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Abstract. One technique to reduce vibrations of low-damped structures is the application of constrained layer damping (CLD) treatments. A unique extension of the CLD is a compressible constrained layer damping (CCLD) where the usually stiff viscoelastic layer is replaced by a compressible one. Due to the adaptable deformation kinematics, as well as compression-dependent material properties, the dynamic behavior of the whole considered structure can be optimized during the operation. One challenge for this application is to determine the optimal compression levels in order to adapt the dynamic behaviour of the component with CCLD to the excitation characteristics. Through the application of finite element model such optimal control strategies can be extensively studied and the dynamics of actuation principle can be included. Herein, low computation times are highly desired, especially for applications where the CCLD undergoes large deformations. This contribution shows how nonlinear model reduction can be applied to accelerate the solution of such finite element models.

1. Introduction and motivation

In order to avoid function impairing or even damage relevant vibrations, efficient solutions for vibration mitigation are required especially for lightweight applications [1]. Besides material-intrinsic design properties, conventional post-manufacturing measures such as supplementary applied noise-absorbent mats as well as different kinds of structure-attached fluidic dampers or tuned mass dampers are used. Such modifications are however often considered as counterproductive with regard to maintain lightweight properties of the manufactured component. Instead, a number of lightweight-compatible solutions for vibration mitigating have been developed [2, 3].

One technique to reduce vibrations of low-damped structures is the application of constrained layer damping (CLD) treatments [4]. Such treatments consist of overall three elements: the main component to be damped, a damping layer made of a material with high energy dissipation properties, and a constraining layer that forces the damping layer to undergo shear deformations rather than bending to induce higher stress levels and hence better energy dissipation. A unique extension of the constrained layer damping consists of the replacement of the incompressible viscoelastic layer of the classical CLD with a compressible one – usually open-cell foam – whose damping properties can be adjusted through thickness adjustment. The thickness changing actuation principle of such compressible CLD (CCLD) bases on structural cavities generating evanescent deformations when supplied with fluidic medium.

However, since the damping properties of the constrained layer damper depend on the excitation frequency and compression stage of the material, the CCLD has to be actuated on-line in order to achieve improvement of the dynamic behaviour. One challenge to set up the optimal compression during operation is the determination of the optimal pressure in function of excitation characteristics like frequency or amplitude. This can be achieved either based on results of experimental investigations or numerical ones by using a finite element model. Experiments can be very cost-intensive because they have to be done for each new CCLD design. A computation of the optimal compression by using a finite element model is usually cheaper but still can take long simulation time due to high dimension of the model and nonlinearities considering potentially large deformations of the structure. Hence, for such applications, it is crucial to significantly reduce the computation time. This contribution shows how nonlinear model reduction can be used in order to speed up design and allow to infer appropriate design parameters for the structure and the control system.

2. Operating principle of compressible constrained layer damping treatment

The general overview of the proposed Compressible Constrained Layer Damping (CCLD) treatment is presented in Figure 1. The analyzed CCLD object is configured as a three layered beam consisting of the load-bearing structure as well as constraining and compressible viscoelastic layer. The unique actuating principle is based on structural cavities generating evanescent deformations when supplied with hydraulic fluid, compressed air or vacuum. The last possibility was used in combination with an open-cell foam architecture ensuring a low bulk modulus and thus a broad deformation range under atmospheric pressure. The evanescent morphing is used to deliberately alter the geometrical and material properties of the viscoelastic elements through compression in order to achieve a damping capacity adaptation. With the decreasing viscoelastic layer thickness, an increase of its damping and stiffness is expected. At the same time, the shear strain becomes higher at a given vibration amplitude. The change in geometry can also affect the structure's stiffness. The combination of these effects should broaden the range of CCLD-adaptation through the compression level setting. The studies regarding experimental characterization of the compression-dependent viscoelastic shear properties are not the main part of these studies and will be presented elsewhere.

