M. Wagner. „*Higgs bosons in heavy supersymmetry with an intermediate m_A*“. Phys. Rev. D92.7 (2015), p. 075032. DOI: `10.1103/PhysRevD.92.075032`. arXiv: `1508.00576 [hep-ph]`.

[40] J. Pardo Vega and G. Villadoro. „*SusyHD: Higgs mass Determination in Supersymmetry*“. JHEP 07 (2015), p. 159. DOI: `10.1007/JHEP07(2015)159`. arXiv: `1504.05200 [hep-ph]`.

[41] E. Bagnaschi, J. Pardo Vega and P. Slavich. „*Improved determination of the Higgs mass in the MSSM with heavy superpartners*“. Eur. Phys. J. C77.5 (2017), p. 334. DOI: `10.1140/epjc/s10052-017-4885-7`. arXiv: `1703.08166 [hep-ph]`.

[42] T. Hahn et al. „*High-Precision Predictions for the Light CP -Even Higgs Boson Mass of the Minimal Supersymmetric Standard Model*“. Phys.Rev.Lett. 112.14 (2014), p. 141801. DOI: `10.1103/PhysRevLett.112.141801`. arXiv: `arXiv:1312.4937 [hep-ph]`.

[43] P. Athron et al. „*Precise Higgs mass calculations in (non-)minimal supersymmetry at both high and low scales*“. JHEP 01 (2017), p. 079. DOI: `10.1007/JHEP01(2017)079`. arXiv: `1609.00371 [hep-ph]`.

[44] F. Staub and W. Porod. „*Improved predictions for intermediate and heavy Supersymmetry in the MSSM and beyond*“. Eur. Phys. J. C77.5 (2017), p. 338. DOI: `10.1140/epjc/s10052-017-4893-7`. arXiv: `1703.03267 [hep-ph]`.

[45] S. Heinemeyer, W. Hollik and G. Weiglein. „*FeynHiggs: A Program for the calculation of the masses of the neutral CP even Higgs bosons in the MSSM*“. Comput.Phys.Commun. 124 (2000), pp. 76–89. DOI: `10.1016/S0010-4655(99)00364-1`. arXiv: `hep-ph/9812320 [hep-ph]`.

[46] T. Hahn et al. „*FeynHiggs: A program for the calculation of MSSM Higgs-boson observables - Version 2.6.5*“. Comput. Phys. Commun. 180 (2009), pp. 1426–1427. DOI: `10.1016/j.cpc.2009.02.014`.

[47] G. Degrassi et al. „*Towards high precision predictions for the MSSM Higgs sector*“. Eur.Phys.J. C28 (2003), pp. 133–143. DOI: `10.1140/epjc/s2003-01152-2`. arXiv: `hep-ph/0212020 [hep-ph]`.

[48] H. Bahl and W. Hollik. „*Precise prediction for the light MSSM Higgs boson mass combining effective field theory and fixed-order calculations*“. Eur. Phys. J. C76.9 (2016), p. 499. DOI: `10.1140/epjc/s10052-016-4354-8`. arXiv: `1608.01880 [hep-ph]`.

[49] H. Bahl et al. „*Reconciling EFT and hybrid calculations of the light MSSM Higgs-boson mass*“. Eur. Phys. J. C78.1 (2018), p. 57. DOI: 10.1140/epjc/s10052-018-5544-3. arXiv: 1706.00346 [hep-ph].

[50] H. Fritzsch, M. Gell-Mann and H. Leutwyler. „*Advantages of the Color Octet Gluon Picture*“. Phys. Lett. 47B (1973), pp. 365–368. DOI: 10.1016/0370-2693(73)90625-4.

[51] D. J. Gross and F. Wilczek. „*Asymptotically Free Gauge Theories - I*“. Phys. Rev. D8 (1973), pp. 3633–3652. DOI: 10.1103/PhysRevD.8.3633.

[52] D. J. Gross and F. Wilczek. „*Ultraviolet Behavior of Nonabelian Gauge Theories*“. Phys. Rev. Lett. 30 (1973), pp. 1343–1346. DOI: 10.1103/PhysRevLett.30.1343.

[53] H. D. Politzer. „*Reliable Perturbative Results for Strong Interactions?*“ Phys. Rev. Lett. 30 (1973). [,274(1973)], pp. 1346–1349. DOI: 10.1103/PhysRevLett.30.1346.

