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Automated Verification and Control of Large-Scale Stochastic Cyber-Physical Systems: Compositional Techniques

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To my wife “***Amy***”
for her endless support, invaluable patience, and unconditional love ...

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Abstract

This dissertation is motivated by the challenges arising in the analysis and synthesis of large-scale stochastic cyber-physical systems (CPSs). In the past few years, stochastic CPSs have received significant attentions as an important modeling framework describing many engineering systems and play significant roles in many real-life applications including traffic networks, transportation systems, power grids, and so on. Automated verification and policy synthesis for this type of complex stochastic systems to achieve some high-level specifications, e.g., those expressed as linear temporal logic (LTL) formulae, are inherently very challenging. In particular, providing automated synthesis of correct-by-design controllers for stochastic CPSs is absolutely crucial in many safety-critical applications such as automated highway driving.

In this respect, decomposition and abstraction are introduced as two key tools to alleviate the computational complexity arising in the analysis of large-scale stochastic CPSs. More specifically, one promising approach to deal with encountered difficulties is to first employ abstractions of subsystems as a replacement of original ones, then synthesize controllers for the abstract interconnected systems, and finally refine the controllers back to the concrete models. Since the mismatch between the output of the overall interconnected system and that of its abstraction is well-quantified, one can guarantee that the concrete systems also satisfy the same specifications as the abstract ones with guaranteed error bounds on their output trajectories.

The computational complexity in synthesizing controllers for large-scale stochastic CPSs can be mitigated via abstractions in two consecutive stages. In the first stage, one can abstract the original system by a simpler one with a lower dimension (infinite abstraction). Then one can construct a finite abstraction (a.k.a. finite Markov decision process (MDP)) as an approximate description of the reduced-order system in which each discrete state corresponds to a collection of continuous states of the reduced-order system. Since the final abstractions are finite, algorithmic machineries from Computer Science are applicable to synthesize controllers enforcing high-level properties over the original systems.

Unfortunately, construction of abstractions for large-scale CPSs in a monolithic manner suffers severely from the so-called *curse of dimensionality*: the complexity exponentially grows as the number of state variables increases. To relieve this issue, one promising solution is to consider the large-scale CPSs as an interconnected system composed of several smaller subsystems, and provide a compositional framework for the construction of abstractions for the given system using abstractions of smaller subsystems.

This dissertation provides novel compositional techniques to analyze and control large-scale stochastic CPSs in an automated as well as formal fashion. In the first part of the thesis, compositional infinite abstractions (model order reductions) of original systems are studied with three different compositional techniques including classic small-gain, max small-gain and dissipativity approaches. We show that the proposed max small-gain approach is more general than the classic one since it does not require linear growth on the gains of subsystems which is the case in the classic small-gain. We also show that the provided approximation error via max small-gain does not change as the number of subsystems grows since the proposed overall error is completely independent of the size of the network, and is computed only based on the maximum error of subsystems instead of being a linear combination of them which is the case in the classic small-gain and dissipativity approaches. On the other hand, we discuss that the proposed dissipativity technique is less conservative than the classic (or max) small-gain approach in the sense that the provided dissipativity-type compositionality condition can enjoy the structure of the interconnection topology and be potentially fulfilled independently of the number or gains of subsystems.

In the second part of the thesis, compositional construction of finite MDPs as finite abstractions of given (reduced-order) systems is studied with the same aforementioned compositionality techniques. We show that if the original system is incremental input-to-state stable (or incrementally passivable in the dissipativity setting), one can construct finite MDPs of original systems for the general setting of nonlinear stochastic control systems. We also extend our results from control systems to switched ones whose switching signals accept a dwell-time condition with multiple Lyapunov-like functions. Moreover, we propose relaxed versions of small-gain and dissipativity approaches in which the stabilizability of individual subsystems for providing the compositionality results is not necessarily required. We then propose a compositional technique for the construction of both infinite and finite abstractions in a unified framework via notions of approximate probabilistic relations. We show that the unified compositional framework is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions. We finally propose a novel model-free reinforcement learning scheme to synthesize policies for unknown, continuous-space MDPs. We provide approximate optimality guarantees between unknown original models and that of their finite MDPs. We discuss that via the proposed model-free learning framework not only one can synthesize controllers for unknown stochastic systems, but also the curse of dimensionality problem is remarkably mitigated.

In the last part of the thesis, we develop a software tool, called AMYTISS, in C++/OpenCL that provides scalable parallel algorithms for (i) constructing finite MDPs from discrete-time stochastic control systems and (ii) synthesizing controllers automatically that satisfy complex logic properties including safety, reachability, and reach-avoid specifications. The software tool is developed based on theoretical results on constructing finite abstractions, and can utilize high-performance computing platforms and cloud-computing services to mitigate the effects of the state-explosion problem, which is always present in analyzing large-scale stochastic systems. This tool significantly improves performances w.r.t. the computation time and memory usage by parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators (e.g., FPGA). We show that this tool outperforms all existing tools available in the literature.

Zusammenfassung

Diese Dissertation ist motiviert durch die Herausforderungen, die sich in der Analyse und Synthese von hochdimensionalen cyber-physikalischen Systemen (CPS) stellen. In den letzten Jahren haben stochastische CPS erhebliche Aufmerksamkeit erhalten als ein wichtiger Modellierungsrahmen, mit dem sich viele ingenieurstechnische Systeme beschreiben lassen und der eine signifikante Rolle in vielen praxisorientierten Anwendungen spielt, u.a. in Verkehrsnetzwerken, Transportsystemen, Stromnetzen usw. Automatische Verifikation und Synthese von Steuerungsstrategien für diese Art von komplexen stochastischen Systemen mit dem Ziel bestimmte High-Level-Spezifikationen zu erfüllen, z.B. solche, die durch Formeln der linearen temporalen Logik (LTL) ausgedrückt werden, ist von Natur aus sehr anspruchsvoll. Speziell in vielen sicherheitskritischen Anwendungen wie z.B. automatischen Verkehrssystemen ist es absolut essentiell, eine automatische Synthese von nachweislich korrekt entworfenen Reglern für stochastische CPS bereitzustellen.

Angesichts dessen werden die Dekomposition und Abstraktion eingeführt als zwei wesentliche Werkzeuge, um die Rechenkomplexität in der Analyse von hochdimensionalen stochastischen CPS zu verringern. Konkreter ist ein vielversprechender Ansatz, um mit den auftretenden Schwierigkeiten umzugehen, dass man zuerst Abstraktionen von Teilsystemen als Ersatz für die ursprünglichen Systeme zum Einsatz bringt, dann Regler für die abstrakten vernetzten Systeme synthetisiert und schließlich die Regler so verfeinert, dass sie auf die ursprünglichen Modelle angewendet werden können. Da der quantitative Unterschied zwischen dem Ausgang des vernetzten Gesamtsystems und dem der Abstraktion genau erfasst werden kann, kann man garantieren, dass die konkreten Systeme und deren Abstraktionen dieselben Spezifikationen erfüllen mit garantierten Fehlerschranken für ihre Ausgangstrajektorien.