In order to maintain optimal cavity pressure for any realistic configuration of excitation frequency and amplitude, a closed-loop control system is necessary. The structure and parameters of such control systems should be matched to the dynamics of the CCLD treatment for any potentially suitable CCLD configuration. Since the application of a finite-element-model with appropriate real-time capabilities could be used for later hardware-in-the-loop tests of the control system a suitable simulation model was developed.

3. Modeling

A finite element model for simulation of the dynamic response and damping properties of the adaptive constrained layer damper is created. Its geometry is a square plate with a side length of 550 mm. The plate consists of 5 layers as illustrated in Figure 2 in order to introduce some kind of symmetric dynamic behavior. Their material properties are summarized in Figure 2 on the right.

The geometry is meshed with quadratic Hexaeder elements with 20 nodes each. The boundaries of the constrained layer damper are fixed in each translational direction. The structure is assumed to be excited at one node of the outer-layer in out-of-plane direction.

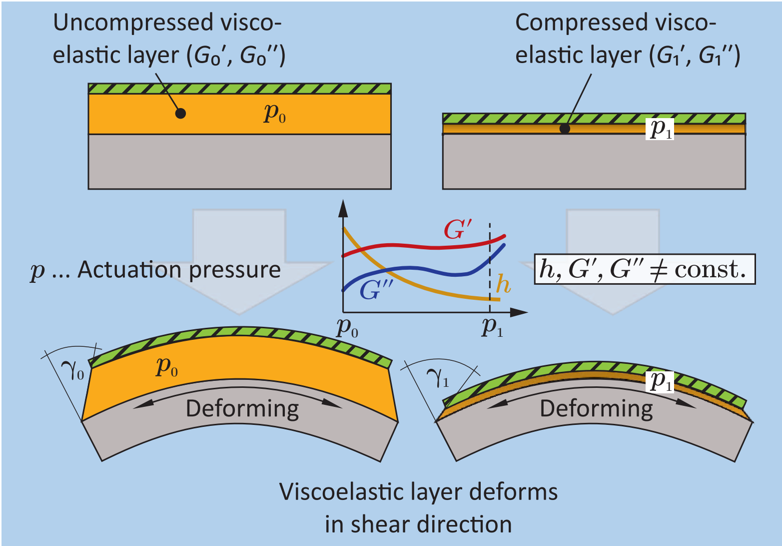


Figure 1. Cross section view of CCLD. The storage and loss modulus of the viscoelastic layer as well as the thickness of the viscolelastic layer depend on the applied pressure.

The full model has 556 308 degrees of freedom. A time-integration requires high computation times due to the geometric nonlinearity and the high number of degrees of freedom. We apply model reduction to reduce computation time.

4. Nonlinear model reduction

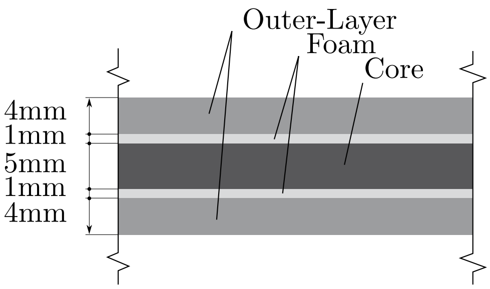
The outcoming equation of motion of the Finite Element model can be described by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{f}_{\text{nl}}(\mathbf{u}(t)) = \mathbf{B}\,\mathbf{f}_{\text{ext}}(t)\ , \tag{1}$$

where $\mathbf{u}(t)$ are the displacements of the nodes, $\mathbf{M}$ is the mass matrix, $\mathbf{D}$ the viscous damping matrix, $\mathbf{f}_{\text{nl}}$ the nonlinear restoring force and $\mathbf{B}$ the matrix for distributing the external forces $\mathbf{f}_{\text{ext}}$. The viscous damping matrix is modeled as

$$\mathbf{D} = a\mathbf{M} + b\left.\frac{\partial \mathbf{f}_{\text{nl}}}{\partial \mathbf{u}}\right|_{\mathbf{u}=0}\ , \tag{2}$$

where the coefficients a and b are chosen in such way that the resonance peaks of the first and fourth eigenfrequencies of the linearized frequency response function at the excitation point match the measured damping ratios.



