Towards Output Krylov Subspace-Based Nonlinear Moment Matching

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1. INTRODUCTION

The transfer of the moment matching reduction concept from linear to nonlinear systems has been pioneered over the last years by Astolfi [2010a]. In this and further publications, the focus lies on the extension of *input Krylov subspace*-based moment matching to the nonlinear case. The generalization is based on the steady-state interpretation of moment matching and involves the difficult solution of a nonlinear partial differential equation (PDE).
Moreover, the time-domain interpretation of *output Krylov subspace*-based moment matching has been also investigated for linear systems (Astolfi [2010b], Ionescu [2016]) using the dual Sylvester equation, and transferred to the nonlinear case in Ionescu and Astolfi [2013, 2016].

The aim of this short contribution is (i) to comprehensively explain the steady-state perception of *output Krylov moment matching* given by Ionescu [2016]. Another goal is (ii) to state our progress concerning a practicable, *projective* output Krylov subspace-based nonlinear moment matching method, so that this and further topics can be discussed at the conference.

2. LINEAR MOMENT MATCHING

Consider a large-scale, linear time-invariant (LTI), asymptotically stable, state-space model of the form

$$\boldsymbol{E}\,\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t), \tag{1}$$

where $\det(\boldsymbol{E}) \neq 0$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{y}(t) \in \mathbb{R}^p$ denote the state, inputs and outputs of the system. The goal of model order reduction is to approximate the full order model (FOM) (1) by a reduced order model (ROM)

$$\boldsymbol{E}_{\mathrm{r}}\,\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \boldsymbol{A}_{\mathrm{r}}\,\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{B}_{\mathrm{r}}\,\boldsymbol{u}(t), \quad \boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{C}_{\mathrm{r}}\,\boldsymbol{x}_{\mathrm{r}}(t), \tag{2}$$

of much lower dimension $r \ll n$ with $\boldsymbol{E}_{\mathrm{r}} = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{V}$, $\boldsymbol{A}_{\mathrm{r}} = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{V}$, $\boldsymbol{B}_{\mathrm{r}} = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{B}$ and $\boldsymbol{C}_{\mathrm{r}} = \boldsymbol{C}\boldsymbol{V}$, such that $\boldsymbol{y}(t) \approx \boldsymbol{y}_{\mathrm{r}}(t)$. The main task in this *projection-based* setting consists in finding suitable (orthogonal) reduction matrices $\boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^{n\times r}$ that span appropriate subspaces.

2.1 Frequency-domain perception of moment matching

The transfer function of (1) is $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{B}$.

Definition 1. The moments $\boldsymbol{m}_i(\sigma)$ of $\boldsymbol{G}(s)$ at a complex expansion point $\sigma \in \mathbb{C}$ are given by

$$\boldsymbol{m}_i(\sigma) = (-1)^i\,\boldsymbol{C}\big((\sigma\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{E}\big)^i(\sigma\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{B} \in \mathbb{R}^{p\times m}.$$

If $\boldsymbol{W}$ is chosen as basis of an *output* Krylov subspace

$$\mathrm{span}\left\{(\mu_1\boldsymbol{E}-\boldsymbol{A})^{-\mathsf{T}}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{l}_1, \ldots, (\mu_r\boldsymbol{E}-\boldsymbol{A})^{-\mathsf{T}}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{l}_r\right\} \subseteq \mathrm{ran}(\boldsymbol{W}), \tag{3}$$

then *left* tangential multipoint interpolation is achieved:

$$\boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{G}(\mu_i) = \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{G}_{\mathrm{r}}(\mu_i) \quad \Leftrightarrow \quad \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_0(\mu_i) = \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_{\mathrm{r},0}(\mu_i). \tag{4}$$

Hereby, the 0-th tangential output moments are defined as $\boldsymbol{m}_0^{\mathsf{T}}(\mu_i, \boldsymbol{l}_i) := \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_0(\mu_i) = \boldsymbol{w}_i^{\mathsf{T}}\,\boldsymbol{B}$ with $\boldsymbol{w}_i^{\mathsf{T}} = \boldsymbol{l}_i^{\mathsf{T}}\boldsymbol{C}(\mu_i\boldsymbol{E}-\boldsymbol{A})^{-1}$. Suitable shifts $\mu_i \in \mathbb{C}\setminus\lambda(\boldsymbol{E}^{-1}\boldsymbol{A})$ and left tangential directions $\boldsymbol{l}_i \in \mathbb{C}^p$ should be chosen for a good approximation.

Any basis of an output Krylov subspace can be interpreted as the solution $\boldsymbol{W}$ of the following Sylvester equation:

$$\boldsymbol{E}^{\mathsf{T}}\,\boldsymbol{W}\,\boldsymbol{S}_w^{\mathsf{T}} - \boldsymbol{A}^{\mathsf{T}}\,\boldsymbol{W} = \boldsymbol{C}^{\mathsf{T}}\,\boldsymbol{L}, \tag{5}$$

where $\boldsymbol{S}_w = \mathrm{diag}(\mu_1, \ldots, \mu_r) \in \mathbb{C}^{r\times r}$ and $\boldsymbol{L} = [\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r] \in \mathbb{C}^{p\times r}$, such that the pair $(\boldsymbol{S}_w, \boldsymbol{L}^{\mathsf{T}})$ is controllable.

2.2 Time-domain perception of moment matching

Theorem 1. Consider the signal generator

$$\dot{\boldsymbol{x}}_{\mathrm{r}}^w(t) = \boldsymbol{S}_w\,\boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{L}^{\mathsf{T}}\,\boldsymbol{y}(t), \quad \boldsymbol{x}_{\mathrm{r}}^w(0) = \boldsymbol{x}_{\mathrm{r},0}^w, \tag{6a}$$

$$\boldsymbol{d}(t) = \boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{W}^{\mathsf{T}}\,\boldsymbol{E}\,\boldsymbol{x}(t). \tag{6b}$$

Consider the interconnection between (1) and (6) as in Fig. 1. Let $\boldsymbol{W}$ be the unique solution of (5) and $\boldsymbol{V}$ such that $\det(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{V}) \neq 0$. Furthermore, let $\boldsymbol{x}_{\mathrm{r},0}^w = \boldsymbol{0}$. Then, the steady-state response of $\boldsymbol{d}(t)$ and $\boldsymbol{\varepsilon}(t)$ match, i.e. $\boldsymbol{d}_{\mathrm{ss}}(t) = \boldsymbol{\varepsilon}_{\mathrm{ss}}(t)$ (see Fig. 1).

Lemma 1. For an asymptotically stable FOM, $\boldsymbol{u}(t) = \boldsymbol{0}$, $\boldsymbol{x}_0 \neq \boldsymbol{0}$ arbitrary and $\boldsymbol{x}_{\mathrm{r},0}^w = \boldsymbol{0}$, the steady-state response $(t \to \infty)$ of $\boldsymbol{x}_{\mathrm{r}}^w(t)$ is given by $\boldsymbol{x}_{\mathrm{r,ss}}^w(t) = -\mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}_0$, where $\boldsymbol{w}_i^{\mathsf{T}} = \boldsymbol{l}_i^{\mathsf{T}}\boldsymbol{C}(\mu_i\boldsymbol{E} - \boldsymbol{A})^{-1}$. Due to (6b) and $\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{E}^{-1}\boldsymbol{A}t}\boldsymbol{x}_0 \overset{t\to\infty}{\to} \boldsymbol{0}$, the steady-state of $\boldsymbol{d}(t)$ is $\boldsymbol{d}_{\mathrm{ss}}(t) = \boldsymbol{x}_{\mathrm{r,ss}}^w(t)$.