Die Rechenkomplexität in der Synthese von Reglern für hochdimensionale stochastische CPS kann durch Abstraktionen in zwei aufeinander folgenden Schritten verringert werden. Im ersten Schritt kann man das ursprüngliche System durch ein einfacheres mit kleinerer Dimension abstrahieren (unendliche Abstraktion). Dann kann man eine endliche Abstraktion (auch als endliches Markov-Entscheidungsproblem (MDP) bekannt) als approximative Beschreibung des ordnungsreduzierten Systems konstruieren, in dem jeder diskrete Zustand mit einer Menge von kontinuierlichen Zuständen des ordnungsreduzierten Systems korrespondiert. Da die abschließenden Abstraktionen endlich sind, lassen sich Algorithmen der Informatik zur Synthetisierung von Reglern anwenden, die High-Level-Spezifikationen der ursprünglichen Systeme erzwingen.

Leider ist die Konstruktion von Abstraktionen für hochdimensionale CPS auf monolithische Art stark vom sogenannten *Fluch der Dimensionalität* betroffen: Die Komplexität wächst mit der Anzahl der Zustandsvariablen exponentiell an. Eine vielver-

sprechende Lösung um dies abzumildern besteht darin, das hochdimensionale CPS als eine Vernetzung von mehreren kleineren Teilsystemen zu betrachten und ein kompositionelles Framework für die Konstruktion von Abstraktionen des gegebenen Systems bereitzustellen, das Abstraktionen kleinerer Teilsysteme verwendet.

Diese Dissertation liefert neue kompositionelle Methoden zur Analyse und Steuerung hochdimensionaler stochastischer CPS auf eine sowohl automatisierte als auch formale Art. Im ersten Teil der Arbeit werden kompositionelle unendliche Abstraktionen (Modellordnungsreduktionen) von Originalsystemen mit Hilfe dreier verschiedener kompositioneller Methoden untersucht, darunter klassisches Small-Gain, max-Small-Gain und Dissipativitätsansätze. Wir beweisen, dass der vorgestellte max-Small-Gain-Ansatz allgemeiner ist als der klassische, da er kein lineares Wachstum der Gains der Teilsysteme erfordert, was beim klassischen Small-Gain-Ansatz der Fall ist. Wir zeigen auch, dass sich der Approximationsfehler beim max-Small-Gain-Ansatz nicht ändert, wenn die Anzahl der Teilsysteme wächst, da der Gesamtfehler vollständig unabhängig von der Größe des Netzwerks ist und allein auf dem maximalen Fehler der Teilsysteme basierend berechnet wird statt als Linearkombination dieser Fehler wie es beim klassischen und beim Dissipationsansatz der Fall ist. Andererseits erörtern wir, dass die vorgeschlagene Dissipationsmethode weniger konservativ ist als der klassische (oder max-) Ansatz in dem Sinne, dass die bereitgestellte dissipativitätsartige Kompositionalitätsbedingung die Struktur der Vernetzungstopologie nutzen und potentiell unabhängig von der Anzahl der Teilsysteme oder deren Gains erfüllt sein kann.

Im zweiten Teil der Arbeit wird die kompositionelle Konstruktion von endlichen MDPs als endliche Abstraktionen von gegebenen (ordnungsreduzierten) Systemen untersucht mit Hilfe derselben zuvor erwähnten kompositionellen Techniken. Unter der Voraussetzung, dass das Ursprungssystem inkrementell Eingangs-Zustands-stabil (oder inkrementell passivierbar in einem Dissipativitätssetting) ist, beweisen wir, dass sich endliche MDPs von den Usprungssystemen im allgemeinen Setting von nichtlinearen stochastischen Kontrollsystemen konstruieren lassen. Wir bauen unsere Resultate außerdem von Kontrollsystemen auf geschaltete Systeme aus, deren Schaltsignale eine Haltezeitbedingung erfüllen, die mehrere Lyapunov-artige Funktionen beinhaltet. Darüberhinaus stellen wir abgeschwächte Versionen von Small-Gain- und Dissipativitätsansätzen vor, in denen die Stabilisierbarkeit von individuellen Teilsystemen nicht notwendigerweise gefordert werden muss, um die kompositionellen Resultate zu erhalten. Dann stellen wir eine kompositionelle Technik vor für die Konstruktion sowohl unendlicher als auch endlicher Abstraktionen in einem vereinheitlichten Rahmen mit Hilfe von Begriffen approximativer probabilistischer Relationen. Wir beweisen, dass das vereinheitlichte kompositionelle Framework weniger konservativ ist als die Zwei-Schritt-Methode, die unabhängig voneinander unendliche und endliche Abstraktionen konstruiert. Schließlich stellen wir einen neuen modellfreien Reinforcement-Learning-Entwurf vor, um Regelungss trategien für unbekannte MDPs auf kontinuierlichen Räumen zu synthetisieren. Wir liefern probabilistische Genauigkeitsgarantien zwischen unbekannten Orginial-Modellen und deren endlichen MDPs. Wir erörtern, dass man mit Hilfe des modellfreien Learning-Frameworks nicht nur Regler für unbekannte stochastische Systeme synthetisieren kann, sondern dass auch das Problem des Fluches der Dimensionalität erheblich abgeschwächt wird.

Im letzten Teil der Arbeit entwickeln wir ein Software-Tool, genannt AMYTISS, in C++/OpenCL, das skalierbare parallele Algorithmen liefert für (i) die Konstruktion endlicher MDPs von zeitdiskreten stochastischen Kontrollsystemen und (ii) die automatische Synthetisierung von Reglern, die komplexe Logikeigenschaften erfüllen, u.a. Sicher heits-, Erreichbarkeits- und Erreichbarkeits-Ausweich-Spezifikationen. Das Software-Tool wird basierend auf theoretischen Resultaten zur Konstruktion endlicher Abstraktionen entwickelt und kann High-Performance-Rechenplattformen und Cloud-Computing-Dienste nutzen, um die Auswirkungen des Zustands-Explosions-Problems abzuschwächen, das in der Analyse von hochdimensionalen stochastischen Systemen stets auftritt. Dieses Werkzeug verbessert die Performance hinsichtlich der Rechenzeit und des Speicherbedarfs signifikant durch paralleles Rechnen auf verschiedenen heterogenen Rechenplattformen, die CPUs, GPUs und Hardware-Beschleuniger beinhalten (z.B. FPGA). Wir zeigen, dass dieses Werkzeug alle existierenden Werkzeuge übertrifft, die in der Literatur zu finden sind.

Publications by the Author during Ph.D.

Journal Papers

1. **A. Lavaei**, S. Soudjani, A. Abate, and M. Zamani, "Automated Verification and Synthesis of Stochastic Hybrid Systems: An Overview". *Automatica*, accepted as a survey paper proposal, 2019.

2. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Abstraction-based Synthesis for Networks of Stochastic Switched Systems". *Automatica*, vol. 114, 2020.

3. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional (In)Finite Abstractions for Large-Scale Interconnected Stochastic Systems". *IEEE Transactions on Automatic Control*, to appear as a full paper, 2020.

4. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Abstraction of Large-Scale Stochastic Systems: A Relaxed Dissipativity Approach". *Nonlinear Analysis: Hybrid Systems*, vol. 36, 2020.

5. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Construction of Infinite Abstractions for Networks of Stochastic Control Systems", *Automatica*, vol. 107, pp. 125-137, 2019.

Book Chapters

6. **A. Lavaei**, S. Soudjani, and M. Zamani, "Approximate Probabilistic Relations for Compositional Synthesis of Stochastic Systems", *Numerical Software Verification*, Lecture Notes in Computer Science 11652, pp. 101–109, Springer, 2019.

Conference Papers

7. **A. Lavaei***, M. Khaled*, S. Soudjani, and M. Zamani, "AMYTISS: A Parallelized Tool on Automated Controller Synthesis for Large-Scale Stochastic Systems", *23rd ACM International Conference on Hybrid Systems: Computation and Control*, to appear, 2020.

8. **A. Lavaei**, F. Somenzi, S. Soudjani, A. Trivedi, and M. Zamani, "Formal Controller Synthesis of Unknown Continuous-Space MDPs via Model-Free Reinforcement Learning", *11th ACM/IEEE International Conference on Cyber-Physical Systems (ICCPS)*, to appear, 2020.

*Both authors have contributed equally.

9. **A. Lavaei**, and M. Zamani, "Compositional Verification of Large-Scale Stochastic Systems via Relaxed Small-Gain Conditions", *58th IEEE Conference on Decision and Control (CDC)*, to appear, 2020.

10. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Synthesis of not Necessarily Stabilizable Stochastic Systems via Finite Abstractions", *18th European Control Conference (ECC)*, pp. 2802–2807, 2019.

11. **A. Lavaei**, and M. Zamani, "Compositional Construction of Finite MDPs for Large-Scale Stochastic Switched Systems: A Dissipativity Approach", *15th IFAC Symposium on Large-Scale Complex Systems: Theory and Applications (LSS)*, vol. 52, no. 3, pp. 31-36, 2019. **(IFAC Young Author Award Finalist)**

12. **A. Lavaei**, and M. Zamani, "Compositional Finite Abstractions for Large-Scale Stochastic Switched Systems", *5th International Workshop on Symbolic-Numeric Methods for Reasoning about CPS and IoT (SNR) in conjunction with Cyber-Physical Systems and Internet-of-Things Week (CPS-IoT Week)*, pp. 3-5, 2019.

13. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Synthesis of Finite Abstractions for Continuous-Space Stochastic Control Systems: A Small-Gain Approach", *6th IFAC Conference on Analysis and Design of Hybrid Systems (ADHS)*, vol. 51, no. 16, pp. 265-270, 2018.

14. **A. Lavaei**, S. Soudjani, and M. Zamani, "From Dissipativity Theory to Compositional Construction of Finite Markov Decision Processes", *21st ACM International Conference on Hybrid Systems: Computation and Control (HSCC)*, pp. 21-30, 2018.

15. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Synthesis of Interconnected Stochastic Control Systems based on Finite MDPs", *21st ACM International Conference on Hybrid Systems: Computation and Control (HSCC)*, pp. 273-274, 2018.

16. **A. Lavaei**, S. Soudjani, R. Majumdar, and M. Zamani, "Compositional Abstractions of Interconnected Discrete-Time Stochastic Control Systems", *56th IEEE Conference on Decision and Control (CDC)*, pp. 3551-3556, 2017.

Under Submission and Review

17. M. Anand*, **A. Lavaei***, and M. Zamani, "Compositional Barrier Certificates for Temporal Logic Synthesis of Large-Scale Stochastic Systems", under submission, 2020.

18. **A. Lavaei**, F. Somenzi, S. Soudjani, A. Trivedi, and M. Zamani, "Compositional Controller Synthesis of Black-box Continuous-Space MDPs: A Model-Free Reinforcement Learning", under submission, 2020.

19. **A. Lavaei**, and M. Zamani, "From Dissipativity Theory to Compositional Synthesis of Large-Scale Stochastic Switched Systems", submitted for publication, 2020.

20. **A. Lavaei***, M. Khaled*, S. Soudjani, and M. Zamani, "AMYTISS: Parallel Automated Controller Synthesis for Large-Scale Stochastic Systems", submitted for publication, 2020.

21. M. Anand*, **A. Lavaei***, and M. Zamani, "Compositional Control Barrier Certificates for Large-Scale Interconnected Stochastic Systems", submitted for publication, 2019.

22. **A. Lavaei**, S. Soudjani, and M. Zamani, "Compositional Abstraction-based Synthesis of General MDPs via Approximate Probabilistic Relations", submitted for publication, 2019.

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List of Abbreviations

AMYTISS	Parallel automated controller synthesis for large-scale stochastic systems
CoG	Center of gravity
dt-SAS	discrete-time stochastic autonomous systems
dt-SCS	discrete-time stochastic control systems
dt-SS	discrete-time stochastic switched systems
DBN	Dynamic Bayesian network
DDPG	Deep deterministic policy gradient
DFA	Deterministic finite-state automata
FPSF	Finite-step stochastic pseudo-simulation functions
FSF	Finite-step stochastic simulation functions
GB	Gigabyte
HPC	High-performance computing
HWAs	Hardware accelerators
HWC	Hardware configuration
i.i.d.	independent and identically distributed
LMI	Linear matrix inequality
MC	Markov chain
MDP	Markov decision process
OFA	On-the-fly abstraction
PDF	Probability density function
PEs	Processing elements
RL	Reinforcement learning
scLTL	syntactically co-safe linear temporal logic
SPSF	Stochastic pseudo-simulation functions
SSF	Stochastic simulation functions
SPStF	Stochastic pseudo-storage functions
SStF	Stochastic storage functions
Tr	Trace
δ-ISS	Incrementally input-to-state stable

List of Symbols

Number Sets

$\mathbb{N} := \{0, 1, 2, \ldots\}$	Set of nonnegative integers
$\mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\}$	Set of positive integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_{\geq 0}$	Set of nonnegative real numbers
$\mathbb{R}_{>0}$	Set of positive real numbers

Stochastic Control Systems

$a, \hat{a}$	Initial conditions
A	System matrix
b	Slope coefficient
B	External input matrix
$\mathcal{B}(X)$	Borel sigma-algebra on state space
C^1	External output matrix
C^2	Internal output matrix
D	Internal input matrix
E	Column vector corresponding to nonlinear term
$\mathbb{E}$	Conditional expectation
f	Transition map
F	Row vector corresponding to nonlinear term
$\mathcal{F}_\Omega$	Sigma-algebra on Ω comprising subsets of Ω as events
h^1	External output map
h^2	Internal output map
$\bar{H}$	Function from $X \to U$
$\mathbb{I}_n$	Identity matrix in $\mathbb{R}^{n\times n}$
$\mathcal{I}_d$	Identity function
k	Time step
K	Feedback matrix
$\tilde{L}$	Laplacian matrix of an undirected graph
L_1, L_2	Column vectors in interface map
m	Integer number as dimension of external input
M	Coupling matrix
$\tilde{M}$	Positive-definite matrix
n	Integer number as dimension of state