[54] S. L. Glashow. „*Partial Symmetries of Weak Interactions*“. Nucl. Phys. 22 (1961), pp. 579–588. DOI: 10.1016/0029-5582(61)90469-2.

[55] S. Weinberg. „*A Model of Leptons*“. Phys. Rev. Lett. 19 (1967), pp. 1264–1266. DOI: 10.1103/PhysRevLett.19.1264.

[56] A. Salam. „*Weak and Electromagnetic Interactions*“. Conf. Proc. C680519 (1968), pp. 367–377.

[57] P. W. Higgs. „*Broken symmetries, massless particles and gauge fields*“. Phys.Lett. 12 (1964), pp. 132–133. DOI: 10.1016/0031-9163(64)91136-9.

[58] P. W. Higgs. „*Broken Symmetries and the Masses of Gauge Bosons*“. Phys.Rev.Lett. 13 (1964), pp. 508–509. DOI: 10.1103/PhysRevLett.13.508.

[59] F. Englert and R. Brout. „*Broken Symmetry and the Mass of Gauge Vector Mesons*“. Phys.Rev.Lett. 13 (1964), pp. 321–323. DOI: 10.1103/PhysRevLett.13.321.

[60] K. A. Olive et al. „*Review of Particle Physics*“. Chin.Phys. C38 (2014), p. 090001. DOI: 10.1088/1674-1137/38/9/090001.

[61] S. R. Coleman and J. Mandula. „*All Possible Symmetries of the S Matrix*“. Phys.Rev. 159 (1967), pp. 1251–1256. DOI: 10.1103/PhysRev.159.1251.

[62] R. Haag, J. T. Lopuszanski and M. Sohnius. „*All Possible Generators of Supersymmetries of the s Matrix*“. Nucl.Phys. B88 (1975), p. 257. DOI: 10.1016/0550-3213(75)90279-5.

[63] S. P. Martin. „*A Supersymmetry primer*“. Adv.Ser.Direct.High Energy Phys. 21 (2010), pp. 1–153. DOI: 10.1142/9789814307505_0001. arXiv: hep-ph/9709356 [hep-ph].

[64] H. P. Nilles. „*Supersymmetry, Supergravity and Particle Physics*“. Phys.Rept. 110 (1984), pp. 1–162. DOI: 10.1016/0370-1573(84)90008-5.

[65] G. F. Giudice and R. Rattazzi. „*Theories with gauge mediated supersymmetry breaking*“. Phys.Rept. 322 (1999), pp. 419–499. DOI: 10.1016/S0370-1573(99)00042-3. arXiv: hep-ph/9801271 [hep-ph].

[66] L. Girardello and M. T. Grisaru. „*Soft Breaking of Supersymmetry*“. Nucl. Phys. B194 (1982), p. 65. DOI: 10.1016/0550-3213(82)90512-0.

[67] H. E. Haber and G. L. Kane. „*The Search for Supersymmetry: Probing Physics Beyond the Standard Model*“. Phys. Rept. 117 (1985), pp. 75–263. DOI: 10.1016/0370-1573(85)90051-1.

[68] T. Takagi. Japanese J.Math. 1 (1927), p. 83.

[69] G. 't Hooft and M. J. G. Veltman. „*Regularization and Renormalization of Gauge Fields*“. Nucl. Phys. B44 (1972), pp. 189–213. DOI: 10.1016/0550-3213(72)90279-9.

[70] C. G. Bollini and J. J. Giambiagi. „*Lowest order divergent graphs in nu-dimensional space*“. Phys. Lett. 40B (1972), pp. 566–568. DOI: 10.1016/0370-2693(72)90483-2.

[71] G. M. Cicuta and E. Montaldi. „*Analytic renormalization via continuous space dimension*“. Lett. Nuovo Cim. 4 (1972), pp. 329–332. DOI: 10.1007/BF02756527.

[72] J. F. Ashmore. „*A Method of Gauge Invariant Regularization*“. Lett. Nuovo Cim. 4 (1972), pp. 289–290. DOI: 10.1007/BF02824407.

[73] W. Siegel. „*Supersymmetric Dimensional Regularization via Dimensional Reduction*“. Phys. Lett. 84B (1979), pp. 193–196. DOI: 10.1016/0370-2693(79)90282-X.

