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A novel derivation for modal derivatives based on Volterra series representation and its use in nonlinear model order reduction

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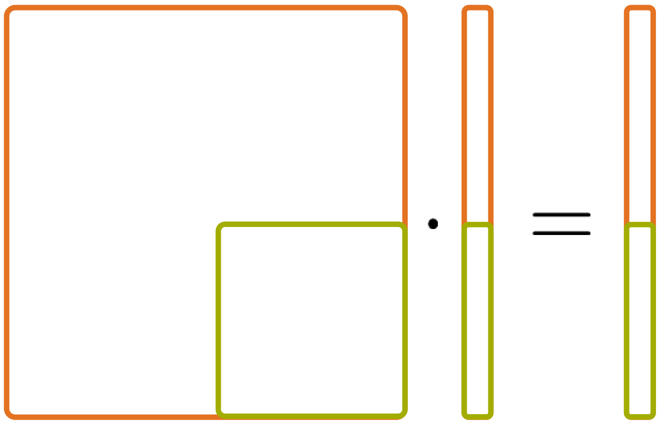
COMPDYN-UNCECOMP 2019

MS 27: Advances in MOR techniques for CSD

Crete, June 26th 2019



Projective Model Order Reduction

Nonlinear second-order full order model (FOM)

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boxed{\boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{f}(\boldsymbol{q}(t))} &= \boxed{\boldsymbol{B}\boldsymbol{F}(t)} \qquad \boldsymbol{q}(0) = \boldsymbol{q}_0,\ \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0, \\ \boldsymbol{y}(t) &= \boxed{\boldsymbol{C}\boldsymbol{q}(t)} \end{aligned}$$

Linear Galerkin projection

MOR

$$\boldsymbol{q}(t) \approx \boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}(t)\,, \quad \boldsymbol{V} \in \mathbb{R}^{n\times r} \qquad r \ll n$$

Reduced order model (ROM)

$$\begin{aligned} \boldsymbol{M}_{\mathrm{r}}\ddot{\boldsymbol{q}}_{\mathrm{r}}(t) + \boldsymbol{D}_{\mathrm{r}}\dot{\boldsymbol{q}}_{\mathrm{r}}(t) + \boldsymbol{V}^{\mathsf{T}}\boldsymbol{f}\big(\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}(t)\big) &= \boldsymbol{B}_{\mathrm{r}}\boldsymbol{F}(t) \qquad \{\boldsymbol{q}_{\mathrm{r}}(0),\, \dot{\boldsymbol{q}}_{\mathrm{r}}(0)\} = (\boldsymbol{V}^{\mathsf{T}}\boldsymbol{V})^{-1}\boldsymbol{V}^{\mathsf{T}}\{\boldsymbol{q}_0,\, \dot{\boldsymbol{q}}_0\}\,, \\ \boldsymbol{y}_{\mathrm{r}}(t) &= \boldsymbol{C}_{\mathrm{r}}\boldsymbol{q}_{\mathrm{r}}(t) \end{aligned}$$

with

$$\{\boldsymbol{M}_{\mathrm{r}}, \boldsymbol{D}_{\mathrm{r}}\} = \boldsymbol{V}^{\mathsf{T}}\{\boldsymbol{M}, \boldsymbol{D}\}\boldsymbol{V}, \qquad \boldsymbol{f}_{\mathrm{r}}(\boldsymbol{q}) = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{f}(\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}})$$

Hyper-reduction!

$$\boldsymbol{B}_{\mathrm{r}} = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{B},$$

$$\boldsymbol{C}_{\mathrm{r}} = \boldsymbol{C}\boldsymbol{V},$$

In this talk: Dimensional reduction
How to choose $\boldsymbol{V}$?

Nonlinear dimensional reduction methods

Simulation-based approaches (e.g. POD)

Take snapshots of the simulated trajectory for typical (training) input force and perform SVD

$$\underset{(n,n_\mathrm{s})}{\boldsymbol{Q}} = \left[\boldsymbol{q}(t_1),\, \boldsymbol{q}(t_2),\, \cdots,\, \boldsymbol{q}(t_{n_\mathrm{s}})\right]$$

$$\boldsymbol{Q} \overset{\mathrm{SVD}}{=} \underset{(n,n)}{\boldsymbol{M}}\; \underset{(n,n_\mathrm{s})}{\boldsymbol{\Sigma}}\; \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}^\mathsf{T}} \approx \underset{(n,r)}{\boldsymbol{M}_\mathrm{r}}\; \underset{(r,n_\mathrm{s})}{\boldsymbol{\Sigma}_\mathrm{r}}\; \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}_\mathrm{r}^\mathsf{T}}$$

Reduction basis: $\boldsymbol{V} = \boldsymbol{M}_\mathrm{r} \in \mathbb{R}^{n\times r}$

$$\boldsymbol{q}(t) \approx \boldsymbol{V}\,\boldsymbol{q}_\mathrm{r}(t) = \sum_{i=1}^{r} \boldsymbol{v}_i\, q_{\mathrm{r},i}(t)$$

Simulation-free / System-theoretic methods

- **Basis augmentation:** Enrichment of a linear basis with nonlinear information

$$\boldsymbol{V}_\mathrm{aug} = \left[\boldsymbol{V}^{(1)},\, \boldsymbol{V}^{(2)}\right] \qquad \boldsymbol{q}(t) \approx \boldsymbol{V}_\mathrm{aug}\,\boldsymbol{q}_{\mathrm{r,aug}}(t)$$