Further note that the interconnected system ((1) and (6)) is equivalent to the system (cf. Fig. 1)

$$\dot{\boldsymbol{d}}(t) = \boldsymbol{S}_w\underbrace{\big(\boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{x}(t)\big)}_{\boldsymbol{d}(t)} - \boldsymbol{W}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{u}(t), \; \boldsymbol{d}(0) = -\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{x}_0.$$

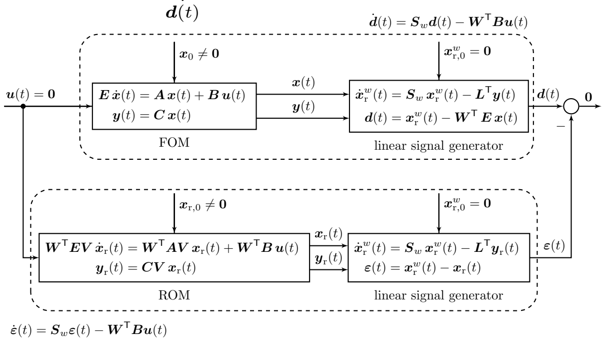

Fig. 1. Time-domain interpretation of $\boldsymbol{W}$-sided moment matching for linear systems ("FOM/ROM drives the signal generator").

Thus, the solution for $\boldsymbol{u}(t)=\boldsymbol{0}$ is given by $\boldsymbol{d}(t)=\mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{d}(0)=-\mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}_0$, which corresponds to the solution before. Moreover, for $\boldsymbol{u}(t)=\boldsymbol{0}$ and $\boldsymbol{x}_\mathrm{r}^w(t)=\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\boldsymbol{x}(t)$ it follows

$$\dot{\boldsymbol{d}}(t)\Big|_{\boldsymbol{x}_\mathrm{r}^w=\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\boldsymbol{x},\,\boldsymbol{u}=\boldsymbol{0}}=\boldsymbol{0}\,. \tag{7}$$

The $r\times n$ Sylvester equation can be derived as follows. First insert the linear approximation ansatz $\boldsymbol{x}_\mathrm{r}(t)=\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}(t)$ with $\boldsymbol{x}_\mathrm{r}(t)\overset{!}{=}\boldsymbol{x}_\mathrm{r}^w(t)$ in (6a). Then, the linear system (1) for $\boldsymbol{u}(t)=\boldsymbol{0}$ and arbitrary $\boldsymbol{x}_0\neq\boldsymbol{0}$ is plugged in, yielding

$$\boldsymbol{0}=\Big(\boldsymbol{S}_w\,\boldsymbol{W}^\mathsf{T}\boldsymbol{E}-\boldsymbol{W}^\mathsf{T}\boldsymbol{A}-\boldsymbol{L}^\mathsf{T}\boldsymbol{C}\Big)\cdot\boldsymbol{x}(t). \tag{8}$$

3. NONLINEAR MOMENT MATCHING

Consider now a large-scale, nonlinear time-invariant, in $\boldsymbol{x}_\mathrm{eq}=\boldsymbol{0}$ exponentially stable, state-space model of the form

$$\begin{aligned}\boldsymbol{E}\,\dot{\boldsymbol{x}}(t)&=\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big),\quad \boldsymbol{x}(0)=\boldsymbol{x}_0,\\ \boldsymbol{y}(t)&=\boldsymbol{h}\big(\boldsymbol{x}(t)\big),\end{aligned} \tag{9}$$

with smooth mappings $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}):\mathbb{R}^n\times\mathbb{R}^m\rightarrow\mathbb{R}^n$ and $\boldsymbol{h}(\boldsymbol{x}):\mathbb{R}^n\rightarrow\mathbb{R}^p$. The reduction is performed by a nonlinear Petrov-Galerkin projection using the mappings $\boldsymbol{x}(t)\approx\boldsymbol{\nu}(\boldsymbol{x}_\mathrm{r}(t))$ and $\boldsymbol{\omega}(\boldsymbol{x}(t))$, together with their corresponding Jacobians $\widetilde{\boldsymbol{V}}(\boldsymbol{x}_\mathrm{r})=\partial\boldsymbol{\nu}(\boldsymbol{x}_\mathrm{r})/\partial\boldsymbol{x}_\mathrm{r}$, $\widetilde{\boldsymbol{W}}(\boldsymbol{x})^\mathsf{T}=(\partial\boldsymbol{\omega}(\boldsymbol{x})/\partial\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{\nu}(\boldsymbol{x}_\mathrm{r})}$. This yields the nonlinear ROM

$$\begin{aligned}\widetilde{\boldsymbol{E}}_\mathrm{r}\,\dot{\boldsymbol{x}}_\mathrm{r}(t)&=\frac{\partial\boldsymbol{\omega}(\boldsymbol{x}(t))}{\partial\boldsymbol{x}(t)}\,\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big)\Big|_{\boldsymbol{x}(t)=\boldsymbol{\nu}(\boldsymbol{x}_\mathrm{r}(t))},\\ \boldsymbol{y}_\mathrm{r}(t)&=\boldsymbol{h}\big(\boldsymbol{\nu}(\boldsymbol{x}_\mathrm{r}(t))\big),\end{aligned} \tag{10}$$

with $\widetilde{\boldsymbol{E}}_\mathrm{r}=\widetilde{\boldsymbol{W}}(\boldsymbol{x})^\mathsf{T}\boldsymbol{E}\,\widetilde{\boldsymbol{V}}(\boldsymbol{x}_\mathrm{r})$ and the initial condition $\boldsymbol{x}_\mathrm{r}(0)=\arg\min_{\boldsymbol{x}_{\mathrm{r},0}}\|\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r},0})-\boldsymbol{x}_0\|_2^2$ (cf. `lsqnonlin` in MATLAB).

3.1 Time-domain perception of $\boldsymbol{W}$-sided moment matching

Theorem 2. Consider the interconnection of the nonlinear system (9) with the nonlinear signal generator

$$\dot{\boldsymbol{x}}_\mathrm{r}^w(t)=\boldsymbol{s}_w\big(\boldsymbol{x}_\mathrm{r}^w(t),\boldsymbol{y}(t)\big),\quad \boldsymbol{x}_\mathrm{r}^w(0)=\boldsymbol{x}_{\mathrm{r},0}^w, \tag{11a}$$

$$\boldsymbol{d}(t)=\boldsymbol{\Omega}\big(\boldsymbol{x}_\mathrm{r}^w(t),\boldsymbol{x}(t)\big), \tag{11b}$$