n_x	Cardinality of state set
n_w	Cardinality of internal input set
n_ν	Cardinality of external input set
N	Number of subsystems
$\bar{p}$	Integer number as dimension of internal input
$\tilde{P}$	Matrix in $\mathbb{R}^{n\times\hat{n}}$ employed for order reduction
$\mathbb{P}_\Omega$	Probability measure
q^1	Integer number as dimension of external output
q^2	Integer number as dimension of internal output
R	Noise matrix
$\tilde{T}$	Room temperature
T_d	Maximum finite time step
$\tilde{T}_{ei}$	Outside temperatures
$\tilde{T}_h$	Heater temperature
T_x	Conditional stochastic kernel
U	External input set
$\mathcal{U}$	Collections of sequences $\{\nu(k) : \Omega \to U,\ k \in \mathbb{N}\}$
V	Stochastic simulation function
w	Internal input variable
$\bar{w}$	Internal input representative point
W	Internal input set
$\mathcal{W}$	Collections of sequences $\{w(k) : \Omega \to W,\ k \in \mathbb{N}\}$
x	State variable
$\bar{x}$	State representative point
X	State set
y^1	External output variable
y^2	Internal output variable
Y^1	External output set
Y^2	Internal output set
$\mathbf{0}_n$	Column vector with all elements equal to zero
$\mathbb{1}_n$	Column vector with all elements equal to one
$\alpha, \hat{\alpha}, \tilde{\alpha}, \underline{\alpha}, \overline{\alpha}$	$\mathcal{K}_\infty$ functions
β	Conduction factor between external environment and room i
$\gamma, \hat{\gamma}_i$	$\mathcal{K}_\infty$ functions
$\hat{\delta}$	Closeness bound
$\bar{\delta}$	State discretization parameter
$\underline{\delta}$	Function of x and $\hat{x}$ taking values in interval $[0, b]$
$\tilde{\delta}_f$	$\mathcal{K}_\infty$ function
$\bar{\Delta}$	Maximum degree of graph
ε	Confidence bound
η	Conduction factor between rooms $i \pm 1$ and room i
θ	External input discretization parameter
$\bar{\theta}$	Conduction factor between heater and room i

$\kappa, \underline{\kappa}$	$\mathcal{K}_\infty$ functions
$\lambda_{\min}(M)$	Minimum eigenvalue of symmetric matrix M
$\lambda_{\max}(M)$	Maximum eigenvalue of symmetric matrix M
$\bar{\lambda}$	$\mathcal{K}_\infty$ function
$\bar{\mu}$	Internal input discretization parameter
ν	External input variable
$\bar{\nu}$	External input representative point
π	Positive contract in Young's inequality
$\bar{\Pi}_M$	Class of all Markov policies
Π_x	Quantization map for state
Π_w	Quantization map for internal input
ρ	Spectral radius
$\rho_{\text{int}}, \rho_{\text{ext}}$	$\mathcal{K}_\infty \cup \{0\}$ functions
$\bar{\rho}_n$	Universally measurable stochastic kernels
σ	$\mathcal{K}_\infty$ function
$\bar{\sigma}$	Standard deviation of noise
ς	Sequence of i.i.d. random variables
Σ	Original (concrete) stochastic systems
$\widehat{\Sigma}$	(In)Finite abstract systems
τ	Sampling time
φ	Nonlinear term
Ω	Sample space

Syntactically Co-Safe LTL

AP	Set of atomic propositions
$\bar{A}$	Avoid set
$\mathcal{A}_\phi$	DFA
F_{a}	Set of accept locations
$\mathcal{L}(\mathcal{A})$	Accepted language of $\mathcal{A}$
L	Labeling function
q_{abs}	Absorbing location
$q_0 \in Q_\ell$	Initial location
Q_ℓ	Finite set of locations
$\bar{\mathcal{S}}$	Safe set
t	Transition function
$\bar{\mathcal{T}}$	Target set
$\Sigma_{\mathsf{a}} := 2^{AP}$	Finite set (a.k.a. alphabet)
ω	Infinite word

Stochastic Switched Systems

$\mathbf{F} = \{f_1, \ldots, f_m\}$	Collection of vector fields

$\mathbb{F}$	Transition function of global MDP
$\mathbb{G}(\Sigma)$	Global MDP
$\widehat{\mathbb{G}}(\widehat{\Sigma})$	Abstract global MDP
$\mathbb{H}^1$	External output map of global MDP
$\mathbb{H}^2$	Internal output map of global MDP
k_d	Dwell-time
l	Switching counter
l_0	Initial switching counter
$\tilde{l}_i$	length of cells
p	Current value of switching signal
p_0	Initial switching signal
P	Finite set of modes
$\mathcal{P}$	Subset of $\mathcal{S}(\mathbb{N}, P)$ denoting set of functions from $\mathbb{N}$ to P
$\tilde{q}$	Ratio of vehicles that goes out on exit of each cell
$\mathbf{p}(k)$	Switching signal
$\tilde{\mathcal{Q}}_p$	Set of symmetric matrices referred to incremental multiplier matrices
$\mathfrak{s}_k$	Elements of switching time instants
$\mathfrak{S}_{\mathbf{p}}$	Switching time instants
$\mathbb{T}_{\mathsf{x}}$	Conditional stochastic kernel of Global MDP
$\mathbb{U}$	External input set of global MDP
v_i	Flow speed of vehicles
$\mathbb{W}$	Internal input set of global MDP
$\mathbb{X}$	State set of global MDP
$\mathbb{Y}^1$	External output set of global MDP
$\mathbb{Y}^2$	Internal output set of global MDP
$\tilde{\mu} \geq 1$	Ratio between multiple Lyapunov functions

Relaxed Compositional Approaches

$\tilde{A}$	System matrix of auxiliary system
$\tilde{B}$	External input matrix of auxiliary system
$\tilde{D}$	Internal input matrix of auxiliary system
$\tilde{f}$	Transition map of auxiliary system
M_a	Coupling matrix of auxiliary system
$\tilde{R}$	Noise matrix of auxiliary system
w	Internal input variable of auxiliary system
$\tilde{W}$	Internal input set of auxiliary system
$\tilde{\eta}$	$\mathcal{K}_\infty$ and concave function
$\tilde{\varsigma}$	Vector containing noise terms
Σ_{aux}	$\mathcal{M}$-sampled (auxiliary) system

Approximate Probabilistic Relations

$c_{\hat{\nu}}$	Positive constant as an upper bound of abstract external input
c_{ς}	Positive constant as an upper bound of $\varsigma^T\varsigma$
$\tilde{F}_{1i}, \tilde{F}_{2i}$	Symmetric matrices in S-procedure
$\tilde{g}_{1i}, \tilde{g}_{2i}$	Vectors in S-procedure
$\tilde{h}_{1i}, \tilde{h}_{2i}$	Real numbers in S-procedure
$\bar{\mathscr{L}}_T$	Borel measurable stochastic kernel
$\mathscr{L}$	Lifting
$\mathcal{N}$	Normal distribution
r	Index for reduced-order version of original model
$\mathscr{R}$	Relation that relates $x \in X$ with $y \in Y$ if $(x, y) \in \mathscr{R}$
$\mathscr{R}_x$	Relation between states
$\mathscr{R}_w$	Relation between internal inputs
$\bar{\mathscr{R}}_\delta$	δ-lifted relation
$\mathcal{X}_2^{-1}$	Chi-square inverse cumulative distribution function with 2 degrees of freedom
$\delta_d(\cdot \mid \mathsf{c})$	Dirac delta distribution centered at c
ϵ	Precision for states
ϵ_w	Precision for internal inputs
μ_c	Mean
$\tilde{\Sigma}$	Covariance matrix