+ : Easy projection
− : Higher reduced order

- **Nonlinear projection (e.g. Quadratic Manifold)**

$$\boldsymbol{V}^{(1)} \in \mathbb{R}^{n\times r}$$
$$\boldsymbol{V}^{(2)} \in \mathbb{R}^{n\times r^2}$$

$$\boldsymbol{q}(t) \approx \boldsymbol{V}^{(1)}\boldsymbol{q}_\mathrm{r}(t) + \boldsymbol{V}^{(2)}\left(\boldsymbol{q}_\mathrm{r}(t) \otimes \boldsymbol{q}_\mathrm{r}(t)\right)$$

+ : Smaller reduced order
− : Difficult projection

Reduced coordinates: $\boldsymbol{q}_\mathrm{r}(t) = \left[q_{\mathrm{r},1}(t), \cdots, q_{\mathrm{r},r}(t)\right]^\mathsf{T} = \left[\eta_1(t), \cdots, \eta_r(t)\right]^\mathsf{T}$

Original derivation for modal derivatives

Idea: Compose reduction basis with both *vibration modes* and *modal derivatives*

$$\boldsymbol{V}_{\text{aug}} = [\boldsymbol{\Phi}_r,\, \boldsymbol{\Theta}_{r^2}] \qquad \boldsymbol{\Phi}_r = [\boldsymbol{\phi}_{1,\text{eq}}, \dots, \boldsymbol{\phi}_{r,\text{eq}}] \in \mathbb{R}^{n\times r} \qquad \boldsymbol{\Theta}_{r^2} = [\boldsymbol{\theta}_{11}, \dots, \boldsymbol{\theta}_{rr}] \in \mathbb{R}^{n\times r^2}$$

1.) Vibration modes of the linearized model:

Linearization point: $\boldsymbol{q}_{\text{eq}}$

$$\boxed{(\boldsymbol{K}_{\text{eq}} - \omega_{i,\text{eq}}^2 \boldsymbol{M})\boldsymbol{\phi}_{i,\text{eq}} = \boldsymbol{0}} \qquad i = 1, \dots, r$$

$$\boldsymbol{K}_{\text{eq}} = \boldsymbol{K}(\boldsymbol{q}_{\text{eq}}) = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{q}(t))}{\partial \boldsymbol{q}(t)}\right|_{\boldsymbol{q}_{\text{eq}}}$$

Normalization condition: $\boxed{\boldsymbol{\phi}_{i,\text{eq}}^{\mathsf{T}} \boldsymbol{M}\, \boldsymbol{\phi}_{i,\text{eq}} = 1}$

2.) Perturbation of eigenmodes:

$$\frac{\partial}{\partial \eta_j(t)} \Big(\boldsymbol{K}(\boldsymbol{q}_{\text{eq}}) - \omega_i^2(\boldsymbol{q}_{\text{eq}})\boldsymbol{M}\Big)\boldsymbol{\phi}_i(\boldsymbol{q}_{\text{eq}}) = \boldsymbol{0}$$

$$\boxed{\left(\frac{\partial \boldsymbol{K}_{\text{eq}}}{\partial \eta_j(t)} - \frac{\partial \omega_{i,\text{eq}}^2}{\partial \eta_j(t)}\boldsymbol{M}\right)\boldsymbol{\phi}_{i,\text{eq}} + \left(\boldsymbol{K}_{\text{eq}} - \omega_{i,\text{eq}}^2 \boldsymbol{M}\right)\frac{\partial \boldsymbol{\phi}_{i,\text{eq}}}{\partial \eta_j(t)} = \boldsymbol{0}} \qquad \begin{array}{l} i = 1, \dots, r \\ j = 1, \dots, r \end{array}$$

Modal derivatives (MDs): $\boldsymbol{\theta}_{ij} = \frac{\partial \boldsymbol{\phi}_{i,\text{eq}}}{\partial \eta_j(t)}$

Modal derivatives

Calculation formula

$$\boldsymbol{\theta}_{ij} = \frac{\partial \boldsymbol{\phi}_{i,\mathrm{eq}}}{\partial \eta_j(t)}$$

$$\boxed{\left(\boldsymbol{K}_{\mathrm{eq}} - \omega_{i,\mathrm{eq}}^2 \boldsymbol{M}\right)\boldsymbol{\theta}_{ij} = \left(\frac{\partial \omega_{i,\mathrm{eq}}^2}{\partial \eta_j(t)}\boldsymbol{M} - \frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\right)\boldsymbol{\phi}_{i,\mathrm{eq}}}$$

$$i = 1, \dots, r \qquad j = 1, \dots, r$$

Singular linear system of equations

A x = b

Right-hand side (rhs)

- Derivative of eigenfrequencies: $\boldsymbol{\phi}_{i,\mathrm{eq}}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{\phi}_{i,\mathrm{eq}} = 1 \implies \boxed{\frac{\partial \omega_{i,\mathrm{eq}}^2}{\partial \eta_j(t)} = \boldsymbol{\phi}_{i,\mathrm{eq}}^{\mathsf{T}} \frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)} \boldsymbol{\phi}_{i,\mathrm{eq}}}$
- Finite difference scheme for tangential stiffness matrix:
  - Forward difference: $\left.\frac{\partial \boldsymbol{K}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} = \frac{\boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}} + \boldsymbol{\phi}_{j,\mathrm{eq}} \cdot h) - \boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}})}{h}$
  - Backward difference: $\left.\frac{\partial \boldsymbol{K}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} = \frac{\boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}}) - \boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}} - \boldsymbol{\phi}_{j,\mathrm{eq}} \cdot h)}{h}$
  - Central difference: $\left.\frac{\partial \boldsymbol{K}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} = \frac{\boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}} + \boldsymbol{\phi}_{j,\mathrm{eq}} \cdot h) - \boldsymbol{K}(\boldsymbol{q}_{\mathrm{eq}} - \boldsymbol{\phi}_{j,\mathrm{eq}} \cdot h)}{2h}$