[74] D. M. Capper, D. R. T. Jones and P. van Nieuwenhuizen. „*Regularization by Dimensional Reduction of Supersymmetric and Nonsupersymmetric Gauge Theories*“. Nucl. Phys. B167 (1980), pp. 479–499. DOI: 10.1016/0550-3213(80)90244-8.

[75] A. Freitas and D. Stöckinger. „*Gauge dependence and renormalization of tan beta*“ (2002), pp. 657–661. arXiv: hep-ph/0210372 [hep-ph].

[76] S. Borowka et al. „*Momentum-dependent two-loop QCD corrections to the neutral Higgs-boson masses in the MSSM*“. Eur. Phys. J. C74.8 (2014), p. 2994. DOI: 10.1140/epjc/s10052-014-2994-0. arXiv: 1404.7074 [hep-ph].

[77] S. Borowka et al. „*Renormalization scheme dependence of the two-loop QCD corrections to the neutral Higgs-boson masses in the MSSM*“. Eur. Phys. J. C75.9 (2015), p. 424. DOI: 10.1140/epjc/s10052-015-3648-6. arXiv: 1505.03133 [hep-ph].

[78] K. E. Williams, H. Rzehak and G. Weiglein. „*Higher order corrections to Higgs boson decays in the MSSM with complex parameters*“. Eur. Phys. J. C71 (2011), p. 1669. DOI: 10.1140/epjc/s10052-011-1669-3. arXiv: 1103.1335 [hep-ph].

[79] D. Buttazzo et al. „*Investigating the near-criticality of the Higgs boson*“. JHEP 1312 (2013), p. 089. DOI: 10.1007/JHEP12(2013)089. arXiv: arXiv:1307.3536 [hep-ph].

[80] P. H. Chankowski et al. „*Delta R in the MSSM*“. Nucl. Phys. B417 (1994), pp. 101–129. DOI: 10.1016/0550-3213(94)90539-8.

[81] S. Heinemeyer et al. „*Precise prediction for M(W) in the MSSM*“. JHEP 08 (2006), p. 052. DOI: 10.1088/1126-6708/2006/08/052. arXiv: hep-ph/0604147 [hep-ph].

[82] S. Heinemeyer et al. „*Implications of LHC search results on the W boson mass prediction in the MSSM*“. JHEP 12 (2013), p. 084. DOI: 10.1007/JHEP12(2013)084. arXiv: 1311.1663 [hep-ph].

[83] O. Stål, G. Weiglein and L. Zeune. „*Improved prediction for the mass of the W boson in the NMSSM*“. JHEP 09 (2015), p. 158. DOI: 10.1007/JHEP09(2015)158. arXiv: 1506.07465 [hep-ph].

[84] G. Degrassi, S. Di Vita and P. Slavich. „*Two-loop QCD corrections to the MSSM Higgs masses beyond the effective-potential approximation*“. Eur. Phys. J. C75.2 (2015), p. 61. DOI: 10.1140/epjc/s10052-015-3280-5. arXiv: 1410.3432 [hep-ph].

[85] H. E. Haber and R. Hempfling. „*The Renormalization group improved Higgs sector of the minimal supersymmetric model*“. Phys.Rev. D48 (1993), pp. 4280–4309. DOI: 10.1103/PhysRevD.48.4280. arXiv: hep-ph/9307201 [hep-ph].

[86] M. Sperling, D. Stöckinger and A. Voigt. „*Renormalization of vacuum expectation values in spontaneously broken gauge theories*“. JHEP 1307 (2013), p. 132. DOI: 10.1007/JHEP07(2013)132. arXiv: arXiv:1305.1548 [hep-ph].

[87] M. Sperling, D. Stöckinger and A. Voigt. „*Renormalization of vacuum expectation values in spontaneously broken gauge theories: Two-loop results*“. JHEP 1401 (2014), p. 068. DOI: 10.1007/JHEP01(2014)068. arXiv: arXiv:1310.7629 [hep-ph].

[88] M. Binger. „*Higgs boson mass in split supersymmetry at two-loops*“. Phys. Rev. D73 (2006), p. 095001. DOI: 10.1103/PhysRevD.73.095001. arXiv: hep-ph/0408240 [hep-ph].

[89] L. N. Mihaila, J. Salomon and M. Steinhauser. „*Gauge coupling beta functions in the Standard Model to three loops*“. Phys. Rev. Lett. 108 (2012), p. 151602. DOI: 10.1103/PhysRevLett.108.151602. arXiv: 1201.5868 [hep-ph].

