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Shape Optimization for Fluid-Structure Interaction

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Abstract

In this thesis, we investigate unsteady fluid-structure interaction (FSI) problems. We prove a new improved regularity result for the linear hyperbolic wave equation. Under the assumption that this result can be adapted to the Lamé system, we show that a linear FSI problem attains a unique solution under weaker assumptions on the geometry of the domain than in previous works. This is also the basis for a local-in-time existence and regularity result for a nonlinear, unsteady FSI problem that couples the Navier-Stokes equations with the Lamé system. Furthermore, we examine shape optimization for unsteady FSI. Since the concept of domain transformations is well-established in the monolithic FSI context due to the necessity of representing the coupled system in a uniform framework, we apply the method of mappings. We develop a general framework for deriving continuity and differentiability for the solution of nonlinear, unsteady, parameter-dependent partial differential equations and apply it to show differentiability of the states of the unsteady FSI problem with respect to domain variations. In order to show the viability of our approach for shape optimization of unsteady FSI, we further do numerical simulations.

Zusammenfassung

Diese Arbeit befasst sich mit instationären Fluid-Struktur Interaktionsproblemen. Wir beweisen eine neue verbesserte Regularitätsabschätzung für die Normalenableitung der Lösung der linearen, hyperbolischen Wellengleichung. Unter der Annahme, dass diese Regulartiätsaussage auf das Lamé-System übertragen werden kann, zeigen wir die Existenz und Eindeutigkeit von Lösungen für ein lineares FSI Problem unter weniger restriktiven Voraussetzungen an die Geometrie des Gebietes als in bisher verfügbaren Resultaten. Ausgehend davon lässt sich auch zeitlokale Existenz und Eindeutigkeit von Lösungen für ein nichtlineares, instationäres FSI Problem, das die Navier-Stokes Gleichungen und das Lamé System koppelt, herleiten. Desweiteren wird Formoptimierung für instationäre FSI mit Hilfe der sogenannten "method of mappings" betrachtet. Dieser Ansatz arbeitet, ähnlich wie die Herleitung der monolithischen Darstellung des FSI Modells, mit Gebietstransformationen. Es wird ein allgemeines Konzept entwickelt, mit dem sich Stetigkeits- und Differenzierbarkeitsaussagen für die Lösungen von nichtlinearen, instationären und parameterabhängigen Differentialgleichungen herleiten lassen. Wir wenden dieses an, um Differenzierbarkeit der Zustände des instationären FSI Problems bezüglich Gebietsvariationen zu zeigen. Numerische Simulationen demonstrieren die Praktikabilität des Formoptimierungsansatzes für instationäre FSI.

Notation

$\mathbb{N}$	natural numbers (without 0)
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	natural numbers with 0
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
d	dimension $d \in \mathbb{N}$
$\emptyset$	empty set
$\mathbf{I}$	identity matrix $\mathbf{I} \in \mathbb{R}^{d \times d}$
I	time interval $I = (0, T)$, $T > 0$
$\Omega, \Omega_f, \Omega_s$	domain/ subset of $\mathbb{R}^d$, fluid part of domain Ω, solid part of domain Ω
$\Gamma, \partial\Omega$	boundary of domain Ω
Γ_i	interface between fluid and solid part of domain, one requires that $\Omega = \Omega_f \cup \Omega_s \cup \Gamma_i$, $\Omega_f \cap \Omega_s = \emptyset$, $\overline{\Omega}_f \cap \overline{\Omega}_s = \Gamma_i$
$\Gamma_f, (\Gamma_s)$	outer, exterior boundary part of Ω_f (Ω_s), $\Gamma_f = \partial\Omega_f \setminus \Gamma_i$, ($\Gamma_s = \partial\Omega_s \setminus \Gamma_i$)
$\Gamma_{fD}, (\Gamma_{sD})$	part of Γ_f (Γ_s), where Dirichlet boundary conditions are imposed
$\Gamma_{fN}, (\Gamma_{sN})$	part of Γ_f (Γ_s), where Neumann boundary conditions are imposed
$\check{\cdot}$	superscript indicating that we are in the physical framework
$\hat{\cdot}$	superscript indicating that we are in the ALE framework
$\tilde{\cdot}$	superscript that indicates that we are in the framework for shape optimization with the method of mappings approach
ξ	coordinates on Ω
x	coordinates on $\check{\Omega}$
y	coordinates on $\hat{\Omega}$
z	coordinates on $\tilde{\Omega}$
t, s	time coordinates
$\cdot_f$	indicates that the quantity is defined on the fluid part of the domain

$\cdot_s$	indicates that the quantity is defined on the solid part of the domain
$\cdot_{\mathsf{h}}$	indicates that the discretized version of the quantity is considered
Q_*^T	space-time cylinder $\Omega_* \times (0,T)$, except for physical domain where it is defined as $\check{Q}_*^T := \bigcup_{\mathsf{t}\in I} \check{\Omega}_*(\mathsf{t}) \times \{\mathsf{t}\}$
Σ_*^T	space-time cylinder $\Gamma_* \times (0,T)$, except for physical domain where it is defined as $\check{\Sigma}_*^T := \bigcup_{\mathsf{t}\in I} \check{\Sigma}_*(\mathsf{t}) \times \{\mathsf{t}\}$
$\mathbf{v}$, $\mathbf{u}$, $\underline{\mathbf{u}}$	velocity
p, q, $\underline{q}$	pressure
$\mathbf{w}$, $\mathbf{z}$, $\underline{\mathbf{z}}$	displacement
$\mathrm{div}(\cdot)$, $(\mathrm{div}_{\mathsf{x}}(\cdot), \mathrm{div}_{\mathsf{y}}(\cdot), \mathrm{div}_{\mathsf{z}}(\cdot))$	divergence, differential operator defined by $\mathrm{div}(\mathbf{v}) = \sum_{i=1}^d \partial_{\xi_i} \mathbf{v}_i$ for a vector valued quantity $\mathbf{v} : \Omega \to \mathbb{R}^d$ and by $\mathrm{div}(\mathbf{A}) = (\sum_{i=1}^d \partial_{\xi_i} \mathbf{A}_{j,i})_j$ for a matrix valued quantity $\mathbf{A} : \Omega \to \mathbb{R}^{d\times d}$, (analogue definition on $\check{\Omega}$, $\hat{\Omega}$, $\tilde{\Omega}$)
$\nabla(\cdot)$, $(\nabla_{\mathsf{x}}(\cdot), \nabla_{\mathsf{y}}(\cdot), \nabla_{\mathsf{z}}(\cdot))$	gradient, differential operator defined by $\nabla p = (\partial_{\xi_i} p)_i$ for a scalar valued quantity $p : \Omega \to \mathbb{R}$ and by $\nabla \mathbf{v} = (\partial_{\xi_i} \mathbf{v}_j)_{i,j}$ for a vector valued quantity $\mathbf{v} : \Omega \to \mathbb{R}^d$, (analogue definition on $\check{\Omega}$, $\hat{\Omega}$, $\tilde{\Omega}$)
$D(\cdot)$, $(D_{\mathsf{x}}(\cdot), D_{\mathsf{y}}(\cdot), D_{\mathsf{z}}(\cdot))$	Jacobian, differential operator defined by $D\mathbf{v} = (\partial_{\xi_j} \mathbf{v}_i)_{i,j} = \nabla \mathbf{v}^\top$ for a vector valued quantity $\mathbf{v} : \Omega \to \mathbb{R}^d$, (analogue definition on $\check{\Omega}$, $\hat{\Omega}$, $\tilde{\Omega}$)
D^α	differential operator of order $\alpha \in \mathbb{N}_0^d$, $D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial \xi_1^{\alpha_1} \ldots \partial \xi_d^{\alpha_d}}$, $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_d$
$\epsilon(\mathbf{v})$, $(\epsilon_{\mathsf{x}}(\mathbf{v}), \epsilon_{\mathsf{y}}(\mathbf{v}), \epsilon_{\mathsf{z}}(\mathbf{v}))$	differential operator defined by $\epsilon(\mathbf{v}) = \frac{1}{2}(D\mathbf{v} + D\mathbf{v}^\top)$, (analogue definition on $\check{\Omega}$, $\hat{\Omega}$, $\tilde{\Omega}$)
$\sigma_f(\mathbf{v}, p), (\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}, \check{p}), \sigma_{f,\mathsf{y}}(\hat{\mathbf{v}}, \hat{p}), \sigma_{f,\mathsf{z}}(\tilde{\mathbf{v}}, \tilde{p}))$	fluid stress tensor, differential operator defined by $\sigma_f(\mathbf{v}, p) = 2\nu\epsilon(\mathbf{v}) - p\mathbf{I}$, (analogue definition on $\check{\Omega}$, $\hat{\Omega}$, $\tilde{\Omega}$)
$\Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})$	solid stress tensor, differential operator
$\mathbf{n}$	outer unit normal vector
$\hat{\boldsymbol{\chi}}$	ALE transformation, maps $\hat{\Omega}_* \times (0,T) \to \check{Q}_*^T$
$\check{\boldsymbol{\Upsilon}}$	inverse of $\hat{\boldsymbol{\chi}}$
$\tilde{\boldsymbol{\tau}}$	transformation for shape optimization with method of mappings, maps $\tilde{\Omega}_* \to \hat{\Omega}_*$, cf., Section 2.7
$\tilde{\mathbf{u}}_\tau$	control for shape optimization with method of mappings, cf., Section 2.7
$\widehat{\mathbf{F}}_{\boldsymbol{\chi}}$	deformation gradient $\widehat{\mathbf{F}}_{\boldsymbol{\chi}} = D_{\mathsf{y}} \hat{\boldsymbol{\chi}}$
$\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}$	inverse deformation gradient $\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} = \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}$
$\hat{J}_{\boldsymbol{\chi}}$	determinant of deformation gradient $\hat{J}_{\boldsymbol{\chi}} = \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})$

$\hat{\boldsymbol{\sigma}}_f$	transformed fluid stress tensor $\sigma_{f,x}(\check{\mathbf{v}}_f, \check{p}_f) \circ \hat{\boldsymbol{\chi}}$
$\hat{\boldsymbol{\sigma}}_s$	appropriate definition to obtain an analogy to the fluid equations, $\hat{\boldsymbol{\sigma}}_s := \hat{J}_{\boldsymbol{\chi}}{}^{-1} \widehat{\mathbf{F}}_{\boldsymbol{\chi}} \Sigma_{s,\mathrm{y}}(\hat{\mathbf{w}}_s) \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^T$
$\mathcal{D}(\Omega)$	space that contains all φ such that $\varphi : \Omega \to \mathbb{R}$ is infinitely differentiable on Ω and has compact support in Ω
$\mathcal{D}'(\mathbb{R}^d)$	space of distributions, dual space of $\mathcal{D}(\mathbb{R}^d)$
$\mathcal{S}'(\mathbb{R}^d)$	space of tempered distributions, subset of $\mathcal{D}'(\mathbb{R}^d)$
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space
$L^p(\Omega)$	Banach space of equivalence classes of measurable, p-integrable ($p \in [1, \infty)$) or essentially bounded ($p = \infty$) functions on Ω
$\mathscr{F} v$	Fourier transform of integrable function $v : \mathbb{R} \to \mathbb{C}$
$\mathcal{C}^m(\overline{\Omega})$	space of m-times differentiable functions on $\overline{\Omega}$, endowed with the norm $\|v\|_{\mathcal{C}^m(\overline{\Omega})} = \sum_{\lvert\alpha\rvert \leq m} \max_{\xi \in \overline{\Omega}} \lvert D^\alpha v(\xi) \rvert$
$H^s(\Gamma)$	cf., Section 2.2.1
$H^s(\Omega)$	cf., Section 2.2.2
$H^s((0,T), H^r(\Omega))$, $H^s((0,T), X)$	cf., Section 2.2.3
$H^{s,r}(Q)$	cf., Section 2.2.3
$[X, Y]_\theta$	complex interpolation of two Hilbert spaces X, Y such that $X \subset Y$ and X dense in Y, $\theta \in [0,1]$ with continuous injection, cf., [90, p. 10, Def. 2.1]
$\mathcal{C}_0^\infty(\Omega)$	$\mathcal{C}_0^\infty(\Omega) = \mathcal{D}(\Omega)$
$L^2((0,T), X)$	$L^2((0,T), X) = H^0((0,T), X)$
$dS(\xi)$	surface measure on $\partial\Omega$
$\mathbf{E}_T$	$(H^{2+\ell, 1+\frac{\ell}{2}}(Q_f^T))^d$, cf., Section 2.2.4
$\mathbf{E}_{T,M_0,\mathbf{v}_0}$	subset of $\mathbf{E}_T$, cf., Section 2.2.4
$\mathbf{F}_T$	$(H^{\ell, \frac{\ell}{2}}(Q_f^T))^d$, cf., Section 2.2.4
G_T	$L^2((0,T), H^{1+\ell}(\Omega_f)) \cap H^{\frac{\ell}{2}}((0,T), H^1(\Omega_f))$, cf., Section 2.2.4
$\mathbf{G}_T$	$H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)$, cf., Section 2.2.4
H_T	$H^{\frac{1}{2}+\ell, \frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)$, cf., Section 2.2.4
$\mathbf{H}_T$	$(H^{\frac{1}{2}+\ell, \frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T))^d$, cf., Section 2.2.4
P_T	$\{p \in L^2(Q_f^T) \ : \ \nabla p \in H^{\ell, \frac{\ell}{2}}(Q_f^T), p\vert_{\Sigma_i^T} \in H^{\frac{1}{2}+\ell, \frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)\}$, cf., Section 2.2.4
$P_{T,M_0,\mathbf{v}_0}$	subset of P_T, cf., Section 2.2.4
S_T	$H^1((0,T), H^{1+\ell}(\Omega_f)) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f))$, cf., Section 2.2.4

$\mathbf{S}_T$	$H^1((0,T), H^{1+\ell}(\Omega_f)^d) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)$, cf., Section 2.2.4
$\underline{\mathbf{S}}_T$	$H^1((0,T), H^{1+\ell}(\Omega_f)^{d\times d}) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^{d\times d})$, cf., Section 2.2.4
$\mathbf{W}_T$	$\mathcal{C}([0,T], H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)^d) \cap \mathcal{C}^1([0,T], H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)^d)$, cf., Section 2.2.4
$\mathscr{B}$	set of first order operators tangential to boundary, cf. (2.15)
B	arbitrary first order operator, $B = \sum_i \mathbf{b}_i(\xi)\partial_{\xi_i}$ for $\mathbf{b}_i \in \mathcal{C}^\infty(\overline{\Omega})^d$
Ext	extension operator in time, cf., Section 2.5
ext	extension operator in space, cf., Proof of Lemma 3.17
R	restriction operator in time, cf., Section 2.5
$\mathcal{P}$	Leray projector, cf., Section 3.3.2
$\mathcal{I}$	identity $L^2(\Omega_f)^d \to L^2(\Omega_f)^d$, cf., Section 3.3.2
R	regularization term in optimization problem, cf., Section 5.3.2
$\hat{\mathcal{O}}_{ad}$	set of admissible domains, cf., Section 2.7
$\tilde{\mathcal{T}}_{ad}$	set of admissible transformations, cf. Section 2.7
$\tilde{\mathbf{U}}_{ad}$	$\{\tilde{\mathbf{u}}_\tau : \mathbb{R}^d \to \mathbb{R}^d \ : \ \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau \in \tilde{\mathcal{T}}_{ad}\}$, cf. Sections 2.7, 4
$\{\mathcal{O}_j, \varphi_j, \alpha_j\}$	system of local maps and partition of unity, cf., Section 2.2.1
$(\cdot,\cdot)_\Omega$	$L^2(\Omega)$ inner product of functions, cf., Section 3.1
$(\cdot,\cdot)_\Gamma$	$L^2(\Gamma)$ inner product of functions, cf., Section 3.1
$(\!(\cdot,\cdot)\!)_{Q^T}$	$L^2(Q^T)$ inner product of functions, cf., Section 3.1
$(\!(\cdot,\cdot)\!)_{\Sigma^T}$	$L^2(\Sigma^T)$ inner product of functions, cf., Section 3.1
$\mathcal{T}_{\mathsf{h}}$	triangulation
$\mathcal{P}^k(\mathcal{T}_{\mathsf{h}})^m$	space of continuous, piecewise polynomial functions up to degree k on the triangulation $\mathcal{T}_{\mathsf{h}}$
$\mathcal{P}_0^k(\mathcal{T}_{\mathsf{h}})^m$	subspace of $\mathcal{P}^k(\mathcal{T}_{\mathsf{h}})^m$, contains functions that have value 0 on the boundary of $\bigcup_{K\in\mathcal{T}_{\mathsf{h}}} K$

Contents

1. Introduction

Fluid-structure interaction (FSI) is a particularly important subclass of multi-physics problems that arise frequently in applications such as wind turbines, bridges, naval architecture or biomedical applications, cf., e.g., [21, 11, 33, 42, 45, 46, 69, 70, 76, 127]. We focus on applications with incompressible flow and consider fluid modeled by the unsteady incompressible Navier-Stokes equations. These equations are formulated in the Eulerian framework, i.e., on the time-dependent physical domain $\check{\Omega}_f(\mathsf{t}) \subset \mathbb{R}^d$ for $\mathsf{t} \in I := (0, T)$, $T > 0$. We divide the fluid boundary $\partial\check{\Omega}_f(\mathsf{t}) = \check{\Gamma}_{fD}(\mathsf{t}) \cup \check{\Gamma}_{fN}(\mathsf{t})$ into two disjoint parts, on which Dirichlet (on $\check{\Gamma}_{fD}(\mathsf{t})$) or Neumann (on $\check{\Gamma}_{fN}(\mathsf{t})$) boundary conditions are imposed. The corresponding space-time cylinders shall be denoted by

$$\check{Q}_f^T := \bigcup_{\mathsf{t}\in I} \check{\Omega}_f(\mathsf{t}) \times \{\mathsf{t}\}, \quad \check{\Sigma}_{fD}^T := \bigcup_{\mathsf{t}\in I} \check{\Gamma}_{fD}(\mathsf{t}) \times \{\mathsf{t}\}, \quad \check{\Sigma}_{fN}^T := \bigcup_{\mathsf{t}\in I} \check{\Gamma}_{fN}(\mathsf{t}) \times \{\mathsf{t}\}.$$

The differential equations are given by

$$\begin{aligned} \rho_f \partial_{\mathsf{t}} \check{\mathbf{v}}_f + (\check{\mathbf{v}}_f \cdot \nabla_{\mathsf{x}})\check{\mathbf{v}}_f - \operatorname{div}_{\mathsf{x}}(\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f)) &= \rho_f \check{\mathbf{f}}_f \quad \text{on } \check{Q}_f^T, \\ \operatorname{div}_{\mathsf{x}}(\check{\mathbf{v}}_f) &= 0 \quad \text{on } \check{Q}_f^T, \\ \check{\mathbf{v}}_f &= \check{\mathbf{v}}_{fD} \quad \text{on } \check{\Sigma}_{fD}^T, \\ \sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f)\check{\mathbf{n}}_f &= \check{\mathbf{g}}_f \quad \text{on } \check{\Sigma}_{fN}^T, \end{aligned}$$

with the initial condition

$$\check{\mathbf{v}}_f(\cdot, 0) = \check{\mathbf{v}}_{0f} \quad \text{on } \check{\Omega}_f(0),$$

where $\check{\mathbf{v}}_f$ denotes the fluid velocity, $\check{p}_f$ the pressure and $\check{\mathbf{n}}_f$ the outer unit normal vector. $\check{\mathbf{f}}_f$, $\check{\mathbf{v}}_{fD}$, $\check{\mathbf{g}}_f$ and $\check{\mathbf{v}}_{0f}$ are right-hand side, boundary and intitial terms. The fluid stress tensor is defined by

$$\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f) = \rho_f \nu_f (D_{\mathsf{x}}\check{\mathbf{v}}_f + D_{\mathsf{x}}\check{\mathbf{v}}_f{}^T) - \check{p}_f \mathbf{I},$$

with unit matrix $\mathbf{I} \in \mathbb{R}^{d\times d}$ and Jacobian $D_{\mathsf{x}}(\cdot) := (\partial_{\mathsf{x}_j}(\cdot)_i)_{i,j}$. The parameters ρ_f and ν_f denote the fluid density and viscosity, respectively. The structure equations, however, are formulated in the Lagrangian framework, i.e., on a fixed reference domain $\hat{\Omega}_s$ with disjoint Dirichlet and Neumann boundary parts $\hat{\Gamma}_{sD}$ and $\hat{\Gamma}_{sN}$ such that $\partial\hat{\Omega}_s = \hat{\Gamma}_{sD} \cup \hat{\Gamma}_{sN}$. The physical domain $\check{\Omega}_s(\mathsf{t})$ for any $\mathsf{t} \in I$ is obtained by the transformation $\hat{\boldsymbol{\chi}}_s(\cdot, \mathsf{t}) : \hat{\Omega}_s \to \check{\Omega}_s(\mathsf{t})$, $\hat{\boldsymbol{\chi}}_s(\mathsf{y}, t) = \mathsf{y} + \hat{\mathbf{w}}_s(\mathsf{y}, \mathsf{t})$, where the deformation $\hat{\mathbf{w}}_s$ solves the hyperbolic equations

$$\begin{aligned} \rho_s \partial_{\mathsf{tt}} \hat{\mathbf{w}}_s - \operatorname{div}_{\mathsf{y}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)) &= \rho_s \hat{\mathbf{f}}_s \quad \text{on } \hat{Q}_s^T := \hat{\Omega}_s \times I, \\ \hat{\mathbf{w}}_s &= \hat{\mathbf{w}}_{sD} \quad \text{on } \hat{\Sigma}_{sD}^T := \hat{\Gamma}_{sD} \times I, \end{aligned}$$

$$\begin{aligned} \widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)\hat{\mathbf{n}}_s &= \hat{\mathbf{g}}_s \quad \text{on } \hat{\Sigma}_{sN}^T := \hat{\Gamma}_{sN} \times I, \\ \hat{\mathbf{w}}_s(\cdot, 0) &= \hat{\mathbf{w}}_{0s} \quad \text{on } \hat{\Omega}_s, \\ \partial_{\mathsf{t}} \hat{\mathbf{w}}_s(\cdot, 0) &= \hat{\mathbf{w}}_1 \quad \text{on } \hat{\Omega}_s, \end{aligned}$$

and we define $\widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} := D_{\mathsf{y}} \hat{\boldsymbol{\chi}}_s$. Here, ρ_s denotes the structure density and $\hat{\mathbf{f}}_s$, $\hat{\mathbf{w}}_{sD}$, $\hat{\mathbf{g}}_s$, $\hat{\mathbf{w}}_{0s}$ and $\hat{\mathbf{w}}_1$ denote right hand side, boundary and intial terms.

- For a linear elastic material the stress tensor $\Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)$ is given by

$$\Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s) := \widehat{\mathbf{F}}_{\boldsymbol{\chi}_s}^{-1}(\mu_s(D_{\mathsf{y}}\hat{\mathbf{w}}_s + D_{\mathsf{y}}\hat{\mathbf{w}}_s^{\top}) + \lambda_s \operatorname{tr}(D_{\mathsf{y}}\hat{\mathbf{w}}_s)\mathbf{I}),$$

  where the so-called Lamé coefficients λ_s and μ_s are chosen such that $\mu_s > 0$ and $\lambda_s + \mu_s > 0$.

- For the nonlinear Saint Venant Kirchhoff type material the stress tensor $\Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)$ is given by

$$\Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s) := \lambda_s \operatorname{tr}(\widehat{\mathbf{E}}_{\boldsymbol{\chi}_s})\mathbf{I} + 2\mu_s \widehat{\mathbf{E}}_{\boldsymbol{\chi}_s},$$

  with $\widehat{\mathbf{E}}_{\boldsymbol{\chi}_s} := \frac{1}{2}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}_s}^{\top} \widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} - \mathbf{I})$.

The first challenge for considering the coupled problem arises from the fact that the above canonical models for the fluid and structure equations are formulated in different frameworks.

For FSI simulations, partitioned as well as monolithic approaches have been proposed. Partitioned methods solve the corresponding models seperately and typically apply fixed point iterations to the coupling interface conditions, which can, e.g., be accelerated by Quasi-Newton [30, 77]. Monolithic approaches [32, 36, 43, 44, 50, 60, 129], such as arbitrary Lagrangian-Eulerian (ALE) [32, 36, 60] and fully Eulerian methods [36, 43, 44, 129, 130], use the same reference frame for fluid and solid. While fully Eulerian approaches use the spatial reference frame, the ALE framework is obtained by introducing an arbitrary but fixed reference domain $\hat{\Omega}_f$ such that the fluid and solid reference domains are disjoint, i.e., $\hat{\Omega}_s \cap \hat{\Omega}_f = \emptyset$, and share the same boundary at the interface $\hat{\Gamma}_i := \overline{\hat{\Omega}_s} \cap \overline{\hat{\Omega}_f}$. Moreover, an extension $\hat{\boldsymbol{\chi}}(\mathsf{t}) : \hat{\Omega}_s \to \hat{\Omega}$ of the solid transformation $\hat{\boldsymbol{\chi}}_s(\mathsf{t})$ to the whole reference domain $\hat{\Omega} := \hat{\Omega}_s \cup \hat{\Omega}_f \cup \hat{\Gamma}_i$ is introduced for any $\mathsf{t} \in I$. It can, e.g., be obtained by choosing a fully Lagrangian setting or an harmonic or biharmonic extension of the solid displacement to the fluid reference domain. Transformation of the fluid equations with the help of $\hat{\boldsymbol{\chi}}$ to the fixed reference domain $\hat{\Omega}_f$ and coupling the fluid and structure equations across the interface $\hat{\Gamma}_i$

yields the system of equations

$$
\begin{aligned}
(\rho_f \partial_{\mathsf{t}} \check{\mathbf{v}}_f + (\check{\mathbf{v}}_f \cdot \nabla_{\mathsf{x}})\check{\mathbf{v}}_f - \operatorname{div}_{\mathsf{x}}(\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f))) \circ \hat{\boldsymbol{\chi}} &= \rho_f \hat{\mathbf{f}}_f && \text{on } \hat{\Omega}_f \times I,\\
\operatorname{div}_{\mathsf{x}}(\check{\mathbf{v}}_f) \circ \hat{\boldsymbol{\chi}} &= 0 && \text{on } \hat{\Omega}_f \times I,\\
\check{\mathbf{v}}_f \circ \hat{\boldsymbol{\chi}} &= \hat{\mathbf{v}}_{fD} && \text{on } \hat{\Gamma}_{fD} \times I,\\
(\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f)\check{\mathbf{n}}_f) \circ \hat{\boldsymbol{\chi}} &= \hat{\mathbf{g}}_f && \text{on } \hat{\Gamma}_{fN} \times I,\\
\check{\mathbf{v}}_f \circ \hat{\boldsymbol{\chi}}(\cdot, 0) &= \hat{\mathbf{v}}_{0f} && \text{on } \hat{\Omega}_f,\\
\rho_s \partial^2_{\mathsf{tt}} \hat{\mathbf{w}}_s - \operatorname{div}_{\mathsf{y}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)) &= \rho_s \hat{\mathbf{f}}_s && \text{on } \hat{\Omega}_s \times I,\\
\hat{\mathbf{w}}_s &= \hat{\mathbf{w}}_{sD} && \text{on } \hat{\Gamma}_{sD} \times I,\\
\widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)\hat{\mathbf{n}}_s &= \hat{\mathbf{g}}_s && \text{on } \hat{\Gamma}_{sN} \times I,\\
\hat{\mathbf{w}}_s(\cdot, 0) &= \hat{\mathbf{w}}_{0s} && \text{on } \hat{\Omega}_s,\\
\partial_{\mathsf{t}} \hat{\mathbf{w}}_s(\cdot, 0) &= \hat{\mathbf{w}}_1 && \text{on } \hat{\Omega}_s
\end{aligned}
\tag{1.1}
$$

with additional coupling conditions

$$
\begin{aligned}
\partial_{\mathsf{t}} \hat{\mathbf{w}}_s &= \check{\mathbf{v}}_f \circ \hat{\boldsymbol{\chi}} && \text{on } \hat{\Gamma}_i \times I,\\
-(\sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f)\check{\mathbf{n}}_f) \circ \hat{\boldsymbol{\chi}} &= \widehat{\mathbf{F}}_{\boldsymbol{\chi}_s} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s)\hat{\mathbf{n}}_s && \text{on } \hat{\Gamma}_i \times I.
\end{aligned}
$$

Here, $\hat{\mathbf{f}}_f := \check{\mathbf{f}}_f \circ \hat{\boldsymbol{\chi}}$ and $\hat{\mathbf{g}}_f$, $\hat{\mathbf{v}}_{fD}$, as well as, $\hat{\mathbf{v}}_{0f}$ are defined analogously. Introducing $\hat{\mathbf{v}}_f = \check{\mathbf{v}}_f \circ \hat{\boldsymbol{\chi}}$, $\hat{p}_f = \check{p}_f \circ \hat{\boldsymbol{\chi}}$, $\hat{\boldsymbol{\sigma}}_f = \sigma_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f) \circ \hat{\boldsymbol{\chi}}$, $\hat{\boldsymbol{\sigma}}_s := \hat{J}_{\boldsymbol{\chi}}^{-1} \widehat{\mathbf{F}}_{\boldsymbol{\chi}} \Sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}_s) \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{\top}$, where $\widehat{\mathbf{F}}_{\boldsymbol{\chi}} := D_{\mathsf{y}} \hat{\boldsymbol{\chi}}$ and $\hat{J}_{\boldsymbol{\chi}} := \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})$, as well as, $\hat{\mathbf{v}}_s = \partial_{\mathsf{t}} \hat{\mathbf{w}}_s$, yields the equivalent formulation

$$
\begin{aligned}
\hat{J}_{\boldsymbol{\chi}} \rho_f \partial_{\mathsf{t}} \hat{\mathbf{v}}_f + \hat{J}_{\boldsymbol{\chi}} \rho_f ((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\hat{\mathbf{v}}_f - \partial_{\mathsf{t}} \hat{\boldsymbol{\chi}})) \cdot \nabla_{\mathsf{y}}) \hat{\mathbf{v}}_f & \\
- \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) &= \hat{J}_{\boldsymbol{\chi}} \rho_f \hat{\mathbf{f}}_f && \text{on } \hat{\Omega}_f \times I,\\
\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\mathbf{v}}_f) &= 0 && \text{on } \hat{\Omega}_f \times I,\\
\hat{\mathbf{v}}_f &= \hat{\mathbf{v}}_{fD} && \text{on } \hat{\Gamma}_{fD} \times I,\\
\hat{J} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top} \hat{\mathbf{n}}_f &= \hat{\mathbf{g}}_f && \text{on } \hat{\Gamma}_{fN} \times I,\\
\hat{\mathbf{v}}_f(\cdot, 0) &= \hat{\mathbf{v}}_{0f} && \text{on } \hat{\Omega}_f,\\
\rho_s \partial_{\mathsf{t}} \hat{\mathbf{v}}_s - \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_s \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) &= \rho_s \hat{\mathbf{f}}_s && \text{on } \hat{\Omega}_s \times I,\\
\rho_s \partial_{\mathsf{t}} \hat{\mathbf{w}}_s - \rho_s \hat{\mathbf{v}}_s &= 0 && \text{on } \hat{\Omega}_s \times I,\\
\hat{\mathbf{w}}_s &= \hat{\mathbf{w}}_{sD} && \text{on } \hat{\Gamma}_{sD} \times I,\\
\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_s \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top} \hat{\mathbf{n}}_s &= \hat{\mathbf{g}}_s && \text{on } \hat{\Gamma}_{sN} \times I,\\
\hat{\mathbf{w}}_s(\cdot, 0) &= \hat{\mathbf{w}}_{0s} && \text{on } \hat{\Omega}_s,\\
\hat{\mathbf{v}}_s(\cdot, 0) &= \hat{\mathbf{w}}_1 && \text{on } \hat{\Omega}_s
\end{aligned}
\tag{1.2}
$$

with additional coupling conditions

$$\begin{aligned} \partial_{\mathsf{t}}\hat{\mathbf{w}}_s = \hat{\mathbf{v}}_s = \hat{\mathbf{v}}_f \quad &\text{on } \hat{\Gamma}_i \times I, \\ -\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_f = \hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_s\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_s \quad &\text{on } \hat{\Gamma}_i \times I, \end{aligned}$$

where the transformed fluid stress tensor is given by

$$\hat{\boldsymbol{\sigma}}_f := \rho_f\nu_f(D_{\mathsf{y}}\hat{\mathbf{v}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} + \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}D_{\mathsf{y}}\hat{\mathbf{v}}_f^{\top}) - \hat{p}_f\mathbf{I}.$$

In the following, for the sake of convencience, we will omit the f, s-indices for the functions $\hat{\mathbf{v}}_f$, $\hat{\mathbf{v}}_s$, $\hat{\mathbf{w}}_s$ and $\hat{p}_f$. Furthermore, we will denote coordinates on the physical domain $\check{\Omega}$ by x and on the reference domain $\hat{\Omega}$ by y and the subscripts of the nabla-operators indicate on which variables they act on.

Remark 1.1. That the system (1.2) coincides with the system (1.1), that is also the basis for the considerations in [113], can be motivated with the following considerations. It holds

$$\int_{\Delta\hat{\Omega}_f} \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})d\mathsf{y} = \int_{\Delta\hat{\Omega}_f} (\operatorname{div}_{\mathsf{y}}(((\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{j,\cdot})^{\top}))_j d\mathsf{y} = \int_{\partial\Delta\hat{\Omega}_f} \hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_f ds_{\mathsf{y}}.$$

With Nanson's formula we obtain

$$\int_{\partial\Delta\hat{\Omega}_f} \hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_f ds_{\mathsf{y}} = \int_{\partial\Delta\check{\Omega}_f(\mathsf{t})} \check{\mathbf{n}}_f ds_{\mathsf{x}},$$

where $\Delta\check{\Omega}_f(\mathsf{t}) = \hat{\boldsymbol{\chi}}(\Delta\hat{\Omega}_f, t)$ and $\partial\Delta\check{\Omega}_f(\mathsf{t}) = \hat{\boldsymbol{\chi}}(\partial\Delta\hat{\Omega}_f, t)$. The latter term is equal to

$$\int_{\partial\Delta\check{\Omega}_f(\mathsf{t})} \check{\mathbf{n}}_f ds_{\mathsf{x}} = \int_{\Delta\check{\Omega}_f(\mathsf{t})} \operatorname{div}_{\mathsf{x}}\mathbf{I}d\mathsf{x} = 0.$$

Thus,

$$\int_{\Delta\hat{\Omega}_f} \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})d\mathsf{y} = 0.$$

Since the test volume $\Delta\hat{\Omega}_f$ is chosen arbitrarily, we have Piola's identity

$$\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) = 0.$$

Therefore, we have

$$\begin{aligned} \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) &= \sum_i \partial_{\mathsf{y}_i}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}))_{j,i} = \sum_{i,k} \partial_{\mathsf{y}_i}(\hat{J}_{\boldsymbol{\chi}}(\hat{\boldsymbol{\sigma}}_f)_{j,k}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i}) \\ &= \sum_{i,k} \partial_{\mathsf{y}_i}(\hat{J}_{\boldsymbol{\chi}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i})(\hat{\boldsymbol{\sigma}}_f)_{j,k} + \partial_{\mathsf{y}_i}((\hat{\boldsymbol{\sigma}}_f)_{j,k})\hat{J}_{\boldsymbol{\chi}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i} \\ &= \sum_k \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_k(\hat{\boldsymbol{\sigma}}_f)_{j,k} + \sum_{i,k} \partial_{\mathsf{y}_i}((\hat{\boldsymbol{\sigma}}_f)_{j,k})\hat{J}_{\boldsymbol{\chi}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i}, \end{aligned}$$

implying that

$$\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) = \sum_{i,k}\partial_{\mathsf{y}_i}((\hat{\boldsymbol{\sigma}}_f)_{j,k})\hat{J}_{\boldsymbol{\chi}}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i}.$$

In addition, it holds that

$$\begin{aligned}(\operatorname{div}_{\mathsf{x}}(\check{\boldsymbol{\sigma}}_f))_j &= \sum_k \partial_{x_k}(\check{\boldsymbol{\sigma}}_f)_{j,k} = \sum_k \partial_{x_k}((\hat{\boldsymbol{\sigma}}_f)_{j,k}\circ\hat{\boldsymbol{\chi}}^{-1}) = \sum_{i,k}(\partial_{\mathsf{y}_i}(\hat{\boldsymbol{\sigma}}_f)_{j,k})\circ\hat{\boldsymbol{\chi}}^{-1}\partial_{x_k}\hat{\boldsymbol{\chi}}_i^{-1}\\ &= \sum_{i,k}(\partial_{\mathsf{y}_i}(\hat{\boldsymbol{\sigma}}_f)_{j,k})\circ\hat{\boldsymbol{\chi}}^{-1}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1})_{i,k}\circ\hat{\boldsymbol{\chi}}^{-1} = (\sum_{i,k}\partial_{\mathsf{y}_i}((\hat{\boldsymbol{\sigma}}_f)_{j,k})(\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{k,i})\circ\hat{\boldsymbol{\chi}}^{-1}.\end{aligned}$$

This shows that

$$\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) = \hat{J}_{\boldsymbol{\chi}}(\operatorname{div}_{\mathsf{x}}(\check{\boldsymbol{\sigma}}_f))\circ\hat{\boldsymbol{\chi}}.$$

Due to its nonlinearity FSI problems are a challenging problem. It can be simplified by reduction of the introduced system to a model that is linear or steady or has a stationary interface. Thus, we obtain an hierarchy of increasing difficulties, ranging from steady, e.g., [51, 132], to unsteady, e.g., [17, 27, 28, 34, 38, 72, 83, 84, 85, 113], from stationary, e.g., [9, 34, 38, 84, 85], to moving interfaces, e.g., [17, 27, 28, 72, 83, 113], and from linear, e.g., [34, 38, 132], to nonlinear models, e.g., [9, 17, 27, 28, 51, 72, 83, 84, 85, 113]. We first focus on an unsteady, linear model with stationary interface. Since the interface is stationary (and we assume the outer fluid boundary to be fixed) there is no distinction between $\check{\Omega}(\mathsf{t})$ and $\hat{\Omega}$ for $\mathsf{t}\in I$. For that reason and for the sake of clarity we omit the superscripts. From an applicational point of view, the model corresponds to a physical situation in which the displacement of the solid is one order of magnitude smaller than its velocity, i.e., the solid oscillates rapidly with small amplitude [9]. From an analytical point of view, it can be seen as the foundation for the analysis of a more difficult setting. In [113] an existence and regularity result for the system with stationary interface

$$\begin{aligned}\partial_{\mathsf{t}}\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} &\quad \text{in } Q_f^T,\\ \operatorname{div}(\mathbf{v}) = g = \operatorname{div}(\mathbf{g}) &\quad \text{in } Q_f^T,\\ \mathbf{v}(\cdot,0) = \mathbf{v}_0 &\quad \text{in } \Omega_f,\\ \mathbf{v} = \mathbf{v}_D &\quad \text{on } \Sigma_f^T,\\ \mathbf{v} = \partial_{\mathsf{t}}\mathbf{w} &\quad \text{on } \Sigma_i^T,\\ \sigma_f(\mathbf{v},p)\mathbf{n}_f = \sigma_s(\mathbf{w})\mathbf{n}_f + \mathbf{h} &\quad \text{on } \Sigma_i^T,\\ \partial_{\mathsf{tt}}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) = 0 &\quad \text{in } Q_s^T,\\ \mathbf{w}(\cdot,0) = 0 &\quad \text{in } \Omega_s,\\ \partial_{\mathsf{t}}\mathbf{w}(\cdot,0) = \mathbf{w}_1 &\quad \text{in } \Omega_s,\\ \mathbf{w} = 0 &\quad \text{on } \Sigma_s^T,\end{aligned} \tag{1.3}$$

is used for the derivation of a local-in-time existence and regularity result for the unsteady, non-linear model with moving interface that couples the Navier-Stokes equations with linear elasticity, the ALE reformulation of which corresponds to system (1.3) if the nonhomogeneities $\mathbf{h}$, $\mathbf{f}$, $\mathbf{g}$ are chosen such that they collect the appearing non-linear terms. We denote the exterior fluid boundary by $\Gamma_f = \partial\Omega_f \backslash \Gamma_i$ and the exterior solid boundary by $\Gamma_s = \partial\Omega_s \backslash \Gamma_i$. The corresponding space-time cylincers are denoted by Σ_s^T and Σ_f^T, respectively. In the considered case we have $\Gamma_{fD} = \Gamma_f$ and $\Gamma_{sD} = \Gamma_s$. For the sake of simplicity we set $\rho_f = \rho_s = 1$ and omit the subscript of ν_f. The fluid stress tensor is given by

$$\sigma_f(\mathbf{v}, p) = 2\nu\epsilon(\mathbf{v}) - p\mathbf{I},$$

where $D\mathbf{v}$ denotes the Jacobian of $\mathbf{v}$ and

$$\epsilon(\mathbf{v}) = \frac{1}{2}(D\mathbf{v} + (D\mathbf{v})^\top).$$

The solid stress tensor is defined by

$$\sigma_s(\mathbf{w}) = \lambda \mathrm{tr}\epsilon(\mathbf{w})\mathbf{I} + 2\mu\epsilon(\mathbf{w}),$$

where the Lamé constants λ, μ are chosen such that $\mu > 0$ and $\lambda + \mu > 0$. In addition, $\mathbf{v}_0$, $\mathbf{w}_1$ denote appropriate initial conditions on the fluid velocity and the time-derivative of the solid displacement.

One of the main difficulties that arise in the analysis of FSI problems is the a-priori mismatch between the regularity of the solutions of the Navier-Stokes equations and the elasticity equations. Possibilities to circumvent this issue are adding a structural damping term that regularizes the hyperbolic dynamics [72], using a finite dimensional approximation for the elasticity [17], or to consider smooth data which yields local-in-time existence of smooth solutions but leads to a loss of regularity, e.g., the regularity of the initial velocity needs to be in $H^5(\Omega_f)^d$ while it is only proven that the regularity of the velocity is in $L^2((0,T), H^3(\Omega_f)^d)$, cf. [27, 28]. These results were improved by [83] but still imply a slight loss of regularity. Another approach is given by establishing improved or hidden regularity results for the normal derivative of the hyperbolic solution [86] which allow to show existence and regularity results without additional damping terms [9, 84, 85, 113]. The way, how these hidden regularity results are established in [113] requires a restriction on the geometry of the domain. Particularly, the interface between the solid and fluid region needs to be flat which also requires periodicity in order to handle the problem analytically on a bounded domain. In this thesis, we show that the same hidden regularity results can be obtained for the wave equation without additional requirements on the domain.

As already mentioned before, from the existence and regularity result for the unsteady, linear setting with stationary interface it is straightforward to derive local-in-time existence and regularity results for the unsteady Navier-Stokes-Lamé system with moving interface following the argumentation of [113], cf. Section 3.4. In the fully Lagrangian setting, this system reads as follows:

$$\begin{aligned}
\partial_{\mathsf{t}}\hat{\mathbf{v}} - \nu\Delta_{\mathsf{y}}\hat{\mathbf{v}} + \nabla_{\mathsf{y}}\hat{p} = \hat{\mathcal{F}}(\hat{\mathbf{v}},\hat{p}) &\quad \text{in } \hat{Q}_f^T,\\
\mathrm{div}_{\mathsf{y}}(\hat{\mathbf{v}}) = \hat{\mathcal{G}}(\hat{\mathbf{v}}) = \mathrm{div}_{\mathsf{y}}(\hat{\mathbf{g}}(\hat{\mathbf{v}})) &\quad \text{in } \hat{Q}_f^T,\\
\hat{\mathbf{v}}(\cdot,0) = \hat{\mathbf{v}}_0 &\quad \text{in } \hat{\Omega}_f,\\
\hat{\mathbf{v}} = \hat{\mathbf{v}}_D &\quad \text{on } \hat{\Sigma}_f^T,\\
\hat{\mathbf{v}} = \partial_{\mathsf{t}}\hat{\mathbf{w}} &\quad \text{on } \hat{\Sigma}_i^T,\\
\sigma_{f,\mathsf{y}}(\hat{\mathbf{v}},\hat{p})\mathbf{n}_f = \sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})\mathbf{n}_f + \hat{\mathcal{H}}(\hat{\mathbf{v}},\hat{p}) &\quad \text{on } \hat{\Sigma}_i^T,\\
\partial_{\mathsf{tt}}\hat{\mathbf{w}} - \mathrm{div}(\sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})) = 0 &\quad \text{in } \hat{Q}_s^T,\\
\hat{\mathbf{w}}(\cdot,0) = 0 &\quad \text{in } \hat{\Omega}_s,\\
\partial_{\mathsf{t}}\hat{\mathbf{w}}(\cdot,0) = \hat{\mathbf{w}}_1 &\quad \text{in } \hat{\Omega}_s,\\
\hat{\mathbf{w}} = 0 &\quad \text{on } \hat{\Sigma}_s^T.
\end{aligned}$$

Here, the fluid and solid stress tensors $\sigma_{f,\mathsf{y}}$ and $\sigma_{s,\mathsf{y}}$ are given by

$$\sigma_{f,\mathsf{y}}(\hat{\mathbf{v}},\hat{p}) := 2\nu\epsilon_{\mathsf{y}}(\hat{\mathbf{v}}) - \hat{p}\mathbf{I}, \quad \text{and} \quad \sigma_{s,\mathsf{y}}(\hat{\mathbf{w}}) = \lambda \mathrm{tr}\epsilon_{\mathsf{y}}(\hat{\mathbf{w}})\mathbf{I} + 2\mu\epsilon_{\mathsf{y}}(\hat{\mathbf{w}}),$$

where $\epsilon_{\mathsf{y}}(\cdot) := \frac{1}{2}(D_{\mathsf{y}}\cdot + (D_{\mathsf{y}}\cdot)^\top)$ and λ, μ are Lamé coefficients that are chosen such that $\mu > 0$ and $\lambda + \mu > 0$. $D_{\mathsf{y}}(\cdot)$ denotes the Jacobian. In the fully Lagrangian setting, the transformation is given by

$$\hat{\boldsymbol{\chi}}(\cdot,\mathsf{t})|_{\hat{\Omega}_f}: \quad \hat{\Omega}_f \to \check{\Omega}_f(\mathsf{t}), \quad \mathsf{y} \to \mathsf{y} + \int_0^{\mathsf{t}} \hat{\mathbf{v}}(\mathsf{y},\mathsf{s})d\mathsf{s}$$

for any $\mathsf{t} \in (0,T)$ and its inverse $\check{\boldsymbol{\Upsilon}}(\cdot,\mathsf{t}) := (\hat{\boldsymbol{\chi}}(\cdot,\mathsf{t}))^{-1}$, which exists if $T > 0$ is sufficiently small and the initial data are smooth enough, cf. [113]. Consequently, the right hand side terms are given by:

$$\begin{aligned}
\hat{\mathcal{F}}(\hat{\mathbf{v}},\hat{p}) &= \nu \sum_{j,k} \frac{\partial^2}{\partial \mathsf{x}_j^2}\check{\boldsymbol{\Upsilon}}_k \circ \hat{\boldsymbol{\chi}} \frac{\partial}{\partial \mathsf{y}_k}\hat{\mathbf{v}} + \nu \sum_{i,j,k} \frac{\partial}{\partial \mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_i \circ \hat{\boldsymbol{\chi}} \frac{\partial}{\partial \mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_k \circ \hat{\boldsymbol{\chi}} \frac{\partial^2}{\partial \mathsf{y}_i \partial \mathsf{y}_k}\hat{\mathbf{v}}\\
&\quad - \nu\Delta_{\mathsf{y}}\hat{\mathbf{v}} + (\mathbf{I} - \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top)\nabla_{\mathsf{y}}\hat{p},\\
\hat{\mathcal{H}}(\hat{\mathbf{v}},\hat{p}) &= -\nu(D_{\mathsf{y}}\hat{\mathbf{v}}\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(D_{\mathsf{y}}\hat{\mathbf{v}})^\top)\mathrm{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\hat{\mathbf{n}}_f + \hat{p}I\mathrm{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\hat{\mathbf{n}}_f\\
&\quad + \nu(D_{\mathsf{y}}\hat{\mathbf{v}} + (D_{\mathsf{y}}\hat{\mathbf{v}})^\top)\hat{\mathbf{n}}_f - \hat{p}I\hat{\mathbf{n}}_f,\\
\hat{\mathcal{G}}(\hat{\mathbf{v}}) &= \mathrm{div}_{\mathsf{y}}\hat{\mathbf{v}} - \det(\hat{J}_X)D_{\mathsf{y}}\hat{\mathbf{v}} : \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top = D_{\mathsf{y}}\hat{\mathbf{v}} : (I - \det(\hat{J}_X)\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top),
\end{aligned}$$

where $\widehat{\mathbf{F}}_{\boldsymbol{\chi}} = D_{\mathsf{y}}\hat{\boldsymbol{\chi}} = (\nabla_{\mathsf{y}}\hat{\boldsymbol{\chi}})^\top$ is the Jacobian of $\hat{\boldsymbol{\chi}}$ and $\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} := \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}$ its inverse. Furthermore, let $\hat{\mathbf{g}}$ be defined by $\hat{\mathbf{g}}(\hat{\mathbf{v}}) := (I - \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}})\hat{\mathbf{v}}$ such that $\mathrm{div}_{\mathsf{y}}(\hat{\mathbf{g}}(\hat{\mathbf{v}})) = \hat{\mathcal{G}}(\hat{\mathbf{v}})$ due to Piola's identity.

Shape optimization can be analyzed with different, yet closely related, techniques. On the one hand, shape calculus [31, 57, 58, 64, 104, 111, 112, 122] can be used to investigate functionals $\hat{J}(\hat{\Omega})$ depending on the domain $\hat{\Omega}$. The Eulerian derivative $d\hat{J}(\hat{\Omega},\hat{V})$ admits a representation by the Hadamard-Zolésio shape gradient, a distribution that is supported on the design boundary and only acts on the normal boundary variation $\hat{V} \cdot \hat{\mathbf{n}}_f$. If a state equation is involved, the Eulerian derivative depends on the shape derivative of the state and can be expressed using an adjoint state. An alternative approach is the method of mappings [13, 20, 55, 78, 79, 89, 105, 120], also called perturbation of identity, which parametrizes the shape by a bi-Lipschitz homeomorphism $\tilde{\boldsymbol{\tau}} : \mathbb{R}^d \to \mathbb{R}^d$ via $\hat{\Omega} = \tilde{\boldsymbol{\tau}}(\tilde{\Omega})$, where $\tilde{\Omega} \subset \mathbb{R}^d$ is a nominal domain (or shape reference domain). Optimization can be performed based on the function $\tilde{J} : \tilde{\boldsymbol{\tau}} \mapsto \hat{J}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))$. An underlying state equation is then transformed to $\tilde{\Omega}$ and derivatives of $\tilde{J}$ can be obtained via sensitivities or adjoints. The Hadamard-Zolésio calculus can be derived from this approach essentially by an integration by parts. The method of mappings directly yields an optimal control setting in Banach spaces. Moreover, it fits well in the theoretical setting of the FSI model that was introduced above since it also employs the idea of domain transformations.

Optimal shape design problems for FSI have mainly been tackled by applied, engineering approaches, see, e.g., [14, 62, 63, 68, 96, 97, 98, 99, 100, 119]. For developing a theoretical foundation continuity and differentiability of the state with respect to domain variations are studied, cf. Chapter 4. So far, in all conscience, differentiability results have only been available for steady FSI models [106, 132].

Throughout the thesis the superscripts over the functions correspond to the superscripts of the domains on which they are defined. Furthermore, the spatial coordinates on the physical domain $\check{\Omega}$ are denoted by x, on $\hat{\Omega}, \tilde{\Omega}$ by y, z, respectively. If a result is valid for a general domain or if it is clear in the context the notation Ω is used and the coordinates are denoted by ξ.

In Chapter 2 the main definitions and concepts are collected, that are then used in Chapter 3 to derive existence and regularity results for a linear and a nonlinear unsteady FSI system and in Chapter 4 to derive differentiability results of the states of an unsteady nonlinear FSI system with respect to domain variations. Chapter 5 is devoted to the numerical realization of shape optimization for unsteady FSI.

2. Preliminaries

In this chapter, the basic definitions, tools and concepts are presented. We start with a short introduction to the function spaces and a collection of useful results (Section 2.2) that will be used in the theoretical analysis of the FSI system (Section 3). Furthermore, the method of successive approximations (Section 2.3) which is the foundation for the considerations in Section 2.4, the concept of extension by continuity (Section 2.5) and the method of mappings (Section 2.7) are introduced. The main contributions are the extension of well-known results on Ω to $\Gamma = \partial\Omega$ under some assumptions on Ω in Section 2.2, as well as, the framework for deriving differentiability results in Section 2.4. Parts of this chapter have already been published [59], including Sections 2.3, 2.4 and 2.2.3 to a great extend.

2.1. Geometric Topology

Let $\mathbf{n}$ be the outer unit normal vector of Ω on Γ. Then, the following holds.

Lemma 2.1. Let Ω be a bounded, smooth domain with boundary Γ of class $\mathcal{C}^\infty$. Then, there exists $\delta > 0$ such that for every $\xi \in B_\delta(\Gamma) \cap \overline{\Omega}$, there exist unique $\alpha \geq 0$ and $\xi_\Gamma \in \Gamma$ such that $\xi = \xi_\Gamma - \alpha\mathbf{n}(\xi_\Gamma)$.

Proof. This holds true due to the tubular neighborhood theorem, cf., e.g., [65, p.109, Thm. 5.1]. □

Definition 2.2. We say that a function $\mathbf{b} \in \mathcal{C}^\infty(\overline{\Omega})^d$ is constant along normal directions in a neighborhood (or locally) around Γ if there exists $\delta > 0$ such that for every $\xi \in B_\delta(\Gamma) \cap \overline{\Omega}$, there exist unique $\alpha \geq 0$ and $\xi_\Gamma \in \Gamma$ such that $\xi = \xi_\Gamma - \alpha\mathbf{n}(\xi_\Gamma)$ and $\mathbf{b}(\xi) = \mathbf{b}(\xi_\Gamma)$.

Corollary 2.3. For every $\mathbf{b}_\Gamma \in \mathcal{C}^\infty(\Gamma)^d$, there exits $\mathbf{b} \in \mathcal{C}^\infty(\overline{\Omega})^d$ such that $\mathbf{b}|_\Gamma = \mathbf{b}_\Gamma$ and $\mathbf{b}$ is constant along normal directions in a small neighborhood around Γ.

Proof. Let $\gamma(\alpha) \in \mathcal{C}^\infty([0, \delta])$ with $\gamma(\alpha) = 1$ for $\alpha \in [0, \frac{\delta^2}{2}]$ and the support of γ is compactly contained in $[0, \delta^2)$. By Lemma 2.1 and [88, p.257, Prop. 10.20], there exists a smooth retraction $\mathrm{r} : B_\delta(\Gamma) \to \Gamma$, $\xi \mapsto \xi_\Gamma$. Then the assertion follows by choosing $\mathbf{b}(\xi) := \mathbf{b}_\Gamma(\mathrm{r}(\xi))\gamma(\|\mathrm{r}(\xi) - \xi\|^2_{\mathbb{R}^d})$ if $\xi \in B_\delta(\Gamma) \cap \Omega$ and 0 on $\Omega \setminus B_\delta(\Gamma)$. □

Corollary 2.4. There exists $\mathbf{h} \in \mathcal{C}^\infty(\overline{\Omega})^d$ such that for all $\mathbf{b} \in \mathcal{C}^\infty(\overline{\Omega})^d$ with $\mathbf{b} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{b}$ being constant along normal directions in a small neighborhood around Γ, there holds

$$\mathbf{h}|_\Gamma = \mathbf{n},$$

$$\begin{aligned}
&\mathbf{h} \text{ is constant along normal directions locally around } \Gamma, \\
&\nabla \mathbf{h}_l \cdot \mathbf{h} = 0, \\
&\nabla \mathbf{b}_l \cdot \mathbf{h} = 0,
\end{aligned}$$

for all $l \in \{1, \dots, d\}$ in a small neighborhood around Γ.

Proof. By Corollary 2.3, we obtain $\mathbf{h}$ such that $\mathbf{h}|_\Gamma = \mathbf{n}$ and $\mathbf{h}$ is constant along normal directions in a small neighborhood around Γ, i.e., there exists $\delta > 0$ such that for every $\xi \in B_\delta(\Gamma) \cap \overline{\Omega}$ there exist $\alpha \in [0, \delta)$ and $\xi_\Gamma \in \Gamma$ such that $\xi = \xi_\Gamma - \alpha \mathbf{n}(\xi_\Gamma)$ and $\mathbf{h}(\xi) = \mathbf{n}(\xi_\Gamma)$ for all $\xi \in J := \{\xi_\Gamma - t\mathbf{n}(\xi_\Gamma) | \ t \in [0, \delta)\}$. Now, for $\xi \in B_\delta(\Gamma) \cap \Omega$,

$$\nabla \mathbf{h}(\xi)^T \mathbf{h}(\xi) = -\partial_t \mathbf{h}(\xi_\Gamma - t\mathbf{n}(\xi_\Gamma))|_{t=\alpha} = -\partial_t \mathbf{n}(\xi_\Gamma)|_{t=\alpha} = 0.$$

Since $\mathbf{b}$ is constant along normal directions in a small neighborhood around Γ, there exists $0 < \epsilon \leq \delta$ such that $\mathbf{b}(\xi_\Gamma - t\mathbf{n}(\xi_\Gamma)) = \mathbf{b}(\xi_\Gamma - t\mathbf{h}(\xi_\Gamma)) = \mathbf{b}(\xi_\Gamma)$ for all $t \in (0, \epsilon)$. The derivative wrt. t at $t = \alpha$ therefore yields $\nabla \mathbf{b}_l(\xi) \cdot \mathbf{h}(\xi) = 0$ for all $\xi \in B_\epsilon(\Gamma) \cap \Omega$ and $l \in \{1, \dots, d\}$. Since $\nabla \mathbf{b}_l \cdot \mathbf{h} \in \mathcal{C}^\infty(\overline{\Omega})$ and $\nabla \mathbf{h}_l \cdot \mathbf{h} \in \mathcal{C}^\infty(\overline{\Omega})$ it holds that $\nabla \mathbf{b}_l(\xi) \cdot \mathbf{h}(\xi) = 0$ and $\nabla \mathbf{h}_l(\xi) \cdot \mathbf{h}(\xi) = 0$ for all $\xi \in B_\epsilon(\Gamma) \cap \overline{\Omega}$. □

2.2. Function Spaces

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded open domain with boundary $\Gamma = \partial\Omega$ of class $\mathcal{C}^\infty$. In the following, useful properties are collected and proved. The presentation is mainly based on Lions and Magenes [90, 91].
Let $s \in \mathbb{R}$. The Hilbert space $H^s(\mathbb{R}^d)$ is defined by

$$H^s(\mathbb{R}^d) = \{v \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|v\|_{H^s(\mathbb{R}^d)} < \infty\},$$

with norm

$$\|v\|_{H^s(\mathbb{R}^d)} := \|(1 + |\xi|^2)^{\frac{s}{2}} \mathscr{F} v(\xi)\|_{L^2(\mathbb{R}^d)},$$

where $\mathscr{F} v$ denotes the Fourier transform of v and $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions.

2.2.1. On the Space $H^s(\Gamma)$

The definition of the fractional order Sobolev spaces $H^s(\Gamma)$, $s \in \mathbb{R}$ is based on the definition of $H^s(\mathbb{R}^d)$, cf. [90, pp. 34-35]. Under the standing assumptions on Ω a system $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$ can be found, which consists of

- a finite family $\{\mathcal{O}_j, \ j \in \{1, \dots, N\}\}$ of open, bounded sets that covers Γ.
- a finite family $\{\varphi_j, \ j \in \{1, \dots, N\}\}$ of infinitely differentiable functions

  $$\varphi_j : \mathcal{O}_j \to B_1(0) := \{\mathsf{y} \in \mathbb{R}^d \ : \ \mathsf{y} < 1\},$$

  that

- locally flatten the boundary, i.e.,

$$\varphi_j(\Gamma \cap \mathcal{O}_j) = B_1(0) \cap \{\mathsf{y}_d = 0\} \subset \mathbb{R}^{d-1}_{\mathsf{y}'} := \{(\mathsf{y}', 0) \ : \ \mathsf{y}' \in \mathbb{R}^{d-1}\},$$

  and $\varphi_j(\Omega \cap \mathcal{O}_j) = B_1(0) \cap \{\mathsf{y}_d > 0\}$.
- have an infinitely differentiable inverse φ_j^{-1}.
- fulfill a compatibility condition for all i, j such that $\mathcal{O}_j \cap \mathcal{O}_i \neq \emptyset$ which requires that there exists an infinitely times differentiable homeomorphism J_{ij} of $\varphi_i(\mathcal{O}_i \cap \mathcal{O}_j)$ with positive jacobian such that $\varphi_j(\xi) = J_{ij}(\varphi_i(\xi))$ for all $\xi \in \mathcal{O}_i \cap \mathcal{O}_j$.

$\{\mathcal{O}_j, \varphi_j\}$ is the system of local maps.

- a partition of unity $\{\alpha_j\}$, where $\alpha_j \in \mathcal{C}^\infty(\Gamma)$ is non-negative, has compact support on $\mathcal{O}_j \cap \Gamma$ and adds up to 1, i.e. $\sum_{j=1}^N \alpha_j = 1$ on Γ.

The main definitions and properties are collected in the following.

- For $s \in \mathbb{R}$, let $H^s(\Gamma)$ be defined by

$$H^s(\Gamma) := \{u \ : \ \varphi_j^*(\alpha_j u) \in H^s(\mathbb{R}^{d-1}_{\mathsf{y}'}), j \in \{1, \dots, N\}\},$$

  with norm

$$\|u\|_{H^s(\Gamma)} = (\sum_{j=1}^N \|\varphi_j^*(\alpha_j u)\|^2_{H^s(\mathbb{R}^{d-1}_{\mathsf{y}'})})^{\frac{1}{2}},$$

  where $\varphi_j^*(\alpha_j u)(\mathsf{y}') := \begin{cases} (\alpha_j u)(\varphi_j^{-1}(\mathsf{y}')) & \text{if } \mathsf{y}' \in B_1(0) \cap \mathbb{R}^{d-1}_{\mathsf{y}'}, \\ 0 & \text{if } \mathsf{y}' \in \mathbb{R}^{d-1}_{\mathsf{y}'} \setminus B_1(0). \end{cases}$
  This definition depends on the choice of $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$, however, it can be verified that all norms are equivalent.

- $\mathcal{C}^\infty(\Gamma)$ is dense in $H^s(\Gamma)$ for $s \geq 0$.

2.2.2. On the Space $H^s(\Omega)$, $s \geq 0$

Let $s \geq 0$. Then, the Hilbert space $H^s(\Omega)$ is defined as an interpolation space of integer-valued Sobolev spaces

$$H^s(\Omega) = [H^m(\Omega), H^0(\Omega)]_\theta,$$

where $m \in \mathbb{N}$ and $\theta \in (0, 1)$ are such that $s = (1-\theta)m$ [90, Ch. 1, Sec. 9] and $H^0(\Omega) = L^2(\Omega)$. The norm $\|\cdot\|_{H^s(\Omega)}$ is defined by [90, p. 10, Def. 2.1]. Under the standing assumptions on Ω all definitions of $H^s(\Omega)$ with respect to different choices of θ and m are equivalent and it can be shown that

- $H^s(\Omega) = \{u \ : \ u = r_\Omega v, \ v \in H^s(\mathbb{R}^d)\}$ can be endowed with the equivalent norm

$$\|u\|_{H^s(\Omega)} \equiv \|u\|_s := \inf\{\|v\|_{H^s(\mathbb{R}^d)} \ : \ v \in H^s(\mathbb{R}^d), \ u = r_\Omega v\},$$

  where r_Ω denotes the restriction on Ω.

- $\mathcal{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$.
- $$[H^{s_1}(\Omega), H^{s_2}(\Omega)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\Omega)$$
  and there exists a constant $C > 0$ that generally depends on Ω such that
  $$\|u\|_{H^{(1-\theta)s_1+\theta s_2}(\Omega)} \leq C\|u\|_{H^{s_1}(\Omega)}^{1-\theta}\|u\|_{H^{s_2}(\Omega)}^{\theta}$$
  for all $u \in H^{s_1}(\Omega)$, $0 < s_2 < s_1$ and $\theta \in (0,1)$.
- if $s > \frac{d}{2} + m$, $m \in \mathbb{N}_0$ it holds that
  $$H^s(\Omega) \subset \mathcal{C}^m(\bar{\Omega})$$
  with continuous injection.
- If the standing assumptions on Ω are fulfilled, the following trace inequality holds true [90, p.41, Thm. 9.4]. Let $s > \frac{1}{2}$ and $u \in H^s(\Omega)$. Then, $u \in H^{s-\frac{1}{2}}(\Gamma)$ and $\|u\|_{H^{s-\frac{1}{2}}(\Gamma)} \leq C\|u\|_{H^s(\Omega)}$, where C depends on Ω.

Furthermore, the following result corresponds to [54, Prop. B.1 (i)]. Even though this lemma will only be used for fractional order Sobolev spaces with positiv order, it is stated in the general setting, which allows for fractional Sobolev spaces with negativ order. A introduction to these spaces can be found in [90, Ch. 1, Sec. 12].

Lemma 2.5. Let the standing assumptions on $\Omega \subset \mathbb{R}^d$ be fulfilled, $\lambda, \mu, \omega \in \mathbb{R}$. Additionally, let $f \in H^{\lambda+\mu}(\Omega)$ and $g \in H^{\lambda+\omega}(\Omega)$. Then, there exists $C > 0$ such that

$$\|fg\|_{H^\lambda(\Omega)} \leq C\|f\|_{H^{\lambda+\mu}(\Omega)}\|g\|_{H^{\lambda+\omega}(\Omega)},$$

1. if $\mu + \omega + \lambda \geq \frac{d}{2}$, $\mu > 0$, $\omega > 0$, and $2\lambda > -\mu - \omega$,
2. or $\mu + \omega + \lambda > \frac{d}{2}$, $\mu \geq 0$, $\omega \geq 0$, and $2\lambda \geq -\mu - \omega$.

From the definition of the Sobolev spaces on the boundary Γ it is straightforward to deduce the following Lemma.

Lemma 2.6. Let the standing assumptions on $\Omega \subset \mathbb{R}^d$ with boundary Γ of class $\mathcal{C}^\infty$ be fulfilled, $\lambda, \mu, \omega \in \mathbb{R}$. Additionally, let $f \in H^{\lambda+\mu}(\Gamma)$ and $g \in H^{\lambda+\omega}(\Gamma)$. Then, there exists $C > 0$ such that

$$\|fg\|_{H^\lambda(\Gamma)} \leq C\|f\|_{H^{\lambda+\mu}(\Gamma)}\|g\|_{H^{\lambda+\omega}(\Gamma)},$$

1. if $\mu + \omega + \lambda \geq \frac{d-1}{2}$, $\mu > 0$, $\omega > 0$, and $2\lambda > -\mu - \omega$,
2. or $\mu + \omega + \lambda > \frac{d-1}{2}$, $\mu \geq 0$, $\omega \geq 0$, and $2\lambda \geq -\mu - \omega$.

Proof. Let the extended system $\{\mathcal{O}_j, \varphi_j, \alpha_j, \hat{\mathcal{O}}_j, \psi_j\}$ be given, which is chosen such that:

- the family $\{\mathcal{O}_j,\ j \in \{1,\dots,N\}\}$ is chosen such that there exists $\{\hat{\mathcal{O}}_j,\ j \in \{1,\dots,N\}\}$ that covers Γ with $\hat{\mathcal{O}}_j$ compactly contained in $\mathcal{O}_j$ for all $j \in \{1,\dots,N\}$.

- the partition of unity $\{\alpha_j\}$ is chosen such that $\alpha_j \in \mathcal{C}^\infty(\Gamma)$ has compact support on $\hat{\mathcal{O}}_j \cap \Gamma$. Additionally, we choose $\{\psi_j,\ j \in \{1,\dots,N\}\}$ such that $\psi_j \in \mathcal{C}^\infty(\Gamma)$ has compact support on $\mathcal{O}_j \cap \Gamma$, $\psi_j(\xi) \geq 0$, and is identical to one on an open neighborhood around $\hat{\mathcal{O}}_j \cap \Gamma$.