Modal derivatives / Static modal derivatives

Handling the singular left-hand side (lhs)

By imposing an additional condition/constraint: $\frac{\partial}{\partial \eta_j(t)}\left(\boldsymbol{\phi}_{i,\mathrm{eq}}^{\mathsf{T}}\,\boldsymbol{M}\,\boldsymbol{\phi}_{i,\mathrm{eq}}\right) = 0 \implies \boxed{\boldsymbol{\phi}_{i,\mathrm{eq}}^{\mathsf{T}}\,\boldsymbol{M}\,\frac{\partial \boldsymbol{\phi}_{i,\mathrm{eq}}}{\partial \eta_j(t)} = 0}$

- Nelson's method: [Nelson '76]
- Direct method:

$$\begin{bmatrix} (\boldsymbol{K}_{\mathrm{eq}} - \omega_{i,\mathrm{eq}}^2 \boldsymbol{M}) & -\boldsymbol{M}\boldsymbol{\phi}_{i,\mathrm{eq}} \\ -(\boldsymbol{M}\boldsymbol{\phi}_{i,\mathrm{eq}})^{\mathsf{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_{ij} \\ \frac{\partial \omega_{i,\mathrm{eq}}^2}{\partial \eta_j(t)} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\boldsymbol{\phi}_{i,\mathrm{eq}} \\ 0 \end{bmatrix} \qquad \begin{matrix} i = 1, \ldots, r \\ j = 1, \ldots, r \end{matrix}$$

Excluding mass consideration

Mass terms are usually neglected, leading to the so-called *static modal derivatives* (SMDs)

$$\left(\boldsymbol{K}_{\mathrm{eq}} - \cancel{\omega_{i,\mathrm{eq}}^2 \boldsymbol{M}}\right)\frac{\partial \boldsymbol{\phi}_{i,\mathrm{eq}}}{\partial \eta_j(t)} = \left(\cancel{\frac{\partial \omega_{i,\mathrm{eq}}^2}{\partial \eta_j(t)}\boldsymbol{M}} - \frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\right)\boldsymbol{\phi}_{i,\mathrm{eq}}$$

$$\Rightarrow \quad \boxed{\boldsymbol{K}_{\mathrm{eq}}\left.\frac{\partial \boldsymbol{\phi}_{i,\mathrm{eq}}}{\partial \eta_j(t)}\right|_{\mathrm{s}} = -\frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\boldsymbol{\phi}_{i,\mathrm{eq}}} \qquad \begin{matrix} i = 1, \ldots, r \\ j = 1, \ldots, r \end{matrix}$$

Properties:

- Only one factorization of $\boldsymbol{K}_{\mathrm{eq}}$ is needed.
- Static modal derivatives are *symmetric:* $\boldsymbol{\theta}_{\mathrm{s},ij} = \left.\frac{\partial \boldsymbol{\phi}_{i,\mathrm{eq}}}{\partial \eta_j(t)}\right|_{\mathrm{s}} = \left.\frac{\partial \boldsymbol{\phi}_{j,\mathrm{eq}}}{\partial \eta_i(t)}\right|_{\mathrm{s}} = \boldsymbol{\theta}_{\mathrm{s},ji}.$

Polynomial system representation

Nonlinear second-order model

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{f}(\boldsymbol{q}(t)) &= \boldsymbol{B}\boldsymbol{F}(t) \qquad \boldsymbol{q}(0) = \boldsymbol{q}_0,\ \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0, \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{q}(t) \end{aligned}$$

Taylor series expansion

$$\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{q}_{\mathrm{eq}}) + \frac{\partial \boldsymbol{f}(\boldsymbol{q}_{\mathrm{eq}})}{\partial \boldsymbol{q}}(\boldsymbol{q}-\boldsymbol{q}_{\mathrm{eq}}) + \frac{1}{2!}\frac{\partial^2 \boldsymbol{f}(\boldsymbol{q}_{\mathrm{eq}})}{\partial \boldsymbol{q}^2}(\boldsymbol{q}-\boldsymbol{q}_{\mathrm{eq}})^{(2)} + \frac{1}{3!}\frac{\partial^3 \boldsymbol{f}(\boldsymbol{q}_{\mathrm{eq}})}{\partial \boldsymbol{q}^3}(\boldsymbol{q}-\boldsymbol{q}_{\mathrm{eq}})^{(3)} + \cdots$$

$\boldsymbol{q}_{\mathrm{eq}} = \boldsymbol{0}$

$$\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{K}^{(1)}\boldsymbol{q} + \boldsymbol{K}^{(2)}(\boldsymbol{q}\otimes\boldsymbol{q}) + \boldsymbol{K}^{(3)}(\boldsymbol{q}\otimes\boldsymbol{q}\otimes\boldsymbol{q}) + \cdots$$

Polynomial (cubic) nonlinear second-order model

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}(t) + \boldsymbol{K}^{(2)}\big(\boldsymbol{q}(t)\otimes\boldsymbol{q}(t)\big) + \boldsymbol{K}^{(3)}\big(\boldsymbol{q}(t)\otimes\boldsymbol{q}(t)\otimes\boldsymbol{q}(t)\big) = \boldsymbol{B}\boldsymbol{F}(t)$$