[90] L. N. Mihaila, J. Salomon and M. Steinhauser. „*Renormalization constants and beta functions for the gauge couplings of the Standard Model to three-loop order*“. Phys. Rev. D86 (2012), p. 096008. DOI: 10.1103/PhysRevD.86.096008. arXiv: 1208.3357 [hep-ph].

[91] K. G. Chetyrkin and M. F. Zoller. „*Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model*“. JHEP 06 (2012), p. 033. DOI: 10.1007/JHEP06(2012)033. arXiv: 1205.2892 [hep-ph].

[92] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin. *„Yukawa coupling beta-functions in the Standard Model at three loops"*. Phys. Lett. B722 (2013), pp. 336–340. DOI: 10.1016/j.physletb.2013.04.038. arXiv: 1212.6829 [hep-ph].

[93] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin. *„Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops"*. JHEP 01 (2013), p. 017. DOI: 10.1007/JHEP01(2013)017. arXiv: 1210.6873 [hep-ph].

[94] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin. *„Higgs self-coupling beta-function in the Standard Model at three loops"*. Nucl. Phys. B875 (2013), pp. 552–565. DOI: 10.1016/j.nuclphysb.2013.07.015. arXiv: 1303.4364 [hep-ph].

[95] K. G. Chetyrkin and M. F. Zoller. *„β-function for the Higgs self-interaction in the Standard Model at three-loop level"*. JHEP 04 (2013). [Erratum: JHEP09,155(2013)], p. 091. DOI: 10.1007/JHEP04(2013)091,10.1007/JHEP09(2013)155. arXiv: 1303.2890 [hep-ph].

[96] M. S. Carena et al. *„Reconciling the two loop diagrammatic and effective field theory computations of the mass of the lightest CP - even Higgs boson in the MSSM"*. Nucl.Phys. B580 (2000), pp. 29–57. DOI: 10.1016/S0550-3213(00)00212-1. arXiv: hep-ph/0001002 [hep-ph].

[97] J. S. Lee et al. *„CPsuperH: A Computational tool for Higgs phenomenology in the minimal supersymmetric standard model with explicit CP violation"*. Comput. Phys. Commun. 156 (2004), pp. 283–317. DOI: 10.1016/S0010-4655(03)00463-6. arXiv: hep-ph/0307377 [hep-ph].

[98] J. F. Gunion and H. E. Haber. *„The CP conserving two Higgs doublet model: The Approach to the decoupling limit"*. Phys. Rev. D67 (2003), p. 075019. DOI: 10.1103/PhysRevD.67.075019. arXiv: hep-ph/0207010 [hep-ph].

[99] J. R. Ellis, J. S. Lee and A. Pilaftsis. *„Higgs Phenomenology with CPsuperH"*. Mod. Phys. Lett. A21 (2006), pp. 1405–1422. DOI: 10.1142/S0217732306020895. arXiv: hep-ph/0605288 [hep-ph].

[100] J. S. Lee et al. *„CPsuperH2.0: an Improved Computational Tool for Higgs Phenomenology in the MSSM with Explicit CP Violation"*. Comput. Phys. Commun. 180 (2009), pp. 312–331. DOI: 10.1016/j.cpc.2008.09.003. arXiv: 0712.2360 [hep-ph].

[101] B. Allanach. *„SOFTSUSY: a program for calculating supersymmetric spectra"*. Comput.Phys.Commun. 143 (2002), pp. 305–331. DOI: 10.1016/S0010-4655(01)00460-X. arXiv: hep-ph/0104145 [hep-ph].

[102] W. Porod. *„SPheno, a program for calculating supersymmetric spectra, SUSY particle decays and SUSY particle production at e+ e- colliders"*. Comput. Phys. Commun. 153 (2003), pp. 275–315. DOI: 10.1016/S0010-4655(03)00222-4. arXiv: hep-ph/0301101 [hep-ph].

[103] W. Porod and F. Staub. *„SPheno 3.1: Extensions including flavour, CP-phases and models beyond the MSSM"*. Comput. Phys. Commun. 183 (2012), pp. 2458–2469. DOI: 10.1016/j.cpc.2012.05.021. arXiv: 1104.1573 [hep-ph].