where $\boldsymbol{s}_w\big(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{y}\big):\mathbb{R}^r\times\mathbb{R}^p\rightarrow\mathbb{R}^r$ and $\boldsymbol{\Omega}\big(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{x}\big):\mathbb{R}^r\times\mathbb{R}^n\rightarrow\mathbb{R}^r$ are smooth mappings such that $\boldsymbol{s}_w(\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}$, $\boldsymbol{\Omega}(\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}$ and $\big(\partial\boldsymbol{\Omega}(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{x})/\partial\boldsymbol{x}_\mathrm{r}^w\big)\big|_{(\boldsymbol{0},\boldsymbol{0})}$ is full rank. Further assume that there exists a smooth mapping $\boldsymbol{\omega}(\boldsymbol{x}):\mathbb{R}^n\rightarrow\mathbb{R}^r$ such that $\boldsymbol{\Omega}(\boldsymbol{\omega}(\boldsymbol{x}),\boldsymbol{x})=\boldsymbol{0}$, i.e. $\boldsymbol{d}$ restricted to the manifold $\boldsymbol{x}_\mathrm{r}^w=\boldsymbol{\omega}(\boldsymbol{x})$ is zero. Then, the steady-state response of $\boldsymbol{d}(t)$ and $\boldsymbol{\varepsilon}(t)$ match (see Fig. 2), where the mapping $\boldsymbol{\Omega}(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{x})$ is the unique solution of the following PDE

$$\frac{\partial\boldsymbol{\Omega}(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{0})=-\frac{\partial\boldsymbol{\Omega}(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{x})}{\partial\boldsymbol{x}_\mathrm{r}^w}\boldsymbol{s}_w\big(\boldsymbol{x}_\mathrm{r}^w,\boldsymbol{h}(\boldsymbol{x})\big)\Big|_{\boldsymbol{x}_\mathrm{r}^w=\boldsymbol{\omega}(\boldsymbol{x})}.$$

3.2 Nonlinear input-affine case

Consider now a large-scale, nonlinear time-invariant system (9) with $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})=\tilde{\boldsymbol{f}}(\boldsymbol{x})+\boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$, where $\tilde{\boldsymbol{f}}(\boldsymbol{x}):\mathbb{R}^n\rightarrow\mathbb{R}^n$ and $\boldsymbol{G}(\boldsymbol{x}):\mathbb{R}^n\rightarrow\mathbb{R}^{n\times m}$ are smooth mappings.

Theorem 3. Consider the interconnection of a nonlinear (input-affine) system with the input-affine generator

$$\dot{\boldsymbol{x}}_\mathrm{r}^w(t)=\overbrace{\tilde{\boldsymbol{s}}_w\big(\boldsymbol{x}_\mathrm{r}^w(t)\big)-\boldsymbol{L}\big(\boldsymbol{x}_\mathrm{r}^w(t)\big)\boldsymbol{y}(t)}^{\boldsymbol{s}_w\left(\boldsymbol{x}_\mathrm{r}^w(t),\,\boldsymbol{y}(t)\right)},\ \boldsymbol{x}_\mathrm{r}^w(0)=\boldsymbol{x}_{\mathrm{r},0}^w, \tag{12a}$$

$$\boldsymbol{d}(t)=\underbrace{\boldsymbol{x}_\mathrm{r}^w(t)-\boldsymbol{\omega}\big(\boldsymbol{x}(t)\big)}_{\boldsymbol{\Omega}\left(\boldsymbol{x}_\mathrm{r}^w(t),\,\boldsymbol{x}(t)\right)}, \tag{12b}$$

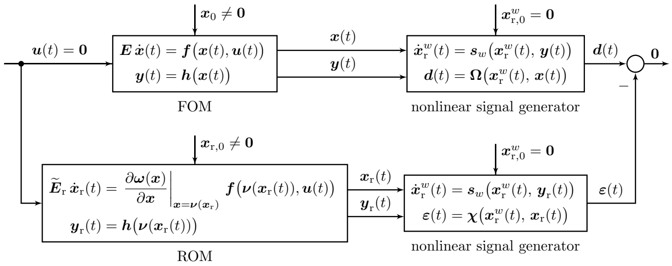

Fig. 2. Time-domain interpretation of $\boldsymbol{W}$-sided moment matching for nonlinear systems (adapted from Ionescu [2016]).

where $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w):\mathbb{R}^r\rightarrow\mathbb{R}^r$, $\boldsymbol{L}(\boldsymbol{x}_\mathrm{r}^w):\mathbb{R}^r\rightarrow\mathbb{R}^{r\times p}$ and $\boldsymbol{\omega}(\boldsymbol{x}):\mathbb{R}^n\rightarrow\mathbb{R}^r$. Then, the steady-state of $\boldsymbol{d}(t)$ and $\boldsymbol{\varepsilon}(t)$ match, where $\boldsymbol{\omega}(\boldsymbol{x})$ is the unique solution of the PDE

$$\frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{0})=\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w)-\boldsymbol{L}(\boldsymbol{x}_\mathrm{r}^w)\boldsymbol{h}(\boldsymbol{x})\Big|_{\boldsymbol{x}_\mathrm{r}^w=\boldsymbol{\omega}(\boldsymbol{x})}. \tag{13}$$

The Sylvester-like PDE (13) can be derived as follows. First, the nonlinear approximation ansatz $\boldsymbol{x}_\mathrm{r}(t)=\boldsymbol{\omega}(\boldsymbol{x}(t))$ with $\boldsymbol{x}_\mathrm{r}(t)\overset{!}{=}\boldsymbol{x}_\mathrm{r}^w(t)$ is substituted in (12a). Afterwards, the nonlinear system $\boldsymbol{E}\dot{\boldsymbol{x}}(t)=\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big)$, $\boldsymbol{y}(t)=\boldsymbol{h}\big(\boldsymbol{x}(t)\big)$ for $\boldsymbol{u}(t)=\boldsymbol{0}$ is plugged in, yielding the $r\times 1$ equation (13).

4. PRACTICABLE $\boldsymbol{W}$-SIDED MODEL REDUCTION BY NONLINEAR MOMENT MATCHING

The first step towards a practical method consists in applying a linear projection $\boldsymbol{x}_\mathrm{r}^w(t)=\boldsymbol{\omega}(\boldsymbol{x}(t))=\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}(t)$. (cf. Cruz Varona et al. [2019]).

Nonlinear signal generator In this case, the PDE (13) becomes the following nonlinear system of equations

$$\boldsymbol{W}^\mathsf{T}\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{0}\big)=\tilde{\boldsymbol{s}}_w\big(\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}(t)\big)-\boldsymbol{L}\big(\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}(t)\big)\,\boldsymbol{h}\big(\boldsymbol{x}(t)\big), \tag{14}$$

where the triple $(\tilde{\boldsymbol{s}}_w,\boldsymbol{L},\boldsymbol{x}_0)$ is user-defined. Note that the underdetermined system consists of r equations for $r\cdot n$ unknowns in $\boldsymbol{W}^\mathsf{T}\in\mathbb{R}^{r\times n}$, and a row-wise consideration for each $\boldsymbol{w}_i^\mathsf{T}\in\mathbb{R}^{1\times n}$, $i=1,\ldots,r$ does not help any further.

Linear signal generator Interconnecting the nonlinear system (9) with the linear signal generator (6), where $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w(t))=\boldsymbol{S}_w\,\boldsymbol{x}_\mathrm{r}^w(t)$ and $\boldsymbol{L}(\boldsymbol{x}_\mathrm{r}^w(t))=\boldsymbol{L}^\mathsf{T}$, yields

$$\boldsymbol{W}^\mathsf{T}\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{0}\big)=\boldsymbol{S}_w\,\boldsymbol{W}^\mathsf{T}\boldsymbol{E}\,\boldsymbol{x}(t)-\boldsymbol{L}^\mathsf{T}\,\boldsymbol{h}\big(\boldsymbol{x}(t)\big), \tag{15}$$

which is a *linear* system of equations.