Symmetric tensors $\boldsymbol{K}^{(1)} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\mathcal{K}}^{(2)} \in \mathbb{R}^{n\times n\times n}$ and $\boldsymbol{\mathcal{K}}^{(3)} \in \mathbb{R}^{n\times n\times n\times n}$

$$K^{(1)}_{ab} = \frac{\partial f_a}{\partial q_b} = \frac{\partial^2 \mathcal{V}}{\partial q_a \partial q_b}, \quad \mathcal{K}^{(2)}_{abc} = \frac{1}{2}\frac{\partial^2 f_a}{\partial q_b \partial q_c} = \frac{1}{2}\frac{\partial^3 \mathcal{V}}{\partial q_a \partial q_b \partial q_c}, \quad \mathcal{K}^{(3)}_{abcd} = \frac{1}{6}\frac{\partial^3 f_a}{\partial q_b \partial q_c \partial q_d} = \frac{1}{6}\frac{\partial^4 \mathcal{V}}{\partial q_a \partial q_b \partial q_c \partial q_d}$$

Volterra theory and variational equations

For an input of the form $\alpha \boldsymbol{F}(t)$, it is assumed that the response is

$$\boldsymbol{q}(t) = \alpha \boldsymbol{q}_1(t) + \alpha^2 \boldsymbol{q}_2(t) + \ldots\,,$$
$$\ddot{\boldsymbol{q}}(t) = \alpha \ddot{\boldsymbol{q}}_1(t) + \alpha^2 \ddot{\boldsymbol{q}}_2(t) + \ldots\,.$$

aka. Poincaré expansion
[Nayfeh ’08], [Rugh ’81]

Inserting the assumed input and the assumed response into the polynomial system, yields

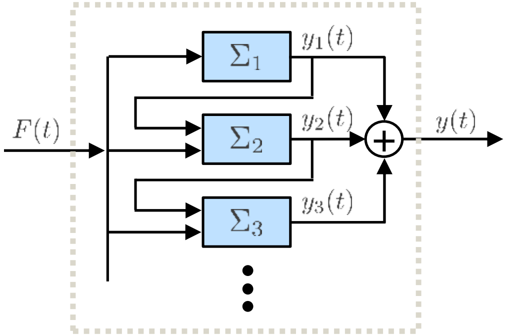
$$\boldsymbol{M}(\alpha \ddot{\boldsymbol{q}}_1(t) + \alpha^2 \ddot{\boldsymbol{q}}_2(t) + \ldots) \;+\; \boldsymbol{K}^{(1)}(\alpha \boldsymbol{q}_1(t) + \alpha^2 \boldsymbol{q}_2(t) + \ldots)$$
$$+\; \boldsymbol{K}^{(2)}\big(\boldsymbol{q}(t) \otimes \boldsymbol{q}(t)\big) \;+\; \boldsymbol{K}^{(3)}\big(\boldsymbol{q}(t) \otimes \boldsymbol{q}(t) \otimes \boldsymbol{q}(t)\big) = \boldsymbol{B}\,\alpha \boldsymbol{F}(t).$$

Equating coefficients of α^k, yields the *variational equations* ($\boldsymbol{D} = \boldsymbol{0}$)

$$\alpha: \quad \boldsymbol{M}\ddot{\boldsymbol{q}}_1(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}_1(t) = \boldsymbol{B}\,\boldsymbol{F}(t)\,, \qquad \boldsymbol{q}_1(0) = \boldsymbol{q}_0,$$

$$\alpha^2: \quad \boldsymbol{M}\ddot{\boldsymbol{q}}_2(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}_2(t) = -\boldsymbol{K}^{(2)}\big(\boldsymbol{q}_1(t) \otimes \boldsymbol{q}_1(t)\big)\,, \qquad \boldsymbol{q}_2(0) = \boldsymbol{0},$$

$$\alpha^3: \quad \boldsymbol{M}\ddot{\boldsymbol{q}}_3(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}_3(t) \qquad \boldsymbol{q}_3(0) = \boldsymbol{0},$$
$$= -\boldsymbol{K}^{(2)}\big(\boldsymbol{q}_1(t) \otimes \boldsymbol{q}_2(t) + \boldsymbol{q}_2(t) \otimes \boldsymbol{q}_1(t)\big) - \boldsymbol{K}^{(3)}\big(\boldsymbol{q}_1(t) \otimes \boldsymbol{q}_1(t) \otimes \boldsymbol{q}_1(t)\big)$$



series of cascaded subsystems

Novel derivation of modal derivatives

Modes (First subsystem)

Subsystem state-equation: $\boldsymbol{M}\ddot{\boldsymbol{q}}_1(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}_1(t) = \boldsymbol{B}\,\boldsymbol{F}(t)$

Ansatz for the homogeneous solution: $\boxed{\boldsymbol{q}_1(t) = \sum_{i=1}^{n} c_i\,\boldsymbol{\phi}_i \cos(\omega_i t)}$

Inserting ansatz (with $\ddot{\boldsymbol{q}}_1(t)$) in state-equation yields: $\left(\boldsymbol{K}^{(1)} - \omega_i^2\boldsymbol{M}\right)\boldsymbol{\phi}_i \sum_{i=1}^{n} c_i \cos(\omega_i t) = \boldsymbol{0},$

Modal derivatives (Second subsystem)

Subsystem state-equation: $\boldsymbol{M}\ddot{\boldsymbol{q}}_2(t) + \boldsymbol{K}^{(1)}\boldsymbol{q}_2(t) = -\boldsymbol{K}^{(2)}\left(\boldsymbol{q}_1(t) \otimes \boldsymbol{q}_1(t)\right)$

Ansatz for the particular solution: $\boxed{\boldsymbol{q}_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \tfrac{1}{2}c_i c_j \left(\widetilde{\boldsymbol{\theta}}_{ij}\cos\left((\omega_i+\omega_j)t\right) + \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij}\cos\left((\omega_i-\omega_j)t\right)\right)}$

(method of undetermined coeff.)