Model-Free Reinforcement Learning

f_a	Transition map
$\bar{g}$	Measurable function
$\mathscr{H}$	Lipschitz constant of stochastic kernel
$\underline{\mathscr{L}}$	Lebesgue measure of specification set
p_r	Learned policy
p_*	Optimal policy
$\tilde{\varepsilon}$	Closeness bound
$\bar{\pi}$	Archimedes' constant

Software Tool: AMYTISS

$C_{S,f}$	Front cornering stiffness coefficient
$C_{S,r}$	Rear cornering stiffness coefficient
h_{cg}	Hight of CoG
I_z	Moment of inertia for entire mass around z axis
l_f	Distance from the front axle to CoG
l_r	Distance from the rear axle to CoG
l_{wb}	Wheelbase

$\bar{m}$	Total mass of vehicle
$\mathrm{Sat}_1(\cdot), \mathrm{Sat}_2(\cdot)$	Input saturation functions
$\mathbb{V}_s$	Optimal satisfaction probability
x_1, x_2	Position coordinates
x_3	Steering angle
x_4	Heading velocity
x_5	Yaw angle
x_6	Yaw rate
x_7	Slip angle
$\hat{X}_{\gamma_t}^{\tilde{\Sigma}}$	Cutting region
γ_t	Cutting probability threshold
μ_c^*	Mean with $w = 0$
$\bar{\mu}_f$	Friction coefficient
ν_1	Control input for steering angle
ν_2	Control input for heading velocity

1 Introduction

1.1 Motivation

Cyber-physical systems (CPSs) are complex networked models combining both cyber (computation and communication) and physical components, which tightly interact with each other in a feedback loop. In the past few years, stochastic CPSs have received significant attentions as an important modeling framework describing many engineering systems and play significant roles in many real-life applications including traffic networks, transportation systems, power grids, and so on. Most stochastic CPSs are of hybrid nature: discrete dynamics model computation parts including hardware and software, and continuous dynamics model control systems. Automated verification and policy synthesis for this type of complex models to achieve some high-level specifications, e.g., those expressed as linear temporal logic (LTL) formulae [Pnu77], are inherently very challenging. In particular, the ability to handle the interaction between continuous and discrete dynamics is a prerequisite for providing a rigorous formal framework for the automated verification and synthesis of stochastic CPSs.

Since the complexity induced by the aforementioned interaction often makes it difficult to obtain analytical results, the verification and policy synthesis for stochastic CPSs are often addressed by methods of (in)finite abstractions. More precisely, since the closed-form characterization of synthesized policies for stochastic CPSs is not available, a suitable approach is to approximate original (concrete) models by simpler ones with possibly lower dimensional or finite state spaces. A crucial step is to provide formal guarantees during this approximation phase such that the analysis or synthesis on abstract models can be refined back over original ones. Stochastic simulation functions are then employed as Lyapunov-like functions defined over the Cartesian product of state spaces of two systems to relate trajectories of abstract systems to those of original ones such that the mismatch between two systems remains within some guaranteed error bounds.

The computational complexity in synthesizing controllers for stochastic CPSs can be alleviated via abstractions in two consecutive stages. In the first stage, original systems can be abstracted by simpler ones with lower dimensions (model order reductions). Then one can employ infinite abstractions as a replacement of concrete systems, perform analysis and synthesis over abstract models, and finally carry the results back (via an interface map) over concrete systems. Since the mismatch between outputs of original systems and those of their infinite abstractions are well-quantified, one can guarantee that concrete systems also satisfy the same specifications as abstract ones with quantified error bounds.

In the second stage of the abstract procedure, one can construct finite abstractions (a.k.a. finite Markov decision processes (MDPs)) as approximate descriptions of (reduced-order) systems in which each discrete state corresponds to a collection of continuous states of (reduced-order) systems. Since the final abstractions are finite, algorithmic machineries from computer science [BK08] are applicable to synthesize controllers over concrete systems enforcing complex logic properties including safety, reachability, reach-avoid, etc.

In order to make the approaches provided by (in)finite abstractions applicable to networks of interacting systems, compositional abstraction-based techniques are proposed in the past few years. In particular, construction of (in)finite abstractions for large-scale stochastic CPSs in a monolithic manner suffers severely from the so-called curse of dimensionality. To mitigate this issue, one promising solution is to consider the large-scale complex system as an interconnected system composed of several smaller subsystems, and provide a compositional framework for the construction of (in)finite abstractions for the given interconnected system using abstractions of smaller subsystems.

1.2 Research Goals and Original Contributions

In this dissertation, we propose novel compositional techniques for automated verification and control of large-scale stochastic CPSs. In the first part of the thesis, we discuss compositional infinite abstractions (model order reductions) of concrete systems using three different compositional techniques including classic small-gain, max small-gain and dissipativity approaches. We demonstrate that the proposed max small-gain approach is more general than the classic one since it does not require linear growth on the gains of subsystems which is the case in the classic small-gain. We also prove that the proposed approximation error using max small-gain does not change as the number of subsystems grows. This issue is due to the fact that the proposed overall error is completely independent of the size of the network, and is computed only based on the maximum error of subsystems instead of being a linear combination of them which is the case in the classic small-gain and dissipativity approaches. On the other hand, we discuss that the proposed dissipativity technique is less conservative than the classic (or max) small-gain approach in the sense that the provided dissipativity-type compositional condition can enjoy the structure of the interconnection topology and be potentially fulfilled independently of the number or gains of subsystems. It should be noted that we do not put any restriction on the sources of uncertainties in concrete and abstract systems meaning that the noise of the abstraction can be completely independent of that of the concrete system. Thus our results in this thesis are more general than the ones available in the literature (e.g., [Zam14, ZRME17]), where the noises in concrete and abstract systems are assumed to be the same. This means the abstraction has access to the noise of the concrete system, which is a strong assumption.

In the second part of the thesis, compositional finite MDPs as finite abstractions of given (reduced-order) systems are studied with the same aforementioned compositional techniques. We show that if the original system is incrementally input-to-state stable (or incrementally passivable in the dissipativity setting), one can construct finite MDPs of original systems for the general setting of nonlinear stochastic control systems. We also propose novel frameworks for the construction of finite MDPs for some particular classes of nonlinear stochastic systems whose nonlinearities satisfy a slope restriction or (in a more general form) an incremental quadratic inequality. We extend our results from control systems to switched ones whose switching signals accept dwell-time condition with multiple Lyapunov-like functions. Moreover, we propose relaxed versions of small-gain and dissipativity approaches in which the stabilizability of individual subsystems for providing the compositionality results is not necessarily required. We also propose a compositional technique for the construction of both infinite and finite abstractions in a unified framework via notions of approximate probabilistic relations. We show that the unified compositional framework is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions. We finally propose a novel model-free reinforcement learning framework to synthesize policies for unknown, continuous-space MDPs. We provide probabilistic closeness guarantees between unknown original models and that of their finite MDPs. We discuss that via the proposed model-free learning framework not only one can synthesize controllers for unknown stochastic systems, but also the curse of dimensionality problem is remarkably mitigated.