Zero signal generator This special (linear) signal generator, where $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w(t))=\boldsymbol{0}$ and $\boldsymbol{L}(\boldsymbol{x}_\mathrm{r}^w(t))=\boldsymbol{L}^\mathsf{T}$, yields

$$\boldsymbol{W}^\mathsf{T}\boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{0}\big)=-\boldsymbol{L}^\mathsf{T}\,\boldsymbol{h}\big(\boldsymbol{x}(t)\big), \tag{16}$$

which is again a *linear* system of equations.

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Abstract

In this contribution, we report our progress concerning a practicable, projective method for output nonlinear moment matching. First, we explain the time-domain interpretation of output Krylov subspace-based moment matching for linear systems. Then, based on [Ionescu and Astolfi 2016], the steady-state perception of output moments and moment matching for nonlinear systems is given. Finally, some simplifications to approximate the solution of the arising partial differential equation (PDE) are proposed towards a practical, numerical algorithm for nonlinear model order reduction.

Linear systems

Linear time-invariant systems

Consider a large-scale ($n \gg 10^3$), linear time-invariant (LTI), asymptotically stable, MIMO state-space model of the form:

$$\boldsymbol{E}\,\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t),$$

with non-singular descriptor matrix, i.e. $\det(\boldsymbol{E}) \neq 0$.

The input-output behavior of LTI systems is characterized in the frequency-domain by the transfer function matrix

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{B} \in \mathbb{C}^{p \times m}.$$

Projective Model Order Reduction

The goal of model order reduction is to find a reduced order model (ROM) of much lower dimension $r \ll n$:

$$\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{V}\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{V}\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{W}^{\mathsf{T}}\boldsymbol{B}\,\boldsymbol{u}(t), \quad \boldsymbol{x}_{\mathrm{r}}(0) = \boldsymbol{x}_{\mathrm{r},0},$$
$$\boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{C}\boldsymbol{V}\boldsymbol{x}_{\mathrm{r}}(t),$$

with $\boldsymbol{x}_{\mathrm{r}}(0) = (\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{V})^{-1}\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(0)$, such that $y(t) \approx y_{\mathrm{r}}(t)$.

In this *Petrov-Galerkin projection* setting, the main task is to find suitable reduction matrices $\boldsymbol{V}, \boldsymbol{W}$. One numerically efficient linear reduction technique relies on the concept of implicit moment matching by *rational Krylov subspaces*.

Frequency-domain Output Moment Matching

If $\boldsymbol{W}$ is chosen as basis of an *output* Krylov subspace

$$\mathrm{span}\left\{(\mu_1\boldsymbol{E} - \boldsymbol{A})^{-\mathsf{T}}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{l}_1, \ldots, (\mu_r\boldsymbol{E} - \boldsymbol{A})^{-\mathsf{T}}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{l}_r\right\} \subseteq \mathrm{ran}(\boldsymbol{W}),$$

then the ROM fulfills the *left* tangential multipoint conditions

$$\boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{G}(\mu_i) = \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{G}_{\mathrm{r}}(\mu_i) \quad \Leftrightarrow \quad \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_0(\mu_i) = \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_{\mathrm{r},0}(\mu_i).$$

Hereby, the 0th tangential output moments are defined as

$$\boldsymbol{m}_0^{\mathsf{T}}(\mu_i, \boldsymbol{l}_i) := \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{m}_0(\mu_i) = \boldsymbol{w}_i^{\mathsf{T}}\,\boldsymbol{B}, \qquad \boldsymbol{w}_i^{\mathsf{T}} = \boldsymbol{l}_i^{\mathsf{T}}\,\boldsymbol{C}(\mu_i\boldsymbol{E} - \boldsymbol{A})^{-1}.$$

$\boldsymbol{W}$ can be also interpreted as solution of the Sylvester equation

$$\boldsymbol{E}^{\mathsf{T}}\,\boldsymbol{W}\,\boldsymbol{S}_w^{\mathsf{T}} - \boldsymbol{A}^{\mathsf{T}}\,\boldsymbol{W} = \boldsymbol{C}^{\mathsf{T}}\,\boldsymbol{L}.$$

Steady-state of interconnected system

Consider the signal generator

$$\dot{\boldsymbol{x}}_{\mathrm{r}}^w(t) = \boldsymbol{S}_w\,\boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{L}^{\mathsf{T}}\,\boldsymbol{y}(t), \quad \boldsymbol{x}_{\mathrm{r}}^w(0) = \boldsymbol{x}_{\mathrm{r},0}^w,$$
$$\boldsymbol{d}(t) = \boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{W}^{\mathsf{T}}\,\boldsymbol{E}\,\boldsymbol{x}(t),$$

with $\boldsymbol{S}_w = \mathrm{diag}(\mu_1, \ldots, \mu_r) \in \mathbb{C}^{r \times r}$ and $\boldsymbol{L} = [\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r] \in \mathbb{C}^{p \times r}$.

The steady-state of the interconnected system from Fig. 1 is

$$\boldsymbol{x}_{\mathrm{r}}^w(t) = \int_0^t -\mathrm{e}^{\boldsymbol{S}_w(t-\tau)}\boldsymbol{L}^{\mathsf{T}}\boldsymbol{C}\,\mathrm{e}^{\boldsymbol{E}^{-1}\boldsymbol{A}\tau}\boldsymbol{x}_0\,\mathrm{d}\tau \;+\; \mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{x}_{\mathrm{r},0}^w,$$

$$x_{\mathrm{r},i}^w(t) = \mathrm{e}^{\mu_i t}\big(\underbrace{x_{\mathrm{r},0,i}^w}_{0} - \boldsymbol{l}_i^{\mathsf{T}}\boldsymbol{C}(\mu_i\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{E}\,\boldsymbol{x}_0\big) + \boldsymbol{l}_i^{\mathsf{T}}\boldsymbol{C}(\mu_i\boldsymbol{E} - \boldsymbol{A})^{-1}\underbrace{\mathrm{e}^{\boldsymbol{E}^{-1}\boldsymbol{A}t}}_{\boldsymbol{0}}\boldsymbol{x}_0$$

$$\overset{t\to\infty}{\longrightarrow} -\mathrm{e}^{\mu_i t}\boldsymbol{w}_i^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}_0 \quad \text{and} \quad \boldsymbol{d}_{\mathrm{ss}}(t) = \boldsymbol{x}_{\mathrm{r,ss}}^w(t) = -\mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}_0.$$

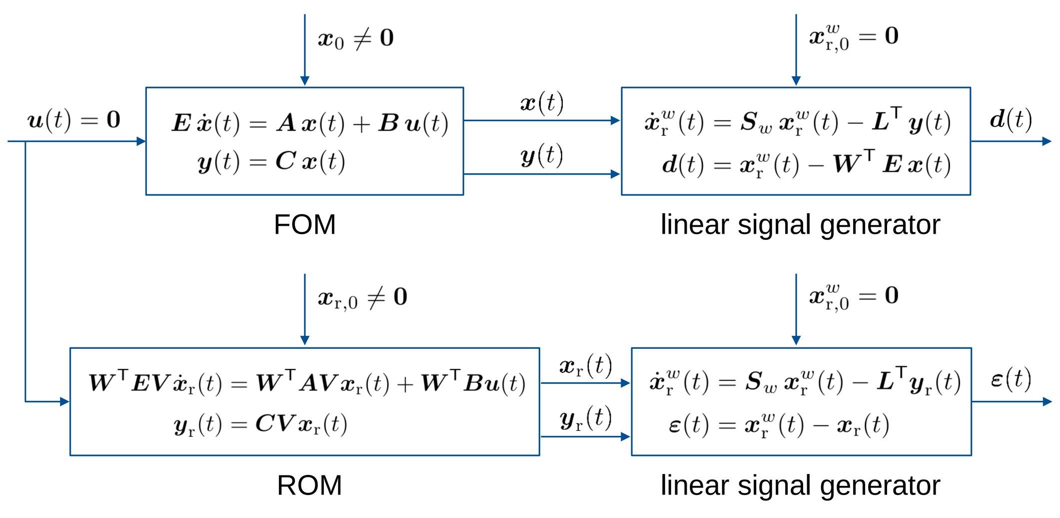

Fig. 1. Time-domain interpretation of output moment matching for linear systems: "System drives the signal generator".