Inserting ansatz (with $\ddot{\boldsymbol{q}}_2(t)$ and $\boldsymbol{q}_1(t)\otimes\boldsymbol{q}_1(t)$) in state-equation yields exemplarily:

$$\boldsymbol{0} = \tfrac{1}{2}c_1^2\cos(2\omega_1 t)\underbrace{\left(\left(-(2\omega_1)^2\boldsymbol{M} + \boldsymbol{K}^{(1)}\right)\widetilde{\boldsymbol{\theta}}_{11} + \boldsymbol{K}^{(2)}\left(\boldsymbol{\phi}_1\otimes\boldsymbol{\phi}_1\right)\right)}_{=\boldsymbol{0}} + \tfrac{1}{2}c_1^2\underbrace{\left(\boldsymbol{K}^{(1)}\widetilde{\widetilde{\boldsymbol{\theta}}}_{11} + \boldsymbol{K}^{(2)}\left(\boldsymbol{\phi}_1\otimes\boldsymbol{\phi}_1\right)\right)}_{=\boldsymbol{0}} + \cdots$$

$$+ \tfrac{1}{2}c_1c_2\cos\left((\omega_1+\omega_2)\,t\right)\underbrace{\left(\left(-(\omega_1+\omega_2)^2\boldsymbol{M} + \boldsymbol{K}^{(1)}\right)\left(\widetilde{\boldsymbol{\theta}}_{12} + \widetilde{\boldsymbol{\theta}}_{21}\right)\boldsymbol{K}^{(2)}\left(\boldsymbol{\phi}_1\otimes\boldsymbol{\phi}_2 + \boldsymbol{\phi}_2\otimes\boldsymbol{\phi}_1\right)\right)}_{=\boldsymbol{0}} + \cdots$$

New modal derivatives

It follows from all brackets:

$$\left(\boldsymbol{K}^{(1)} - (\omega_i + \omega_j)^2 \boldsymbol{M}\right) \widetilde{\boldsymbol{\theta}}_{ij} = -\boldsymbol{K}^{(2)} \left(\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_j\right), \qquad i, j = 1, \dots, r\,,$$

$$\left(\boldsymbol{K}^{(1)} - (\omega_i - \omega_j)^2 \boldsymbol{M}\right) \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} = -\boldsymbol{K}^{(2)} \left(\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_j\right), \qquad i, j = 1, \dots, r\,.$$

Equivalent description for the right-hand side

$$\frac{\partial K^{(1)}_{ab}}{\partial \eta_j(t)} \, (\boldsymbol{\phi}_i)_b = \frac{\partial K^{(1)}_{ab}}{\partial q_c} \frac{\partial q_c}{\partial \eta_j(t)} \, (\boldsymbol{\phi}_i)_b := \frac{\partial^2 f_a}{\partial q_b \, \partial q_c} \, (\boldsymbol{\phi}_j)_c \, (\boldsymbol{\phi}_i)_b = 2 \, \mathcal{K}^{(2)}_{abc} \, (\boldsymbol{\phi}_j)_c \, (\boldsymbol{\phi}_i)_b.$$

$$\left.\frac{\partial \boldsymbol{K}^{(1)}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} \boldsymbol{\phi}_i := 2\, \boldsymbol{K}^{(2)} \left(\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_j\right) = \boldsymbol{K}^{(2)} \left(\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_j + \boldsymbol{\phi}_j \otimes \boldsymbol{\phi}_i\right).$$

Thus, the *new modal derivatives* are given by:

$$\boxed{\begin{aligned} \left(\boldsymbol{K}^{(1)} - (\omega_i + \omega_j)^2 \boldsymbol{M}\right) \widetilde{\boldsymbol{\theta}}_{ij} &= -\frac{1}{2} \left.\frac{\partial \boldsymbol{K}^{(1)}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} \boldsymbol{\phi}_i, \qquad i, j = 1, \dots, r\,, \\ \left(\boldsymbol{K}^{(1)} - (\omega_i - \omega_j)^2 \boldsymbol{M}\right) \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} &= -\frac{1}{2} \left.\frac{\partial \boldsymbol{K}^{(1)}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}} \boldsymbol{\phi}_i, \qquad i, j = 1, \dots, r\,. \end{aligned}}$$

Comments on the gained new derivatives

Conventional modal derivatives

$$\left(\boldsymbol{K}_{\mathrm{eq}} - \omega_{i,\mathrm{eq}}^2 \boldsymbol{M}\right)\boldsymbol{\theta}_{ij} = \left(\frac{\partial \omega_{i,\mathrm{eq}}^2}{\partial \eta_j(t)}\boldsymbol{M} - \frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\right)\boldsymbol{\phi}_{i,\mathrm{eq}}$$

$$\boldsymbol{K}_{\mathrm{eq}}\,\boldsymbol{\theta}_{\mathrm{s},ij} = -\frac{\partial \boldsymbol{K}_{\mathrm{eq}}}{\partial \eta_j(t)}\boldsymbol{\phi}_{i,\mathrm{eq}}$$

Properties:

- Single eigenfrequency
- Rhs of MDs and SMDs are different
- Neglection of mass → SMDs
- SMDs only obtained when neglecting mass
- MDs are (in general) NOT symmetric
- Singular linear system of equations