The existence of such a system is ensured by the following considerations. Let $\{\tilde{\mathcal{O}}_i, \tilde{\varphi}_i,\ i \in \{1,\dots,\tilde{N}\}\}$ be a system of local maps of Γ. Hence, for every $\xi \in \Gamma$ there exists $\epsilon_\xi > 0$ and $i_\xi \in \{1,\dots,\tilde{N}\}$ such that $B_{\epsilon_\xi}(\xi)$ is compactly contained in $\mathcal{O}_{i_\xi}$. Since the system of local maps exists, Γ is compact and there exists a finite subcover $\{\hat{\mathcal{O}}_j,\ j \in \{1,\dots,N\}$, where $N \in \mathbb{N}$ and $\hat{\mathcal{O}}_j = B_{\epsilon_{\xi_j}}(\xi_j)$ for $\xi_j \in \Gamma$. Furthermore, let $\varphi_j := \tilde{\varphi}_{i_{\xi_j}}$, $\mathcal{O}_j := \tilde{\mathcal{O}}_{i_{\xi_j}}$ and $\{\alpha_j\}$ be the partition of unity constructed on the finite subcover $\{\hat{\mathcal{O}}_j,\ j \in \{1,\dots,N\}\}$. In addition, $\{\psi_j,\ j \in \{1,\dots,N\}\}$ is chosen such that it fulfills the requirements.
For $j \in \{1,\dots,N\}$, choose $\hat{D}_j \subset \mathbb{R}^{d-1}_{y'}$ such that $\text{supp}(\alpha_j) \subset \hat{D}_j \subset \varphi_j^*(\Gamma \cap \hat{\mathcal{O}}_j)$ and $\hat{D}_j$ is a domain with a smooth boundary of class $\mathcal{C}^\infty$. Then, due to [90, p.60, Thm. 11.4], there exists $C > 0$ such that

$$\begin{aligned}\|fg\|_{H^\lambda(\Gamma)} &= \Big(\sum_{j=1}^N \|\varphi_j^*(\alpha_j fg)\|^2_{H^\lambda(\mathbb{R}^{d-1}_{y'})}\Big)^{\frac{1}{2}} \\ &\leq C\Big(\sum_{j=1}^N \|\varphi_j^*(\alpha_j f)\varphi_j^*(\psi_j g)\|^2_{H^\lambda(\hat{D}_j)}\Big)^{\frac{1}{2}}.\end{aligned}$$

Now, since either 1 or 2 is fulfilled, we can apply Lemma 2.5 in order to obtain

$$\|fg\|_{H^\lambda(\Gamma)} \leq C\Big(\sum_{j=1}^N \|\varphi_j^*(\alpha_j f)\|^2_{H^{\lambda+\mu}(\hat{D}_j)} \|\varphi_j^*(\psi_j g)\|^2_{H^{\lambda+\omega}(\hat{D}_j)}\Big)^{\frac{1}{2}}.$$

In order to estimate the second factor, we see that with

$$\mathcal{K} := \{k \in \{1,\dots,N\} \text{ such that } \mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset\}$$

and $\Psi \in \mathcal{C}^\infty(\Gamma)$ with compact support on $\mathcal{O}_j \cap \Gamma$, we have

$$\begin{aligned}&\|\varphi_j^*(\Psi)\|_{H^m(\mathbb{R}^{d-1}_{y'})} \\ &\leq C\|\sum_{k\in\mathcal{K}} \varphi_j^*(\alpha_k \Psi)\|_{H^m(\mathbb{R}^{d-1}_{y'})} = C\|\sum_{k\in\mathcal{K}} (\alpha_k\Psi) \circ \varphi_k^{-1} \circ \varphi_k \circ \varphi_j^{-1}\|_{H^m(\mathbb{R}^{d-1}_{y'})} \\ &= C\|\sum_{k\in\mathcal{K}} (\alpha_k\Psi) \circ \varphi_k^{-1} \circ J_{jk}\|_{H^m(\mathbb{R}^{d-1}_{y'})} \leq C\sum_{k\in\mathcal{K}} \|(\alpha_k\Psi) \circ \varphi_k^{-1}\|_{H^m(\mathbb{R}^{d-1}_{y'})} \\ &= C\sum_{k\in\mathcal{K}} \|\varphi_k^*(\alpha_k\Psi)\|_{H^m(\mathbb{R}^{d-1}_{y'})},\end{aligned} \tag{2.1}$$

where we used that $\|\cdot \circ J_{jk}\|_{H^m(\mathbb{R}^{d-1}_{y'})} \leq C\|\cdot\|_{H^m(\mathbb{R}^{d-1}_{y'})}$ for the infinitely differentiable homeomorphism J_{jk}. Hence,

$$\|\varphi_j^*(\psi_j g)\|_{H^{\lambda+\omega}(\hat{D}_j)} \leq C\|g\|_{H^{\lambda+\omega}(\Gamma)}$$

and, therefore,

$$\|fg\|_{H^\lambda(\Gamma)} \leq C\|f\|_{H^{\lambda+\mu}(\Gamma)}\|g\|_{H^{\lambda+\omega}(\Gamma)}.$$

□

2.2.3. On the Spaces $H^s((0,T), H^r(\Omega))$ and $H^{s,r}(\Omega \times (0,T))$

Let $s, r \in [0,\infty)$, $\theta \in (0,1)$, $X, \tilde{X}, Y, \tilde{Y}, Z$ be Hilbert spaces, Ω be a bounded open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, with smooth boundary $\partial\Omega = \Gamma$ of class $\mathcal{C}^\infty$. For $T > 0$, $Q^T := \Omega \times (0,T)$ and $\Sigma^T := \Gamma \times (0,T)$ denote space-time cylinders. The analysis is carried out in fractional order Sobolev spaces $H^s((0,T), H^r(\Omega))$ and in anisotropic Sobolev spaces $H^{r,s}(Q^T)$. The vector-valued versions are denoted by $H^s((0,T), H^r(\Omega)^d)$ and $(H^{r,s}(Q^T))^d$. For more details on these spaces the reader is referred to [90, Ch. 1, Sec. 9], [91, Ch. 4, Sec. 2] and [53, Sec. 2].

$H^s((0,T), X)$

The fractional order Sobolev spaces $H^s((0,T), X)$ can be endowed with the norm

$$|\cdot|_{H^s((0,T),X)} = (\|\cdot\|^2_{H^m((0,T),X)} + |\partial_{\mathsf{t}}^m \cdot|^2_{\sigma,(0,T),X})^{\frac{1}{2}}, \tag{2.2}$$

where m, σ are chosen such that $s = m + \sigma$, $m \in \mathbb{N}_0$ and for $0 < \sigma < 1$ the semi-norm $|\cdot|_{\sigma,(0,T),X}$ is defined by

$$|\cdot|^2_{\sigma,(0,T),X} = \int_0^T \int_0^T \frac{\|\cdot(\mathsf{t}) - \cdot(\mathsf{s})\|^2_X}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s} d\mathsf{t}.$$

Remark 2.7. This norm is equivalent to the norm introduced in [90, p.10, Def. 2.1], which is equivalent to the complex interpolation norm due to [90, p. 92, Thm. 14.1 and p. 23, Remark 3.6]. In the Hilbert space setting, complex and real interpolation norms are equivalent due to [26, Thm. 3.3 and Rem. 3.6]. [8, (3.4), (3.5) and (3.7)] concludes the argumentation. More details, e.g., the definition of '$\doteq$', can be found in [7, Sec. 5]. For the equivalence of Besov and Sobolev-Slobodeckij spaces for $\sigma \in (0,1)$, the reader is also refered to [121, Prop. 2], for the interpolation of Besov spaces to [121, Proof of Thm. 30] and the references therin, [24, p. 194, Theo. 3.4.2].

The theoretical analysis requires knowledge about the T-dependency of appearing constants since fixed point type arguments are used for small time horizons. Hence, the choice of the norm on the spaces $H^s((0,T), X)$ is crucial. More precisely, for $-\infty < T_1 < T_2 < \infty$

and $T_f \geq T$, the spaces $H^s((T_1,T_2),X)$ and the subspaces

$$Y^s_{(T_1,T_2)} := \begin{cases} \{u \in H^s((T_1,T_2),X)\} & \text{if } s \in [0,\frac{1}{2}),\\ \{u \in H^s((T_1,T_2),X) \ : \ u(T_1)=0\} & \text{if } s \in (\frac{1}{2},1],\\ \{u \in H^s((T_1,T_2),X) \ : \ u(T_1)=0,\ \partial_{\mathsf{t}} u \in Y^{s-1}_{(T_1,T_2)}\} & \text{if } s>1,\ s+\frac{1}{2}\notin \mathbb{N}, \end{cases}$$

are endowed with a norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ such that

P1 for all $s \geq 1$ such that $s+\frac{1}{2} \notin \mathbb{N}$,

$$\|\cdot\|_{H^s((T_1,T_2),X)} = (\|\cdot\|^2_{L^2((T_1,T_2),X)} + \|\partial_{\mathsf{t}}(\cdot)\|^2_{H^{s-1}((T_1,T_2),X)})^{\frac{1}{2}}.$$

and $\|\cdot\|_{H^0((T_1,T_2),X)} = \| \cdot\|_{L^2((T_1,T_2),X)}$, where $\| \cdot\|_{L^2((T_1,T_2),X)}$ denotes the standard $L^2((T_1,T_2),X)$-norm.

P2 for all $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$, there exist constants $c_{\Delta T},\ C_{\Delta T} > 0$ depending on $\Delta T = T_2 - T_1$ such that

$$c_{\Delta T}|\cdot|_{H^s((T_1,T_2),X)} \leq \|\cdot\|_{H^s((T_1,T_2),X)} \leq C_{\Delta T}|\cdot|_{H^s((T_1,T_2),X)}.$$

P3 for all $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$, the extension operator Ext defined by

$$\mathrm{Ext}(u)(\mathsf{t}) := \begin{cases} u(\mathsf{t}) & \text{if } \mathsf{t} \in (0,T),\\ 0 & \text{if } \mathsf{t} \in (T-T_f,0), \end{cases}$$

is continuous as a mapping $Y^s_{(0,T)} \to Y^s_{(T-T_f,T)}$ with a continuity constant that does not depend on T.

P4 for all $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$, we have

$$\|u\|_{H^s((0,T),X)} \leq C\|u\|_{H^s((T-T_f,T),X)}$$

for all $u \in H^s((T-T_f,T),X)$ such that $u|_{(T-T_f,0)} = 0$ with a constant C independent of T.

P5 for all $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$, the restriction operator R defined by

$$\mathrm{R}(u)(\mathsf{t}) := u(\mathsf{t})$$

is continuous as a mapping $H^s((0,T_f),X) \to H^s((0,T),X)$ with a continuity constant that does not depend on T.

P6 for $s \in [0,1) \setminus \{\frac{1}{2}\}$ and $\epsilon > 0$ such that $s+\epsilon \in (0,1] \setminus \{\frac{1}{2}\}$, we have

$$\|u\|_{H^s((0,T),X)} \leq CT^{\epsilon}\|u\|_{H^{s+\epsilon}((0,T),X)}$$

for all $u \in Y^s_{(0,T)}$ with a constant C that does not depend on T.

P7 for $s \in [0,1] \setminus \{\frac{1}{2}\}$, real, separable Hilbert spaces X_1, X_2 and a linear operator K that is continuous as a mapping from X_1 to X_2, we have

$$\|K(u)\|_{H^s((0,T),X_2)} \leq C\|u\|_{H^s((0,T),X_1)}$$

for all $u \in H^s((0,T),X_1)$ with a constant C that does not depend on T.

P8 for all $s \geq 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$, $T_1 < T_2$,

$$\|u\|_{H^s((T_1,T_2),X)} = \|\tilde{u}\|_{H^s((0,T_2-T_1),X)},$$

for all $u \in H^s((T_1,T_2),X)$, where $\tilde{u}(\mathsf{t}) := u(\mathsf{t}+T_1)$ for (a.e.) $\mathsf{t} \in (0, T_2 - T_1)$.

Lemma 2.8. Let X be a separable Hilbert space, $-\infty < T_1 < T_2 < \infty$. There exists a norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ on $H^s((T_1,T_2),X)$ that fulfills **P1 - P8**.

Proof. cf. Section A.1. □

Let X, Y and $\tilde{X}, \tilde{Y}$, respectively, be continuously embedded in a Hausdorff topological vector space V and $\tilde{V}$, respectively. For $s_0, s_1 \in [0,\infty)$, $s_0 > s_1$, by [8, (3.5)–(3.7), Thm. 3.1, Cor. 4.3], [26, Rem. 3.6], and [16, Thm. 3.4.1], there holds

$$[H^{s_0}((0,T),X), H^{s_1}((0,T),Y)]_\theta = H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta)$$

and by the interpolation inequality we know

$$\|\cdot\|_{H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta)} \leq C\|\cdot\|^{1-\theta}_{H^{s_0}((0,T),X)}\|\cdot\|^{\theta}_{H^{s_1}((0,T),Y)} \tag{2.3}$$

for a constant C that might depend on T, cf., e.g., [90, p.19, Prop. 2.3]. Let, in addition, $\theta \in (0,1)$ and $\tilde{s}_0, \tilde{s}_1 \in [0,\infty)$, $\tilde{s}_0 > \tilde{s}_1$. If

$$\mathcal{A} \in \mathcal{L}(H^{s_0}((0,T),X), H^{\tilde{s}_0}((0,T),\tilde{X})) \cap \mathcal{L}(H^{s_1}((0,T),Y), H^{\tilde{s}_1}((0,T),\tilde{Y})),$$

then, $\mathcal{A} \in \mathcal{L}(H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta), H^{(1-\theta)\tilde{s}_0+\theta\tilde{s}_1}((0,T),[\tilde{X},\tilde{Y}]_\theta))$ and

$$\begin{aligned}
&\|\mathcal{A}\|_{\mathcal{L}(H^{(1-\theta)s_0+\theta s_1}((0,T),[X,Y]_\theta),H^{(1-\theta)\tilde{s}_0+\theta\tilde{s}_1}((0,T),[\tilde{X},\tilde{Y}]_\theta))} \\
&\leq C\|\mathcal{A}\|^{1-\theta}_{\mathcal{L}(H^{s_0}((0,T),X),H^{\tilde{s}_0}((0,T),\tilde{X}))}\|\mathcal{A}\|^{\theta}_{\mathcal{L}(H^{s_1}((0,T),Y),H^{\tilde{s}_1}((0,T),\tilde{Y}))},
\end{aligned}$$

for a constant C that might depend on T, cf., e.g., [25, p.115, 4.].

Lemma 2.9. ([59, Lem. 2]) Let X, Y, Z be real, separable Hilbert spaces and m be a bounded bilinear mapping from $X \times Y$ into Z. Furthermore, let $f \in H^{s_1}((0,T),X)$ and $g \in H^{s_2}((0,T),Y)$ with $s_1, s_2 \geq 0$. Then the following holds.

1. If $\frac{1}{2} < s_1 \leq 1$, $0 \leq s_2 < \frac{1}{2}$, then $m(f,g)$ belongs to $H^{s_2}((0,T),Z)$ and

$$\|m(f,g)\|_{H^{s_2}((0,T),Z)} \leq C_{s_1,s_2}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)\|g\|_{H^{s_2}((0,T),Y)},$$

for all $0 \leq T \leq T_f$, where C_{s_1,s_2} is independent of T.

2. If $\frac{1}{2} < s_1 \leq s_2 \leq 1$, then $m(f,g)$ belongs to $H^{s_1}((0,T),Z)$ and
$$\|m(f,g)\|_{H^{s_1}((0,T),Z)} \leq C_{s_1,s_2}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)(\|g\|_{H^{s_2}((0,T),Y)} + \|g(0)\|_Y),$$
for all $0 \leq T \leq T_f$, where C_{s_1,s_2} is independent of T.

Proof. We prove 2., 1. follows with similar arguments. Let $f_0 \in H^1((-\infty,\infty),X)$ and $g_0 \in H^1((-\infty,\infty),Y)$ be such that $f_0(0) = f(0)$, $g_0(0) = g(0)$ and for $-\infty < a < b < \infty$,
$$\begin{aligned}\|f_0\|_{H^1((a,b),X)} &\leq C_0\|f(0)\|_X,\\ \|g_0\|_{H^1((a,b),Y)} &\leq C_0\|g(0)\|_Y,\end{aligned}$$
with a constant C_0 independent of $(b-a)$ (extension to $H^1((0,\infty),X)$ and mirroring at $\mathtt{t} = 0$). Let C and C_{T_f} denote generic constants (C_{T_f} is used if the constant might depend on T_f). Using property **P2** of the norm and [113, Lem. A.1] yields
$$\|m(f,g)\|_{H^{s_1}((a,a+T_f),Z)} \leq C_{T_f}\|f\|_{H^{s_1}((a,a+T_f),X)}\|g\|_{H^{s_2}((a,a+T_f),Y)}, \tag{2.4}$$
(use equivalence of norms with T_f-dependent constants). Now,
$$\|m(f,g)\|_{H^{s_1}((0,T),Z)} \leq (\|m(f,g) - m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} + \|m(f_0,g_0)\|_{H^{s_1}((0,T),Z)}).$$
Due to Property **P5** of the norm and [113, Lem. A.1],
$$\begin{aligned}\|m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} &\leq C\|m(f_0,g_0)\|_{H^{s_1}((0,T_f),Z)} \leq C_{T_f}\|f_0\|_{H^{s_1}((0,T_f),X)}\|g_0\|_{H^{s_2}((0,T_f),Y)}\\ &\leq C_{T_f}\|f(0)\|_X\|g(0)\|_Y.\end{aligned}$$
Furthermore,
$$\begin{aligned}\|m(f,g) - m(f_0,g_0)\|_{H^{s_1}((0,T),Z)} &\leq \|m(f-f_0,g-g_0)\|_{H^{s_1}((0,T),Z)}\\ &\quad + \|m(f-f_0,g_0)\|_{H^{s_1}((0,T),Z)} + \|m(f_0,g-g_0)\|_{H^{s_1}((0,T),Z)}.\end{aligned}$$
We know that $(f-f_0)|_{\mathtt{t}=0} = 0$. Due to properties **P3** and **P4** of the norm and with (2.4),
$$\begin{aligned}&\|m(f-f_0,g-g_0)\|_{H^{s_1}((0,T),Z)}\\ &= \|\mathrm{Ext}(m(f-f_0,g-g_0))\|_{H^{s_1}((0,T),Z)}\\ &\leq C\|\mathrm{Ext}(m(f-f_0,g-g_0))\|_{H^{s_1}((T-T_f,T),Z)}\\ &\leq C\|m(\mathrm{Ext}(f-f_0),\mathrm{Ext}(g-g_0))\|_{H^{s_1}((T-T_f,T),Z)}\\ &\leq C_{T_f}\|\mathrm{Ext}(f-f_0)\|_{H^{s_1}((T-T_f,T),X)}\|\mathrm{Ext}(g-g_0)\|_{H^{s_2}((T-T_f,T),Y)}\\ &\leq C_{T_f}\|f-f_0\|_{H^{s_1}((0,T),X)}\|g-g_0\|_{H^{s_2}((0,T),Y)}\\ &\leq C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f_0\|_{H^{s_1}((0,T_f),X)})(\|g\|_{H^{s_2}((0,T),Y)} + \|g_0\|_{H^{s_2}((0,T_f),Y)})\\ &\leq C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f_0\|_{H^1((0,T_f),X)})(\|g\|_{H^{s_2}((0,T),Y)} + \|g_0\|_{H^1((0,T_f),Y)})\\ &\leq C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f_0\|_{H^1((0,T_f),X)})(\|g\|_{H^{s_2}((0,T),Y)} + \|g_0\|_{H^1((0,T_f),Y)})\end{aligned}$$

$$\leq C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)(\|g\|_{H^{s_2}((0,T),Y)} + \|g(0)\|_Y).$$

We now estimate $m(f - f_0, g_0)$ using the norm properties **P3**, **P4**:

$$\begin{aligned}\|m(f - f_0, g_0)\|_{H^{s_1}((0,T),Z)} &= \|\mathrm{Ext}(m(f - f_0, g_0))\|_{H^{s_1}((0,T),Z)} \\ &= \|m(\mathrm{Ext}(f - f_0), g_0)\|_{H^{s_1}((0,T),Z)} \\ &\leq C\|m(\mathrm{Ext}(f - f_0), g_0)\|_{H^{s_1}((T-T_f,T),Z)} \\ &\leq C_{T_f}\|\mathrm{Ext}(f - f_0)\|_{H^{s_1}((T-T_f,T),X)}\|g_0\|_{H^{s_2}((T-T_f,T),Y)} \\ &\leq C_{T_f}\|f - f_0\|_{H^{s_1}((0,T),X)}\|g_0\|_{H^{s_2}((-T_f,T_f),Y)} \\ &\leq C_{T_f}(\|f\|_{H^{s_1}((0,T),X)} + \|f(0)\|_X)\|g(0)\|_Y.\end{aligned}$$

Since $m(f_0, g - g_0)$ can be estimated in the same way, this concludes the proof of 2. □

Lemma 2.10. ([59, Lem. 3]) Let X be a real, separable Hilbert space and $\alpha \in [0,1) \setminus \{\frac{1}{2}\}$. Furthermore, let $\beta > 0$ be such that $\alpha + \beta \in (\frac{1}{2}, 1]$, $c \in X$ and $g \in H^{\alpha+\beta}((0,T), X)$ be such that $g(0) = c$. Then, there exists a constant C independent of T such that

$$\|g\|_{H^\alpha((0,T),X)} \leq C(T^\beta\|g\|_{H^{\alpha+\beta}((0,T),X)} + \|c\|_X).$$

Proof. Let C denote a generic constant independent of T, where $0 < T \leq T_f$. There exists $h \in H^1((0,T_f), X)$ such that $h(0) = c$ and $\|h\|_{H^1((0,T_f),X)} \leq C\|c\|_X$ e.g., $h(\mathsf{t}) := cT_f^{-1}(T_f - \mathsf{t})$ for $\mathsf{t} \in (0, T_f)$. Set $\tilde{g} = g - h$.

Properties **P5**, **P2**, the definition of h and **P6** yield

$$\begin{aligned}\|g\|_{H^\alpha((0,T),X)} &\leq \|\tilde{g}\|_{H^\alpha((0,T),X)} + \|h\|_{H^\alpha((0,T),X)} \leq \|\tilde{g}\|_{H^\alpha((0,T),X)} + C\|h\|_{H^1((0,T_f),X)} \\ &\leq \|\tilde{g}\|_{H^\alpha((0,T),X)} + C\|c\|_X \leq CT^\beta\|\tilde{g}\|_{H^{\alpha+\beta}((0,T),X)} + C\|c\|_X \\ &\leq C(T^\beta\|g\|_{H^{\alpha+\beta}((0,T),X)} + \|c\|_X).\end{aligned}$$

□

Lemma 2.11. ([59, Lem. 4]) Let X be a real, separable Hilbert space and $s \geq 0$. Let $c \in X$ and $g(\mathsf{t}) = c$ for a.e. $\mathsf{t} \in (0,T)$. Then, $g \in H^s((0,T), X)$ and there exists a constant C independent of T such that $\|g\|_{H^s((0,T),X)} \leq C\|c\|_X$.

Proof. Let $T_f \geq T$ and C denote a generic constant independent of T. For $s \geq 1$ we have, due to **P1** and $\partial_{\mathsf{t}} g = 0$,

$$\|g\|_{H^s((0,T),X)} = \|g\|_{L^2((0,T),X)} \leq T^{\frac{1}{2}}\|c\|_X \leq C\|c\|_X. \tag{2.5}$$

For $s \in [0,1)$, Lemma 2.10 and (2.5) yield

$$\|g\|_{H^s((0,T),X)} \leq C(T^{1-s}\|g\|_{H^1((0,T),X)} + \|c\|_X) \leq C\|c\|_X.$$

□

The following Lemma is a helpful tool.

Lemma 2.12. ([59, Lem. 9]) Let $T > 0$, $k \in \mathbb{N}$, $k \geq 2$, X, X_j, Y, W_n, Z be real, separable Hilbert spaces, $1 \leq j \leq k$, $2 \leq n \leq k-1$, $s_1 \in [0,1]\backslash\{\frac{1}{2}\}$, $s_i \in (\frac{1}{2}, 1]$ for $2 \leq i \leq k$ and $0 \leq s \leq \min_j s_j$. Let $m_1 : X_1 \times W_2 \to X$, $m_l : X_l \times W_{l+1} \to W_l$ for $2 \leq l \leq k-2$ and $m_{k-1} : X_{k-1} \times X_k \to W_{k-1}$ be continuous bilinear forms, $m : \bigtimes_{j=1}^k X_j \to X$ be defined by $m(x_1, \ldots, x_k) = m_1(x_1, m_2(x_2, \ldots))$ and $\mathcal{T}_j : Y \times Z \to S_j$, where $S_j := H^{s_j}((0,T), X_j)$ is endowed with the norm

- $\|\cdot\|_{S_j} := \|\cdot\|_{H^{s_j}((0,T),X_j)}$, if $s_j \in [0, \frac{1}{2})$,
- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^{s_j}((0,T),X_j)} + \|\cdot(0)\|^2_{X_j})^{\frac{1}{2}}$, if $s_j \in (\frac{1}{2}, 1]$,

and $S := H^s((0,T), X)$ be endowed with the analogously defined norm $\|\cdot\|_S$. Furthermore, let $\mathcal{T} : Y \times Z \to S$ be defined by

$$\mathcal{T}(y,z) = m(\mathcal{T}_1(y,z), \ldots, \mathcal{T}_k(y,z)).$$

1. Let $M_j > 0$, $\tilde{Y} \subset Y$ and $\tilde{Z} \subset Z$ be such that $\|\mathcal{T}_j(y,z)\|_{S_j} \leq M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$, $1 \leq j \leq k$. Then, there exists a constant $C > 0$ that is independent of T such that $\|\mathcal{T}(y,z)\|_S \leq C\Pi_j M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$.
2. Let in addition to 1. $\mathcal{T}_j : Y \times Z \to S_j$ be Lipschitz continuous on $\tilde{Y} \times \tilde{Z}$ for all $1 \leq j \leq k$, i.e., there exist $M_{j,1}, M_{j,2} > 0$ such that $\|\mathcal{T}_j(y_2,z_2) - \mathcal{T}_j(y_1,z_1)\|_{S_j} \leq M_{j,1}\|y_2 - y_1\|_Y + M_{j,2}\|z_2 - z_1\|_Z$ for arbitrary $y_1, y_2 \in \tilde{Y}$ and $z_1, z_2 \in \tilde{Z}$. Then, $\|\mathcal{T}(y_2,z_2) - \mathcal{T}(y_1,z_1)\|_{H^s((0,T),X)} \leq C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2 - y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2 - z_1\|_Z)$ with a constant $C > 0$ that is independent of T.
3. Let (y_1, z_1) be an element of the relative interior of $\tilde{Y} \times \tilde{Z}$ and $\mathcal{T}_j : \tilde{Y} \times \tilde{Z} \to S_j$ be Fréchet differentiable in (y_1, z_1) for all $1 \leq j \leq k$. Then, $\mathcal{T} : \tilde{Y} \times \tilde{Z} \to S$ is Fréchet differentiable in (y_1, z_1).

Proof. By recursively applying Lemmas 2.5 and 2.9 it can be verified that $m : \Pi_j S_j \to H^s((0,T), X)$ is a continuous multilinear form that fulfills

$$\|m(x_1, \ldots, x_k)\|_{H^s((0,T),X)} \leq C\Pi_j\|x_j\|_{S_j},$$

where C is a constant independent of T. Assertion 1 follow immediately if one directly uses the continuity properties of m in order to estimate the norms at the initial value $\mathtt{t} = 0$. Further, for $y_1, y_2 \in \tilde{Y}$, $z_1, z_2 \in \tilde{Z}$ we have

$$m(\mathcal{T}_1(y_2,z_2), \ldots, \mathcal{T}_k(y_2,z_2)) - m(\mathcal{T}_1(y_1,z_1), \ldots, \mathcal{T}_k(y_1,z_1))$$

$$= m((\mathcal{T}_1(y_2,z_2) - \mathcal{T}_1(y_1,z_1)), \mathcal{T}_2(y_2,z_2), \mathcal{T}_3(y_2,z_2) \dots, \mathcal{T}_k(y_2,z_2))$$
$$+ m(\mathcal{T}_1(y_1,z_1), (\mathcal{T}_2(y_2,z_2) - \mathcal{T}_2(y_1,z_1)), \mathcal{T}_3(y_2,z_2), \dots, \mathcal{T}_k(y_2,z_2))$$
$$+ \dots + m(\mathcal{T}_1(y_1,z_1), \dots, \mathcal{T}_{k-1}(y_1,z_1), (\mathcal{T}_k(y_2,z_2) - \mathcal{T}_k(y_1,z_1))),$$

which implies

$$\begin{aligned}
&\|m(\mathcal{T}_1(y_2,z_2),\dots,\mathcal{T}_k(y_2,z_2)) - m(\mathcal{T}_1(y_1,z_1),\dots,\mathcal{T}_k(y_1,z_1))\|_S \\
&\leq C\sum_{j=1}^{k}((\Pi_{n<j}\|\mathcal{T}_n(y_1,z_1)\|_{S_n})(\Pi_{n>j}\|\mathcal{T}_n(y_2,z_2)\|_{S_n})\|\mathcal{T}_j(y_2,z_2) - \mathcal{T}_j(y_1,z_1)\|_{S_j}) \\
&\leq C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2 - y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2 - z_1\|_Z)
\end{aligned}$$

for a generic constant C independent of T and therefore assertion 2. Since a continuous multilinear form is infinitely differentiable 3 follows with the chain rule.

□

Lemma 2.13. ([59, Lem. 10]) Let $T > 0$, $k \in \mathbb{N}$, $X_1, X_2, X_{j,1}, X_{j,2}, Y, Z$ be real, separable Hilbert spaces, $1 \leq j \leq k$, $s_1 \in [0,1] \setminus \{\frac{1}{2}\}$, $s_i \in (\frac{1}{2}, 1]$ for $2 \leq i \leq k$. Let m be a k-linear form that is recursively constructed via bilinear forms as in Lemma 2.12 such that $m : \bigtimes_{j=1}^{k} X_{j,1} \to X_1$ and $m : \bigtimes_{j=1}^{k} X_{j,1+\delta_{jl}} \to X_2$ are continuous for all $1 \leq l \leq k$, where δ_{jl} denotes the Kronecker delta. Let $0 \leq s \leq \min_j s_j$ and

$$S_j := H^1((0,T), X_{j,1}) \cap H^{1+s_j}((0,T), X_{j,2})$$

be endowed with the norm

- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})\cap H^{1+s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,1}})^{\frac{1}{2}}$, if $s_j \in [0, \frac{1}{2})$.
- $\|\cdot\|_{S_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})\cap H^{1+s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,1}} + \|\partial_t(\cdot)(0)\|^2_{X_{j,2}})^{\frac{1}{2}}$, if $s_j \in (\frac{1}{2}, 1]$.

and $S := H^1((0,T), X_1) \cap H^{1+s}((0,T), X_2)$ be endowed with the analogously defined norm $\|\cdot\|_S$. Further, let $\mathcal{T}_j : Y \times Z \to S_j$ and $\mathcal{T} : Y \times Z \to S$ be defined by

$$\mathcal{T}(y,z) = m(\mathcal{T}_1(y,z), \dots, \mathcal{T}_k(y,z)).$$

Then,

1. Let $M_j > 0$, $\tilde{Y} \subset Y$ and $\tilde{Z} \subset Z$ be such that $\|\mathcal{T}_j(y,z)\|_{S_j} \leq M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$, $1 \leq j \leq k$. Then, there exists a constant $C > 0$ that is independent of T such that $\|\mathcal{T}(y,z)\|_S \leq C\Pi_j M_j$ for all $(y,z) \in \tilde{Y} \times \tilde{Z}$.
2. Let in addition to 1. $\mathcal{T}_j : Y \times Z \to S_j$ be Lipschitz continuous on $\tilde{Y} \times \tilde{Z}$ for all $1 \leq j \leq k$, i.e., there exist $M_{j,1}, M_{j,2} > 0$ such that

$$\|\mathcal{T}_j(y_2,z_2) - \mathcal{T}_j(y_1,z_1)\|_{S_j} \leq M_{j,1}\|y_2 - y_1\|_Y + M_{j,2}\|z_2 - z_1\|_Z$$

for arbitrary $y_1, y_2 \in \tilde{Y}$ and $z_1, z_2 \in \tilde{Z}$. Then,

$$\begin{aligned} &\|\mathcal{T}(y_2, z_2) - \mathcal{T}(y_1, z_1)\|_S \\ &\leq C(\max_j(M_{j,1}\Pi_{n\neq j}M_n)\|y_2 - y_1\|_Y + \max_j(M_{j,2}\Pi_{n\neq j}M_n)\|z_2 - z_1\|_Z) \end{aligned}$$

with a constant $C > 0$ that is independent of T.

3. Let (y_1, z_1) be an element of the relative interior of $\tilde{Y} \times \tilde{Z}$ and $\mathcal{T}_j : \tilde{Y} \times \tilde{Z} \to S_j$ be Fréchet differentiable in (y_1, z_1) for all $1 \leq j \leq k$. Then, $\mathcal{T} : \tilde{Y} \times \tilde{Z} \to S$ is Fréchet differentiable in (y_1, z_1).

Proof. We recursively apply Lemma 2.9 in order to get continuity of

$$\begin{aligned} m &: \bigtimes_{j=1}^{k} S_j \to L^2((0,T), X_1), \\ \partial_t m &: \bigtimes_{j=1}^{k} S_j \to L^2((0,T), X_1), \text{ as well as,} \\ \partial_t m &: \bigtimes_{j=1}^{k} S_j \to H^s((0,T), X_2), \end{aligned}$$

and use that

$$\partial_t m(x_1, \ldots, x_k) = m(\partial_t x_1, x_2, \ldots, x_k) + m(x_1, \partial_t x_2, \ldots, x_k) + \cdots + m(x_1, x_2, \ldots, \partial_t x_k).$$

It holds

$$\|m(x_1, \ldots, x_k)\|_{L^2((0,T),X_1)} \leq C\|x_1\|_{L^2((0,T),X_{1,1})}\Pi_{i=2}^k\|x_i\|_{\hat{S}_i},$$

$$\|m(x_1, \ldots, \partial_t x_j, \ldots, x_k)\|_{L^2((0,T),X_1)} \leq C\|\partial_t x_j\|_{L^2((0,T),X_{j,1})}\Pi_{i\neq j}\|x_i\|_{\hat{S}_i},$$

where $\hat{S}_j := H^1((0,T), X_{j,1})$ is endowed with the norm

$$\|\cdot\|_{\hat{S}_j} := (\|\cdot\|^2_{H^1((0,T),X_{j,1})} + \|\cdot(0)\|^2_{X_{j,1}})^{\frac{1}{2}}$$

for $1 \leq j \leq k$. Furthermore, there holds

$$\|m(x_1, \ldots, \partial_t x_j, \ldots, x_k)\|_{H^s((0,T),X_2)} \leq C\|\partial_t x_j\|_{\tilde{S}_j}\Pi_{i\neq j}\|x_i\|_{\hat{S}_i},$$

where $\tilde{S}_j := H^{s_j}((0,T), X_{j,2})$ is endowed with the norm

- $\|\cdot\|_{\tilde{S}_j} := \|\cdot\|_{H^{s_j}((0,T),X_{j,2})}$, if $s_j \in [0, \frac{1}{2})$.
- $\|\cdot\|_{\tilde{S}_j} := (\|\cdot\|^2_{H^{s_j}((0,T),X_{j,2})} + \|\cdot(0)\|^2_{X_{j,2}})^{\frac{1}{2}}$, if $s_j \in (\frac{1}{2}, 1]$.

In order to show the boundedness in the norm $\|\cdot\|_S$ the initial values have to be bounded appropriately. However, this is ensured by the continuity properties of the multilinear form m. Moreover, property **P1** of the norm is used. The assertions now follow directly as in Lemma 2.12. □

$H^{s,r}(\Omega \times (0,T))$

For $r, s > 0$, the spaces $H^{r,s}(Q^T)$ are defined by

$$H^{r,s}(Q^T) = L^2((0,T), H^r(\Omega)) \cap H^s((0,T), L^2(\Omega))$$

and endowed with the norm

$$\|\cdot\|_{H^{r,s}(Q^T)} = (\|\cdot\|^2_{L^2((0,T),H^r(\Omega))} + \|\cdot\|^2_{H^s((0,T),L^2(\Omega))})^{\frac{1}{2}}.$$

For $0 \leq r' \leq r$, $s' = s(r-r')/r$, the inequality

$$\|\cdot\|_{H^{s'}((0,T),H^{r'}(\Omega))} \leq C\|\cdot\|_{H^{r,s}(Q^T)}$$

holds true for a constant $C > 0$ that might depend on T, cf. [53, (2.9)] or [54, (2.7)], which implies

$$\|\cdot\|_{H^{(1-\theta)s}((0,T),H^{\theta r}(\Omega))} \leq C\|\cdot\|_{H^{r,s}(Q^T)}$$

for $\theta \in (0,1)$. Trace theorems for the Sobolev-type spaces $H^{r,s}(Q^T)$ imply

$$\|\cdot|_{\Sigma_i^T}\|_{H^{r',s'}(\Sigma_i^T)} \leq C\|\cdot\|_{H^{r,s}(Q^T)},$$

where $C > 0$ is dependent on T, $r > \frac{1}{2}$, $s \geq 0$, $r' = r - \frac{1}{2}$ and $s' = (r - \frac{1}{2})\frac{s}{r}$, cf. [91, Ch. 4, Thm. 2.1], [54, Prop. 2.2] or [39, Thm. 3].

2.2.4. Setting for the Theoretical Analysis of the FSI Problem

For $\ell \in (\frac{1}{2}, 1)$ the analysis of the FSI problem is conducted on the function spaces

$$\begin{aligned}
\mathbf{E}_T &:= L^2((0,T), H^{2+\ell}(\Omega_f)^d) \cap H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d) = (H^{2+\ell,1+\frac{\ell}{2}}(Q_f^T))^d, \\
\mathbf{F}_T &:= L^2((0,T), H^{\ell}(\Omega_f)^d) \cap H^{\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d) = (H^{\ell,\frac{\ell}{2}}(Q_f^T))^d, \\
G_T &:= L^2((0,T), H^{1+\ell}(\Omega_f)) \cap H^{\frac{\ell}{2}}((0,T), H^1(\Omega_f)), \\
\mathbf{G}_T &:= H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d), \\
H_T &:= L^2((0,T), H^{\frac{1}{2}+\ell}(\Gamma_i)) \cap H^{\frac{1}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i)) = H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T), \\
\mathbf{H}_T &:= L^2((0,T), H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{1}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i)^d) = (H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T))^d, \\
\mathbf{N}_T &:= H^1((0,T), H^{\frac{3}{2}+\ell}(\Gamma_s \cup \Gamma_i)^d) \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s \cup \Gamma_i)^d), \\
P_T &:= \{p \in L^2(Q_f^T) \ : \ \nabla p \in H^{\ell,\frac{\ell}{2}}(Q_f^T),\ p|_{\Sigma_i^T} \in H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)\},
\end{aligned}$$

$$
\begin{aligned}
S_T &:= H^1((0,T), H^{1+\ell}(\Omega_f)) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f)),\\
\mathbf{S}_T &:= H^1((0,T), H^{1+\ell}(\Omega_f)^d) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d),\\
\underline{\mathbf{S}}_T &:= H^1((0,T), H^{1+\ell}(\Omega_f)^{d\times d}) \cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^{d\times d}),\\
\mathbf{W}_T &:= \mathcal{C}([0,T], H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)^d) \cap \mathcal{C}^1([0,T], H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)^d),\\
\mathbf{V}_0 &:= H^{1+\ell}(\Omega_f)^d,\\
\mathbf{W}_1 &:= H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d.
\end{aligned}
$$

For $\mathbf{v}_0 \in \mathbf{V}_0$ and $M_0 > 0$, we define the metric spaces

$$
\begin{aligned}
\mathbf{E}_{T,M_0,\mathbf{v}_0} &:= \{\mathbf{v} \in \mathbf{E}_T \ : \ \mathbf{v}(\cdot,0) = \mathbf{v}_0,\ \|\mathbf{v}\|_{\mathbf{E}_T} \le M_0\},\\
P_{T,M_0,\mathbf{v}_0} &:= \{p \in P_T \ : \ \|\nabla p\|_{\mathbf{F}_T} \le M_0,\ \|p|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)} \le M_0,\\
&\qquad\qquad p|_{\Gamma_i\times\{0\}} = 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f\cdot\mathbf{n}_f|_{\Gamma_i}\}.
\end{aligned}
\tag{2.6}
$$

Moreover, let

$$
\begin{aligned}
\|\cdot\|_{\mathbf{E}_T} := (\|\cdot\|^2_{(H^{2+\ell,1+\frac{\ell}{2}}(Q_f^T))^d} &+ \|\cdot\|^2_{H^1((0,T),H^\ell(\Omega_f)^d))} + \|\cdot\|^2_{H^{\frac{\ell}{2}}((0,T),H^2(\Omega_f)^d)}\\
&+ \|\cdot|_{\Sigma_i^T}\|^2_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),H^1(\Gamma_i)^d)} + \|\cdot|_{\Sigma_i^T}\|^2_{(H^{\frac{3}{2}+\ell,\frac{3}{4}+\frac{\ell}{2}}(\Sigma_i^T))^d}\\
&+ \|\cdot\|^2_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),H^1(\Omega_f)^d)} + \|\cdot\|^2_{H^{\frac{1}{4}+\frac{\ell}{4}}((0,T),H^{1+\ell}(\Omega_f)^d)})^{\frac{1}{2}},
\end{aligned}
$$

as well as,

$$
\|\cdot\|_{S_T} := (\|\cdot\|^2_{H^1((0,T),H^{1+\ell}(\Omega_f))\cap H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\Omega_f))} + \|\cdot(0)\|^2_{H^{1+\ell}(\Omega_f)} + \|\partial_t\cdot(0)\|^2_{L^2(\Omega_f)})^{\frac{1}{2}},
\tag{2.7}
$$

and do an analogous definition for the spaces $\mathbf{S}_T$ and $\underline{\mathbf{S}}_T$. Due to trace theorems and interpolation theorems the modified norms on $\mathbf{E}_T$ and $S_T, \mathbf{S}_T, \underline{\mathbf{S}}_T$ are equivalent to the standard norms on these function spaces. However, the appearing equivalence constant might depend on T without further knowledge about this dependency. Since the dependency of the appearing constants on T is a key point in the theoretical analysis it is therefore necessary to work with the modified norms defined above. The other function spaces are endowed with the canonical choice for the norm, i.e., e.g., $\|\cdot\|_{\mathbf{F}_T} = \|\cdot\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d}$. Furthermore, the following results is useful.

Lemma 2.14. ([59, Lem. 11])

1. Let $\tilde{f}, \tilde{g} \in \tilde{S}_T$. Then, $\tilde{f}\tilde{g} \in \tilde{S}_T$ and

$$
\|\tilde{f}\tilde{g}\|_{\tilde{S}_T} \le C\|\tilde{f}\|_{\tilde{S}_T}\|\tilde{g}\|_{\tilde{S}_T}
$$

with a constant C that is independent of T.

2. Let $\tilde{f} \in \tilde{S}_T$. If $\tilde{f} \geq \omega > 0$ holds a.e. on $\tilde{Q}_f^T$ with a constant $\omega > 0$ then $\tilde{f}^{-1} \in \tilde{S}_T$ and

$$\|\tilde{f}^{-1}\|_{\tilde{S}_T} \leq C(1 + \|\tilde{f}\|_{\tilde{S}_T})^{10} \|\tilde{f}\|_{\tilde{S}_T}$$

for a constant C that is independent of T.

Proof. 1. The bilinear form $m(x_1, x_2) := x_1 \cdot x_2$ is by Lemma 2.5 continuous as a mapping $L^2(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \to L^2(\tilde{\Omega}_f)$ and as a mapping $H^{1+\ell}(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \to H^{1+\ell}(\tilde{\Omega}_f)$. Therefore, Lemma 2.13 implies $\|\tilde{f}\tilde{g}\|_{\tilde{S}_T} \leq C\|\tilde{f}\|_{\tilde{S}_T}\|\tilde{g}\|_{\tilde{S}_T}$ for a constant C that is independent of T. Here, we recall that the norm on $\tilde{S}_T$ is defined by (2.7).

2. By [113, Lem. A.7] we know that

$$\begin{aligned}
&\|\tilde{f}^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f)}\\
&\leq C(1 + \|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} + \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})\|\tilde{f}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))},\\
&\|\partial_t \tilde{f}^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f)}\\
&\leq C(1 + \|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} + \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})^4\|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))}.
\end{aligned}$$

for a constant C independent of T. The proof of this Lemma also shows that

$$\|\tilde{f}^{-1}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)} \leq C(1 + \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})\|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)}.$$

Let C now denote a generic constant independent of T.
In order to bound $\|\partial_t \tilde{f}^{-1}(\cdot,0)\|_{L^2(\tilde{\Omega}_f)}$, we consider $G \in \mathcal{C}^\infty(\mathbb{R})$ such that $G(0) = 0$ and $G(x) = x^{-1}$ for all $x \geq \omega$. Then,

$$\begin{aligned}
&\|\partial_t \tilde{f}^{-1}(\cdot,0)\|^2_{L^2(\tilde{\Omega}_f)} = \|\partial_t G(\tilde{f})(\cdot,0)\|^2_{L^2(\tilde{\Omega}_f)} = \|G'(\tilde{f})(\cdot,0)\partial_t \tilde{f}(\cdot,0)\|^2_{L^2(\tilde{\Omega}_f)}\\
&= \int_{\tilde{\Omega}_f} (G'(\tilde{f})(\mathsf{z},0)\partial_t \tilde{f}(\mathsf{z},0))^2 \,\mathrm{d}\mathsf{z}\\
&\leq \sup_{\mathsf{z}\in\tilde{\Omega}_f} |G'(\tilde{f})(\mathsf{z},0)|\|\partial_t \tilde{f}(\cdot,0)\|^2_{L^2(\tilde{\Omega}_f)} \leq C\|\partial_t \tilde{f}(\cdot,0)\|^2_{L^2(\tilde{\Omega}_f)}.
\end{aligned}$$

These estimates imply

$$\begin{aligned}
&\|\tilde{f}^{-1}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)}\\
&\leq C(1 + \|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} + \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})^4\|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))}\\
&\leq C(1 + \|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} + \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})^5.
\end{aligned}$$

Now,

$$\|\tilde{f}^{-1}\|_{L^2((0,T),L^2(\tilde{\Omega}_f))} \leq C\|\tilde{f}^{-1}\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))}$$

for a constant independent of T and it remains to estimate $\|\partial_t \tilde{f}^{-1}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))}$.

We obtain with Lemma 2.9, 2.

$$
\begin{aligned}
\|\partial_t \tilde{f}^{-1}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))} &= \|\tilde{f}^{-2}\partial_t \tilde{f}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))} \\
\leq C(\|\tilde{f}^{-2}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} &+ \|\tilde{f}^{-2}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)}) \\
&\times (\|\partial_t \tilde{f}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))} + \|\partial_t \tilde{f}(\cdot,0)\|_{L^2(\tilde{\Omega}_f)}) \\
\leq C(\|\tilde{f}^{-1}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} &+ \|\tilde{f}^{-1}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})^2 \\
&\times (\|\partial_t \tilde{f}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))} + \|\partial_t \tilde{f}(\cdot,0)\|_{L^2(\tilde{\Omega}_f)}) \\
\leq C(1 + \|\tilde{f}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f))} &+ \|\tilde{f}(\cdot,0)\|_{H^{1+\ell}(\tilde{\Omega}_f)})^{10} \\
&\times (\|\tilde{f}\|_{H^{\frac{3}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f))} + \|\partial_t \tilde{f}(\cdot,0)\|_{L^2(\tilde{\Omega}_f)}).
\end{aligned}
$$

Combining the estimates implies the assertion.

□

2.3. Method of Successive Approximations

The method of successive approximations is a well known approach for establishing existence and uniqueness results for nonlinear equations

$$A(y) = 0,$$

where $y \in Y$ and Y is a Banach space. We write this in the form $By = \mathcal{F}(y)$, where $\mathcal{F}(y) := By - A(y)$ and B is a linear operator such that the system $By = f$ has a unique solution $y = Sf$, where $S \in \mathcal{L}(W, Y)$ and W is a Banach space. Existence and uniqueness of solutions is now studied via the fixed point equation

$$y = S\mathcal{F}(y). \tag{2.8}$$

Unique solvability of (2.8) on a closed subset $\tilde{Y} \subset Y$ is ensured if $y \mapsto S\mathcal{F}(y)$ maps $\tilde{Y}$ into itself and is a contraction on $\tilde{Y}$. This is the case if, e.g., $\|S\|_{\mathcal{L}(W,Y)} \leq L_S$ and if $\mathcal{F} : \tilde{Y} \to W$ is Lipschitz continuous with a constant $L_{\mathcal{F}} < \frac{1}{L_S}$.

2.4. Framework for Continuity and Differentiability Results

The considerations of Section 2.3 can be extended to parameter-dependent equations

$$A(y, z) = 0$$

with parameter or control z in a Banach space Z. Let B be chosen as in Section 2.3 and $S \in \mathcal{L}(W, Y)$ be the solution operator of $By = f$. As before, we consider solutions of the fixed point equation

$$y = S\,\mathcal{F}(y, z), \tag{2.9}$$

where $\mathcal{F}(y,z) := By - A(y,z)$.

Theorem 2.15. ([59, Thm. 1]) Let $\tilde{W}, W, Y, Z$ be Banach spaces, $\tilde{W}$ continuously embedded in W, $S \in \mathcal{L}(\tilde{W}, Y)$, and $L_S > 0$ a constant such that $\|Sf\|_Y \leq L_S \|f\|_W$ for all $f \in \tilde{W}$. Let $\tilde{Z} \subset Z$ be open, $\tilde{Y} \subset Y$ be closed and $\mathcal{F} : \tilde{Y} \times \tilde{Z} \to \tilde{W}$ be an operator. Let there exist constants $L_{\mathcal{F}} \in (0, \frac{1}{L_S})$ and $C > 0$ such that, for all $y, y_1, y_2 \in \tilde{Y}$, $z, z_1, z_2 \in \tilde{Z}$, there hold

$$\|\mathcal{F}(y_2, z_2) - \mathcal{F}(y_1, z_1)\|_W \leq L_{\mathcal{F}} \|y_2 - y_1\|_Y + C\|z_2 - z_1\|_Z, \tag{2.10}$$

$$S\mathcal{F}(y,z) \in \tilde{Y}. \tag{2.11}$$

Then, for all $z \in \tilde{Z}$, the system (2.9) has a unique solution $y(z)$ and $z \mapsto y(z)$ is Lipschitz continuous on $\tilde{Z}$:

$$\|y(z_2) - y(z_1)\|_Y \leq \frac{CL_S}{1 - L_S L_{\mathcal{F}}} \|z_2 - z_1\|_Z \quad \forall\, z_1, z_2 \in \tilde{Z}. \tag{2.12}$$

In addition, let $y(z)$ lie in the relative interior of $\tilde{Y}$ and denote by $\tilde{Y}_L$ the linear subspace parallel to the affine hull $\mathit{aff}(\tilde{Y})$. Assume that $\mathcal{F}$ is Fréchet differentiable at $(y(z), z)$, where (y,z)-variations are taken in $\tilde{Y}_L \times Z$.
Then $y(\cdot)$ is Fréchet differentiable at z. The derivative is given by $y'(z)(h) = \delta_h y(z)$, where $h \in Z$ and $\delta_h y(z) \in \tilde{Y}_L \subset Y$ solves the formally linearized equation

$$\delta_h y(z) = S\delta\mathcal{F}(y(z), z)(\delta_h y(z), h), \tag{2.13}$$

where $\delta\mathcal{F}(y(z),z)(\delta_h y(z), h) := \mathcal{F}_y(y(z),z)\delta_h y(z) + \mathcal{F}_z(y(z),z)h$.

Proof. For any fixed $z \in \tilde{Z}$, (2.10) implies the Lipschitz continuity of the mapping $\mathcal{F}(\cdot, z) : \tilde{Y} \to W$. Using (2.10), (2.11), and the properties of $\mathcal{F}$, $L_{\mathcal{F}}$ and L_S shows that the map $y \in \tilde{Y} \mapsto S\mathcal{F}(y,z) \in \tilde{Y}$ is a well-defined contraction. The existence of a unique solution $y(z) \in \tilde{Y}$ is thus ensured by the method of successive approximations. Now (2.12) follows from $\|y(z_2) - y(z_1)\|_Y = \|S(\mathcal{F}(y(z_2), z_2) - \mathcal{F}(y(z_1), z_1))\|_Y \leq L_S \|\mathcal{F}(y(z_2), z_2) - \mathcal{F}(y(z_1), z_1)\|_W$ and (2.10).
For showing differentiability, we fix $z \in \tilde{Z}$ and assume that $\mathcal{F}$ is differentiable at $(y(z), z)$ in the way stated in the theorem. Let $h \in Z$ be arbitrarily fixed. Since $y(z)$ is a relative interior point of $\tilde{Y}$, we obtain from (2.10) that, for all $d_1, d_2 \in \tilde{Y}_L$, there holds:

$$\|\delta\mathcal{F}(y(z),z)(d_2,h) - \delta\mathcal{F}(y(z),z)(d_1,h)\|_W = \|\mathcal{F}_y(y(z),z)(d_2 - d_1)\|_W \leq L_{\mathcal{F}} \|d_2 - d_1\|_Y. \tag{2.14}$$

Thus, since $L_{\mathcal{F}} < \frac{1}{L_S}$, the method of successive approximations applied to the fixed point equation $\delta_h y(z) = S\delta\mathcal{F}(y(z),z)(\delta_h y(z), h)$ posed in $\tilde{Y}_L$, see (2.13), yields a unique solution $\delta_h y(z) \in \tilde{Y}_L \subset Y$ which by linearity of (2.13) depends linearly on h. Let $\|h\|_Z$ be sufficiently small. Then $z + h \in \tilde{Z}$ and, as $h \to 0$,

$$\begin{aligned}
&\|\mathcal{F}(y(z+h), z+h) - \mathcal{F}(y(z),z) - \delta\mathcal{F}(y(z),z)(\delta_h y(z), h)\|_W \\
&\leq \|\delta\mathcal{F}(y(z),z)(y(z+h) - y(z), h) - \delta\mathcal{F}(y(z),z)(\delta_h y(z), h)\|_W
\end{aligned}$$

$$
\begin{aligned}
&+ o(\|y(z+h) - y(z)\|_Y + \|h\|_Z) \\
&\leq L_{\mathcal{F}} \|y(z+h) - y(z) - \delta_h y(z)\|_Y + o(\|h\|_Z),
\end{aligned}
$$

where (2.14) is used. Now

$$
\begin{aligned}
&T\|y(z+h) - y(z) - \delta_h y(z)\|_Y \\
&= \|S\mathcal{F}(y(z+h), z+h) - S\mathcal{F}(y(z), z) - S\delta\mathcal{F}(y(z), z)\delta_h y(z)\|_Y \\
&\leq L_S \|\mathcal{F}(y(z+h), z+h) - \mathcal{F}(y(z), z) - \delta\mathcal{F}(y(z), z)\delta_h y(z)\|_W \\
&\leq L_S L_{\mathcal{F}} \|y(z+h) - y(z) - \delta_h y(z)(z)\|_Y + L_S o(\|h\|_Z) \quad (\|h\|_Z \to 0).
\end{aligned}
$$

Therefore,

$$
\|y(z+h) - y(z) - \delta_h y(z)\|_Y \leq \frac{L_S}{1 - L_S L_{\mathcal{F}}} o(\|h\|_Z) = o(\|h\|_Z) \quad (\|h\|_Z \to 0),
$$

which proves the Fréchet differentiability of $z \mapsto y(z)$ at z with $y'(z)h = \delta_h y(z)$. □

2.5. Extension by Continuity

One technique, that is a common tool, cf., e.g., [86], is extension by continuity that takes advantage of the fact, that under some additional assumptions linear operators inherit continuity properties on a dense subset.

Let Z, Y be Banach spaces and S be a linear operator that is continuous as a mapping $Z \to Y$. Let $\hat{Z} \subset Z$ and $\hat{Y} \subset Y$ with continous injection and $\tilde{Z} \subset \hat{Z}$ be a dense subset of $\hat{Z}$. Additionally, assume that there exists $C > 0$ such that

$$
\|S(z)\|_{\hat{Y}} \leq C\|z\|_{\hat{Z}}, \quad \forall z \in \tilde{Z}.
$$

Then, we know that there exists a unique continuous linear operator $\hat{S} : \hat{Z} \to \hat{Y}$ such that

$$
\|\hat{S}(z)\|_{\hat{Y}} \leq C\|z\|_{\hat{Z}}, \quad \forall z \in \hat{Z},
$$

and $\hat{S}(z) = S(z)$ for all $z \in \tilde{Z}$. Since $\tilde{Z}$ is dense in $\hat{Z}$, for every $z \in \hat{Z}$, there exists a sequence $\{z_n\} \subset \tilde{Z}$ and $\|z - z_n\|_{\hat{Z}} \to 0$ for $n \to \infty$. This implies $\|z - z_n\|_Z \to 0$ for $n \to \infty$ and due to the continuity of $\hat{S}$, we additionally know that

$$
\|\hat{S}(z) - \hat{S}(z_n)\|_{\hat{Y}} = \|\hat{S}(z) - S(z_n)\|_{\hat{Y}} \to 0
$$

for $n \to \infty$. Thus, since

$$
\begin{aligned}
\|\hat{S}(z) - S(z)\|_Y &\leq \|\hat{S}(z) - S(z_n)\|_Y + \|S(z_n) - S(z)\|_Y \\
&\leq \|\hat{S}(z) - S(z_n)\|_Y + C\|z_n - z\|_Z
\end{aligned}
$$

for all $n \in \mathbb{N}$, we have that $\hat{S}(z) = S(z)$ in Y and therefore,

$$\|S(z)\|_{\hat{Y}} \leq C\|z\|_{\hat{Z}}, \quad \forall z \in \hat{Z}.$$

2.6. First Order Differential Operators

First order differential operators that are tangential to the boundary are one of the key tools to derive improved regularity results for hyperbolic equations in [86]. In this section, some properties of these operators are proven. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded open domain with boundary $\Gamma = \partial\Omega$ of class $\mathcal{C}^\infty$. We define

$$\begin{aligned}\mathscr{B} := \{B = \sum_{i=1}^{d} \mathbf{b}_i \partial_{\xi_i} \ : \mathbf{b} \in C^\infty(\overline{\Omega})^d,\ \mathbf{b}\cdot\mathbf{n} = 0 \text{ on } \Gamma, \quad &\text{and} \\ \mathbf{b} \text{ is constant along normal directions locally around } \Gamma\}.&\end{aligned} \tag{2.15}$$

Let $m \in \mathbb{N}_0$. The operators $B \in \mathscr{B}$ are well-defined as mappings from $H^{m+1}(\Omega) \to H^m(\Omega)$. Using the system of local maps $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$, $B \in \mathscr{B}$ can also be represented as an operator from $H^{m+1}(\Gamma) \to H^m(\Gamma)$ with

$$B\Phi = \sum_{j=1}^{N} \alpha_j \sum_{i=1}^{d} \mathbf{b}_i \sum_{k=1}^{d-1} \partial_{\xi_i}(\varphi_j)_k \partial_{\mathrm{y}'_k}(\Phi \circ \varphi_j^{-1}) \circ \varphi_j \tag{2.16}$$

for $\Phi \in H^{m+1}(\Gamma)$. The following lemmas provide some helpful properties.

Lemma 2.16. Let $w \in \mathcal{C}^\infty(\overline{\Omega})$ and $B \in \mathscr{B}$. Then, $B(w|_\Gamma) = (Bw)|_\Gamma$.

Proof. Let $\xi \in \Gamma$ and $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$ be the system of local maps of Γ. It holds that

$$\begin{aligned} Bw(\xi) &= ((\sum_{j=1}^{N} \alpha_j \sum_{i=1}^{d} \mathbf{b}_i \partial_{\xi_i} w) \circ \varphi_j^{-1} \circ \varphi_j)(\xi) \\ &= (\sum_{j=1}^{N} \alpha_j \sum_{i=1}^{d} \mathbf{b}_i \sum_{k=1}^{d} \partial_{\mathrm{y}'_k}(w \circ \varphi_j^{-1}) \circ \varphi_j \partial_{\xi_i}(\varphi_j)_k)(\xi). \end{aligned}$$

Since $(\varphi_j)_d$ is constantly zero on Γ and the gradient is perpendicular on the level sets, there exists $c \in \mathbb{R}$ such that $\partial_{\xi_k}(\varphi_j)_d = c\mathbf{n}_k$ for all $k \in \{1, \ldots, d\}$, where $\mathbf{n}$ denotes the outer unit normal vector. Hence, $\sum_{i=1}^{d} \mathbf{b}_i \partial_{\xi_i}(\varphi_j)_d = 0$ and

$$Bw(\xi) = (\sum_{j=1}^{N} \alpha_j \sum_{i=1}^{d} \mathbf{b}_i \sum_{k=1}^{d-1} \partial_{\mathrm{y}'_k}(w \circ \varphi_j^{-1}) \circ \varphi_j \partial_{\xi_i}(\varphi_j)_k)(\xi).$$

□

Lemma 2.17. Let $d = 3$, $m \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^d$ be a bounded, open domain with boundary Γ of class $\mathcal{C}^\infty$. Let $\Phi \in H^{m+1}(\Gamma)$ and $B \in \mathscr{B}$. Then, there exists $C > 0$ such that

$$\|B\Phi\|_{H^m(\Gamma)} \leq C\|\Phi\|_{H^{m+1}(\Gamma)}.$$

Proof. Let $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$ be the system of local maps of Γ. With (2.16) and (2.1) we obtain

$$\begin{aligned}\|B\Phi\|_{H^m(\Gamma)} &= (\sum_{j=1}^N \|\varphi_j^*(\alpha_j B\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})})^{\frac{1}{2}} \\ &\leq C(\sum_{j=1}^N \|\sum_{i=1}^d \sum_{k=1}^{d-1} \varphi_j^*(\alpha_j \mathbf{b}_i \partial_{\xi_i}(\varphi_j)_k)\partial_{\mathrm{y}'_k}(\Phi \circ \varphi_j^{-1})\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})})^{\frac{1}{2}} \\ &\leq C(\sum_{j=1}^N \sum_{k=1}^{d-1} \|\varphi_j^*(\alpha_j)\partial_{\mathrm{y}'_k}(\Phi \circ \varphi_j^{-1})\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})})^{\frac{1}{2}}\end{aligned}$$

where C is a generic constant. Moreover,

$$\varphi_j^*(\alpha_j)\partial_{\mathrm{y}'_k}(\Phi \circ \varphi_j^{-1}) = \partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi) - \partial_{\mathrm{y}'_k}(\varphi_j^*(\alpha_j))\varphi_j^*(\Phi).$$

Hence, with (2.1) and a generic constant C,

$$\begin{aligned}\|B\Phi\|_{H^m(\Gamma)} &\leq C(\sum_{j=1}^N \sum_{k=1}^{d-1}(\|\partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})} + \|\partial_{\mathrm{y}'_k}(\varphi_j^*(\alpha_j))\varphi_j^*(\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})}))^{\frac{1}{2}} \\ &\leq C(\sum_{j=1}^N \sum_{k=1}^{d-1}(\|\partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})} + \|\varphi_j^*(\alpha_j\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})}))^{\frac{1}{2}} \\ &\leq C(\sum_{j=1}^N \|\varphi_j^*(\alpha_j\Phi)\|^2_{H^{m+1}(\mathbb{R}^{d-1}_{\mathrm{y}'})})^{\frac{1}{2}} = C\|\Phi\|_{H^{m+1}(\Gamma)}.\end{aligned}$$

□

Lemma 2.18. Let $d = 3$, $m \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^d$ a bounded, open domain with boundary Γ of class $\mathcal{C}^\infty$. Let $\Phi \in H^m(\Gamma)$ such that $B\Phi \in H^m(\Gamma)$ for all $B \in \mathscr{B}$. Then, $\Phi \in H^{m+1}(\Gamma)$ and there exists a set of finitely many first-order operators $\mathcal{B} \subset \mathscr{B}$ such that

$$\|\Phi\|_{H^{m+1}(\Gamma)} \leq C(\|\Phi\|_{H^m(\Gamma)} + \sup_{B\in\mathcal{B}} \|B\Phi\|_{H^m(\Gamma)})$$

for a constant $C > 0$.

Proof. Let the system $\{\mathcal{O}_j, \varphi_j, \alpha_j\}$ be given. We know that $\mathcal{C}^\infty(\Gamma)$ is dense in $H^m(\Gamma)$ for $m \in \mathbb{N}_0$ and therefore consider $\Phi \in \mathcal{C}^\infty(\Gamma)$ and then extend by continuity, see Section 2.5.

We have

$$\|\Phi\|_{H^{m+1}(\Gamma)} = (\sum_{j=1}^{N} \|\varphi_j^*(\alpha_j\Phi)\|^2_{H^{m+1}(\mathbb{R}^{d-1}_{\mathrm{y}'})})^{\frac{1}{2}}. \tag{2.17}$$

Furthermore,

$$\|\varphi_j^*(\alpha_j\Phi)\|^2_{H^{m+1}(\mathbb{R}^{d-1}_{\mathrm{y}'})} = \|\varphi_j^*(\alpha_j\Phi)\|^2_{L^2(\mathbb{R}^{d-1}_{\mathrm{y}'})} + \sum_{k=1}^{d-1} \|\partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi)\|^2_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})} \tag{2.18}$$

and, on $B_1(0) \cap \mathbb{R}^{d-1}_{\mathrm{y}'}$,

$$\begin{aligned}\partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi) = \partial_{\mathrm{y}'_k}((\alpha_j\Phi)\circ\varphi_j^{-1}) &= \sum_{m=1}^{d}(\partial_{\xi_m}(\alpha_j\Phi))\circ\varphi_j^{-1}\partial_{\mathrm{y}'_k}(\varphi_j^{-1})_m \\ &= \varphi_j^*((\sum_{m=1}^{d}\mathbf{a}_{j,k,m})\Phi) + \varphi_j^*(\alpha_j(\sum_{m=1}^{d}\mathbf{b}_{j,k,m}\partial_{\xi_m}\Phi)),\end{aligned}$$

where $\mathbf{a}_{j,k,m} = \partial_{\xi_m}\alpha_j\partial_{y'_k}(\varphi_j^{-1})_m\circ\varphi_j$ and $\mathbf{b}_{j,k,m} = \partial_{y'_k}(\varphi_j^{-1})_m\circ\varphi_j$.
Since $(\varphi_j)_d$ is constant on Γ and the gradient is perpendicular on the level sets, there exists $c \in \mathbb{R}$ such that $\partial_{\xi_m}(\varphi_j)_d = c\mathbf{n}_m$ for all $m \in \{1,2,3\}$, where $\mathbf{n}$ denotes the outer unit normal vector. Thus, on $B_1(0) \cap \mathbb{R}^{d-1}_{\mathrm{y}'}$, we have

$$\begin{aligned}0 &= (\nabla_{\mathrm{y}'}\varphi_j^*(\varphi_j))_{k,d} = \partial_{\mathrm{y}'_k}(\varphi_j^*(\varphi_j))_d \\ &= \sum_{m=1}^{d}\partial_{\xi_m}(\varphi_j)_d\circ\varphi_j^{-1}\partial_{\mathrm{y}'_k}(\varphi_j^{-1})_m = c\varphi_j^*(\sum_{m=1}^{d}\mathbf{b}_{j,k,m}\mathbf{n}_m).\end{aligned}$$

since $(\varphi_j)_d = 0$. By the choice of α_j and φ_j we know that $\mathbf{a}_{j,k,m} \in \mathcal{C}^\infty(\Gamma)$ and $\mathbf{b}_{j,k,m} \in \mathcal{C}^\infty(\Gamma)$. Due to Corollary 2.3, there exists a $\mathcal{C}^\infty(\Omega)$-extension $\tilde{\mathbf{b}}_{j,k,m}$ of $\mathbf{b}_{j,k,m}$ such that $\tilde{\mathbf{b}}_{j,k,m}$ is constant along normal directions locally around Γ and let

$$\mathcal{B} := \{B = \sum_{m=1}^{d}\tilde{\mathbf{b}}_{j,k,m}(\xi)\partial_{\xi_m}, \quad k \in \{1,2\}, \quad j \in \{1,\dots,N\}\}.$$

It holds

$$\begin{aligned}\|\partial_{\mathrm{y}'_k}\varphi_j^*(\alpha_j\Phi)\|_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})} \le \|\varphi_j^*((\sum_{m=1}^{d}\mathbf{a}_{j,k,m})\Phi)\|_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})} \\ + \|\varphi_j^*(\alpha_j(\sum_{m=1}^{d}\mathbf{b}_{j,k,m}\partial_{\xi_m}\Phi))\|_{H^m(\mathbb{R}^{d-1}_{\mathrm{y}'})}.\end{aligned} \tag{2.19}$$

With (2.1) we obtain

$$\begin{aligned} \|\varphi_j^*((\sum_{m=1}^d \mathbf{a}_{j,k,m})\Phi)\|_{H^m(\mathbb{R}^{d-1}_{y'})} &\le C \sum_{k\in\mathcal{K}} \|\varphi_k^*(\alpha_k(\sum_{m=1}^d \mathbf{a}_{j,k,m})\Phi)\|_{H^m(\mathbb{R}^{d-1}_{y'})} \\ &\le C\|(\sum_{m=1}^d \mathbf{a}_{j,k,m})\Phi\|_{H^m(\Gamma)} \le C\|\Phi\|_{H^m(\Gamma)}, \end{aligned} \tag{2.20}$$

where the last inequality holds due to the fact that $(\sum_{m=1}^d \mathbf{a}_{j,k,m}) \in \mathcal{C}^\infty(\Gamma)$. Additionally, we have

$$\|\varphi_j^*(\alpha_j(\sum_{m=1}^d \mathbf{b}_{j,k,m}\partial_{\xi_m}\Phi))\|_{H^m(\mathbb{R}^{d-1}_{y'})} \le \sup_{B\in\mathcal{B}} \|B\Phi\|_{H^m(\Gamma)}. \tag{2.21}$$

Combination of (2.17), (2.18), (2.19), (2.20) and (2.21) yields

$$\|\Phi\|_{H^{m+1}(\Gamma)} \le C(\|\Phi\|_{H^m(\Gamma)} + \sup_{B\in\mathcal{B}} \|B\Phi\|_{H^m(\Gamma)}).$$

□

Corollary 2.19. Let $d = 3$, $m \in \mathbb{N}_0$, Ω be an open, bounded domain with boundary Γ of class $\mathcal{C}^\infty$. Let $v \in L^2((0,T), H^m(\Gamma))$ and $Bv \in L^2((0,T), H^m(\Gamma))$ for all $B \in \mathscr{B}$. Then, $v \in L^2((0,T), H^{m+1}(\Gamma))$ and there exists a set of finitely many first-order operators $\mathcal{B} \subset \mathscr{B}$ such that

$$\|v\|_{L^2((0,T),H^{m+1}(\Gamma))} \le C(\|v\|_{L^2((0,T),H^m(\Gamma))} + \sup_{B\in\mathcal{B}} \|Bv\|_{L^2((0,T),H^m(\Gamma))}).$$

Proof. Follows from Lemma 2.18. □

2.7. Method of Mappings

A detailed and general discussion of the method of mappings can, e.g., be found in [20, 41, 105]. Here, a short comprehensive motivation of the method is given. We start at observing that a general, abstract and intuitive formulation for shape optimization problems is given by

$$\min_{\hat{\Omega}\in\hat{\mathcal{O}}_{ad}} \hat{j}(\hat{\Omega}),$$

where $\hat{\mathcal{O}}_{ad}$ denotes an appropriate set of admissible domains and $\hat{j} : \hat{\mathcal{O}}_{ad} \to \mathbb{R}$ a shape functional [31, Def. 4.3.1].