Time-domain Output Moment Matching

The 0th tangential output moments at $\{\mu_i, \boldsymbol{l}_i\}$ are related to the (well-defined) steady-state of $\boldsymbol{d}(t)$ from Fig. 1:

$$\boldsymbol{m}_0^{\mathsf{T}}(\mu_i, \boldsymbol{l}_i) = \boldsymbol{w}_i^{\mathsf{T}}\,\boldsymbol{B} \;\Leftrightarrow\; \boldsymbol{d}_{\mathrm{ss}}(t) = -\mathrm{e}^{\boldsymbol{S}_w t}\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}_0 = -\begin{bmatrix}\mathrm{e}^{\mu_1 t}\boldsymbol{w}_1^{\mathsf{T}}\boldsymbol{E}\boldsymbol{x}_0\\ \vdots \\ \mathrm{e}^{\mu_r t}\boldsymbol{w}_r^{\mathsf{T}}\boldsymbol{E}\boldsymbol{x}_0\end{bmatrix},$$

where $\boldsymbol{W}$ is the unique solution of the Sylvester equation.

Thus, output moment matching in time-domain corresponds to the interpolation of the steady-state of $\boldsymbol{d}(t)$ and $\boldsymbol{\varepsilon}(t)$:

$$d_{\mathrm{ss},i}(t) = -\mathrm{e}^{\mu_i t}\boldsymbol{w}_i^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}_0 \equiv -\mathrm{e}^{\mu_i t}\boldsymbol{l}_i^{\mathsf{T}}\boldsymbol{C}_{\mathrm{r}}(\mu_i\boldsymbol{E}_{\mathrm{r}} - \boldsymbol{A}_{\mathrm{r}})^{-1}\boldsymbol{E}_{\mathrm{r}}\,\boldsymbol{x}_{\mathrm{r},0} = \varepsilon_{\mathrm{ss},i}(t),$$

where $\boldsymbol{V}$ is arbitrary but such that $\det(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{V}) \neq 0$.

Nonlinear systems

Nonlinear time-invariant systems

Consider now a large-scale, nonlinear time-invariant (NLTI), in $\boldsymbol{x}_{\mathrm{eq}} = \boldsymbol{0}$ exponentially stable, MIMO state-space model of the form:

$$\boldsymbol{E}\,\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$
$$\boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{x}(t)\big),$$

with mappings $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\boldsymbol{h}(\boldsymbol{x}): \mathbb{R}^n \to \mathbb{R}^p$. For the sake of simplicity, we consider the input-affine case later on, where

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{x})\,u_j.$$

Nonlinear Petrov-Galerkin projection

One way to reduce NLTI systems is to apply a nonlinear Petrov-Galerkin projection using the mappings $\boldsymbol{x} \approx \boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}})$ and $\boldsymbol{\omega}(\boldsymbol{x})$

$$\widetilde{\boldsymbol{E}}_{\mathrm{r}}\,\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \frac{\partial\boldsymbol{\omega}(\boldsymbol{x}(t))}{\partial\boldsymbol{x}(t)}\,\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))\bigg|_{\boldsymbol{x}(t) = \boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}}(t))}, \qquad \boldsymbol{x}_{\mathrm{r}}(0) = \boldsymbol{x}_{\mathrm{r},0},$$
$$\boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{h}\big(\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}}(t))\big),$$

with $\widetilde{\boldsymbol{E}}_{\mathrm{r}} = \widetilde{\boldsymbol{W}}(\boldsymbol{x})^{\mathsf{T}}\,\boldsymbol{E}\,\widetilde{\boldsymbol{V}}(\boldsymbol{x}_{\mathrm{r}})$ and $\boldsymbol{x}_{\mathrm{r}}(0) = \arg\min_{\boldsymbol{x}_{\mathrm{r},0}} \|\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r},0}) - \boldsymbol{x}_0\|_2^2$. In this setting, the main task is to find suitable nonlinear reduction mappings $\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}})$, $\boldsymbol{\omega}(\boldsymbol{x})$ and their Jacobians

$$\widetilde{\boldsymbol{V}}(\boldsymbol{x}_{\mathrm{r}}) = \partial\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}})/\partial\boldsymbol{x}_{\mathrm{r}}, \quad \widetilde{\boldsymbol{W}}(\boldsymbol{x})^{\mathsf{T}} = (\partial\boldsymbol{\omega}(\boldsymbol{x})/\partial\boldsymbol{x})|_{\boldsymbol{x} = \boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}})}.$$

Steady-state of interconnected system

Consider the nonlinear, input-affine signal generator

$$\dot{\boldsymbol{x}}_{\mathrm{r}}^w(t) = \tilde{\boldsymbol{s}}_w\big(\boldsymbol{x}_{\mathrm{r}}^w(t)\big) - \boldsymbol{L}\big(\boldsymbol{x}_{\mathrm{r}}^w(t)\big)\boldsymbol{y}(t), \quad \boldsymbol{x}_{\mathrm{r}}^w(0) = \boldsymbol{x}_{\mathrm{r},0}^w,$$
$$\boldsymbol{d}(t) = \boldsymbol{x}_{\mathrm{r}}^w(t) - \boldsymbol{\omega}\big(\boldsymbol{x}(t)\big),$$

with user-defined $\tilde{\boldsymbol{s}}_w\big(\boldsymbol{x}_{\mathrm{r}}^w(t)\big): \mathbb{R}^r \to \mathbb{R}^r$, $\boldsymbol{L}\big(\boldsymbol{x}_{\mathrm{r}}^w(t)\big): \mathbb{R}^r \to \mathbb{R}^{r \times p}$.