Layer	E	ν
Core	85 MPa	0.1
Foam	0.1 MPa	0.0
Outer-Layer	23 GPa[1]	0.3

[1] Orthotropic material, Young's modulus is given in direction of fibers

Figure 2. Left: Lay-up, Right: Material properties.

To reduce computation time for solving equation (1), the displacements $\mathbf{u}(t)$ are approximated by $\mathbf{u} \approx \mathbf{V}\mathbf{q}(t)$ with $\mathbf{V} \in \mathbb{R}^{N\times n}$ and $n \ll N$. Inserting this approximation into (1) and applying a Galerkin projection leads to the reduced equation of motion

$$\mathbf{V}^\mathsf{T}\mathbf{M}\mathbf{V}\ddot{\mathbf{q}} + \mathbf{V}^\mathsf{T}\mathbf{D}\mathbf{V}\dot{\mathbf{q}} + \mathbf{V}^\mathsf{T}\mathbf{f}_{\text{nl}}(\mathbf{V}\mathbf{q}) = \mathbf{V}^\mathsf{T}\mathbf{B}\,\mathbf{f}_{\text{ext}} \ . \tag{3}$$

This projection is similar to linear model reduction techniques (cf. [5, 6]). It reduces the dimension of the problem from N to n. However, no matter how small one chooses the reduced dimension n, the mere dimensional reduction does not reduce computation time because the nonlinear restoring force term $\mathbf{f}_{\text{nl}}$ still needs to be evaluated in the full element domain. It is computed by evaluating the element forces $\mathbf{f}_e$ for each element of the mesh and assembling

$$\mathbf{V}^\mathsf{T}\mathbf{f}_{\text{nl}} = \sum_{e\in E} \mathbf{V}^\mathsf{T}\mathbf{L}_e^\mathsf{T}\mathbf{f}_e(\mathbf{L}_e\mathbf{V}\mathbf{q}) \ , \tag{4}$$

where $\mathbf{L}_e$ is a Boolean operator that maps the local degrees of freedom of element e to the global ones.

To accelerate the evaluation of the nonlinear term, a hyperreduction must be performed. Within this study, we use the Energy Conserving Sampling and Weighting method (ECSW) [7] for hyperreduction. This method only evaluates a subset of all element forces and assembles $\mathbf{V}^T\mathbf{f}_{\text{nl}}$ by weighting their contributions with a positive scalar $\xi_e^\star$, such that

$$\mathbf{V}^\mathsf{T}\mathbf{f}_{\text{nl}}(\mathbf{V}\mathbf{q}(t)) \approx \sum_{e\in\tilde{E}\subset E} \xi_e^\star \mathbf{V}^\mathsf{T}\mathbf{L}_e^\mathsf{T}\mathbf{f}_e(\mathbf{L}_e\mathbf{V}\mathbf{q}) \ . \tag{5}$$

The weights $\xi_e^\star$ and element subset $\tilde{E}$ are chosen such that the hyperreduced nonlinear force term matches the projected nonlinear force vector $\mathbf{V}^\mathsf{T}\mathbf{f}_{\text{nl}}(\mathbf{V}\mathbf{q})$ within a certain error bound for N_τ training displacements $\mathbf{u}_\tau = \mathbf{V}\mathbf{q}_\tau$. This relation can be written as follows: Let