Remark 2.20. In the case of PDE constrained optimization $\hat{j}(\hat{\Omega})$ usually denotes a reduced cost functional defined by $\hat{j}(\hat{\Omega}) = \hat{J}(\hat{y}, \hat{\Omega})$, where

$$\hat{J} : \{(\hat{y}, \hat{\Omega}) : \hat{y} \in Y(\hat{\Omega}), \hat{\Omega} \in \hat{\mathcal{O}}_{ad}\} \to \mathbb{R}$$

and $\hat{y}$ denotes the solution of a partial differential equation given by $\hat{E}(\hat{y}, \hat{\Omega}) = 0$ with

$$\hat{E} : \{(\hat{y}, \hat{\Omega}) : \hat{y} \in Y(\hat{\Omega}), \hat{\Omega} \in \hat{\mathcal{O}}_{ad}\} \to \{\hat{z} : \hat{z} \in Z(\hat{\Omega}), \hat{\Omega} \in \hat{\mathcal{O}}_{ad}\}.$$

Here, $Y(\hat{\Omega})$ and $Z(\hat{\Omega})$ denote Banach spaces.

One of the main challenges in considering this optimization problem addresses the topological structure , more precisely, the definition of an appropriate metric on $\hat{\mathcal{O}}_{ad}$. Besides the consideration of characteristic functions (which motivates, e.g., phase field approaches, cf., e.g., [47, 48]) or distance functions [31], $\hat{\mathcal{O}}_{ad}$ can be endowed with a metric that is defined via transformations [102, 31]. Similarly to the FSI problem the Lagrangian or Eulerian perspective can be chosen to work with transformations. The latter leads to the notion of shape derivatives, cf., e.g., [1, 31, 122], and to level set methods, cf., e.g., [1, 22, 23, 110]. The Lagrangian perspective is known as method of mappings or perturbation of the identity [20, 105]. The main idea is the introduction of a reference domain $\tilde{\Omega}$ and the choice

$$\hat{\mathcal{O}}_{ad} = \{\tilde{\boldsymbol{\tau}}(\tilde{\Omega}) \ : \ \tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}\}$$

as the set of shapes that can be obtained by transforming a nominal or shape reference domain $\tilde{\Omega}$ with $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad} \subset \mathcal{T}(\tilde{\Omega})$, where $\mathcal{T}(\tilde{\Omega})$ denotes the Banach space of bicontinuous transformation of $\tilde{\Omega}$. This allows for a reformulation of the shape optimization problem in an optimal control setting defined on $\tilde{\Omega}$ with control $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$ which is given by

$$\min_{\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}} \tilde{j}(\tilde{\boldsymbol{\tau}}),$$

where $\tilde{j}(\tilde{\boldsymbol{\tau}}) := \hat{j}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))$ for all $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$. In order for the optimization problem to be equivalend to the original problem we have to ensure a one-to-one correspondance between transformations and shapes.

Remark 2.21. Any bi-Lipschitz transformation $\tilde{\boldsymbol{\tau}}_0$ that just transforms the interior of $\tilde{\Omega}$ can be added to $\tilde{\boldsymbol{\tau}}$ and it holds $\tilde{\boldsymbol{\tau}}(\tilde{\Omega}) = (\tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\tau}}_0)(\tilde{\Omega})$. The reformulation of the shape optimization problem has to take care of these kernel spaces, which motivates the consideration of shape optimization problems on appropriate linear subspaces or manifolds.

Remark 2.22. In the case of PDE constrained optimization, it holds

$$\tilde{j}(\tilde{\boldsymbol{\tau}}) = \hat{j}(\hat{y}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})) = \hat{J}(\hat{y}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})),$$

where $\hat{y}$ solves $\hat{E}(\hat{y}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})) = 0$. Under the assumption that $\tilde{\boldsymbol{\tau}}$ is smooth enough such that for $\hat{y} \in Y(\tilde{\boldsymbol{\tau}}(\Omega))$ it holds that $\hat{y} \circ \tilde{\boldsymbol{\tau}} \in Y(\tilde{\Omega})$ for all $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$ and such that for $\tilde{y} \in Y(\tilde{\Omega})$ it holds that $\tilde{y} \circ \tilde{\boldsymbol{\tau}}^{-1} \in Y(\tilde{\boldsymbol{\tau}}^{-1}(\tilde{\Omega}))$ for all $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$, we have

$$\hat{J}(\hat{y}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})) = \hat{J}(\hat{y} \circ \tilde{\boldsymbol{\tau}}, \tilde{\Omega})$$

and

$$\hat{E}(\hat{y}, \tilde{\boldsymbol{\tau}}(\tilde{\Omega})) = \hat{E}(\hat{y} \circ \tilde{\boldsymbol{\tau}}, \tilde{\Omega}),$$

which yields an reformulation of the shape optimization problem on the reference domain $\tilde{\Omega}$. The assumption is fulfilled, if $Y(\tilde{\Omega}) = \{\hat{y} \circ \tilde{\boldsymbol{\tau}} : \hat{y} \in Y(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\}$ and the mapping

$$\hat{y} \in Y(\tilde{\boldsymbol{\tau}}(\tilde{\Omega})) \to \tilde{y} := \hat{y} \circ \tilde{\boldsymbol{\tau}} \in Y(\tilde{\Omega})$$

is a homeomorphism for all $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$. In that case, there are two possibilities to compute the gradient of the reduced cost functional. For the iterate $\tilde{\boldsymbol{\tau}}$ one either solves the with $\tilde{\boldsymbol{\tau}}$ transformed state and adjoint equations on $\tilde{\Omega}$ or one solves the untransformed state and adjoint equations on the transformed domain $\tilde{\boldsymbol{\tau}}(\tilde{\Omega})$.

It is convenient to define $\tilde{\mathbf{u}}_\tau := \tilde{\boldsymbol{\tau}} - \mathrm{id}_{\mathbf{z}}$ to ensure that 0 is admissible in the optimization process, as well as,

$$\tilde{\mathbf{U}}_{ad} := \{\tilde{\mathbf{u}}_\tau : \mathbb{R}^d \to \mathbb{R}^d \;:\quad \mathrm{id}_{\mathbf{z}} + \tilde{\mathbf{u}}_\tau \in \tilde{\mathcal{T}}_{ad}\},$$

and optimize over $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}}_{ad}$ instead of $\boldsymbol{\tau} \in \tilde{\mathcal{T}}_{ad}$. Thus, we arrive at the optimization problem

$$\min_{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}}_{ad}} j(\tilde{\mathbf{u}}_\tau),$$

where $j(\tilde{\mathbf{u}}_\tau) := \tilde{j}(\mathrm{id}_{\mathbf{z}} + \tilde{\mathbf{u}}_\tau)$.

3. Fluid-Structure Interaction

This chapter collects theoretical results for FSI problems under simplifying assumptions. In Section 3.1 it is shown for an unsteady, linear FSI model that the adjoint equation has the same structure as the forward model but reverses the temporal flow of information. A new improved regularity result for linear hyperbolic equations is derived in Section 3.2, which is the basis for a regularity result for an unsteady, linear FSI problem (Section 3.3). These considerations are the foundation for a local-in-time regularity result for an unsteady, nonlinear FSI problem in Section 3.4. The main contribution is the new improved regulartiy results for linear hyperbolic equations in Section 3.2.

3.1. Adjoint Considerations for a Linear Unsteady FSI Problem with Stationary Interface

For computing the gradients in optimal control settings, the adjoint equations have to be solved. Especially in cases, where no automatic differentiation can be applied, it is crucial to derive an explicit formula for the adjoint equations. Even though the FSI model is modified for performing shape optimization, the adjoint equations to the unmodified model can be used to drive the optimization (Section 2.7), basically when every iteration is performed on the current ALE reference domain instead of the nominal domain, cf. [20, Sec. 2.2.2].

We consider the adjoint of a linear version of the fluid-structure interaction model (1.2). More precisely, we consider Stokes flow for the fluid and linear elasticity for the solid equation. Additionally, we restrict ourselves to the case with a stationary interface $\hat{\Gamma}_i$ and homogeneous Dirichlet boundary conditions, i.e., $\hat{\Omega} = \check{\Omega}(\mathsf{t}) = \Omega$ and $\partial\Omega(\mathsf{t}) = \overline{\hat{\Gamma}_{fD}} \cup \overline{\hat{\Gamma}_{sD}}$ for any $\mathsf{t} \in I$. This also implies that $\hat{\boldsymbol{\chi}} = \mathrm{id}_{\mathsf{y}}$, $\hat{J}_{\boldsymbol{\chi}} = 1$ and $\widehat{\mathbf{F}}_{\boldsymbol{\chi}} = \mathbf{I}$. The resulting fluid-structure interaction problem (for the sake of clarity without superscripts) reads as follows

$$\begin{aligned}
\rho_f \partial_{\mathsf{t}} \mathbf{v} - \operatorname{div}(\boldsymbol{\sigma}_f) &= \rho_f \mathbf{f}_f && \text{in } \Omega_f \times I, \\
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega_f \times I, \\
\mathbf{v} &= 0 && \text{on } \Gamma_f \times I, \\
\mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\
\rho_s \partial_{\mathsf{t}} \mathbf{v} - \operatorname{div}(\boldsymbol{\sigma}_s) &= \rho_s \mathbf{f}_s && \text{in } \Omega_s \times I, \\
\rho_s (\partial_{\mathsf{t}} \mathbf{w} - \mathbf{v}) &= 0 && \text{in } \Omega_s \times I, \\
\mathbf{w} &= 0 && \text{on } \Gamma_s \times I, \\
\mathbf{w}(\cdot, 0) &= \mathbf{w}_0 && \text{in } \Omega_s, \\
\mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_s,
\end{aligned} \tag{3.1}$$

with the additional coupling conditions

$$\begin{aligned} \partial_t \mathbf{w} = \mathbf{v} \quad & \text{on } \Gamma_i \times I, \\ -\boldsymbol{\sigma}_f \mathbf{n}_f = \boldsymbol{\sigma}_s \mathbf{n}_s \quad & \text{on } \Gamma_i \times I, \end{aligned}$$

where $\boldsymbol{\sigma}_f = \mu_f(D\mathbf{v} + D\mathbf{v}^\top) - p\mathbf{I}$, $\mu_f = \rho_f \nu_f$, $\boldsymbol{\sigma}_s = \mu_s(D\mathbf{w} + D\mathbf{w}^\top) + \lambda_s \mathrm{tr}(D\mathbf{w})\mathbf{I}$ and, for the sake of convenience, we introduced $\mathbf{v}_0$ defined by $\mathbf{v}_0|_{\Omega_f} = \mathbf{v}_{0f}$ and $\mathbf{v}_0|_{\Omega_s} = \mathbf{w}_1$. For compatibility reasons there holds $\mathbf{w}_0|_{\Gamma_s} = 0$. This corresponds to the setting considered in [34].

We are interested in the adjoint equations and therefore do calculations on a formal level in order to derive a formulation for the adjoint system. In particular, we do not analyze the regularity of solutions but only assume that all functions are smooth enough such that the appearing terms and operations are well-defined. For the analysis of (3.1) we refer to [34, 35, 38]. Since we consider a linear unsteady partial differential equation, we aim for a weak formulation for which the adjoint attains the same structure as the forward model but reverses the temporal flow of information. In [38], it is shown that an unsophisticated straightforward weak formulation does not have the desired property, basically due to the term $\partial_t \mathbf{w} - \mathbf{v} = 0$. As a remedy, it is proposed to work with $\nabla \partial_t \mathbf{w} - \nabla \mathbf{v} = 0$ instead. In the following, we apply ideas from [34] to reformulate the weak formulation and obtain an analogous result.

Since $\partial_t \mathbf{w} = \mathbf{v}$ on Ω_s it follows that $\mathbf{w}(\cdot, t) = \mathbf{w}_0 + \int_0^t \mathbf{v}(s) ds$ on Ω_s.
Let $\mathbf{W}(\mathbf{v})(\cdot, t) = \mathbf{w}_0 + \int_0^t \mathbf{v}(\cdot, s) ds$, then the problem reads as follows:

$$\begin{aligned} \rho_f \partial_t \mathbf{v} - \mu_f \mathrm{div}(D\mathbf{v} + D\mathbf{v}^\top) + \nabla p = \rho_f \mathbf{f}_f \quad & \text{in } Q_f^T, \\ \mathrm{div}(\mathbf{v}) = 0 \quad & \text{in } Q_f^T, \\ \mathbf{v} = 0 \quad & \text{on } \Sigma_f^T, \\ \rho_s \partial_t \mathbf{v} - \mu_s \mathrm{div}(D\mathbf{W}(\mathbf{v}) + D\mathbf{W}(\mathbf{v})^\top) - \lambda_s \nabla(\mathrm{div}(\mathbf{W}(\mathbf{v}))) = \rho_s \mathbf{f}_s \quad & \text{in } Q_s^T, \\ \mathbf{v} = 0 \quad & \text{on } \Sigma_s^T, \\ \mathbf{v}(\cdot, 0) = \mathbf{w}_1 \quad & \text{in } \Omega, \end{aligned}$$

with the additional coupling condition

$$p\mathbf{n}_f - \mu_f(D\mathbf{v} + D\mathbf{v}^\top)\mathbf{n}_f = \mu_s(D\mathbf{W}(\mathbf{v}) + D\mathbf{W}(\mathbf{v})^\top)\mathbf{n}_s + \lambda_s \mathrm{div}(\mathbf{W}(\mathbf{v}))\mathbf{n}_s \quad \text{on } \Sigma_i^T.$$

One can check that this formulation is equivalent to the previous one since $\rho_s(\partial_t \mathbf{W}(\mathbf{v}) - \mathbf{v}) = 0$ on Q_s^T, $\mathbf{W}(\mathbf{v}) = 0$ on Σ_s^T and $\mathbf{W}(\mathbf{v})(\cdot, 0) = \mathbf{w}_0$ are satisfied by the definition of $\mathbf{W}(\mathbf{v})$ as well as $\partial_t \mathbf{W}(\mathbf{v}) = \mathbf{v}$ on Σ_i^T is satisfied if we require $\mathbf{v} \in H_0^1(\Omega)$ for almost all $t \in I$, which implies uniqueness of the trace.

The following notation is used:

- $(p, q)_\Omega := \int_\Omega pq d\xi$ for all $p, q \in L^2(\Omega)$, $(\mathbf{v}, \mathbf{u})_\Omega := \int_\Omega \mathbf{v} \cdot \mathbf{u} d\xi$ for all $\mathbf{v}, \mathbf{u} \in L^2(\Omega)^d$ and $(\mathbf{A}, \mathbf{B})_\Omega := \int_\Omega \mathbf{A} : \mathbf{B} d\xi$ for all $\mathbf{A}, \mathbf{B} \in L^2(\Omega)^{d \times d}$.
- $(\mathbf{v}, \mathbf{u})_\Gamma := \int_\Gamma \mathbf{v} \cdot \mathbf{u} dS(\xi)$ for all $\mathbf{v}, \mathbf{u} \in L^2(\Omega)^d$, where $dS(\xi)$ denotes the surface measure on Γ.

- $(\!(p,q)\!)_{Q^T} := \int_0^T (p(\cdot,\mathsf{t}), q(\cdot,\mathsf{t}))_\Omega d\mathsf{t}$ for all $p, q \in L^2((0,T), L^2(\Omega))$,
  $(\!(\mathbf{v},\mathbf{u})\!)_{Q^T} := \int_0^T (\mathbf{v}(\cdot,\mathsf{t}), \mathbf{u}(\cdot,\mathsf{t}))_\Omega d\mathsf{t}$ for all $\mathbf{v}, \mathbf{u} \in L^2((0,T), L^2(\Omega)^d)$ and
  $(\!(\mathbf{A},\mathbf{B})\!)_{Q^T} := \int_0^T (\mathbf{A},\mathbf{B})_\Omega d\mathsf{t}$ for all $\mathbf{A}, \mathbf{B} \in L^2((0,T), L^2(\Omega)^{d\times d})$.

- $a_f(\mathbf{v},\mathbf{z}) = \frac{\mu_f}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top)_{\Omega_f}$,
  $a_s(\mathbf{v},\mathbf{z}) = \frac{\mu_s}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top\rangle_{\Omega_s} + \lambda_s(\mathrm{div}(\mathbf{v}), \mathrm{div}(\mathbf{z}))_{\Omega_s}$, for $\mathbf{v}, \mathbf{z} \in H_0^1(\Omega)^d$.

The corresponding weak formulation reads as follows:

$$\begin{aligned}
&\rho_f(\!(\partial_{\mathsf{t}}\mathbf{v}, \boldsymbol{\psi}^v)\!)_{Q_f^T} - \mu_f(\!(\mathrm{div}(D\mathbf{v} + D\mathbf{v}^\top), \boldsymbol{\psi}^v)\!)_{Q_f^T} \\
&+ (\!(\nabla p, \boldsymbol{\psi}^v)\!)_{Q_f^T} - (\!(\rho_f \mathbf{f}_f, \boldsymbol{\psi}^v)\!)_{Q_f^T} + \rho_s(\!(\partial_{\mathsf{t}}\mathbf{v}, \boldsymbol{\psi}^v)\!)_{Q_s^T} \\
&- ((\rho_s \mathbf{f}_s, \boldsymbol{\psi}^v))_{Q_s^T} - (\!(\mathrm{div}(\mathbf{v}), \psi^p)\!)_{Q_f^T} + (\mathbf{v}(\cdot,0) - \mathbf{v}_0, \boldsymbol{\psi}^v(\cdot,0))_\Omega \\
&- \mu_s(\!(\mathrm{div}(D\mathbf{W}(\mathbf{v}) + D\mathbf{W}(\mathbf{v})^\top), \boldsymbol{\psi}^v)\!)_{Q_s^T} - \lambda_s(\!(\nabla(\mathrm{div}(\mathbf{W}(\mathbf{v}))), \boldsymbol{\psi}^v)\!)_{Q_s^T} = 0.
\end{aligned}$$

Integration by parts yields the formulas

$$\begin{aligned}
&a_f(\mathbf{v},\mathbf{z}) = \mu_f \sum_{j,k} \int_{\Omega_f} (\partial_k \mathbf{v}_j \partial_k \mathbf{z}_j + \partial_k \mathbf{v}_j \partial_j \mathbf{z}_k) d\xi \\
&= -\mu_f \sum_{j,k} \int_{\Omega_f} (\partial_k \partial_k \mathbf{v}_j \mathbf{z}_j + \partial_j \partial_k \mathbf{v}_j \mathbf{z}_k) d\xi + \mu_f \sum_{j,k} \int_{\partial\Omega_f} (\partial_k \mathbf{v}_j \mathbf{z}_j \mathbf{n}_{f,k} + \partial_k \mathbf{v}_j \mathbf{z}_k \mathbf{n}_{f,i}) dS(\xi) \\
&= -\mu_f \sum_{j,k} \int_{\Omega_f} (\partial_k \partial_k \mathbf{v}_j \mathbf{z}_j + \partial_j \partial_k \mathbf{v}_j \mathbf{z}_k) d\xi + \mu_f \sum_{j,k} \int_{\Gamma_i} (\partial_k \mathbf{v}_j \mathbf{z}_j \mathbf{n}_{f,k} + \partial_k \mathbf{v}_j \mathbf{z}_k \mathbf{n}_{f,i}) dS(\xi) \\
&= -\mu_f (\mathrm{div}(D\mathbf{v} + D\mathbf{v}^\top), \mathbf{z})_{\Omega_f} + \mu_f ((D\mathbf{v} + D\mathbf{v}^\top)\mathbf{n}_f, \mathbf{z})_{\Gamma_i},
\end{aligned}$$

$$(\nabla p, \mathbf{z})_{\Omega_f} = -(p, \mathrm{div}(\mathbf{z}))_{\Omega_f} + (p\mathbf{n}_f, \mathbf{z})_{\partial\Omega_f} = -(p, \mathrm{div}(\mathbf{z}))_{\Omega_f} + (p\mathbf{n}_f, \mathbf{z})_{\Gamma_i},$$

and

$$\begin{aligned}
&a_s(\mathbf{v},\mathbf{z}) = \frac{\mu_s}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top)_{\Omega_s} + \lambda_s \sum_{j,k} \int_{\Omega_s} \partial_j \mathbf{v}_j \partial_k \mathbf{z}_k d\mathsf{x} \\
&= \frac{\mu_s}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top)_{\Omega_s} - \lambda_s \sum_{j,k} \int_{\Omega_s} \partial_k \partial_j \mathbf{v}_j \mathbf{z}_k d\mathsf{x} + \lambda_s \sum_{j,k} \int_{\partial\Omega_s} \partial_j \mathbf{v}_j \mathbf{z}_k \mathbf{n}_{s,k} dS(\mathsf{x}) \\
&= \frac{\mu_s}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top)_{\Omega_s} - \lambda_s \sum_{j,k} \int_{\Omega_s} \partial_k \partial_j \mathbf{v}_j \mathbf{z}_k d\mathsf{x} + \lambda_s \sum_{j,k} \int_{\Gamma_i} \partial_j \mathbf{v}_j \mathbf{z}_k \mathbf{n}_{s,k} dS(\mathsf{x}) \\
&= \frac{\mu_s}{2}(D\mathbf{v} + D\mathbf{v}^\top, D\mathbf{z} + D\mathbf{z}^\top)_{\Omega_s} - \lambda_s(\nabla(\mathrm{div}(\mathbf{v})), \mathbf{z})_{\Omega_s} + \lambda_s((\mathrm{div}(\mathbf{v}))\mathbf{n}_s, \mathbf{z})_{\Gamma_i} \\
&= -\mu_s(\mathrm{div}(D\mathbf{v} + D\mathbf{v}^\top), \mathbf{z})_{\Omega_s} + \mu_s((D\mathbf{v} + D\mathbf{v}^\top)\mathbf{n}_f, \mathbf{z})_{\Gamma_i} \\
&\quad - \lambda_s(\nabla(\mathrm{div}(\mathbf{v})), \mathbf{z})_{\Omega_s} + \lambda_s((\mathrm{div}(\mathbf{v}))\mathbf{n}_s, \mathbf{z})_{\Gamma_i}.
\end{aligned}$$

Thus, the weak formulation can be reformulated as

$$
\begin{aligned}
&\rho_f(\!(\partial_\mathsf{t}\mathbf{v},\boldsymbol{\psi}^v)\!)_{Q_f^T}+\int_0^T a_f(\mathbf{v},\boldsymbol{\psi}^v)d\mathsf{t}-(\!(p,\nabla\cdot\boldsymbol{\psi}^v)\!)_{Q_f^T}\\
&+\rho_s(\!(\partial_\mathsf{t}\mathbf{v},\boldsymbol{\psi}^v)\!)_{Q_s^T}+\int_0^T a_s(\mathbf{W}(\mathbf{v}),\boldsymbol{\psi}^v)d\mathsf{t}-(\!(\nabla\cdot\mathbf{v},\psi^p)\!)_{Q_f^T}\\
&-(\!(\rho_f\mathbf{f}_f,\boldsymbol{\psi}^v)\!)_{Q_f^T}-(\!(\rho_s\mathbf{f}_s,\boldsymbol{\psi}^v)\!)_{Q_s^T}+(\mathbf{v}(\cdot,0)-\mathbf{v}_0,\boldsymbol{\psi}^v(\cdot,0))_\Omega\\
&-\int_0^T(\mu_f(\nabla\mathbf{v}+\nabla\mathbf{v}^\top)\mathbf{n}_f-p\mathbf{n}_f+\mu_s(\nabla\mathbf{W}(\mathbf{v})+\nabla\mathbf{W}(\mathbf{v})^\top)\mathbf{n}_s\\
&+\lambda_s(\nabla\cdot\mathbf{W}(\mathbf{v}))\mathbf{n}_s,\boldsymbol{\psi}^v)_{\Gamma_i}d\mathsf{t}=0,
\end{aligned}
$$

which can be simplified by using the interface condition:

$$
\begin{aligned}
&\rho_f(\!(\partial_\mathsf{t}\mathbf{v},\boldsymbol{\psi}^v)\!)_{Q_f^T}+\int_0^T a_f(\mathbf{v},\boldsymbol{\psi}^v)d\mathsf{t}-(\!(p,\nabla\cdot\boldsymbol{\psi}^v)\!)_{Q_f^T}\\
&+\rho_s(\!(\partial_\mathsf{t}\mathbf{v},\boldsymbol{\psi}^v)\!)_{Q_s^T}+\int_0^T a_s(\mathbf{W}(\mathbf{v}),\boldsymbol{\psi}^v)d\mathsf{t}-(\!(\nabla\cdot\mathbf{v},\psi^p)\!)_{Q_f^T}\\
&-(\!(\rho_f\mathbf{f}_f,\boldsymbol{\psi}^v)\!)_{Q_f^T}-(\!(\rho_s\mathbf{f}_s,\boldsymbol{\psi}^v)\!)_{Q_s^T}+(\mathbf{v}(\cdot,0)-\mathbf{v}_0,\boldsymbol{\psi}^v(\cdot,0))_\Omega=0.
\end{aligned}
$$

Linearization of this equation yields an operator A, which is defined by

$$
\begin{aligned}
\langle A(\boldsymbol{\eta}^v,\eta^p),(\boldsymbol{\psi}^v,\psi^p)\rangle&=\rho_f(\!(\partial_\mathsf{t}\boldsymbol{\eta}^v,\boldsymbol{\psi}^v)\!)_{Q_f^T}+\int_0^T a_f(\boldsymbol{\eta}^v,\boldsymbol{\psi}^v)d\mathsf{t}-(\!(\eta^p,\nabla\cdot\boldsymbol{\psi}^v)\!)_{Q_f^T}\\
&+\rho_s(\!(\partial_\mathsf{t}\boldsymbol{\eta}^v,\boldsymbol{\psi}^v)\!)_{Q_s^T}+\int_0^T a_s(\int_0^\mathsf{t}\boldsymbol{\eta}^v(\cdot,\mathsf{s})d\mathsf{s},\boldsymbol{\psi}^v)d\mathsf{t}-(\!(\nabla\cdot\boldsymbol{\eta}^v,\psi^p)\!)_{Q_f^T}\\
&+(\boldsymbol{\eta}^v(\cdot,0),\boldsymbol{\psi}^v(\cdot,0))_\Omega.
\end{aligned}
$$

The term which destroys the symmetry of the operator is given by a_s. Closer consideration of this term yields (under the assumption that we are on spaces where Fubini's theorem is valid) yields:

$$
\begin{aligned}
&\int_0^T a_s(\int_0^\mathsf{t}\boldsymbol{\eta}^v(\cdot,\mathsf{s})d\mathsf{s},\boldsymbol{\psi}^v)d\mathsf{t}=\int_0^T a_s(\int_0^\mathsf{t}\boldsymbol{\eta}^v(\cdot,\mathsf{s})d\mathsf{s},\boldsymbol{\psi}^v(\cdot,\mathsf{t}))d\mathsf{t}\\
&=\int_0^T\int_0^\mathsf{t} a_s(\boldsymbol{\eta}^v(\cdot,\mathsf{s}),\boldsymbol{\psi}^v(\cdot,\mathsf{t}))d\mathsf{s}d\mathsf{t}=\int_0^T\int_s^T a_s(\boldsymbol{\eta}^v(\cdot,\mathsf{s}),\boldsymbol{\psi}^v(\cdot,\mathsf{t}))d\mathsf{t}d\mathsf{s}\\
&=\int_0^T a_s(\boldsymbol{\eta}^v(\cdot,\mathsf{s}),\int_\mathsf{s}^T\boldsymbol{\psi}^v(\cdot,\mathsf{t})d\mathsf{t})d\mathsf{s}=\int_0^T a_s(\boldsymbol{\eta}^v(\cdot,\mathsf{s}),\int_0^{T-\mathsf{s}}\boldsymbol{\psi}^v(\cdot,T-\mathsf{t})d\mathsf{t})d\mathsf{s}\\
&=\int_0^T a_s(\boldsymbol{\eta}^v(\cdot,T-\mathsf{s}),\int_0^\mathsf{s}\boldsymbol{\psi}^v(\cdot,T-\mathsf{t})d\mathsf{t})d\mathsf{s}
\end{aligned}
$$

Introducing $\overline{\boldsymbol{\eta}}^v(\xi,\mathsf{t})=\boldsymbol{\eta}^v(\xi,T-\mathsf{t})$, $\overline{\eta}^p(\xi,\mathsf{t})=\eta^p(\xi,T-\mathsf{t})$, $\overline{\boldsymbol{\psi}}^v(\xi,\mathsf{t})=\boldsymbol{\psi}^v(\xi,T-\mathsf{t})$,

$\overline{\psi}^p(\xi, \mathsf{t}) = \psi^p(\xi, T - \mathsf{t})$ yields

$$\int_0^T a_s\Big(\int_0^{\mathsf{t}} \boldsymbol{\eta}^v(\cdot, \mathsf{s}) d\mathsf{s}, \boldsymbol{\psi}^v\Big) d\mathsf{t} = \int_0^T a_s\Big(\int_0^{\mathsf{t}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{s}) d\mathsf{s}, \overline{\boldsymbol{\eta}}^v\Big) d\mathsf{t}.$$

We have $\partial_{\mathsf{t}} \boldsymbol{\psi}^v(\cdot, \mathsf{t}) = -\partial_{\mathsf{s}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{s})$ for $\mathsf{s} = T - \mathsf{t}$ and the following equation holds true:

$$\begin{aligned}
(\!(\boldsymbol{\eta}^v, \partial_{\mathsf{t}} \boldsymbol{\psi}^v)\!)_{Q_f^T} &= \int_0^T (\boldsymbol{\eta}^v(\cdot, \mathsf{t}), \partial_{\mathsf{t}} \boldsymbol{\psi}^v(\cdot, \mathsf{t}))_{\Omega_f} d\mathsf{t} \\
&= \int_0^T (\overline{\boldsymbol{\eta}}^v(\cdot, T - \mathsf{t}), -\partial_{\mathsf{s}} \overline{\boldsymbol{\psi}}^v(\cdot, T - \mathsf{t}))_{\Omega_f} d\mathsf{t} \\
&= -\int_T^0 (\overline{\boldsymbol{\eta}}^v(\cdot, \mathsf{s}), -\partial_{\mathsf{s}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{s}))_{\Omega_f} d\mathsf{s} = -\int_0^T (\overline{\boldsymbol{\eta}}^v(\cdot, \mathsf{s}), \partial_{\mathsf{s}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{s}))_{\Omega_f} d\mathsf{s} \\
&= -\int_0^T (\overline{\boldsymbol{\eta}}^v(\cdot, \mathsf{t}), \partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{t}))_{\Omega_f} d\mathsf{t} = -(\!(\overline{\boldsymbol{\eta}}^v, \partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v)\!)_{Q_f^T}.
\end{aligned}$$

This is the reason why partial integration yields

$$\begin{aligned}
(\!(\partial_{\mathsf{t}} \boldsymbol{\eta}^v, \boldsymbol{\psi}^v)\!)_{Q_f^T} &= \int_0^T \partial_{\mathsf{t}} (\boldsymbol{\eta}^v, \boldsymbol{\psi}^v)_{\Omega_f} d\mathsf{t} - (\!(\boldsymbol{\eta}^v, \partial_{\mathsf{t}} \boldsymbol{\psi}^v)\!)_{Q_f^T} \\
&= (\boldsymbol{\eta}^v(\cdot, T), \boldsymbol{\psi}^v(\cdot, T))_{\Omega_f} - (\boldsymbol{\eta}^v(\cdot, 0), \boldsymbol{\psi}^v(\cdot, 0))_{\Omega_f} + (\!(\overline{\boldsymbol{\eta}}^v, \partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v)\!)_{Q_f^T}.
\end{aligned}$$

Analogoulsy,

$$\begin{aligned}
(\!(\partial_{\mathsf{t}} \boldsymbol{\eta}^v, \boldsymbol{\psi}^v)\!)_{Q_s^T} &= \int_0^T \partial_{\mathsf{t}} (\boldsymbol{\eta}^v, \boldsymbol{\psi}^v)_{\Omega_s} d\mathsf{t} - (\!(\boldsymbol{\eta}^v, \partial_{\mathsf{t}} \boldsymbol{\psi}^v)\!)_{Q_s^T} \\
&= (\boldsymbol{\eta}^v(\cdot, T), \boldsymbol{\psi}^v(\cdot, T))_{\Omega_s} - (\boldsymbol{\eta}^v(\cdot, 0), \boldsymbol{\psi}^v(\cdot, 0))_{\Omega_s} + (\!(\overline{\boldsymbol{\eta}}^v, \partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v)\!)_{Q_s^T}.
\end{aligned}$$

Combining these results, rewriting the terms in $\overline{\boldsymbol{\eta}}^v$, $\overline{\eta}^p$, $\overline{\boldsymbol{\psi}}^v$, $\overline{\psi}^p$ and using that $(\boldsymbol{\eta}^v(\cdot, T), \boldsymbol{\psi}^v(\cdot, T))_\Omega = (\overline{\boldsymbol{\eta}}^v(\cdot, 0), \overline{\boldsymbol{\psi}}^v(\cdot, 0))_\Omega$ yields:

$$\begin{aligned}
\langle A(\boldsymbol{\eta}^v, \eta^p), (\boldsymbol{\psi}^v, \psi^p) \rangle &= \rho_f (\!(\partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v, \overline{\boldsymbol{\eta}}^v)\!)_{Q_f^T} + \int_0^T a_f(\overline{\boldsymbol{\psi}}^v, \overline{\boldsymbol{\eta}}^v) d\mathsf{t} - (\!(\nabla \cdot \overline{\boldsymbol{\psi}}^v, \overline{\eta}^p)\!)_{Q_f^T} \\
&\quad + \rho_s (\!(\partial_{\mathsf{t}} \overline{\boldsymbol{\psi}}^v, \overline{\boldsymbol{\eta}}^v)\!)_{Q_s^T} + \int_0^T a_s\Big(\int_0^{\mathsf{t}} \overline{\boldsymbol{\psi}}^v(\cdot, \mathsf{s}) d\mathsf{s}, \overline{\boldsymbol{\eta}}^v\Big) d\mathsf{t} - (\!(\overline{\psi}^p, \nabla \cdot \overline{\boldsymbol{\eta}}^v)\!)_{Q_f^T} \\
&\quad + (\overline{\boldsymbol{\psi}}^v(\cdot, 0), \overline{\boldsymbol{\eta}}^v(\cdot, 0))_\Omega = \langle A(\overline{\boldsymbol{\psi}}^v, \overline{\psi}^p), (\overline{\boldsymbol{\eta}}^v, \overline{\eta}^p) \rangle.
\end{aligned}$$

Thus, the adjoint has the same structure as the forward model, but reverses the temporal flow of information.

3.2. Improved Regularity Result for Linear Hyperbolic Equations

In order to motivate improved regularity results for the Lamé system, we first consider the classical hyperbolic system

$$\begin{aligned} \partial_{\mathsf{tt}} w - \Delta w &= f \quad \text{in } Q_s^T, \\ w &= G \quad \text{on } \Sigma_s^T, \\ w(\cdot, \mathsf{0}) &= w_0 \quad \text{in } \Omega_s, \\ \partial_{\mathsf{t}} w(\cdot, \mathsf{0}) &= w_1 \quad \text{in } \Omega_s, \end{aligned} \tag{3.2}$$

and derive improved regularity results for this system.

3.2.1. Available Existence and Regularity Results

The theory is built on the following existence and regularity result, that already contains an improved regularity result for the normal derivative on the boundary. Defining lifting operators as in [113] and using [86, Rem. 2.2, Thm. 2.2, Rem. 2.10] yields in an analogous way to [113, p. 560, Thm. 3.2]. For the time-independency of the constants compare also Theorem 3.12.

Theorem 3.1. 1. Let $f \in L^1((0,T), L^2(\Omega_s))$, $G \in H^1(\Sigma_s^T)$, $w_0 \in H^1(\Omega_s)$ and $w_1 \in L^2(\Omega_s)$ be such that

$$G|_{\mathsf{t=0}} = w_0|_{\Gamma_s}.$$

Then the solution w of system (3.2) satisfies

$$w \in \mathcal{C}([0,T], H^1(\Omega_s)) \cap \mathcal{C}^1([0,T], L^2(\Omega_s))$$

and

$$\nabla w \cdot \mathbf{n}_s \in L^2(\Sigma_s^T) = L^2((0,T), L^2(\Gamma_s)).$$

In addition,

$$\begin{aligned} &\|w\|_{\mathcal{C}([0,T],H^1(\Omega_s))\cap\mathcal{C}^1([0,T],L^2(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} \\ &\leq C(\|f\|_{L^1((0,T),L^2(\Omega_s))} + \|w_0\|_{H^1(\Omega_s)} + \|w_1\|_{L^2(\Omega_s)} + \|G\|_{H^1(\Sigma_s^T)}), \end{aligned}$$

where the constant C is independent of T.

2. Let $f \in L^1((0,T), H^1(\Omega_s))$, $\partial_{\mathsf{t}} f \in L^1((0,T), L^2(\Omega_s))$, $G \in H^2(\Sigma_s^T)$, $w_0 \in H^2(\Omega_s)$ and $w_1 \in H^1(\Omega_s)$ be such that

$$G|_{\mathsf{t=0}} = w_0|_{\Gamma_s}, \quad \partial_{\mathsf{t}} G|_{\mathsf{t=0}} = w_1|_{\Gamma_s}.$$

Then the solution w of system (3.2) satisfies

$$w \in \mathcal{C}([0,T], H^2(\Omega_s)) \cap \mathcal{C}^1([0,T], H^1(\Omega_s)) \cap \mathcal{C}^2([0,T], L^2(\Omega_s))$$

and
$$\nabla w \cdot \mathbf{n}_s \in H^1(\Sigma_s^T) = L^2((0,T),H^1(\Gamma_s)) \cap H^1((0,T),L^2(\Gamma_s)).$$
Furthermore,
$$\begin{aligned}&\|w\|_{\mathcal{C}([0,T],H^2(\Omega_s))\cap\mathcal{C}^1([0,T],H^1(\Omega_s))\cap\mathcal{C}^2([0,T],L^2(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{H^1(\Sigma_s^T)}\\ &\leq C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|\partial_{\mathsf{t}} f\|_{L^1((0,T),L^2(\Omega_s))}\\ &\qquad + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)} + \|G\|_{H^2(\Sigma_s^T)}),\end{aligned}$$
where the constant C is independent of T.

3. Let $f \in L^1((0,T),H^2(\Omega_s))$, $\partial_{\mathsf{t}} f \in L^1((0,T),H^1(\Omega_s))$, $\partial_{\mathsf{tt}} f \in L^1((0,T),L^2(\Omega_s))$, $G \in H^3(\Sigma_s^T)$, $w_0 \in H^3(\Omega_s)$ and $w_1 \in H^2(\Omega_s)$ be such that
$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathsf{t}} G|_{\mathsf{t}=0} = w_1|_{\Gamma_s}, \quad \partial_{\mathsf{tt}} G|_{\mathsf{t}=0} = (\Delta w_0 + f(\cdot,0))|_{\Gamma_s}.$$
Then the solution w of system (3.2) satisfies
$$w \in \mathcal{C}([0,T],H^3(\Omega_s)) \cap \mathcal{C}^1([0,T],H^2(\Omega_s)) \cap \mathcal{C}^2([0,T],H^1(\Omega_s))$$
and
$$\nabla w \cdot \mathbf{n}_s \in H^2(\Sigma_s^T) = L^2((0,T),H^2(\Gamma_s)) \cap H^2((0,T),L^2(\Gamma_s)).$$
Moreover,
$$\begin{aligned}&\|w\|_{\mathcal{C}([0,T],H^3(\Omega_s))\cap\mathcal{C}^1([0,T],H^2(\Omega_s))\cap\mathcal{C}^2([0,T],H^1(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{H^2(\Sigma_s^T)}\\ &\leq C(\|f\|_{L^1((0,T),H^2(\Omega_s))} + \|\partial_{\mathsf{t}} f\|_{L^1((0,T),H^1(\Omega_s))} + \|\partial_{\mathsf{tt}} f\|_{L^1((0,T),L^2(\Omega_s))}\\ &\qquad + \|w_0\|_{H^3(\Omega_s)} + \|w_1\|_{H^2(\Omega_s)} + \|f(\cdot,0)\|_{H^1(\Omega_s)} + \|G\|_{H^3(\Sigma_s^T)}),\end{aligned}$$
where the constant C is independent of T.

3.2.2. Local-in-Time Results

Applying the above results directly in the FSI setting results in a loss of information due to the anisotropic regularity of the solution of the fluid equations in space and time, cf. [59, Assumption 1]. Using a fixed point argument on the coupling conditions motivates to work with boundary conditions $G \in L^2((0,T),H^{r+1}(\Gamma_s)) \cap H^r((0,T),H^1(\Gamma_s))$, $r \in \{1,2\}$. The starting point for our considerations is a technique of [86] which allows to consider a modified linear hyperbolic equation, where the regularity of the boundary term is compatible to the regularity results in Theorem 3.1 without losing information as it is the case when we embed the space with higher regularity into a space with lower regularity. More precisely, we consider a first order operator $B \in \mathscr{B}$ that is tangential to the boundary in order to obtain the system

$$\begin{aligned} \partial_{\mathsf{tt}}(Bw) - \Delta(Bw) = (B\Delta - \Delta B)w + Bf \quad & \text{in } Q_s^T, \\ Bw = BG \quad & \text{on } \Sigma_s^T, \\ Bw(\cdot, 0) = Bw_0 \quad & \text{in } \Omega_s, \\ \partial_{\mathsf{t}} Bw(\cdot, 0) = Bw_1 \quad & \text{in } \Omega_s. \end{aligned}$$

Now, $BG \in L^2((0,T), H^r(\Gamma_s)) \cap H^r((0,T), L^2(\Gamma_s))$. However, the regularity of the right hand side of the hyperbolic equation, more precisely the term $(B\Delta - \Delta B)w$ prevents us from directly applying Theorem 3.1, since $(B\Delta - \Delta B)$ is, in general, a differential operator of order two. Thus, no bootstrapping argument can be used except for geometries where $(B\Delta - \Delta B)$ is a differential operator of order strictly less than two, which is, e.g., the case if Σ_s^T is flat as in [113]. In that special case it holds that $B\Delta - \Delta B = 0$ and Theorem 3.1 can directly be applied. From the regularity of Bw we can then improve the estimates on the regularity of the normal derivative of w on the boundary, cf. Corollary 2.19. The following interchangeability property is useful.

Lemma 3.2. Let $d \in \{2, 3\}$, Ω_s be a domain with smooth boundary Γ_s, $B = \sum_i \mathbf{b}_i(\xi)\partial_{\xi_i}$ be a time-independent first order operator with smooth coefficients $\mathbf{b}_i(\xi)$ such that

$$\sum_{i=1}^{d} \mathbf{b}_i(\xi)(\mathbf{n}_s)_i(\xi) = 0 \quad \text{on } \Gamma_s,$$

and

$$\nabla \mathbf{b}_i \cdot \mathbf{n}_s = 0 \quad \text{on } \Gamma_s,$$

for all $i \in \{1, \dots, d\}$, where $\mathbf{n}_s$ denotes the outer normal unit vector of Ω_s on Γ_s. Then,

$$B(\nabla\Phi \cdot \mathbf{n}_s) = \nabla B\Phi \cdot \mathbf{n}_s + \nabla\Phi \cdot (B\mathbf{n}_s)$$

on Γ_s and for any smooth Φ.

Proof. Follows from the following two identities:

$$\begin{aligned} \nabla B\Phi \cdot \mathbf{n}_s = \nabla(\sum_i \mathbf{b}_i \partial_{\xi_i}\Phi)\mathbf{n}_s &= \sum_{i,j}(\mathbf{b}_i \partial_{\xi_j}\partial_{\xi_i}\Phi(\mathbf{n}_s)_j + \partial_{\xi_j}\mathbf{b}_i \partial_{\xi_i}\Phi(\mathbf{n}_s)_j) \\ &= \sum_{i,j} \mathbf{b}_i \partial_{\xi_j}\partial_{\xi_i}\Phi(\mathbf{n}_s)_j, \end{aligned}$$

$$\begin{aligned} B(\nabla\Phi \cdot \mathbf{n}_s) &= \sum_{i,j}(\mathbf{b}_i \partial_{\xi_j}\partial_{\xi_i}\Phi(\mathbf{n}_s)_j + \mathbf{b}_i \partial_{\xi_j}\Phi\partial_{\xi_i}(\mathbf{n}_s)_j) \\ &= \sum_{i,j} \mathbf{b}_i \partial_{\xi_j}\partial_{\xi_i}\Phi(\mathbf{n}_s)_j + \nabla\Phi \cdot (B\mathbf{n}_s). \end{aligned}$$

□

The main idea to extend these considerations to smooth domains is inspired by techniques in [113]. We consider local-in-time solutions and use the fact that the constants of the estimates for right hand side terms that depend on w show well-behaved T-dependencies. Therefore, choosing T sufficiently small allows us to eliminate these terms. Then, globalization strategies as in [113] can be applied.

Lemma 3.3. Let $T^* > 0$,

$$G \in L^2((0,T^*), H^2(\Gamma_s)) \cap H^1((0,T^*), H^1(\Gamma_s)),$$

$f \in L^2((0,T^*), H^1(\Omega_s))$, $w_0 \in H^2(\Omega_s)$ and $w_1 \in H^1(\Omega_s)$ be such that

$$G|_{\mathsf{t=0}} = w_0|_{\Gamma_s}.$$

Then, there exists $\hat{T} \in (0, T^*]$ such that for all $T \in (0, \hat{T}]$ the solution w of system (3.2) satisfies

$$\nabla w \cdot \mathbf{n}_s \in L^2((0,T), H^1(\Gamma_s))$$

and

$$\begin{aligned}\|\nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^1(\Gamma_s))} \le C(T^{\frac{1}{2}}\|f\|_{L^2((0,T),H^1(\Omega_s))} + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\ + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)}),\end{aligned}$$

with a constant C independent of T.

Proof. We first assume the data to be smooth, i.e.,

$$w_0,\, w_1 \in C^\infty(\overline{\Omega}) \text{ and } f,\, G \in \mathcal{C}^\infty(\overline{\Omega} \times [0,T^*]),$$

and use extension by continuity, see Section 2.5, to conclude the argumentation.
By Theorem 3.1.1 we know that

$$\begin{aligned}&\|w\|_{\mathcal{C}([0,T],H^1(\Omega_s))\cap\mathcal{C}^1([0,T],L^2(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} \\ &\le C(\|f\|_{L^1((0,T),L^2(\Omega_s))} + \|w_0\|_{H^1(\Omega_s)} + \|w_1\|_{L^2(\Omega_s)} + \|G\|_{H^1(\Sigma_s^T)}),\end{aligned} \tag{3.3}$$

with a constant C independent of T. Let $C_B \in (0,\infty)$ be chosen such that

$$\begin{aligned}\mathcal{B} \subset \mathcal{B}_{C_B} := \{B \in \mathscr{B} \,:\, B = \sum_i \mathbf{b}_i \partial_{\xi_i},\ \text{s.t. } \mathbf{b}\cdot\mathbf{n}_s = 0 \text{ on } \Gamma_s,\ \sup_{1\le i\le d}\|\mathbf{b}_i\|_{\mathcal{C}^\infty(\overline{\Omega}_s)} \le C_B \text{ and} \\ \mathbf{b} \text{ is constant along normal directions locally around } \Gamma_s\},\end{aligned}$$

where $\mathcal{B}$ is the set of finitely many first-order operators defined in Corollary 2.19. Due to

Corollary 2.4, there exists $\mathbf{h} \in \mathcal{C}^\infty(\overline{\Omega}_s, \mathbb{R}^d)$ such that for all $B \in \mathcal{B}_{C_B}$ there holds

$$\begin{aligned} &\mathbf{h}|_{\Gamma_s} = \mathbf{n}_s, \\ &\mathbf{h} \text{ is constant along normal directions locally around } \Gamma_s, \\ &\nabla \mathbf{h}_l \cdot \mathbf{h} = 0, \\ &\nabla \mathbf{b}_l \cdot \mathbf{h} = 0, \end{aligned} \tag{3.4}$$

in a small neighborhood of Γ_s for all $l \in \{1, \dots, d\}$. Consider the system

$$\begin{aligned} \partial_{\mathsf{tt}}(Bw) - \Delta(Bw) = (B\Delta - \Delta B)w + Bf \quad & \text{in } Q_s^T, \\ Bw = BG \quad & \text{on } \Sigma_s^T, \\ Bw(\cdot, 0) = Bw_0 \quad & \text{in } \Omega_s, \\ \partial_{\mathsf{t}} Bw(\cdot, 0) = Bw_1 \quad & \text{in } \Omega_s. \end{aligned}$$

Standard estimates, Lemma 2.17 and $\|Bf\|_{L^1((0,T),L^2(\Omega_s))} \leq T^{\frac{1}{2}} \|Bf\|_{L^2((0,T),L^2(\Omega_s))}$ yield

$$\begin{aligned} &\|Bf\|_{L^1((0,T),L^2(\Omega_s))} \leq CT^{\frac{1}{2}} \|f\|_{L^2((0,T),H^1(\Omega_s))}, \\ &\|Bw_0\|_{H^1(\Omega_s)} \leq C\|w_0\|_{H^2(\Omega_s)}, \\ &\|Bw_1\|_{L^2(\Omega_s)} \leq C\|w_1\|_{H^1(\Omega_s)}, \\ &\|BG\|_{H^1(\Sigma_s^T)} \leq C\|G\|_{L^2((0,T),H^2(\Gamma_s)) \cap H^1((0,T),H^1(\Gamma_s))}, \end{aligned}$$

with constants C that depend on C_B but not on T. We have

$$\begin{aligned} (B\Delta - \Delta B)w &= \sum_{i,j} \mathbf{b}_j \partial_{\xi_j} \partial_{\xi_i} \partial_{\xi_i} w - \sum_i \partial_{\xi_i} \partial_{\xi_i} (\sum_j \mathbf{b}_j \partial_{\xi_j} w) \\ &= -2 \sum_{i,j} \partial_{\xi_i} \mathbf{b}_j \partial_{\xi_j} \partial_{\xi_i} w - \sum_{i,j} \partial_{\xi_i} \partial_{\xi_i} \mathbf{b}_j \partial_{\xi_j} w. \end{aligned} \tag{3.5}$$

We aim at proving the following estimate

$$\begin{aligned} &\|(B\Delta - \Delta B)w\|_{L^1((0,T),L^2(\Omega_s))} \\ &\leq C\|w\|_{L^1((0,T),H^1(\Omega_s))} + CT^{\frac{1}{2}} \sup_{\tilde{B} \in \mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}, \end{aligned} \tag{3.6}$$

for a constant C independent of T where we use that

$$\|\tilde{B}w\|_{L^1((0,T),H^1(\Omega_s))} \leq T^{\frac{1}{2}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}.$$

It holds that

$$\|\sum_{i,j} \partial_{\xi_i} \partial_{\xi_i} \mathbf{b}_j \partial_{\xi_j} w\|_{L^1((0,T),L^2(\Omega_s))} \leq C\|w\|_{L^1((0,T),H^1(\Omega_s))} \tag{3.7}$$

for a constant C independent of T.

However, for the first term of (3.5) we have $\sum_i \partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j} \notin \mathcal{B}_{C_B}$. The idea is to split the operator ∂_{ξ_i} on w into a normal and tangential part. Therefore, we define

$$\tilde{\mathbf{h}}_i := (\mathbf{h}\cdot\mathbf{e}_i)\mathbf{h}, \quad \tilde{\mathbf{b}}_i := \mathbf{e}_i - \tilde{\mathbf{h}}_i,$$

where $\mathbf{e}_i$ denotes the ith unit vector. We obtain

$$\sum_{i,j}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}\partial_{\xi_i}w = \sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}((\tilde{\mathbf{h}}_i)_k\partial_{\xi_k}w) + \sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}((\tilde{\mathbf{b}}_i)_k\partial_{\xi_k}w). \tag{3.8}$$

By definition of $\tilde{\mathbf{b}}_i$ we have $\tilde{\mathbf{b}}_i\cdot\mathbf{h}|_{\Gamma_s} = 0$ and due to (3.4) we know that $(\tilde{\mathbf{b}}_i)_k$ are constant along normal directions in a small neighborhood around Γ_s for all $k \in \{1,\dots,d\}$. Therefore, $\sum_k(\tilde{\mathbf{b}}_i)_k\partial_{\xi_k} \in \mathcal{B}_{C_B}$ and the second term of the right hand side of (3.8) can be estimated

$$\begin{aligned}\|\sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}((\tilde{\mathbf{b}}_i)_k\partial_{\xi_k}w)\|_{L^1((0,T),L^2(\Omega_s))} &\leq \sum_{i,j} C_B\|\partial_{\xi_j}(\sum_k(\tilde{\mathbf{b}}_i)_k\partial_{\xi_k}w)\|_{L^1((0,T),L^2(\Omega_s))}\\ &\leq CT^{\frac{1}{2}}\sup_{\tilde{B}\in\mathcal{B}_{C_B}}\|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}.\end{aligned} \tag{3.9}$$

The first summand of the right hand side of (3.8) splits into

$$\sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}((\tilde{\mathbf{h}}_i)_k\partial_{\xi_k}w) = \sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j(\tilde{\mathbf{h}}_i)_k\partial_{\xi_j}\partial_{\xi_k}w + \sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}(\tilde{\mathbf{h}}_i)_k\partial_{\xi_k}w. \tag{3.10}$$

The second summand of the right hand side of (3.10) is easy to handle and we obtain

$$\|\sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\partial_{\xi_j}(\tilde{\mathbf{h}}_i)_k\partial_{\xi_k}w\|_{L^1((0,T),L^2(\Omega_s))} \leq C\|w\|_{L^1((0,T),H^1(\Omega_s))}. \tag{3.11}$$

The first term of the right hand side of (3.10) reads as

$$\sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j(\tilde{\mathbf{h}}_i)_k\partial_{\xi_j}\partial_{\xi_k}w = \sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j\mathbf{h}_i\mathbf{h}_k\partial_{\xi_j}\partial_{\xi_k}w.$$

By (3.4), $\sum_i\partial_{\xi_i}\mathbf{b}_j\mathbf{h}_i\mathbf{h}_k = (\nabla\mathbf{b}_j\cdot\mathbf{h})\mathbf{h}_k$ is 0 in a small neighborhood around Γ_s. Consequently, $\gamma\sum_{i,j}\partial_{\xi_i}\mathbf{b}_j\mathbf{h}_i\mathbf{h}_k\partial_{\xi_j} \in \mathcal{B}_{B_C}$ for $\gamma = \frac{1}{d\|h\|^2_{\mathcal{C}^\infty(\overline{\Omega}))}}$ and

$$\begin{aligned}&\|\sum_{i,j,k}\partial_{\xi_i}\mathbf{b}_j(\tilde{\mathbf{h}}_i)_k\partial_{\xi_j}\partial_{\xi_k}w\|_{L^1((0,T),L^2(\Omega))}\\ &\leq \sum_k(\|\sum_{i,j}\partial_{\xi_k}(\partial_{\xi_i}\mathbf{b}_j(\tilde{\mathbf{h}}_i)_k)\partial_{\xi_j}w\|_{L^1((0,T),L^2(\Omega))} + \|\partial_{\xi_k}(\sum_{i,j}\partial_{\xi_i}\mathbf{b}_j(\tilde{\mathbf{h}}_i)_k\partial_{\xi_j}w)\|_{L^1((0,T),L^2(\Omega))})\\ &\leq C(\|w\|_{L^1((0,T),H^1(\Omega))} + T^{\frac{1}{2}}\sup_{\tilde{B}\in\mathcal{B}_{C_B}}\|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}).\end{aligned} \tag{3.12}$$

(3.5), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) yield estimate (3.6). Due to Theorem 3.1.1

there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))\cap\mathcal{C}^1([0,T],L^2(\Omega_s))} + \|\nabla Bw \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} \\
&\leq C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)} + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\
&\quad + \|w\|_{L^1((0,T),H^1(\Omega_s))} + T^{\frac{1}{2}} \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}).
\end{aligned}
$$

Using that $B \in \mathcal{B}_{C_B}$ yields

$$
\begin{aligned}
&\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))} + \sup_{B\in\mathcal{B}_{C_B}} \|\nabla Bw \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} \\
&\leq 2C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)} \\
&\quad + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} + \|w\|_{L^1((0,T),H^1(\Omega_s))} \\
&\quad + T^{\frac{1}{2}} \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}).
\end{aligned}
\tag{3.13}
$$

To apply Corollary 2.19, we need an estimate for $\|B(\nabla w \cdot \mathbf{n}_s)\|_{L^2(\Sigma_s^T)}$. By Lemma 3.2, we know that

$$
\|B(\nabla w \cdot \mathbf{n}_s)\|_{L^2(\Sigma_s^T)} \leq \|\nabla Bw \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} + \|\nabla w \cdot B\mathbf{n}_s\|_{L^2(\Sigma_s^T)}. \tag{3.14}
$$

The first summand can be bounded with (3.13). The second term can be written as

$$
\nabla w \cdot B\mathbf{n}_s = \sum_i \partial_{\xi_i} w \sum_j \mathbf{b}_j \partial_{\xi_j} (\mathbf{n}_s)_i = \sum_i \hat{\mathbf{b}}_i \partial_{\xi_i} w,
$$

where $\hat{\mathbf{b}}_i := \sum_j \mathbf{b}_j \partial_{\xi_j} \mathbf{h}_i$. We split $\hat{\mathbf{b}} := (\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_d)$ in a normal part $\hat{\mathbf{b}}_n := (\hat{\mathbf{b}} \cdot \mathbf{h})\mathbf{h}$ and a tangential part $\hat{\mathbf{b}}_t := \hat{\mathbf{b}} - \hat{\mathbf{b}}_n$ with $\|\hat{\mathbf{b}}_n\|_{\mathcal{C}^\infty(\overline{\Omega}_s,\mathbb{R}^d)} \leq \alpha C_B$ and $\|\hat{\mathbf{b}}_t\|_{\mathcal{C}^\infty(\overline{\Omega}_s,\mathbb{R}^d)} \leq \beta C_B$, where the constants $\alpha := d^2\|\mathbf{h}\|^3_{\mathcal{C}^\infty(\overline{\Omega})}$ and $\beta := d(1 + d\|\mathbf{h}\|^2_{\mathcal{C}^\infty(\overline{\Omega})})\|\mathbf{h}\|_{\mathcal{C}^\infty(\overline{\Omega})}$ do not depend on B. Therefore, $\tilde{B} := \sum_i \beta^{-1}(\hat{\mathbf{b}}_t)_i \partial_{\xi_i} \in \mathcal{B}_{C_B}$ and there exists a constant C independent of T and B such that

$$
\begin{aligned}
\|\nabla w \cdot B\mathbf{n}_s\|_{L^2(\Sigma_s^T)} &\leq \|\nabla w \cdot \hat{\mathbf{b}}_n\|_{L^2(\Sigma_s^T)} + \beta\|\tilde{B}w\|_{L^2(\Sigma_s^T)} \\
&\leq C(\|\nabla w \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} + \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2(\Sigma_s^T)}) \\
&\leq C(\|\nabla w \cdot \mathbf{n}_s\|_{L^2(\Sigma_s^T)} + \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}),
\end{aligned}
$$

where we use the trace inequality in space. Combining this result with (3.3), (3.13) and

(3.14) yields a constant C independent of T such that

$$\begin{aligned}
&\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))} + \sup_{B\in\mathcal{B}_{C_B}} \|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)} \\
&\leq C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)} \\
&\quad + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\
&\quad + \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))}).
\end{aligned} \tag{3.15}$$

We have

$$\sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{L^2((0,T),H^1(\Omega_s))} \leq CT^{\frac{1}{2}} \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{\mathcal{C}([0,T],H^1(\Omega_s))}, \tag{3.16}$$

which implies for $T > 0$ sufficiently small

$$\begin{aligned}
\sup_{B\in\mathcal{B}_{C_B}} \|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)} &\leq C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|w_0\|_{H^2(\Omega_s)} \\
&\quad + \|w_1\|_{H^1(\Omega_s)} + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))}).
\end{aligned}$$

(3.3) and Corollary 2.19 and using extension by continuity yields the assertion. □

Likewise we can show the following lemma.

Lemma 3.4. Let $T^* > 0$,

$$G \in L^2((0,T^*),H^3(\Gamma_s)) \cap H^2((0,T^*),H^1(\Gamma_s)),$$

$f \in L^2((0,T^*),H^2(\Omega_s)) \cap H^1((0,T),H^1(\Omega_s))$, $w_0 \in H^3(\Omega_s)$ and $w_1 \in H^2(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathsf{t}} G|_{\mathsf{t}=0} = w_1|_{\Gamma_s}.$$

Then, there exists $\hat{T} \in (0,T^*]$ such that for all $T \in (0,\hat{T}]$ the solution w of system (3.2) satisfies

$$\nabla w\cdot\mathbf{n}_s \in L^2((0,T),H^2(\Gamma_s)) \cap H^1((0,T),H^1(\Gamma_s)).$$

$$\begin{aligned}
&\|\nabla w\cdot\mathbf{n}_s\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\
&\leq C(T^{\frac{1}{2}}\|f\|_{L^2((0,T),H^2(\Omega_s))\cap H^1((0,T),H^1(\Omega_s))} + \|G\|_{L^2((0,T),H^3(\Gamma_s))\cap H^2((0,T),H^1(\Gamma_s))} \\
&\quad + \|w_0\|_{H^3(\Omega_s)} + \|w_1\|_{H^2(\Omega_s)}),
\end{aligned}$$

with a constant C independent of T.

Proof. This is obtained with the same arguments as in Lemma 3.3, but on the basis of Lemma 3.1.2 instead of 3.1.1. □

3.2.3. Global-in-Time Results

Since the local-in-time improved regularity results are shown for linear operators, a globalization is straightforward using ideas of [107, Prop. 2.7].

Lemma 3.5. Let $T^* > 0$ and $0 < T \leq T^*$,

$$G \in L^2((0,T^*), H^2(\Gamma_s)) \cap H^1((0,T^*), H^1(\Gamma_s)),$$

$f \in L^2((0,T^*), H^1(\Omega_s))$, $w_0 \in H^2(\Omega_s)$ and $w_1 \in H^1(\Omega_s)$ be such that

$$G|_{\mathsf{t=0}} = w_0|_{\Gamma_s}.$$

Then, the solution w of system (3.2) satisfies

$$\nabla w \cdot \mathbf{n}_s \in L^2((0,T), H^1(\Gamma_s))$$

and

$$\begin{aligned}\|\nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^1(\Gamma_s))} \leq C(T^{\frac{1}{2}}\|f\|_{L^2((0,T),H^1(\Omega_s))} + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\ + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)}),\end{aligned}$$

with a constant C independent of T.

Proof. Let the data f, G, w_0 and w_1 be smooth and conclude the argumentation with extension by continuity. Combination of (3.15) and (3.16) yields a constant C independent of T such that

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))} + \sup_{B\in\mathcal{B}_{C_B}} \|B(\nabla w \cdot \mathbf{n}_s)\|_{L^2(\Sigma_s^T)} \\ &\leq C(\|f\|_{L^1((0,T),H^1(\Omega_s))} + \|w_0\|_{H^2(\Omega_s)} + \|w_1\|_{H^1(\Omega_s)} \\ &\quad + \|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\ &\quad + T^{\frac{1}{2}} \sup_{\tilde{B}\in\mathcal{B}_{C_B}} \|\tilde{B}w\|_{\mathcal{C}([0,T],H^1(\Omega_s))})\end{aligned} \tag{3.17}$$

for all $T \in (0,T^*]$. Let $0 \leq T_0 < T_1 < T_2$, then,

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([0,T_2],H^1(\Omega_s))} \geq \sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([T_1,T_2],H^1(\Omega_s))}, \\ &\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{L^2((0,T_2),H^1(\Omega_s))} \leq \sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{L^2((0,T_1),H^1(\Omega_s))} + \sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{L^2((T_1,T_2),H^1(\Omega_s))}, \\ &\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{L^2((T_0,T_2),H^1(\Omega_s))} \leq (T_2-T_0)^{\frac{1}{2}} \sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([T_0,T_2],H^1(\Omega_s))},\end{aligned} \tag{3.18}$$

if $\sup_{B\in\mathcal{B}_{C_B}} \|Bw\|_{\mathcal{C}([0,T_2],H^1(\Omega_s))} < \infty$. Let $\bar{T} > 0$ be chosen such that $C\bar{T}^{\frac{1}{2}} \leq \frac{1}{2}$. Then, (3.17) implies

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}}\|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))}+\sup_{B\in\mathcal{B}_{C_B}}\|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)}\\&\le 2C(\|f\|_{L^1((0,T),H^1(\Omega_s))}+\|w_0\|_{H^2(\Omega_s)}\\&\quad+\|w_1\|_{H^1(\Omega_s)}+\|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))})\end{aligned}\tag{3.19}$$

for all $T\in(0,\bar{T}]$. Choose $\hat{T}=\bar{T}$ and $\Delta T=\min(\hat{T}+\bar{T},T^*)$. Then, for $T\in(\hat{T},\hat{T}+\Delta T]$, (3.17) and (3.18) imply

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}}\|Bw\|_{\mathcal{C}([\hat{T},T],H^1(\Omega_s))})+\sup_{B\in\mathcal{B}_{C_B}}\|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)}\\&\le C(\|f\|_{L^1((0,T),H^1(\Omega_s))}+\|w_0\|_{H^2(\Omega_s)}+\|w_1\|_{H^1(\Omega_s)}\\&\quad+\|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))}\\&\quad+\hat{T}^{\frac12}\sup_{\tilde{B}\in\mathcal{B}_{C_B}}\|\tilde{B}w\|_{\mathcal{C}([0,\hat{T}],H^1(\Omega_s))}+(T-\hat{T})^{\frac12}\sup_{\tilde{B}\in\mathcal{B}_{C_B}}\|\tilde{B}w\|_{\mathcal{C}([\hat{T},T],H^1(\Omega_s))}).\end{aligned}$$

Since $C\bar{T}^{\frac12}\le\frac12$, and $T-\hat{T}\le\bar{T}$ for $T\in(\hat{T},\hat{T}+\Delta T]$, there exists a constant $C>0$ independent of T such that

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}}\|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))}+\sup_{B\in\mathcal{B}_{C_B}}\|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)}\\&\le C(\|f\|_{L^1((0,T),H^1(\Omega_s))}+\|w_0\|_{H^2(\Omega_s)}+\|w_1\|_{H^1(\Omega_s)}\\&\quad+\|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))}+\hat{T}^{\frac12}\sup_{\tilde{B}\in\mathcal{B}_{C_B}}\|\tilde{B}w\|_{\mathcal{C}([0,\hat{T}],H^1(\Omega_s))})\end{aligned}$$

for $T\in(\hat{T},\hat{T}+\Delta T]$. Due to (3.19) there exists a constant $C>0$ independent of T such that

$$\begin{aligned}&\sup_{B\in\mathcal{B}_{C_B}}\|Bw\|_{\mathcal{C}([0,T],H^1(\Omega_s))}+\sup_{B\in\mathcal{B}_{C_B}}\|B(\nabla w\cdot\mathbf{n}_s)\|_{L^2(\Sigma_s^T)}\\&\le C(\|f\|_{L^1((0,T),H^1(\Omega_s))}+\|w_0\|_{H^2(\Omega_s)}+\|w_1\|_{H^1(\Omega_s)}\\&\quad+\|G\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))})\end{aligned}$$

for $T\in(0,\hat{T}+\Delta T]$. Replacing $\hat{T}$ with $\hat{T}+\Delta T$ and recursively applying this argumentation, (3.3), Corollary 2.19 and extension by continuity yield the result. □

With the same arguments we obtain the globalized version of Lemma 3.4.

Lemma 3.6. Let $T^*>0$ and $0<T\le T^*$,

$$G\in L^2((0,T^*),H^3(\Gamma_s))\cap H^2((0,T^*),H^1(\Gamma_s)),$$

$f\in L^2((0,T^*),H^2(\Omega_s))\cap H^1((0,T),H^1(\Omega_s))$, $w_0\in H^3(\Omega_s)$ and $w_1\in H^2(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0}=w_0|_{\Gamma_s},\quad \partial_{\mathsf{t}}G|_{\mathsf{t}=0}=w_1|_{\Gamma_s}.$$

Then, the solution w of system (3.2) satisfies

$$\nabla w \cdot \mathbf{n}_s \in L^2((0,T), H^2(\Gamma_s)) \cap H^1((0,T), H^1(\Gamma_s)).$$

Furthermore,

$$\begin{aligned}
&\|\nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^2(\Gamma_s))\cap H^1((0,T),H^1(\Gamma_s))} \\
&\leq C(T^{\frac{1}{2}}\|f\|_{L^2((0,T),H^2(\Omega_s))\cap H^1((0,T),H^1(\Omega_s))} + \|G\|_{L^2((0,T),H^3(\Gamma_s))\cap H^2((0,T),H^1(\Gamma_s))} \\
&\quad + \|w_0\|_{H^3(\Omega_s)} + \|w_1\|_{H^2(\Omega_s)}),
\end{aligned}$$

with a constant C independent of T.

Let, for the sake of convenience, $f = 0$ and consider the system

$$\begin{aligned}
\partial_{\mathsf{tt}} w - \Delta w &= 0 && \text{in } Q_s^T, \\
w &= G && \text{on } \Sigma_s^T, \\
w(\cdot, \mathsf{0}) &= w_0 && \text{in } \Omega_s, \\
\partial_{\mathsf{t}} w(\cdot, \mathsf{0}) &= w_1 && \text{in } \Omega_s,
\end{aligned} \tag{3.20}$$

The argumentation for obtaining an estimate that is compatible to the fluid equations is motivated by [113] and presented in a slightly modified manner.

Lemma 3.7. Let $T^* > 0$ and $0 < T \leq T^*$, $\ell \in (\frac{1}{2}, 1)$,

$$G \in L^2((0,T^*), H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{1}{2}+\ell}((0,T^*), H^1(\Gamma_s)),$$

$w_0 \in H^{\frac{3}{2}+\ell}(\Omega_s)$ and $w_1 \in H^{\frac{1}{2}+\ell}(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$\nabla w \cdot \mathbf{n}_s \in L^2((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\ell-\frac{1}{2}}((0,T), H^1(\Gamma_s)),$$

and

$$\begin{aligned}
&\|\nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\ell-\frac{1}{2}}((0,T),H^1(\Gamma_s))} \\
&\leq C(\|G\|_{L^2((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell}((0,T),H^1(\Gamma_s))} + \|w_0\|_{H^{\frac{3}{2}+\ell}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell}(\Omega_s)}),
\end{aligned}$$

where the constant C might depend on T.