In the last part of the thesis, we develop a software tool in C++/OpenCL, called AMYTISS, for designing correct-by-construction controllers of large-scale discrete-time stochastic systems. This software tool provides scalable parallel algorithms that allow to (i) construct finite MDPs from discrete-time stochastic control systems, and (ii) synthesize controllers satisfying complex logic properties including safety, reachability, and reach-avoid specifications. AMYTISS is developed based on theoretical results on constructing finite abstractions by employing high-performance computing platforms and cloud-computing services to alleviate the effects of the state-explosion problem, which is always the case in analyzing large-scale stochastic systems. This tool significantly improves performances w.r.t. the computation time and memory usage by parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators (e.g., FPGA). To the best of our knowledge, AMYTISS is the only tool of this kind for the stochastic systems that is able to utilize these types of compute units, simultaneously. We show that this tool outperforms all existing tools available in the literature.

It should be noted that in different parts throughout the thesis, to demonstrate the effectiveness of our proposed results, we apply the proposed techniques to *real-world* applications. In particular, we apply our results to the *temperature regulation* in a circular building and construct compositionally a finite abstraction of a big network containing many rooms. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies regulating the temperature in each room for a bounded time horizon. We also apply our approaches to a *road traffic network* in a circular cascade ring composed of different cells, and construct compositionally a finite MDP of the network. We utilize the constructed abstraction as a substitute to compositionally synthesize policies keeping the density of the traffic lower than some bounded level per cell. We also apply our proposed model-free reinforcement learning as well as parallel algorithms in the last chapter to a 3-dimensional autonomous vehicle and a 7-dimensional *nonlinear* model of a BMW 320i car by synthesizing autonomous parking controllers.

1.3 Outline of the Thesis

This dissertation is divided into 6 chapters, the first of which is the current introduction. The rest is structured as follows:

Chapter 2 presents some mathematical notations and preliminaries, and also basic notions from control theory that will be frequently used throughout the thesis.

Chapter 3 studies compositional infinite abstractions with three different compositionality approaches including classic small-gain, max small-gain, and dissipativity approaches. The results of this chapter are respectively presented based on [LSMZ17, LSZ20c, LSZ19c].

Chapter 4 discusses compositional construction of finite abstractions with the same compositional techniques (as the previous chapter) as well as their relaxed versions. This chapter also includes the results extended to stochastic switched systems. Compositional infinite and finite abstractions in a unified framework are also proposed in this chapter using approximate probabilistic relations. Finally, a novel reinforcement learning scheme to synthesize policies for unknown continuous-space MDPs is proposed. The results of this chapter are respectively presented based on [LSZ18b, LSZ18a, LSZ20c, LZ19b, LSZ20a, LSZ18c, LZ19a, LZ20, LSZ19d, LZ19c, LSZ20b, LSZ19a, LSZ19b, LSS$^+$20].

Chapter 5 provides a software tool by proposing novel scalable parallel algorithms and efficient distributed data structures for constructing finite MDPs of large-scale discrete-time stochastic systems and automating the computation of their correct-by-construction controllers, given high level specification such as safety, reachability, and reach-avoid. The results of this chapter are presented based on [LKSZ20a, LKSZ20b].

Chapter 6 summarizes the results of this thesis and outlines potential directions for the future research.

For more clarity of exposition, Chapters 3, 4, 5 follow a common structure. They start with an introduction including a description of the problem addressed, a brief literature review, and a statement of the contributions made. The developed techniques are detailed in subsequent sections, followed by a section illustrating their efficiency on different case studies. The chapters are concluded with a summary section.

2 Mathematical Notations, Preliminaries and Basic Notions in Control Theory

2.1 Notations

The following notations are employed throughout the thesis. The sets of nonnegative and positive integers are denoted by $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\}$, respectively. Moreover, the symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote, respectively, the sets of real, positive and nonnegative real numbers. For any set X we denote by 2^X the power set of X that is the set of all subsets of X. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1, \ldots, N\}$, we use $x = [x_1; \ldots; x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$. Any n-dimensional hyper-rectangle (a.k.a. hyper interval) is characterized by two corner vectors $x_{lb}, x_{ub} \in \mathbb{R}^n$ and we denote it by $[\![x_{lb}, x_{ub}]\!] := [x_{lb,1}, x_{ub,1}] \times [x_{lb,2}, x_{ub,2}] \times \cdots \times [x_{lb,n}, x_{ub,n}]$. We denote by $\| \cdot \|$ and $\| \cdot \|_2$ the infinity and Euclidean norms, respectively. Given any $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a. Symbols $\mathbb{I}_n$, $\mathbf{0}_n$, and $\mathbb{1}_n$ denote the identity matrix in $\mathbb{R}^{n \times n}$ and the column vector in $\mathbb{R}^{n \times 1}$ with all elements equal to zero and one, respectively. The identity function and composition of functions are denoted by $\mathcal{I}_d$ and symbol $\circ$, respectively.

Given a symmetric matrix M, the minimum and maximum eigenvalues of M are respectively denoted by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$. We also denote by $\mathsf{diag}(a_1, \ldots, a_N)$ a diagonal matrix in $\mathbb{R}^{N \times N}$ with diagonal matrix entries $a_1, \ldots, a_N$ starting from the upper left corner. Given a matrix A in $\mathbb{R}^{n \times m}$, $A(:, b)$ denotes the b-th column of A including the all rows, and $A(b, :)$ the other way around. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \ldots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \ldots, x_N) = [f_1(x_1); \ldots; f_N(x_N)]$. For any set A, we denote by $\mathsf{A}^{\mathbb{N}}$ the Cartesian product of a countable number of copies of A, i.e., $\mathsf{A}^{\mathbb{N}} = \prod_{k=0}^{\infty} \mathsf{A}$. Given sets X and Y, a relation $\mathscr{R} \subseteq X \times Y$ is a subset of the Cartesian product $X \times Y$ that relates $x \in X$ with $y \in Y$ if $(x, y) \in \mathscr{R}$, which is equivalently denoted by $x\mathscr{R}y$. Given a measurable function $f : \mathbb{N} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := \text{(ess)sup}\{\|f(k)\|, k \geq 0\}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\gamma(r) \to \infty$ as $r \to \infty$.