The steady-state of the interconnected system (cf. Fig. 1) is

$$\boldsymbol{x}_{\mathrm{r}}^w(t) = \boldsymbol{x}_{\mathrm{r,h}}^w(t) \;+\; \boldsymbol{x}_{\mathrm{r,p}}^w(t)$$
$$= \tilde{\boldsymbol{s}}_w\Big(\underbrace{\boldsymbol{x}_{\mathrm{r},0}^w}_{\boldsymbol{0}} - \frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{E}\,\boldsymbol{x}_0\Big) + \frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\,\underbrace{\boldsymbol{\tau}(\boldsymbol{x}_0)}_{\boldsymbol{0}}$$
$$\overset{t\to\infty}{\longrightarrow} \tilde{\boldsymbol{s}}_w\left(-\frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{E}\,\boldsymbol{x}_0\right) \quad \text{and} \quad \boldsymbol{d}_{\mathrm{ss}}(t) = \boldsymbol{x}_{\mathrm{r,ss}}^w(t).$$

Output Nonlinear Moments

The 0th output nonlinear moments at $(\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_{\mathrm{r}}^w(t)), \boldsymbol{L}(\boldsymbol{x}_{\mathrm{r}}^w(t)), \boldsymbol{x}_0)$ are related to the (well-defined) steady-state of $\boldsymbol{d}(t)$

$$\boldsymbol{m}_0^{\mathsf{T}}\big(\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_{\mathrm{r}}^w(t)),\, \boldsymbol{L}(\boldsymbol{x}_{\mathrm{r}}^w(t)),\, \boldsymbol{x}_0\big) \quad\Leftrightarrow\quad \boldsymbol{d}_{\mathrm{ss}}(t) = \tilde{\boldsymbol{s}}_w\left(-\frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{E}\,\boldsymbol{x}_0\right),$$

where $\boldsymbol{\omega}(\boldsymbol{x})$ is the unique solution of the Sylvester-like PDE

$$\frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{0}) = \tilde{\boldsymbol{s}}_w(\boldsymbol{x}_{\mathrm{r}}^w) - \boldsymbol{L}(\boldsymbol{x}_{\mathrm{r}}^w)\boldsymbol{h}(\boldsymbol{x})\bigg|_{\boldsymbol{x}_{\mathrm{r}}^w = \boldsymbol{\omega}(\boldsymbol{x})}.$$

This PDE represents the nonlinear counterpart of

$$\boldsymbol{W}^{\mathsf{T}}\boldsymbol{A}\,\boldsymbol{x}(t) = \boldsymbol{S}_w\,\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(t) - \boldsymbol{L}^{\mathsf{T}}\boldsymbol{C}\,\boldsymbol{x}(t).$$

Note that the PDE has the dimension $r \times 1$, and not $r \times n$.

Output Nonlinear Moment Matching

The PDE can be derived as follows:

1.) Insert the ansatz $\boldsymbol{x}_{\mathrm{r}}^w(t) = \boldsymbol{\omega}(\boldsymbol{x}(t))$ in the signal generator.

2.) Substitute the nonlinear system with $\boldsymbol{u}(t) = \boldsymbol{0}$.

Output nonlinear moment matching can be interpreted in time-domain as the interpolation of the steady-state of $\boldsymbol{d}(t)$ and $\boldsymbol{\varepsilon}(t)$ (cf. Fig. 1):

$$\boldsymbol{d}_{\mathrm{ss}}(t) = \tilde{\boldsymbol{s}}_w\left(-\frac{\partial\boldsymbol{\omega}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{E}\,\boldsymbol{x}_0\right) \equiv \boldsymbol{\varepsilon}_{\mathrm{ss}}(t),$$

where $\boldsymbol{\omega}(\boldsymbol{x})$ is the unique solution of the Sylvester-like PDE and $\boldsymbol{\nu}(\boldsymbol{x}_{\mathrm{r}})$ is arbitrary but such that $\det(\widetilde{\boldsymbol{W}}^{\mathsf{T}}\boldsymbol{E}\widetilde{\boldsymbol{V}}) \neq 0$.

Practicable W-sided Model Reduction by Approximated Nonlinear Moment Matching

Since the PDE is difficult to solve, we propose here some simplifications to approximate its solution and achieve a practical method for output nonlinear moment matching.

The first simplification step consists in applying a linear projection $\boldsymbol{x}_{\mathrm{r}}^w(t) = \boldsymbol{\omega}(\boldsymbol{x}(t)) = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(t)$ instead of the nonlinear reduction mapping $\boldsymbol{\omega}(\boldsymbol{x}(t))$. In the following, we distinguish three different signal generator cases.

Nonlinear signal generator: In this case, the PDE becomes the following *algebraic* nonlinear system of equations

$$\boldsymbol{W}^{\mathsf{T}}\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{0}) = \tilde{\boldsymbol{s}}_w\big(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(t)\big) - \boldsymbol{L}\big(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(t)\big)\,\boldsymbol{h}(\boldsymbol{x}(t)),$$

where the triple $(\tilde{\boldsymbol{s}}_w, \boldsymbol{L}, \boldsymbol{x}_0)$ is user-defined. Note that the above system is underdetermined, consisting of r equations for $r \cdot n$ unknowns in $\boldsymbol{W}^{\mathsf{T}} \in \mathbb{R}^{r \times n}$. A row-wise consideration for each $\boldsymbol{w}_i^{\mathsf{T}} \in \mathbb{R}^{1 \times n}$, $i = 1, \ldots, r$ does – at first – not help any further.

Linear signal generator: Interconnecting the nonlinear system with a linear signal generator, where $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_{\mathrm{r}}^w(t)) = \boldsymbol{S}_w\,\boldsymbol{x}_{\mathrm{r}}^w(t)$ and $\boldsymbol{L}(\boldsymbol{x}_{\mathrm{r}}^w(t)) = \boldsymbol{L}^{\mathsf{T}}$, yields

$$\boldsymbol{W}^{\mathsf{T}}\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{0}) = \boldsymbol{S}_w\,\boldsymbol{W}^{\mathsf{T}}\boldsymbol{E}\,\boldsymbol{x}(t) - \boldsymbol{L}^{\mathsf{T}}\,\boldsymbol{h}(\boldsymbol{x}(t)).$$

This is a *linear* system of equations, since the searched solution *does not* enter nonlinearly.

Zero signal generator: For this special case, where $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_{\mathrm{r}}^w(t)) = \boldsymbol{0}$, $\boldsymbol{L}(\boldsymbol{x}_{\mathrm{r}}^w(t)) = \boldsymbol{L}^{\mathsf{T}}$, it follows

$$\boldsymbol{W}^{\mathsf{T}}\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{0}) = -\boldsymbol{L}^{\mathsf{T}}\,\boldsymbol{h}(\boldsymbol{x}(t)),$$

which is again a linear system of equations.

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Towards Output Krylov Subspace-Based Nonlinear Moment Matching

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Abstract

In this contribution, we report our progress concerning a practicable, projective method for output nonlinear moment matching. First, we explain the time-domain interpretation of output Krylov subspace-based moment matching for linear systems. Then, based on [Ionescu and Astolfi 2016], the steady-state perception of output moments and moment matching for nonlinear systems is given. Finally, some simplifications to approximate the solution of the arising partial differential equation (PDE) are proposed towards a practical, numerical algorithm for nonlinear model order reduction.

Nonlinear systems (cont.)