Volterra series-based modal derivatives

$$\left(\boldsymbol{K}^{(1)} - (\omega_i + \omega_j)^2 \boldsymbol{M}\right)\widetilde{\boldsymbol{\theta}}_{ij} = -\frac{1}{2}\left.\frac{\partial \boldsymbol{K}^{(1)}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}}\boldsymbol{\phi}_i,$$

$$\left(\boldsymbol{K}^{(1)} - (\omega_i - \omega_j)^2 \boldsymbol{M}\right)\widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} = -\frac{1}{2}\left.\frac{\partial \boldsymbol{K}^{(1)}(\boldsymbol{q})}{\partial \eta_j(t)}\right|_{\boldsymbol{q}_{\mathrm{eq}}}\boldsymbol{\phi}_i,$$

Properties:

- Sum/subtraction of eigenfrequencies
- Rhs for new MDs is the same as for SMDs!
- Neglection of mass → $\widetilde{\boldsymbol{\theta}}_{ij} = \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} = \boldsymbol{\theta}_{\mathrm{s},ij}$
- Cancelation of eigenfrequencies: $\widetilde{\widetilde{\boldsymbol{\theta}}}_{ii} = \boldsymbol{\theta}_{\mathrm{s},ij}$
- New MDs are symmetric!
- Regular linear system of equations
  - only singular, if sum/subtraction of eigenfrequencies is again eigenfrequency

Possible applications of the new derivatives

1.) Assess approximation quality of Volterra approximation

Compare analytical solution given by the first and second subsystem

$$\boldsymbol{q}_{1..2}(t) = \underbrace{\sum_{i=1}^{n} c_i \boldsymbol{\phi}_i \cos(\omega_i t)}_{\boldsymbol{q}_1(t)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \tfrac{1}{2} c_i c_j \left(\widetilde{\boldsymbol{\theta}}_{ij} \cos\left((\omega_i + \omega_j)t\right) + \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} \cos\left((\omega_i - \omega_j)t\right)\right)}_{\boldsymbol{q}_2(t)}$$

with simulated solution $\boldsymbol{q}_{\mathrm{sim}}(t)$ or solution given by superposing nonlinear normal modes

Reformulated analytical solution using trigonometric identities

$$\begin{aligned}\boldsymbol{q}_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \tfrac{1}{2} c_i c_j \left(\widetilde{\boldsymbol{\theta}}_{ij} \cos((\omega_i + \omega_j)t) + \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij} \cos((\omega_i - \omega_j)t)\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(\overline{\boldsymbol{\theta}}_{ij} \cos(\omega_i t)\cos(\omega_j t) - \widehat{\boldsymbol{\theta}}_{ij} \sin(\omega_i t)\sin(\omega_j t)\right)\end{aligned}$$

$$\boxed{\begin{aligned}\overline{\boldsymbol{\theta}}_{ij} &= \tfrac{1}{2}(\widetilde{\boldsymbol{\theta}}_{ij} + \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij}) \\ \widehat{\boldsymbol{\theta}}_{ij} &= \tfrac{1}{2}(\widetilde{\boldsymbol{\theta}}_{ij} - \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij})\end{aligned}}$$

$$\Rightarrow \boxed{\boldsymbol{q}_{1..2}(t) = \underbrace{\sum_{i=1}^{n} c_i \boldsymbol{\phi}_i \cos(\omega_i t)}_{\boldsymbol{q}_1(t)} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \left(\overline{\boldsymbol{\theta}}_{ij} \cos(\omega_i t)\cos(\omega_j t) - \widehat{\boldsymbol{\theta}}_{ij} \sin(\omega_i t)\sin(\omega_j t)\right)}_{\boldsymbol{q}_2(t)}}$$

Possible applications of the new derivatives

2.) Novel quadratic manifold approaches for model order reduction

- Common quadratic manifold approach

$$\boldsymbol{q}(t) \approx \sum_{i=1}^{r} \boldsymbol{\phi}_i q_{\mathrm{r},i}(t) \; + \sum_{i=1}^{r}\sum_{j=1}^{r} \boldsymbol{\theta}_{ij}\, q_{\mathrm{r},i}(t) q_{\mathrm{r},j}(t)$$
$$= \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\mathrm{r}}(t) + \boldsymbol{\Theta}_{r^2}\left(\boldsymbol{q}_{\mathrm{r}}(t) \otimes \boldsymbol{q}_{\mathrm{r}}(t)\right)$$

Using symmetrized MDs $\boldsymbol{\theta}_{ij}$
or static MDs (SMDs) $\boldsymbol{\theta}_{\mathrm{s},ij}$

- Novel quadratic manifold approach (1)

$$\boldsymbol{q}(t) \approx \sum_{i=1}^{r} \boldsymbol{\phi}_i q_{\mathrm{r},i}(t) \; + \sum_{i=1}^{r}\sum_{j=1}^{r} \overline{\boldsymbol{\theta}}_{ij}\, q_{\mathrm{r},i}(t) q_{\mathrm{r},j}(t)$$
$$= \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\mathrm{r}}(t) + \overline{\boldsymbol{\Theta}}_{r^2}\left(\boldsymbol{q}_{\mathrm{r}}(t) \otimes \boldsymbol{q}_{\mathrm{r}}(t)\right)$$

Reduced coordinates:

$$q_{\mathrm{r},i}(t) = c_i \cos(\omega_i t),$$
$$q_{\mathrm{r},i}(t)\, q_{\mathrm{r},j}(t) = c_i\, c_j \cos(\omega_i t) \cos(\omega_j t)$$

$$\boxed{\begin{aligned} \overline{\boldsymbol{\theta}}_{ij} &= \tfrac{1}{2}(\widetilde{\boldsymbol{\theta}}_{ij} + \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij}) \\ \widehat{\boldsymbol{\theta}}_{ij} &= \tfrac{1}{2}(\widetilde{\boldsymbol{\theta}}_{ij} - \widetilde{\widetilde{\boldsymbol{\theta}}}_{ij}) \end{aligned}}$$