$$\mathbf{g}_{le}(\mathbf{V}\mathbf{q}_{\tau_l}) = \mathbf{V}^\mathsf{T}\mathbf{L}_e^\mathsf{T}\mathbf{f}_e(\mathbf{L}_e\mathbf{V}\mathbf{q}_{\tau_l}) \ \in \mathbb{R}^n \qquad \mathbf{b}_l = \mathbf{V}^\mathsf{T}\mathbf{f}_{\text{nl}}(\mathbf{V}\mathbf{q}_{\tau_l}) = \sum_{e=1}^{N_e} \mathbf{g}_{le}(\mathbf{V}\mathbf{q}_{\tau_l}) \ \in \mathbb{R}^n$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_{11} & \cdots & \mathbf{g}_{1N_e} \\ \vdots & \ddots & \vdots \\ \mathbf{g}_{N_\tau 1} & \cdots & \mathbf{g}_{N_\tau N_e} \end{bmatrix} \in \mathbb{R}^{nN_\tau \times N_e} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{N_\tau} \end{bmatrix} \in \mathbb{R}^{nN_\tau} \ .$$

The minimization problem

$$\xi^\star = \arg\min_{\xi\in\Phi} \|\xi\|_0 \qquad \Phi = \{\xi \in \mathbb{R}^{N_e} : \|\mathbf{G}\,\xi - \mathbf{b}\|_2 \le \varepsilon\|\mathbf{b}\|_2, \ \xi \ge 0\} \ ,$$

where $\|\xi\|_0$ is the number of all non-zero coordinates, returns the weights $\xi_e^\star$. The element subset $\tilde{E}$, whose element forces $\mathbf{f}_e$ are evaluated, consists of those elements that have a non-zero $\xi_e^\star$. It is noteworthy that this kind of problem is NP-hard. Therefore, it has been replaced by a similar problem that approximates the formulation above as proposed in [7].

Two entities that highly depend on the investigated system need to be determined to perform the nonlinear model reduction described above: first, a reduction basis $\mathbf{V}$ for Galerkin projection and, second, a training set $\mathbf{u}_\tau = \mathbf{V}\mathbf{q}_\tau$ for hyperreduction.

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4.1. Determination of reduction bases $\mathbf{V}$

One option to determine a reduction basis is using the first n left singular vectors of a matrix with column-stacked displacement snapshots coming from a full simulation. This method is called Proper Orthogonal Decomposition [8]. Its characteristic is the dependence on full simulation results. These are typically very costly to compute. In this contribution, we make use of simulation-free methods that do not depend on solutions to the full model.

We use a combination of static solutions, vibration modes of the linearized system and their static modal derivatives. Thus, the reduction basis consists of three parts:

$$\mathbf{V} = [\mathbf{V}_\mathrm{S}\, \mathbf{V}_\mathrm{VM}\, \mathbf{V}_\mathrm{SD}] \; . \tag{6}$$

$\mathbf{V}_\mathrm{S}$ contains the static solutions of the model for different magnitudes of the external force. $\mathbf{V}_\mathrm{VM}$ consists of mode shapes of the linearized system that are computed by solving the eigenproblem

$$(\mathbf{K}_\mathsf{T} - \omega_i^2 \mathbf{M})\phi_i = \mathbf{0} \qquad \mathbf{V}_\mathrm{VM} = [\phi_1, \phi_2, \ldots, \phi_k] \; , \tag{7}$$

with tangent stiffness matrix $\mathbf{K}_\mathsf{T} = \left.\frac{\partial \mathbf{f}_\mathrm{nl}}{\partial \mathbf{u}}\right|_{\mathbf{u}_0}$ where $\mathbf{u}_0$ is a static solution to an actuated state $\mathbf{u}_0$ where a constant pressure has been applied to the surfaces that are connected with the foam. $\mathbf{V}_\mathrm{SD}$ contains static modal derivatives [9, 10]

$$\mathbf{K}_\mathsf{T}\mathbf{v}_{ij} = \frac{\partial \mathbf{K}_\mathsf{T}}{\partial \phi_i}\phi_j \qquad \mathbf{V}_\mathrm{SD} = [\mathbf{v}_{ij_1}, \mathbf{v}_{ij_2}, \ldots, \mathbf{v}_{ij_m}] \tag{8}$$

that insert nonlinear information into the reduction basis $\mathbf{V}$.