Proof. Interpolation of Lemmas 3.6 and 3.5 with $\theta = \frac{3}{2} - \ell$ yields

$$\begin{aligned}
&\|\!|\nabla w \cdot \mathbf{n}_s|\!\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\ell-\frac{1}{2}}((0,T),H^1(\Gamma_s))} \\
&\leq C(\|\!|G|\!\|_{L^2((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell}((0,T),H^1(\Gamma_s))} + \|\!|w_0|\!\|_{H^{\frac{3}{2}+\ell}(\Omega_s)} + \|\!|w_1|\!\|_{H^{\frac{1}{2}+\ell}(\Omega_s)}),
\end{aligned}$$

with a constant $C > 0$ independent of T. The equivalence constants between the interpolation norms $|||\cdot|||$ and the norms $\|\cdot\|$ on the time interval $(0,T)$ might depend on T. $\square$

Lemma 3.8. Let $T^* > 0$ and $0 < T \leq T^*$, $\ell \in (\frac{1}{2}, 1)$,

$$G \in H^1((0,T^*), H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{3}{2}+\ell}((0,T^*), H^1(\Gamma_s)),$$

$w_0 \in H^{\frac{5}{2}+\ell}(\Omega_s)$ and $w_1 \in H^{\frac{3}{2}+\ell}(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathsf{t}} G|_{\mathsf{t}=0} = w_1|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$\nabla w \cdot \mathbf{n}_s \in H^1((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\ell+\frac{1}{2}}((0,T), H^1(\Gamma_s)),$$

and

$$\begin{aligned} &\|\nabla w \cdot \mathbf{n}_s\|_{H^1((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\ell+\frac{1}{2}}((0,T),H^1(\Gamma_s))} \\ &\leq C(\|G\|_{H^1((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{2}+\ell}((0,T),H^1(\Gamma_s))} + \|w_0\|_{H^{\frac{5}{2}+\ell}(\Omega_s)} + \|w_1\|_{H^{\frac{3}{2}+\ell}(\Omega_s)}), \end{aligned}$$

where the constant C might depend of T.

Proof. Lemma 3.7 yields

$$\begin{aligned} &\|\nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\ell-\frac{1}{2}}((0,T),H^1(\Gamma_s))} \\ &\leq C(\|G\|_{L^2((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell}((0,T),H^1(\Gamma_s))} + \|w_0\|_{H^{\frac{3}{2}+\ell}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell}(\Omega_s)}). \end{aligned} \tag{3.21}$$

$\partial_{\mathsf{t}} w$ is a solution of the system

$$\begin{aligned} \partial_{\mathsf{tt}}(\partial_{\mathsf{t}} w) - \Delta(\partial_{\mathsf{t}} w) = 0 \quad & \text{in } Q_s^T, \\ \partial_{\mathsf{t}} w = \partial_{\mathsf{t}} G \quad & \text{on } \Sigma_s^T, \\ \partial_{\mathsf{t}} w(\cdot, 0) = w_1 \quad & \text{in } \Omega_s, \\ \partial_{\mathsf{t}}(\partial_{\mathsf{t}} w)(\cdot, 0) = \Delta w_0 \quad & \text{in } \Omega_s. \end{aligned} \tag{3.22}$$

Lemma 3.7 applied to system (3.22) yields

$$\begin{aligned} &\|\partial_{\mathsf{t}} \nabla w \cdot \mathbf{n}_s\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\ell-\frac{1}{2}}((0,T),H^1(\Gamma_s))} \\ &\leq C(\|\partial_{\mathsf{t}} G\|_{L^2((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell}((0,T),H^1(\Gamma_s))} \\ &\quad + \|w_1\|_{H^{\frac{3}{2}+\ell}(\Omega_s)} + \|\Delta w_0\|_{H^{\frac{1}{2}+\ell}(\Omega_s)}) \\ &\leq C(\|G\|_{H^1((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{2}+\ell}((0,T),H^1(\Gamma_s))} \\ &\quad + \|w_0\|_{H^{\frac{5}{2}+\ell}(\Omega_s)} + \|w_1\|_{H^{\frac{3}{2}+\ell}(\Omega_s)}). \end{aligned} \tag{3.23}$$

Combination of (3.21) and (3.23) yields the assertion. □

Lemma 3.9. Let $T^* > 0$ and $0 < T \leq T^*$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$,

$$G \in H^{\beta}((0,T^*), H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{1}{2}+\ell+\beta}((0,T^*), H^1(\Gamma_s)),$$

$w_0 \in H^{\frac{3}{2}+\ell+\beta}(\Omega_s)$ and $w_1 \in H^{\frac{1}{2}+\ell+\beta}(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$\nabla w \cdot \mathbf{n}_s \in H^{\beta}((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\ell-\frac{1}{2}+\beta}((0,T), H^1(\Gamma_s)),$$

and

$$\begin{aligned} &\|\nabla w \cdot \mathbf{n}_s\|_{H^{\beta}((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\ell-\frac{1}{2}+\beta}((0,T),H^1(\Gamma_s))} \\ &\leq C(\|G\|_{H^{\beta}((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))} + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}), \end{aligned}$$

where the constant C might depend on T.

Proof. Is obtained by interpolation of Lemmas 3.8 and 3.7 with $\theta = 1 - \beta$. □

Lemma 3.10. Let $T^* > 0$ and $0 < T \leq T^*$, $\ell \in (\frac{1}{2}, 1)$,

$$G \in H^{\frac{7}{4}+\frac{\ell}{2}}(\Sigma_s^{T^*}),$$

$w_0 \in H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)$ and $w_1 \in H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)$ be such that

$$G|_{\mathsf{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathsf{t}} G|_{\mathsf{t}=0} = w_1|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$w \in \mathcal{C}([0,T], H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)) \cap \mathcal{C}^1([0,T], H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)), \quad \nabla w \cdot \mathbf{n}_s \in H^{\frac{3}{4}+\frac{\ell}{2}}(\Sigma_s^T),$$

and

$$\begin{aligned} &\|w\|_{\mathcal{C}([0,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)) \cap \mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{H^{\frac{3}{4}+\frac{\ell}{2}}(\Sigma_s^T)} \\ &\leq C(\|G\|_{H^{\frac{7}{4}+\frac{\ell}{2}}(\Sigma_s^T)} + \|w_0\|_{H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)} + \|w_1\|_{H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)}), \end{aligned}$$

where the constant C might depend on T.

Proof. The assertion is obtained by interpolation of Theorem 3.1.2 and 3.1.3 with $\theta = \frac{5}{4} - \frac{\ell}{2}$. □

Theorem 3.11. Let $T > 0$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$,

$$G \in H^{\beta}((0,T), H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{1}{2}+\ell+\beta}((0,T), H^1(\Gamma_s)) \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s)),$$

$w_0 \in H^{\frac{3}{2}+\ell+\beta}(\Omega_s)$ and $w_1 \in H^{\frac{1}{2}+\ell+\beta}(\Omega_s)$ be such that

$$G|_{\mathrm{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathrm{t}} G|_{\mathrm{t}=0} = w_1|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$\begin{aligned} &w \in \mathcal{C}([0,T], H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)) \cap \mathcal{C}^1([0,T], H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)), \\ &\nabla w \cdot \mathbf{n}_s \in H^{\beta}((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s)), \end{aligned}$$

and

$$\begin{aligned} &\|w\|_{\mathcal{C}([0,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{H^{\beta}((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\le C(\|G\|_{H^{\beta}((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\quad + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}), \end{aligned}$$

where the constant C might depend on T.

Proof. Combining Lemmas 3.9 and 3.10 yields the assertion using the fact that $\frac{11}{8} + \frac{\ell}{4} \ge \frac{1}{2} + \ell + \beta$ for $\beta \in (0, 1-\ell)$ and

$$[H^1((0,T), H^{\frac{3}{2}+\ell}(\Gamma_s)), H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s))]_\theta = H^{\frac{11}{8}+\frac{\ell}{4}}((0,T), H^{\frac{3}{4}+\frac{\ell}{2}}(\Gamma_s))$$

for $\theta = \frac{1}{2}$. □

Theorem 3.12. Let $T > 0$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$,

$$G \in H^{\beta}((0,T), H^{\frac{3}{2}+\ell}(\Gamma_s)) \cap H^{\frac{1}{2}+\ell+\beta}((0,T), H^1(\Gamma_s)) \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s)),$$

$w_0 \in H^{\frac{3}{2}+\ell+\beta}(\Omega_s)$ and $w_1 \in H^{\frac{1}{2}+\ell+\beta}(\Omega_s)$ be such that

$$G|_{\mathrm{t}=0} = w_0|_{\Gamma_s}, \quad \partial_{\mathrm{t}} G|_{\mathrm{t}=0} = w_1|_{\Gamma_s}.$$

Then, the solution w of system (3.20) satisfies

$$\begin{aligned} &w \in \mathcal{C}([0,T], H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s)) \cap \mathcal{C}^1([0,T], H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s)), \\ &\nabla w \cdot \mathbf{n}_s \in H^{\beta}((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s)), \end{aligned}$$

and

$$\|w\|_{\mathcal{C}([0,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} + \|\nabla w \cdot \mathbf{n}_s\|_{H^{\beta}((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))}$$

$$\leq C(\|G\|_{H^\beta((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}),$$

where the constant C is independent of T.

Proof. Due to [90, p.41, Thm. 9.4] and Remark 2.7 there exists a continuous lifting operator

$$H^{\frac{3}{2}+\ell+\beta}(\Omega_s) \times H^{\frac{1}{2}+\ell+\beta}(\Omega_s) \to H^{\frac{3}{2}+\ell+\beta}(\Gamma_s \times (0,\infty)),$$

$(w_0, w_1) \mapsto G_0$ such that $G_0(\cdot,0)|_{\Gamma_s} = w_0$, $\partial_t G_0(\cdot,0)|_{\Gamma_s} = w_1$, i.e.,

$$|G_0|_{H^{\frac{3}{2}+\ell+\beta}(\Gamma_s\times(0,\infty))} \leq C(\|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}) \tag{3.24}$$

for a constant C independent of T, where $|\cdot|_{H^{\frac{3}{2}+\ell+\beta}(\Gamma_s\times(0,\infty))}$ denotes the Sobolev-Slobodeckij norm, see (2.2). Consider the systems

$$\begin{aligned} \partial_{tt}\tilde{w} - \Delta\tilde{w} &= 0 && \text{in } Q_s^T, \\ \tilde{w} &= G_0 && \text{on } \Sigma_s^T, \\ \tilde{w}(\cdot,0) &= w_0 && \text{in } \Omega_s, \\ \partial_t\tilde{w}(\cdot,0) &= w_1 && \text{in } \Omega_s, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \partial_{tt}\hat{w} - \Delta\hat{w} &= 0 && \text{in } Q_s^T, \\ \hat{w} &= G - G_0 && \text{on } \Sigma_s^T, \\ \hat{w}(\cdot,0) &= 0 && \text{in } \Omega_s, \\ \partial_t\hat{w}(\cdot,0) &= 0 && \text{in } \Omega_s. \end{aligned} \tag{3.26}$$

Due to the linearity of the hyperbolic equations, $w = \tilde{w} + \hat{w}$. Consider the system (3.26). We know that $(G - G_0)(\cdot,0) = 0$ and $\partial_t(G - G_0)(\cdot,0) = 0$. Furthermore, for $T_f > T$,

$$\begin{aligned} {}[H^{\frac{3}{2}+\ell+\beta}((0,T_f),L^2(\Gamma_s)), L^2((0,T_f),H^{\frac{3}{2}+\ell+\beta}(\Gamma_s))]_{\frac{\frac{3}{2}+\ell}{\frac{3}{2}+\ell+\beta}} &= H^\beta((0,T_f),H^{\frac{3}{2}+\ell}(\Gamma_s)), \\ {}[H^{\frac{3}{2}+\ell+\beta}((0,T_f),L^2(\Gamma_s)), L^2((0,T_f),H^{\frac{3}{2}+\ell+\beta}(\Gamma_s))]_{\frac{1}{\frac{3}{2}+\ell+\beta}} &= H^{\frac{1}{2}+\ell+\beta}((0,T_f),H^1(\Gamma_s)) \end{aligned} \tag{3.27}$$

and hence with **P5**, **P2**, Remark 2.7, (3.27) and (3.24)

$$\begin{aligned} &\|G_0\|_{H^\beta((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\leq C\|G_0\|_{H^\beta((0,T_f),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T_f),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T_f),L^2(\Gamma_s))} \\ &\leq C|G_0|_{H^{\frac{3}{2}+\ell+\beta}(\Gamma_s\times(0,T_f))} \leq C|G_0|_{H^{\frac{3}{2}+\ell+\beta}(\Gamma_s\times(0,\infty))} \end{aligned}$$

$$\leq C(\|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}) \tag{3.28}$$

with a generic constant C independent of T. Therefore, **P3** and (3.28) yield

$$\begin{aligned} &\|\mathrm{Ext}(G-G_0)\|_{H^\beta((T-T_f,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((T-T_f,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((T-T_f,T),L^2(\Gamma_s))} \\ &\leq C(\|G\|_{H^\beta((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\quad + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}) \end{aligned} \tag{3.29}$$

for a constant C independent of T. In addition, $\mathrm{Ext}(\hat{w})$ solves

$$\begin{aligned} \partial_{\mathsf{tt}}\mathrm{Ext}(\hat{w}) - \Delta\mathrm{Ext}(\hat{w}) &= 0 \quad \text{in } Q_s^{(T-T_f,T)}, \\ \mathrm{Ext}(\hat{w}) &= \mathrm{Ext}(G-G_0) \quad \text{on } \Sigma_s^{(T-T_f,T)}, \\ \mathrm{Ext}(\hat{w})(\cdot,T-T_f) &= 0 \quad \text{in } \Omega_s, \\ \partial_{\mathsf{t}}\mathrm{Ext}(\hat{w})(\cdot,T-T_f) &= 0 \quad \text{in } \Omega_s. \end{aligned}$$

Therefore, with **P4**, Theorem 3.11 and (3.29) we obtain

$$\begin{aligned} &\|\hat{w}\|_{\mathcal{C}([0,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} + \|\nabla\hat{w}\cdot\mathbf{n}_s\|_{H^\beta((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\leq \|\mathrm{Ext}(\hat{w})\|_{\mathcal{C}([T-T_f,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} \\ &\quad + \|\nabla\mathrm{Ext}(\hat{w})\cdot\mathbf{n}_s\|_{H^\beta((T-T_f,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((T-T_f,T),L^2(\Gamma_s))} \\ &\leq C\|\mathrm{Ext}(G-G_0)\|_{H^\beta((T-T_f,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((T-T_f,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((T-T_f,T),L^2(\Gamma_s))} \\ &\leq C(\|G\|_{H^\beta((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\ &\quad + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}), \end{aligned} \tag{3.30}$$

where the generic constant C is independent of T. The solution of the system

$$\begin{aligned} \partial_{\mathsf{tt}}\bar{w} - \Delta\bar{w} &= 0 \quad \text{in } Q_s^{(0,T_f)}, \\ \bar{w} &= G_0 \quad \text{on } \Sigma_s^{(0,T_f)}, \\ \bar{w}(\cdot,0) &= w_0 \quad \text{in } \Omega_s, \\ \partial_{\mathsf{t}}\bar{w}(\cdot,0) &= w_1 \quad \text{in } \Omega_s, \end{aligned}$$

fulfills $\mathrm{R}(\bar{w}) = \tilde{w}$, which is the solution of (3.25), and thus **P5**, Theorem 3.11 and (3.28) yield

$$\begin{aligned}
&\|\tilde{w}\|_{\mathcal{C}([0,T],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} + \|\nabla\tilde{w}\cdot\mathbf{n}_s\|_{H^\beta((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))} \\
&\le C(\|\bar{w}\|_{\mathcal{C}([0,T_f],H^{\frac{7}{4}+\frac{\ell}{2}}(\Omega_s))\cap\mathcal{C}^1([0,T_f],H^{\frac{3}{4}+\frac{\ell}{2}}(\Omega_s))} \\
&\qquad + \|\nabla\bar{w}\cdot\mathbf{n}_s\|_{H^\beta((0,T_f),H^{\frac{1}{2}+\ell}(\Gamma_s))\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T_f),L^2(\Gamma_s))}) \\
&\le C(\|G_0\|_{H^\beta((0,T_f),H^{\frac{3}{2}+\ell}(\Gamma_s))\cap H^{\frac{1}{2}+\ell+\beta}((0,T_f),H^1(\Gamma_s))\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T_f),L^2(\Gamma_s))} \\
&\qquad + \|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)}), \\
&\le C(\|w_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)} + \|w_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)})
\end{aligned} \tag{3.31}$$

where the generic constant C depends on T_f but is independent of T. The assertion follows from (3.30) and (3.31) since $w = \tilde{w} + \hat{w}$. □

3.3. Existence and Regularity for Unsteady Stokes-Lamé System with Stationary Interface

In this section, the linear unsteady FSI problem (1.3) is considered, which is given by

$$\begin{aligned}
\partial_t\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} && \text{in } Q_f^T, \\
\operatorname{div}(\mathbf{v}) = g &= \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\
\mathbf{v}(\cdot,0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\
\mathbf{v} &= \mathbf{v}_D && \text{on } \Sigma_f^T, \\
\mathbf{v} &= \partial_t\mathbf{w} && \text{on } \Sigma_i^T, \\
\sigma_f(\mathbf{v},p)\mathbf{n}_f &= \sigma_s(\mathbf{w})\mathbf{n}_f + \mathbf{h} && \text{on } \Sigma_i^T, \\
\partial_{tt}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) &= 0 && \text{in } Q_s^T, \\
\mathbf{w}(\cdot,0) &= 0 && \text{in } \Omega_s, \\
\partial_t\mathbf{w}(\cdot,0) &= \mathbf{w}_1 && \text{in } \Omega_s, \\
\mathbf{w} &= 0 && \text{on } \Sigma_s^T.
\end{aligned}$$

We require that the bounded domain $\Omega = \Omega_f \cup \Omega_s \cup \Gamma_i \subset \mathbb{R}^d$, $d = 3$ is such that

- Γ_i denotes the interface between Ω_s and Ω_f, i.e. $\overline{\Gamma}_i = \overline{\Omega}_s \cap \overline{\Omega}_f$.
- the solid domain Ω_s is a domain with boundary $\partial\Omega_s$ of class $\mathcal{C}^\infty$ such that $\partial\Omega_s = \Gamma_i \cup \Gamma_s$, where Γ_s denotes the outer boundary solid boundary and $\overline{\Gamma}_i \cap \overline{\Gamma}_s = \emptyset$.
- the fluid domain Ω_f is a Lipschitz domain with boundary $\partial\Omega_f = \Gamma_i \cup \Gamma_f$, where Γ_f denotes the outer boundary fluid boundary and $\overline{\Gamma}_i \cap \overline{\Gamma}_f = \emptyset$.

3.3.1. Lamé System

Improved regularity results for hyperbolic equations play an important role in the existence and regularity theory for FSI problems in order to overcome the a-priori mismatch between the regularity of parabolic and hyperbolic equations. The improved regularity result in Section 3.2 was derived with the purpose of motivating existence and regularity of solutions of the Lamé system. We assume that the results can be adapted to the Lamé system.

Assumption 3.13. Let $T > 0$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$,

$$\boldsymbol{\eta} \in H^{\beta}((0,T), H^{\frac{3}{2}+\ell}(\Gamma_s))^d \cap H^{\frac{1}{2}+\ell+\beta}((0,T), H^1(\Gamma_s))^d \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s))^d,$$

$\mathbf{w}_0 \in H^{\frac{3}{2}+\ell+\beta}(\Omega_s)^d$ and $\mathbf{w}_1 \in H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d$ be such that

$$\boldsymbol{\eta}|_{\mathsf{t=0}} = \mathbf{w}_0|_{\Gamma_s}, \; \partial_{\mathsf{t}}\boldsymbol{\eta}|_{\mathsf{t=0}} = \mathbf{w}_1|_{\Gamma_s}.$$

Then, the solution w of system

$$\begin{aligned} \partial_{\mathsf{tt}}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) &= 0 && \text{in } Q_s^T, \\ \mathbf{w} &= \boldsymbol{\eta} && \text{on } \Sigma_s^T, \\ \mathbf{w}(\cdot, 0) &= \mathbf{w}_0 && \text{in } \Omega_s, \\ \partial_{\mathsf{t}}\mathbf{w}(\cdot, 0) &= \mathbf{w}_1 && \text{in } \Omega_s, \end{aligned}$$

satisfies $\mathbf{w} \in \mathbf{W}_T$ and

$$\sigma_s(\mathbf{w})\mathbf{n}_f \in H^{\beta}((0,T), H^{\frac{1}{2}+\ell}(\Gamma_s)^d) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_s)^d).$$

Furthermore,

$$\begin{aligned} &\|\mathbf{w}\|_{\mathbf{W}_T} + \|\sigma_s(\mathbf{w})\mathbf{n}_f\|_{H^{\beta}((0,T),H^{\frac{1}{2}+\ell}(\Gamma_s)^d)\cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s)^d)} \\ &\leq C(\|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d} + \|\mathbf{w}_0\|_{H^{\frac{3}{2}+\ell+\beta}(\Omega_s)^d} \\ &\quad + \|\boldsymbol{\eta}\|_{H^{\beta}((0,T),H^{\frac{3}{2}+\ell}(\Gamma_s))^d\cap H^{\frac{1}{2}+\ell+\beta}((0,T),H^1(\Gamma_s))^d\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_s))^d}), \end{aligned}$$

where the constant C is independent of T.

Remark 3.14. The theorem is analogous to Theorem 3.12. [12, Sec. 2 Thm. 1, Sec. 2 Prop.1, Comments 2.5] and [113, Sec. 3 Thm. 3.2] indicate that the Lemmas 3.5, 3.6 and Theorem 3.1 also hold true for the Lamé system. Also the Lemmas in Section 2.2 hold true for vector valued functions, however, a complete argumentation is beyond the scope of this work.

3.3.2. Stokes Equations

As parabolic system the Stokes equations are considered. Thus, we need to consider the system

$$\begin{aligned} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\mathbf{v}) = g &= \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\ \mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \mathbf{v} &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\mathbf{v}, p)\mathbf{n}_f &= \mathbf{h} && \text{on } \Sigma_i^T. \end{aligned} \tag{3.32}$$

We give a proof in a slightly modified setting compared to [113, Sec. 4]. The basis for the theoretical analysis of that system is [54, Thm. 7.5]. The Dirichlet boundary term is kept in this theorem since it appears in (3.34).

Theorem 3.15. Let $d \in \{2, 3\}$, $\Omega_f \subset \mathbb{R}^d$ be a domain with smooth boundary such that $\partial\Omega_f = \Gamma_f \cup \Gamma_i$ and $\overline{\Gamma}_f \cap \overline{\Gamma}_i = \emptyset$. Let $\ell \in (\frac{1}{2}, 1)$, $\mathbf{f} \in \mathbf{F}_T$, $\mathbf{h} \in \mathbf{H}_T$, $\mathbf{v}_0 \in H^{1+\ell}(\Omega_f)^d$ and $\mathbf{v}_D \in (H^{\frac{3}{2}+\ell, \frac{3}{4}+\frac{\ell}{2}}(\Sigma_f^T))^d$. Let the compatibility conditions

$$\begin{aligned} \operatorname{div}(\mathbf{v}_0) &= 0 && \text{in } \Omega_f, \\ \mathbf{v}_0(\cdot) &= \mathbf{v}_D(\cdot, t) && \text{on } \Gamma_f, \\ \mathbf{h}(\cdot, 0) &= 0 && \text{on } \Gamma_i, \\ 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f \cdot \boldsymbol{\tau} &= 0 && \text{on } \Gamma_i, \end{aligned}$$

for any unit vector $\boldsymbol{\tau}$ tangential to Γ_i be fulfilled. Then, the solution to system

$$\begin{aligned} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\mathbf{v}) &= 0 && \text{in } Q_f^T, \\ \mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \mathbf{v} &= \mathbf{v}_D && \text{on } \Sigma_f^T, \\ \sigma_f(\mathbf{v}, p)\mathbf{n}_f &= \mathbf{h} && \text{on } \Sigma_i^T. \end{aligned}$$

satisfies

$$\begin{aligned} &\|\mathbf{v}\|_{(H^{2+\ell, 1+\frac{\ell}{2}}(Q_f^T))^d} + \|\nabla p\|_{(H^{\ell, \frac{\ell}{2}}(Q_f^T))^d} + \|p\|_{H^{1+\ell, \frac{\ell}{2}}(Q_f^T)} \\ &\leq C(\|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{h}\|_{\mathbf{H}_T} + \|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{v}_D\|_{(H^{\frac{3}{2}+\ell, \frac{3}{4}+\frac{\ell}{4}}(\Sigma_f^T))^d}). \end{aligned}$$

Proof. c.f [54, Def. 7.2, Thm. 7.5]. □

This theorem allows us to directly handle all inhomogeneities except for the nonhomogeneous divergence condition. Therefore, we split the linear system (3.32) in two subsystems such that $(\mathbf{v}, p) = (\tilde{\mathbf{v}}, \tilde{p}) + (\hat{\mathbf{v}}, \hat{p})$, where $(\hat{\mathbf{v}}, \hat{p})$ solves

$$\begin{aligned} \partial_t \hat{\mathbf{v}} - \nu \Delta \hat{\mathbf{v}} + \nabla \hat{p} &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\hat{\mathbf{v}}) &= 0 && \text{in } Q_f^T, \\ \hat{\mathbf{v}}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \hat{\mathbf{v}} &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\hat{\mathbf{v}}, \hat{p})\mathbf{n}_f &= \mathbf{h} && \text{on } \Sigma_i^T, \end{aligned}$$

and $(\tilde{\mathbf{v}}, \tilde{q})$ solves

$$\begin{aligned} \partial_t \tilde{\mathbf{v}} - \nu \Delta \tilde{\mathbf{v}} + \nabla \tilde{p} &= 0 && \text{in } Q_f^T, \\ \operatorname{div}(\tilde{\mathbf{v}}) &= g = \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\ \tilde{\mathbf{v}}(\cdot, 0) &= 0 && \text{in } \Omega_f, \\ \tilde{\mathbf{v}} &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\tilde{\mathbf{v}}, \tilde{p})\mathbf{n}_f &= 0 && \text{on } \Sigma_i^T. \end{aligned} \tag{3.33}$$

The first system can directly be treated with Theorem 3.15.

As already pointed out in [113] the main difficulty is the derivation of an similiar estimate for the case of a non-homogeneous divergence condition as it appears in system (3.33). The statements are slightly modified compared to [113] and included for the sake of completeness.

Leray Projector

The Leray projector $\mathcal{P}$ is defined as the orthogonal projector form $L^2(\Omega_f)^d$ to

$$V^0_{\Gamma_f}(\Omega_f)^d := \{\mathbf{v} \in L^2(\Omega_f)^d \ : \ \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega_f, \ \mathbf{v} \cdot \mathbf{n}_f = 0 \text{ on } \Gamma_f\}.$$

It can be precisely defined as

$$\mathcal{P} : \ L^2(\Omega_f)^d \to V^0_{\Gamma_f}(\Omega_f)^d, \ \mathbf{v} \mapsto \mathbf{v} - \nabla(\zeta + \pi),$$

cf. [113, Sec. 4.1]. Here, ζ is the solution to the elliptic equation

$$\Delta \zeta = \operatorname{div}(\mathbf{v}) \text{ in } \Omega_f, \quad \zeta = 0 \text{ on } \Gamma_f, \quad \zeta = 0 \text{ on } \Gamma_i.$$

Since $\operatorname{div}\mathbf{v} \in H^{-1}(\Omega_f)$ we know due to standard regularity theory for elliptic equations that $\zeta \in H^1(\Omega_f)$. Moreover, π is the solution to the elliptic equation

$$\Delta \pi = 0 \text{ in } \Omega_f, \quad \nabla \pi \cdot \mathbf{n}_f = (\mathbf{v} - \nabla \zeta) \cdot \mathbf{n}_f \text{ on } \Gamma_f, \quad \pi = 0 \text{ on } \Gamma_i.$$

Due to $(\mathbf{v} - \nabla \zeta) \in L^2(\Omega_f)^d$ and $\operatorname{div}(\mathbf{v} - \nabla \zeta) = 0$ it follows that $(\mathbf{v} - \nabla \zeta) \cdot \mathbf{n}_f \in H^{-\frac{1}{2}}(\Gamma_f)$, and the solution theory for elliptic equations gives $\pi \in H^1(\Omega_f)$. Now, there holds

$$\operatorname{div}(\mathcal{P}\mathbf{v}) = \operatorname{div}(\mathbf{v}) - \Delta \zeta - \Delta \pi = 0 \text{ in } \Omega_f, \quad \mathcal{P}\mathbf{v} \cdot \mathbf{n}_f = 0 \text{ on } \Gamma_f,$$

and

$$\mathbf{v} - \mathcal{P}\mathbf{v} \in \nabla H^1_{\Gamma_i}(\Omega_f)^d$$
$$:= \{\mathbf{v} \in L^2(\Omega_f)^d \ : \ \exists v \in H^1(\Omega_f) \text{ s.t. } \nabla v = \mathbf{v} \text{ in } \Omega_f, \ v = 0 \text{ on } \Gamma_i\}.$$

For $\mathbf{v}_1 \in V^0_{\Gamma_f}(\Omega_f)^d$ and $\mathbf{v}_2 \in \nabla H^1_{\Gamma_i}(\Omega_f)^d$ there exists $v_2 \in H^1(\Omega_f)$ such that $\nabla v_2 = \mathbf{v}_2$ in Ω_f and $v_2 = 0$ on Γ_i and due to the divergence theorem, there holds

$$\begin{aligned}(\mathbf{v}_1, \mathbf{v}_2)_{L^2(\Omega_f)^d} &= \int_{\Omega_f} \mathbf{v}_1 \cdot \mathbf{v}_2 d\xi = \int_{\Omega_f} \mathbf{v}_1 \cdot \nabla v_2 d\xi = \int_{\Omega_f} \operatorname{div}(v_2 \mathbf{v}_1) d\xi - \int_{\Omega_f} v_2 \operatorname{div} \mathbf{v}_1 d\xi \\ &= \int_{\Gamma_i \cup \Gamma_f} v_2 \mathbf{v}_1 \cdot \mathbf{n}_f ds - \int_{\Omega_f} v_2 \operatorname{div} \mathbf{v}_1 d\xi = 0.\end{aligned}$$

The above considerations imply that

$$L^2(\Omega_f)^d = V^0_{\Gamma_f}(\Omega_f)^d \oplus \nabla H^1_{\Gamma_i}(\Omega_f)^d.$$

Regularity Results for the Stokes Equations with Nonhomogeneous Divergence Condition

Consider the system (3.33). Using the Leray projector $\mathcal{P}$ and the relation $\tilde{\mathbf{v}} = \mathcal{P}\tilde{\mathbf{v}} + \nabla(\zeta + \pi)$ with the parametrized solutions

$$\Delta\zeta(\cdot, \mathsf{t}) = g(\cdot, \mathsf{t}) = \operatorname{div}(\mathbf{g}(\cdot, \mathsf{t})) \text{ in } \Omega_f, \quad \zeta(\cdot, \mathsf{t}) = 0 \text{ on } \Gamma_f, \quad \zeta(\cdot, \mathsf{t}) = 0 \text{ on } \Gamma_i,$$

and

$$\Delta\pi(\cdot, \mathsf{t}) = 0 \text{ in } \Omega_f, \quad \nabla\pi(\cdot, \mathsf{t}) \cdot \mathbf{n}_f = -\nabla\zeta(\cdot, \mathsf{t}) \cdot \mathbf{n}_f \text{ on } \Gamma_f, \quad \pi(\cdot, \mathsf{t}) = 0 \text{ on } \Gamma_i,$$

the system (3.33) can be reformulated as

$$\begin{aligned} \partial_{\mathsf{t}} \mathcal{P}\tilde{\mathbf{v}} - \nu\Delta\mathcal{P}\tilde{\mathbf{v}} + \nabla\tilde{p} &= \nu\nabla g - \nabla\partial_{\mathsf{t}}\pi - \nabla\partial_{\mathsf{t}}\zeta && \text{in } Q^T_f, \\ \operatorname{div}(\mathcal{P}\tilde{\mathbf{v}}) &= 0 && \text{in } Q^T_f, \\ \mathcal{P}\tilde{\mathbf{v}}(\cdot, 0) &= 0 && \text{in } \Omega_f, \\ \mathcal{P}\tilde{\mathbf{v}} &= -\nabla_{\boldsymbol{\tau}}\pi && \text{on } \Sigma^T_f, \\ \sigma_f(\mathcal{P}\tilde{\mathbf{v}}, \tilde{p})\mathbf{n}_f &= -2\nu\epsilon(\nabla(\zeta + \pi))\mathbf{n}_f && \text{on } \Sigma^T_s, \end{aligned} \tag{3.34}$$

where $\nabla_{\boldsymbol{\tau}}\pi \cdot \mathbf{n}_f = 0$ and $\nabla_{\boldsymbol{\tau}}\pi \cdot \boldsymbol{\tau} = \nabla\pi \cdot \boldsymbol{\tau}$ for all $\boldsymbol{\tau}$ that are tangential to the boundary. The corresponding condition on Σ^T_f is motivated by the fact that $\mathcal{P}\tilde{\mathbf{v}} \cdot \mathbf{n}_f = \tilde{\mathbf{v}} \cdot \mathbf{n}_f - \nabla(\zeta + \pi) \cdot \mathbf{n}_f = 0$, $\tilde{\mathbf{v}}|_{\Gamma_f} = 0$ and $\zeta|_{\Gamma_f} = 0$ for which reason $\nabla\zeta \cdot \boldsymbol{\tau} = 0$ for all $\boldsymbol{\tau}$ tangential to the boundary. Furthermore, we have the relation

$$(\mathcal{I} - \mathcal{P})\tilde{\mathbf{v}}(\cdot, \mathsf{t}) = \nabla\zeta(\cdot, \mathsf{t}) + \nabla\pi(\cdot, \mathsf{t}), \qquad \text{for all } \mathsf{t} \in (0, T).$$

System (3.34) can be handled with Theorem 3.15 if an estimate for $\nabla_{\boldsymbol{\tau}}\pi$ is established. For estimating $(\mathcal{I} - \mathcal{P})$ we have to bound $\nabla\zeta$ and $\nabla\pi$.

This is done in the following collection of lemmas.

Lemma 3.16. If $\mathbf{g} \in H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)$ and $g \in L^2((0,T), H^{1+\ell}(\Omega_f))$, then

$$\|\zeta\|_{L^2((0,T),H^{3+\ell}(\Omega_f))} \leq C\|g\|_{L^2((0,T),H^{1+\ell}(\Omega_f))}$$

and

$$\|\zeta\|_{H^{1+\frac{\ell}{2}}((0,T),H^1(\Omega_f))} \leq C\|\mathbf{g}\|_{\mathbf{G}_T}.$$

In particular,

$$\|\nabla\zeta \cdot \mathbf{n}_f\|_{H^{\frac{3}{2}+\ell,\frac{3}{4}+\frac{\ell}{2}}(\Sigma_f^T)} \leq C(\|g\|_{L^2((0,T),H^{1+\ell}(\Omega_f))} + \|\mathbf{g}\|_{\mathbf{G}_T}).$$

Proof. cf. [113, Lem. 4.1]. □

Lemma 3.17. Let $\mathbf{g} \in H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d) \cap H^1((0,T), H^\ell(\Omega_f)^d)$ and $\mathbf{g}|_{\Sigma_f^T} = 0$. Then,

$$\|\nabla\zeta \cdot \mathbf{n}_f\|_{H^{\frac{1}{2}+\ell}((0,T),H^{-\frac{1}{2}}(\Gamma_f))} \leq C\|\mathbf{g}\|_{\mathbf{G}_T}.$$

Proof. We consider the system

$$\begin{aligned} \Delta\,\zeta(\cdot,\mathsf{t}) &= \operatorname{div}(\mathbf{g}(\cdot,\mathsf{t})) \quad \text{in } \Omega_f,\\ \zeta(\cdot,\mathsf{t}) &= 0 \quad \text{on } \Gamma_f \cup \Gamma_i, \end{aligned} \tag{3.35}$$

for a.e. $\mathsf{t} \in (0,T)$. Testing the first equation of (3.35) with functions ϕ such that $\phi|_{\Gamma_i} = 0$ yields, since we are in a setting that allows us to use integration by parts,

$$\begin{aligned} &\int_{\Gamma_f} \nabla\,\zeta(\cdot,\mathsf{t}) \cdot \mathbf{n}_f \phi \,\mathrm{d}S(\xi) - \int_{\Omega_f} \nabla\,\zeta(\cdot,\mathsf{t}) \cdot \nabla\,\phi \,\mathrm{d}\xi = \int_{\Omega_f} \Delta\,\zeta(\cdot,\mathsf{t})\phi \,\mathrm{d}\xi \\ &= \int_{\Omega_f} \operatorname{div}\mathbf{g}(\cdot,\mathsf{t})\phi \,\mathrm{d}\xi = \int_{\Gamma_f} \mathbf{g}(\cdot,\mathsf{t}) \cdot \mathbf{n}_f \phi \,\mathrm{d}S(\xi) - \int_{\Omega_f} \mathbf{g}(\cdot,\mathsf{t}) \cdot \nabla\,\phi \,\mathrm{d}\xi \end{aligned}$$

for a.e. $\mathsf{t} \in (0,T)$. Due to the fact that there exists a bounded extension operator ext : $H^{\frac{1}{2}}(\Gamma_f) \to H^1(\Omega_f)$ and due to $\int_{\Gamma_f} \mathbf{g}(\cdot,\mathsf{t}) \cdot \mathbf{n}_f \phi \,\mathrm{d}S(\xi) = 0$ it holds

$$\int_{\Gamma_f} \nabla\,\zeta(\cdot,\mathsf{t}) \cdot \mathbf{n}_f \phi \,\mathrm{d}S(\xi) = \int_{\Omega_f} \nabla\,\zeta(\cdot,\mathsf{t}) \cdot \nabla\,\phi \,\mathrm{d}\xi - \int_{\Omega_f} \mathbf{g}(\cdot,\mathsf{t}) \cdot \nabla\,\phi \,\mathrm{d}\xi, \tag{3.36}$$

and there exists $C > 0$ such that

$$\|\nabla\,\zeta(\cdot,\mathsf{t}) \cdot \mathbf{n}_f\|_{H^{-\frac{1}{2}}(\Gamma_f)} \leq C \sup_{\phi \in H^1(\Omega_f), \|\phi\|_{H^1(\Omega_f)} \leq 1} \int_{\Gamma_f} \nabla\,\zeta(\cdot,\mathsf{t}) \cdot \mathbf{n}_f \phi \,\mathrm{d}S(\xi)$$

$$\leq C \sup_{\phi \in H^1(\Omega_f), \|\phi\|_{H^1(\Omega_f)} \leq 1} \left(\int_{\Omega_f} \nabla \zeta(\cdot, \mathsf{t}) \cdot \nabla \phi \, \mathrm{d}\xi - \int_{\Omega_f} \mathbf{g}(\cdot, \mathsf{t}) \cdot \nabla \phi \, \mathrm{d}\xi \right)$$
$$\leq C(\|\zeta(\cdot, \mathsf{t})\|_{H^1(\Omega_f)} + \|\mathbf{g}(\cdot, \mathsf{t})\|_{L^2(\Omega_f)^d})$$

for a.e. $\mathsf{t} \in (0, T)$. The proof of [113, Lem. 4.1] implies that $\|\zeta(\cdot, \mathsf{t})\|_{H^1(\Omega_f)} \leq C\|\mathbf{g}\|_{L^2(\Omega)^d}$. Thus,

$$\|\nabla \zeta(\cdot, \mathsf{t}) \cdot \mathbf{n}_f\|_{H^{-\frac{1}{2}}(\Gamma_f)} \leq C\|\mathbf{g}(\cdot, \mathsf{t})\|_{L^2(\Omega_f)^d}$$

for $\mathsf{t} \in (0, T)$. Differentiation of (3.36) with respect to t and using exactly the same argumentation yields

$$\|\partial_{\mathsf{t}} \nabla \zeta(\cdot, \mathsf{t}) \cdot \mathbf{n}_f\|_{H^{-\frac{1}{2}}(\Gamma_f)} \leq C\|\partial_{\mathsf{t}} \mathbf{g}(\cdot, \mathsf{t})\|_{L^2(\Omega_f)^d}$$

$\mathsf{t} \in (0, T)$.
Direct computations involving the explicit representation of the Sobolev-Slobodeckij seminorm as it is done in [113, p. 569] yield

$$\|\nabla \zeta \cdot \mathbf{n}_f\|_{H^{1+\frac{\ell}{2}}((0,T), H^{-\frac{1}{2}}(\Gamma_f))} \leq C\|\mathbf{g}\|_{\mathbf{G}_T}.$$

□

Lemma 3.18. Let $\mathbf{g} \in H^{1+\frac{\ell}{2}}((0, T), L^2(\Omega_f)^d) \cap H^1((0, T), H^\ell(\Omega_f)^d)$ such that $\mathbf{g}|_{\Sigma_f^T} = 0$ and $g \in L^2((0, T), H^{1+\ell}(\Omega_f))$. Then,

$$\|\pi\|_{H^{1+\frac{\ell}{2}}((0,T), H^1(\Omega_f)) \cap L^2((0,T), H^{3+\ell}(\Omega_f))} \leq C(\|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)} + \|g\|_{L^2((0,T), H^{1+\ell}(\Omega_f))}).$$

Moreover we have

$$\|\nabla \pi\|_{H^{\frac{3}{2}+\ell, \frac{3}{4}+\frac{\ell}{2}}(\Sigma_f^T)} \leq C(\|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)} + \|g\|_{L^2((0,T), H^{1+\ell}(\Omega_f))}),$$

and

$$\|\nabla_{\boldsymbol{\tau}} \pi\|_{H^{\frac{3}{2}+\ell, \frac{3}{4}+\frac{\ell}{2}}(\Sigma_f^T)} \leq C(\|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d)} + \|g\|_{L^2((0,T), H^{1+\ell}(\Omega_f))}).$$

Proof. cf. [113, Lem. 4.2]. □

These lemmas yield the following result.

Lemma 3.19. Let $\mathbf{g} \in \mathbf{G}_T \cap H^1((0, T), H^\ell(\Omega_f)^d)$, $g \in G_T$, $\mathbf{g}(\cdot, 0) = 0$, $g(\cdot, 0) = 0$ and $\mathbf{g}|_{\Sigma_f^T} = 0$. Then, $\tilde{\mathbf{v}} \in \mathbf{E}_T$ and

$$\|\mathcal{P}\tilde{\mathbf{v}}\|_{\mathbf{E}_T} + \|\nabla \tilde{p}\|_{\mathbf{F}_T} \leq C(\|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d))} + \|g\|_{G_T}),$$
$$\|(\mathcal{I} - \mathcal{P})\tilde{\mathbf{v}}\|_{\mathbf{E}_T} \leq C(\|\mathbf{g}\|_{H^{1+\frac{\ell}{2}}((0,T), L^2(\Omega_f)^d))} + \|g\|_{L^2((0,T), H^{1+\ell}(\Omega_f))}).$$

Proof. Since $\pi(\cdot, 0) = 0$ the compatibility conditions are satisfied and Theorem 3.15 can be applied. The rest follows as in [113, Lem. 4.3]. □

Now, having estimated all subsystems, we obtain the following theorem.

Theorem 3.20. Let $d \in \{2, 3\}$, $\Omega_f \subset \mathbb{R}^d$ be a domain with smooth boundary such that $\partial\Omega_f = \Gamma_f \cup \Gamma_i$ and $\overline{\Gamma}_f \cap \overline{\Gamma}_i = \emptyset$. Let $\ell \in (\frac{1}{2}, 1)$, $\mathbf{f} \in \mathbf{F}_T$, $\mathbf{h} \in \mathbf{H}_T$ and $\mathbf{v}_0 \in H^{1+\ell}(\Omega_f, \mathbb{R}^d)$. Let $\mathbf{g} \in \mathbf{G}_T \cap H^1((0,T), H^\ell(\Omega_f)^d)$, $g \in G_T$, $\mathbf{g}(\cdot, 0) = 0$, $g(\cdot, 0) = 0$ and $\mathbf{g}|_{\Sigma_f^T} = 0$. Let the compatibility conditions

$$\begin{aligned} \operatorname{div}(\mathbf{v}_0) &= 0 && \text{in } \Omega_f, \\ \mathbf{v}_0(\cdot) &= 0 && \text{on } \Gamma_f, \\ \mathbf{h}(\cdot, 0) &= 0 && \text{on } \Gamma_i, \\ 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f \cdot \boldsymbol{\tau} &= 0 && \text{on } \Gamma_i, \end{aligned}$$

for any unit vector $\boldsymbol{\tau}$ tangential to Γ_i be fulfilled. Then, the solution to system (3.32) satisfies

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{E}_T} + \|\nabla p\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d} + \|p|_{\Sigma_i^T}\|_{H_T} \leq C(&\|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{h}\|_{\mathbf{H}_T} + \|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} \\ &+ \|\mathbf{g}\|_{\mathbf{G}_T} + \|g\|_{G_T}). \end{aligned}$$

Proof. This follows directly from the above considerations. The only thing that is left to be shown is the estimate for $p|_{\Sigma_i^T}$, which follows from the trace inequality

$$\|p|_{\Sigma_i^T}\|_{H_T} \leq C\|p\|_{H^{1+\ell,\frac{\ell}{2}}(Q_f^T)}.$$

□

Corollary 3.21. Let $d \in \{2, 3\}$, $\Omega_f \subset \mathbb{R}^d$ be a Lipschitz domain such that $\partial\Omega_f = \Gamma_f \cup \Gamma_i$ and $\Gamma_f \cap \Gamma_i = \emptyset$. Let $\ell \in (\frac{1}{2}, 1)$, $\mathbf{f} \in \mathbf{F}_T$ and $\mathbf{h} \in \mathbf{H}_T$ and $\mathbf{v}_0 \in H^{1+\ell}(\Omega_f)^d$. Let the compatibility conditions

$$\begin{aligned} \operatorname{div}(\mathbf{v}_0) &= 0 && \text{in } \Omega_f, \\ \mathbf{v}_D(\cdot, 0) &= \mathbf{v}_0(\cdot) && \text{on } \Gamma_f, \\ \mathbf{h}(\cdot, 0) &= 0 && \text{on } \Gamma_i, \\ 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f \cdot \boldsymbol{\tau} &= 0 && \text{on } \Gamma_i, \end{aligned}$$

for any unit vector $\boldsymbol{\tau}$ tangential to Γ_i be fulfilled. Then, the solution to system (3.32) satisfies

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{E}_T} + \|\nabla p\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d} + \|p|_{\Sigma_i^T}\|_{H_T} \leq C(&\|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{h}\|_{\mathbf{H}_T} + \|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} \\ &+ \|\mathbf{g}\|_{\mathbf{G}_T} + \|g\|_{G_T}), \end{aligned}$$

where the constant C is independent of T.

Proof. Let $T_f > T$. Consider the systems

$$\begin{aligned} \partial_t \hat{\mathbf{v}}_1 - \nu\Delta\hat{\mathbf{v}}_1 + \nabla\hat{p}_1 &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\hat{\mathbf{v}}_1) = g &= \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\ \hat{\mathbf{v}}_1(\cdot,0) &= 0 && \text{in } \Omega_f, \\ \hat{\mathbf{v}}_1 &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\hat{\mathbf{v}}_1,\hat{p}_1)\mathbf{n}_f &= \mathbf{h} && \text{on } \Sigma_i^T, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} \partial_t \hat{\mathbf{v}}_2 - \nu\Delta\hat{\mathbf{v}}_2 + \nabla\hat{p}_2 &= 0 && \text{in } Q_f^T, \\ \operatorname{div}(\hat{\mathbf{v}}_2) &= 0 && \text{in } Q_f^T, \\ \hat{\mathbf{v}}_2(\cdot,0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \hat{\mathbf{v}}_2 &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\hat{\mathbf{v}}_2,\hat{p}_2)\mathbf{n}_f &= 0 && \text{on } \Sigma_i^T. \end{aligned} \tag{3.38}$$

Due to the linearity of the Stokes equations, $(\hat{\mathbf{v}},\hat{p}) = (\hat{\mathbf{v}}_1,\hat{p}_1)+(\hat{\mathbf{v}}_2,\hat{p}_2)$. In system (3.37), the temporal fractional order of the right hand side terms is smaller than $\frac{1}{2}$ or in $(\frac{1}{2},\frac{3}{2})$ but with additional zero initial conditions. Note that $p|_{\Gamma_i\times\{0\}} = 2\nu\epsilon(\mathbf{v}(\cdot,0))\mathbf{n}_f\cdot\mathbf{n}_f|_{\Gamma_i} + \mathbf{h}(\cdot,0)\cdot\mathbf{n}_f = 0$ due to [54, p. 242, (4.7)]). Property **P3** of the norm yields that $\operatorname{Ext} : Y^s_{(0,T)} \to Y^s_{(T-T_f,T)}$ is continuous with a continuity constant independent of T for $s \in [0,\frac{3}{2}) \setminus \{\frac{1}{2}\}$. Furthermore, $\bar{\mathbf{v}}_1 = \operatorname{Ext}(\hat{\mathbf{v}}_1)$ and $\bar{p}_1 = \operatorname{Ext}(\hat{p}_1)$ solve

$$\begin{aligned} \partial_t \bar{\mathbf{v}}_1 - \nu\Delta\bar{\mathbf{v}}_1 + \nabla\bar{p}_1 &= \bar{\mathbf{f}} && \text{in } \Omega_f \times (T-T_f,T), \\ \operatorname{div}(\bar{\mathbf{v}}_1) = \bar{g} &= \operatorname{div}(\bar{\mathbf{g}}) && \text{in } \Omega_f \times (T-T_f,T), \\ \bar{\mathbf{v}}_1(\cdot,T-T_f) &= 0 && \text{in } \Omega_f, \\ \bar{\mathbf{v}}_1 &= 0 && \text{on } \Gamma_f \times (T-T_f,T), \\ \sigma_f(\bar{\mathbf{v}}_1,\bar{p}_1)\mathbf{n}_f &= \bar{\mathbf{h}} && \text{on } \Gamma_i \times (T-T_f,T), \end{aligned}$$

where $\bar{\mathbf{h}} = \operatorname{Ext}(\mathbf{h})$, $\bar{\mathbf{f}} = \operatorname{Ext}(\mathbf{f})$, $\bar{g} = \operatorname{Ext}(g)$ and $\bar{\mathbf{g}} = \operatorname{Ext}(\mathbf{g})$. Applying Theorem 3.20 yields

$$\begin{aligned} &\|\bar{\mathbf{v}}_1\|_{(H^{2+\ell,1+\frac{\ell}{2}}(\Omega_f\times(T-T_f,T)))^d} + \|\nabla\bar{p}_1\|_{(H^{\ell,\frac{\ell}{2}}(\Omega_f\times(T-T_f,T)))^d} + \|\bar{p}_1|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Gamma_i\times(T-T_f,T))} \\ &\le C(\|\bar{\mathbf{f}}\|_{(H^{\ell,\frac{\ell}{2}}(\Omega_f\times(T-T_f,T)))^d} + \|\bar{\mathbf{h}}\|_{(H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Gamma_i\times(T-T_f,T)))^d} \\ &\quad + \|\bar{g}\|_{L^2((T-T_f,T),H^{1+\ell}(\Omega_f))\cap H^{\frac{\ell}{2}}((T-T_f,T),H^1(\Omega_f))} + \|\bar{\mathbf{g}}\|_{H^{1+\frac{\ell}{2}}((T-T_f,T),L^2(\Omega_f)^d)}). \end{aligned} \tag{3.39}$$

C is independent of T but might depend on T_f. Using interpolation and trace inequalities with constants that depend on T_f, not on T, $\|\cdot\|_{(H^{2+\ell,1+\frac{\ell}{2}}(Q_f^{(T-T_f,T)}))^d}$ can be replaced by

$$
\begin{aligned}
(\|\cdot\|^2_{L^2((T-T_f,T),H^{2+\ell}(\Omega_f)^d)\cap H^{1+\frac{\ell}{2}}((T-T_f,T),L^2(\Omega_f)^d)} \\
+\|\cdot\|^2_{H^{\frac{\ell}{2}}((T-T_f,T),H^2(\Omega_f)^d)} + \|\cdot|_{\Sigma_i^T}\|^2_{H^{\frac{1}{4}+\frac{\ell}{2}}((T-T_f,T),H^1(\Gamma_i)^d)} \\
+\|\cdot\|^2_{H^1((T-T_f,T),H^\ell(\Omega_f)^d)} + \|\cdot\|^2_{H^{\frac{1}{2}+\frac{\ell}{2}}((T-T_f,T),H^1(\Omega_f)^d)} \\
+\|\cdot\|^2_{H^{\frac{1}{4}+\frac{\ell}{4}}((T-T_f,T),H^{1+\ell}(\Omega_f)^d)} + \|\cdot|_{\Sigma_i^T}\|^2_{H^{\frac{3}{4}+\frac{\ell}{2}}((T-T_f,T),L^2(\Gamma_i)^d)\cap L^2((T-T_f,T),H^{\frac{3}{2}+\ell}(\Gamma_i)^d)})^{\frac{1}{2}}.
\end{aligned}
\tag{3.40}
$$

Using properties **P4** and **P3**, yields

$$
\begin{aligned}
&\|\hat{\mathbf{v}}_1\|_{\mathbf{E}_T} + \|\nabla\hat{p}_1\|_{(H^{\ell,\frac{\ell}{2}}(\Sigma_i^T))^d} + \|\hat{p}_1|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(Q_f^T)} \\
&\leq C(\|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{h}\|_{\mathbf{H}_T} + \|g\|_{G_T} + \|\mathbf{g}\|_{\mathbf{G}_T})
\end{aligned}
$$

with a constant C independent of T. Consider the system

$$
\begin{aligned}
\partial_t\bar{\mathbf{v}}_2 - \nu\Delta\bar{\mathbf{v}}_2 + \nabla\bar{p}_2 &= 0 \quad \text{in } \Omega_f\times(0,T_f), \\
\operatorname{div}(\hat{\mathbf{v}}_2) &= 0 \quad \text{in } \Omega_f\times(0,T_f), \\
\hat{\mathbf{v}}_2(\cdot,0) &= \mathbf{v}_0 \quad \text{in } \Omega_f, \\
\hat{\mathbf{v}}_2 &= 0 \quad \text{on } \Gamma_f\times(0,T_f), \\
\sigma_f(\hat{\mathbf{v}}_2,\hat{p}_2)\mathbf{n}_f &= 0 \quad \text{on } \Gamma_i\times(0,T_f)
\end{aligned}
$$

for which we know with property **P5** that for $\mathrm{R}: H^s((0,T_f),X)\to H^s((0,T),X)$, $\mathrm{R}(\bar{\mathbf{v}}_2)=\hat{\mathbf{v}}_2$ and $\mathrm{R}(\bar{p}_2)=\hat{p}_2$, where $(\hat{\mathbf{v}}_2,\hat{p}_2)$ is the solution of system (3.38). Applying Theorem 3.20 yields

$$
\begin{aligned}
&\|\bar{\mathbf{v}}_2\|_{(H^{2+\ell,1+\frac{\ell}{2}}(\Omega_f\times(0,T_f)))^d} + \|\nabla\bar{p}_2\|_{(H^{\ell,\frac{\ell}{2}}(\Omega_f\times(0,T_f)))^d} + \|\bar{p}_2|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Gamma_i\times(0,T_f))} \\
&\leq C(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d})
\end{aligned}
$$

with a constant C independent of T. Thus, by property **P5**, and due to the fact that we can replace $\|\cdot\|_{(H^{2+\ell,1+\frac{\ell}{2}}(\Omega_f\times(0,T_f)))^d}$ analogous to (3.40),

$$
\|\hat{\mathbf{v}}_2\|_{\mathbf{E}_T} + \|\nabla\hat{p}_2\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d} + \|\hat{p}_2|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)} \leq C\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d}. \tag{3.41}
$$

Combination of the estimates (3.39) and (3.41) yields the assertion, since

$$
\begin{aligned}
&\|\hat{\mathbf{v}}\|_{\mathbf{E}_T} \leq \|\hat{\mathbf{v}}_1\|_{\mathbf{E}_T} + \|\hat{\mathbf{v}}_2\|_{\mathbf{E}_T}, \\
&\|\hat{p}|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)} \leq \|\hat{p}_1|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)} + \|\hat{p}_2|_{\Sigma_i^T}\|_{H^{\frac{1}{2}+\ell,\frac{1}{4}+\frac{\ell}{2}}(\Sigma_i^T)}, \\
&\|\nabla\hat{p}\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d} \leq \|\nabla\hat{p}_1\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d} + \|\nabla\hat{p}_2\|_{(H^{\ell,\frac{\ell}{2}}(Q_f^T))^d}.
\end{aligned}
$$

□

3.3.3. Coupled System

The following theorem is a generalization of [113, Thm. 5.1] to a larger class of domains. The proof is essentially the same.

Theorem 3.22. Let $d = 3$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$, $T_f > 0$ and $0 < T \leq T_f$. Assume that

- $\Omega = \Omega_f \cup \Omega_s \cup \Gamma_i \subset \mathbb{R}^d$ is bounded and such that
  - Γ_i denotes the interface between Ω_s and Ω_f, i.e. $\overline{\Gamma}_i = \overline{\Omega}_s \cap \overline{\Omega}_f$.
  - the solid domain Ω_s is a domain with boundary $\partial\Omega_s = \Gamma_i \cup \Gamma_s$ of class $\mathcal{C}^\infty$, where Γ_s denotes the outer boundary solid boundary and $\overline{\Gamma}_i \cap \overline{\Gamma}_s = \emptyset$.
  - the fluid domain Ω_f is a Lipschitz domain with boundary $\partial\Omega_f = \Gamma_i \cup \Gamma_f$, where Γ_f denotes the outer boundary fluid boundary and $\overline{\Gamma}_i \cap \overline{\Gamma}_f = \emptyset$.

  The corresponding space-time-cylinders are denoted by $Q_f^T := \Omega_f \times (0,T)$, $Q_s^T := \Omega_s \times (0,T)$, $\Sigma_f^T := \Gamma_f \times (0,T)$, $\Sigma_s^T := \Gamma_s \times (0,T)$, $\Sigma_i^T := \Gamma_i \times (0,T)$.

- the initial conditions
$$\mathbf{v}_0 \in H^{1+\ell}(\Omega_f)^d \quad \text{and} \quad \mathbf{w}_1 \in H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d,$$
  are chosen such that
$$\operatorname{div}(\mathbf{v}_0) = 0$$
  and the compatibility conditions
$$\mathbf{v}_0|_{\Gamma_f}(\cdot) = 0, \quad \mathbf{v}_0|_{\Gamma_i} = \mathbf{w}_1|_{\Gamma_i}, \quad 2\nu\epsilon(\mathbf{v}_0)\mathbf{n}_f \cdot \boldsymbol{\tau} = 0 \quad \text{on } \Gamma_i,$$
  for any unit vector $\boldsymbol{\tau}$ tangential to Γ_i.

- the right hand side terms
$$\mathbf{f} \in \mathbf{F}_T, \quad \mathbf{h} \in \mathbf{H}_T, \quad g \in G_T, \quad \mathbf{g} \in \mathbf{G}_T \cap H^1((0,T), H^\ell(\Omega_f)^d)$$
  are chosen such that the compatibility conditions
$$\mathbf{g}(\cdot, 0) = 0 \quad \text{and} \quad \mathbf{h}(\cdot, 0) = 0$$
  and
$$\mathbf{g}|_{\Sigma_f^T} = 0$$
  are satisfied.

Then, the system

$$
\begin{aligned}
\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} &\quad \text{in } Q_f^T, \\
\operatorname{div}(\mathbf{v}) = g = \operatorname{div}(\mathbf{g}) &\quad \text{in } Q_f^T, \\
\mathbf{v}(\cdot, 0) = \mathbf{v}_0 &\quad \text{in } \Omega_f, \\
\mathbf{v} = 0 &\quad \text{on } \Sigma_f^T, \\
\mathbf{v} = \partial_t \mathbf{w} &\quad \text{on } \Sigma_i^T, \\
\sigma_f(\mathbf{v}, p)\mathbf{n}_f = \sigma_s(\mathbf{w})\mathbf{n}_f + \mathbf{h} &\quad \text{on } \Sigma_i^T, \\
\partial_{tt} \mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) = 0 &\quad \text{in } Q_s^T, \\
\mathbf{w}(\cdot, 0) = 0 &\quad \text{in } \Omega_s, \\
\partial_t \mathbf{w}(\cdot, 0) = \mathbf{w}_1 &\quad \text{in } \Omega_s, \\
\mathbf{w} = 0 &\quad \text{on } \Sigma_s^T,
\end{aligned}
$$

admits a unique solution $(\mathbf{v}, p, \mathbf{w}) \in \mathbf{E}_T \times P_T \times \mathbf{W}_T$ and the states dependent continuously on the initial data and the right hand sides

$$
\begin{aligned}
&\|\mathbf{v}\|_{\mathbf{E}_T} + \|\nabla p\|_{\mathbf{F}_T} + \|\sigma(\mathbf{w})\mathbf{n}_f\|_{\mathbf{H}_T} + \|p|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{w}\|_{\mathbf{W}_T} \\
&\leq C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d} + \|\mathbf{f}\|_{\mathbf{F}_T} + \|\mathbf{g}\|_{\mathbf{G}_T} \\
&\quad + \|g\|_{G_T} + \|\mathbf{h}\|_{\mathbf{H}_T}),
\end{aligned}
$$

for all $0 < T \leq T_f$. The constant C_S depends on T_f but is independent of T.

Remark 3.23. One could question the usefulness of this result for practical situations, especially in 3D, where fixing the solid structure leads to violation of the condition $\overline{\Gamma}_i \cap \overline{\Gamma}_s = \emptyset$. Therefore, a next step in the analysis of this problem might be the generalization of Theorem 3.15 (and also Theorem 3.20) to a setting where this condition does not have to be fulfilled. Further, one could also think about weakening the regularity assumptions on the domains Ω_f and Ω_s. It is also desirable to work with nonlinear elasticity. Using analgous techniques as for the fluid equations, one would consider the appearing nonlinear terms as right hand side terms, which, in general, contain the second order derivatives of the state $\mathbf{w}$. Thus, the regularity requirements of hyperbolic equations on the right hand sides prevent us from performing fixed point arguments. To this end, more elaborate techniques have to be used to work with nonlinear elasticity.