2.2 Preliminaries

We consider a probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$, where Ω is the sample space, $\mathcal{F}_\Omega$ is a sigma-algebra on Ω comprising subsets of Ω as events, and $\mathbb{P}_\Omega$ is a probability measure that assigns probabilities to events. We assume that random variables introduced in the thesis are measurable functions of the form $X : (\Omega, \mathcal{F}_\Omega) \rightarrow (S_X, \mathcal{F}_X)$. Any random variable X induces a probability measure on its space $(S_X, \mathcal{F}_X)$ as $Prob\{A\} = \mathbb{P}_\Omega\{X^{-1}(A)\}$ for any $A \in \mathcal{F}_X$. We often directly discuss the probability measure on $(S_X, \mathcal{F}_X)$ without explicitly mentioning the underlying probability space and the function X itself.

A topological space $\mathcal{S}$ is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Examples of a Borel space are Euclidean spaces $\mathbb{R}^n$, its Borel subsets endowed with a subspace topology as well as hybrid spaces. Any Borel space $\mathcal{S}$ is assumed to be endowed with a Borel sigma-algebra, which is denoted by $\mathcal{B}(\mathcal{S})$. We say that a map $f : \mathcal{S} \rightarrow Y$ is measurable whenever it is Borel measurable.

2.3 Discrete-Time Stochastic Control Systems

In this thesis, we consider stochastic *control* systems in discrete time (dt-SCS) defined formally as follows.

Definition 2.3.1. *A discrete-time stochastic control system (dt-SCS) is characterized by the tuple*

$$\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2), \tag{2.3.1}$$

where:

- $X \subseteq \mathbb{R}^n$ *is a Borel space as the state space of the system. We denote by* $(X, \mathcal{B}(X))$ *the measurable space with* $\mathcal{B}(X)$ *being the Borel sigma-algebra on the state space;*
- $U \subseteq \mathbb{R}^m$ *is a Borel space as the* external *input space of the system;*
- $W \subseteq \mathbb{R}^{\bar{p}}$ *is a Borel space as the* internal *input space of the system;*
- ς *is a sequence of independent and identically distributed (i.i.d.) random variables from a sample space* Ω *to the measurable space* $(\mathcal{V}_\varsigma, \mathcal{F}_\varsigma)$

$$\varsigma := \{\varsigma(k) : (\Omega, \mathcal{F}_\Omega) \rightarrow (\mathcal{V}_\varsigma, \mathcal{F}_\varsigma),\ k \in \mathbb{N}\},$$

- $f : X \times U \times W \times \mathcal{V}_\varsigma \rightarrow X$ *is a measurable function characterizing the state evolution of the system;*
- $Y^1 \subseteq \mathbb{R}^{q^1}$ *is a Borel space as the* external *output space of the system;*
- $Y^2 \subseteq \mathbb{R}^{q^2}$ *is a Borel space as the* internal *output space of the system;*
- $h^1 : X \rightarrow Y^1$ *is a measurable function as the* external *output map that maps a state* $x \in X$ *to its* external *output* $y^1 = h^1(x)$*;*
- $h^2 : X \rightarrow Y^2$ *is a measurable function as the* internal *output map that maps a state* $x \in X$ *to its* internal *output* $y^2 = h^2(x)$.

For given initial state $x(0) \in X$ and input sequences $\nu(\cdot) : \mathbb{N} \to U$ and $w(\cdot) : \mathbb{N} \to W$, the evolution of the state of dt-SCS Σ can be written as

$$\Sigma : \begin{cases} x(k+1) = f(x(k), \nu(k), w(k), \varsigma(k)), \\ y^1(k) = h^1(x(k)), \\ y^2(k) = h^2(x(k)), \end{cases} \qquad k \in \mathbb{N}. \tag{2.3.2}$$

We associate respectively to U and W the sets $\mathcal{U}$ and $\mathcal{W}$ to be collections of sequences $\{\nu(k) : \Omega \to U,\ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to W,\ k \in \mathbb{N}\}$, in which $\nu(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. For any initial state $a \in X$, $\nu(\cdot) \in \mathcal{U}$, and $w(\cdot) \in \mathcal{W}$, the random sequences $x_{a\nu w} : \Omega \times \mathbb{N} \to X$, $y^1_{a\nu w} : \Omega \times \mathbb{N} \to Y^1$ and $y^2_{a\nu w} : \Omega \times \mathbb{N} \to Y^2$ that satisfy (2.3.2) are respectively called the solution process and external and internal output trajectories of Σ under an external input ν, an internal input w, and an initial state a. System Σ is called finite if X, U, W are finite sets and infinite otherwise.

Remark 2.3.2. *The above definition can be generalized by allowing the set of valid external inputs to depend on the current state and internal input of the system, i.e., to include $\{U(x,w) \,|\, x \in X, w \in W\}$ in the definition of dt-SCS, which is a family of non-empty measurable subsets of U with the property that*

$$K := \{(x, \nu, w) : x \in X,\ w \in W,\ \nu \in U(x,w)\},$$

is measurable in $X \times U \times W$. For the succinct presentation of the results, we assume in this thesis that the set of valid external inputs is the whole external input space: $U(x,w) = U$ for all $x \in X$ and $w \in W$, but the obtained results are generally applicable.

Remark 2.3.3. *Note that we employ the term "internal" for inputs and outputs of subsystems that are affecting each other in the interconnection: an internal output of a subsystem affects an internal input of another subsystem. We utilize the term "external" for inputs and outputs that are not employed for the sake of constructing the interconnection. Properties of the interconnected system are specified over external outputs. The main goal is to synthesize external inputs to satisfy desired properties over external outputs.*

In this thesis, we are ultimately interested in investigating discrete-time stochastic control systems without internal inputs and outputs. In this case, the tuple (2.3.1) reduces to $(X, U, \varsigma, f, Y, h)$ and dt-SCS (2.3.2) can be re-written as

$$\Sigma : \begin{cases} x(k+1) = f(x(k), \nu(k), \varsigma(k)), \\ y(k) = h(x(k)), \end{cases} \qquad k \in \mathbb{N}, \tag{2.3.3}$$

where $f : X \times U \times \mathcal{V}_\varsigma \to X$. The interconnected control systems, defined later, are also a class of control systems without internal signals, resulting from the interconnection of dt-SCSs having both internal and external inputs and outputs.

2.4 Infinite Markov Decision Processes

A dt-SCS Σ in (2.3.1) can be *equivalently* represented as an infinite Markov decision process (MDP) [Kal97, Proposition 7.6, pp. 122]

$$\Sigma = (X, U, W, T_{\mathsf{x}}, Y^1, Y^2, h^1, h^2), \tag{2.4.1}$$

where the map $T_{\mathsf{x}} : \mathcal{B}(X) \times X \times U \times W \rightarrow [0,1]$, is a conditional stochastic kernel that assigns to any $x \in X$, $\nu \in U$, and $w \in W$, a probability measure $T_{\mathsf{x}}(\cdot \,|\, x, \nu, w)$ on the measurable space $(X, \mathcal{B}(X))$ so that for any set $\mathcal{A} \in \mathcal{B}(X)$,

$$\mathbb{P}(x(k+1) \in \mathcal{A} \,|\, x(k), \nu(k), w(k)) = \int_{\mathcal{A}} T_{\mathsf{x}}(dx(k+1) \,|\, x(k), \nu(k), w(k)).$$

For given inputs $\nu(\cdot), w(\cdot)$, the stochastic kernel T_{x} captures the evolution of the state of Σ and can be uniquely determined by the pair (ς, f) from (2.3.2).