4.2. Determination of training sets for hyperreduction

Usually, some displacement vectors from the solution of the full model are used as training displacements. But we want to circumvent their high computation costs and use a simulation-free method. We use a method called Nonlinear Stochastic Krylov Training Sets (NSKTS) that has been proposed by Rutzmoser in [11].

The first step is to build a subspace that is able to approximate the nonlinear force vector $\mathbf{f}_\mathrm{nl}$. If the viscous damping term is neglected, one finds

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}_\mathrm{nl}(\mathbf{u}(t)) = \mathbf{B}\,\mathbf{f}_\mathrm{ext}(t) \quad \Rightarrow \quad \mathbf{f}_\mathrm{nl} \in \mathrm{span}(\mathbf{B},\, \mathbf{M}\ddot{\mathbf{u}}(t_1),\, \mathbf{M}\ddot{\mathbf{u}}(t_2),\, \ldots,\, \mathbf{M}\ddot{\mathbf{u}}(t_n)) \; . \tag{9}$$

As the accelerations $\ddot{\mathbf{u}}(t_i)$ are unknown, we need an approximation for the subspace described above. According to [11], it is approximated by the Krylov subspace

$$\mathbf{F}_\mathrm{kry} = \mathrm{span}\{\mathbf{B},\; \mathbf{M}\mathbf{K}^{-1}\mathbf{B},\; (\mathbf{M}\mathbf{K}^{-1})^2\mathbf{B}, \ldots\} = \mathcal{K}(\mathbf{M}\mathbf{K}^{-1},\, \mathbf{B}) \tag{10}$$

that is orthogonalized and normalized such that $\mathbf{F}_\mathrm{kry}^\mathsf{T}\mathbf{K}^{-1}\mathbf{F}_\mathrm{kry} = \mathbf{I}$.

Afterwards, some random vectors $\mathbf{f}^\tau_\mathrm{NSKTS}$ living in this Krylov subspace are generated and applied as external force to the nonlinear static problem $\mathbf{f}_\mathrm{nl}(\mathbf{u}_\tau) = \mathbf{f}^\tau_\mathrm{NSKTS}$. This equation must be solved by a nonlinear solver such as Newton-Raphson. The training set is then built by saving the solution $\mathbf{u}_\tau^{(k)}$ of each Newton iteration k for each applied external force $\mathbf{f}^\tau_\mathrm{NSKTS}$ as training vector.

5. Results

The finite element model is reduced with different hyperreduction tolerances ε and different reduction bases $\mathbf{V}$ that differ in size and combination of vectors to compare the performance of the reduction. We consider four different bases:

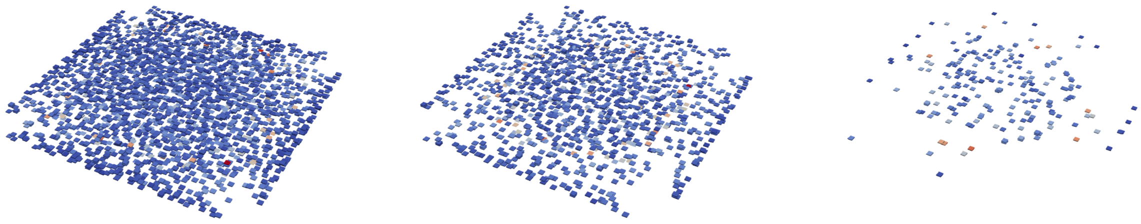

Figure 3. Reduced mesh of the plate with $\varepsilon = 0.001$ (left), $\varepsilon = 0.01$ (middle), $\varepsilon = 0.1$ (right)

$\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ full: 50 vibration modes (with smallest eigenfrequencies) and 1274 static derivatives

$\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ 600: 50 vibration modes and 550 static derivatives (truncated by using a SVD)

$\mathbf{V}_{\mathrm{S}} + \mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ full: Augmentation of $\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **full** by 20 Newton iterations of the static solution

$\mathbf{V}_{\mathrm{S}} + \mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ 600 Augmentation of $\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **600** augmented by 20 Newton iterations of the static solution

The model also is hyperreduced with ECSW method using different values for the tolerance ε. 160 Nonlinear Stochastic Training Vectors are used as training set for all hyperreductions.