Possible approach to achieve a practical algorithm

Nonlinear signal generator: The afore obtained nonlinear system is underdetermined

$$\boldsymbol{W}^\top \boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{0}\big) = \tilde{\boldsymbol{s}}_w\big(\boldsymbol{W}^\top \boldsymbol{E}\,\boldsymbol{x}(t)\big) - \boldsymbol{L}\big(\boldsymbol{W}^\top \boldsymbol{E}\,\boldsymbol{x}(t)\big)\,\boldsymbol{h}\big(\boldsymbol{x}(t)\big),$$

consisting of r equations for $r\cdot n$ unknowns in $\boldsymbol{W}^\top \in \mathbb{R}^{r\times n}$. A row-wise consideration for each $\boldsymbol{w}_i^\top \in \mathbb{R}^{1\times n},\ i=1,\dots,r$ might help, if n (relevant) initial states $\boldsymbol{x}_{0,\ell}\in\mathbb{R}^n,\ \ell=1,\dots,n$ are taken, yielding well-determined systems of equations of the form:

$$\boldsymbol{w}_i^\top \left\{\boldsymbol{f}\big(\boldsymbol{x}_{0,\ell},\boldsymbol{0}\big)\right\}_{\ell=1}^n = \left\{\tilde{s}_{w_i}\big(\boldsymbol{w}_i^\top \boldsymbol{E}\,\boldsymbol{x}_{0,\ell}\big)\right\}_{\ell=1}^n - \left\{\boldsymbol{l}_i^\top\big(\boldsymbol{w}_i^\top \boldsymbol{E}\,\boldsymbol{x}_{0,\ell}\big)\,\boldsymbol{h}\big(\boldsymbol{x}_{0,\ell}\big)\right\}_{\ell=1}^n, \quad i=1,\dots,r.$$

Herein, $x_{\mathrm{r},i}^w(\ell) = \boldsymbol{w}_i^\top \boldsymbol{E}\,\boldsymbol{x}_{0,\ell}\in\mathbb{R},\ \ \tilde{s}_{w_i}\big(x_{\mathrm{r},i}^w(\ell)\big):\mathbb{R}\to\mathbb{R},\ \ \boldsymbol{l}_i^\top\big(x_{\mathrm{r},i}^w(\ell)\big):\mathbb{R}\to\mathbb{R}^{1\times p}$ is used instead of $\boldsymbol{x}_\mathrm{r}^w(t) = \boldsymbol{W}^\top\boldsymbol{E}\,\boldsymbol{x}(t)\in\mathbb{R}^r,\ \ \tilde{\boldsymbol{s}}_w\big(\boldsymbol{x}_\mathrm{r}^w(t)\big):\mathbb{R}^r\to\mathbb{R}^r,\ \ \boldsymbol{L}\big(\boldsymbol{x}_\mathrm{r}^w(t)\big):\mathbb{R}^r\to\mathbb{R}^{r\times p}$.

Linear signal generator: With the proposed approach, the underdetermined *linear* system

$$\boldsymbol{W}^\top \boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{0}\big) = \boldsymbol{S}_w\,\boldsymbol{W}^\top \boldsymbol{E}\,\boldsymbol{x}(t) - \boldsymbol{L}^\top\,\boldsymbol{h}\big(\boldsymbol{x}(t)\big)$$

becomes

$$\boldsymbol{w}_i^\top \left\{\mu_i \boldsymbol{E}\,\boldsymbol{x}_{0,\ell} - \boldsymbol{f}\big(\boldsymbol{x}_{0,\ell},\boldsymbol{0}\big)\right\}_{\ell=1}^n = \left\{\boldsymbol{l}_i^\top\,\boldsymbol{h}\big(\boldsymbol{x}_{0,\ell}\big)\right\}_{\ell=1}^n, \qquad i=1,\dots,r$$

which consists of n equations for n unknowns in $\boldsymbol{w}_i^\top$, for each $i=1,\dots,r$.

Zero signal generator: The underdetermined *linear* system from before becomes

$$\boldsymbol{w}_i^\top \left\{-\boldsymbol{f}\big(\boldsymbol{x}_{0,\ell},\boldsymbol{0}\big)\right\}_{\ell=1}^n = \left\{\boldsymbol{l}_i^\top\,\boldsymbol{h}\big(\boldsymbol{x}_{0,\ell}\big)\right\}_{\ell=1}^n, \qquad i=1,\dots,r$$

which is also well-determined (*n* equations for *n* unknowns).

Simulation-free Output Nonlinear Moment Matching Algorithm

Based on the proposed simplifications – i.e. (i) linear projection, (ii) row-wise consideration and (iii) *n* initial states $\{\boldsymbol{x}_{0,\ell}\}_{\ell=1}^n$ – we are now ready to state our practical algorithm for projective, output nonlinear moment matching:

Algorithm 1 Output Nonlinear Moment Matching (O-NLMM)

Input: $\boldsymbol{E}$, $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$, $\tilde{s}_{w_i}(\cdot)$, $\boldsymbol{l}_i^\top(\cdot)$, initial states $\{\boldsymbol{x}_{0,\ell}\}_{\ell=1}^n$, initial guesses $\boldsymbol{w}_{0,i}$
Output: orthogonal basis $\boldsymbol{W}$

1: **for** `i = 1 : r` **do**
2: **for** `l = 1 : n` **do**
3: $\boldsymbol{A} = \big[\boldsymbol{A},\ \boldsymbol{f}\big(\boldsymbol{x}_{0,\ell},\boldsymbol{0}\big)\big]$
4: `xril=@(w) w'` $*\,\boldsymbol{E}\,\boldsymbol{x}_{0,\ell}$
5: `swil=@(w)` $\tilde{s}_{w_i}$`(xril)`, `yil=@(w)` $\boldsymbol{l}_i^\top$`(xril)` $\boldsymbol{h}\big(\boldsymbol{x}_{0,\ell}\big)$
6: $\tilde{\boldsymbol{s}}_w^\top(\mathtt{w}) = \big[\tilde{\boldsymbol{s}}_w^\top(\mathtt{w}),\ \mathtt{swil}\big]$, $\boldsymbol{y}_w^\top(\mathtt{w}) = \big[\boldsymbol{y}_w^\top(\mathtt{w}),\ \mathtt{yil}\big]$
7: **end for**
8: `fun=@(w)` $\boldsymbol{A}^\top * \mathtt{w} - \tilde{\boldsymbol{s}}_w(\mathtt{w}) + \boldsymbol{y}_w(\mathtt{w})$
9: `Jfun=@(w)` $\boldsymbol{A}^\top - \partial\tilde{\boldsymbol{s}}_w(\mathtt{w})/\partial\mathtt{w} + \partial\boldsymbol{y}_w(\mathtt{w})/\partial\mathtt{w}$
10: $\boldsymbol{w}_i = \mathtt{Newton}(\mathtt{fun},\ \boldsymbol{w}_{0,i},\ \mathtt{Jfun})$
11: `W(:,i)` $\leftarrow\ \boldsymbol{w}_i$
12: `W = gramSchmidt(`$\boldsymbol{w}_i$`, W)` ▷ optional
13: **end for**

The inner for-loop is used to iteratively construct the (sparse) matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, as well as the row-vector functions $\tilde{\boldsymbol{s}}_w^\top(\mathtt{w})\in\mathbb{R}^{1\times n}$ and $\boldsymbol{y}_w^\top(\mathtt{w})\in\mathbb{R}^{1\times n}$, which depend on $\boldsymbol{w}_i \mathrel{\widehat{=}} \mathtt{w}\in\mathbb{R}^{n\times 1}$.

Computational Aspects

The above algorithm is given for the most general case of a nonlinear signal generator. In this case, nonlinear systems of equations (NLSEs) of full order dimension *n* have to be solved (cf. line 8-10). These NLSEs can be solved using either a self-programmed Newton-Raphson scheme:

Algorithm 1 Newton-Raphson

Input: `fun`$(\boldsymbol{x})$, `x0`, `Jfun`, `Opts`
Output: root $\boldsymbol{x}$

```
1: tol = Opts.AbsTol
2: xcurr = x0,  fcurr = fun(xcurr)
3: iter = 0
4: while norm(fcurr) > tol do
5:    iter = iter + 1
6:    if iter > Opts.MaxIter then
7:       break
8:    end if
9:    fcurr = fun(xcurr)
10:   dxcurr = Jfun(xcurr) \ fcurr
11:   xcurr = xcurr - dxcurr
12:   tol = Opts.RelTol*norm(fcurr) + Opts.AbsTol
13: end while
14: x = xcurr
```

or a built-in `fsolve` routine (same holds for our proposed Input NLMM algorithm).