- Novel quadratic manifold approach (2)

$$\boldsymbol{q}(t) \approx \sum_{i=1}^{r} \boldsymbol{\phi}_i\, q_{\mathrm{r},i}(t) + \sum_{i=1}^{r}\sum_{j=1}^{r} \overline{\boldsymbol{\theta}}_{ij}\, q_{\mathrm{r},i}(t) q_{\mathrm{r},j}(t) - \frac{1}{\omega_i \omega_j} \widehat{\boldsymbol{\theta}}_{ij}\, \dot{q}_{\mathrm{r},i}(t) \dot{q}_{\mathrm{r},j}(t)$$
$$= \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\mathrm{r}}(t) + \overline{\boldsymbol{\Theta}}_{r^2}\left(\boldsymbol{q}_{\mathrm{r}}(t) \otimes \boldsymbol{q}_{\mathrm{r}}(t)\right) - \widehat{\boldsymbol{\Theta}}_{r^2}\left(\dot{\boldsymbol{q}}_{\mathrm{r}}(t) \otimes \dot{\boldsymbol{q}}_{\mathrm{r}}(t)\right)$$

Reduced velocities:

$$\dot{q}_{\mathrm{r},i}(t) = -c_i\, \omega_i \sin(\omega_i t),$$
$$\dot{q}_{\mathrm{r},i}(t)\, \dot{q}_{\mathrm{r},j}(t) = c_i\, c_j\, \omega_i\, \omega_j \sin(\omega_i t) \sin(\omega_j t)$$

Possible applications of the new derivatives

Dimensional reduction – Roadmap and Workflow

Basis augmentation

$$\boldsymbol{q}(t) \approx \boldsymbol{V}_{\text{aug}}\, \boldsymbol{q}_{\text{r,aug}}(t)$$

$$\boldsymbol{V}_{\text{aug}} = [\boldsymbol{\Phi}_r,\, \boldsymbol{\Theta}_{r^2}]\,, \qquad \boldsymbol{V}_{\text{aug}} = [\boldsymbol{\Phi}_r,\, \boldsymbol{\Theta}_{\text{s}}]$$

$$\boldsymbol{V}_{\text{aug}} = \left[\boldsymbol{\Phi}_r,\, \widetilde{\boldsymbol{\Theta}},\, \widetilde{\widetilde{\boldsymbol{\Theta}}}\right], \qquad \boldsymbol{V}_{\text{aug}} = \left[\boldsymbol{\Phi}_r,\, \overline{\boldsymbol{\Theta}},\, \widehat{\boldsymbol{\Theta}}\right]$$

Quadratic manifold

$$\boldsymbol{q}(t) \approx \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\text{r}}(t) + \boldsymbol{\Theta}_{r^2}\left(\boldsymbol{q}_{\text{r}}(t) \otimes \boldsymbol{q}_{\text{r}}(t)\right)$$

$$\boldsymbol{q}(t) \approx \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\text{r}}(t) + \overline{\boldsymbol{\Theta}}_{r^2}\left(\boldsymbol{q}_{\text{r}}(t) \otimes \boldsymbol{q}_{\text{r}}(t)\right)$$

$$\boldsymbol{q}(t) \approx \boldsymbol{\Phi}_r\, \boldsymbol{q}_{\text{r}}(t) + \overline{\boldsymbol{\Theta}}_{r^2}\left(\boldsymbol{q}_{\text{r}}(t) \otimes \boldsymbol{q}_{\text{r}}(t)\right) - \widehat{\boldsymbol{\Theta}}_{r^2}\left(\dot{\boldsymbol{q}}_{\text{r}}(t) \otimes \dot{\boldsymbol{q}}_{\text{r}}(t)\right)$$

Application to nonlinear system

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{B}\boldsymbol{F}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q}$$

Application to polynomial system

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}^{(1)}\boldsymbol{q} + \boldsymbol{K}^{(2)}(\boldsymbol{q} \otimes \boldsymbol{q}) + \boldsymbol{K}^{(3)}(\boldsymbol{q} \otimes \boldsymbol{q} \otimes \boldsymbol{q}) = \boldsymbol{B}\boldsymbol{F}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q}$$

Evaluation in time-domain

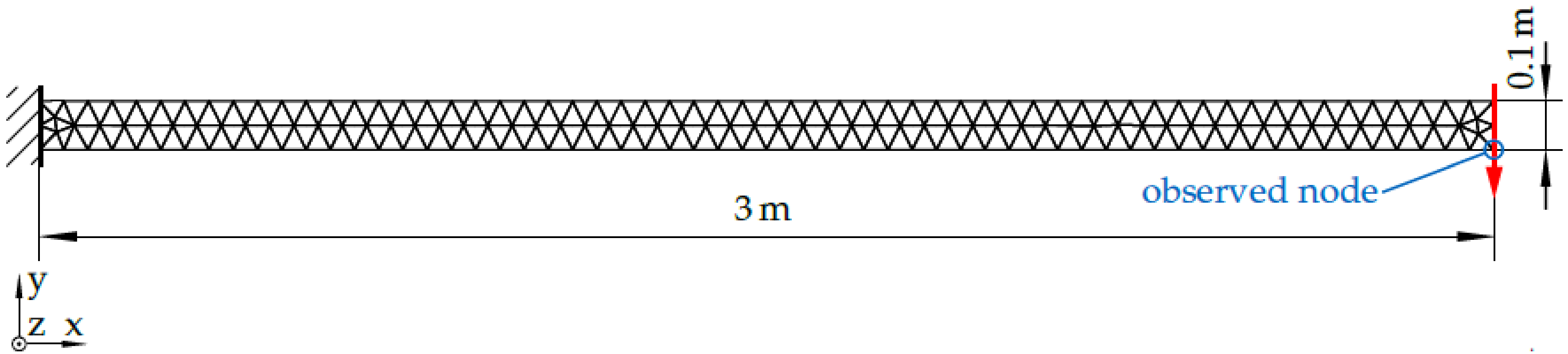
- Compare FOM and ROM via simulation runs for different inputs
- Compare ROMs with POD-ROM