A constant pressure of -0.1 bar has been applied to the surfaces that are connected with the foam as prestress. This simulates an adaption of the CCLD through compression. All necessary linearizations (e.g. for determining vibration modes) are made relative to the static deformation due to the constant pressure.

Figure 3 shows the reduced meshes of the hyperreduced model for different values ε. One can see that higher values for ε lead to less elements that have to be evaluated. The number of evaluated elements are listed in Table 1.

Table 1. Amount of elements in element set computed by ECSW method depending on reduction basis $\mathbf{V}$ and tolerance ε. The full model has 45 375 elements

reduction basis $\mathbf{V}$	tolerance ε			
	0.01	0.005	0.001	0.0005
$\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **full**	1290	1622	2385	2670
$\mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **600**	1073	1286	1840	2067
$\mathbf{V}_{\mathrm{S}} + \mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **full**	1211	1550	2293	2625
$\mathbf{V}_{\mathrm{S}} + \mathbf{V}_{\mathrm{VM}} + \mathbf{V}_{\mathrm{SD}}$ **600**	1090	1337	1895	2095

We define the relative error

$$RE = \frac{\sqrt{\sum_{i=0}^{N_t} \Delta\mathbf{u}(t_i)^T \, \Delta\mathbf{u}(t_i)}}{\sqrt{\sum_{t=0}^{N_t} \mathbf{u}_{\mathrm{ref}}(t_i)^T \, \mathbf{u}_{\mathrm{ref}}(t_i)}} \qquad \text{with} \quad \Delta\mathbf{u}(t_i) = \mathbf{u}(t_i) - \mathbf{u}_{\mathrm{ref}}(t_i) \tag{11}$$

as measure for the accuracy of the reduced models where N_t is the number of timesteps, $\mathbf{u}$ is solution to the reduced model and $\mathbf{u}_{\mathrm{ref}}$ is the solution to the full model. Table 2 shows relative errors and simulation times for a simulation with 220 timesteps.

The simulation time can be reduced significantly by applying nonlinear model reduction. But acceptable error measures can only be achieved with small tolerance values ε. Interestingly,

Table 2. Relative errors RE and simulation times t_{simul} depending on reduction basis $\mathbf{V}$ and tolerance ε. A linear full solution leads to $RE = 10\,\%$ and $t_{\text{simul}} = 0.5\,\text{h}$

	reduction basis $\mathbf{V}$	tolerance ε 0.01	0.005	0.001	0.0005
$RE[\%]$	$\mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **full**	115.4	34.1	9.4	3.5
	$\mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **600**	177.9	80.5	83.3	82.1
	$\mathbf{V}_{\text{S}} + \mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **full**	58.6	34.1	7.3	1.4
	$\mathbf{V}_{\text{S}} + \mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **600**	89.4	72.2	67.4	67.2
$t_{\text{simul}}[h]$	**full solution**	23.70			
	$\mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **full**	6.44	5.78	5.81	6.18
	$\mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **600**	5.17	4.43	5.06	4.45
	$\mathbf{V}_{\text{S}} + \mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **full**	5.85	5.90	6.15	6.40
	$\mathbf{V}_{\text{S}} + \mathbf{V}_{\text{VM}} + \mathbf{V}_{\text{SD}}$ **600**	4.64	4.92	4.62	4.47

smaller tolerances ε for hyperreduction do not increase the simulation time significantly. The reason is that the Newton Raphson solver that is called during time-integration needs fewer iterations for smaller ε which compensates the higher costs for evaluating more elements.

6. Conclusion

Nonlinear model reduction can reduce computation time to estimate best values for the actuated pressure in a compressible constrained layer damper treatment. However, the computation time is still high which prohibits to use the method for model predictive control of the CCLD. Further investigations are needed to figure out if bases with smaller dimensions and equal accuracies can be determined or if other nonlinear reduction methods could improve performance.

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