Proof. The proof corresponds to [113, Proof of Thm. 5.1] and is included for the sake of completeness. Let us denote

$$
\mathbf{H}_{T,0} = \{\boldsymbol{\zeta} \in \mathbf{H}_T \ : \ \boldsymbol{\zeta}(\cdot, 0) = 0 \text{ on } \Gamma_i\} \subset \mathbf{H}_T
$$

and equip the closed subspace $\mathbf{H}_{T,0}$ with the norm of $\mathbf{H}_T$. Moreover,

$$
\mathcal{A}: \quad \mathbf{H}_{T,0} \to \mathbf{H}_{T,0}, \quad \boldsymbol{\zeta} \mapsto \sigma_s(\mathbf{w}_\zeta)\mathbf{n}_f,
$$

denotes an affine mapping, where $\mathbf{w}_\zeta$ solves

$$\begin{aligned} \partial^2_{\mathsf{tt}}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) &= 0 && \text{in } Q_s^T, \\ \mathbf{w} &= \int_0^{\mathsf{t}} \mathbf{v}_\zeta(\mathsf{s}) d\mathsf{s} && \text{on } \Sigma_i^T, \\ \mathbf{w} &= 0 && \text{on } \Sigma_s^T, \\ \mathbf{w}(\cdot, 0) &= 0 && \text{in } \Omega_s, \\ \partial_{\mathsf{t}}\mathbf{w}(\cdot, 0) &= \mathbf{w}_1 && \text{in } \Omega_s, \end{aligned}$$

and $(\mathbf{v}_\zeta, p_\zeta)$ is the solution of the system

$$\begin{aligned} \partial_{\mathsf{t}}\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} && \text{in } Q_f^T, \\ \operatorname{div}(\mathbf{v}) = g &= \operatorname{div}(\mathbf{g}) && \text{in } Q_f^T, \\ \mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f, \\ \mathbf{v} &= 0 && \text{on } \Sigma_f^T, \\ \sigma_f(\mathbf{v}, p)\mathbf{n}_f &= \boldsymbol{\zeta} + \mathbf{h} && \text{on } \Sigma_i^T. \end{aligned}$$

This mapping is well defined, i.e. $\mathcal{A}(\mathbf{H}_{T,0}) \subset \mathbf{H}_{T,0}$. In order to see this, choose $\boldsymbol{\zeta} \in \mathbf{H}_{T,0}$. By Corollary 3.21 we know that $\eta(\cdot, \mathsf{t}) := \int_0^{\mathsf{t}} \mathbf{v}_\zeta(\cdot, \mathsf{s}) d\mathsf{s}$ for all $\mathsf{t} \in (0, T)$ fulfills

$$\boldsymbol{\eta} \in H^1((0,T), H^{\frac{3}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i)^d) \cap H^{\frac{5}{4}+\frac{\ell}{2}}((0,T), H^1(\Gamma_i)^d).$$

and thus,

$$\boldsymbol{\eta} \in H^\beta((0,T), H^{\frac{3}{2}+\ell}(\Gamma_i))^d \cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i))^d \cap H^{\frac{1}{2}+\ell+\beta}((0,T), H^1(\Gamma_i))^d.$$

Since $\boldsymbol{\eta}|_{\mathsf{t}=0} = 0$ and $\partial_{\mathsf{t}}\boldsymbol{\eta}|_{\mathsf{t}=0} = \mathbf{v}_0|_{\Gamma_i} = \mathbf{w}_1|_{\Gamma_i}$, by Assumption 3.13, we obtain that

$$\sigma_s(\mathbf{w}_\zeta)\mathbf{n}_f \in H^\beta((0,T), H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i)^d) \subset \mathbf{H}_{T,0},$$

since $\mathbf{w}_\zeta(\cdot, 0) = 0$.
In order to show the assertion via a fixed point argument, we show that there exists a constant C_A independent of T such that

$$\|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{\tilde{\mathbf{H}}_T} \le C_A \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathbf{H}_T} \tag{3.42}$$

for all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbf{H}_{T,0}$, $\tilde{\mathbf{H}}_T := H^\beta((0,T), H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((0,T), L^2(\Gamma_i)^d)$. This is due to Theorem 3.20 which implies that there exists a constant $C > 0$ such that

$$\|\mathbf{v}_{\zeta_1} - \mathbf{v}_{\zeta_2}\|_{\mathbf{E}_T} \le C\|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathbf{H}_T},$$

and, therefore, with $\boldsymbol{\eta}_{\zeta_1}(\cdot) - \boldsymbol{\eta}_{\zeta_2}(\cdot) := \int_0^{\cdot} (\mathbf{v}_{\zeta_1}(\mathsf{s}) - \mathbf{v}_{\zeta_2}(\mathsf{s})) d\mathsf{s}$,

$$\|\boldsymbol{\eta}_{\zeta_1} - \boldsymbol{\eta}_{\zeta_2}\|_{H^1((0,T),H^{\frac{3}{2}+\ell}(\Gamma_i)^d)\cap H^{\frac{7}{4}+\frac{\ell}{2}}((0,T),L^2(\Gamma_i)^d)\cap H^{\frac{5}{4}+\frac{\ell}{2}}((0,T),H^1(\Gamma_i)^d)} \le C\|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathbf{H}_T},$$

and Assumption 3.13. Property **P6** of the norm implies that there exists a constant C_H independent of T such that $\|\cdot\|_{\mathbf{H}_T} \leq C_H T^\alpha \|\cdot\|_{\tilde{\mathbf{H}}_T}$ for $\alpha = \min(\frac{1}{2}, \beta)$. Now, for small enough $T > 0$ it can be seen that $\mathcal{A}$ is a contraction. Thus, there exists $\bar{T} > 0$ and $\boldsymbol{\zeta}_0 \in \mathbf{H}_{\bar{T},0}$ such that $\mathcal{A}(\boldsymbol{\zeta}_0) = \boldsymbol{\zeta}_0$. In order to obtain the global-in-time result we use a similar argumentation as in [107, Prop. 2.7]. Let C_E denote the constant in **P3** and C_R denote the constant in **P5**. Let w.l.o.g. $2\bar{T} < T_f$ (otherwise choose T_f instead of $2\bar{T}$) and $C_R C_E C_H C_A \bar{T}^\alpha < 1$. Consider

$$\mathbf{H}_{T,1} := \{\boldsymbol{\zeta} \in \mathbf{H}_{2\bar{T}} \ : \ \boldsymbol{\zeta}(\cdot, \mathsf{t}) = \boldsymbol{\zeta}_0(\cdot, \mathsf{t}) \text{ for a.e. } \mathsf{t} \in (0, \bar{T})\}.$$

Let $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbf{H}_{T,1}$ be arbitrarily chosen. Due to **P3** and **P8** there holds

$$\|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathbf{H}_{2\bar{T}}} \leq C_E \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{L^2((\bar{T},2\bar{T}),H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{1}{4}+\frac{\ell}{2}}((\bar{T},2\bar{T}),L^2(\Gamma_i)^d)}. \tag{3.43}$$

Due to **P5** and **P8**

$$\|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{H^\beta((\bar{T},2\bar{T}),H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((\bar{T},2\bar{T}),L^2(\Gamma_i)^d)} \leq C_R \|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{\tilde{\mathbf{H}}_{2T}}, \tag{3.44}$$

since $(\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2)(\cdot, \mathsf{t}) = 0$ for all $\mathsf{t} \in (0, \bar{T})$. Thus, with **P8**, **P6**, (3.44), (3.42) and (3.43),

$$\begin{aligned}
&\|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{L^2((\bar{T},2\bar{T}),H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{1}{4}+\frac{\ell}{2}}((\bar{T},2\bar{T}),L^2(\Gamma_i)^d)} \\
&= \|\widetilde{\mathcal{A}\boldsymbol{\zeta}_1} - \widetilde{\mathcal{A}\boldsymbol{\zeta}_2}\|_{\mathbf{H}_T} \leq C_H T^\alpha \|\widetilde{\mathcal{A}\boldsymbol{\zeta}_1} - \widetilde{\mathcal{A}\boldsymbol{\zeta}_2}\|_{\tilde{\mathbf{H}}_T} \\
&= C_H T^\alpha \|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{H^\beta((\bar{T},2\bar{T}),H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{3}{4}+\frac{\ell}{2}}((\bar{T},2\bar{T}),L^2(\Gamma_i)^d)} \\
&\leq C_R C_H \bar{T}^\alpha \|\mathcal{A}\boldsymbol{\zeta}_1 - \mathcal{A}\boldsymbol{\zeta}_2\|_{\tilde{\mathbf{H}}_{2\bar{T}}} \\
&\leq C_A C_R C_H \bar{T}^\alpha \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{H_{2\bar{T}}} \\
&\leq C_E C_A C_R C_H \bar{T}^\alpha \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{L^2((\bar{T},2\bar{T}),H^{\frac{1}{2}+\ell}(\Gamma_i)^d) \cap H^{\frac{1}{4}+\frac{\ell}{2}}((\bar{T},2\bar{T}),L^2(\Gamma_i)^d)},
\end{aligned}$$

where $\tilde{\cdot}(\mathsf{t}) = \cdot(\mathsf{t} + \bar{T})$ for all $\mathsf{t} \in (0, \bar{T})$ and the mapping is a contraction on the metric space $\mathbf{H}_{T,1}$. Recursively applying this argumentation yields the global-in-time result. □

3.4. Local-in-Time Existence and Regularity for Unsteady Navier-Stokes-Lamé System

Following the argumentation of [113] the result for the linear unsteady FSI system with stationary interface that was analyzed above can be used as a basis for deriving local-in-time existence and regularity results for the nonlinear unsteady Navier-Stokes-Lamé system with moving interface in the fully Lagrangian setting. As already motivated in the introduction the system is given by

$$
\begin{aligned}
\partial_{\mathsf{t}}\hat{\mathbf{v}} - \nu\Delta_{\mathsf{y}}\hat{\mathbf{v}} + \nabla_{\mathsf{y}}\hat{p} = \hat{\mathcal{F}}(\hat{\mathbf{v}},\hat{p}) &\quad \text{in } \hat{Q}_f^T,\\
\operatorname{div}_{\mathsf{y}}(\hat{\mathbf{v}}) = \hat{\mathcal{G}}(\hat{\mathbf{v}}) = \operatorname{div}_{\mathsf{y}}(\hat{\mathbf{g}}(\hat{\mathbf{v}})) &\quad \text{in } \hat{Q}_f^T,\\
\hat{\mathbf{v}}(\cdot,0) = \hat{\mathbf{v}}_0 &\quad \text{in } \hat{\Omega}_f,\\
\hat{\mathbf{v}} = 0 &\quad \text{on } \hat{\Sigma}_f^T,\\
\hat{\mathbf{v}} = \partial_{\mathsf{t}}\hat{\mathbf{w}} &\quad \text{on } \hat{\Sigma}_i^T,\\
\sigma_{f,\mathsf{y}}(\hat{\mathbf{v}},\hat{p})\mathbf{n}_f = \sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})\mathbf{n}_f + \hat{\mathcal{H}}(\hat{\mathbf{v}},\hat{p}) &\quad \text{on } \hat{\Sigma}_i^T,\\
\partial_{\mathsf{tt}}\hat{\mathbf{w}} - \operatorname{div}(\sigma_{s,\mathsf{y}}(\hat{\mathbf{w}})) = 0 &\quad \text{in } \hat{Q}_s^T,\\
\hat{\mathbf{w}}(\cdot,0) = 0 &\quad \text{in } \hat{\Omega}_s,\\
\partial_{\mathsf{t}}\hat{\mathbf{w}}(\cdot,0) = \hat{\mathbf{w}}_1 &\quad \text{in } \hat{\Omega}_s,\\
\hat{\mathbf{w}} = 0 &\quad \text{on } \hat{\Sigma}_s^T,
\end{aligned}
\tag{3.45}
$$

with right hand side terms

$$
\begin{aligned}
\hat{\mathcal{F}}(\hat{\mathbf{v}},\hat{p}) &= \nu\sum_{j,k}\frac{\partial^2}{\partial \mathsf{x}_j^2}\check{\boldsymbol{\Upsilon}}_k\circ\hat{\boldsymbol{\chi}}\frac{\partial}{\partial \mathsf{y}_k}\hat{\mathbf{v}} + \nu\sum_{i,j,k}\frac{\partial}{\partial \mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_i\circ\hat{\boldsymbol{\chi}}\frac{\partial}{\partial \mathsf{x}_j}\check{\boldsymbol{\Upsilon}}_k\circ\hat{\boldsymbol{\chi}}\frac{\partial^2}{\partial \mathsf{y}_i\partial \mathsf{y}_k}\hat{\mathbf{v}}\\
&\quad - \nu\Delta_{\mathsf{y}}\hat{\mathbf{v}} + (\mathbf{I} - \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top)\nabla_{\mathsf{y}}\hat{p},\\
\hat{\mathcal{H}}(\hat{\mathbf{v}},\hat{p}) &= -\nu(D_{\mathsf{y}}\hat{\mathbf{v}}\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top(D_{\mathsf{y}}\hat{\mathbf{v}})^\top)\operatorname{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\hat{\mathbf{n}}_f + \hat{p}\operatorname{cof}(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\hat{\mathbf{n}}_f\\
&\quad + \nu(D_{\mathsf{y}}\hat{\mathbf{v}} + (D_{\mathsf{y}}\hat{\mathbf{v}})^\top)\hat{\mathbf{n}}_f - \hat{p}\hat{\mathbf{n}}_f,\\
\hat{\mathcal{G}}(\hat{\mathbf{v}}) &= \operatorname{div}_{\mathsf{y}}\hat{\mathbf{v}} - \det(\hat{J}_X)D_{\mathsf{y}}\hat{\mathbf{v}} : \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top = D_{\mathsf{y}}\hat{\mathbf{v}} : (\mathbf{I} - \det(\hat{J}_X)\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top),
\end{aligned}
\tag{3.46}
$$

and transformation

$$
\hat{\boldsymbol{\chi}}(\cdot,\mathsf{t})|_{\hat{\Omega}_f} : \quad \hat{\Omega}_f \to \check{\Omega}_f(\mathsf{t}), \quad \mathsf{y} \to \mathsf{y} + \int_0^{\mathsf{t}} \hat{\mathbf{v}}(\mathsf{y},\mathsf{s})d\mathsf{s}
$$

with inverse $\check{\boldsymbol{\Upsilon}}$. $\widehat{\mathbf{F}}_{\boldsymbol{\chi}} = D_{\mathsf{y}}\hat{\boldsymbol{\chi}} = (\nabla_{\mathsf{y}}\hat{\boldsymbol{\chi}})^\top$ is the Jacobian of $\hat{\boldsymbol{\chi}}$ and $\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} := \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}$ its inverse. Additionally, $\hat{\mathbf{g}}$ is defined by $\hat{\mathbf{g}}(\hat{\mathbf{v}}) := (\mathbf{I} - \det(\widehat{\mathbf{F}}_{\boldsymbol{\chi}})\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}})\hat{\mathbf{v}}$, such that $\operatorname{div}_{\mathsf{y}}(\hat{\mathbf{g}}(\hat{\mathbf{v}})) = \hat{\mathcal{G}}(\hat{\mathbf{v}})$.
Since the following theorem is not only applied on the ALE reference domain $\hat{\Omega}$ but also on the shape reference domain $\tilde{\Omega}$ in Theorem 4.6 it is formulated without superscripts.

Theorem 3.24. Let $d = 3$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$ and $T_f > 0$. Assume that

- $\Omega = \Omega_f \cup \Omega_s \cup \Gamma_i$, as well as, $\mathbf{v}_0$, $\mathbf{w}_1$ and $\mathbf{v}_D$ fulfill the requirements of Lemma 3.22 and let $K_0 := C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d})$, where C_S is the constant from Lemma 3.22.
- there exists $0 < T^* < T_f$ such that for all $0 < T \leq T^*$ and for arbitrary $M_0 > K_0$, $\mathbf{v}, \mathbf{v}^1, \mathbf{v}^2 \in \mathbf{E}_{T,M_0,\mathbf{v}_0}$ and $p, p^1, p^2 \in P_{T,M_0,\mathbf{v}_0}$ the right hand side terms fulfill

$$
\mathcal{F}(\mathbf{v},p) \in \mathbf{F}_T, \quad \mathcal{H}(\mathbf{v},p) \in \mathbf{H}_T, \quad \mathcal{G}(\mathbf{v}) \in G_T, \quad \mathbf{g}(\mathbf{v}) \in \mathbf{G}_T \cap H^1((0,T), H^\ell(\Omega_f)^d).
$$

and there exist positive constants $K_f, K_h, K_g, K_{\mathbf{g}}$ that do not depend on T such that

$$\begin{aligned}
&\|\mathcal{F}(\mathbf{v},p)\|_{\mathbf{F}_T} \le K_f \chi(M_0),\\
&\|\mathcal{H}(\mathbf{v},p)\|_{\mathbf{H}_T} \le K_h \chi(M_0),\\
&\|\mathcal{G}(\mathbf{v})\|_{G_T} \le K_g \chi(M_0),\\
&\|\mathbf{g}(\mathbf{v})\|_{\mathbf{G}_T} \le K_{\mathbf{g}} \chi(M_0),
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
&\|\mathcal{F}(\mathbf{v}^2,p^2) - \mathcal{F}(\mathbf{v}^1,p^1)\|_{\mathbf{F}_T} \le K_f T^\alpha \chi(M_0)(\|\mathbf{v}^2-\mathbf{v}^1\|_{\mathbf{E}_T} + \|\nabla p^2 - \nabla p^1\|_{\mathbf{F}_T}),\\
&\|\mathcal{H}(\mathbf{v}^2,p^2) - \mathcal{H}(\mathbf{v}^1,p^1)\|_{\mathbf{H}_T} \le K_h T^\alpha \chi(M_0)(\|\mathbf{v}^2-\mathbf{v}^1\|_{\mathbf{E}_T} + \|p^1|_{\Sigma_s^T} - p^2|_{\Sigma_s^T}\|_{H_T}),\\
&\|\mathcal{G}(\mathbf{v}^2) - \mathcal{G}(\mathbf{v}^1)\|_{G_T} \le K_g T^\alpha \chi(M_0)(\|\mathbf{v}^2-\mathbf{v}^1\|_{\mathbf{E}_T}),\\
&\|\mathbf{g}(\mathbf{v}^2) - \mathbf{g}(\mathbf{v}^1)\|_{\mathbf{G}_T} \le K_{\mathbf{g}} T^\alpha \chi(M_0)(\|\mathbf{v}^2-\mathbf{v}^1\|_{\mathbf{E}_T}),
\end{aligned} \tag{3.48}$$

for some $\alpha > 0$ and some polynomial χ. Furthermore, the compatibility conditions

$$\mathbf{g}(\mathbf{v})(\cdot,0) = 0 \quad \text{and} \quad \mathcal{H}(\mathbf{v},p)(\cdot,0) = 0$$

and

$$\mathbf{g}(\mathbf{v})|_{\Sigma_f^T} = 0$$

are satisfied.

Then, there exists $T > 0$ and $0 < M_0 < \infty$ such that the system

$$\begin{aligned}
\partial_{\mathsf{t}}\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathcal{F}(\mathbf{v},p) &\quad \text{in } Q_f^T,\\
\operatorname{div}(\mathbf{v}) = \mathcal{G}(\mathbf{v}) = \operatorname{div}(\mathbf{g}(\mathbf{v})) &\quad \text{in } Q_f^T,\\
\mathbf{v}(\cdot,0) = \mathbf{v}_0 &\quad \text{in } \Omega_f,\\
\mathbf{v} = 0 &\quad \text{on } \Sigma_f^T,\\
\mathbf{v} = \partial_{\mathsf{t}}\mathbf{w} &\quad \text{on } \Sigma_i^T,\\
\sigma_f(\mathbf{v},p)\mathbf{n}_f = \sigma_s(\mathbf{w})\mathbf{n}_f + \mathcal{H}(\mathbf{v},p) &\quad \text{on } \Sigma_i^T,\\
\partial_{\mathsf{tt}}\mathbf{w} - \operatorname{div}(\sigma_s(\mathbf{w})) = 0 &\quad \text{in } Q_s^T,\\
\mathbf{w}(\cdot,0) = 0 &\quad \text{in } \Omega_s,\\
\partial_{\mathsf{t}}\mathbf{w}(\cdot,0) = \mathbf{w}_1 &\quad \text{in } \Omega_s,\\
\mathbf{w} = 0 &\quad \text{on } \Sigma_s^T,
\end{aligned} \tag{3.49}$$

admits a unique solution

$$(\mathbf{v},p,\mathbf{w}) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T.$$

Proof. This theorem corresponds to a large extent to [113, Theorem 2.1], where the requirements (3.47) and (3.48) replace [113, Proposition 6.1]. For the sake of completeness, the proof of this theorem is repeated at this point.
As a first step the system (3.49) is reformulated as a fixed point system. To this end, $(\mathbf{v}^0, p^0, \mathbf{w}^0)$ is introduced as the solution of the system

$$\begin{aligned}
\partial_t \mathbf{v}^0 - \nu\Delta\mathbf{v}^0 + \nabla p^0 &= 0 && \text{in } Q_f^T,\\
\operatorname{div}(\mathbf{v}^0) &= 0 && \text{in } Q_f^T,\\
\mathbf{v}^0(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega_f,\\
\mathbf{v}^0 &= 0 && \text{on } \Sigma_f^T,\\
\mathbf{v}^0 &= \partial_t \mathbf{w}^0 && \text{on } \Sigma_i^T,\\
\sigma_f(\mathbf{v}^0, p^0)\mathbf{n}_f &= \sigma_s(\mathbf{w}^0)\mathbf{n}_f && \text{on } \Sigma_i^T,\\
\partial_{tt}\mathbf{w}^0 - \operatorname{div}(\sigma_s(\mathbf{w}^0)) &= 0 && \text{in } Q_s^T,\\
\mathbf{w}^0(\cdot, 0) &= 0 && \text{in } \Omega_s,\\
\partial_t \mathbf{w}^0(\cdot, 0) &= \mathbf{w}_1 && \text{in } \Omega_s,\\
\mathbf{w} &= 0 && \text{on } \Sigma_s^T,
\end{aligned}$$

that due to Lemma 3.22 admits for $0 < T \leq T_f$ a solution that fulfills

$$\begin{aligned}
&\|\mathbf{v}^0\|_{\mathbf{E}_T} + \|\nabla p^0\|_{\mathbf{F}_T} + \|p^0|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{w}^0\|_{\mathbf{W}_T}\\
&\leq C_S(\|\mathbf{v}_0\|_{H^{1+\ell}(\Omega_f)^d} + \|\mathbf{w}_1\|_{H^{\frac{1}{2}+\ell+\beta}(\Omega_s)^d}) \leq K_0,
\end{aligned}$$

where $K_0 > 0$ is a constant that does not depend on T but on T_f. The solution $(\mathbf{v}, p, \mathbf{w})$ of the system (3.49) then fulfills $\mathbf{v} = \mathbf{u} + \mathbf{v}^0$, $p = q + p^0$ and $\mathbf{w} = \mathbf{z} + \mathbf{w}^0$, where $(\mathbf{u}, q, \mathbf{z})$ is the solution to

$$\begin{aligned}
\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla q &= \mathcal{F}(\mathbf{u} + \mathbf{v}^0, q + p^0) && \text{in } Q_f^T,\\
\operatorname{div}(\mathbf{u}) = \mathcal{G}(\mathbf{u} + \mathbf{v}^0) &= \operatorname{div}(\mathbf{g}(\mathbf{u} + \mathbf{v}^0)) && \text{in } Q_f^T,\\
\mathbf{u}(\cdot, 0) &= 0 && \text{in } \Omega_f,\\
\mathbf{u} &= 0 && \text{on } \Sigma_f^T,\\
\mathbf{u} &= \partial_t \mathbf{z} && \text{on } \Sigma_i^T,\\
\sigma_f(\mathbf{u}, q)\mathbf{n}_f &= \sigma_s(\mathbf{z})\mathbf{n}_f + \mathcal{H}(\mathbf{u} + \mathbf{v}^0, q + p^0) && \text{on } \Sigma_i^T,\\
\partial_{tt}\mathbf{z} - \operatorname{div}(\sigma_s(\mathbf{z})) &= 0 && \text{in } Q_s^T,\\
\mathbf{z}(\cdot, 0) &= 0 && \text{in } \Omega_s,\\
\partial_t \mathbf{z}(\cdot, 0) &= 0 && \text{in } \Omega_s,\\
\mathbf{z} &= 0 && \text{on } \Sigma_s^T.
\end{aligned}$$

To prove the existence of solutions of the system (3.49) or the equivalent system (3.4), the method of successive approximations is used.

Therefore, we show that there exists some $M_0 > K_0$ such that the mapping

$$\mathcal{M}: \quad \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T \to \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T, \quad (\mathbf{u}, q, \mathbf{z}) \to (\underline{\mathbf{u}}, \underline{q}, \underline{\mathbf{z}}),$$

is well-defined and a contraction with respect to the norm

$$\|(\mathbf{u}, q, \mathbf{z})\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} := \|\mathbf{u}\|_{\mathbf{E}_T} + \|\nabla q\|_{\mathbf{F}_T} + \|q|_{\Sigma_i^T}\|_{H_T} + \|\mathbf{z}\|_{\mathbf{W}_T},$$

if we choose $T \leq T_f$ small enough. Here, $(\underline{\mathbf{u}}, \underline{q}, \underline{\mathbf{z}})$ is defined as the solution of

$$\begin{aligned}
\partial_t \underline{\mathbf{u}} - \nu \Delta \underline{\mathbf{u}} + \nabla \underline{q} = \mathcal{F}(\mathbf{u} + \mathbf{v}^0, q + p^0) &\quad \text{in } Q_f^T, \\
\operatorname{div}(\underline{\mathbf{u}}) = \mathcal{G}(\mathbf{u} + \mathbf{v}^0) = \operatorname{div}(\mathbf{g}(\mathbf{u} + \mathbf{v}^0)) &\quad \text{in } Q_f^T, \\
\underline{\mathbf{u}}(\cdot, 0) = 0 &\quad \text{in } \Omega_f, \\
\underline{\mathbf{u}} = 0 &\quad \text{on } \Sigma_f^T, \\
\underline{\mathbf{u}} = \partial_t \underline{\mathbf{z}} &\quad \text{on } \Sigma_i^T, \\
\sigma_f(\underline{\mathbf{u}}, \underline{q}) \mathbf{n}_f = \sigma_s(\underline{\mathbf{z}}) \mathbf{n}_f + \mathcal{H}(\mathbf{u} + \mathbf{v}^0, q + p^0) &\quad \text{on } \Sigma_i^T, \\
\partial_{tt} \underline{\mathbf{z}} - \operatorname{div}(\sigma_s(\underline{\mathbf{z}})) = 0 &\quad \text{in } Q_s^T, \\
\underline{\mathbf{z}}(\cdot, 0) = 0 &\quad \text{in } \Omega_s, \\
\partial_t \underline{\mathbf{z}}(\cdot, 0) = 0 &\quad \text{in } \Omega_s, \\
\underline{\mathbf{z}} = 0 &\quad \text{on } \Sigma_s^T.
\end{aligned}$$

In order to show the contraction property we consider arbitrary

$$(\mathbf{u}^1, q^1, \mathbf{z}^1), (\mathbf{u}^2, q^2, \mathbf{z}^2) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T.$$

Due to Lemma 3.22 and the inequalities (3.48) we know that

$$\begin{aligned}
&\|\mathcal{M}(\mathbf{u}^2, q^2, \mathbf{z}^2) - \mathcal{M}(\mathbf{u}^1, q^1, \mathbf{z}^1)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leq C_S(\|\mathcal{F}(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0) - \mathcal{F}(\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0)\|_{\mathbf{F}_T} + \|\mathbf{g}(\mathbf{u}^2 + \mathbf{v}^0) - \mathbf{g}(\mathbf{u}^1 + \mathbf{v}^0)\|_{\mathbf{G}_T} \\
&\qquad + \|\mathcal{G}(\mathbf{u}^2 + \mathbf{v}^0) - \mathcal{G}(\mathbf{u}^1 + \mathbf{v}^0)\|_{G_T} + \|\mathcal{H}(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0) - \mathcal{H}(\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0)\|_{\mathbf{H}_T}) \\
&\leq C_S C T^\alpha \chi(M_0) \|(\mathbf{u}^2 + \mathbf{v}^0, q^2 + p^0, 0) - (\mathbf{u}^1 + \mathbf{v}^0, q^1 + p^0, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leq C_S C T^\alpha \chi(M_0) \|(\mathbf{u}^2, q^2, 0) - (\mathbf{u}^1, q^1, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T},
\end{aligned}$$

where C is a constant independent of T. If we define $K_1 > 0$ as the constant that bounds

$$\|\mathcal{M}(0, 0, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \leq K_1,$$

and choose $M_0 > K_1$, than T can be chosen such that $C_S C T^\alpha \chi(M_0) < 1$, as well as,

$$\begin{aligned}
\|\mathcal{M}(\mathbf{u}, q, \mathbf{z})\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} &\leq \|\mathcal{M}(0, 0, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} + C_S C T^\alpha \chi(M_0) \|(\mathbf{u}, q, 0)\|_{\mathbf{E}_T \times P_T \times \mathbf{W}_T} \\
&\leq K_1 + 3 C_S C T^\alpha \chi(M_0) M_0 \leq M_0
\end{aligned}$$

for any $(\mathbf{u}, q, \mathbf{z}) \in \mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T$. Thus, $\mathcal{M}$ is a well-defined contraction and we can apply the fixed point theorem of Banach in order to show existence and uniqueness of the solution to the fixed point equation $\mathcal{M}(\mathbf{u}, q, \mathbf{z}) = (\mathbf{u}, q, \mathbf{z})$ in $\mathbf{E}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\mathbf{v}_0} \times \mathbf{W}_T$. □

It can be show that the requirements on the right hand side terms are indeed fulfilled for the choice (3.46) of the right hand side terms. The proof of this is postponed to Section 4 since it is a special case of Lemma 4.5.

Corollary 3.25. Let $d = 3$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1 - \ell)$, $T_f > 0$ and $0 < T \leq T_f$. Assume that

- $\hat{\Omega} = \hat{\Omega}_f \cup \hat{\Omega}_s \cup \hat{\Gamma}_i \subset \mathbb{R}^d$ such that
  - $\hat{\Gamma}_i$ denotes the interface between $\hat{\Omega}_s$ and $\hat{\Omega}_f$, i.e. $\overline{\hat{\Gamma}_i} = \overline{\hat{\Omega}_s} \cap \overline{\hat{\Omega}_f}$.
  - the solid domain $\hat{\Omega}_s$ is a domain with boundary $\partial\hat{\Omega}_s = \overline{\hat{\Gamma}_i} \cup \hat{\Gamma}_s$ of class $\mathcal{C}^\infty$, where $\hat{\Gamma}_s$ denotes the outer boundary solid boundary and $\overline{\hat{\Gamma}_i} \cap \overline{\hat{\Gamma}_s} = \emptyset$.
  - the fluid domain $\hat{\Omega}_f$ is a Lipschitz domain with boundary $\partial\hat{\Omega}_f = \hat{\Gamma}_i \cup \hat{\Gamma}_f$, where $\hat{\Gamma}_f$ denotes the outer boundary fluid boundary and $\overline{\hat{\Gamma}_i} \cap \overline{\hat{\Gamma}_f} = \emptyset$.
- the initial conditions

$$\hat{\mathbf{v}}_0 \in H^{1+\ell}(\hat{\Omega}_f)^d \quad \text{and} \quad \hat{\mathbf{w}}_1 \in H^{\frac{1}{2}+\ell+\beta}(\hat{\Omega}_s)^d,$$

  are chosen such that

$$\operatorname{div}(\hat{\mathbf{v}}_0) = 0$$

  and the compatibility conditions

$$\hat{\mathbf{v}}_0|_{\hat{\Gamma}_f}(\cdot) = 0, \quad \hat{\mathbf{v}}_0|_{\hat{\Gamma}_i} = \hat{\mathbf{w}}_1|_{\hat{\Gamma}_i}, \quad 2\nu(\epsilon(\hat{\mathbf{v}}_0)\mathbf{n}_f) \cdot \boldsymbol{\tau} = 0 \quad \text{on } \hat{\Gamma}_i,$$

  are satisfied for any unit vector $\boldsymbol{\tau}$ tangential to $\hat{\Gamma}_i$.

Then, there exists $0 < T_0$ and $0 < M_0 < \infty$ such that the system (3.45) admits a unique solution

$$(\hat{\mathbf{v}}(\tilde{\boldsymbol{\tau}}), \hat{p}(\tilde{\boldsymbol{\tau}}), \hat{\mathbf{w}}(\tilde{\boldsymbol{\tau}})) \in \hat{\mathbf{E}}_{T,M_0,\hat{\mathbf{v}}_0} \times \hat{P}_{T,M_0,\hat{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$$

for all $0 < T \leq T_0$.

Proof. Follows from Theorem 3.24 and Lemma 4.5 for $\tilde{\boldsymbol{\tau}} = \mathrm{id}_{\mathbf{z}}$, i.e., $\tilde{\mathbf{u}}_{\mathcal{T}} = 0$. □

4. Theoretical Analysis of Shape Optimization for Unsteady FSI

In this chapter, we apply the general framework for continuity and differentiability results that was introduced in Section 2.4 to shape optimization for unsteady FSI. The method of mappings, cf. Section 2.7, is used in order to reformulate the shape optimization problem. The main contribution are the differentiability results of the state of the unsteady FSI system with respect to domain variations. This chapter has already been published in [59, Sec. 4.1].

In order to maintain the structure required to apply Theorem 3.24 we have to ensure that the source term of the transformed elasiticity equation remains 0. For this purpose, the set of admissible transformations is chosen such that $\tilde{\boldsymbol{\tau}}|_{\tilde{\Omega}_s} = \mathrm{id}_{\mathsf{z}}$ for all $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$, i.e. $\tilde{\mathbf{u}}_{\tau}|_{\tilde{\Omega}_s} = 0$ for all $\tilde{\mathbf{u}}_{\tau} \in \tilde{\mathbf{U}}_{ad}$. The transformation of the Navier-Stokes-Lamé system (3.45) from the reference domain $\hat{\Omega}$ to the shape reference domain $\tilde{\Omega}$ via $\tilde{\boldsymbol{\tau}}$ yields the following system:

$$\begin{aligned}
\partial_{\mathsf{t}}\,\tilde{\mathbf{v}} - \nu\Delta_{\mathsf{z}}\,\tilde{\mathbf{v}} + \nabla_{\mathsf{z}}\,\tilde{p} &= \tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_{\tau}) && \text{in } \tilde{Q}_f^T,\\
\mathrm{div}_{\mathsf{z}}\,(\tilde{\mathbf{v}}) &= \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_{\tau}) && \text{in } \tilde{Q}_f^T,\\
\tilde{\mathbf{v}}(\cdot, 0) &= \tilde{\mathbf{v}}_0 && \text{in } \tilde{\Omega}_f,\\
\tilde{\mathbf{v}} &= 0 && \text{on } \tilde{\Sigma}_f^T,\\
\tilde{\mathbf{v}} &= \partial_{\mathsf{t}}\,\tilde{\mathbf{w}} && \text{on } \tilde{\Sigma}_i^T,\\
\sigma_{f,\mathsf{z}}(\tilde{\mathbf{v}}, \tilde{p})\tilde{\mathbf{n}}_f &= \sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}})\tilde{\mathbf{n}}_f + \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_{\tau}) && \text{on } \tilde{\Sigma}_i^T,\\
\partial_{\mathsf{tt}}\,\tilde{\mathbf{w}} - \mathrm{div}_{\mathsf{z}}\,(\sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}})) &= 0 && \text{in } \tilde{Q}_s^T,\\
\tilde{\mathbf{w}} &= 0 && \text{on } \tilde{\Sigma}_s^T,\\
\tilde{\mathbf{w}}(\cdot, 0) = 0, \quad \partial_{\mathsf{t}}\,\tilde{\mathbf{w}}(\cdot, 0) &= \tilde{\mathbf{w}}_1 && \text{in } \tilde{\Omega}_s,
\end{aligned} \tag{4.1}$$

where

$$\sigma_{f,\mathsf{z}}(\tilde{\mathbf{v}}, \tilde{p}) := 2\nu\epsilon_{\mathsf{z}}(\tilde{\mathbf{v}}) - \tilde{p}\mathbf{I}, \quad \sigma_{s,\mathsf{z}}(\tilde{\mathbf{w}}) := \lambda \mathrm{tr}(\epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}))\mathbf{I} + 2\mu\epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}), \quad \epsilon_{\mathsf{z}}(\tilde{\mathbf{w}}) := \frac{1}{2}(D_{\mathsf{z}}\,\tilde{\mathbf{w}} + (D_{\mathsf{z}}\,\tilde{\mathbf{w}})^\top),$$

$\tilde{\mathbf{v}}_0 = \hat{\mathbf{v}}_0 \circ \tilde{\boldsymbol{\tau}}$, $\tilde{\mathbf{w}}_1 = \hat{\mathbf{w}}_1 \circ \tilde{\boldsymbol{\tau}}$ and the nonlinear terms $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$ are defined by

$$\begin{aligned}
\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_{\tau}) = \nu &\sum_{j,k,l} (\partial_{\mathsf{x}_j\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l}\tilde{\mathbf{v}}\\
&+ \nu \sum_{i,k,l} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_i\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l}\tilde{\mathbf{v}}\\
&+ \nu \sum_{i,k,l,m} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})((\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_i}(\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}})\partial_{\mathsf{z}_l\mathsf{z}_m}\tilde{\mathbf{v}} - \nu\Delta_{\mathsf{z}}\,\tilde{\mathbf{v}}
\end{aligned}$$

$$
\begin{aligned}
&\quad + (\mathbf{I} - \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^{\top}((D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1})^{\top} \circ \tilde{\boldsymbol{\tau}}))\nabla_{\mathsf{z}}\,\tilde{p},\\
\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_{\tau}) &= -\nu(D_{\mathsf{z}}\,\tilde{\mathbf{v}}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^{\top}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-T}D_{\mathsf{z}}\,\tilde{\mathbf{v}}^{\top})\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f\\
&\quad + \nu(D_{\mathsf{z}}\,\tilde{\mathbf{v}} + (D_{\mathsf{z}}\,\tilde{\mathbf{v}})^{\top})\tilde{\mathbf{n}}_f - \tilde{p}(\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}))\tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_{\tau}) &= D_{\mathsf{z}}\,\tilde{\mathbf{v}} : (\mathbf{I} - \det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^{\top}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-T}) = D_{\mathsf{z}}\,\tilde{\mathbf{v}} : (\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})),
\end{aligned}
$$

where

$$
\tilde{\boldsymbol{\tau}} = \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_{\tau}, \quad \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \hat{\boldsymbol{\chi}} \circ \tilde{\boldsymbol{\tau}}, \quad \widetilde{\mathbf{F}}_{\boldsymbol{\chi}} = \widehat{\mathbf{F}}_{\boldsymbol{\chi}} \circ \tilde{\boldsymbol{\tau}}, \quad \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} = \widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}} \circ \tilde{\boldsymbol{\tau}} \tag{4.2}
$$

and thus $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t}) := \mathbf{I} + \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}(\mathsf{z},\mathsf{s})(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}(\mathsf{z}))^{-1}\,\mathrm{d}\mathsf{s}$.
Moreover, the function $\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_{\tau}) = (\mathbf{I} - \operatorname{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{\top}\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^{\top})\tilde{\mathbf{v}}$ satisfies

$$
\operatorname{div}_{\mathsf{z}}(\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_{\tau})) = \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_{\tau}).
$$

Assumption 4.1. Let $d = 3$, $\ell \in (\frac{1}{2}, 1)$, $\beta \in (0, 1-\ell)$, $T_f > 0$ and $0 < T \leq T_f$. Assume that

- $\tilde{\Omega} = \tilde{\Omega}_f \cup \tilde{\Omega}_s \cup \tilde{\Gamma}_i \subset \mathbb{R}^d$ such that
  - $\tilde{\Gamma}_i$ denotes the interface between $\tilde{\Omega}_s$ and $\tilde{\Omega}_f$, i.e. $\overline{\tilde{\Gamma}_i} = \overline{\tilde{\Omega}_s} \cap \overline{\tilde{\Omega}_f}$.
  - the solid domain $\tilde{\Omega}_s$ is a domain with boundary $\partial\tilde{\Omega}_s = \tilde{\Gamma}_i \cup \tilde{\Gamma}_s$ of class $\mathcal{C}^\infty$, where $\tilde{\Gamma}_s$ denotes the outer boundary solid boundary and $\overline{\tilde{\Gamma}_i} \cap \overline{\tilde{\Gamma}_s} = \emptyset$.
  - the fluid domain $\tilde{\Omega}_f$ is a Lipschitz domain with boundary $\partial\tilde{\Omega}_f = \tilde{\Gamma}_i \cup \tilde{\Gamma}_f$, where $\tilde{\Gamma}_f$ denotes the outer boundary fluid boundary and $\overline{\tilde{\Gamma}_i} \cap \overline{\tilde{\Gamma}_f} = \emptyset$.
- the initial conditions

  $$
  \hat{\mathbf{v}}_0 \in H^{1+\ell}(\tilde{\Omega}_f)^d \quad \text{and} \quad \tilde{\mathbf{w}}_1 \in H^{\frac{1}{2}+\ell+\beta}(\tilde{\Omega}_s)^d,
  $$

  are chosen such that

  $$
  \operatorname{div}(\tilde{\mathbf{v}}_0) = 0
  $$

  and the compatibility conditions

  $$
  \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_f}(\cdot) = 0, \quad \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_i} = \tilde{\mathbf{w}}_1|_{\tilde{\Gamma}_i}, \quad 2\nu(\epsilon(\hat{\mathbf{v}}_0)\mathbf{n}_f)\cdot\boldsymbol{\tau} = 0 \quad \text{on } \tilde{\Gamma}_i,
  $$

  are satisfied for any unit vector $\boldsymbol{\tau}$ tangential to $\tilde{\Gamma}_i$.

Let

$$
\tilde{\mathbf{U}} := \{\tilde{\mathbf{u}}_{\tau} \in H^{2+\ell}(\tilde{\Omega})^d \ : \operatorname{supp}(\tilde{\mathbf{u}}_{\tau}) \cap \operatorname{supp}(\tilde{\mathbf{v}}_0) = \emptyset,\ \tilde{\mathbf{u}}_{\tau}|_{\tilde{\Omega}_s} = 0\},
$$

which is a closed linear subspace of $H^{2+\ell}(\tilde{\Omega})^d$, be endowed with the norm

$$
\|\cdot\|_{\tilde{\mathbf{U}}} = \|\cdot\|_{H^{2+\ell}(\tilde{\Omega})^d}.
$$

Furthermore, let $\alpha_1 > \|\mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}$. We consider solutions of the FSI problem for transformations $\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau$ induced by displacements $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, where

$$\begin{aligned}\tilde{\mathbf{V}} := \{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}} \ :\ & \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau \text{ can be extended to an orientation-preserving } C^1\text{-diffeomorphism} \\ & \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^d \text{ with } \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} - \mathrm{id}_{\mathsf{z}} \in H^{2+\ell}(\mathbb{R}^d)^d, \\ & \|D_{\mathsf{z}}\,(\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1,\ \|(D_{\mathsf{z}}\,(\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau))^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1\},\end{aligned} \tag{4.3}$$

which by Lemma 4.3 is an open subset of $\tilde{\mathbf{U}}$. In particular, if $\tilde{\mathbf{U}}_{ad} \subset \tilde{\mathbf{V}}$, then our results will hold at any admissible design displacement. Alternatively, the current design of the ALE domain could be viewed as the reference shape domain, making it correspond to $\tilde{\mathbf{u}}_\tau = 0$, and our results then can be applied to study continuity and differentiability w.r.t. variations of this domain.

Remark 4.2. ([59, Remark 3])

1. In [67, Thm. 4.1] it is shown that C^1-diffeomorphisms map bounded Lipschitz domains to bounded Lipschitz domains. Therefore, for all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, $(\mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau)(\tilde{\Omega})$ is a bounded Lipschitz domain.

2. The requirements on the $\mathbb{R}^d$-extended transformations in the definition of the set on the right hand side of (4.3) allow to apply [73, Lem. B.5, B.6] showing that they map $H^s(\mathbb{R}^d)$-functions to $H^s(\mathbb{R}^d)$-functions for all $0 \le s \le 2 + \ell$. Furthermore, by [73, Cor. 2.1], there exist constants $M > 0$ and $\omega > 0$ such that

$$\begin{aligned}&\|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} < M, \quad \|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})^{-1}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} < M, \\ &\inf_{\mathsf{z}\in\mathbb{R}^d} \det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}(\mathsf{z})) > \omega.\end{aligned} \tag{4.4}$$

Lemma 4.3. ([59, Lem. 8]) For any $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ there exists $\rho = \rho(\tilde{\mathbf{u}}_\tau) > 0$ such that $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ holds for all $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{U}}$, $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \le \rho$.

Proof. Let $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ be arbitrary and set $\tilde{\boldsymbol{\tau}} = \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau$. For $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ we use the notation $\tilde{\boldsymbol{\tau}}_v = \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{v}}_\tau$. It has to be verified that there exists $\rho > 0$ such that for all $\tilde{\mathbf{v}}_\tau \in \tilde{\mathbf{V}}$ with $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \le \rho$ the following holds: $\tilde{\boldsymbol{\tau}}_v$ can be extended to an orientation-preserving C^1-diffeomorphism $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}$ of $\mathbb{R}^d$ satisfying $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - \mathrm{id}_{\mathsf{z}} \in H^{2+\ell}(\mathbb{R}^d)^d$, $\|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$, and $\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$.

The set $\Omega_{\tilde{\mathbf{u}}_\tau} := \tilde{\boldsymbol{\tau}}(\tilde{\Omega})$ is a bounded Lipschitz domain by the definition of $\tilde{\mathbf{V}}$ and Remark 4.2. Using, e.g., [123, Thm. 5, p. 181] combined with interpolation, there exists a bounded linear extension operator $H^{2+\ell}(\Omega_{\tilde{\mathbf{u}}_\tau})^d \to H^{2+\ell}(\mathbb{R}^d)^d$.
Moreover, the embeddings $H^{2+\ell}(\Omega_{\tilde{\mathbf{u}}_\tau})^d \subset W^{1,\infty}(\Omega_{\tilde{\mathbf{u}}_\tau})^d$ and $H^{2+\ell}(\mathbb{R}^d)^d \subset W^{1,\infty}(\mathbb{R}^d)^d$ are continuous.

Now $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ implies $\|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} =: \alpha_1' < \alpha_1$. Hence, we obtain as required $\|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \le \|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} + \|D_{\mathsf{z}}\,(\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \le \alpha_1' + \rho < \alpha_1$ for ρ sufficiently small.

Denote by $\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} \in H^{2+\ell}(\mathbb{R}^d)^d$ the orientation-preserving C^1-diffeomorphism that extends $\tilde{\boldsymbol{\tau}}$. Then, by part 2 of Remark 4.2, there exist constants $M > 0$ and $\omega > 0$ such that (4.4) holds. With the extension operator we obtain $\mathbf{h}_\tau \in H^{2+\ell}(\mathbb{R}^d)^d$ with $\mathbf{h}_\tau|_{\tilde{\Omega}} = \tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau$, $\|\mathbf{h}_\tau\|_{H^{2+\ell}(\mathbb{R}^d)^d} \leq C\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega})^d}$, and $\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \leq C\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega})^d}$. Setting $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} = \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} + \mathbf{h}_\tau$, there holds $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}|_{\tilde{\Omega}} = \tilde{\boldsymbol{\tau}}_v$ and $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - \mathrm{id}_{\mathsf{z}} = (\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d} - \mathrm{id}_{\mathsf{z}}) + \mathbf{h}_\tau \in H^{2+\ell}(\mathbb{R}^d)^d$. By a Sobolev embedding we obtain also that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}$ is C^1.

Since $W^{1,\infty}(\mathbb{R}^d)$ and $C^{0,1}(\mathbb{R}^d)$, are equal with equivalent norms, see [61, Thm. 4.1, Rem. 4.2], there there exists $c' > 0$ such that any $f \in W^{1,\infty}(\mathbb{R}^d)^d$ has a Lipschitz continuous representative with modulus $\leq c'\|f\|_{W^{1,\infty}(\mathbb{R}^d)^d}$.

We now show that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^d$ is bijective. In fact for any fixed $\mathsf{z}' \in \mathbb{R}^d$, the equation $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z}) = \mathsf{z}'$ can be written as

$$\mathsf{z} = \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z})) =: \mathcal{A}(\mathsf{z}'; \mathsf{z}).$$

For sufficiently small ρ, the map $\mathcal{A}(\mathsf{z}'; \cdot)$ is a contraction since, for any $\mathsf{z}_1, \mathsf{z}_2 \in \mathbb{R}^d$, by using (4.4)

$$\begin{aligned} \|\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z}_1)) - \tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}^{-1}(\mathsf{z}' - \mathbf{h}_\tau(\mathsf{z}_2))\| &\leq c'\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})^{-1}\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\|\mathbf{h}_\tau(\mathsf{z}_1) - \mathbf{h}_\tau(\mathsf{z}_2)\| \\ &\leq Mc'\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d}\|\mathsf{z}_1 - \mathsf{z}_2\| \leq CMc'\rho\|\mathsf{z}_1 - \mathsf{z}_2\|. \end{aligned}$$

Hence, by the Banach fixed point theorem, if ρ is sufficiently small, then for any $\mathsf{z}' \in \mathbb{R}^d$ there exists a unique $\mathsf{z} \in \mathbb{R}^d$ with $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z}) = \mathsf{z}'$.

We show next that $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}^{-1}$ is C^1. From (4.4) and $\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \leq C\rho$ we obtain a constant $C' > 0$ with

$$\begin{aligned} \inf_{\mathsf{z}\in\mathbb{R}^d} \det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z})) &\geq \omega - \|\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}) - \det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d})\|_{L^\infty(\mathbb{R}^d)} \\ &\geq \omega - C'\|D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d} - D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{\mathbb{R}^d}\|_{L^\infty(\mathbb{R}^d)^{d\times d}} \\ &\geq \omega - C'\|\mathbf{h}_\tau\|_{W^{1,\infty}(\mathbb{R}^d)^d} \geq \omega - CC'\rho. \end{aligned}$$

Hence, for $\rho > 0$ small enough we obtain $\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}(\mathsf{z})) > \omega/2$ for all $\mathsf{z} \in \mathbb{R}^d$ and thus $\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}^{-1}$ is C^1 by the inverse function theorem.

We have shown that for $\rho > 0$ small enough it holds that $\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_{v,\mathbb{R}^d}) \geq \omega/2$. Now $(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} = 1/\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)\mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^\top$.

Since by Lemma 2.5 products of functions in $H^{1+\ell}(\tilde{\Omega}_f)$ are again in $H^{1+\ell}(\tilde{\Omega}_f)$, we have $\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v), \mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v) \in H^{1+\ell}(\tilde{\Omega}_f)$ and since $\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v) \geq \omega/2 > 0$ by [116, pp. 336 and 297] also $1/\det(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v) \in H^{1+\ell}(\tilde{\Omega}_f)$. Hence, $(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} \in H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$ for $\|\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} \leq \rho$.

Finally, with a constant $C' > 0$ we obtain

$$\begin{aligned} &\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \\ &= \|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}} - D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \\ &\leq C'\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\|D_{\mathsf{z}}\,(\tilde{\mathbf{v}}_\tau - \tilde{\mathbf{u}}_\tau)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \\ &\leq \alpha_1 C'(\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} + \alpha_1)\rho, \end{aligned}$$

from which $\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}_v)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} < \alpha_1$ follows if ρ is chosen sufficiently small.

□

Let with $\rho = \rho(0)$ according to Lemma 4.3

$$\tilde{\mathbf{V}}_\rho := \{\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}} \;:\; \|\tilde{\mathbf{u}}_\tau\|_{\tilde{\mathbf{U}}} < \rho\}. \tag{4.5}$$

Then $\tilde{\mathbf{V}}_\rho$ is by Lemma 4.3 an open subset of $\tilde{\mathbf{U}}$ and we will study the differentiability of the solution of (4.1) on $\tilde{\mathbf{V}}_\rho$ at $\tilde{\mathbf{u}}_\tau = 0$.

The choice of the space of admissible transformations restricts the shape optimization to the optimal design of the fluid domain, but keeps the interface in the Lagrangian frame fixed. The boundedness properties of $\tilde{\mathbf{V}}$ allow us to establish estimates of the right hand sides in (4.1). The following Lemma is a helpful tool that takes the special structure of the right hand side terms into account.

Since $\mathsf{z} = \tilde{\boldsymbol{\tau}}^{-1}(\tilde{\boldsymbol{\tau}}(\mathsf{z}))$, it follows that $\mathbf{I} = (D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\tau}})D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}$ and

$$D_{\mathsf{y}}\,\tilde{\boldsymbol{\tau}}^{-1} \circ \tilde{\boldsymbol{\tau}} = (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}. \tag{4.6}$$

Furthermore, for arbitrary invertible matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ one has

$$\begin{aligned}
&\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1},\\
&\mathbf{A}^{-1} - \mathbf{B}^{-1} - \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})(\mathbf{A}^{-1} - \mathbf{B}^{-1})\\
&\qquad\qquad\qquad\qquad = \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1}.
\end{aligned} \tag{4.7}$$

Let $\tilde{\mathbf{u}}_\tau^i \in \tilde{\mathbf{V}}$, then $\tilde{\boldsymbol{\tau}}^i := \mathrm{id}_{\mathsf{z}} + \tilde{\mathbf{u}}_\tau^i$, $i = 1, 2$, satisfy by Lemma 2.5, (4.7) and the definition of $\tilde{\mathbf{V}}$

$$\begin{aligned}
&\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \le C\|\tilde{\boldsymbol{\tau}}^1 - \tilde{\boldsymbol{\tau}}^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} = C\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d},\\
&\|(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1)^{-1} - (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1} + (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^1 - D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^2)^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}\\
&\le C\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|^2_{H^{2+\ell}(\tilde{\Omega}_f)^d}.
\end{aligned} \tag{4.8}$$

We define analogously to $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ in (4.2)

$$\widetilde{\mathbf{F}}^i_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t}) = \widetilde{\mathbf{F}}^i_{\boldsymbol{\chi}}(\mathsf{z},\mathsf{t};\tilde{\mathbf{v}}^i,\tilde{\mathbf{u}}^i_\tau) := \mathbf{I} + \int_0^{\mathsf{t}} D_{\mathsf{z}}\,\tilde{\mathbf{v}}^i(\mathsf{z},\mathsf{s})(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^i(\mathsf{z}))^{-1}\,\mathrm{d}\mathsf{s},\;\; i \in \{1,2\}. \tag{4.9}$$

Lemma 4.4. ([59, Lemma 12]) Let Assumption 4.1 be satisfied. Let $M_0 > 0$, $\alpha \in (0,1)$ and $\alpha_1 > 0$. Then, there exists $T_\alpha > 0$ such that $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})$ is invertible, and $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot,\mathsf{t})) \ge \alpha$ for all $\mathsf{t} \in (0, T_\alpha)$ and for all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$ and $\tilde{\mathbf{v}} \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$. In addition, for each of the following terms, there exists a constant $C > 0$ independent of T such that for all $0 < T < T_\alpha$, $\tilde{\mathbf{v}}, \tilde{\mathbf{v}}^1, \tilde{\mathbf{v}}^2 \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$, $\tilde{\mathbf{u}}_\tau, \tilde{\mathbf{u}}^1_\tau, \tilde{\mathbf{u}}^2_\tau \in \tilde{\mathbf{V}}$ we have

1. a) $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} \le C(1 + M_0)$,

b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}} - \widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} \le C\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + C(1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

2. a) $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) \in \underline{\tilde{\mathbf{S}}}_T$, $\|\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \le C(1+M_0^2)$,

b) $\|\operatorname{cof}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \operatorname{cof}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \le C(1+M_0)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

3. a) $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) \in \tilde{S}_T$, $\|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T} \le C(1+M_0^3)$,

b) $\|\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\tilde{S}_T}$
$\le C(1+M_0^2)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

4. a) $(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1} \in \tilde{S}_T$, $\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \le C(1+M_0^{33})$,

b) $\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}$
$\le C(1+M_0^{68})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

5. a) $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{\underline{\tilde{\mathbf{S}}}_T} \le C(1+M_0^{35})$,

b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}} - \widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}}\|_{\underline{\tilde{\mathbf{S}}}_T}$
$\le C(1+M_0^{70})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

6. a) $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T \in \underline{\tilde{\mathbf{S}}}_T$, $\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T\|_{\underline{\tilde{\mathbf{S}}}_T} \le C(1+M_0^{70})$,

b) $\|\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\Upsilon}})^T - \widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\Upsilon}})^T\|_{\underline{\tilde{\mathbf{S}}}_T}$
$\le C(1+M_0^{105})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \underline{\tilde{\mathbf{S}}}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^T$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

7. a) $(\partial_{\mathsf{x}_j}\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} \in H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$,
$\|(\partial_{\mathsf{x}_j}\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}\|_{H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)} \le C(1+M_0^{70})$,

b) $\|(\partial_{\mathsf{x}_j}\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}^2 - \partial_{\mathsf{x}_j}\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}^1) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}\|_{H^1((0,T),H^\ell(\tilde{\Omega}_f)^d)}$
$\le C(1+M_0^{105})(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$,

c) The mapping $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto (\partial_{\mathsf{x}_j} \partial_{\mathsf{x}_k} \check{\mathbf{\Upsilon}}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$ is Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

Proof. In order to show the existence of the required $T_\alpha > 0$ we consider

$$\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I} = \int_0^{\mathsf{t}} D_{\mathsf{z}} \tilde{\mathbf{v}}(\mathsf{z}, \mathsf{s})(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}(\mathsf{z}))^{-1} \, \mathsf{ds}$$

and estimate with Lemma 2.5

$$\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot, \mathsf{t}) - \mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d \times d}} \le C\alpha_1 \int_0^{\mathsf{t}} \|D_{\mathsf{z}} \tilde{\mathbf{v}}(\cdot, \mathsf{s})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d \times d}} \, \mathsf{ds}.$$

Thus, since $H^{1+\ell}(\tilde{\Omega}_f)^{d \times d} \hookrightarrow \mathcal{C}(\overline{\tilde{\Omega}}_f)^{d \times d}$, we have

$$\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I}\|_{\mathcal{C}(\overline{\tilde{Q}_f^T})^{d \times d}} \le C T^{\frac{1}{2}} \alpha_1 M_0$$

for a constant C independent of T. Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot, 0)) = \det(\mathbf{I}) = 1$, we can find T_α such that $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot, \mathsf{t})$ is invertible and $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\cdot, \mathsf{t})) \ge \alpha$ for all $\mathsf{t} \in [0, T_\alpha]$, all $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}$, and all $\tilde{\mathbf{v}} \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$.

Now, let $0 < T < T_\alpha$. Consider the multilinear form $m(x_1, \dots, x_k) = x_1 \cdot \ldots \cdot x_k$ for $k \in \mathbb{N}$, which is by Lemma 2.5 continuous as a mapping

$$L^2(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \times \cdots \times H^{1+\ell}(\tilde{\Omega}_f) \to L^2(\tilde{\Omega}_f)$$

and as a mapping

$$H^{1+\ell}(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \times \cdots \times H^{1+\ell}(\tilde{\Omega}_f) \to H^{1+\ell}(\tilde{\Omega}_f).$$

The terms we have to estimate are obtained by inserting operators $\mathcal{T}_j : \tilde{\mathbf{E}}_T \times \tilde{\mathbf{V}} \to \tilde{S}_T$, $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \mapsto \mathcal{T}_j(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ in the multilinear form. If they are bounded, continuous and Fréchet differentiable for $1 \le j \le k$ and arbitrary $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$, we can use Lemma 2.13 to show the claims of the lemma. If we have to estimate vector or matrix valued quantities, we use the argumentation for every component. In the following, C denotes a generic constant independent of T.

1. Consider $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I} = m(\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau), \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau))$ with

$$\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \int_0^{\mathsf{t}} D_{\mathsf{z}} \tilde{\mathbf{v}}(\mathsf{s}) \, \mathsf{ds} \text{ and } \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-1}.$$

We have $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{S}}_T} \le C(1 + \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T})$, since $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0)\|_{H^{1+\ell}(\Omega)^{d \times d}} = 0$ and $\|\partial_{\mathsf{t}} \mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0)\|_{L^2(\Omega)^{d \times d}} = \|D_{\mathsf{z}} \tilde{\mathbf{v}}_0\|_{L^2(\Omega)^{d \times d}}$, as well as, with **P7**,

$$\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(\cdot, \mathsf{t})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d \times d}} \le T^{\frac{1}{2}} \|D_{\mathsf{z}} \tilde{\mathbf{v}}\|_{L^2((0,T), H^{1+\ell}(\tilde{\Omega}_f)^{d \times d})} \le T^{\frac{1}{2}} \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},$$
$$\|\partial_{\mathsf{t}} \mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T), H^{1+\ell}(\tilde{\Omega}_f)^{d \times d})} = \|D_{\mathsf{z}} \tilde{\mathbf{v}}\|_{L^2((0,T), H^{1+\ell}(\tilde{\Omega}_f)^{d \times d})} \le \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},$$

$$\begin{aligned}\|\partial_{\mathtt{t}}\, \mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^{d\times d})} &= \|D_{\mathtt{z}}\, \tilde{\mathbf{v}}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^{d\times d})} \\ &\leq \|\tilde{\mathbf{v}}\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f)^{d\times d})} \leq \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T},\end{aligned}$$

for almost every $\mathtt{t} \in (0,T)$ due to the definition of $\|\cdot\|_{\tilde{\mathbf{E}}_T}$. Boundedness follows with property **P1** of the norm. Fréchet differentiability and continuity now follow by linearity of $\mathcal{T}_1$ and due to

$$(\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1))(0) = \partial_{\mathtt{t}}\,(\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1))(0) = 0$$

for all $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}$. Note that $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ is independent of $\tilde{\mathbf{v}}$ and depends linearly on $(D_{\mathtt{z}}\, \tilde{\boldsymbol{\tau}})^{-1}$ with $\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\underline{\mathbf{S}}}_T} \leq C\|(D_{\mathtt{z}}\, \tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}}$. Hence, boundedness, continuity and differentiability follow from the definition of $\tilde{\mathbf{V}}$, (4.5) and (4.8).

2. Each component of the cofactor matrix $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ can be written as a finite sum of terms $a \cdot x_1 \cdot x_2$, where x_1, x_2 denote components of the matrix $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ and $a \in \{-1, 1\}$. Therefore, $\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a sum of bilinear forms with factors $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := a(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})_{i,j}$ and $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})_{k,l}$ for $i, j, k, l \in \{1, 2, 3\}$. Due to the estimates in 1.(a) we know that $\|\mathcal{T}_i(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{S}_T} \leq C(1 + M_0)$ for $i \in \{1, 2\}$, and, therefore, $\|\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{\underline{\mathbf{S}}}_T} \leq C(1 + M_0^2)$. 1.(b) yields $\|\mathcal{T}_i(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_i(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{S}_T} \leq C\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + C(1+M_0)\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}$, $i \in \{1, 2\}$. Therefore, the continuity estimate and Fréchet differentiability follow from Lemma 2.13.

3. Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a polynomial of degree 3 in the components of the matrix $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$, the assertions can be proved similar to 2.

4. a) Since $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ is a cubic polynomial in the components of $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}$ and we know that $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})(\cdot, \mathtt{t}) \geq \alpha > 0$ for all $\mathtt{t} \in [0, T_\alpha]$, the assertion follows from Lemma 2.14, 2., which implies

$$\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leq C(1 + \|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T})^{10}\|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\tilde{S}_T}.$$

Now, 3.(a) implies that

$$\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leq C(1 + M_0^3)^{10}(1 + M_0^3) \leq C(1 + M_0^{33}).$$

   b) The difference

$$(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2))^{-1} - (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))^{-1} = -(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))^{-1}(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2))^{-1}(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2) - \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))$$

is a 3-linear form with factors $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))^{-1}$, $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2))^{-1}$ and $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) := -(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2) - \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))$. Lemma 2.14, 1. now yields

$$\begin{aligned}&\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2))^{-1} - (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))^{-1}\|_{\tilde{S}_T} \\ &\leq C\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1))^{-1}\|_{\tilde{S}_T}\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2))^{-1}\|_{\tilde{S}_T}\|\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^2) - \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^1)\|_{\tilde{S}_T}.\end{aligned}$$

The estimates 3.(b) and 4.(a) now imply

$$\begin{aligned}&\|(\det(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}))^{-1} - (\det(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}\\ &\leq C(1+M_0^{33})(1+M_0^{33})(1+M_0^2)(\|\tilde{\mathbf{v}}^2-\tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}\\ &\quad + (1+M_0)\|\tilde{\mathbf{u}}^2_\tau - \tilde{\mathbf{u}}^1_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}).\end{aligned}$$

c) Let $(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$ be arbitrary. Then by 2. and 4.(a) we have $\|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T} \leq C(1+M_0^{33})$ and $\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leq C(1+M_0^2)$. Hence, Lemma 2.14 yields

$$\begin{aligned}\|\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} &= \|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T}\\ &\leq \|(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\|_{\tilde{S}_T}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{\underline{\tilde{\mathbf{S}}}_T} \leq C(1+M_0^{35}).\end{aligned} \tag{4.10}$$

Now $\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^{-1} = \det(\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}})$, thus it suffices by 1., 3. and the chain rule to show that $(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}^1_\tau) \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \mapsto (\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})^{-1} \in \underline{\tilde{\mathbf{S}}}_T$ is Fréchet differentiable at $(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}^1_\tau)$. This follows from (4.7), (4.10) and Lemma 2.14, since with $\mathbf{A} = \widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}}$

$$\begin{aligned}\|\mathbf{A}^{-1} - \widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}} - \widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}\|_{\underline{\tilde{\mathbf{S}}}_T} &= \|\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\widetilde{\mathbf{F}}^{-1}_{\boldsymbol{\chi}}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A})\mathbf{A}^{-1}\|_{\underline{\tilde{\mathbf{S}}}_T}\\ &\leq C(1+M_0^{35})^3\|\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{A}\|^2_{\underline{\tilde{\mathbf{S}}}_T},\end{aligned}$$

which yields with 1. the Fréchet differentiability.

5. Since $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} = (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^{-1} = (\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))^{-1}\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top$, we can prove the result via multilinear forms and use Lemma 2.14, 1. .

6. Again, the assertions can be shown via multilinear forms.

7. From $\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \mathrm{id}_z$, it follows that

$$\mathbf{I} = D_{\mathsf{z}}\,(\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}) = D_{\mathsf{x}}\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}.$$

Therefore, since $D_{\mathsf{z}}\,\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \widetilde{\mathbf{F}}_{\boldsymbol{\chi}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}$, we have

$$D_{\mathsf{x}}\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = (\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1} = D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}. \tag{4.11}$$

Furthermore, we have $(\widehat{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k} = (\widehat{\mathbf{F}}^{-1}_{\boldsymbol{\chi}})_{l,k} = (\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l) \circ \hat{\boldsymbol{\chi}}$, which implies

$$(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k} = (\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}. \tag{4.12}$$

Thus, $(\partial_{\mathsf{x}_j}\partial_{\mathsf{x}_k}\check{\boldsymbol{\Upsilon}}_l) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \partial_{\mathsf{x}_j}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1})_{l,k} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$ and with (4.11) we obtain

$$\partial_{\mathsf{x}_j}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1})_{l,k} \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}} = \sum_m (\partial_{\mathsf{z}_m}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k}\partial_{\mathsf{x}_j}(\tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}{}^{-1})_m \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}$$

$$= \sum_{m,i} (\partial_{\mathbf{z}_m} \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{l,k} (D_{\mathbf{z}}\, \tilde{\boldsymbol{\tau}}^{-1})_{m,i} (\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})_{i,j}$$

and each summand is the composition of a multilinear form $m(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3$, which is by Lemma 2.5 continuous as a mapping

$$H^\ell(\tilde{\Omega}_f) \times H^{1+\ell}(\tilde{\Omega}_f) \times \cdots \times H^{1+\ell}(\tilde{\Omega}_f) \to H^\ell(\tilde{\Omega}_f)$$

with an operator

$$\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}} \to H^1((0,T), H^\ell(\tilde{\Omega}_f)) \times H^1((0,T), H^{1+\ell}(\tilde{\Omega}_f)) \times H^1((0,T), H^{1+\ell}(\tilde{\Omega}_f))$$

that by (4.8), **P7** and 5. is bounded and continuous on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$ as well as Fréchet differentiable on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$. Now, we can apply Lemma 2.12 to conclude the proof. □

With the above Lemmas the required right hand side estimates can be established.

Lemma 4.5. ([59, Lemma 13]) Let Assumption 4.1 be satisfied. Let $T_f > 0$ and $\rho = \rho(0)$ be given by Lemma 4.3. Then, there exist $0 < T^* \le T_f$, $\alpha_1 > 0$, as well as, for each of the following terms, a constant $C > 0$ independent of T but dependent on T_f and a polynomial χ such that for all $0 < T < T^*$, $0 < M_0$, $\tilde{\mathbf{v}}, \tilde{\mathbf{v}}^1, \tilde{\mathbf{v}}^2 \in \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0}$, $\tilde{p}, \tilde{p}^1, \tilde{p}^2 \in \tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0}$ and $\tilde{\mathbf{u}}_\tau, \tilde{\mathbf{u}}_\tau^1, \tilde{\mathbf{u}}_\tau^2 \in \tilde{\mathbf{V}}_\rho$ we have

$$\begin{aligned} &\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{F}}_T, \quad \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{H}}_T, \quad \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{G}_T, \\ &\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in \tilde{\mathbf{G}}_T \cap H^1((0,T), H^\ell(\tilde{\Omega}_f)^d), \end{aligned}$$

$$\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)|_{\Sigma_f^T} = 0, \quad \tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(\cdot, 0) = 0, \quad \text{and} \quad \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)(\cdot, 0) = 0, \tag{4.13}$$

as well as,

$$\begin{aligned} &\|\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T} \le C\chi(M_0)(T^{1-\ell} + \rho), \qquad \|\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T} \le C\chi(M_0)(T^{1-\ell} + \rho), \\ &\|\tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{G}_T} \le C\chi(M_0)(T^{1-\ell} + \rho), \qquad \|\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{G}}_T} \le C\chi(M_0)(1 + \rho), \end{aligned}$$

and

$$\begin{aligned} &\|\tilde{\mathcal{F}}(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathcal{F}}(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{F}}_T} \\ &\quad \le C\chi(M_0)((T^{1-\ell} + \rho)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\nabla_{\mathbf{z}} \tilde{p}^2 - \nabla_{\mathbf{z}} \tilde{p}^1\|_{\tilde{\mathbf{F}}_T}) + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}), \\ &\|\tilde{\mathcal{H}}(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathcal{H}}(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{H}}_T} \\ &\quad \le C\chi(M_0)((T^{1-\ell} + \rho)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\tilde{p}^2|_{\tilde{\Sigma}_i^T} - \tilde{p}^1|_{\tilde{\Sigma}_i^T}\|_{H_T}) + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}), \\ &\|\tilde{\mathcal{G}}(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathcal{G}}(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{G}_T} \le C\chi(M_0)((T^{1-\ell} + \rho)\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}), \\ &\|\tilde{\mathbf{g}}(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathbf{g}}(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{G}}_T} \le C\chi(M_0)((T^{\frac{1}{4} - \frac{\ell}{4}} + \rho)\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}), \end{aligned}$$

and

$$\tilde{\mathcal{F}}: \quad \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \tilde{\mathbf{F}}_T, \qquad \tilde{\mathcal{H}}: \quad \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \tilde{\mathbf{H}}_T,$$
$$\tilde{\mathcal{G}}: \quad \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \tilde{G}_T, \qquad \tilde{\mathbf{g}}: \quad \tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho \to \tilde{\mathbf{G}}_T$$

are Fréchet differentiable on the relative interior of $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$ and $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$, respectively.

Proof. The compatibility conditions (4.13) are fulfilled, due to the choice of $\tilde{\mathbf{V}}$, which ensures that $\operatorname{supp} \tilde{\mathbf{u}}_\tau \cap \operatorname{supp} \tilde{\mathbf{v}}_0 = \emptyset$ and therefore $\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)(\cdot, 0) = 0$ and $\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(\cdot, 0) = 0$. The boundary condition on Σ_f^T ensures that $\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)|_{\Sigma_f^T} = 0$. The right hand side terms $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$ are sums of multilinear forms as introduced in Lemma 2.12 and 2.13. In Lemma 4.4 boundedness, continuity and Fréchet differentiability of the corresponding factors are shown. Thus, it suffices to establish an appropriate boundedness estimate such that the product of the appearing M_j in Lemma 2.12 have the structure $\tilde{C}(T^\alpha + \|\tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$ for a suitable $\alpha \geq 0$ and $\tilde{C}$ which is independent of T. The explicit time dependency is obtained by using the extension and restriction properties **P3**, **P4** and **P5** of the norm and by using **P6**. The time dependency for the corresponding constants $M_{j,1}$ and $M_{j,2}$ follows with similar arguments. The desired continuity estimates, as well as, Fréchet differentiability can be deduced from Lemma 2.12 if (4.6) and thus

$$(\partial_{\mathsf{y}_i} D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1}) \circ \tilde{\boldsymbol{\tau}} = (\partial_{\mathsf{y}_i}(D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1} - \mathbf{I})) \circ \tilde{\boldsymbol{\tau}} = \sum_m \partial_{\mathsf{z}_m}((D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I})(\partial_{\mathsf{y}_i} \tilde{\boldsymbol{\tau}}_m^{-1}) \circ \tilde{\boldsymbol{\tau}},$$

are kept in mind, which by Lemma 2.5 and the definition of $\tilde{\mathbf{V}}$ implies

$$\|(\partial_{\mathsf{y}_i} D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1}) \circ \tilde{\boldsymbol{\tau}}\|_{H^\ell(\tilde{\Omega}_f)} \leq C\|(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)} \|(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I}\|_{H^{1+\ell}(\tilde{\Omega}_f)} \leq C\alpha_1(1 + \alpha_1). \tag{4.14}$$

Moreover, since for arbitrary matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ the cofactor-matrix is a polynomial of degree $d-1$ in every entry, we have that

$$\operatorname{cof}(\mathbf{A}) - \operatorname{cof}(\mathbf{B}) \leq \sum_{i,j} \chi_{i,j}(\mathbf{A}, \mathbf{B})(\mathbf{A} - \mathbf{B})_{i,j},$$

where $\chi_{i,j}$ is a polynomial of degree $d-2$ in the entries of $\mathbf{A}$ and $\mathbf{B}$ for $1 \leq i, j \leq 3$. Thus,

$$\|\operatorname{cof}(\mathbf{A}) - \operatorname{cof}(\mathbf{B})\|_{H^{1+\ell}(\Omega)^{d\times d}} \leq C(\|\mathbf{A}\|^{d-2}_{H^{1+\ell}(\Omega)^{d\times d}} + \|\mathbf{B}\|^{d-2}_{H^{1+\ell}(\Omega)^{d\times d}})\|\mathbf{A} - \mathbf{B}\|_{H^{1+\ell}(\Omega)^{d\times d}},$$

and for $\tilde{\mathbf{u}}_\tau, \tilde{\mathbf{u}}_\tau^1, \tilde{\mathbf{u}}_\tau^2 \in \tilde{\mathbf{V}}$ we have

$$\|\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \leq C \tag{4.15}$$

$$\begin{aligned} \|\operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}^1) - \operatorname{cof}(D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}}^2)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} &\leq C\alpha_1^{d-2}\|\tilde{\boldsymbol{\tau}}^1 - \tilde{\boldsymbol{\tau}}^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} \\ &\leq C\alpha_1^{d-2}\|\tilde{\mathbf{u}}_\tau^1 - \tilde{\mathbf{u}}_\tau^2\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}. \end{aligned} \tag{4.16}$$

We show boundedness of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$. In order to obtain the estimates we have to split the terms such that the initial values of selected factors vanish at $\mathsf{t} = 0$. To this end, we decompose

$$
\begin{aligned}
\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= \tilde{\mathcal{F}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_5(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau),\\
\tilde{\mathcal{F}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{j,k,l} (\partial_{\mathsf{x}_j \mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}}) \partial_{\mathsf{z}_l} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{i,k,l} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})(\partial_{\mathsf{y}_i \mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}}) \partial_{\mathsf{z}_l} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{i,k,l,m} ((\sum_j \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_i \partial_{\mathsf{x}_j} \check{\boldsymbol{\Upsilon}}_k - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}})((\partial_{\mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_i} (\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}}) \partial_{\mathsf{z}_l \mathsf{z}_m} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \nu \sum_{k,l} (\partial_{\mathsf{y}_k \mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}}) \partial_{\mathsf{z}_l} \tilde{\mathbf{v}} + \nu \sum_{k,l,m} ((\partial_{\mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_l \partial_{\mathsf{y}_k} (\tilde{\boldsymbol{\tau}}^{-1})_m) \circ \tilde{\boldsymbol{\tau}} - \delta_{l,m}) \partial_{\mathsf{z}_l \mathsf{z}_m} \tilde{\mathbf{v}},\\
\tilde{\mathcal{F}}_5(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= (\mathbf{I} - \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top) \nabla_{\mathsf{z}} \tilde{p},\\
\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top (\mathbf{I} - (D_{\mathsf{y}} \tilde{\boldsymbol{\tau}}^{-1})^\top \circ \tilde{\boldsymbol{\tau}}) \nabla_{\mathsf{z}} \tilde{p},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &+ \tilde{\mathcal{H}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_5(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\\
&+ \tilde{\mathcal{H}}_6(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_7(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_8(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_9(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{H}}_{10}(\tilde{p}, \tilde{\mathbf{u}}_\tau),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}} \tilde{\mathbf{v}} (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-1} \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} (\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}) \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}} \tilde{\mathbf{v}}^\top (\operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}) \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}} \tilde{\mathbf{v}} (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-1} (\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \mathbf{I}) \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu (\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \mathbf{I})^\top (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}} \tilde{\mathbf{v}}^\top \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_5(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}} \tilde{\mathbf{v}} (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-1} (\operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) - \mathbf{I}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_6(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu (D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{z}} \tilde{\mathbf{v}}^\top (\operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) - \mathbf{I}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_7(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu D_{\mathsf{z}} \tilde{\mathbf{v}} ((D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-1} - \mathbf{I}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_8(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= -\nu ((D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^{-\top} - \mathbf{I}) D_{\mathsf{z}} \tilde{\mathbf{v}}^\top \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_9(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) &= -\tilde{p} (\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})) \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}) \tilde{\mathbf{n}}_f,\\
\tilde{\mathcal{H}}_{10}(\tilde{p}, \tilde{\mathbf{u}}_\tau) &= -\tilde{p} (\mathbf{I} - \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})) \tilde{\mathbf{n}}_f,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= D_{\mathsf{z}} \tilde{\mathbf{v}} : ((\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})) \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})) + D_{\mathsf{z}} \tilde{\mathbf{v}} : (\mathbf{I} - \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}}))\\
&=: \tilde{\mathcal{G}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathcal{G}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) &= \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^\top (\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top) \tilde{\mathbf{v}} + (\mathbf{I} - \operatorname{cof}(D_{\mathsf{z}} \tilde{\boldsymbol{\tau}})^\top) \tilde{\mathbf{v}}\\
&=: \tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) + \tilde{\mathbf{g}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau).
\end{aligned}
$$

Since the ideas for the estimates for the different summands of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathbf{g}}$ are similar we just present the proofs for $\tilde{\mathcal{F}}_2$, $\tilde{\mathcal{F}}_6$, $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{G}}_1$ and $\tilde{\mathbf{g}}_1$. Let C denote a generic constant independent of T. In the following argumentation we frequently use Lemma 2.5 in order to ensure that $X_1, \dots, X_k$ are chosen such that multilinear forms $m(x_1, \dots, x_k) := x_1 \cdot \ldots \cdot x_k$ fulfill the requirements of Lemma 2.12. The notation S_i, M_i, $M_{i,1}$, $M_{i,2}$, s_i for $i \in \{1, \dots, k\}$ is defined by Lemma 2.12.