Evaluation in frequency-domain

- Compute NNMs and NLFRF via shooting and path continuation
- Compare NNMs and NLFRFs of FOM and differently obtained ROMs

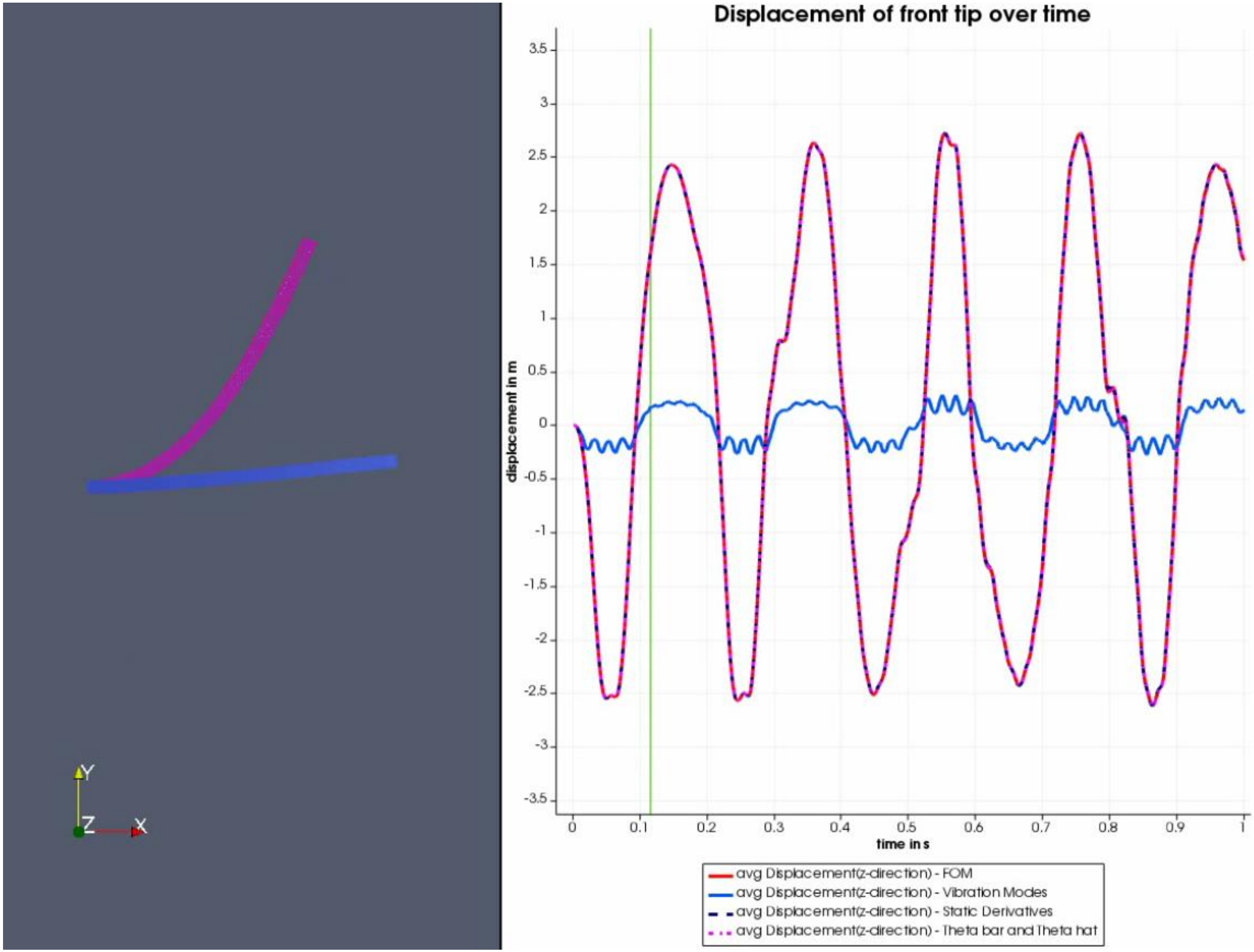
Preliminary simulation – Cantilever Beam

2D model of a cantilever beam



- 246 triangular Tri6 elements; 1224 dofs
- linear St. Venant-Kirchhoff material
- geometric nonlinear behaviour
- loading force at the tip in negative y-direction
- simulation conducted with open-source AMfe-code
- reduction via basis augmentation with new MDs

Preliminary simulation – Cantilever Beam



Summary & Outlook

Take-Home Messages:

- Model reduction with modal derivatives to capture (geometric) nonlinear behaviour
- Classical derivation of MDs is based on perturbation of the linearized eigenvalue problem
- Novel derivation based on Volterra series yields slightly different expressions for MDs
- Novel MDs are inherently symmetric; static derivatives can be retrieved from the new MDs
- Possible promising applications in nonlinear model order reduction

Ongoing / Future Work:

- Implementation of novel quadratic manifold approaches
- Validation of ROMs in time-, but also in frequency-domain (NLFRFs)
- Hyper-Reduction

Thank you for your attention!