For the Newton-Raphson scheme, it is highly recommended to supply the analytical Jacobian `Jfun(w)` of the residual `fun(w)`, in order to achieve a faster computation and avoid the approximation via finite differences. Moreover, good initial guesses $\boldsymbol{w}_{0,i}$ can considerably speed-up the convergence of the Newton method.
In line 10, a *direct* solver (e.g. "\" in MATLAB) or an *iterative* solver (e.g. pcg) can be used.

Output Nonlinear Moment Matching for linear/zero signal generator

As seen above, the proposed simplifications yield *linear* systems of equations (LSEs) in case of a linear or zero signal generator. Therefore, the O-NLMM algorithm can be simplified as follows:

Algorithm 1 O-NLMM for linear/zero signal generator

Input: $\boldsymbol{E}$, $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$, shifts μ_i, tangential directions $\boldsymbol{l}_i^\top$, initial states $\{\boldsymbol{x}_{0,\ell}\}_{\ell=1}^n$
Output: orthogonal basis $\boldsymbol{W}$

1: **for** `i = 1 : r` **do**
2: **for** `l = 1 : n` **do**
3: $\boldsymbol{A}_{\mu_i} = \big[\boldsymbol{A}_{\mu_i},\ \mu_i\boldsymbol{E}\,\boldsymbol{x}_{0,\ell} - \boldsymbol{f}\big(\boldsymbol{x}_{0,\ell},\boldsymbol{0}\big)\big]$
4: $y_i^w(\ell) = \boldsymbol{l}_i^\top\,\boldsymbol{h}\big(\boldsymbol{x}_{0,\ell}\big)$
5: $\boldsymbol{y}_w^\top = \big[\boldsymbol{y}_w^\top,\ y_i^w(\ell)\big]$
6: **end for**
7: Solve $\boldsymbol{w}_i^\top\boldsymbol{A}_{\mu_i} = \boldsymbol{y}_w^\top$ or $\boldsymbol{A}_{\mu_i}^\top\boldsymbol{w}_i = \boldsymbol{y}_w$: $\boldsymbol{w}_i^\top = \boldsymbol{y}_w^\top/\boldsymbol{A}_{\mu_i} \ \Leftrightarrow\ \boldsymbol{w}_i = \boldsymbol{A}_{\mu_i}^\top\backslash\boldsymbol{y}_w$
8: `W(:,i)` $\leftarrow\ \boldsymbol{w}_i$
9: `W = gramSchmidt(`$\boldsymbol{w}_i$`, W)` ▷ optional
10: **end for**

Due to the linearity of $\tilde{s}_{w_{i,\ell}}(\boldsymbol{w}_i) = \boldsymbol{w}_i^\top\mu_i\boldsymbol{E}\,\boldsymbol{x}_{0,\ell}$ and $y_{i,\ell}^w(\boldsymbol{w}_i) = \boldsymbol{l}_i^\top\,\boldsymbol{h}(\boldsymbol{x}_{0,\ell})$ w.r.t. the unknown $\boldsymbol{w}_i$, the (sparse, possibly complex) "shifted" matrix $\boldsymbol{A}_{\mu_i}\in\mathbb{C}^{n\times n}$ (cf. line 3) as well as the *constant* row-vector $\boldsymbol{y}_w^\top\in\mathbb{C}^{1\times n}$ (cf. line 5) can be constructed, yielding the well-known LSEs $\boldsymbol{A}_{\mu_i}^\top\boldsymbol{w}_i = \boldsymbol{y}_w$. These LSEs can be solved using a *direct* ("\" in MATLAB) or an *iterative* solver (e.g. pcg). Remarkably, no Newton-Raphson scheme is required at all in this case.

Analysis, Discussion and Limitations

At this point, we want to briefly discuss the proposed simplifications, as well as the degrees of freedom and limitations of the presented O-NLMM algorithm.

Adequate selection of the projection ansatz. The simple choice of a linear projection $\boldsymbol{x}_\mathrm{r}^w(t) = \boldsymbol{\omega}(\boldsymbol{x}(t)) = \boldsymbol{W}^\top\boldsymbol{E}\,\boldsymbol{x}(t)$ is motivated by its easy and frequent use in comparison to nonlinear projections. Nevertheless, a more sophisticated projection, like e.g. a polynomial series expansion ansatz, might be superior and even indispensable in certain cases.

Appropriate choice of the signal generator. The signal generator $\big(\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w(t)),\ \boldsymbol{L}\big(\boldsymbol{x}_\mathrm{r}^w(t)\big)\big)$ determines (1) the ansatz for the dynamics $\boldsymbol{x}_\mathrm{r}^w(t)$ and (2) the "weighting" of the output. In case of a linear signal generator, $\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w(t)) = \boldsymbol{S}_w\,\boldsymbol{x}_\mathrm{r}^w(t)$ and $\boldsymbol{L}(\boldsymbol{x}_\mathrm{r}^w(t)) = \boldsymbol{L}^\top$ holds, meaning that complex exponentials $\mathrm{e}^{\mu_i t}$ and left tangential directions $\boldsymbol{l}_i^\top$ are being employed. Regardless of their (questionable) suitability, an expansion-based generator ansatz can be also used.

Obtaining a state-independent matrix equation. In the Sylvester-like PDE from above, the state vector $\boldsymbol{x}(t)$ cannot be factored out so easily like in the linear case. In fact, the key to obtain a state-independent matrix equation of dimension $r\times n$ lies on both the choice of an adequate projection ansatz and signal generator, customized for the nonlinear system.

Limitations of the row-wise consideration. If a linear projection is applied and the factorization of $\boldsymbol{x}(t)$ does not succeed, then the underdetermined system from above is obtained. The proposed row-wise consideration, together with the initial states $\{\boldsymbol{x}_{0,\ell}\}_{\ell=1}^n$, has the limitation that the underdetermined equation is generally not fulfilled, since the (nonlinear) couplings in $\boldsymbol{W}^\top\boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{0}),\ \tilde{\boldsymbol{s}}_w(\boldsymbol{W}^\top\boldsymbol{E}\,\boldsymbol{x}(t)),\ \boldsymbol{L}(\boldsymbol{W}^\top\boldsymbol{E}\,\boldsymbol{x}(t))$ are not being considered.

Approximating output nonlinear moments. What moments are being matched, when we apply the O-NLMM algorithm? Since the PDE is not being solved, the "true" output nonlinear moments $\boldsymbol{m}_0^\top\big(\tilde{\boldsymbol{s}}_w(\boldsymbol{x}_\mathrm{r}^w(t)),\ \boldsymbol{L}\big(\boldsymbol{x}_\mathrm{r}^w(t)\big),\ \boldsymbol{x}_0\big)$ are not exactly matched. Instead, we are *approximately* matching these moments at the chosen data $\big(\tilde{s}_{w_i}(x_{\mathrm{r},i}^w(\ell)),\ \boldsymbol{l}_i^\top(x_{\mathrm{r},i}^w(\ell)),\ \boldsymbol{x}_{0,\ell}\big)$.

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