- Estimation of $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$:

  To apply Lemma 2.12 we use property **P1**, which implies

  $$\| \cdot \|^2_{\tilde{\mathbf{G}}_T} = \| \cdot \|^2_{L^2((0,T),L^2(\tilde{\Omega}_f)^d)} + \|\partial_\mathsf{t} \cdot \|^2_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)},$$

  and estimate $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ and $\partial_\mathsf{t}\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ separately.

  1. $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors

     $$\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \tilde{\mathbf{v}}, \; \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top, \; \mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}})^\top.$$

     With Lemma 2.12, $s = s_1 = 0$, $s_2 = \ell$, $s_3 = 1$, $X = L^2(\tilde{\Omega}_f)^d$, $X_1 = H^{2+\ell}(\tilde{\Omega}_f)^d$, $X_2 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$, we obtain

     $$\|\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),L^2(\Omega_f)^d)} \leq CM_0(1 + M_0)T^{1-\ell},$$

     since by **P6**, (4.15) and Lemmas 2.11, 2.5, 4.4

     $$\begin{aligned} \|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_1} &\leq \|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T} \leq M_0, \\ \|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_2} &\leq CT^{1-\ell}\|\mathbf{I} - \operatorname{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\|_{H^1((0,T),X_2)} \leq CT^{1-\ell}(1 + M_0^2), \\ \|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_3} &= (\|\operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}})\|^2_{X_3} + \|\operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}})\|^2_{L^2((0,T),X_3)})^{\frac{1}{2}} \\ &\leq C\|\operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}})\|_{X_3} \leq C, \end{aligned} \tag{4.17}$$

     i.e., $M_1 = M_0$, $M_2 = CT^{1-\ell}(1+M_0^2)$ and $M_3 = C$ in the notation of Lemma 2.12. Using in addition (4.16) gives

     $$\begin{aligned} &\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leq \|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}, \\ &\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_2} \leq CT^{1-\ell}\|\operatorname{cof}(\widetilde{\mathbf{F}}^2_{\boldsymbol{\chi}}) - \operatorname{cof}(\widetilde{\mathbf{F}}^1_{\boldsymbol{\chi}})\|_{\tilde{\mathbf{E}}_T} \\ &\qquad \leq CT^{1-\ell}(1 + M_0)(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + (1 + M_0)\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}), \\ &\|\mathcal{T}_3(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_3} \\ &\qquad \leq C\|\operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}}^2) - \operatorname{cof}(D_\mathsf{z}\, \tilde{\boldsymbol{\tau}}^1)\|_{X_3} \leq C\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}. \end{aligned}$$

     Hence, $M_{1,1} = 1$, $M_{1,2} = 0$, $M_{2,1} = CT^{1-\ell}(1 + M_0)$, $M_{2,2} = CT^{1-\ell}(1 + M_0)^2$, $M_{3,2} = C$, $M_{3,1} = 0$ and Lemma 2.12 yields for a polynomial χ

     $$\|\tilde{\mathbf{g}}^1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathbf{g}}^1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{L^2((0,T),L^2(\Omega_f)^d)}$$

$$\leq C\chi(M_0)T^{1-\ell}(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d})$$

and Fréchet differentiability of $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) : \tilde{\mathbf{E}}_T \times \tilde{\mathbf{V}} \to L^2((0,T), L^2(\tilde{\Omega}_f)^d)$ on $\tilde{\mathbf{E}}_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{V}}_\rho$.

2. $\partial_\mathsf{t}\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = -\mathrm{cof}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^\top \partial_\mathsf{t}\,\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top \tilde{\mathbf{v}} + \mathrm{cof}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^\top(\mathbf{I} - \mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)\partial_\mathsf{t}\,\tilde{\mathbf{v}}$ is a sum of multilinear forms. We exemplarily estimate the first term. Here, $\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \tilde{\mathbf{v}}$, $\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = -\partial_\mathsf{t}\,\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top$ and $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \mathrm{cof}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^\top$. Choose $s = s_1 = \frac{\ell}{2}$, $s_2 = \frac{1}{2} + \frac{\ell}{4}$, $s_3 = 1$, $X = X_2 = L^2(\tilde{\Omega}_f)^d$, $X_1 = H^{1+\ell}(\tilde{\Omega}_f)^d$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)^d$. With Lemmas 2.10, 4.4 we obtain

$$\begin{aligned}
&\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_1} \leq C(\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^d} + T^{\frac{1}{4}-\frac{\ell}{4}}\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{E}}_T}) \leq C(1 + M_0),\\
&\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_2} \leq C(\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0))\|_{X_2} + T^{\frac{\ell}{4}}\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)})\\
&\qquad\qquad\qquad \leq C(1 + M_0^2),\\
&\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_3} \leq C,
\end{aligned} \tag{4.18}$$

where we use for the second term that $0 = \partial_\mathsf{t}\,(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}) = \partial_\mathsf{t}\,\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}\partial_\mathsf{t}\,\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}$ and thus with (4.9)

$$\begin{aligned}
\partial_\mathsf{t}(\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top)(0) &= \partial_\mathsf{t}\,(\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0) = (\partial_\mathsf{t}\det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} + \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\partial_\mathsf{t}\,\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0)\\
&= (\mathrm{tr}(\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})^\top \partial_\mathsf{t}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} - \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}} D_\mathsf{z}\,\tilde{\mathbf{v}}(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})(0)\\
&= \mathrm{tr}(D_\mathsf{z}\,\tilde{\mathbf{v}}_0(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^{-1})\mathbf{I} - D_\mathsf{z}\,\tilde{\mathbf{v}}_0(D_\mathsf{z}\,\tilde{\boldsymbol{\tau}})^{-1}.
\end{aligned}$$

Since $(\mathcal{T}_i(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_i(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1))(0) = 0$ for $i \in \{1, 2, 3\}$, analogous to (4.18), we obtain with (4.16)

$$\begin{aligned}
&\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leq CT^{\frac{1}{4}-\frac{\ell}{4}}\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{E}}_T},\\
&\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_2}\\
&\quad \leq CT^{\frac{\ell}{4}}\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{H^{\frac{1}{2}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)},\\
&\|\mathcal{T}_3(\tilde{\mathbf{v}}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_3} \leq C\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}.
\end{aligned}$$

Continuity and Fréchet differentiability follow now by Lemmas 2.12, 4.4. Finally, $\tilde{\mathbf{g}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) \in H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$, since $\mathbf{v} \in H^1((0,T), H^\ell(\tilde{\Omega}_f)^d)$, $\tilde{\boldsymbol{\tau}} \in H^{2+\ell}(\tilde{\Omega}_f)^d$ and $(\mathbf{I} - \mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})) \in H^1((0,T), H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})$.

- Bound for $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T}$:

  $\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors

$$\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \nabla_{\mathsf{z}} \tilde{p},\ \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \mathbf{I} - (D_{\mathsf{y}}\, \tilde{\boldsymbol{\tau}}^{-1})^\top \circ \tilde{\boldsymbol{\tau}} = \mathbf{I} - (D_{\mathsf{z}}\, \tilde{\boldsymbol{\tau}})^{-\top}$$

  due to (4.6) and $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau) = \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}^\top$.

  1. $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^\ell(\tilde{\Omega}_f)^d)}$:
     Choose $s = s_1 = 0$, $s_2 = 1$, $s_3 = \ell$, $X = X_1 = H^\ell(\tilde{\Omega}_f)^d$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$. $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_1} \leq M_0$ follows by (2.6). With (4.8) and Lemma 2.11 we obtain

$$\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_2} \leq C\|\tilde{\mathbf{u}}_\tau\|_{H^{2+\ell}(\tilde{\Omega}_f)^d} \leq C\rho \tag{4.19}$$

     since $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}_\rho$. $\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0) = \mathbf{I}$ and Lemmas 2.10, 4.4 imply

$$\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{S_3} \leq C(1 + T^{1-\ell}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),X_3)}) \leq C(1 + M_0^{35}). \tag{4.20}$$

     With (4.8) and Lemma 4.4 we have

$$\begin{aligned}
&\|\mathcal{T}_1(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_1(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_1} \leq \|\nabla_{\mathsf{z}} \tilde{p}^2 - \nabla_{\mathsf{z}} \tilde{p}^1\|_{\tilde{\mathbf{F}}_T},\\
&\|\mathcal{T}_2(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_2(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_2} \leq C\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}},\\
&\|\mathcal{T}_3(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \mathcal{T}_3(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{S_3} \leq C(1 + M_0^{70})T^{1-\ell}(\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T}\\
&\qquad\qquad + (1 + M_0)\|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}).
\end{aligned}$$

  2. $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}$:
     Let $s = s_1 = \frac{\ell}{2}$, $s_2 = 1$, $s_3 = \ell$, $X = X_1 = L^2(\tilde{\Omega}_f)^d$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}$. With (2.6), (4.19), (4.20) and Lemmas 2.5, 4.4 we obtain the same bounds as before.
     Thus, with Lemma 2.12, $\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T} \leq C(1 + M_0^{36})\rho$ and

$$\begin{aligned}
&\|\tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}^2, \tilde{p}^2, \tilde{\mathbf{u}}_\tau^2) - \tilde{\mathcal{F}}_6(\tilde{\mathbf{v}}^1, \tilde{p}^1, \tilde{\mathbf{u}}_\tau^1)\|_{\tilde{\mathbf{F}}_T}\\
&\leq C\chi(M_0)(T^{1-\ell}\|\tilde{\mathbf{v}}^2 - \tilde{\mathbf{v}}^1\|_{\tilde{\mathbf{E}}_T} + \rho\|\nabla_{\mathsf{z}} \tilde{p}^2 - \nabla_{\mathsf{z}} \tilde{p}^1\|_{\tilde{\mathbf{F}}_T} + \|\tilde{\mathbf{u}}_\tau^2 - \tilde{\mathbf{u}}_\tau^1\|_{H^{2+\ell}(\tilde{\Omega}_f)^d}),
\end{aligned}$$

     where χ is a polynomial.

As seen in the previous estimates, due to Lemma 2.12, the derivation of the continuity estimates and Fréchet differentiability is straightforward if one knows how to show boundedness of the multilinear forms. We thus only address boundedness in the following.

- Bound for $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T}$:

  $\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ is a sum of multilinear forms with factors

$$\begin{aligned}&\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \partial_{\mathsf{z}_l}\tilde{\mathbf{v}}, \quad \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \nu\partial_{\mathsf{y}_i}\partial_{\mathsf{y}_k}(\tilde{\boldsymbol{\tau}}^{-1})_l \circ \tilde{\boldsymbol{\tau}},\\ &\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (\sum_j(\partial_{\mathsf{x}_j}(\check{\boldsymbol{\Upsilon}})_i\partial_{\mathsf{x}_j}(\check{\boldsymbol{\Upsilon}})_k) - \delta_{i,k}) \circ \tilde{\boldsymbol{\chi}}_{\boldsymbol{\tau}}\end{aligned}$$

  for $i, k, l \in \{1, \ldots, d\}$ with $\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0) = 0$.

  1. $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^\ell(\tilde{\Omega}_f)^d)}$:
     Boundedness, continuity and Fréchet differentiability are obtained with Lemma 2.12 for $s = s_1 = 0$, $s_2 = 1$, $s_3 = \ell$, and $X = H^\ell(\tilde{\Omega}_f)^d$, $X_1 = H^{1+\ell}(\tilde{\Omega}_f)^d$, $X_2 = H^\ell(\tilde{\Omega}_f)$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)$ and Lemma 4.4. By **P6** and (4.12) we obtain

$$\begin{aligned}\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_3} &\leq CT^{1-\ell}\|\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^1((0,T),X_3)}\\ &\leq CT^{1-\ell}(1 + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}})^\top\|_{\underline{\tilde{\mathbf{S}}}_T}) \leq CT^{1-\ell}(1 + M_0^{70}),\end{aligned} \quad (4.21)$$

     (4.14) and Lemma 2.11 imply $\|\mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_2} \leq C$ and, with **P7**,

$$\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{s_1}((0,T),X_1)} \leq C\|\tilde{\mathbf{v}}\|_{L^2((0,T),H^{2+\ell}(\tilde{\Omega}_f)^d)} \leq CM_0.$$

  2. $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),L^2(\tilde{\Omega}_f)^d)}$:
     Choose $s = s_1 = \frac{\ell}{2}$, $s_2 = 1$, $s_3 = \ell$, $X = L^2(\tilde{\Omega}_f)^d$, $X_1 = H^1(\tilde{\Omega}_f)^d$, $X_2 = H^\ell(\tilde{\Omega}_f)$, $X_3 = H^{1+\ell}(\tilde{\Omega}_f)$ and use (4.14), (4.21), **P6**, **P7** and Lemmas 2.11, 2.5, 2.12 and 4.4.

  We obtain $\|\tilde{\mathcal{F}}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{F}}_T} \leq CT^{1-\ell}(1 + M_0^{71})$.

- Bound for $\|\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T}$:

  $\tilde{\mathcal{H}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)$ is a multilinear form with factors

$$\begin{aligned}&\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = D_{\mathsf{z}}\,\tilde{\mathbf{v}}, \quad \mathcal{T}_2(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})^{-1},\\ &\mathcal{T}_3(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}, \quad \mathcal{T}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = (\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}),\\ &\mathcal{T}_5(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau) = \nu\,\mathrm{cof}(D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}})\tilde{\mathbf{n}}_f,\end{aligned}$$

  with $\mathcal{T}_4(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(0) = 0$.

  Due to Lemma 2.5 on Γ_i, which can locally be mapped to bounded open domains on $\mathbb{R}^{d-1}$, Lemma 2.12 can be applied. **P7** and boundedness of the trace operator yield

$$\|\mathcal{T}_j(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^\alpha((0,T),H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d})} \leq C\|\mathcal{T}_j(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^\alpha((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})}$$

  for $\alpha \in [0, 1] \setminus \{\frac{1}{2}\}$, $j \in \{1, \ldots, 5\}$.

1. $\|\widetilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d)}$:
   Choose $s = s_1 = 0$, $s_2 = s_5 = 1$, $s_3 = s_4 = \ell$, and $X = X_5 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d$, and $X_1 = X_2 = X_3 = X_4 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d}$.
   We have with **P7** and the definition of $\tilde{\mathbf{V}}$
   $$\begin{aligned}\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} &\le C\|\tilde{\mathbf{v}}\|_{L^2((0,T),H^{2+\ell}(\tilde{\Omega}_f)^d)} \le CM_0,\\ \|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2} &\le C\|(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})^{-1}\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \le C\alpha_1.\end{aligned}$$
   Lemma 2.10 and 4.4 imply
   $$\begin{aligned}\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} &\le C(T^{1-\ell}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}})\\ &\le C(1+M_0^{35}).\end{aligned}$$
   Moreover, (4.15) yields $\|\mathcal{T}_5(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_5} \le C\|\mathrm{cof}(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}})\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}} \le C$. Finally, **P6** implies
   $$\begin{aligned}\|\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_4} &\le C\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}\|_{H^\ell((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})}\\ &\le CT^{1-\ell}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \le CT^{1-\ell}(1+M_0^2).\end{aligned}$$

2. $\|\widetilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Gamma}_i)^d)}$:
   Let $s = s_1 = s_3 = s_4 = \frac{1}{4}+\frac{\ell}{2}$, $s_2 = s_5 = 1$, $X = L^2(\tilde{\Gamma}_i)^d$, $X_1 = L^2(\tilde{\Gamma}_i)^{d\times d}$, $X_2 = X_3 = X_4 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^{d\times d}$, $X_5 = H^{\frac{1}{2}+\ell}(\tilde{\Gamma}_i)^d$.
   The estimates for $\|\mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_2}$ and $\|\mathcal{T}_5(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_5}$ are as above.
   Since $\|D_{\mathbf{z}}\cdot|_{\tilde{\Sigma}_i^T}\|_{H^{\frac{1}{4}+\frac{\ell}{2}}((0,T),L^2(\tilde{\Gamma}_i)^{d\times d})}$ appears in the definition of $\|\cdot\|_{\tilde{\mathbf{E}}_T}$ we have $\|\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_1} \le CM_0$. Lemmas 2.10 and 4.4 yield
   $$\begin{aligned}\|\mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_3} &\le C(T^{\frac{3}{4}-\frac{\ell}{2}}\|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} + \|\widetilde{\mathbf{F}}_{\boldsymbol{\Upsilon}}(0)\|_{H^{1+\ell}(\tilde{\Omega}_f)^{d\times d}})\\ &\le C(1+M_0^{35}).\end{aligned}$$
   Lemma 2.10 and **P6** imply
   $$\|\mathcal{T}_4(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{S_4} \le CT^{\frac{3}{4}-\frac{\ell}{2}}\|\mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}) - \mathbf{I}\|_{H^1((0,T),H^{1+\ell}(\tilde{\Omega}_f)^{d\times d})} \le CT^{\frac{3}{4}-\frac{\ell}{2}}(1+M_0^2).$$

Hence, application of Lemma 2.12 in both cases yields
$$\|\widetilde{\mathcal{H}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)\|_{\tilde{\mathbf{H}}_T} \le CT^{1-\ell}(1+M_0^{38}).$$

- Estimation of $\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$:
  $\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau)$ is a sum of multilinear forms with factors
  $$\mathcal{T}_1(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (D_{\mathbf{z}}\,\tilde{\mathbf{v}})_{i,j},\ \mathcal{T}_2(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (\mathbf{I} - \mathrm{cof}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}))_{i,k},\ \mathcal{T}_3(\tilde{\mathbf{v}},\tilde{\mathbf{u}}_\tau) = (\mathrm{cof}(D_{\mathbf{z}}\,\tilde{\boldsymbol{\tau}}))_{k,j}$$

with $i, k, j \in \{1, \dots, d\}$.

1. $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))}$:
   Choose $s = s_1 = 0$, $s_2 = \ell$, $s_3 = 1$, $X = X_1 = X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)$. $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))} \leq C\|\tilde{\mathbf{v}}\|_{\tilde{\mathbf{E}}_T} \leq CM_0$ due to **P7** and (2.6), and with (4.17) we obtain the bound $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{L^2((0,T),H^{1+\ell}(\tilde{\Omega}_f))} \leq CT^{1-\ell}M_0(1 + M_0^2)$.

2. $\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f))}$:
   We choose $s = s_1 = \frac{\ell}{2}$, $s_2 = \ell$, $s_3 = 1$, $X = X_1 = H^1(\tilde{\Omega}_f)$, $X_2 = X_3 = H^{1+\ell}(\tilde{\Omega}_f)$. **P7** and (2.6) yield $\|\mathcal{T}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{S_1} \leq C\|\tilde{\mathbf{v}}\|_{H^{\frac{\ell}{2}}((0,T),H^2(\tilde{\Omega}_f)^d)} \leq CM_0$. Thus, with (4.17) and Lemmas 2.5, 2.12, 4.4 we obtain

$$\|\tilde{\mathcal{G}}_1(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau)\|_{H^{\frac{\ell}{2}}((0,T),H^1(\tilde{\Omega}_f))} \leq CT^{1-\ell}M_0(1 + M_0^2).$$

□

Theorem 4.6. ([59, Theorem 3]) Let Assumption 4.1 be fulfilled. Then, there exist $\epsilon_l > 0$, $T_l > 0$ and $M_l > 0$ such that for all $0 < T \leq T_l$ and for arbitrary $\tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{V}}_{\epsilon_l}$ the system (4.1) admits a unique solution $\tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau) := (\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{w}}(\tilde{\mathbf{u}}_\tau))$ on the relative interior of $\tilde{\mathbf{E}}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_l,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$. The mapping $\tilde{\mathbf{u}}_\tau \mapsto \tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau)$ is continuous and Fréchet differentiable on the interior of $\tilde{\mathbf{V}}_{\epsilon_l}$ and, for $\mathbf{h} \in \tilde{\mathbf{U}}$ the derivative $\tilde{\mathbf{y}}(\tilde{\mathbf{u}}_\tau)'\mathbf{h} := \delta_h\tilde{\mathbf{y}} = (\delta_h\tilde{\mathbf{v}}, \delta_h\tilde{p}, \delta_h\tilde{\mathbf{w}})$ is given as the solution of the system

$$\begin{aligned}
\partial_{\mathsf{t}}\, \delta_h\tilde{\mathbf{v}} - \nu\Delta_{\mathbf{z}}\, \delta_h\tilde{\mathbf{v}} + \nabla_{\mathbf{z}}\, \delta_h\tilde{p} = (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} & \\
+(\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_p\delta_h\tilde{p} + (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} & \quad \text{in } \tilde{Q}_f^T, \\
\operatorname{div}_{\mathbf{z}}(\delta_h\tilde{\mathbf{v}}) = (\tilde{\mathcal{G}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} + (\tilde{\mathcal{G}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} & \quad \text{in } \tilde{Q}_f^T, \\
\delta_h\tilde{\mathbf{v}}(\cdot, 0) = 0 & \quad \text{in } \tilde{\Omega}_f, \\
\delta_h\tilde{\mathbf{v}} = 0 & \quad \text{on } \tilde{\Sigma}_f^T \\
\delta_h\tilde{\mathbf{v}} = \partial_{\mathsf{t}}\, \delta_h\tilde{\mathbf{w}} & \quad \text{on } \tilde{\Sigma}_i^T, \\
\sigma_{f,\mathbf{z}}(\delta_h\tilde{\mathbf{v}}, \delta_h\tilde{p})\tilde{\mathbf{n}}_f = \sigma_{s,\mathbf{z}}(\delta_h\tilde{\mathbf{w}})\tilde{\mathbf{n}}_f + (\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\mathbf{v}}\delta_h\tilde{\mathbf{v}} & \\
+(\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_p\delta_h\tilde{p} + (\tilde{\mathcal{H}}(\tilde{\mathbf{v}}(\tilde{\mathbf{u}}_\tau), \tilde{p}(\tilde{\mathbf{u}}_\tau), \tilde{\mathbf{u}}_\tau))_{\boldsymbol{u}_\tau}\tilde{\mathbf{h}} & \quad \text{on } \tilde{\Sigma}_i^T, \\
\partial_{\mathsf{tt}}\, \delta_h\tilde{\mathbf{w}} - \operatorname{div}_{\mathbf{z}}(\sigma_{s,\mathbf{z}}(\delta_h\tilde{\mathbf{w}})) = 0 & \quad \text{in } \tilde{Q}_s^T, \\
\delta_h\tilde{\mathbf{w}} = 0 & \quad \text{on } \tilde{\Sigma}_s^T, \\
\delta_h\tilde{\mathbf{w}}(\cdot, 0) = 0, \quad \partial_{\mathsf{t}}\, \delta_h\tilde{\mathbf{w}}(\cdot, 0) = 0 & \quad \text{in } \tilde{\Omega}_s.
\end{aligned}$$

Proof. In the notation of Theorem 2.15, choose

$$\begin{aligned}
& y = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}), \quad z = \tilde{\mathbf{u}}_\tau, \\
& Y = \tilde{\mathbf{E}}_T \times \tilde{P}_T \times \tilde{\mathbf{W}}_T, \quad Z = \tilde{\mathbf{U}}, \\
& \mathcal{F}(y, z) = (\tilde{\mathcal{F}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau), \tilde{\mathcal{H}}(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{u}}_\tau), \tilde{\mathcal{G}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau), \tilde{\mathbf{g}}(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}_\tau), \tilde{\mathbf{v}}_0, \tilde{\mathbf{w}}_1).
\end{aligned}$$

Furthermore, let $W := \tilde{\mathbf{F}}_T \times \tilde{\mathbf{H}}_T \times \tilde{G}_T \times \tilde{\mathbf{G}}_T \times \tilde{\mathbf{V}}_0 \times \tilde{\mathbf{W}}_1$, and

$$\begin{aligned}\tilde{W} := \tilde{\mathbf{F}}_T &\times \{\tilde{\mathbf{h}} \in \tilde{\mathbf{H}}_T \ : \ \tilde{\mathbf{h}}(0) = 0\} \times \tilde{G}_T \\ &\times \{\tilde{\mathbf{g}} \in \tilde{\mathbf{G}}_T \cap H^1((0,T), H^\ell(\tilde{\Omega}_f)^d) \ : \ \tilde{\mathbf{g}}|_{\tilde{\Sigma}_f^T} = 0, \ \mathbf{g}(0) = 0\} \\ &\times \{(\tilde{\mathbf{v}}_0, \tilde{\mathbf{w}}_1) \in \tilde{\mathbf{V}}_0 \times \tilde{\mathbf{W}}_1 \ : \ \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_f} = 0, \ \ \mathrm{div}_{\mathbf{z}}(\tilde{\mathbf{v}}_0) = 0, \ \tilde{\mathbf{v}}_0|_{\tilde{\Gamma}_i} = \tilde{\mathbf{w}}_1|_{\tilde{\Gamma}_i}, \\ &\qquad 2\nu(\epsilon_{\mathbf{z}}(\tilde{\mathbf{v}}_0)) \cdot \boldsymbol{\tau} = 0 \text{ on } \tilde{\Gamma}_i \text{ for any unit vector } \boldsymbol{\tau} \text{ tangential to } \tilde{\Gamma}_i\},\end{aligned}$$

let $\rho = \rho(0)$ be given by Lemma 4.3. Lemma 3.22 defines the operator S and yields $T_f > 0$ and $L_S = C_S > 0$ such that $S \in \mathcal{L}(\tilde{W}, \tilde{\mathbf{E}}_T \times \tilde{P}_T \times \tilde{\mathbf{W}}_T)$ for $0 < T \leq T_f$ and

$$\|Sf\|_Y \ \leq L_S \|f\|_W$$

for all $f \in \tilde{W}$.
Theorem 3.24 and Lemma 4.5 yield constants $M_0 > 0$, $T_0 \in (0, T_f)$ and $\epsilon = \min(\rho, T_0^{\frac{1}{4} - \frac{l}{4}})$ such that, for all $0 < T \leq T_0$ and $z \in \tilde{\mathbf{V}}_\epsilon$ there exists a unique solution

$$y_0(z) \in \tilde{\mathbf{E}}_{T,M_0,\mathbf{v}_0} \times P_{T,M_0,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T,$$

which is a subset of the relative interior of $\tilde{\mathbf{E}}_{T,M_\ell,\tilde{\mathbf{v}}_0} \times P_{T,M_\ell,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ for $M_l > M_0$.
Furthermore, Theorem 3.24 and Lemma 4.5 yield $T_l \in (0, T_0]$ and $\epsilon_l \leq \min(\rho, T_l^{\frac{1}{4} - \frac{l}{4}})$ such that for all $z \in \tilde{\mathbf{V}}_{\epsilon_l}$ there exists a unique solution

$$y_l(z) \in \tilde{\mathbf{E}}_{T,M_\ell,\tilde{\mathbf{v}}_0} \times P_{T,M_\ell,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$$

and the required boundedness, continuity and Fréchet differentiability results are fulfilled for the choices $\tilde{Y} = \tilde{\mathbf{E}}_{T,M_\ell,\tilde{\mathbf{v}}_0} \times \tilde{P}_{T,M_\ell,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ and $\tilde{Z} = \tilde{\mathbf{V}}_{\epsilon_l}$ and $0 < T \leq T_l$.
Moreover, the proof of Theorem 3.24 implies that $S\mathcal{F}(y,z) \in \tilde{Y}$ for $(y,z) \in \tilde{Y} \times \tilde{Z}$. Since $y_0(z) \in \tilde{\mathbf{E}}_{T,M_\ell,\tilde{\mathbf{v}}_0} \times P_{T,M_\ell,\tilde{\mathbf{v}}_0} \times \tilde{\mathbf{W}}_T$ and the solution is unique, we have $y_0(z) = y_l(z)$ for all $z \in \tilde{Z}$. Thus, $y_l(z)$ is in the relative interior of $\tilde{Y}$ and Theorem 2.15 can be applied. □

5. Numerical Simulation of Shape Optimization for Unsteady FSI

In order to show the applicability of the method of mappings (cf. Section 2.7) for FSI problems we present numerical results (Section 5.6) for this approach applied to the FSI benchmark 2 (Sections 5.1, 5.2, 5.3), cf. [125]. Details on the numerical realization can be found in Sections 5.4 and 5.5.

5.1. FSI Model for Numerical Simulations

We consider the model (1.2), which was introduced in Section 1 and is fully described if the ALE transformation $\hat{\boldsymbol{\chi}} : \hat{\Omega} \times I \to \bigcup_{\mathsf{t}\in I} \check{\Omega}(\mathsf{t}) \times \{\mathsf{t}\}$ is defined.

5.1.1. ALE Transformation

There are several possibilities to choose the ALE transformation, e.g., using a fully Lagrangian approach or extending the solid displacement to the fluid domain.

Fully Lagrangian Approach

For the theoretical analysis the fully Lagrangian approach is chosen, i.e., the reference domain $\hat{\Omega}$ is given by the initial domain $\check{\Omega}(0)$ and the transformation is induced by the velocity field $\hat{\mathbf{v}}$, i.e.,

$$\hat{\boldsymbol{\chi}}(\cdot, \mathsf{t}) := \mathsf{y} + \int_0^{\mathsf{t}} \hat{\mathbf{v}}(\cdot, \mathsf{s}) d\mathsf{s}$$

for all $\mathsf{t} \in I$. This has several advantages. On the one hand, the contributions of the nonlinear term of the Navier-Stokes equations vanish on $\hat{\Omega}$. Additionally, no deformation variable on the fluid domain has to be introduced. However, it has drawbacks for numerical simulations, e.g., vortices in the flow might lead to mesh degeneration even though no solid displacement takes place. Therefore, we do not use the fully Lagrangian approach in the numerical implementation and dwell on other extension techniques, which are presented below.

Harmonic Extension

An approach to construct the ALE transformation is the extension of the solid displacement $\hat{\mathbf{w}}_s$ to the fluid reference domain, denoted by $\hat{\mathbf{w}}_f$. We define

$$\hat{\boldsymbol{\chi}}(\mathsf{y}, \mathsf{t}) := \mathsf{y} + \hat{\mathbf{w}}_f(\mathsf{y}, \mathsf{t})$$

for every $\mathsf{t} \in I$. One choice is given by the harmonic extension which is defined by

$$\begin{aligned} -\Delta_{\mathsf{y}} \hat{\mathbf{w}}_f &= 0 && \text{on } \hat{\Omega}_f \times I, \\ \hat{\mathbf{w}}_f &= 0 && \text{on } (\hat{\Gamma}_{fD} \cup \hat{\Gamma}_{fN}) \times I, \\ \hat{\mathbf{w}}_f &= \hat{\mathbf{w}}_s && \text{on } \hat{\Gamma}_i \times I. \end{aligned}$$

Numerical tests indicate that the harmonic extension is not robust if large mesh displacements occur [128]. Thus, smoother extensions have to be found.

Biharmonic Extension

A biharmonic extension, cf. e.g., [56], of the solid displacement to the fluid domain is given by

$$\begin{aligned} \Delta_{\mathsf{y}}^2 \hat{\mathbf{w}}_f &= 0 && \text{on } \hat{\Omega}_f \times I, \\ \hat{\mathbf{w}}_f &= 0 && \text{on } \left(\hat{\Gamma}_{fD} \cup \hat{\Gamma}_{fN}\right) \times I, \\ \nabla_{\mathsf{y}} \hat{\mathbf{w}}_f \cdot \mathbf{n}_f &= 0 && \text{on } \left(\hat{\Gamma}_{fD} \cup \hat{\Gamma}_{fN}\right) \times I, \\ \hat{\mathbf{w}}_f &= \hat{\mathbf{w}}_s && \text{on } \hat{\Gamma}_i \times I, \\ \nabla_{\mathsf{y}} \hat{\mathbf{w}}_f \cdot \mathbf{n}_f &= 0 && \text{on } \hat{\Gamma}_i \times I. \end{aligned}$$

For the solution of the discretized equations $H^2(\hat{\Omega}_f)$-conforming finite elements are needed. However, these elements are not necessarily implemented in standard finite element toolboxes. One way to circumvent this is the weak imposition of the continuity of normal derivates accross the finite element faces using a discontinuous Galerkin approach [49]. Another approach is the consideration of a mixed formulation of the biharmonic equation

$$\begin{aligned} -\Delta_{\mathsf{y}} \hat{\mathbf{w}}_f &= \hat{\mathbf{z}}_f && \text{on } \hat{\Omega}_f \times I, \\ -\alpha \Delta_{\mathsf{y}} \hat{\mathbf{z}}_f &= 0 && \text{on } \hat{\Omega}_f \times I, \\ \hat{\mathbf{w}}_f &= 0 && \text{on } \left(\hat{\Gamma}_{fD} \cup \hat{\Gamma}_{fN}\right) \times I, \\ \hat{\mathbf{w}}_f &= \hat{\mathbf{w}}_s && \text{on } \hat{\Gamma}_i \times I, \\ \nabla_{\mathsf{y}} \hat{\mathbf{w}}_f \cdot \mathbf{n}_f &= 0 && \text{on } \partial\hat{\Omega}_f \times I, \end{aligned}$$

where $\alpha > 0$ is chosen arbitrarily, see [128].

Remark 5.1. For numerical simulations, the extension has to be chosen sophisticatedly because of the discretization of the domain and the necessity of avoiding mesh degeneration. Therefore, it is an area of active research how to choose extensions that preserve mesh quality, see, e.g., [10, 37]. In general, one can also construct extension operators by hand that are not represented by (discretized) partial differential equations.

5.1.2. Strong ALE Formulation

A full description of the FSI equations with mixed biharmonic extension of the solid deformation to the fluid domain is given by

$$
\begin{aligned}
\hat{J}_{\boldsymbol{\chi}}\rho_f\partial_{\mathsf{t}}\hat{\mathbf{v}}_f + \hat{J}_{\boldsymbol{\chi}}\rho_f((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\hat{\mathbf{v}}_f - \partial_{\mathsf{t}}\hat{\mathbf{w}}_f))\cdot\nabla_{\mathsf{y}})\hat{\mathbf{v}}_f & \\
-\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) = \hat{J}_{\boldsymbol{\chi}}\rho_f\hat{\mathbf{f}}_f &\quad \text{on } \hat{\Omega}_f\times I,\\
\operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}\hat{\mathbf{v}}_f) = 0 &\quad \text{on } \hat{\Omega}_f\times I,\\
\hat{\mathbf{v}}_f = \hat{\mathbf{v}}_{fD} &\quad \text{on } \hat{\Gamma}_{fD}\times I,\\
\hat{J}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_f = \hat{\mathbf{g}}_f &\quad \text{on } \hat{\Gamma}_{fN}\times I,\\
\hat{\mathbf{v}}_f(\cdot,0) = \hat{\mathbf{v}}_{0f} &\quad \text{on } \hat{\Omega}_f,\\
\rho_s\partial_t\hat{\mathbf{v}}_s - \operatorname{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_s\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) = \rho_s\hat{\mathbf{f}}_s &\quad \text{on } \hat{\Omega}_s\times I,\\
\rho_s\partial_t\hat{\mathbf{w}}_s - \rho_s\hat{\mathbf{v}}_s = 0 &\quad \text{on } \hat{\Omega}_s\times I,\\
\hat{\mathbf{w}}_s = \hat{\mathbf{w}}_{sD} &\quad \text{on } \hat{\Gamma}_{sD}\times I,\\
\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_s\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_s = \hat{\mathbf{g}}_s &\quad \text{on } \hat{\Gamma}_{sN}\times I,\\
\hat{\mathbf{w}}_s(\cdot,0) = \hat{\mathbf{w}}_{0s} &\quad \text{on } \hat{\Omega}_s,\\
\hat{\mathbf{v}}_s(\cdot,0) = \hat{\mathbf{w}}_1 &\quad \text{on } \hat{\Omega}_s,\\
-\Delta_{\mathsf{y}}\hat{\mathbf{w}}_f = \hat{\mathbf{z}}_f &\quad \text{on } \hat{\Omega}_f\times I,\\
-\Delta_{\mathsf{y}}\hat{\mathbf{z}}_f = 0 &\quad \text{on } \hat{\Omega}_f\times I,\\
\hat{\mathbf{w}}_f = 0 &\quad \text{on } (\hat{\Gamma}_{fD}\cup\hat{\Gamma}_{fN})\times I,\\
\nabla_{\mathsf{y}}\hat{\mathbf{w}}_f\cdot\mathbf{n}_f = 0 &\quad \text{on } \partial\hat{\Omega}_f\times I,
\end{aligned}
\tag{5.1}
$$

with additional coupling conditions

$$
\begin{aligned}
\partial_{\mathsf{t}}\hat{\mathbf{w}}_s = \hat{\mathbf{v}}_s = \hat{\mathbf{v}}_f &\quad \text{on } \hat{\Gamma}_i\times I,\\
-\hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_f\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_f = \hat{J}_{\boldsymbol{\chi}}\hat{\boldsymbol{\sigma}}_s\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}\hat{\mathbf{n}}_s &\quad \text{on } \hat{\Gamma}_i\times I,\\
\hat{\mathbf{w}}_f = \hat{\mathbf{w}}_s &\quad \text{on } \hat{\Gamma}_i\times I.
\end{aligned}
$$

5.1.3. Weak ALE Formulation

We define the function spaces

$$
\begin{aligned}
&\hat{\mathbf{V}} = \mathbf{V}(\hat{\Omega}) \subset \{\hat{\mathbf{v}}\in H^1(\hat{\Omega},\mathbb{R}^d) \;:\; \hat{\mathbf{v}}|_{\hat{\Gamma}_{fD}} = \hat{\mathbf{v}}_{fD}\},\\
&\hat{\mathbf{V}}_0 = \mathbf{V}_0(\hat{\Omega}) \subset \{\hat{\mathbf{v}}\in H^1(\hat{\Omega},\mathbb{R}^d) \;:\; \hat{\mathbf{v}}|_{\hat{\Gamma}_{fD}} = 0\},\\
&\hat{\mathbf{W}} = \mathbf{W}(\hat{\Omega}) \subset \{\hat{\mathbf{w}}\in H^1(\hat{\Omega},\mathbb{R}^d) \;:\; \hat{\mathbf{w}}|_{\hat{\Gamma}_{fD}\cup\hat{\Gamma}_{fN}} = 0,\; \hat{\mathbf{w}}|_{\hat{\Gamma}_{sD}} = \hat{\mathbf{w}}_{sD}\},\\
&\hat{\mathbf{W}}_0 = \mathbf{W}_0(\hat{\Omega}) \subset \{\hat{\mathbf{w}}\in H^1(\hat{\Omega},\mathbb{R}^d) \;:\; \hat{\mathbf{w}}|_{\hat{\Gamma}_{fD}\cup\hat{\Gamma}_{fN}} = 0,\; \hat{\mathbf{w}}|_{\hat{\Gamma}_{sD}} = 0\},\\
&\hat{\mathbf{Z}}_f = \mathbf{Z}_f(\hat{\Omega}) \subset H^1(\hat{\Omega}_f,\mathbb{R}^d),
\end{aligned}
$$

$$\hat{\mathbf{Z}} = \mathbf{Z}(\hat{\Omega}) \subset H^1(\hat{\Omega}, \mathbb{R}^d),$$
$$\hat{P}_f = P_f(\hat{\Omega}) \subset \{\hat{p} \in L^2(\hat{\Omega}_f) \ : \ \int_{\hat{\Omega}_f} \hat{p} d\mathsf{y} = 0\},$$
$$\hat{P} = P(\hat{\Omega}) \subset \{\hat{p} \in L^2(\hat{\Omega}) \ : \ \int_{\hat{\Omega}} \hat{p} d\mathsf{y} = 0\}.$$

In addition, let $W_{2,q}(I, \hat{\mathbf{V}}) := \{\hat{\mathbf{v}} \in L^2(I, \hat{\mathbf{V}}) \ : \ \hat{\mathbf{v}}_\mathsf{t} \in L^q(I, \hat{\mathbf{V}}^*)\}$, where $q > 0$ and $\hat{\mathbf{V}}^*$ denotes the dual space of $\hat{\mathbf{V}}$. The weak formulation of (5.1) is given by:
Find $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}) \in W_{2,2}(I, \hat{\mathbf{V}}) \times L^2(I, \hat{P}_f) \times W_{2,2}(I, \hat{\mathbf{W}}) \times L^2(I, \hat{\mathbf{Z}}_f)$ such that $\hat{\mathbf{v}}(\cdot, 0) = \hat{\mathbf{v}}_0$, $\hat{\mathbf{w}}(\cdot, 0) = \hat{\mathbf{w}}_0$ and

$$\begin{aligned}
&\langle A(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}), (\hat{\boldsymbol{\psi}^v}, \hat{\psi^p}, \hat{\boldsymbol{\psi}^w}, \hat{\boldsymbol{\psi}^z})\rangle := (\hat{J}_{\boldsymbol{\chi}} \rho_f \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} \\
&\quad + (\hat{J}_{\boldsymbol{\chi}} \rho_f ((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} (\hat{\mathbf{v}} - \partial_\mathsf{t} \hat{\mathbf{w}})) \cdot \nabla_\mathsf{y}) \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} + (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} - (\hat{\mathbf{g}}_f, \hat{\boldsymbol{\psi}^v})_{\hat{\Gamma}_{fN}} \\
&\quad - (\hat{J}_{\boldsymbol{\chi}} \rho_f \hat{\mathbf{f}}_f, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} + (\rho_s \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} + (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_s \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} - (\hat{\mathbf{g}}_s, \hat{\boldsymbol{\psi}^v})_{\hat{\Gamma}_{sN}} \\
&\quad - (\hat{J}_{\boldsymbol{\chi}} \rho_s \hat{\mathbf{f}}_s, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} + (\rho_s (\partial_\mathsf{t} \hat{\mathbf{w}} - \hat{\mathbf{v}}), \hat{\boldsymbol{\psi}^w})_{\hat{\Omega}_s} + \alpha_w (D_\mathsf{y} \hat{\mathbf{z}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^w})_{\hat{\Omega}_f} + (D_\mathsf{y} \hat{\mathbf{w}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^z})_{\hat{\Omega}_f} \\
&\quad - (\hat{\mathbf{z}}, \hat{\boldsymbol{\psi}^z})_{\hat{\Omega}_f} + (\mathrm{div}_\mathsf{y} (\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\mathbf{v}}), \hat{\psi^p})_{\hat{\Omega}_f} = 0,
\end{aligned}$$

for all $(\hat{\boldsymbol{\psi}^v}, \hat{\psi^p}, \hat{\boldsymbol{\psi}^w}, \hat{\boldsymbol{\psi}^z}) \in \hat{\mathbf{V}}_0 \times \hat{P}_f \times \hat{\mathbf{W}}_0 \times \hat{\mathbf{Z}}_f$ and a.e. $\mathsf{t} \in I$. Here, $\alpha_w > 0$ is a small constant. Since we want to work with functions defined on the whole domain, we consider the modified weak formulation:
Find $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}) \in W_{2,2}(I, \hat{\mathbf{V}}) \times L^2(I, \hat{P}) \times W_{2,2}(I, \hat{\mathbf{W}}) \times L^2(I, \hat{\mathbf{Z}})$ such that $\hat{\mathbf{v}}(\cdot, 0) = \hat{\mathbf{v}}_0$, $\hat{\mathbf{w}}(\cdot, 0) = \hat{\mathbf{w}}_0$ and

$$\begin{aligned}
&\langle \hat{A}_\Omega(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}), (\hat{\boldsymbol{\psi}^v}, \hat{\psi^p}, \hat{\boldsymbol{\psi}^w}, \hat{\boldsymbol{\psi}^z})\rangle := (\hat{J}_{\boldsymbol{\chi}} \rho_f \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} \\
&\quad + (\hat{J}_{\boldsymbol{\chi}} \rho_f ((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} (\hat{\mathbf{v}} - \partial_\mathsf{t} \hat{\mathbf{w}})) \cdot \nabla_\mathsf{y}) \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} + (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} - (\hat{\mathbf{g}}_f, \hat{\boldsymbol{\psi}^v})_{\hat{\Gamma}_{fN}} \\
&\quad - (\hat{J}_{\boldsymbol{\chi}} \rho_f \hat{\mathbf{f}}_f, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_f} + (\rho_s \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} + (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_s \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} - (\hat{\mathbf{g}}_s, \hat{\boldsymbol{\psi}^v})_{\hat{\Gamma}_{sN}} \\
&\quad - (\hat{J}_{\boldsymbol{\chi}} \rho_s \hat{\mathbf{f}}_s, \hat{\boldsymbol{\psi}^v})_{\hat{\Omega}_s} + (\rho_s (\partial_\mathsf{t} \hat{\mathbf{w}} - \hat{\mathbf{v}}), \hat{\boldsymbol{\psi}^w})_{\hat{\Omega}_s} + \alpha_w (D_\mathsf{y} \hat{\mathbf{z}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^w})_{\hat{\Omega}_f} + (D_\mathsf{y} \hat{\mathbf{w}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^z})_{\hat{\Omega}} \\
&\quad - (\hat{\mathbf{z}}, \hat{\boldsymbol{\psi}^z})_{\hat{\Omega}} + \alpha_p (\nabla_\mathsf{y} \hat{p}, \nabla_\mathsf{y} \hat{\psi^p})_{\hat{\Omega}_s} + (\mathrm{div}_\mathsf{y} (\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\mathbf{v}}), \hat{\psi^p})_{\hat{\Omega}_f} = 0,
\end{aligned} \tag{5.2}$$

for all $(\hat{\boldsymbol{\psi}^v}, \hat{\psi^p}, \hat{\boldsymbol{\psi}^w}, \hat{\boldsymbol{\psi}^z}) \in \hat{\mathbf{V}}_0 \times \hat{P} \times \hat{\mathbf{W}}_0 \times \hat{\mathbf{Z}}$ and a.e. $\mathsf{t} \in I$. $\alpha_p > 0$ denotes a small constant. The corresponding formulation on the space-time cylinder reads as follows:
Find $(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}) \in W_{2,2}(I, \hat{\mathbf{V}}) \times L^2(I, \hat{P}) \times W_{2,2}(I, \hat{\mathbf{W}}) \times L^2(I, \hat{\mathbf{Z}})$ such that $\hat{\mathbf{v}}(\cdot, 0) = \hat{\mathbf{v}}_0$, $\hat{\mathbf{w}}(\cdot, 0) = \hat{\mathbf{w}}_0$ and

$$\begin{aligned}
&\langle \hat{A}_{Q^T}(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{w}}, \hat{\mathbf{z}}), (\hat{\boldsymbol{\psi}^v}, \hat{\psi^p}, \hat{\boldsymbol{\psi}^w}, \hat{\boldsymbol{\psi}^z})\rangle := (\!(\hat{J}_{\boldsymbol{\chi}} \rho_f \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_f^T} \\
&\quad + (\!(\hat{J}_{\boldsymbol{\chi}} \rho_f ((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} (\hat{\mathbf{v}} - \partial_\mathsf{t} \hat{\mathbf{w}})) \cdot \nabla_\mathsf{y}) \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_f^T} + (\!(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_f^T} - (\!(\hat{\mathbf{g}}_f, \hat{\boldsymbol{\psi}^v})\!)_{\hat{\Sigma}_{fN}^T} \\
&\quad - (\!(\hat{J}_{\boldsymbol{\chi}} \rho_f \hat{\mathbf{f}}_f, \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_f^T} + (\!(\rho_s \partial_\mathsf{t} \hat{\mathbf{v}}, \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_s^T} + (\!(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_s \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_\mathsf{y} \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_s^T} - (\!(\hat{\mathbf{g}}_s, \hat{\boldsymbol{\psi}^v})\!)_{\hat{\Sigma}_{sN}^T} \\
&\quad - (\!(\hat{J}_{\boldsymbol{\chi}} \rho_s \hat{\mathbf{f}}_s, \hat{\boldsymbol{\psi}^v})\!)_{\hat{Q}_s^T} + (\!(\rho_s (\partial_\mathsf{t} \hat{\mathbf{w}} - \hat{\mathbf{v}}), \hat{\boldsymbol{\psi}^w})\!)_{\hat{Q}_s^T} + \alpha_w (\!(D_\mathsf{y} \hat{\mathbf{z}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^w})\!)_{\hat{Q}_f^T} + (\!(D_\mathsf{y} \hat{\mathbf{w}}, D_\mathsf{y} \hat{\boldsymbol{\psi}^z})\!)_{\hat{Q}^T}
\end{aligned}$$

$$- (\!(\hat{\mathbf{z}}, \hat{\boldsymbol{\psi}}^z)\!)_{\hat{Q}^T} + \alpha_p (\!(\nabla_{\mathsf{y}} \hat{p}, \nabla_{\mathsf{y}} \hat{\psi}^p)\!)_{\hat{Q}_s^T} + (\!(\mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\mathbf{v}}), \hat{\psi}^p)\!)_{\hat{Q}_f^T} = 0,$$

for all $(\hat{\boldsymbol{\psi}}^v, \hat{\psi}^p, \hat{\boldsymbol{\psi}}^w, \hat{\boldsymbol{\psi}}^z) \in W_{2,2}(I, \hat{\mathbf{V}}_0) \times L^2(I, \hat{P}) \times W_{2,2}(I, \hat{\mathbf{W}}_0) \times L^2(I, \hat{\mathbf{Z}})$.

Remark 5.2. Since the definition of ∇_{y} on vector-valued functions and $\mathrm{div}_{\mathsf{y}}$ on matrix-valued functions is not uniform in the literature (which is the reason why we introduced the operator D_{y} in the introduction, see Chapter 1), we verify a well-known result for our specific choice of these operators. More precisely, we derive the weak formulation of the vector-valued divergence term. We have

$$\int_{\partial\hat{\Omega}_f} \boldsymbol{\psi}^{v\top} \hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top} \hat{\mathbf{n}}_f ds_{\mathsf{y}} = \int_{\hat{\Omega}_f} \mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\boldsymbol{\sigma}}_f^\top \boldsymbol{\psi}^v) d\mathsf{y}$$

and it holds that

$$\begin{aligned}
&\mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\boldsymbol{\sigma}}_f^\top \boldsymbol{\psi}^v) = \sum_i \partial_{\mathsf{y}_i} (\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\boldsymbol{\sigma}}_f^\top \boldsymbol{\psi}^v)_i = \sum_{i,j} \partial_{\mathsf{y}_i} (\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\boldsymbol{\sigma}}_f^\top)_{i,j} \boldsymbol{\psi}_j^v \\
&+ \sum_{i,j} (\hat{J}_{\boldsymbol{\chi}} \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} \hat{\boldsymbol{\sigma}}_f^\top)_{i,j} \partial_{\mathsf{y}_i} \boldsymbol{\psi}_j^v = \sum_j (\mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}))_j \boldsymbol{\psi}_j^v + \sum_{i,j} (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top})_{j,i} \partial_{\mathsf{y}_i} \boldsymbol{\psi}_j^v \\
&= \sum_j (\mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}))_j \boldsymbol{\psi}_j^v + (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}) : D_{\mathsf{y}} \boldsymbol{\psi}^v.
\end{aligned}$$

Therefore, we obtain that

$$(\mathrm{div}_{\mathsf{y}}(\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}), \boldsymbol{\psi}^v)_{\hat{\Omega}_f} = (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top} \hat{\mathbf{n}}_f, \boldsymbol{\psi}^v)_{\partial\hat{\Omega}_f} - (\hat{J}_{\boldsymbol{\chi}} \hat{\boldsymbol{\sigma}}_f \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}, D_{\mathsf{y}} \boldsymbol{\psi}^v)_{\hat{\Omega}_f}.$$

5.2. FSI benchmark 2

To be able to validate our numerical implementation, we work on the FSI benchmark 2, which was proposed in [125]. This benchmark considers the coupling of the Navier-Stokes equations and Saint Venant-Kirchhoff type material equations in a two-dimensional rectangular domain of length $l = 2.5$ and height $h = 0.41$, the bottom left corner of which is located at the origin $(0, 0)^\top$. On the left boundary $\hat{\Gamma}_{fDi}$ we have a parabolic inflow given by

$$\hat{\mathbf{v}}_{fD}((0, \mathsf{z}_2)^\top, \mathsf{t}) = \begin{cases} (3\bar{v}h^{-2}\mathsf{z}_2(h - \mathsf{z}_2)(1 - \cos(\frac{\pi}{2}\mathsf{t})), 0)^\top & \text{if } \mathsf{t} < 2.0, \\ (6\bar{v}h^{-2}\mathsf{z}_2(h - \mathsf{z}_2), 0)^\top & \text{otherwise,} \end{cases}$$

with mean inflow velocity $\bar{v}$. Moreover, no slip condition on the bottom and top $\hat{\Gamma}_{fD0}$, as well as, do-nothing boundary conditions on the right boundary $\hat{\Gamma}_{fN}$, i.e., $\hat{\mathbf{g}}_f = 0$, are imposed. In this pipe, there is a circular obstacle with radius $r = 0.05$ centered at $(0.2, 0.2)$ to which an elastic beam of length 0.4 and width 0.02 is attached as illustrated in Figure 5.1. The boundary $\hat{\Gamma}_d = \hat{\Gamma}_{fDc} \cup \hat{\Gamma}_{sDc}$ of the obstacle with $\hat{\Gamma}_{fDc} = \hat{\Gamma}_d \cap \partial\hat{\Omega}_f$ and $\hat{\Gamma}_{sDc} = \hat{\Gamma}_d \cap \partial\hat{\Omega}_s$ serves as design boundary. More precisely, we want to optimize its shape such that the drag is minimized. On $\hat{\Gamma}_d$ homogeneous Dirichlet boundary conditions are imposed on the fluid

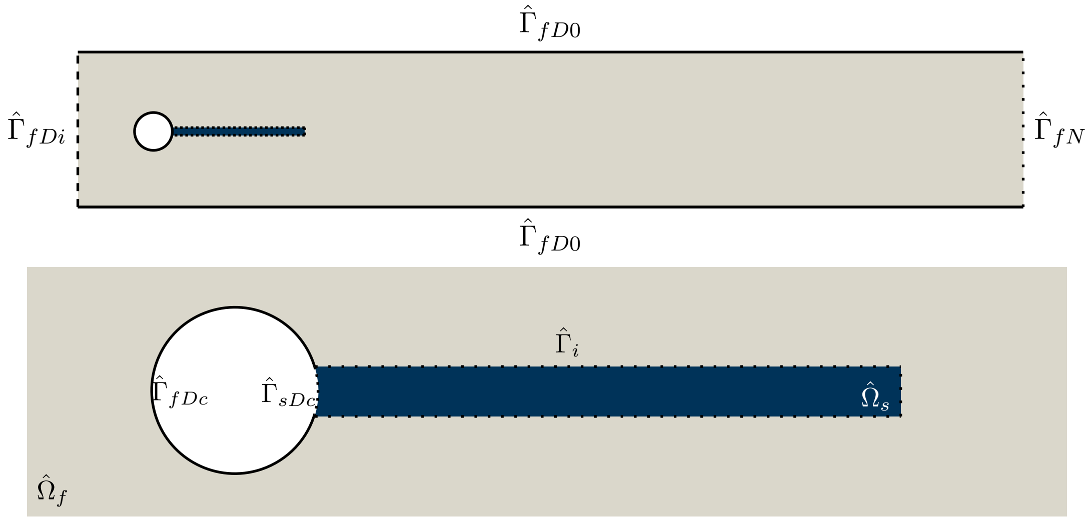


Figure 5.1.: Shape reference domain $\hat{\Omega}$

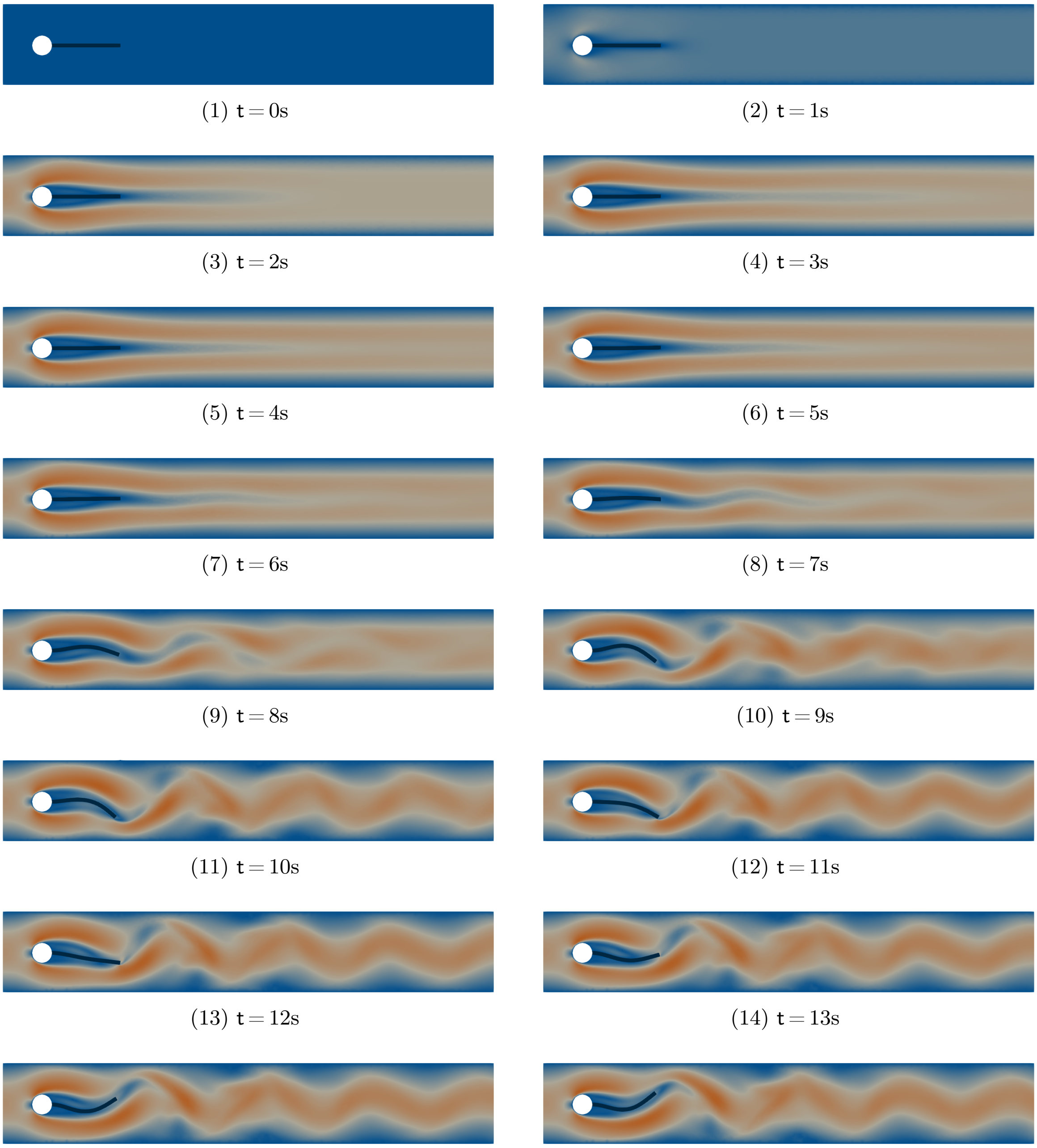
velocity and solid displacement. The initial conditions are set to 0. Thus, the fluid-structure system is completely determined by the parameters $\rho_s = 1 \cdot 10^4$, $\lambda_s = 2 \cdot 10^6$, $\mu_s = 5 \cdot 10^5$, $\rho_f = 1 \cdot 10^3$, $\nu_f = 1 \cdot 10^{-3}$, $\bar{v} = 1$ and $\alpha_p = \alpha_w = 1 \cdot 10^{-9}$. Figure 5.2 shows snapshots of the simulation of the FSI benchmark 2. Details on the implementation are given in the following sections.

5.3. Model of Shape Optimization Problem for FSI

In this section we model the shape optimization problem (Section 5.3.4) for the unsteady, nonlinear FSI system (5.2) via the method of mappings. For this purpose, we transform the FSI equations to the nominal domain (Section 5.3.1), choose a set of admissible transformations (Section 5.3.3) and an objective function (Section 5.3.2).

5.3.1. Transformation of FSI Equations to Nominal Domain

The method of mappings, which was introduced in Section 2.7, can be applied to shape optimization problems that are governed by the FSI equations formulated on the ALE reference domain $\hat{\Omega}$. This reference domain is obtained from the actual physical domain via the homeomorphism $\hat{\boldsymbol{\chi}}^{-1}(\cdot, \mathsf{t}) : \check{\Omega}(\mathsf{t}) \to \hat{\Omega}$. For the method of mappings we have to apply an additional transformation $\tilde{\boldsymbol{\tau}} : \tilde{\Omega} \to \hat{\Omega}$, which is a bi-Lipschitz transformation from the nominal domain $\tilde{\Omega}$ to the ALE reference domain $\hat{\Omega}$. Thus, the physical domain $\check{\Omega}$ can be obtained from the shape reference domain $\tilde{\Omega}$ by the composition of the transformations $\tilde{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{\chi}}$, which is visualized in Figure 5.3. Since we want to optimize the shape of the domain and not the initial conditions or boundary conditions, we assume that the considered transformations in $\tilde{\mathcal{T}}_{ad}$ do not change these conditions. In particular, this means that we do not have to transform the appearing boundary integrals in the weak formulation of the monolithic ALE formulation for the FSI problem. For the sake of convenience, and in correspondence with



(1) t = 0s
(2) t = 1s
(3) t = 2s
(4) t = 3s
(5) t = 4s
(6) t = 5s
(7) t = 6s
(8) t = 7s
(9) t = 8s
(10) t = 9s
(11) t = 10s
(12) t = 11s
(13) t = 12s
(14) t = 13s
(15) t = 14s
(16) t = 15s

Figure 5.2.: Snapshots of the simulation of FSI benchmark 2

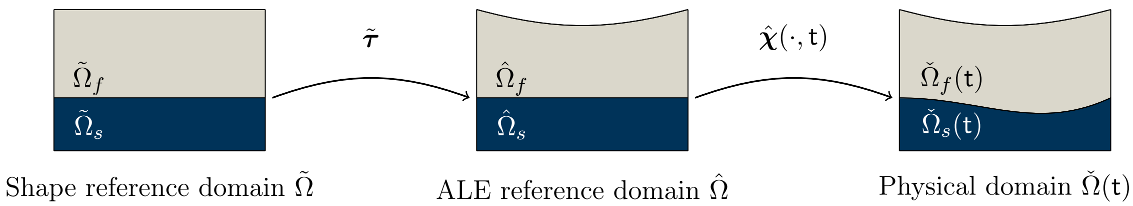


Figure 5.3.: Schematic illustration of the method of mappings combined with an ALE transformation

our numerical setting, we choose $\hat{\mathbf{g}}_f = 0$ and $\hat{\mathbf{g}}_s = 0$. Additionally, we have to ensure that the transformation is equal to the identity on the support of the initial conditions.

For fixed $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$, we introduce the spaces on the shape reference domain

$$\begin{aligned}
\tilde{\mathbf{V}}_{\tilde{\boldsymbol{\tau}}} &= \{\hat{\mathbf{v}} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{\mathbf{v}} \in \mathbf{V}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\},\\
\tilde{\mathbf{V}}_{0,\tilde{\boldsymbol{\tau}}} &= \{\hat{\mathbf{v}} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{\mathbf{v}} \in \mathbf{V}_0(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\},\\
\tilde{\mathbf{W}}_{\tilde{\boldsymbol{\tau}}} &= \{\hat{\mathbf{w}} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{\mathbf{w}} \in \mathbf{W}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\},\\
\tilde{\mathbf{W}}_{0,\tilde{\boldsymbol{\tau}}} &= \{\hat{\mathbf{w}} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{\mathbf{w}} \in \mathbf{W}_0(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\},\\
\tilde{P}_{\tilde{\boldsymbol{\tau}}} &= \{\hat{p} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{p} \in P(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\},\\
\tilde{\mathbf{Z}}_{\tilde{\boldsymbol{\tau}}} &= \{\hat{\mathbf{z}} \circ \tilde{\boldsymbol{\tau}} \;:\; \hat{\mathbf{z}} \in \mathbf{Z}(\tilde{\boldsymbol{\tau}}(\tilde{\Omega}))\}.
\end{aligned}$$

The additional transformation with $\tilde{\boldsymbol{\tau}}$ yields the following weak formulation on the shape reference domain $\tilde{\Omega}$.

For fixed $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$, find $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}) \in W_{2,2}(I, \tilde{\mathbf{V}}_{\tilde{\boldsymbol{\tau}}}) \times L^2(I, \tilde{P}_{\tilde{\boldsymbol{\tau}}}) \times W_{2,2}(I, \tilde{\mathbf{W}}_{\tilde{\boldsymbol{\tau}}}) \times L^2(I, \tilde{\mathbf{Z}}_{\tilde{\boldsymbol{\tau}}})$ such that $\tilde{\mathbf{v}}(\cdot, 0) = 0$, $\tilde{\mathbf{w}}(\cdot, 0) = 0$ and

$$\begin{aligned}
&\langle \tilde{A}_\Omega((\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}), \tilde{\boldsymbol{\tau}}), (\tilde{\boldsymbol{\psi}}^v, \tilde{\psi}^p, \tilde{\boldsymbol{\psi}}^w, \tilde{\boldsymbol{\psi}}^z)\rangle\\
&:= (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f \partial_{\mathsf{t}}\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f}\\
&\quad + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f(((D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\tilde{\mathbf{v}} - \partial_{\mathsf{t}}\tilde{\mathbf{w}})) \cdot \nabla_{\mathsf{z}})\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f}\\
&\quad + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\tilde{\boldsymbol{\sigma}}_f\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} - (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f\tilde{\mathbf{f}}_f, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f}\\
&\quad + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho_s\partial_{\mathsf{t}}\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\tilde{\boldsymbol{\sigma}}_s\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s}\\
&\quad - (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_s\tilde{\mathbf{f}}_s, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho_s(\partial_{\mathsf{t}}\tilde{\mathbf{w}} - \tilde{\mathbf{v}}), \tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_s}\\
&\quad + \alpha_w(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})D_{\mathsf{z}}\tilde{\mathbf{z}}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_f}\\
&\quad + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})D_{\mathsf{y}}\tilde{\mathbf{w}}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top}, D_{\mathsf{y}}\tilde{\boldsymbol{\psi}}^z)_{\tilde{\Omega}}\\
&\quad - (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{\mathbf{z}}, \tilde{\boldsymbol{\psi}}^z)_{\tilde{\Omega}} + \alpha_p(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top}\nabla_{\mathsf{z}}\tilde{p}, \nabla_{\mathsf{z}}\tilde{\psi}^p)_{\tilde{\Omega}_s}\\
&\quad + (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\mathrm{tr}(D_{\mathsf{z}}(\tilde{J}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}\tilde{\mathbf{v}})(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}), \tilde{\psi}^p)_{\tilde{\Omega}_f} = 0,
\end{aligned} \tag{5.3}$$

for all $(\tilde{\boldsymbol{\psi}}^v, \tilde{\psi}^p, \tilde{\boldsymbol{\psi}}^w, \tilde{\boldsymbol{\psi}}^z) \in \tilde{\mathbf{V}}_0 \times \tilde{P} \times \tilde{\mathbf{W}}_0 \times \tilde{\mathbf{Z}}$ and any $\mathtt{t} \in I$, where $\tilde{J}_{\boldsymbol{\chi}} = \det(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}})$ and $\widetilde{\mathbf{F}}_{\boldsymbol{\chi}} = D_{\mathsf{z}}\hat{\boldsymbol{\chi}}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}$,

$$\tilde{\boldsymbol{\sigma}}_f := \rho_f \nu_f (D_{\mathsf{z}}\tilde{\mathbf{v}}_f(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} + \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-\top} D_{\mathsf{y}}\tilde{\mathbf{v}}_f^\top) - \tilde{p}_f \mathbf{I}$$

denotes the transformed fluid stress tensor and $\tilde{\sigma}_s$ the corresponding transformed solid stress tensor. For Saint Venant-Kirchhoff type material, it is given by

$$\tilde{\boldsymbol{\sigma}}_s = \tilde{J}_{\boldsymbol{\chi}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}(\lambda_s \mathrm{tr}(\widetilde{\mathbf{E}}_{\boldsymbol{\chi}})\mathbf{I} + 2\mu_s \widetilde{\mathbf{E}}_{\boldsymbol{\chi}})\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^\top$$

with $\widetilde{\mathbf{E}}_{\boldsymbol{\chi}} := \frac{1}{2}(\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^\top \widetilde{\mathbf{F}}_{\boldsymbol{\chi}} - \mathbf{I})$. The corresponding operator is denoted by

$$\tilde{A}_\Omega(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}}) = 0, \tag{5.4}$$

where $\tilde{\mathbf{y}} = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}})$. Moreover, let

$$\tilde{\mathbf{Y}}_{\tilde{\boldsymbol{\tau}}} := W_{2,2}(I, \tilde{\mathbf{V}}_{\tilde{\boldsymbol{\tau}}}) \times L^2(I, \tilde{P}_{\tilde{\boldsymbol{\tau}}}) \times W_{2,2}(I, \tilde{\mathbf{W}}_{\tilde{\boldsymbol{\tau}}}) \times L^2(I, \tilde{\mathbf{Z}}_{\tilde{\boldsymbol{\tau}}}).$$

5.3.2. Choice of Objective Function

As objective function, we choose the mean fluid drag which is given by

$$-\frac{1}{T}\int_0^T \int_{\check{\Gamma}_o(\mathtt{t})} \check{\boldsymbol{\psi}}^\top \boldsymbol{\sigma}_{f,\mathsf{x}}(\check{\mathbf{v}}_f, \check{p}_f)\check{\mathbf{n}}_f dS(\mathsf{x})d\mathtt{t}$$

where $\check{\Gamma}_o = \check{\Gamma}_{fDc} \cup \check{\Gamma}_i$, $\check{\Gamma}_{fDc} = \hat{\boldsymbol{\chi}}(\hat{\Gamma}_{fDc})$, $\check{\Gamma}_i = \hat{\boldsymbol{\chi}}(\hat{\Gamma}_i)$, $\check{\boldsymbol{\psi}} = (1,0)^\top$ and $\check{\mathbf{n}}_f$ denotes the outwards pointing normal vector, e.g., [20, 74, 75]. This can be reformulated as a volume integral given by

$$-\frac{1}{T}\int_0^T ((\rho(\partial_\mathtt{t}\check{\mathbf{v}}_f + \check{\mathbf{v}}_f \cdot \nabla_\mathsf{x}\check{\mathbf{v}}), \check{\boldsymbol{\Psi}})_{\check{\Omega}_f(\mathtt{t})} - (\check{p}_f, \mathrm{div}(\check{\boldsymbol{\Psi}}))_{\check{\Omega}_f(\mathtt{t})} + (2\nu\epsilon_\mathsf{x}(\check{\mathbf{v}}_f), \epsilon_\mathsf{x}(\check{\boldsymbol{\Psi}}))_{\check{\Omega}_f(\mathtt{t})})d\mathtt{t},$$

where $\epsilon_\mathsf{x}(\cdot) = \frac{1}{2}(\nabla_\mathsf{x}\cdot + (\nabla_\mathsf{x}\cdot)^\top)$, and $\check{\boldsymbol{\Psi}}$ is an arbitrary function such that $\check{\boldsymbol{\Psi}}|_{\check{\Gamma}_o} = \check{\boldsymbol{\psi}}$ and $\check{\boldsymbol{\Psi}}|_{\partial\check{\Omega}_f \setminus \check{\Gamma}_o} = 0$, cf. [18, 66, 74, 75].