[104] A. Djouadi, J.-L. Kneur and G. Moultaka. *„SuSpect: A Fortran code for the supersymmetric and Higgs particle spectrum in the MSSM"*. Comput. Phys. Commun. 176 (2007), pp. 426–455. DOI: 10.1016/j.cpc.2006.11.009. arXiv: hep-ph/0211331 [hep-ph].

[105] G. Lee and C. Wagner. „MhEFT *package"*. http://gabrlee.com/code (2016).

[106] P. Athron et al. *„FlexibleSUSY – A spectrum generator generator for supersymmetric models"*. Comput.Phys.Commun. 190 (2015), pp. 139–172. DOI: 10.1016/j.cpc.2014.12.020. arXiv: 1406.2319 [hep-ph].

[107] P. Athron et al. *„FlexibleSUSY 2.0: Extensions to investigate the phenomenology of SUSY and non-SUSY models"* (2017). arXiv: 1710.03760 [hep-ph].

[108] F. Staub. *„From Superpotential to Model Files for FeynArts and CalcHep/CompHep"*. Comput.Phys.Commun. 181 (2010), pp. 1077–1086. DOI: 10.1016/j.cpc.2010.01.011. arXiv: 0909.2863 [hep-ph].

[109] F. Staub. *„Automatic Calculation of supersymmetric Renormalization Group Equations and Self Energies“*. Comput.Phys.Commun. 182 (2011), pp. 808–833. DOI: `10.1016/j.cpc.2010.11.030`. arXiv: `1002.0840 [hep-ph]`.

[110] F. Staub. *„SARAH 3.2: Dirac Gauginos, UFO output, and more“*. Computer Physics Communications 184 (2013), pp. 1792–1809. DOI: `10.1016/j.cpc.2013.02.019`. arXiv: `1207.0906 [hep-ph]`.

[111] F. Staub. *„SARAH 4 : A tool for (not only SUSY) model builders“*. Comput. Phys. Commun. 185 (2014), pp. 1773–1790. DOI: `10.1016/j.cpc.2014.02.018`. arXiv: `1309.7223 [hep-ph]`.

[112] P. Draper and H. Rzehak. *„A Review of Higgs Mass Calculations in Supersymmetric Models“*. Phys. Rept. 619 (2016), pp. 1–24. DOI: `10.1016/j.physrep.2016.01.001`. arXiv: `1601.01890 [hep-ph]`.

[113] E. Bagnaschi et al. *„Vacuum stability and supersymmetry at high scales with two Higgs doublets“*. JHEP 03 (2016), p. 158. DOI: `10.1007/JHEP03(2016)158`. arXiv: `1512.07761 [hep-ph]`.

[114] S. Heinemeyer, W. Hollik and G. Weiglein. *„Constraints on* $\tan\beta$ *in the MSSM from the upper bound on the mass of the lightest Higgs boson“*. JHEP 06 (2000), p. 009. DOI: `10.1088/1126-6708/2000/06/009`. arXiv: `hep-ph/9909540 [hep-ph]`.

[115] E. Bagnaschi et al. *„Benchmark scenarios for low* $\tan\beta$ *in the MSSM“* (2015). eprint: `LHCHXSWG-2015-002`.

[116] P. Bechtle et al. *„The Light and Heavy Higgs Interpretation of the MSSM“*. Eur. Phys. J. C77.2 (2017), p. 67. DOI: `10.1140/epjc/s10052-016-4584-9`. arXiv: `1608.00638 [hep-ph]`.

[117] H. E. Haber, S. Heinemeyer and T. Stefaniak. *„The Impact of Two-Loop Effects on the Scenario of MSSM Higgs Alignment without Decoupling“* (2017). arXiv: `1708.04416 [hep-ph]`.

[118] M. Carena et al. *„Effective Lagrangian for the* $\bar{t}bH^+$ *interaction in the MSSM and charged Higgs phenomenology“*. Nucl. Phys. B577 (2000), pp. 88–120. DOI: `10.1016/S0550-3213(00)00146-2`. arXiv: `hep-ph/9912516 [hep-ph]`.

[119] M. Gorbahn et al. *„The supersymmetric Higgs sector and* $B - \bar{B}$ *mixing for large tan* β*“*. Phys. Rev. D84 (2011), p. 034030. DOI: `10.1103/PhysRevD.84.034030`. arXiv: `0901.2065 [hep-ph]`.

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