Remark 5.3. Analogously to the observation in [64], where the evaluation of the shape gradient via surface integrals is compared to the evaluation via volume integrals, using the volume integral formulation for the drag is expected to provide better numerical results. Our numerical tests confirm this.

The corresponding transformed formulation on the ALE domain $\hat{\Omega}_f$ reads as

$$\begin{aligned}\widehat{F}_D(\hat{\boldsymbol{y}}) = &-\frac{1}{T}\int_0^T ((\hat{J}_{\boldsymbol{\chi}}\rho(\partial_\mathtt{t}\hat{\mathbf{v}}_f + ((\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\hat{\mathbf{v}}_f - \partial_\mathtt{t}\hat{\boldsymbol{\chi}})) \cdot \nabla_\mathsf{y})\hat{\mathbf{v}}_f, \hat{\boldsymbol{\Psi}}))_{\hat{\Omega}_f} - (\hat{J}_{\boldsymbol{\chi}}\hat{p}_f, \mathrm{tr}(D_\mathsf{y}\hat{\boldsymbol{\Psi}}\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}))_{\hat{\Omega}_f} \\ &+ (2\nu\hat{J}_{\boldsymbol{\chi}}\epsilon_\mathsf{y}(\hat{\mathbf{v}}_f), \epsilon_\mathsf{y}(\hat{\mathbf{v}}_f))_{\hat{\Omega}_f})d\mathtt{t}\end{aligned}$$

with $\hat{J}_{\boldsymbol{\chi}} = \det \widehat{\mathbf{F}}_{\boldsymbol{\chi}}$, and the transformation on the shape reference domain $\tilde{\Omega}_f$ yields

$$\begin{aligned}\tilde{F}_D(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}}) = &- \frac{1}{T}\int_0^T ((\tilde{J}_{\boldsymbol{\chi}} \det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho(\partial_{\mathsf{t}}\tilde{\mathbf{v}}_f + ((D_{\mathsf{z}}\tilde{\boldsymbol{\tau}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\tilde{\mathbf{v}}_f - \partial_{\mathsf{t}}\tilde{\boldsymbol{\chi}})) \cdot \nabla_{\mathsf{z}})\tilde{\mathbf{v}}_f), \tilde{\boldsymbol{\Psi}})_{\tilde{\Omega}_f} \\ &- (\tilde{J}_{\boldsymbol{\chi}} \det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{p}, \operatorname{tr}(D_{\mathsf{z}}\tilde{\boldsymbol{\Psi}} D_{\mathsf{z}}\tilde{\boldsymbol{\tau}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}))_{\tilde{\Omega}_f} \\ &+ (2\nu \tilde{J}_{\boldsymbol{\chi}} \det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\epsilon_{\mathsf{z}}(\tilde{\mathbf{v}}_f), \epsilon_{\mathsf{z}}(\tilde{\boldsymbol{\Psi}}))_{\tilde{\Omega}_f})d\mathsf{t}\end{aligned}$$

with $\epsilon_{\mathsf{y}}(\cdot) = \frac{1}{2}((D_{\mathsf{y}}\,\cdot)\widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} + \widehat{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{y}}\,\cdot)^{\top})$ and $\epsilon_{\mathsf{z}}(\cdot) = \frac{1}{2}((D_{\mathsf{z}}\,\cdot)D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1} + \widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}D_{\mathsf{z}}\,\tilde{\boldsymbol{\tau}}^{-\top}(D_{\mathsf{z}}\,\cdot)^{\top})$.

5.3.3. Choice of Admissible Shape Transformations

As nominal domain $\tilde{\Omega}$ we choose the domain introduced in Section 5.2. As design part of the boundary we choose the boundary of the obstacle, i.e, $\tilde{\Gamma}_d := \tilde{\Gamma}_{fDc} \cup \tilde{\Gamma}_{sDc}$. The choice of admissible transformations should be deliberate.

- As already mentioned in the previous section, it is important to choose the transformations such that they do not change initial conditions, boundary conditions or source terms, i.e., the support of the deformation $\tilde{\boldsymbol{\tau}} - \mathrm{id}_{\mathsf{z}}$ is disjoint from the support of the initial conditions, boundary conditions and source terms. In case that the design part $\tilde{\Gamma}_d$ is a subset of the Dirichlet boundary, the boundary conditions have to be homogeneous on $\tilde{\Gamma}_d$.

- Since standard existence theory for PDEs requires Lipschitz regularity of the domain, it is straightforward to require the domains to be Lipschitzian during the optimization process. This can be ensured by choosing $\tilde{\Omega}$ as a Lipschitz domain and transformations $\tilde{\boldsymbol{\tau}} \in W^{1,\infty}(\tilde{\Omega})^d$ close to the identity [13, Lem. 2].

- Remark 2.22 motivates another restriction on the regularity of $\tilde{\boldsymbol{\tau}}$. To be able to work with the same function spaces on the shape reference domain independently of the control $\tilde{\boldsymbol{\tau}}$, it is desirable that the spaces for the transformed functions are isomorphic to the spaces on the transformed domain. This means, $\tilde{\mathbf{Y}}_{\tilde{\boldsymbol{\tau}}} = \tilde{\mathbf{Y}}(\tilde{\Omega}) = \tilde{\mathbf{Y}}$ can be considered independently of $\tilde{\boldsymbol{\tau}}$ for $\tilde{\boldsymbol{\tau}} \in \tilde{\mathcal{T}}_{ad}$. Thus, the regularity requirement on $\tilde{\boldsymbol{\tau}}$ depends on the regularity of the state of the partial differential equations and $\tilde{\mathcal{T}}_{ad} \subset \tilde{\mathbf{D}}_{\Omega} \subset W^{1,\infty}(\tilde{\Omega})^d$ for a function space $\tilde{\mathbf{D}}_{\Omega}$ with sufficiently high regularity.

  Remark 5.4. For this reason, we need $\tilde{\boldsymbol{\tau}} \in H^{2+\ell}(\tilde{\Omega}_f, \mathbb{R}^d)$ in the theoretical analysis in Chapter 4 to ensure that $\tilde{\mathbf{v}} \in \tilde{\mathbf{E}}_T$ if and only if $\hat{\mathbf{v}} = \tilde{\mathbf{v}} \circ \tilde{\boldsymbol{\tau}}^{-1} \in \hat{\mathbf{E}}_T$, cf. [73, Lem. B.5, B.6].

- Transformations that only change the interior of the domain but not the boundaries do not change the shape of the domain. To ensure a one-to-one correspondence, shape optimization problems are often considered as optimization problems on manifolds, see, e.g., [103, 115, 117], or on appropriate subsets of linear subspaces, see, e.g., [20]. In order to be in the latter setting, we consider a scalar valued quantitiy $\tilde{d} \in \tilde{D}_{\Gamma_d} \subset W^{1,\infty}(\tilde{\Gamma}_d)$ on the design boundary $\tilde{\Gamma}_d$ and identify it with a shape via a transformation of the form $\mathrm{id}_\mathsf{z} + \mathrm{B}(\tilde{d})$. Classical results show that the Eulerian shape derivative is a distribution that is supported on the design boundary and only acts on the normal boundary variation, cf. [31, Thm. 9.3.6]. In order to maintain this property we restrict to transformations that do not transform the solid domain and the interface by requiring $\tilde{\boldsymbol{\tau}}|_{\tilde{\Omega}_s} = \mathrm{id}_\mathsf{z}$, in correspondence with the analytical setting (Section 4). This constraint is approximated by using a penalization method and we introduce $\tilde{\alpha}_s = \alpha_s > 0$ on $\tilde{\Omega}_s$ and 0 else, where α_s is chosen sufficiently large in order to ensure $\tilde{\boldsymbol{\tau}}|_{\tilde{\Omega}_s} \approx \mathrm{id}_\mathsf{z}$ or by choosing an appropriate deformation field.

Strategy 1. Let $\mathrm{B}(\tilde{d}) := \mathrm{B}_1(\tilde{d})$, where B_1 extends $\tilde{d}\tilde{\mathbf{n}}$ to $\tilde{\Omega}$ and $\tilde{\mathbf{n}}$ denotes the outer unit normal vector of $\tilde{\Omega}$. More precisely, B_1 is the solution operator of

$$\begin{aligned} -\Delta_\mathsf{z}\tilde{\mathbf{w}}^d &= \tilde{\mathbf{z}}^d && \text{in } \tilde{\Omega},\\ -\Delta_\mathsf{z}\tilde{\mathbf{z}}^d + \tilde{\alpha}_s\tilde{\mathbf{w}}^d &= 0 && \text{in } \tilde{\Omega},\\ \tilde{\mathbf{w}}^d &= \tilde{d}\tilde{\mathbf{n}} && \text{on } \tilde{\Gamma}_d,\\ \tilde{\mathbf{w}}^d &= 0 && \text{on } \partial\tilde{\Omega}\setminus\tilde{\Gamma}_d,\\ \nabla_\mathsf{z}\tilde{\mathbf{w}}^d\cdot\tilde{\mathbf{n}} &= 0 && \text{on } \partial\tilde{\Omega}, \end{aligned} \tag{5.5}$$

which is the mixed formulation of

$$\begin{aligned} \Delta_\mathsf{z}^2\tilde{\mathbf{w}}^d + \tilde{\alpha}_s\tilde{\mathbf{w}}^d &= 0 && \text{in } \tilde{\Omega},\\ \tilde{\mathbf{w}}^d &= \tilde{d}\tilde{\mathbf{n}} && \text{on } \tilde{\Gamma}_d,\\ \tilde{\mathbf{w}}^d &= 0 && \text{on } \partial\tilde{\Omega}\setminus\tilde{\Gamma}_d,\\ \nabla_\mathsf{z}\tilde{\mathbf{w}}^d\cdot\tilde{\mathbf{n}} &= 0 && \text{on } \partial\tilde{\Omega}, \end{aligned}$$

cf. Section 5.1.1.

Strategy 2. Let $\tilde{\mathbf{n}}_\tau \in \tilde{\mathbf{E}}_\Omega \subset W^{1,\infty}(\tilde{\Omega})^d$ be an arbitrary transformation vector field, e.g. obtained by a biharmonic extension analogous to (5.5) of a vector field on $\tilde{\Gamma}_d$ which points in normal direction on $\tilde{\Gamma}_{fDc}$ and is zero on $\tilde{\Omega}_s$, and $\mathrm{B}(\tilde{d}) := \mathrm{B}_2(\tilde{d})\tilde{\mathbf{n}}_\tau$. We consider B_2 as the solution operator of

$$\begin{aligned} -\Delta_\mathsf{z}\tilde{w} &= \tilde{z} && \text{in } \tilde{\Omega},\\ -\Delta_\mathsf{z}\tilde{z} + \tilde{\alpha}_s\tilde{w} &= 0 && \text{in } \tilde{\Omega},\\ \tilde{w} &= \tilde{d} && \text{on } \tilde{\Gamma}_d,\\ \tilde{w} &= 0 && \text{on } \partial\tilde{\Omega}\setminus\tilde{\Gamma}_d,\\ \nabla_\mathsf{z}\tilde{w}\cdot\tilde{\mathbf{n}} &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned} \tag{5.6}$$

which corresponds to the mixed formulation of the fourth order partial differential equation

$$\begin{aligned} \Delta_{\mathbf{z}}^2 \tilde{w} + \tilde{\alpha}_s \tilde{w} &= 0 \quad \text{in } \tilde{\Omega}, \\ \tilde{w} &= \tilde{d} \quad \text{on } \tilde{\Gamma}_d, \\ \tilde{w} &= 0 \quad \text{on } \partial\tilde{\Omega} \setminus \tilde{\Gamma}_d, \\ \nabla_{\mathbf{z}} \tilde{w} \cdot \tilde{\mathbf{n}} &= 0 \quad \text{on } \partial\tilde{\Omega}. \end{aligned}$$

To fulfill the regularity requirements, we have to ensure that $\tilde{D}_{\tilde{\Gamma}_d} \subset W^{1,\infty}(\tilde{\Gamma}_d)$ and $\tilde{\mathbf{E}}_\Omega \subset W^{1,\infty}(\tilde{\Omega})^d$ are chosen such that $\mathrm{id}_{\mathbf{z}} + \mathrm{B}(\tilde{d}) \in \tilde{\mathbf{D}}_\Omega$ for all $\tilde{d} \in \tilde{D}_{\Gamma_d}$.

Remark 5.5. If $\tilde{\mathbf{D}}_\Omega = W^{1,\infty}(\tilde{\Omega})^d$, the following result is helpful. [101, Thm. 9] implies that for a smooth reference domain $\tilde{\Omega}$ with disjoint boundary parts $\tilde{\Gamma}_d$ and $\partial\tilde{\Omega} \setminus \tilde{\Gamma}_d$ such that $\overline{\partial\tilde{\Omega} \setminus \tilde{\Gamma}_d} \cap \overline{\tilde{\Gamma}_d} = \emptyset$ the solution operator that maps $\tilde{d}$ to the solution $\tilde{w}$ of the biharmonic equation

$$\begin{aligned} \Delta_{\mathbf{z}}^2 \tilde{w} &= 0 \quad \text{in } \tilde{\Omega}, \\ \tilde{w} &= \tilde{d} \quad \text{on } \tilde{\Gamma}_d, \\ \tilde{w} &= 0 \quad \text{on } \partial\tilde{\Omega} \setminus \tilde{\Gamma}_d, \\ \nabla_{\mathbf{z}} \tilde{w} \cdot \tilde{\mathbf{n}} &= 0 \quad \text{on } \partial\tilde{\Omega}, \end{aligned}$$

is a continuous operator from $W^{1,\infty}(\tilde{\Gamma}_d) \to W^{1,\infty}(\tilde{\Omega})$.

The above considerations motivate the following choice for the set of admissible transformations. We consider sets of admissible transformations

$$\tilde{\mathcal{T}}_{ad} \subset \{\tilde{\boldsymbol{\tau}} = \mathrm{id}_{\mathbf{z}} + \tilde{\mathbf{u}}_\tau,\ \tilde{\mathbf{u}}_\tau \in \tilde{\mathbf{U}}_{ad}\},$$

where $\tilde{\mathbf{U}}_{ad}$ is chosen such that

$$\tilde{\mathbf{U}}_{ad} = \{\tilde{\mathbf{u}}_\tau \ :\ \tilde{\mathbf{u}}_\tau = \mathrm{B}(\tilde{d}),\ \tilde{d} \in \tilde{D}_{\Gamma_d},\ \|\tilde{d}\|_{\tilde{D}_{\Gamma_d}} \leq c\},$$

with a sufficiently small constant $c > 0$.

Furthermore, it is often relevant for practical applications to have an additional constraint on the volume of the domain, e.g., the volume of the obstacle shall not become smaller. This motivates the restriction

$$\tilde{g}_\Omega(\tilde{\mathbf{u}}_\tau) = \int_{\tilde{\Omega}} \det(\mathbf{I} + D_{\mathbf{z}} \tilde{\mathbf{u}}_\tau) d\mathbf{z} - V \leq 0 \tag{5.7}$$

for a constant $V > 0$, e.g., $V = \int_{\tilde{\Omega}} 1 d\mathbf{z}$.

5.3.4. Shape Optimization Problem

The shape optimization problem is given by

$$\begin{aligned} \min_{\tilde{d} \in \tilde{D}_{\Gamma_d}} \ & \tilde{j}_\Omega(\tilde{\mathbf{u}}_\tau) \\ \text{s.t.} \quad & \tilde{g}_\Omega(\tilde{\mathbf{u}}_\tau) \leq 0, \\ & \tilde{\mathbf{u}}_\tau = \mathrm{B}(\tilde{d}), \end{aligned} \tag{5.8}$$

where $\tilde{j}_\Omega(\tilde{\mathbf{u}}_\tau) = \tilde{F}_D(\tilde{\mathbf{y}}, \mathrm{id}_\mathsf{z} + \tilde{\mathbf{u}}_\tau)$, $\tilde{\mathbf{y}}$ is given as the solution to the partial differential equation $\tilde{A}_\Omega(\tilde{\mathbf{y}}, \mathrm{id}_\mathsf{z} + \tilde{\mathbf{u}}_\tau) = 0$, see (5.4). Furthermore, B and $\tilde{g}_\Omega$ are defined in Section 5.3.3 and $\tilde{F}_D$ is defined in Section 5.3.2.

5.4. Discretization

In this section, we discretize the FSI system (5.3) in time (Section 5.4.1) and space (Section 5.4.2). To obtain a discrete formulation (Section 5.4.5) of the optimization problem (5.8), the objective function (Section 5.4.3) and the shape transformations (Section 5.4.4) have to be discretized.

5.4.1. Temporal Discretization

In order to solve the time-dependent problem numerically we need to introduce an appropriate time-stepping technique. We consider a One-Step-θ scheme, cf. [128], and, therefore, divide the terms that appear in the weak formulation into different categories. The first group $\tilde{A}_T(\tilde{\mathbf{y}}, \tau)(\tilde{\boldsymbol{\psi}})$ collects all terms which include time derivatives:

$$\begin{aligned} \tilde{A}_T(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) := {} & (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f(\partial_\mathsf{t}\tilde{\mathbf{v}} - (((D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}\partial_\mathsf{t}\tilde{\mathbf{w}}) \cdot \nabla_\mathsf{z})\tilde{\mathbf{v}}), \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} \\ & + (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\rho_s\partial_\mathsf{t}\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} + (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\rho_s\partial_\mathsf{t}\tilde{\mathbf{w}}, \tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_s}. \end{aligned}$$

The group $\tilde{A}_I(\tilde{\mathbf{y}}, \tau)(\tilde{\boldsymbol{\psi}})$ gathers all implicit terms, i.e., all terms that should be fulfilled exactly by the new iterate such as the incompressibility condition for the fluid:

$$\begin{aligned} \tilde{A}_I(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) := {} & (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\mathrm{tr}(D_\mathsf{z}(\tilde{J}_{\boldsymbol{\chi}}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}\tilde{\mathbf{v}})(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-1}), \tilde{\psi}^p)_{\tilde{\Omega}_f} \\ & + \alpha_p(\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-1}(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-T}\nabla_\mathsf{z}\tilde{p}, \nabla_\mathsf{z}\tilde{\psi}^p)_{\tilde{\Omega}_s} - (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\tilde{\mathbf{z}}, \tilde{\boldsymbol{\psi}}^z)_{\tilde{\Omega}} \\ & + (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})D_\mathsf{z}\tilde{\mathbf{w}}(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-1}(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-T}, D_\mathsf{z}\tilde{\boldsymbol{\psi}}^z)_{\tilde{\Omega}}. \end{aligned}$$

Another group $\tilde{A}_P(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}})$, which is also treated implicitly, collects the pressure terms:

$$\tilde{A}_P(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) := (\det(D_\mathsf{z}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\tilde{\boldsymbol{\sigma}}_{f,p}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_\mathsf{z}\tilde{\boldsymbol{\tau}})^{-T}, D_\mathsf{z}\tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f},$$

where $\tilde{\boldsymbol{\sigma}}_{f,p} = -\tilde{p}\mathbf{I}$. This can be motivated by the fact that the pressure serves as Lagrange multiplier for the incompressibility condition. The remaining terms are collected in the fourth group $\tilde{A}_E(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}})$:

$$\begin{aligned}\tilde{A}_E(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) := &(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f(((D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}\tilde{\mathbf{v}}) \cdot \nabla_{\mathsf{z}})\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} \\ &+ \alpha_w(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})D_{\mathsf{z}}\tilde{\mathbf{z}}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-T}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_f} \\ &+ (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\tilde{\boldsymbol{\sigma}}_{f,v}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-T}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} \\ &+ (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\tilde{\boldsymbol{\sigma}}_s\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-\top}(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-T}, D_{\mathsf{z}}\tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} \\ &- (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_f\tilde{\mathbf{f}}_f, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} - (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}\rho_s\tilde{\mathbf{f}}_s, \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} \\ &- (\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho_s\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_s},\end{aligned}$$

where $\tilde{\boldsymbol{\sigma}}_{f,v} = \tilde{\boldsymbol{\sigma}}_f - \tilde{\boldsymbol{\sigma}}_{f,p}$. The time-stepping scheme can thus be summarized as follows. Let a transformation $\tilde{\boldsymbol{\tau}}$ be given, $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$ be a discretization of $\overline{I} = [0, T]$ and $\theta \in [0, 1]$. Let, for $j \in \{1, 2, \ldots, N\}$, $\tilde{\mathbf{y}}^{j-1}$ be the solution at the time t_{j-1} and the time step size be constant, i.e., $k := k_j = t_j - t_{j-1}$ for all $n \in \{1, \ldots, N\}$. Then, the solution at t_j is computed by:
Find $\tilde{\mathbf{y}}^j$ such that

$$\tilde{A}_T^{j,k}(\tilde{\mathbf{y}}^j, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) + \theta\tilde{A}_E(\tilde{\mathbf{y}}^j, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) + \tilde{A}_P(\tilde{\mathbf{y}}^j, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) + \tilde{A}_I(\tilde{\mathbf{y}}^j, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) = -(1-\theta)\tilde{A}_E(\tilde{\mathbf{y}}^{j-1}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}),$$

for all test functions $\tilde{\boldsymbol{\psi}}$. Here, $\tilde{A}_T^{j,k}(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}})$ is defined as the approximation of $\tilde{A}_T(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}})$ given by

$$\begin{aligned}\tilde{A}_T^{j,k}(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\tau}})(\tilde{\boldsymbol{\psi}}) := &\frac{1}{k}(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\tilde{J}_{\boldsymbol{\chi}}^{j,\theta}\rho_f((\tilde{\mathbf{v}} - \tilde{\mathbf{v}}^{j-1}) - (((D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})^{-1}\widetilde{\mathbf{F}}_{\boldsymbol{\chi}}^{-1}(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{j-1}) \cdot \nabla_{\mathsf{z}})\tilde{\mathbf{v}}), \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_f} \\ &+ \frac{1}{k}(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho_s(\tilde{\mathbf{v}} - \tilde{\mathbf{v}}^{j-1}), \tilde{\boldsymbol{\psi}}^v)_{\tilde{\Omega}_s} + \frac{1}{k}(\det(D_{\mathsf{z}}\tilde{\boldsymbol{\tau}})\rho_s(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{j-1}), \tilde{\boldsymbol{\psi}}^w)_{\tilde{\Omega}_s},\end{aligned}$$

where $\tilde{J}_{\boldsymbol{\chi}}^{j,\theta} := \theta\tilde{J}_{\boldsymbol{\chi}} + (1-\theta)\tilde{J}_{\boldsymbol{\chi}}^{j-1}$ and the time derivatives are approximated by backwards different quotients.
The parameter θ is chosen as $\theta = \frac{1}{2} + \mathcal{O}(k)$, which corresponds to a shifted Crank-Nicolson scheme. By this choice one obtains second order accuracy in time and additionally recovers global stability [114, Sec. 5.3]. The latter is is important for stable behavior for long-term computations and not guaranteed by the standard Crank-Nicolson scheme, see [131].

5.4.2. Spatial Discretization

For the spatial discretization, we use a triangulation $\mathcal{T}_{\mathsf{h}}$ of the domain $\tilde{\Omega}$ with 4451 nodes and 8621 cells K. For the sake of clarity, and since we focus on presenting the main ideas and do not consider variational crimes in the scope of this thesis, we denote the discretized domains also by $\tilde{\Omega}$, $\tilde{\Omega}_f$ and $\tilde{\Omega}_s$. In order to have a stable discretization of the Navier-Stokes part of the FSI equations, we choose Taylor-Hood elements $(\tilde{\mathbf{v}}_{\mathsf{h}}, \tilde{p}_{\mathsf{h}}) \in (\mathcal{P}^2(\mathcal{T}_{\mathsf{h}})^d, \mathcal{P}^1(\mathcal{T}_{\mathsf{h}}))$, where

$$\mathcal{P}^l(\mathcal{T}_{\mathsf{h}})^m := \{\tilde{\mathbf{v}}_{\mathsf{h}} \in \mathcal{C}(\bigcup_{K \in \mathcal{T}_{\mathsf{h}}} K)^m \ : \ \tilde{\mathbf{v}}_{\mathsf{h}}|_K \text{ is a polynomial up to degree } l, \ \forall K \in \mathcal{T}_{\mathsf{h}}\}$$

for $l \geq 0$ and $m \in \mathbb{N}$, i.e., $\tilde{\mathbf{v}}_\mathsf{h}$ is continuous and element-wise quadratic and $\tilde{p}_\mathsf{h}$ is continuous and linear on every element. Since $\tilde{\mathbf{v}}_\mathsf{h}$ is equal to the temporal derivative of $\tilde{\mathbf{w}}_\mathsf{h}$ on $\tilde{\Omega}_s$, $\tilde{\mathbf{w}}_\mathsf{h}$ is chosen such that it has the same degrees of freedom as $\tilde{\mathbf{v}}_\mathsf{h}$. Therefore, we choose $(\tilde{\mathbf{w}}_\mathsf{h}, \tilde{\mathbf{z}}_\mathsf{h}) \in (\mathcal{P}^2(\mathcal{T}_\mathsf{h})^d, \mathcal{P}^2(\mathcal{T}_\mathsf{h})^d)$. The boundary of the circular obstacle $\tilde{\Gamma}_d$ is discretized as a polygonal chain $\tilde{\Gamma}_{d,\mathsf{h}}$ of 47 nodes, 6 of which are part of the boundary of the solid domain. For the sake of clarity we simplify the notation denoting $\tilde{\Gamma}_{d,\mathsf{h}}$ by $\tilde{\Gamma}_d$.

5.4.3. Discretization of Objective Function

The spatial discretization of the objective function is determined by the discretization of the state of the FSI problem. In order to discretize the appearing time derivative terms, we use a finite difference scheme, more precisely, the time derivative $\partial_\mathsf{t}\mathbf{v}_\mathsf{h}(t_j)$ is approximated by $(t_j - t_{j-1})^{-1}(\mathbf{v}_\mathsf{h}(t_j) - \mathbf{v}_\mathsf{h}(t_{j-1}))$. The time integral is approximated using the trapezoidal rule.

5.4.4. Discretization of Shape Transformations

In Section 5.3.3, it is motivated that the choice of admissible shape transformations is delicate and requires available existence and regularity theory for the governing partial differential equations. However, existence and regularity theory for FSI systems is only available for special cases and under additional restrictions or assumptions. In particular, there are no theoretical results concerning existence and regularity of solutions available for the model (5.1). Thus, we restrict the considerations to the discretized problem. Here, the main requirements for choosing admissible shape transformations reduces to ensure

- that the source term, the boundary and initial conditions remain untouched by admissible shape transformations.
- that $\tilde{\boldsymbol{\tau}}_\mathsf{h}(\mathcal{T}_\mathsf{h})$ is the discretization of a Lipschitz domain, which means that mesh degeneration is prevented. This is a delicate task that gained attention in several publications. In the context of shape optimization see, e.g., [71] and the references therein, in the context of ALE transformations see, e.g., [10, 37]. Mesh degeneration is not seen directly since all computations are performed on the fixed shape reference $\tilde{\Omega}$ domain, however, it is the main bottleneck in the performance of the optimization. In particular, it appears
  - for large displacements of the design boundary $\tilde{\Gamma}_d$, in our example particularly in the area around the fixed flap.
  - for oscillatory displacements of the design boundary $\tilde{\Gamma}_d$.
  - if the extension of the design boundary information to $\tilde{\Omega}$ is chosen in an unsophisticated way.

  To penalize oscillatory behaviour of the design boundary displacement, we add a regularization term $R(\tilde{d}_\mathsf{h}) = \|\tilde{d}_\mathsf{h}\|^2_{H^1(\tilde{\Gamma}_d)}$ with a factor $\gamma > 0$. To prevent large displacements, for $\mathsf{z} = (\mathsf{z}_1, \mathsf{z}_2)^\top$, we introduce the bounds

$$c_u(\mathsf{z}) = 0.004(1 + \tfrac{0.25-\mathsf{z}_1}{0.1}(\tfrac{0.015}{0.004} - 1)),$$

$$c_l(\mathsf{z}) = -0.004(1 + \tfrac{0.25-\mathsf{z}_1}{0.1}(\tfrac{0.015}{0.004} - 1)),$$

which are more restrictive close to the fixed flap. Nodal evaluation of these bounds yields vectors $\mathbf{c}_u$, $\mathbf{c}_l \in \mathbb{R}^{n_d}$.

Remark 5.6. One could also think about incorporating the definition of $\tilde{D}_{\Gamma_d}$. In case $\tilde{D}_{\Gamma_d} = W^{1,\infty}(\tilde{\Gamma}_d)$, boundedness of $\tilde{d}$ in $\tilde{D}_{\Gamma_d}$ is ensured by introducing simple bound constraints on $\tilde{d}$ and its gradient. This, however, is a topic for future research.

- that $\tilde{\mathbf{Y}}_\mathsf{h}(\mathcal{T}_\mathsf{h}) \circ \tilde{\boldsymbol{\tau}}_h$ is isomorphic to $\tilde{\mathbf{Y}}_h(\tilde{\boldsymbol{\tau}}_h(\mathcal{T}_\mathsf{h}))$ for all $\tilde{\boldsymbol{\tau}}_\mathsf{h} \in \tilde{\mathcal{T}}_{ad,\mathsf{h}}$, where $\tilde{\mathbf{Y}}_\mathsf{h}$ denotes the discrete state space and $\tilde{\mathcal{T}}_{ad,\mathsf{h}}$ the discrete set of admissible transformations. To do so, we choose $\tilde{\mathcal{T}}_{ad,\mathsf{h}} \subset \mathcal{P}^1(\mathcal{T}_\mathsf{h})$.
- a one-to-one-correspondance between transformations and shapes. Analogously to the continuous case, we choose a scalar valued variable $\tilde{d}_\mathsf{h} \in \tilde{D}_{\Gamma_d,\mathsf{h}}$, where $\tilde{D}_{\Gamma_d,\mathsf{h}}$ denotes the space of piecewise linear functions on Γ_d, in addition, require that $\tilde{\boldsymbol{\tau}}_\mathsf{h}$ is equal to the identity on $\tilde{\Omega}_s$ and consider the discretized version of the operator B presented in Section 5.3.3.

Strategy 1. Discretizing (5.5) gives us the weak form

$$a^1_{\text{ext}}(\tilde{d}_\mathsf{h}, (\tilde{\mathbf{w}}^d_\mathsf{h}, \tilde{\mathbf{z}}^d_\mathsf{h}), (\tilde{\boldsymbol{\psi}}^w_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})) = 0,$$

where a^1_{ext} maps

$$\tilde{D}_{\Gamma_d,\mathsf{h}} \times (P^1_{d0}(\mathcal{T}_\mathsf{h}, \mathbb{R}^d) \times P^1_{d0}(\mathcal{T}_\mathsf{h}, \mathbb{R}^d)) \times (P^1_{d0}(\mathcal{T}_\mathsf{h}, \mathbb{R}^d) \times P^1_{d0}(\mathcal{T}_\mathsf{h}, \mathbb{R}^d)) \to \mathbb{R}$$

with

$$P^1_{d0}(\mathcal{T}_\mathsf{h}, \mathbb{R}^d) := \{\mathbf{v}_h \in P^1(\mathcal{T}_\mathsf{h}, \mathbb{R}^d) \ : \ \mathbf{v}_h|_{\partial\tilde{\Omega}\setminus\tilde{\Gamma}_d} = 0\}$$

and is defined by

$$\begin{aligned}
&a^1_{\text{ext}}(\tilde{d}_\mathsf{h}, (\tilde{\mathbf{w}}^d_\mathsf{h}, \tilde{\mathbf{z}}^d_\mathsf{h}), (\tilde{\boldsymbol{\psi}}^w_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})) \\
&:= (D_\mathsf{z}\tilde{\mathbf{w}}^d_\mathsf{h}, D_\mathsf{z}\tilde{\boldsymbol{\psi}}^z_\mathsf{h})_{\tilde{\Omega}} - (\tilde{\mathbf{z}}^d_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})_{\tilde{\Omega}} + (D_\mathsf{z}\tilde{\mathbf{z}}^d_\mathsf{h}, D_\mathsf{z}\tilde{\boldsymbol{\psi}}^w_\mathsf{h})_{\tilde{\Omega}} \\
&\quad - (\nabla\tilde{\mathbf{w}}^d_\mathsf{h} \cdot \tilde{\mathbf{n}}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})_{\tilde{\Gamma}_d} - (\tilde{\mathbf{z}}^d_\mathsf{h}, \nabla\tilde{\boldsymbol{\psi}}^w_\mathsf{h} \cdot \tilde{\mathbf{n}})_{\tilde{\Gamma}_d} + \alpha_n(\tilde{\mathbf{w}}^d_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})_{\tilde{\Gamma}_d} \\
&\quad - (\nabla\tilde{\mathbf{z}}^d_\mathsf{h} \cdot \tilde{\mathbf{n}}, \tilde{\boldsymbol{\psi}}^w_\mathsf{h})_{\tilde{\Gamma}_d} - (\tilde{\mathbf{w}}^d_\mathsf{h}, \nabla\tilde{\boldsymbol{\psi}}^z_\mathsf{h} \cdot \tilde{\mathbf{n}})_{\tilde{\Gamma}_d} + \alpha_n(\tilde{\mathbf{z}}^d_\mathsf{h}, \tilde{\boldsymbol{\psi}}^w_\mathsf{h})_{\tilde{\Gamma}_d} \\
&\quad + \alpha_s(\tilde{\mathbf{w}}^d_\mathsf{h}, \tilde{\boldsymbol{\psi}}^w_\mathsf{h})_{\tilde{\Omega}_s} + (\tilde{d}_\mathsf{h}\tilde{\mathbf{n}}, \nabla\tilde{\boldsymbol{\psi}}^z_\mathsf{h} \cdot \tilde{\mathbf{n}})_{\tilde{\Gamma}_d} - \alpha_n(\tilde{d}_\mathsf{h}\tilde{\mathbf{n}}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})_{\tilde{\Gamma}_d}.
\end{aligned} \tag{5.9}$$

Here, we take care of the requirement $\tilde{\boldsymbol{\tau}}_\mathsf{h}|_{\tilde{\Omega}_s} = \text{id}_\mathsf{z}$ by adding a penalization term with penalty parameter $\alpha_s > 0$. In addition, we use Nitsche's method [108] with $\alpha_n > 0$ for imposing the Dirichlet boundary conditions on $\tilde{\Gamma}_d$ since on Lipschitz domains with kinks, as it is, e.g., the case for a discretized domain, the normal is only defined almost everywhere, especially not in kinks that correspond to vertices of our discretization. The corresponding extension operator is denoted by $\text{B}_{1,\mathsf{h}} : \tilde{d} \mapsto \tilde{\mathbf{w}}^d_\mathsf{h}$, where $\tilde{\mathbf{w}}^d_\mathsf{h}$ is uniquely defined by the solution of $a^1_{\text{ext}}(\tilde{d}, (\tilde{\mathbf{w}}^d_\mathsf{h}, \tilde{\mathbf{z}}^d_\mathsf{h}), (\tilde{\boldsymbol{\psi}}^w_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h})) = 0$ for all $(\tilde{\boldsymbol{\psi}}^w_\mathsf{h}, \tilde{\boldsymbol{\psi}}^z_\mathsf{h}) \in P^1_{d0}(\mathcal{T}_\mathsf{h})^d \times P^1_{d0}(\mathcal{T}_\mathsf{h})^d$. We choose $\alpha_s = 1 \cdot 10^8$ and $\alpha_n = 1 \cdot 10^4$.

Strategy 2. Discretizing (5.6) yields the weak form

$$a((\tilde{w}_{\mathsf{h}}, \tilde{z}_{\mathsf{h}}), (\tilde{\psi}^w_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}})) = (D_{\mathsf{z}}\tilde{w}_{\mathsf{h}}, D_{\mathsf{z}}\tilde{\psi}^z_{\mathsf{h}})_{\tilde{\Omega}} - (\tilde{z}_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}})_{\tilde{\Omega}} + (D_{\mathsf{z}}\tilde{z}_{\mathsf{h}}, D_{\mathsf{z}}\tilde{\psi}^w_{\mathsf{h}})_{\tilde{\Omega}} + \alpha_s(\tilde{w}_{\mathsf{h}}, \tilde{\psi}^w_{\mathsf{h}})_{\tilde{\Omega}_s} = 0.$$

with a penalty parameter $\alpha_s > 0$. Here, the boundary conditions $\tilde{w}_{\mathsf{h}}|_{\tilde{\Gamma}_d} = \tilde{d}_{\mathsf{h}}$, $\tilde{z}_{\mathsf{h}}|_{\tilde{\Gamma}_d} = 0$ and homogeneous Dirichlet boundary conditions on the rest of the discretized boundary are imposed. Equivalently, one could also rewrite the weak form on the subspace

$$P^1_0(\mathcal{T}_{\mathsf{h}}) := \{v_h \in P^1(\mathcal{T}_{\mathsf{h}}) \ : \ v_h|_{\partial\tilde{\Omega}} = 0\}$$

of functions that vanish on the boundary of the discretized domain. To do so, $a^2_{\text{ext}} : (P^1_0(\mathcal{T}_{\mathsf{h}}) \times P^1_0(\mathcal{T}_{\mathsf{h}})) \times (P^1_0(\mathcal{T}_{\mathsf{h}}) \times P^1_0(\mathcal{T}_{\mathsf{h}})) \to \mathbb{R}$ defined by

$$a^2_{\text{ext}}(\tilde{d}_{\mathsf{h}}, (\tilde{w}^0_{\mathsf{h}}, \tilde{z}_{\mathsf{h}}), (\tilde{\psi}^w_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}})) := a((\tilde{w}^0_{\mathsf{h}} + \text{ext}_{\mathsf{h}}\tilde{d}_h, \tilde{z}_{\mathsf{h}}), (\tilde{\psi}^w_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}})) \tag{5.10}$$

is introduced, where $\text{ext}_{\mathsf{h}} : \tilde{D}_{\Gamma_d,\mathsf{h}} \to P^1(\mathcal{T}_{\mathsf{h}})$ is an arbitrary linear extension operator. The biharmonic extension operator corresponding to (5.10) is denoted by $\mathrm{B}_{2,\mathsf{h}} : \tilde{d} \mapsto \tilde{w}^0_{\mathsf{h}} + \text{ext}_{\mathsf{h}}\tilde{d}_{\mathsf{h}}$, where $\tilde{w}^0_{\mathsf{h}}$ is given as the solution of $a^2_{\text{ext}}(\tilde{d}, (\tilde{w}^0_{\mathsf{h}}, \tilde{z}_{\mathsf{h}}), (\tilde{\psi}^w_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}})) = 0$ for all $(\tilde{\psi}^w_{\mathsf{h}}, \tilde{\psi}^z_{\mathsf{h}}) \in P^1_0(\mathcal{T}_{\mathsf{h}}) \times P^1_0(\mathcal{T}_{\mathsf{h}})$. In order to have the discretized version of B, we have to perform a projection $\mathrm{P}_{\mathbf{n}_\tau,\mathsf{h}}$ of $\tilde{w}_{\mathsf{h}}\tilde{\mathbf{n}}_\tau$ to the finite element space of piecewise linear functions.

The discretization of the volume constraint can be done in different ways. A first possibility is given by discretizing the integral formulation (5.7). This approach is advantageous if, e.g, the explicit boundary displacement is not a-priorily given as it is the case if $\tilde{d}_{\mathsf{h}}$ is imposed via Nitsche's method or as Neumann boundary condition. For the two dimensional case and if $\tilde{d}_{\mathsf{h}}$ is imposed as Dirichlet boundary condition for the extension equations one can also directly compute the area via Gauss' area formula, which is also known as shoelace formula or surveyor's area formula. Assume the discretized $\tilde{\Gamma}_d$ to be a non-self-intersecting polygonal chain $\{\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_n\}$ with nodes $\tilde{\mathbf{p}}_i = (\mathsf{z}_{i,1}, \mathsf{z}_{i,2})^\top$. Let $\tilde{d}_{\mathsf{h}}(\mathsf{z}) = \sum_{i=1}^n \tilde{d}_i\Phi_i(\mathsf{z})$, where $\{\Phi_i(\mathsf{z}), i \in \{1, \ldots, n\}\}$ denotes the set of nodal basis functions on the design boundary, i.e., $\tilde{d}_{\mathsf{h}}$ can be identified with $\tilde{\mathbf{d}} = (\tilde{d}_1, \ldots, \tilde{d}_n) \in \mathbb{R}^n$ via an operator Ψ^{-1}, where

$$\Psi(\tilde{\mathbf{d}}) := \sum_{i=1}^n (\tilde{\mathbf{d}})_i\Phi_i.$$

If one identifies $\tilde{\mathbf{p}}_1$ with $\tilde{\mathbf{p}}_{n+1}$ we obtain, see [19],

$$\tilde{A} = \frac{1}{2}\sum_{i=1}^n (\mathsf{z}_{i,2} + \mathsf{z}_{i+1,2})(\mathsf{z}_{i+1,1} - \mathsf{z}_{i,1}).$$

Transformation of the polygonal chain with nodes $\{\tilde{\mathbf{p}}_0, \ldots, \tilde{\mathbf{p}}_n\}$ yields a polygonal chain with nodes $\{\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_n\}$, where $\hat{\mathbf{p}}_i = (\mathsf{y}_{1,i}, \mathsf{y}_{2,i})^\top = \tilde{\mathbf{p}}_i + \tilde{d}_i\tilde{\mathbf{n}}_{\tau,i}$ and $\tilde{\mathbf{n}}_{\tau,i}$ corresponds to the evaluation of $\tilde{\mathbf{n}}_\tau$ at $\tilde{\mathbf{p}}_i$. Here, for the sake of clarity, we write $\mathsf{y}_{j,i}$ instead of $\mathsf{y}_{j,i}(\tilde{\mathbf{d}})$ for $j \in \{1, 2\}$

and $i \in \{1, \dots, n\}$. The transformed volume is thus given by

$$\hat{A}(\tilde{\mathbf{d}}) = \frac{1}{2} \sum_{i=1}^{n} (\mathsf{y}_{i,2} + \mathsf{y}_{i+1,2})(\mathsf{y}_{i+1,1} - \mathsf{y}_{i,1}).$$

Recall that, if the volume of the circular obstacle shall not become smaller during the optimization process, we are aiming at $\tilde{A} \le \hat{A}$. Therefore, we define

$$\tilde{g}(\tilde{\mathbf{d}}) := \tilde{A} - \hat{A}(\tilde{\mathbf{d}}).$$

We want to enforce the nodes, which are attached to the solid, not to be transformed, i.e., we want to enforce $\tilde{d}_j = 0$ for $j \in J$, where $J \subset \{1, \dots, n\}$. This is carried out by performing a linear transformation which is defined by $\mathbf{A} \in \mathbb{R}^{n \times n_d}$ with $n_d = n - |J|$ and $\mathbf{A}$ is obtained by deleting the jth columns of the $n \times n$-identity matrix for all $j \in J$. For $\mathbf{d} \in \mathbb{R}^{n_d}$ we define $g_2(\mathbf{d}) := \tilde{g}(\mathbf{A}\mathbf{d})$.

5.4.5. Discretized Version of the Shape Optimization Problem

Let $n_d \in \mathbb{N}$ and $\mathbf{A}$, Ψ, $\tilde{j}_\Omega$, $\tilde{g}_\Omega$, g_2, $\mathrm{B}_{1,\mathsf{h}}$, $\mathrm{B}_{2,\mathsf{h}}$, and $\mathrm{P}_{\mathbf{n}_\tau,\mathsf{h}}$ be defined as in Sections 5.3.3, 5.3.4 and 5.4.4. The discretized shape optimization problem attains the form

$$\begin{aligned} &\min_{\mathbf{d} \in \mathbb{R}^{n_d}} f(\mathbf{d}) \\ &\text{s.t.} \quad g(\mathbf{d}) \le 0, \\ &\qquad\ \mathbf{c}_l \le \mathbf{d} \le \mathbf{c}_u. \end{aligned} \tag{5.11}$$

Here, $\mathbf{d} \in \mathbb{R}^{n_d}$ is bounded by $\mathbf{c}_l$, $\mathbf{c}_u \in \mathbb{R}^{n_d}$, $\mathbf{c}_l \le \mathbf{c}_u$. The control $\mathbf{d}$ can be identified with a transformation via the following chain of compositions

$$\mathbf{d} \xmapsto{\ \mathbf{A}\ } \tilde{\mathbf{d}} \xmapsto{\ \Psi\ } \tilde{d}_\mathsf{h} \xmapsto{\ \mathrm{B}_\mathsf{h}\ } \tilde{\mathbf{u}}_{\tau,\mathsf{h}} \xmapsto{\ \mathrm{id}_\mathsf{z}+\cdot\ } \tilde{\boldsymbol{\tau}}_\mathsf{h},$$

where $\mathbf{A}$ and Ψ are defined in Section 5.4.4. The objective is defined by $f(\mathbf{d}) := \tilde{j}_\Omega(\tilde{\mathbf{u}}_{\tau,\mathsf{h}})$, where $\tilde{\mathbf{u}}_{\tau,\mathsf{h}} = \tilde{\mathbf{u}}_{\tau,\mathsf{h}}(\mathbf{d})$.

Strategy 1. $\mathrm{B}_\mathsf{h} := \mathrm{B}_{1,\mathsf{h}}$ and $g(\mathbf{d}) := \tilde{g}_\Omega(\tilde{\mathbf{u}}_{\tau,\mathsf{h}})$.

Strategy 2. $\mathrm{B}_\mathsf{h} := \mathrm{P}_{\mathbf{n}_\tau,\mathsf{h}} \circ \mathrm{B}_{2,\mathsf{h}}$, where $\tilde{d}_\mathsf{h} \xmapsto{\ \mathrm{B}_{2,\mathsf{h}}\ } \tilde{w}_\mathsf{h} \xmapsto{\ \mathrm{P}_{\mathbf{n}_\tau,\mathsf{h}}\ } \tilde{\mathbf{u}}_{\tau,\mathsf{h}}$, and $g(\mathbf{d}) := g_2(\mathbf{d})$.

5.5. Numerical Realization Using FEniCS, dolfin-adjoint and IPOPT

The numerical tests presented here are implemented in FEniCS [3, 92], a collection of free software for the automated solution of partial differential equations combining the software packages dolfin [94, 95], FFC [81, 93, 109], UFL [2, 6], FIAT [80, 82], and UFC [4, 5]. For the computation of the gradients the additional package dolfin-adjoint [40] is used, which provides the automated differentiation of the reduced cost functional based on adjoint computations on the discrete system. It is based on a checkpointing strategy [52], meaning that the forward solution is, to save memory space, not saved for every time-step but only on several checkpoints. In order to solve the backwards equations, the forward equation is solved starting from the checkpoints and then used to compute the adjoint. Additionally, the software package IPOPT [126] is used for solving the constrained optimization problem on the shape reference domain $\tilde{\Omega}$.

5.5.1. Computing Objective Function Value and Gradient with FEniCS and dolfin-adjoint

This section explains how objective function values and gradients are evaluated using FEniCS and dolfin-adjoint to obtain an optimization problem that we can pass over to IPOPT. If one has automated differentiation available for distributed controls, e.g., by using FEniCS and dolfin- adjoint, one can not directly compute the derivative w.r.t. $\mathbf{d}$ but has to apply the chain rule.

Strategy 1

We define $\tilde{j}_{1,\Omega}(\tilde{d}_\mathsf{h}) := \tilde{j}_\Omega(\mathrm{B}_\mathsf{h}(\tilde{d}_\mathsf{h}))$. The derivative $\tilde{j}'_{1,\Omega}(\tilde{d}_\mathsf{h})$ can directly be computed using FEniCS and dolfin-adjoint since $\tilde{d}_\mathsf{h}$ appears in the weak formulation (5.9). The vector representation $\tilde{\mathbf{j}} \in \mathbb{R}^n$ of the derivative w.r.t. the degrees of freedom on the design boundary is formally defined by

$$\tilde{\mathbf{j}}_i := \tilde{j}'_{1,\Omega}(\tilde{d}_h)(\Psi(\mathbf{e}_i))$$

for $i \in \{1, \ldots, n\}$. Considering the mapping $j_1 : \mathbb{R}^{n_d} \to \mathbb{R}$, $j_1(\mathbf{d}) := \tilde{j}_{1,\Omega}(\mathrm{B}_{1,\mathsf{h}}(\Psi(\mathbf{A}\mathbf{d})))$, its gradient is given by $\nabla j_1 = \mathbf{A}^\top \tilde{\mathbf{j}}$.

Strategy 2

We consider $\tilde{j}_{2,\Omega}(\tilde{w}_\mathsf{h}) := \tilde{j}_\Omega(\mathrm{P}_{\mathbf{n}_\tau,\mathsf{h}}(\tilde{w}_\mathsf{h}))$ and compute the gradient $j'_{2,\Omega}(\tilde{w}_\mathsf{h})$ using FEniCS and dolfin-adjoint. In order to obtain the derivative of $\tilde{j}_2(\tilde{d}_\mathsf{h}) := \tilde{j}_{2,\Omega}(\mathrm{B}_{2,\mathsf{h}}(\tilde{d}_\mathsf{h}))$ the chain rule has to be applied. We have

$$\begin{aligned}\tilde{j}_2(\tilde{d}_\mathsf{h}) &= \tilde{j}_{2,\Omega}(\mathrm{B}_{2,\mathsf{h}}(\tilde{d}_\mathsf{h})) = \tilde{j}_{2,\Omega}(\mathrm{ext}_\mathsf{h}\tilde{d}_\mathsf{h} + \tilde{w}^0_\mathsf{h}) \\ &= \tilde{j}_{2,\Omega}(\mathrm{ext}_\mathsf{h}\tilde{d}_\mathsf{h} + \tilde{w}^0_\mathsf{h}) - a^2_{\mathrm{ext}}(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h})) = \mathcal{L}(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h})),\end{aligned}$$

where $(\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}) \in \mathcal{P}^1_0(\mathcal{T}_\mathsf{h}) \times \mathcal{P}^1_0(\mathcal{T}_\mathsf{h})$ solves

$$a^2_{\mathrm{ext}}(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h})) = 0$$

for all $(\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h}) \in \mathcal{P}^1_0(\mathcal{T}_\mathsf{h}) \times \mathcal{P}^1_0(\mathcal{T}_\mathsf{h})$. Testing with the solution $(\tilde{w}^*_\mathsf{h}, \tilde{z}^*_\mathsf{h})$ of the adjoint equation $\tilde{j}'_{2,\Omega}(\tilde{w}_\mathsf{h})(\delta\tilde{w}_\mathsf{h}) - a^2_{\mathrm{ext}}(\tilde{d}_\mathsf{h}, (\delta\tilde{w}_\mathsf{h}, \delta\tilde{z}_\mathsf{h}), (\tilde{w}^*_\mathsf{h}, \tilde{z}^*_\mathsf{h})) = 0$ yields

$$\begin{aligned}\tilde{j}'_2(\tilde{d}_\mathsf{h})(\delta\tilde{d}_\mathsf{h}) &= \mathcal{L}_d(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{w}^*_\mathsf{h}, \tilde{z}^*_\mathsf{h}))(\delta\tilde{d}_\mathsf{h}) \\ &= \tilde{j}'_{2,\Omega}(\mathrm{ext}_\mathsf{h}\tilde{d}_\mathsf{h} + \tilde{w}^0_\mathsf{h})(\mathrm{ext}_\mathsf{h}\delta\tilde{d}_\mathsf{h}) - a^2_{\mathrm{ext},d}(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{w}^*_\mathsf{h}, \tilde{z}^*_\mathsf{h}))(\delta\tilde{d}_\mathsf{h})\end{aligned}$$

with

$$a^2_{\text{ext},d}(\tilde{d}_\mathsf{h}, (\tilde{w}^0_\mathsf{h}, \tilde{z}_\mathsf{h}), (\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h}))(\delta \tilde{d}_\mathsf{h}) = a((\text{ext}_\mathsf{h} \delta \tilde{d}_h, 0), (\tilde{\psi}^w_\mathsf{h}, \tilde{\psi}^z_\mathsf{h})),$$

due to (5.10). Numerically, one can work with the linear extension ext_h which sets the value of all interior nodes to 0. Let $\tilde{\mathbf{j}} \in \mathbb{R}^n$ be defined by

$$\tilde{\mathbf{j}}_i := \tilde{j}'_{2,\Omega}(\tilde{d}_h)(\Psi(\mathbf{e}_i))$$

for $i \in \{1, \dots, n\}$. The gradient of the mapping $j_2 : \mathbb{R}^{n_d} \to \mathbb{R}$, $j_2(\mathbf{d}) := \tilde{j}_{2,\Omega}(\mathrm{B}_{2,\mathsf{h}}(\Psi(\mathbf{Ad})))$ is given by $\nabla j_2 = \mathbf{A}^\top \tilde{\mathbf{j}}$.

5.5.2. Computing Function Value and Gradient of the Constraint with FEniCS and dolfin-adjoint

Strategy 1

Here, one could either compute the gradient of $\tilde{g}_\Omega(\mathrm{B}_{1,\mathsf{h}}(\tilde{d}_\mathsf{h}))$ via automated differentiation or use the gradient $\tilde{g}'_\Omega(\tilde{\mathbf{u}}_{\tau,\mathsf{h}})$ and apply the chain rule. Hence, we can compute the function value and gradient of the function $g_1(\mathbf{d}) := \tilde{g}_\Omega(\mathrm{B}_{1,\mathsf{h}}(\Psi(A\mathbf{d})))$.

Strategy 2

The functions $\tilde{g}(\tilde{\mathbf{d}})$ and $g_2(\mathbf{d})$ are explicitely given in Section 5.4.4. Direct calculus yields

$$(\nabla \tilde{g}(\tilde{\mathbf{d}}))_i = -\frac{1}{2}(\tilde{\mathbf{n}}_\tau)_{i,2}(\mathsf{y}_{i+1,1} - \mathsf{y}_{i-1,1}) - \frac{1}{2}(\tilde{\mathbf{n}}_\tau)_{i,1}(\mathsf{y}_{i-1,2} - \mathsf{y}_{i+1,2})$$

for $i \in \{1, \dots, n\}$, with $\mathsf{y}_{n+1,j} := \mathsf{y}_{1,j}$ and $\mathsf{y}_{0,j} := \mathsf{y}_{n,j}$ for $j \in \{1, 2\}$, where $\mathsf{y}_{i,j} = \mathsf{y}_{i,j}(\tilde{\mathbf{d}})$ depends on $\tilde{\mathbf{d}}$. Furthermore, we obtain $\nabla g_2(\mathbf{d}) := \mathbf{A}^\top \nabla \tilde{g}(\tilde{\mathbf{d}})$.

5.5.3. Solving the Discretized Optimization Problem Using IPOPT

In the previous sections we have seen that the optimization problem that has to be solved attains the form (5.11). Many existing implementations of optimization methods, such as IPOPT, assume that the problem is posed in the Euclidean space. Therefore, handing the discretized optimization problem directly to IPOPT leads to a loss of information since it is no longer taken into account that $\tilde{d}_\mathsf{h}$ is the discretization of a function, that has H^1-regularity if the regularization term R is chosen correspondingly. We have the representation $\tilde{d}_\mathsf{h}(\mathsf{z}) := \sum_{i=1}^n \tilde{\mathbf{d}}_i \Phi_i(\mathsf{z})$, where $\tilde{\mathbf{d}} := (\tilde{d}_1, ..., \tilde{d}_n)^\top \in \mathbb{R}^n$ denotes the vector of degrees of freedom and $\Phi_i(\mathsf{z}) \in H^1(\tilde{\Gamma}_d)$ are appropriate basis functions on the design part of the boundary. The correct discrete inner product is thus given by

$$(\tilde{d}_{1,\mathsf{h}}, \tilde{d}_{2,\mathsf{h}})_{H^1(\tilde{\Gamma}_d)} = \tilde{\mathbf{d}}_1^\top \tilde{\mathbf{S}} \tilde{\mathbf{d}}_2 = \mathbf{d}_1^\top \mathbf{S} \mathbf{d}_2,$$

where $\tilde{\mathbf{d}}_\mathbf{i} = \mathbf{A}\mathbf{d}_i$ for $i \in \{1, 2\}$, $\tilde{\mathbf{S}} = ((\Phi_i, \Phi_j)_{H^1(\tilde{\Gamma}_d)})_{i,j}$ and $\mathbf{S} = \mathbf{A}^\top \tilde{\mathbf{S}} \mathbf{A}$. Working on the space of transformed coordinates

$$\breve{\mathbf{d}} = \breve{\mathbf{S}} \mathbf{d},$$

where $\breve{\mathbf{S}}$ is chosen such that $\breve{\mathbf{S}}^\top\breve{\mathbf{S}} = \mathbf{S}$, e.g., $\breve{\mathbf{S}} = \mathbf{S}^{\frac{1}{2}}$ (which is impracticle if the size of $\mathbf{S}$ is large) or obtained by a (sparse) Cholesky decomposition, takes the above considerations into account. We pass the following functions to IPOPT

$$\breve{f} : \mathbb{R}^{n_d} \to \mathbb{R}, \quad \breve{\mathbf{d}} \mapsto f(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}),$$

where

$$\breve{f}(\breve{\mathbf{d}}) = j(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}) + \gamma\breve{\mathbf{d}}^\top\breve{\mathbf{d}},$$

as well as,

$$\nabla\breve{f} : \mathbb{R}^{n_d} \to \mathbb{R}^{n_d}, \quad \breve{\mathbf{d}} \mapsto \breve{\mathbf{S}}^{-\top}\nabla f(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}).$$

This has several advantages in the numerical solution process of the optimization problem. For the steepest descent method it results in

$$\mathbf{d}_{k+1} = \breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}_{k+1} = \breve{\mathbf{S}}^{-1}(\breve{\mathbf{d}}_k + \breve{\mathbf{S}}^{-\top}\nabla f(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}_k)) = \mathbf{d}_k + \mathbf{S}^{-1}\nabla f(\mathbf{d}_k)$$

if $\mathbf{d}_k = \breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}_k$. This is advantageous since in the function space setting, we have to work with the Riesz-representation of the gradient of f, which is approximated by $\mathbf{S}^{-1}\nabla f$. In [118] it is shown that this leads to mesh-independent convergence rates for some examples. This is also expected for other optimization algorithms. Hence, we consider the optimization problem

$$\begin{aligned} \min_{\breve{\mathbf{d}}\in\mathbb{R}^{n_d}} \; & f(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}) \\ \text{s.t.} \quad & \breve{g}(\breve{\mathbf{d}}) \leq 0, \\ & \mathbf{c}_l \leq \breve{\mathbf{g}}(\breve{\mathbf{d}}) \leq \mathbf{c}_u, \end{aligned}$$

where $\breve{g}(\breve{\mathbf{d}}) = g(\breve{\mathbf{S}}^{-1}\breve{\mathbf{d}})$ and $\breve{\mathbf{g}}(\breve{\mathbf{d}}) = \breve{\mathbf{S}}^{-1}\breve{\mathbf{d}}$.

Remark 5.7. This procedure is computationally justified in our setting since the degrees of freedom of the discretized control is small compared to the size of the discretized systems that are solved to evaluate the objective. This is due to the fact that the control is time independent and lives on the design part of the domain's boundary. In this case the effort for the computation of $\breve{\mathbf{S}}$ and the application of its inverse to $\breve{\mathbf{d}}$ is negligible compared to the effort for solving the FSI equations and its adjoint equations. In other situations, it might be preferable to apply an algorithm which directly works with the correct inner product.

5.6. Numerical Results

As time horizon for the optimization of the mean drag we choose $T = 15s$. Additionally, we use the regularization parameter $\gamma = 10$. For Strategy 1 IPOPT converges after 27 iterations and for Strategy 2 after 23 iterations with an overall NLP error (cf. [126, p. 3, (5)]) smaller than $1\cdot10^{-4}$, see Tables 5.1 and 5.2. The objective function value is reduced about 35 percent. Figure 5.5 shows the initial configuration compared to the optimized configurations for both strategies. On the one hand, the bounds are active close to the flap, which is motivated by

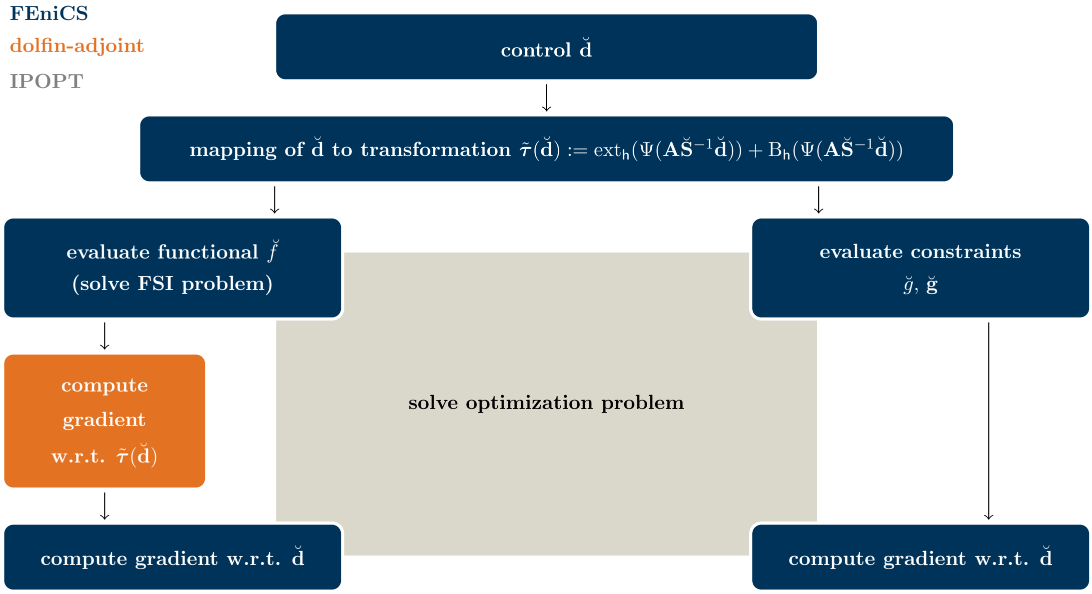
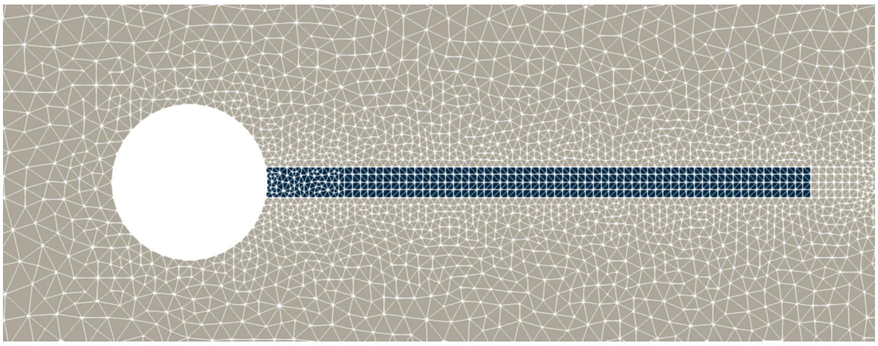


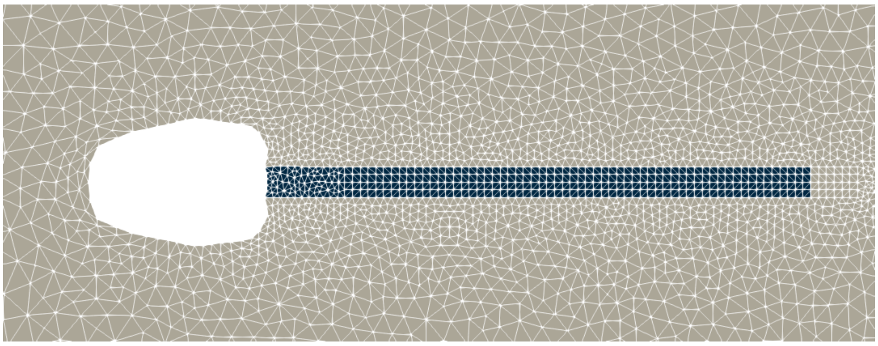
Figure 5.4.: Numerical realization using FEniCS, dolfin-adjoint and IPOPT

the fact that the optimal solutions aims at enclosing the flap. This, however, leads to mesh degeneration if the normal design boundary displacement is too large, which motivates the choice of the bound constraints. On the other hand, the bounds are active on the opposite site of the flap. Figure 5.6 shows the time-dependence of the vertical displacement of the tip of the flap and Figures 5.7 and 5.8 compare the corresponding snapshots for different times. Even though the optimization is only done on the first 15 seconds, Figure 5.6 illustrates that the amplitude of the vertical displacement of the tip of the flap is also smaller for long-term simulations.

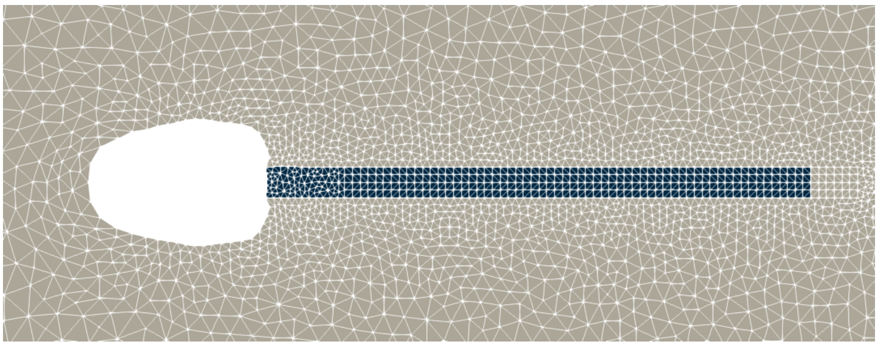
Remark 5.8. The computation times of the forward equations and the gradients do not significantly differ for the two strategies. However, Strategy 2 performs slightly better.



(1)



(2)



(3)

Figure 5.5.: (1) Initial design. (2) Optimized design with Strategy 1. (3) Optimized design with Strategy 2.

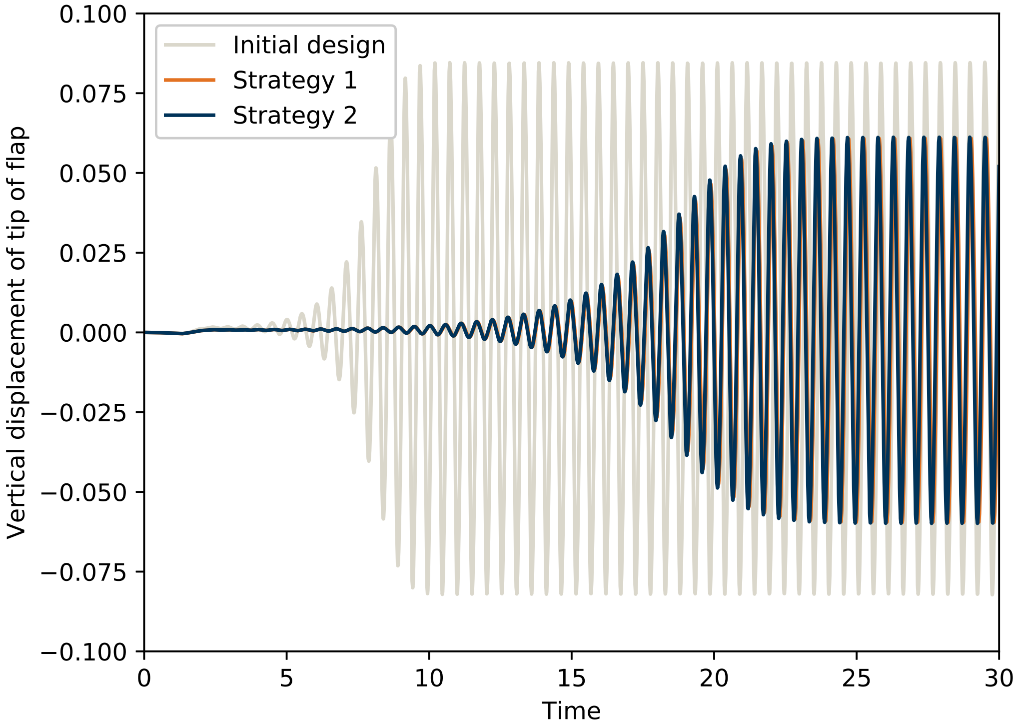


Figure 5.6.: Comparison of vertical displacement of the tip of the flap for the initial and optimized designs

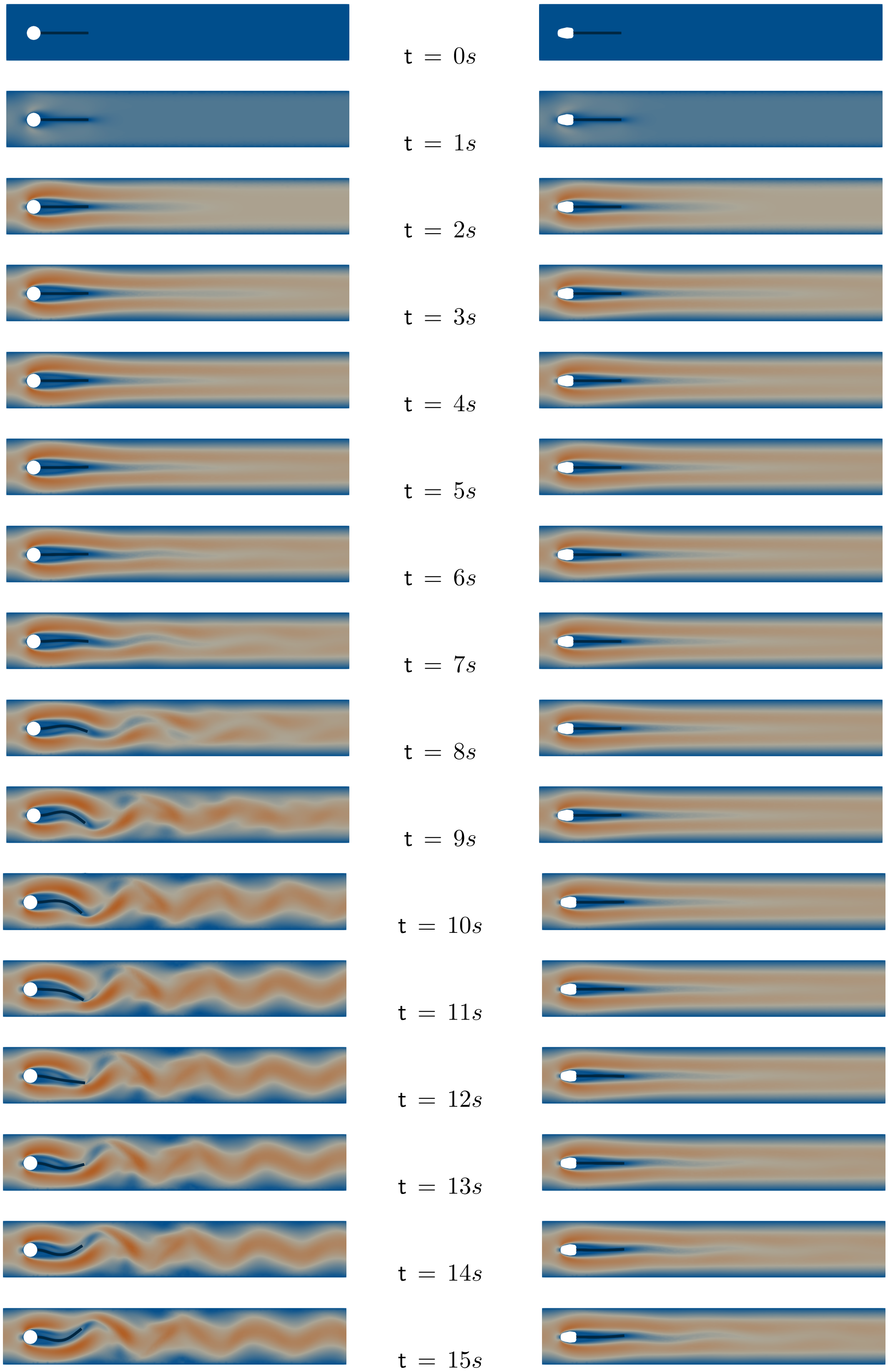


Figure 5.7.: Comparison of initial and optimized setting for Strategy 1

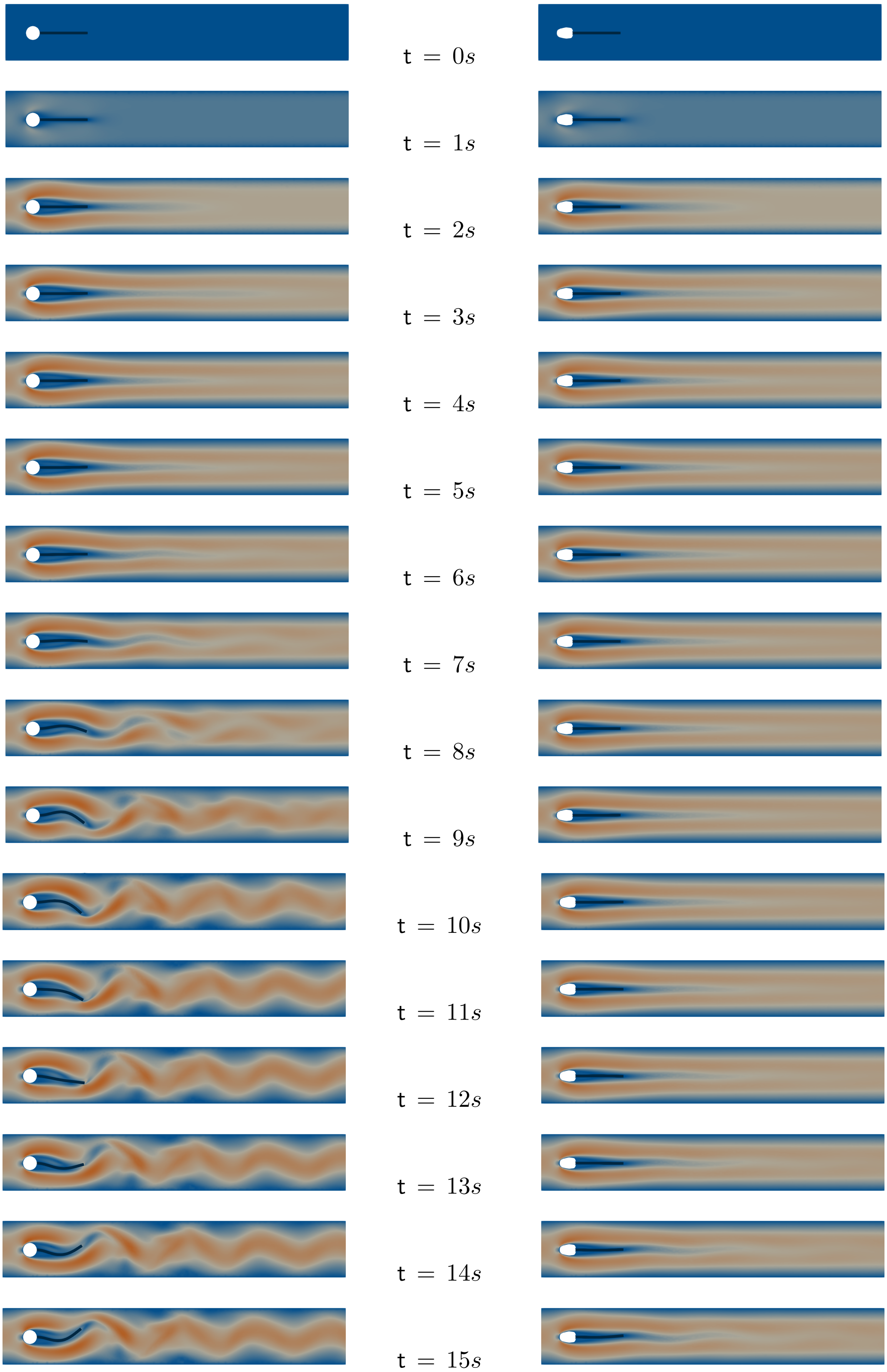


Figure 5.8.: Comparison of initial and optimized setting for Strategy 2

iteration	objective	dual infeasibility	linesearch-steps
0	$1.5998013 \cdot 10^2$	$6.47 \cdot 10^1$	0
1	$1.4272447 \cdot 10^2$	$4.68 \cdot 10^1$	2
2	$1.2470326 \cdot 10^2$	$2.61 \cdot 10^1$	1
3	$1.1414835 \cdot 10^2$	$1.40 \cdot 10^2$	1
4	$1.1486292 \cdot 10^2$	$1.35 \cdot 10^2$	1
5	$1.0696289 \cdot 10^2$	$1.29 \cdot 10^1$	1
6	$1.0554535 \cdot 10^2$	$1.73 \cdot 10^1$	1
7	$1.0453094 \cdot 10^2$	$1.32 \cdot 10^1$	1
8	$1.0347757 \cdot 10^2$	$1.04 \cdot 10^1$	1
9	$1.0281481 \cdot 10^2$	$8.69 \cdot 10^0$	1
10	$1.0231228 \cdot 10^2$	$2.26 \cdot 10^0$	1
11	$1.0208171 \cdot 10^2$	$2.35 \cdot 10^0$	1
12	$1.0197863 \cdot 10^2$	$1.57 \cdot 10^0$	1
13	$1.0193006 \cdot 10^2$	$1.33 \cdot 10^0$	1
14	$1.0191378 \cdot 10^2$	$7.75 \cdot 10^{-1}$	1
15	$1.0190267 \cdot 10^2$	$2.90 \cdot 10^{-1}$	1
16	$1.0190156 \cdot 10^2$	$7.15 \cdot 10^{-2}$	1
17	$1.0189694 \cdot 10^2$	$3.47 \cdot 10^{-2}$	1
18	$1.0189604 \cdot 10^2$	$2.33 \cdot 10^{-1}$	1
19	$1.0189587 \cdot 10^2$	$1.27 \cdot 10^{-1}$	1
20	$1.0189574 \cdot 10^2$	$2.65 \cdot 10^{-2}$	1
21	$1.0189570 \cdot 10^2$	$6.45 \cdot 10^{-3}$	1
22	$1.0189570 \cdot 10^2$	$5.21 \cdot 10^{-3}$	1
23	$1.0189570 \cdot 10^2$	$7.59 \cdot 10^{-4}$	1
24	$1.0189570 \cdot 10^2$	$4.83 \cdot 10^{-4}$	1
25	$1.0189570 \cdot 10^2$	$4.92 \cdot 10^{-4}$	1
26	$1.0189570 \cdot 10^2$	$1.54 \cdot 10^{-3}$	1
27	$1.0189570 \cdot 10^2$	$8.74 \cdot 10^{-5}$	1

Table 5.1.: Optimization results for Strategy 1

iteration	objective	dual infeasibility	linesearch-steps
0	$1.5998013 \cdot 10^2$	$6.46 \cdot 10^1$	0
1	$1.4293307 \cdot 10^2$	$4.43 \cdot 10^1$	2
2	$1.2493011 \cdot 10^2$	$2.70 \cdot 10^1$	1
3	$1.1440306 \cdot 10^2$	$1.38 \cdot 10^2$	1
4	$1.1183690 \cdot 10^2$	$1.04 \cdot 10^2$	1
5	$1.0606273 \cdot 10^2$	$1.03 \cdot 10^1$	1
6	$1.0554501 \cdot 10^2$	$3.85 \cdot 10^0$	1
7	$1.0401135 \cdot 10^2$	$1.22 \cdot 10^1$	1
8	$1.0330660 \cdot 10^2$	$1.02 \cdot 10^1$	1
9	$1.0292723 \cdot 10^2$	$5.76 \cdot 10^0$	1
10	$1.0280065 \cdot 10^2$	$1.15 \cdot 10^0$	1
11	$1.0270233 \cdot 10^2$	$3.85 \cdot 10^{-1}$	1
12	$1.0267593 \cdot 10^2$	$1.39 \cdot 10^{-1}$	1
13	$1.0267218 \cdot 10^2$	$1.66 \cdot 10^0$	1
14	$1.0266923 \cdot 10^2$	$1.55 \cdot 10^{-1}$	1
15	$1.0266877 \cdot 10^2$	$3.18 \cdot 10^{-2}$	1
16	$1.0266874 \cdot 10^2$	$5.14 \cdot 10^{-3}$	1
17	$1.0266873 \cdot 10^2$	$3.37 \cdot 10^{-3}$	1
18	$1.0266873 \cdot 10^2$	$2.60 \cdot 10^{-3}$	1
19	$1.0266972 \cdot 10^2$	$4.40 \cdot 10^{-3}$	1
20	$1.0266872 \cdot 10^2$	$6.06 \cdot 10^{-4}$	1
21	$1.0266872 \cdot 10^2$	$1.14 \cdot 10^{-3}$	1
22	$1.0266872 \cdot 10^2$	$1.03 \cdot 10^{-4}$	1
23	$1.0266872 \cdot 10^2$	$6.59 \cdot 10^{-5}$	1

Table 5.2.: Optimization results for Strategy 2

6. Conclusion and Outlook

In this thesis, we have extended the existence and regularity results of [113] for a linear and a nonlinear unsteady FSI system under the assumption that the new improved regularity for the linear hyperbolic wave equation can be adapted to the Lamé system. More precisely, we considered the coupling of the (Navier-)Stokes-Lamé system and obtained (local-in-time) existence and regularity results without the geometric constraint that the interface between the fluid and the solid is flat. Based on the method of successive approximations, which is the foundation for the theoretical analysis in [113], we developed a general framework for deriving continuity and differentiability results for unsteady nonlinear systems. Applying this framework to shape optimization of an unsteady nonlinear FSI problem via the method of mappings approach allowed us to prove differentiability of the states with respect to shape variations. Numerical tests showed the viability of the method of mappings for solving shape optimization problems governed by a nonlinear unsteady FSI model that couples the Navier-Stokes equations with nonlinear elasticity.

A remaining task is the adaption of the improved regularity result for the linear hyperbolic wave equation to the Lamé system. In addition, instead of considering the coupling with linear elastic material and either using a linear model for the fluid or just guaranteeing local-in-time results, practical applications are based on nonlinear elasticity and long-term simulations. Closing this gap is a difficult task and closely linked to advances in the analysis of hyperbolic equations and the Navier-Stokes equations. Another issue, which we circumvented by restriction to the optimiziation of the fluid part of the domain, is the extension of the differentiability results such that optimization of the interface is covered. The regularity requirements for the source term of the linear hyperbolic system requires more elaborate techniques to tackle this task. Also from a numerical point of view it is interesting to realize shape optimization of the interface, besides applying the presented methods to realistic 3D applications. Moreover, the consideration of these shape optimization problems as optimization problems on manifolds is left for future research.

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Last but not least I want to thank my family and friends for their support.

A. Appendix

A.1. On the Choice of the Norm on $H^s((0,T),X)$

A.1.1. Definition via Interpolation

Based on the following statements from interpolation theory, we construct a norm that satisfies properties **P1 - P8**. If X_0, X_1 are linear subspaces of a larger vector space V, then $\{X_0, X_1\}$ is said to be a compatible pair of Hilbert spaces. Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be compatible pairs of Hilbert spaces. Furthermore, for $s \in (0,1)$, let $[X_0, X_1]_s$ denote the complex interpolation space, cf. [25], [124], [15, p.166], [87, Sec. 0.2.1].

I1 In the Hilbert space setting, the definition of interpolation in [90, p.10, Def. 2.1] is equivalent to complex interpolation with equivalence of norms, cf., Remark 2.7.

I2 Interpolation theorem: Let $T : X_0 \to Y_0$ be a bounded linear operator with norm N_0 and $T : X_1 \to Y_1$ be a bounded linear operator with norm N_1.
Then, $T : [X_0, X_1]_s \to [Y_0, Y_1]_s$ is a bounded linear operator with norm smaller than or equal to $N_0^{1-s} N_1^s$ [25, p.115, 4.].

I3 Reiteration theorem: Let $0 \leq \alpha < \beta \leq 1$. Set $Y_\alpha = [X_0, X_1]_\alpha$ and $Y_\beta = [X_0, X_1]_\beta$ for Banach spaces X_0 and X_1. If $X_0 \cap X_1$ is dense in X_0, X_1 and $Y_\alpha \cap Y_\beta$, then, $[Y_\alpha, Y_\beta]_s = [X_0, X_1]_{(1-s)\alpha + s\beta}$, with equal norms [16, p.101, Theo. 4.6.1], [29].

Let X be a Hilbert space, $-\infty \leq a < b \leq \infty$. Let, for $\sigma \in [0,1]$, $\|\cdot\|_{H^\sigma((a,b),X)}$ be the norm induced by

$$[H^1((a,b),X), L^2((a,b),X)]_{1-\sigma}.$$

For $s = m + \sigma$, $m \in \mathbb{N}$, $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$, define

$$\|\cdot\|_{H^s((a,b),X)} = \begin{cases} \left(\|\cdot\|^2_{H^{m-1}((a,b),X)} + \|\partial_{\mathrm{t}}^m(\cdot)\|^2_{H^\sigma((a,b),X)}\right)^{\frac{1}{2}} & \text{if } \sigma \in (0,1) \setminus \{\frac{1}{2}\}, \\ \|\cdot\|_{H^m((a,b),X)} & \text{if } \sigma = 0. \end{cases}$$

Furthermore, let for $-\infty < T_1 < T_2 < -\infty$,

$$
\begin{aligned}
X^0_{(T_1,T_2)} &:= \{u \in L^2((T_1,T_2),X)\} \\
&\qquad \text{be endowed with the norm } \|\cdot\|_{X^0_{(T_1,T_2)}} := \|\cdot\|_{L^2((T_1,T_2),X)}, \\
X^1_{(T_1,T_2)} &:= \{u \in H^1((T_1,T_2),X) | \ u(T_1) = 0\} \\
&\qquad \text{be endowed with the norm } \|\cdot\|_{X^1_{(T_1,T_2)}} := \|\cdot\|_{H^1((T_1,T_2),X)}, \text{ and,} \\
X^s_{(T_1,T_2)} &:= \begin{cases} \{u \in H^s((T_1,T_2),X)\} & \text{if } s \in (0,\frac{1}{2}), \\ \{u \in H^s((T_1,T_2),X) \ : \ u(T_1) = 0\} & \text{if } s \in (\frac{1}{2},1), \\ \{u \in H^s((T_1,T_2),X) \ : \ u(T_1) = 0, \ \partial_t u \in X^{s-1}_{(T_1,T_2)}\} & \text{if } s \in (1,2] \setminus \{\frac{3}{2}\}, \end{cases} \\
&\qquad \text{be endowed with the norm } \|\!|\cdot|\!\|_{H^s((T_1,T_2),X)}.
\end{aligned}
\tag{A.1}
$$

Proposition A.1. Let $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$ and $-\infty < T_1 < T_2 < \infty$. Then, there exist $c_{\Delta T},\ C_{\Delta T} > 0$ that depend on $\Delta T = T_2 - T_1$ such that

$$c_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)} \leq \|\!|u|\!\|_{H^\sigma((T_1,T_2),X)} \leq C_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)}$$

for all $u \in H^\sigma((T_1,T_2),X)$.

Proof. Holds true due to Remark 2.7. □

Proposition A.2. Let $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$ and $-\infty < T_1 < T_2 < \infty$. Then, the extension-by-zero operator Ext is continuous as a mapping

$$X^\sigma_{(T_1,T_2)} \to H^\sigma((-\infty,T_2),X)$$

with a norm that, in general, depends on $\Delta T = T_2 - T_1$.

Proof. The proof builds on [90, p. 60, Theo. 11.4]. [90, p. 60, Theo. 11.4 and p. 62, Theo. 11.5] are formulated for the scalar valued case, however, [90, p. 47, Remark 9.5] and the proofs of the theorems imply the validity of the assertions for X-valued spaces. C and $C_{\Delta T}$ denote generic constants, where the subscript ΔT indicates the dependence on ΔT. Let $H^\sigma((-\infty,T_2),X)$ be endowed with $\|\!|\cdot|\!\|_{H^\sigma((-\infty,T_2),X)}$. The assertion also holds true for any equivalent norm on $H^\sigma((-\infty,T_2),X)$.

- Let $\sigma \in (0,\frac{1}{2})$.
  For $f \in X^\sigma_{(T_1,T_2)}$ we know by definition that $f \in H^\sigma((T_1,T_2),X)$. By [90, p. 60, Theo. 11.4], there exists a constant $C_{\Delta T} > 0$ such that

$$\|\!|\tilde{f}|\!\|_{H^\sigma(\mathbb{R},X)} \leq C_{\Delta T}\|\!|f|\!\|_{H^\sigma((T_1,T_2),X)}, \tag{A.2}$$

for $\tilde{f}(\mathsf{t}) := \begin{cases} f(\mathsf{t}) & \text{if } \mathsf{t} \in (T_1,T_2), \\ 0 & \text{else.} \end{cases}$

Since the restriction operator $\mathrm{R} : H^m(\mathbb{R},X) \to H^m((-\infty,T_2),X)$, $\mathrm{R}(\tilde{f}) := \tilde{f}|_{(-\infty,T_2)}$, is continuous with norm 1 for $m \in \{0,1\}$, **I2** yields

$$\|\mathrm{Ext}(f)\|_{H^\sigma((-\infty,T_2),X)} = \|R(\tilde{f})\|_{H^\sigma((-\infty,T_2),X)} \leq \|\tilde{f}\|_{H^\sigma(\mathbb{R},X)}. \tag{A.3}$$

Combining (A.2) and (A.3) yields the assertion.

- Let $\sigma \in (\frac{1}{2},1)$.
  Let $f \in X^\sigma_{(T_1,T_2)}$ and $\bar{f}(\mathsf{t}) := (T_2-T_1)^{-1}(\mathsf{t}-T_1)f(T_2)$ for $\mathsf{t} \in (T_1,T_2)$. Then, we know that $\bar{f} \in X^1_{(T_1,T_2)}$ such that $\bar{f}(T_2) = f(T_2)$ and

$$\|\bar{f}\|_{H^1((T_1,T_2),X)} \leq C_{\Delta T}\|f(T_2)\|_X. \tag{A.4}$$

By [90, p. 62, Theo. 11.5], $g := f - \bar{f} \in H^\sigma_0((T_1,T_2),X)$ and, therefore, by [90, p. 60, Theo. 11.4],

$$\begin{aligned}\|\tilde{g}\|_{H^\sigma(\mathbb{R},X)} &\leq C_{\Delta T}\|g\|_{H^\sigma((T_1,T_2),X)} \\ &\leq C_{\Delta T}(\|f\|_{H^\sigma((T_1,T_2),X)} + \|\bar{f}\|_{H^\sigma((T_1,T_2),X)}),\end{aligned} \tag{A.5}$$

where $\tilde{g} = \begin{cases} g(\mathsf{t}) & \text{for } \mathsf{t} \in (T_1,T_2), \\ 0 & \text{else.} \end{cases}$

Due to the interpolation inequality (2.3), (A.4), **I1** and [90, p.41, Proof of Thm. 9.4], we obtain

$$\begin{aligned}\|\bar{f}\|_{H^\sigma((T_1,T_2),X)} &\leq C_{\Delta T}\|\bar{f}\|_{H^1((T_1,T_2),X)} \leq C_{\Delta T}\|f(T_2)\|_X \\ &\leq C_{\Delta T}\|f\|_{H^\sigma((T_1,T_2),X)}.\end{aligned} \tag{A.6}$$

Thus, using the properties of the restriction operator R, cf. (A.3), yields

$$\|\mathrm{Ext}(g)\|_{H^\sigma((-\infty,T_2),X)} = \|\mathrm{R}(\tilde{g})\|_{H^\sigma((-\infty,T_2),X)} \leq \|\tilde{g}\|_{H^\sigma(\mathbb{R},X)}. \tag{A.7}$$

Additionally, we know that

$$\begin{aligned}\|\mathrm{Ext}(\bar{f})\|_{H^\sigma((-\infty,T_2),X)} &\leq C\|\mathrm{Ext}(\bar{f})\|_{H^1((-\infty,T_2),X)} \leq C\|\bar{f}\|_{H^1((T_1,T_2),X)} \\ &\leq C_{\Delta T}\|f\|_{H^\sigma((T_1,T_2),X)},\end{aligned} \tag{A.8}$$

where the last inequality follows from the same considerations as in (A.6). Combining (A.5), (A.6), (A.7) and (A.8) yields

$$\begin{aligned}\|\mathrm{Ext}(f)\|_{H^\sigma((-\infty,T_2),X)} &\leq \|\mathrm{Ext}(\bar{f})\|_{H^\sigma((-\infty,T_2),X)} + \|\mathrm{Ext}(g)\|_{H^\sigma((-\infty,T_2),X)} \\ &\leq C_{\Delta T}\|f\|_{H^\sigma((T_1,T_2),X)},\end{aligned}$$

from which the assertion follows.

□

Lemma A.3. Let $\sigma \in (0,1)\backslash\{\frac{1}{2}\}$. Then, there exists $C_{\Delta T} > 0$ that depends on $\Delta T = T_2 - T_1$ such that

$$\|u\|_{H^\sigma((T_1,T_2),X)} \leq \|u\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-\sigma}} \leq C_{\Delta T}\|u\|_{H^\sigma((T_1,T_2),X)} \tag{A.9}$$

for all $u \in X^\sigma_{(T_1,T_2)}$.

Proof. The mapping $\iota : u \mapsto u$ is continuous with norm 1 as a mapping $X^1_{(T_1,T_2)} \to H^1((T_1,T_2),X)$ and as a mapping $X^0_{(T_1,T_2)} \to L^2((T_1,T_2),X)$. **I2** yields that the mapping

$$\iota : \; [X^1_{(T_1,T_2)}, X^0_{(T_1,T_2)}]_{1-\sigma} \to H^\sigma((T_1,T_2),X)$$

is continuous with a constant smaller than or equal to 1 if $H^\sigma((T_1,T_2),X)$ is endowed with the norm $\|\cdot\|_{H^\sigma((T_1,T_2),X)}$. The extension operator Ext by 0 operator is continuous as a mapping $X^\sigma_{(T_1,T_2)} \to H^\sigma((-\infty,T_2),X)$ with constants that depend on ΔT due to Proposition A.2. The operator $\hat{\mathrm{R}} : u \mapsto u(\cdot) - u(2T_1 - \cdot)$ is continuous as a mapping

$$H^m((-\infty,T_2),X) \to X^m_{(T_1,T_2)},$$

$m \in \{0,1\}$ with norm 1, therefore, by **I2**, also as a mapping

$$H^\sigma((-\infty,T_2),X) \to [X^1_{(T_1,T_2)}, X^0_{(T_1,T_2)}]_{1-\sigma},$$

with norm 1 if $H^\sigma((-\infty,T_2),X)$ is endowed with the norm $\|\cdot\|_{H^\sigma((-\infty,T_2),X)}$. Using the continuity properties of Ext and $\hat{\mathrm{R}}$,

$$\begin{aligned}\|u\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-\sigma}} &= \|\hat{\mathrm{R}}(\mathrm{Ext}(u))\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-\sigma}} \\ &\leq \|\mathrm{Ext}(u)\|_{H^\sigma((-\infty,T_2),X)} \leq C_{\Delta T}\|u\|_{H^\sigma((T_1,T_2),X)}\end{aligned}$$

for all $u \in X^\sigma_{(T_1,T_2)}$. □

Let, for $\sigma \in (\frac{1}{2}, 1)$,

$$L : \quad H^\sigma((T_1,T_2),X) \to H^\sigma((T_1,\infty),X),$$

be a linear, continuous operator such that there exists a constant C independent of $\Delta T := T_2 - T_1$ such that

$$\|L(u)\|_{H^\sigma((T_1,\infty),X)} \leq C\|u(T_1)\|_X, \tag{A.10}$$

and for arbitrary but fixed $T_f > 0$ $L(u)(T_1) = u(T_1)$, more precisely,

$$L(u)(\mathsf{t}) = \begin{cases} u(T_1)T_f^{-1}(T_f + T_1 - \mathsf{t}) & \text{for } \mathsf{t} \in (T_1, T_1 + T_f), \\ 0 & \text{for } \mathsf{t} \in [T_1 + T_f, \infty). \end{cases} \tag{A.11}$$

Let $-\infty < T_1 < T_2 < \infty$. For $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$, define

$$\|\cdot\|_{H^\sigma((T_1,T_2),X)} := \begin{cases} \|\cdot\|_{L^2((T_1,T_2),X)} & \text{if } \sigma = 0,\\ \|\cdot\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-\sigma}} & \text{if } \sigma \in (0,\frac{1}{2}),\\ (\|\cdot - L(\cdot)\|^2_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-\sigma}} + \|\!|L(\cdot)|\!\|^2_{H^\sigma((T_1,\infty),X)})^{\frac{1}{2}} & \text{if } \sigma \in (\frac{1}{2},1), \end{cases} \tag{A.12}$$

and for $s = m+\sigma$, $m \in \mathbb{N}$, $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$,

$$\|\cdot\|_{H^s((T_1,T_2),X)} = \begin{cases} (\|\cdot\|^2_{H^{m-1}((T_1,T_2),X)} + \|\partial_t^m(\cdot)\|^2_{H^\sigma((T_1,T_2),X)})^{\frac{1}{2}} & \text{if } \sigma \in (0,1)\setminus\{\frac{1}{2}\},\\ \|\cdot\|_{H^m((T_1,T_2),X)} & \text{if } \sigma = 0. \end{cases} \tag{A.13}$$

Let, for $s \in [0,2] \setminus \{\frac{1}{2},\frac{3}{2}\}$, $H^s((T_1,T_2),X)$ be endowed with the norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ and

$$Y^s_{(T_1,T_2)} := \begin{cases} \{u \in H^s((T_1,T_2),X)\} & \text{if } s \in [0,\frac{1}{2}),\\ \{u \in H^s((T_1,T_2),X)|\ u(T_1) = 0\} & \text{if } s \in (\frac{1}{2},1],\\ \{u \in H^s((T_1,T_2),X)|\ u(T_1) = 0,\ \partial_t u \in Y^{s-1}_{(T_1,T_2)}\} & \text{if } s \in (1,2]\setminus\{\frac{3}{2}\}, \end{cases} \tag{A.14}$$

be endowed with the norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ with $L(\cdot) = 0$ for $s \in (\frac{1}{2},1)$.

Lemma A.4. Let X be a Hilbert space and $s \in [0,2]\setminus\{\frac{1}{2},\frac{3}{2}\}$. Then, the norm $\|\cdot\|_{H^s((T_1,T_2),X)}$ is equivalent to the norm $\|\!|\cdot|\!\|_{H^s((T_1,T_2),X)}$ with constants depending on $\Delta T = T_2 - T_1$.

Proof. The cases $s \in \{0,1,2\}$ are trivial.

- $s \in (0,\frac{1}{2})$: By (A.12) and (A.9)

$$\begin{aligned} \|u\|_{H^s((T_1,T_2),X)} &= \|u\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-s}} \le C_{\Delta T}\|\!|u|\!\|_{H^s((T_1,T_2),X)}\\ &\le C_{\Delta T}\|u\|_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-s}} = C_{\Delta T}\|u\|_{H^s((T_1,T_2),X)}. \end{aligned}$$

- $s \in (\frac{1}{2},1)$: By (A.10), (A.12) and (A.9)

$$\begin{aligned} \|u\|^2_{H^s((T_1,T_2),X)} &= \|u - L(u)\|^2_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-s}} + \|\!|L(u)|\!\|^2_{H^s((T_1,\infty),X)}\\ &\le C_{\Delta T}\|\!|u - L(u)|\!\|^2_{H^s((T_1,T_2),X)} + \|\!|L(u)|\!\|^2_{H^s((T_1,\infty),X)}\\ &\le C_{\Delta T}(\|\!|u|\!\|^2_{H^s((T_1,T_2),X)} + \|\!|L(u)|\!\|^2_{H^s((T_1,\infty),X)})\\ &\le C_{\Delta T}(\|\!|u|\!\|^2_{H^s((T_1,T_2),X)} + \|u(T_1)\|^2_X)\\ &\le C_{\Delta T}\|\!|u|\!\|^2_{H^s((T_1,T_2),X)}, \end{aligned}$$

  where the last inequality is due to **I1** and [90, p.41, Proof of Thm. 9.4]. The estimate for $\|L(u)\|_{H^s((T_1,T_2),X)}$ is obtained by interpolation of the restriction operator

$$\tilde{\mathrm{R}}: \; H^m((T_1,\infty),X) \to H^m((T_1,T_2),X), \; \tilde{\mathrm{R}}(u)(\mathsf{t}) = u(\mathsf{t}) \; \forall \mathsf{t} \in (T_1,T_2),$$

which, by **I2**, is continuous with constant 1 for $m \in \{0,1\}$.

- $s \in (1,2] \setminus \{\frac{3}{2}\}$: Follows directly from the results obtained for $s \in [0,1] \setminus \{\frac{1}{2}\}$. By (A.9),

$$\begin{aligned} \|u\|^2_{H^s((T_1,T_2),X)} &\leq C(\|u - L(u)\|^2_{H^s((T_1,T_2),X)} + \|L(u)\|^2_{H^s((T_1,T_2),X)}) \\ &\leq C(\|u - L(u)\|^2_{[X^1_{(T_1,T_2)},X^0_{(T_1,T_2)}]_{1-s}} + \|L(u)\|^2_{H^s((T_1,\infty),X)}). \end{aligned}$$

□

Corollary A.5. Let X be a Hilbert space and $s \in [0,2] \setminus \{\frac{1}{2}, \frac{3}{2}\}$. Then, $\|\cdot\|_{H^s((T_1,T_2),X)}$ is equivalent to the norm $|\cdot|_{H^s((T_1,T_2),X)}$ with constants depending on $\Delta T = T_2 - T_1$.

Proof. Follows by combining Proposition A.1 and Lemma A.4. □

Lemma A.6. Let $T_f \geq T$ and $s \geq 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$. Then, the extension operator

$$\mathrm{Ext}(u) := \begin{cases} 0 & \text{for } \mathsf{t} \in (T - T_f, 0) \\ u(\mathsf{t}) & \text{for } \mathsf{t} \in (0,T) \end{cases},$$

is continuous as a mapping $Y^s_{(0,T)} \to Y^s_{(T-T_f,T)}$ with norm bounded by 1.

Proof. For $s \in \mathbb{N}_0$ the assertion is trivial.

- For $m \in \{0,1\}$, Ext is continuous as a mapping $X^m_{(0,T)} \to X^m_{(T-T_f,T)}$ with norm 1, thus, by **I2**, continuous as a mapping

$$[X^1_{(0,T)}, X^0_{(0,T)}]_{1-s} \to [X^1_{(T-T_f,T)}, X^0_{(T-T_f,T)}]_{1-s} \tag{A.15}$$

  with norm bounded by 1.

- Let $s \in (0, \frac{1}{2})$. Due to (A.15), $Y^s_{(0,T)} = X^s_{(0,T)}$ (with different norms), cf. (A.14) and (A.1), and (A.12) we obtain

$$\|\mathrm{Ext}(u)\|_{H^s((T-T_f,T),X)} \leq \|u\|_{H^s((0,T),X)}$$

  for all $u \in Y^s_{(0,T)}$.

- Let $s \in (\frac{1}{2}, 1)$. For $u \in Y^s_{(0,T)}$, we have $u(0) = 0$ and $\mathrm{Ext}(u)(T - T_f) = 0$. Thus, by (A.12),

$$\|u\|_{H^s((0,T),X)} = \|u\|_{[X^1_{(0,T)},X^0_{(0,T)}]_{1-s}},$$

and

$$\|\mathrm{Ext}(u)\|_{H^s((T-T_f,T),X)} = \|\mathrm{Ext}(u)\|_{[X^1_{(T-T_f,T)},X^0_{(T-T_f,T)}]_{1-s}}.$$

Due to (A.15), we obtain

$$\|\mathrm{Ext}(u)\|_{H^s((T-T_f,T),X)} \leq \|u\|_{H^s((0,T),X)}.$$

- For $s \geq 1$ such that $s+\frac{1}{2} \notin \mathbb{N}$, the assertion follows from the results obtained for $s \in [0,1] \setminus \{\frac{1}{2}\}$ and $s \in \mathbb{N}$.

□

Lemma A.7. Let $0 < T \leq T_f$ and $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$. Then, the extension operator

$$\widehat{\mathrm{Ext}}(u) := \begin{cases} 0 & \text{for } \mathsf{t} \in (T-2T_f, T-T_f) \\ u(\mathsf{t}) & \text{for } \mathsf{t} \in (T-T_f, T) \end{cases},$$

is continuous as a mapping $Y^s_{(T-T_f,T)} \to Y^s_{(T-2T_f,T)}$ with norm bounded by 1.

Proof. Completely analogous to the proof of Lemma A.6. □

Lemma A.8. Let $0 < T \leq T_f$, X be a Hilbert space and $s \geq 0$ such that $s+\frac{1}{2} \notin \mathbb{N}$. Then, the restriction operator

$$\mathrm{R}(\cdot)(\mathsf{t}) = \cdot(\mathsf{t})$$

is continuous as a mapping $H^s((0,T_f),X) \to H^s((0,T),X)$ with norm bounded by 1 if $H^s((0,T_f),X)$ and $H^s((0,T),X)$ are endowed with the norm defined by (A.12) and (A.13).

Proof. For $s \in \mathbb{N}_0$ the assertion is trivial.

- Let $s \in (0,\frac{1}{2})$. The restriction operator R is continuous as a mapping $X^m_{(0,T_f)} \to X^m_{(0,T)}$ for $m \in \{0,1\}$ with norm 1. Therefore, **I2** yields continuity as a mapping

$$[X^1_{(0,T_f)}, X^0_{(0,T_f)}]_{1-s} \to [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s}, \tag{A.16}$$

and, using

$$H^s((0,T_2),X) = Y^s((0,T_2),X) = X^s((0,T_2),X) = [X^1_{(0,T_2)}, X^0_{(0,T_2)}]_{1-s}$$

(with different norms) for $T_2 \in \{T, T_f\}$, cf. (A.1), (A.14), Lemma A.3, and the definition of the norm (A.12), as a mapping

$$H^s((0,T_f),X) \to H^s((0,T),X)$$

with norm bounded by 1.

- Let $s \in (\frac{1}{2}, 1)$. It holds $u - L(u) \in Y^s_{(0,T_f)}$, thus, by (A.16) (which holds also true for $s \in (\frac{1}{2}, 1)$),

$$\|u - L(u)\|_{[X^1_{(0,T)}, X^0_{(0,T)}]_{1-s}} \leq \|u - L(u)\|_{[X^1_{(0,T_f)}, X^0_{(0,T_f)}]_{1-s}}.$$

  This estimate and the definition of the norm (A.12) yield the assertion.

- For $s \geq 1$ such that $s + \frac{1}{2} \notin \mathbb{N}$, the assertion follows from the results for $s \in [0, 1] \setminus \{\frac{1}{2}\}$ and $s \in \mathbb{N}$.

□

Lemma A.9. Let $s \in (0, 1) \setminus \{\frac{1}{2}\}$, $T > 0$, $\hat{T} \leq -T$ and X be a Hilbert space. Then, the operator

$$\hat{\mathrm{R}}(\cdot)(\mathsf{t}) = \cdot(\mathsf{t}) - \cdot(-\mathsf{t})$$

is continuous as a mapping $Y^s_{(\hat{T},T)} \to Y^s_{(0,T)}$ with norm bounded by 1.

Proof. The operator $\hat{\mathrm{R}}$ is continuous as a mapping $X^m_{(\hat{T},T)} \to X^m_{(0,T)}$, $m \in \{0, 1\}$, with norm at most 1. Therefore, using **I2**, it is also continuous as a mapping

$$[X^1_{(\hat{T},T)}, X^0_{(\hat{T},T)}]_{1-s} \to [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s},$$

with norm at most 1. Thus, by definitions (A.12) and (A.14), it is also continuous as a mapping $Y^s_{(\hat{T},T)} \to Y^s_{(0,T)}$ with norm bounded by 1. □

Lemma A.10. Let $T^* \geq 2T > 0$, X be a Hilbert space and $s \geq 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$. Furthermore, let $u \in H^s((T - T^*, T), X)$ be such that $u|_{(T-T^*,0)} = 0$. Then,

$$\|u\|_{H^s((0,T),X)} \leq C\|u\|_{H^s((T-T^*,T),X)}$$

with a constant C independent of T.

Proof. It holds $\|u\|_{H^m((0,T),X)} \leq \|u\|_{H^m((T-T^*,T),X)}$ for $m \in \mathbb{N}_0$. Let s be non-integer. Then, there exist $m \in \mathbb{N}_0$ and $\sigma \in (0, 1) \setminus \{\frac{1}{2}\}$ such that $s = m + \sigma$. By (A.14), $\partial^m_{\mathsf{t}} u \in Y^\sigma_{(T-T^*,T)}$. Using $u|_{(T-T^*,0)} = 0$, we obtain $\hat{\mathrm{R}}(\partial^m_{\mathsf{t}} u) = \partial^m_{\mathsf{t}} u|_{(0,T)}$ for $\hat{\mathrm{R}}$ in Lemma A.9 with $\hat{T} = T - T^*$ yields

$$\|\partial^m_{\mathsf{t}} u\|_{H^\sigma((0,T),X)} \leq \|\partial^m_{\mathsf{t}} u\|_{H^\sigma((T-T^*,T),X)}$$

This implies the assertion. □

Lemma A.11. Let $T_f \geq T > 0$, X be a Hilbert space and $s \geq 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$. Furthermore, let $u \in H^s((T - T_f, T), X)$ be such that $u|_{(T-T_f,0)} = 0$. Then,

$$\|u\|_{H^s((0,T),X)} \leq C\|u\|_{H^s((T-T_f,T),X)}$$

with a constant C independent of T.

Proof. By Lemmas A.10 and A.7,

$$\begin{aligned}\|u\|_{H^s((0,T),X)} &= \|\widehat{\mathrm{Ext}}(u)\|_{H^s((0,T),X)}\\ &\leq C\|\widehat{\mathrm{Ext}}(u)\|_{H^s((T-2T_f,T),X)} \leq C\|u\|_{H^s((T-T_f,T),X)}.\end{aligned}$$

□

Lemma A.12. Let X be a Hilbert space, $s \in [0,1) \setminus \{\frac{1}{2}\}$ and $\alpha > 0$ be chosen such that $s + \alpha \in (0,1]$. Then,

$$\|u\|_{H^s((0,T),X)} \leq T^\alpha \|u\|_{H^{s+\alpha}((0,T),X)},$$

for all $u \in Y^s_{(0,T)}$.

Proof. The mapping ι defined by $\iota(u) = u$ is continous as a mapping

$$\iota : X^0_{(0,T)} \to X^0_{(0,T)}, \tag{A.17}$$

$$\iota : X^1_{(0,T)} \to X^1_{(0,T)}, \tag{A.18}$$

with norm 1 and

$$\iota : X^1_{(0,T)} \to X^0_{(0,T)} \tag{A.19}$$

with norm bounded by T. Due to **I2**, interpolation of (A.17) and (A.18) yields continuity of

$$\iota : [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s} \to [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s} \tag{A.20}$$

with norm at most 1 (equal to 1 since the mapping corresponds to the identity) and interpolation of (A.18) and (A.19) yields continuity of

$$\iota : X^1_{(0,T)} \to [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s} \tag{A.21}$$

with norm bounded by T^{1-s}. Interpolating (A.20) and (A.21) yields continuity as a mapping

$$\iota : [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s-\alpha} \to [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s}$$

with norm bounded by T^α, since reiteration gives

$$[X^1_{(0,T)}, [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s}]_{\frac{1-s-\alpha}{1-s}} = [X^1_{(0,T)}, X^0_{(0,T)}]_{1-s-\alpha}$$

with equal norms due to **I3**. □

Lemma A.13. Let X_1, X_2 be real, separable Hilbert spaces and $s \in (0,1) \setminus \{\frac{1}{2}\}$. Let K be a linear operator that is continuous as a mapping $X_1 \to X_2$ and $f \in H^s((0,T), X_1)$. Then,

$$\|K(f)\|_{H^s((0,T),X_2)} \leq C\|f\|_{H^s((0,T),X_1)}$$

with a constant C independent of T.

Proof. Let $-\infty < T_1 < T_2 \leq \infty$ The operator $\tilde{K}$ defined by $\tilde{K}(f)(\mathsf{t}) = K(f(\mathsf{t}))$ for a.e. $\mathsf{t} \in (T_1, T_2)$ is continuous as a mapping $L^2((T_1,T_2),X_1) \to L^2((T_1,T_2),X_2)$ and as a mapping $H^1((T_1,T_2),X_1) \to H^1((T_1,T_2),X_2)$ with norms bounded by C independent of T, since $\partial_{\mathsf{t}}\tilde{K}(f) = \tilde{K}(\partial_{\mathsf{t}} f)$. For $f \in H^1((T_1,T_2),X_1)$ with $f(T_1) = 0$ we obtain $\tilde{K}(f)(T_1) = K(f(T_1)) = 0$. Therefore, $\tilde{K}$ is continuous as a mapping $(X_1)^0_{(T_1,T_2)} \to (X_2)^0_{(T_1,T_2)}$ and as a mapping $(X_1)^1_{(T_1,T_2)} \to (X_2)^1_{(T_1,T_2)}$ with norms at most C. By **I2**,

$$\begin{aligned} \tilde{K} :\ & H^s((T_1,T_2),X_1) \to H^s((T_1,T_2),X_2), \\ \tilde{K} :\ & [(X_1)^1_{(T_1,T_2)}, (X_1)^0_{(T_1,T_2)}]_{1-s} \to [(X_2)^1_{(T_1,T_2)}, (X_2)^0_{(T_1,T_2)}]_{1-s}, \end{aligned} \tag{A.22}$$

are continuous with norms at most C. By definitions (A.1) and (A.12) the assertion is shown for $s \in (0, \frac{1}{2})$. Moreover, the definition (A.11) of L on $(0,T)$ implies for $s \in (\frac{1}{2}, 1)$

$$L(\tilde{K}(f)) = (\tilde{K}(f))(0)T_f^{-1}(T_f - \mathsf{t}) = K(f(0)T_f^{-1}(T_f - \mathsf{t})) = K(L(f)(\mathsf{t})) = \tilde{K}(L(f)). \tag{A.23}$$

By (A.22) and (A.23),

$$\begin{aligned} \|\tilde{K}(f) - L(\tilde{K}(f))\|_{[(X_2)^1_{(0,T)},(X_2)^0_{(0,T)}]_{1-s}} &= \|\tilde{K}(f - L(f))\|_{[(X_2)^1_{(0,T)},(X_2)^0_{(0,T)}]_{1-s}} \\ &\leq C\|f - L(f)\|_{[(X_1)^1_{(0,T)},(X_1)^0_{(0,T)}]_{1-s}}, \\ \|L(\tilde{K}(f))\|_{H^s((0,\infty),X_2)} = \|\tilde{K}(L(f))\|_{H^s((0,\infty),X_2)} &\leq C\|L(f)\|_{H^s((0,\infty),X_1)}, \end{aligned}$$

with a constant C independent of T. □

Lemma A.14. Let X be a real, separable Hilbert space, $s \geq 0$ such that $s + \frac{1}{2} \notin \mathbb{N}$ and $T_1 < T_2$. Then,

$$\|u\|_{H^s((T_1,T_2),X)} = \|\tilde{u}\|_{H^s((0,T_2-T_1),X)},$$

for all $u \in H^s((T_1,T_2),X)$, where $\tilde{u}(\mathsf{t}) := u(\mathsf{t} + T_1)$ for (a.e.) $\mathsf{t} \in (0, T_2 - T_1)$.

Proof. For $s \in \mathbb{N}_0$ the assertion holds true. Let $s \in (0,1) \setminus \{\frac{1}{2}\}$. Let $\tilde{T}_2 > T_1$ ($\tilde{T}_2 = \infty$ not excluded). It holds

$$\begin{aligned} \|u\|_{H^1((T_1,\tilde{T}_2),X)} &= \|\tilde{u}\|_{H^1((0,\tilde{T}_2-T_1),X)}, \\ \|u\|_{L^2((T_1,\tilde{T}_2),X)} &= \|\tilde{u}\|_{L^2((0,\tilde{T}_2-T_1),X)}, \end{aligned}$$

for all $u \in H^1((T_1,\tilde{T}_2),X)$, and $u \in L^2((T_1,\tilde{T}_2),X)$ respectively, and $\tilde{u}(\mathsf{t}) = u(\mathsf{t}+T_1)$ for (a.e.) $\mathsf{t} \in (0, \tilde{T}_2 - T_1)$. Interpolation and **I2** yields continuity of the mapping $u \mapsto \tilde{u}$ as a mapping from $H^s((T_1,\tilde{T}_2),X) \to H^s((0,\tilde{T}_2 - T_1),X)$ and as a mapping from $[X^1_{(T_1,\tilde{T}_2)}, X^0_{(T_1,\tilde{T}_2)}]_{1-s} \to [X^1_{(0,\tilde{T}_2-T_1)}, X^0_{(0,\tilde{T}_2-T_1)}]_{1-s}$ with continuity constants bounded by 1. Analogously, we obtain continuity of the mapping $\tilde{u} \mapsto u$ as a mapping from $H^s((0,\tilde{T}_2 - T_1),X) \to H^s((T_1,\tilde{T}_2),X)$

and as a mapping from $[X^1_{(0,\tilde{T}_2-T_1)}, X^0_{(0,\tilde{T}_2-T_1)}]_{1-s} \to [X^1_{(T_1,\tilde{T}_2)}, X^0_{(T_1,\tilde{T}_2)}]_{1-s}$ with continuity constants bounded by 1. Hence,

$$\|u\|_{H^s((T_1,\tilde{T}_2),X)} = \|\tilde{u}\|_{H^s((0,\tilde{T}_2-T_1),X)},$$
$$\|u\|_{[X^1_{(T_1,\tilde{T}_2)},X^0_{(T_1,\tilde{T}_2)}]_{1-s}} = \|\tilde{u}\|_{[X^1_{(0,\tilde{T}_2-T_1)},X^0_{(0,\tilde{T}_2-T_1)}]_{1-s}}.$$

Furthermore, $\widetilde{L(u)} = L(\tilde{u})$ in case $s \in (\frac{1}{2}, 1)$, where $\widetilde{L(u)}(\mathsf{t}) := L(u)(\mathsf{t}+T_1)$ for $\mathsf{t} \in (0, T_2-T_1)$. This yields the result for $s \in (0,1) \setminus \{\frac{1}{2}\}$. The proof for $s \geq 1$ such that $s + \frac{1}{2} \notin \mathbb{N}$ is straightforward. □

With these Lemmas it is straightforward to verify properties **P1 - P8**.

A.1.2. Definition via Sobolev-Slobodeckij-Norm

Alternatively, one can also construct a norm that satisfies properties **P1 - P8** by using the Sobolev-Slobodeckij norm. Let $-\infty < T_1 < T_2 \leq \infty$. Let $\|\cdot\|_{H^0((T_1,T_2),X)} := \|\cdot\|_{L^2((T_1,T_2),X)}$. Furthermore, let for $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$,

$$\|\cdot\|_{H^\sigma((T_1,T_2),X)} := \begin{cases} (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|\cdot(\mathsf{t})\|_X^2 d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (0,\frac{1}{2}), \\ (|\cdot|^2_{H^\sigma((T_1,T_2),X)} + \frac{1}{\sigma}\int_{T_1}^{T_2}(\mathsf{t}-T_1)^{-2\sigma}\|(\cdot - L(\cdot))(\mathsf{t})\|_X^2 d\mathsf{t})^{\frac{1}{2}} & \text{if } \sigma \in (\frac{1}{2},1), \end{cases} \tag{A.24}$$

where, for $\sigma \in (\frac{1}{2}, 1)$, L is chosen as a linear operator that is continuous as a mapping

$$L: \quad H^\sigma((T_1,T_2),X) \to H^\sigma((T_1,\infty),X),$$

and such that there exists a constant C independent of $\Delta T := T_2 - T_1$ such that

$$|L(u)|_{H^\sigma((T_1,\infty),X)} \leq C\|u(T_1)\|_X,$$

and $L(u)(T_1) = u(T_1)$, e.g., for fixed $T_f > 0$,

$$L(u)(\mathsf{t}) = \begin{cases} u(T_1)T_f^{-1}(T_f + T_1 - \mathsf{t}) & \text{for } \mathsf{t} \in (T_1, T_1 + T_f), \\ 0 & \text{for } \mathsf{t} \in [T_1 + T_f, \infty). \end{cases}$$

Lemma A.15. Let $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$. Then, there exist $C_{\Delta T} > 0$ that depends on $\Delta T = T_2 - T_1$ such that

$$|u|_{H^\sigma((T_1,T_2),X)} \leq \|u\|_{H^\sigma((T_1,T_2),X)} \leq C_{\Delta T}|u|_{H^\sigma((T_1,T_2),X)}$$

for all $u \in H^\sigma((T_1,T_2),X)$.

Proof. The first inequality follows directly from the definition of $\|\cdot\|_{H^\sigma((T_1,T_2),X)}$. The second inequality is a direct consequence of [90, p. 57, Thm. 11.2 and p. 59, Thm. 11.3]. Even though the theorems [90, Thm. 11.2 - Thm. 11.3] are formulated in the scalar valued spaces, the proofs of the theorems, as well as, [90, p. 47, Remark 9.5] imply the validity for X-valued spaces. More precisely, the following holds:

- Let $\sigma \in (0, \frac{1}{2})$ and $u \in H^\sigma((T_1, T_2), X)$:
Due to [90, p. 60, Thm. 11.4] we know that

$$|\tilde{u}|_{H^\sigma(\mathbb{R},X)} \le C_{\Delta T} |u|_{H^\sigma((T_1,T_2),X)}$$

for $\tilde{u}(\mathsf{t}) := \begin{cases} u(\mathsf{t}) & \text{if } \mathsf{t} \in (T_1, T_2), \\ 0 & \text{else.} \end{cases}$ Thus, for $T_3 > T_2$, using [90, p.57, Thm. 11.2],

$$\begin{aligned}\int_{T_1}^{T_2} (\mathsf{t} - T_1)^{-2\sigma} \|u(\mathsf{t})\|_X^2 d\mathsf{t} &\le \int_{T_1}^{T_3} (\mathsf{t} - T_1)^{-2\sigma} \|\tilde{u}(\mathsf{t})\|_X^2 d\mathsf{t} \\ &\le C |\tilde{u}|^2_{H^\sigma((T_1,T_3),X)} \le C |\tilde{u}|^2_{H^\sigma(\mathbb{R},X)} \le C_{\Delta T} |u|^2_{H^\sigma((T_1,T_2),X)}.\end{aligned}$$

- Let $\sigma \in (\frac{1}{2}, 1)$ and $u \in H^\sigma((T_1, T_2), X)$:
Let $v := u - L(u)$, $v_2(\mathsf{t}) := (T_2 - T_1)^{-1}(\mathsf{t} - T_1) v(T_2)$ for $\mathsf{t} \in (T_1, T_2)$ and define $v_1 := v - v_2$. Due to [90, p. 62, Thm. 11.5], $v_1 \in H_0^\sigma((T_1, T_2), X)$ and, by [90, p. 60, Thm. 11.4],

$$|\tilde{v}_1|_{H^\sigma(\mathbb{R},X)} \le C_{\Delta T} |v_1|_{H^\sigma((T_1,T_2),X)},$$

where $\tilde{v}_1(\mathsf{t}) := \begin{cases} v_1(\mathsf{t}) & \text{if } \mathsf{t} \in (T_1, T_2), \\ 0 & \text{else.} \end{cases}$ Furthermore, for $T_3 > T_2$, using [90, p. 59, Thm. 11.3],

$$\begin{aligned}&\int_{T_1}^{T_2} (\mathsf{t} - T_1)^{-2\sigma} \|v_1(\mathsf{t})\|_X^2 d\mathsf{t} \le \int_{T_1}^{T_3} (\mathsf{t} - T_1)^{-2\sigma} \|\tilde{v}_1(\mathsf{t})\|_X^2 d\mathsf{t} \\ &\le C |\tilde{v}_1|^2_{H^\sigma((T_1,T_3),X)} \le C |\tilde{v}_1|^2_{H^\sigma(\mathbb{R},X)} \le C_{\Delta T} |v_1|^2_{H^\sigma((T_1,T_2),X)}\end{aligned} \tag{A.25}$$

and

$$\begin{aligned}\int_{T_1}^{T_2} (\mathsf{t} - T_1)^{-2\sigma} \|v_2(\mathsf{t})\|_X^2 d\mathsf{t} &= \int_{T_1}^{T_2} (\mathsf{t} - T_1)^{2-2\sigma} (T_2 - T_1)^{-2} \|v(T_2)\|_X^2 d\mathsf{t} \\ &= (3 - 2\sigma)^{-1} (T_2 - T_1)^{1-2\sigma} \|v(T_2)\|_X^2 \\ &\le C_{\Delta T} |v|^2_{H^\sigma((T_1,T_2),X)}.\end{aligned} \tag{A.26}$$

Since $|v_2|_{H^\sigma((T_1,T_2);X)} \le C_{\Delta T} \|v(T_2)\|_X \le C_{\Delta T} |v|_{H^\sigma((T_1,T_2),X)}$ and

$$\begin{aligned}|v|_{H^\sigma((T_1,T_2),X)} &\le |u|_{H^\sigma((T_1,T_2),X)} + |L(u)|_{H^\sigma((T_1,T_2),X)} \\ &\le |u|_{H^\sigma((T_1,T_2),X)} + C \|u(T_1)\|_X \le C_{\Delta T} |u|_{H^\sigma((T_1,T_2),X)},\end{aligned} \tag{A.27}$$

where the last inequality follows from the equivalence of the Sobolev-Slobodeckij norm to interpolation norms (Remark 2.7) and [90, p.41, Proof of Thm. 9.4]. (A.25), (A.26) and (A.27) imply

$$\int_{T_1}^{T_2} (\mathsf{t}-T_1)^{-2\sigma}\|v(\mathsf{t})\|_X^2 d\mathsf{t} \le C_{\Delta T}|u|^2_{H^\sigma((T_1,T_2),X)}.$$

□

Lemma A.16. Let $T_f \geq T$ and $\sigma \in (0,1)\setminus\{\frac{1}{2}\}$. Then, the extension operator

$$\mathrm{Ext}u := \begin{cases} 0 & \text{for } \mathsf{t} \in (T-T_f,0), \\ u(\mathsf{t}) & \text{for } \mathsf{t} \in (0,T), \end{cases}$$

is continuous as a mapping $Y^\sigma_{(0,T)} \to Y^\sigma_{(T-T_f,T)}$ with norm bounded by 1.

Proof. Direct computations show that for $u \in Y^\sigma_{(0,T)}$,

$$\|u\|_{H^\sigma((0,T),X)} = |\mathrm{Ext}(u)|_{H^\sigma((-\infty,T),X)} = \|\mathrm{Ext}(u)\|_{H^\sigma((T-T_f,T),X)}. \tag{A.28}$$

This can be verified as follows: It holds that

$$|\mathrm{Ext}(u)|^2_{H^\sigma((-\infty,T),X)} = \int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t})-\mathrm{Ext}(u)(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + \|\mathrm{Ext}(u)\|^2_{L^2((-\infty,T),X)}.$$

We know that

$$\|\mathrm{Ext}(u)\|_{L^2((-\infty,T),X)} = \|\mathrm{Ext}(u)\|_{L^2((T-T_f,T),X)} = \|u\|_{L^2((0,T),X)},$$

further,

$$\begin{aligned}
&\int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t})-\mathrm{Ext}(u)(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\
&= \int_0^T\int_0^T \frac{\|u(\mathsf{t})-u(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_0^T\int_{-\infty}^0 \frac{\|u(\mathsf{t})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\
&= \int_0^T\int_0^T \frac{\|u(\mathsf{t})-u(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_0^T \|u(\mathsf{t})\|_X^2 \int_{\mathsf{t}}^{\infty} \mathsf{q}^{-2\sigma-1} d\mathsf{q}d\mathsf{t} \\
&= \int_0^T\int_0^T \frac{\|u(\mathsf{t})-u(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + \frac{1}{\sigma}\int_0^T \mathsf{t}^{-2\sigma}\|u(\mathsf{t})\|_X^2 d\mathsf{t},
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&\int_{-\infty}^{T}\int_{-\infty}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t})-\mathrm{Ext}(u)(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} \\
&= \int_{T-T_f}^{T}\int_{T-T_f}^{T} \frac{\|\mathrm{Ext}(u)(\mathsf{t})-\mathrm{Ext}(u)(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t} + 2\int_{T-T_f}^{T}\int_{-\infty}^{T-T_f} \frac{\|\mathrm{Ext}(u)(\mathsf{t})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s}d\mathsf{t}
\end{aligned}$$

$$= \int_0^T \int_0^T \frac{\|\mathrm{Ext}(u)(\mathsf{t}) - \mathrm{Ext}(u)(\mathsf{s})\|_X^2}{|\mathsf{t}-\mathsf{s}|^{2\sigma+1}} d\mathsf{s} d\mathsf{t} + \frac{1}{\sigma} \int_0^T (\mathsf{t} - (T - T_f))^{-2\sigma} \|\mathrm{Ext}(u)(\mathsf{t})\|_X^2 d\mathsf{t}.$$

□

Lemma A.17. Let $0 < T \leq T_f$, $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$ and X be a Hilbert space. Then, the restriction operator

$$\mathrm{R}(\cdot)(\mathsf{t}) = \cdot(\mathsf{t})$$

is continuous as a mapping $H^\sigma((0,T_f),X) \to H^\sigma((0,T),X)$ with norm bounded by 1.

Proof. Follows from the definition of the norm (A.24). □

Lemma A.18. Let $T_f \geq T > 0$, X be a Hilbert space and $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$. Furthermore, let $u \in H^\sigma((T - T_f, T), X)$ be such that $u|_{(T-T_f,0)} = 0$. Then,

$$\|u\|_{H^\sigma((0,T),X)} = \|u\|_{H^\sigma((T-T_f,T),X)}.$$

Proof. Follows due to (A.28). □

Lemma A.19. Let X be a Hilbert space, $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$ and $\alpha > 0$ be chosen such that $\sigma + \alpha \in (0,1) \setminus \{\frac{1}{2}\}$. Then, there exists a constant C independent of T such that

$$\|u\|_{H^\sigma((0,T),X)} \leq C T^\alpha \|u\|_{H^{\sigma+\alpha}((0,T),X)},$$

for all $u \in Y^{\sigma+\alpha}_{(0,T)}$.

Proof. We have

$$\|u\|^2_{L^2((0,T),X)} \leq T^{2(\sigma+\alpha)} \int_0^T \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|_X^2 d\mathsf{t} \leq T^{2(\sigma+\alpha)} \|u\|^2_{H^{\sigma+\alpha}((0,T),X)}. \quad \text{(A.29)}$$

In addition,

$$\int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|_X^2}{|\mathsf{s}-\mathsf{t}|^{2\sigma+1}} d\mathsf{s} d\mathsf{t} \leq T^{2\alpha} \int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|_X^2}{|\mathsf{s}-\mathsf{t}|^{2(\sigma+\alpha)+1}} d\mathsf{s} d\mathsf{t} \leq T^{2\alpha} \|u\|^2_{H^{\sigma+\alpha}((0,T),X)} \quad \text{(A.30)}$$

and

$$\int_0^T \mathsf{t}^{-2\sigma} \|u(\mathsf{t})\|_X^2 dt = \int_0^T \mathsf{t}^{2\alpha} \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|_X^2 dt \leq T^{2\alpha} \int_0^T \mathsf{t}^{-2(\sigma+\alpha)} \|u(\mathsf{t})\|_X^2 d\mathsf{t}. \quad \text{(A.31)}$$

Combining (A.29), (A.30) and (A.31) yields the assertion. □

Lemma A.20. Let X be a Hilbert space, $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$. Then, there exists a constant C independent of T such that

$$\|u\|_{H^\sigma((0,T),X)} \leq CT^{1-\sigma}\|u\|_{H^1((0,T),X)},$$

for all $u \in Y^1_{(0,T)}$.

Proof. Let $u \in Y^1_{(0,T)}$. Since $u(0)=0$ and due to Hölders' inequality,

$$\|u\|^2_{L^2((0,T),X)} = \int_0^T \| \int_0^{\mathsf{t}} \partial_{\mathsf{t}} u(\tau) d\tau\|^2_X d\mathsf{t} \leq \int_0^T \mathsf{t} \int_0^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \leq \frac{1}{2} T^2 \|u\|^2_{H^1((0,T),X)}. \tag{A.32}$$

Moreover, for $\sigma \in (0, \frac{1}{2})$,

$$\begin{aligned}
&\int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} = 2\int_0^T \int_0^{\mathsf{t}} \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} \\
&\leq 2\int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} \int_s^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{s}d\mathsf{t} \\
&\leq 2\|u\|^2_{H^1((0,T),X)} \int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} d\mathsf{s}d\mathsf{t} = \frac{2}{(1-2\sigma)(2-2\sigma)} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}.
\end{aligned} \tag{A.33}$$

and for $\sigma \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
&\int_0^T \int_0^T \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} = 2\int_0^T \int_0^{\mathsf{t}} \frac{\|u(\mathsf{s}) - u(\mathsf{t})\|^2_X}{|\mathsf{s}-\mathsf{t}|^{1+2\sigma}} d\mathsf{s}d\mathsf{t} \\
&\leq 2\int_0^T \int_0^{\mathsf{t}} (\mathsf{t}-\mathsf{s})^{-2\sigma} \int_s^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{s}d\mathsf{t} = 2\int_0^T \int_0^{\mathsf{t}} \int_0^{\tau} (\mathsf{t}-\mathsf{s})^{-2\sigma} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\mathsf{s}d\tau d\mathsf{t} \\
&= 2\int_0^T \int_0^{\mathsf{t}} (2\sigma-1)^{-1}((\mathsf{t}-\tau)^{-2\sigma+1} - \mathsf{t}^{-2\sigma+1})\|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \\
&= 2(2\sigma-1)^{-1} \int_0^T \int_\tau^T ((\mathsf{t}-\tau)^{-2\sigma+1} - \mathsf{t}^{-2\sigma+1})\|\partial_{\mathsf{t}} u(\tau)\|^2_X d\mathsf{t} d\tau \\
&= 2(2\sigma-1)^{-1}(2-2\sigma)^{-1} \int_0^T ((T-\tau)^{2-2\sigma} - T^{2-2\sigma} + \tau^{2-2\sigma})\|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau \\
&\leq 2(2\sigma-1)^{-1}(2-2\sigma)^{-1} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}.
\end{aligned} \tag{A.34}$$

Furthermore, for $\sigma \in (0,1)$,

$$\begin{aligned}
&\int_0^T \mathsf{t}^{-2\sigma} \|u(\mathsf{t})\|^2_X d\mathsf{t} = \int_0^T \mathsf{t}^{-2\sigma} \| \int_0^{\mathsf{t}} \partial_{\mathsf{t}} u(\tau) d\tau\|^2_X d\mathsf{t} \\
&\leq \int_0^T \mathsf{t}^{1-2\sigma} \int_0^{\mathsf{t}} \|\partial_{\mathsf{t}} u(\tau)\|^2_X d\tau d\mathsf{t} \leq \frac{1}{2-2\sigma} T^{2-2\sigma} \|u\|^2_{H^1((0,T),X)}.
\end{aligned} \tag{A.35}$$

(A.32), (A.33), (A.34) and (A.35) imply the assertion. □

Lemma A.21. Let X_1, X_2 be real, separable Hilbert spaces and $\sigma \in (0,1) \setminus \{\frac{1}{2}\}$. Let K be a linear operator that is continuous as a mapping $X_1 \to X_2$ and $f \in H^\sigma((0,T), X_1)$. Then,

$$\|K(f)\|_{H^\sigma((0,T),X_2)} \leq C\|f\|_{H^\sigma((0,T),X_1)}$$

with a constant C independent of T.

Proof. Follows directly from the definition (A.24) of the norm. □

Lemma A.22. Let X be a real, separable Hilbert space, $\sigma \in [0,1] \setminus \{\frac{1}{2}\}$ and $T_1 < T_2$. Then,

$$\|u\|_{H^\sigma((T_1,T_2),X)} = \|\tilde{u}\|_{H^\sigma((0,T_2-T_1),X)},$$

for all $u \in H^\sigma((T_1,T_2), X)$, where $\tilde{u}(\mathsf{t}) := u(\mathsf{t} + T_1)$ for (a.e.) $\mathsf{t} \in (0, T_2 - T_1)$.

Proof. Follows from the definition (A.24) of the norm and substitution $\tilde{\mathsf{t}} := \mathsf{t} - T_1$. □

For $s = m + \sigma$, $m \in \mathbb{N}$, $\sigma \in [0,1) \setminus \{\frac{1}{2}\}$, define

$$\|\cdot\|_{H^s((T_1,T_2),X)} = \begin{cases} (\|\cdot\|^2_{H^{m-1}((T_1,T_2),X)} + \|\partial_{\mathsf{t}}^m(\cdot)\|^2_{H^\sigma((T_1,T_2),X)})^{\frac{1}{2}} & \text{if } \sigma \in (0,1) \setminus \{\frac{1}{2}\}, \\ \|\cdot\|_{H^m((T_1,T_2),X)} & \text{if } \sigma = 0. \end{cases}$$

and with Lemmas A.15 - A.21 it is straightforward to verify properties **P1** - **P8** of the norm.

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