Remark 2.4.1. *All the dynamical models we are using in this thesis (the original model, the abstract model with a lower-dimensional state space, and the abstract model with a finite space) can be seen as MDPs. The first two are MDPs with continuous spaces, and the last one is a finite state MDP. We always use finite MDP to refer to a constructed abstract model with a finite state space.*

2.5 Markov Policy

Given the dt-SCS in (2.3.1), we are interested in *Markov policies* to control the system defined as follows.

Definition 2.5.1. *A Markov policy for the dt-SCS Σ in* (2.3.1) *is a sequence $\bar{\rho} = (\bar{\rho}_0, \bar{\rho}_1, \bar{\rho}_2, \ldots)$ of universally measurable stochastic kernels $\bar{\rho}_n$ [BS96], each defined on the input space U given $X \times W$ and such that for all $(x_n, w_n) \in X \times W$, $\rho_n(U(x_n, w_n) \,|\, (x_n, w_n))$ $= 1$. The class of all Markov policies is denoted by $\bar{\Pi}_M$.*

2.6 Discrete-Time Stochastic Switched Systems

We consider stochastic *switched* systems in discrete-time (dt-SS) throughout the thesis formalized in the following definition.

Definition 2.6.1. *A discrete-time stochastic switched system (dt-SS) is characterized here by the tuple*

$$\Sigma = (X, P, \mathcal{P}, W, \varsigma, \mathbf{F}, Y^1, Y^2, h^1, h^2), \tag{2.6.1}$$

where:

- *$X \subseteq \mathbb{R}^n$ is a Borel space as the state space of the system. We denote by $(X, \mathcal{B}(X))$ the measurable space with $\mathcal{B}(X)$ being the Borel sigma-algebra on the state space;*

- *$P = \{1, \ldots, m\}$ is the finite set of modes;*
- *$\mathcal{P}$ is a subset of $\mathcal{S}(\mathbb{N}, P)$ which denotes the set of functions from $\mathbb{N}$ to P;*
- *$W \subseteq \mathbb{R}^{\bar{p}}$ is a Borel space as the* internal *input space of the system;*
- *ς is a sequence of independent and identically distributed (i.i.d.) random variables from a sample space Ω to the measurable space $(\mathcal{V}_\varsigma, \mathcal{F}_\varsigma)$*

$$\varsigma := \{\varsigma(k) : (\Omega, \mathcal{F}_\Omega) \to (\mathcal{V}_\varsigma, \mathcal{F}_\varsigma),\ k \in \mathbb{N}\},$$

- *$\mathbf{F} = \{f_1, \ldots, f_m\}$ is a collection of vector fields indexed by p. For all $p \in P$, the map $f_p : X \times W \times \mathcal{V}_\varsigma \to X$ is a measurable function characterizing the state evolution of the system;*
- *$Y^1 \subseteq \mathbb{R}^{q^1}$ is a Borel space as the* external *output space of the system;*
- *$Y^2 \subseteq \mathbb{R}^{q^2}$ is a Borel space as the* internal *output space of the system;*
- *$h^1 : X \to Y^1$ is a measurable function as the* external *output map that maps a state $x \in X$ to its* external *output $y^1 = h^1(x)$;*
- *$h^2 : X \to Y^2$ is a measurable function as the* internal *output map that maps a state $x \in X$ to its* internal *output $y^2 = h^2(x)$.*

The evolution of the state of Σ, for a given initial state $x(0) \in X$, an input sequence $w(\cdot) : \mathbb{N} \to W$ and a switching signal $\mathbf{p}(k) : \mathbb{N} \to P$, is described by

$$\Sigma : \begin{cases} x(k+1) = f_{\mathbf{p}(k)}(x(k), w(k), \varsigma(k)), \\ y^1(k) = h^1(x(k)), \\ y^2(k) = h^2(x(k)), \end{cases} \qquad k \in \mathbb{N}. \tag{2.6.2}$$

2.7 Incremental Input-to-State Stability

Definition 2.7.1. *A dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2)$ is called* incrementally input-to-state stable *if there exists a function $V : X \times X \to \mathbb{R}_{\geq 0}$ such that $\forall x, x' \in X$, $\forall \nu, \nu' \in U$, $\forall w, w' \in W$, the following two inequalities hold:*

$$\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|), \tag{2.7.1}$$

and

$$\begin{aligned} &\mathbb{E}\Big[V(f(x, \nu, w, \varsigma), f(x', \nu', w', \varsigma)) \,\Big|\, x, x', \nu, \nu', w, w'\Big] - V(x, x') \\ &\leq -\bar{\kappa}(V(x, x')) + \bar{\rho}_{\text{int}}(\|w - w'\|) + \bar{\rho}_{\text{ext}}(\|\nu - \nu'\|), \end{aligned} \tag{2.7.2}$$

for some $\underline{\alpha}, \overline{\alpha}, \bar{\kappa} \in \mathcal{K}_\infty$,and $\bar{\rho}_{\text{int}}$, $\bar{\rho}_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$.

Remark 2.7.2. *Note that Definition 2.7.1 is a stochastic counterpart of the incremental ISS Lyapunov functions defined for discrete-time deterministic systems in [TRK18].*

2.8 Incremental Passivability

Definition 2.8.1. *A dt-SCS* $\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2)$ *is called* incrementally passivable *if there exist functions* $\bar{H} : X \to U$ *and* $V : X \times X \to \mathbb{R}_{\geq 0}$ *such that* $\forall x, x' \in X$*,* $\forall \nu \in U$*,* $\forall w, w' \in W$*, the inequalities*

$$\underline{\alpha}(\|h^1(x) - h^1(x')\|) \leq V(x, x'), \tag{2.8.1}$$

and

$$\begin{aligned} &\mathbb{E}\Big[V(f(x, \bar{H}(x) + \nu, w, \varsigma), f(x', \bar{H}(x') + \nu, w', \varsigma)) \,\big|\, x, x', \nu, w, w'\Big] - V(x, x') \\ &\leq -\bar{\kappa}(V(x, x')) + \begin{bmatrix} w - w' \\ h^2(x) - h^2(x') \end{bmatrix}^T \overbrace{\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix}}^{\bar{X}:=} \begin{bmatrix} w - w' \\ h^2(x) - h^2(x') \end{bmatrix}, \end{aligned} \tag{2.8.2}$$

hold for some $\underline{\alpha}, \bar{\kappa} \in \mathcal{K}_\infty$*, and the matrix* $\bar{X}$ *of an appropriate dimension.*

3 Infinite Abstractions (Reduced-Order Models)

3.1 Introduction

Computational complexity in synthesizing controllers for stochastic CPSs can be alleviated via abstractions in two consecutive stages. In the first phase, original systems can be abstracted by simpler ones with lower dimensions (model order reductions). Then one can employ infinite abstractions as a replacement of original systems, perform analysis and synthesis over abstract models, and finally carry the results back (via an interface map) over concrete systems. Since the mismatch between outputs of original systems and those of their infinite abstractions is well-quantified, one can guarantee that concrete systems also satisfy the same specifications as abstract ones with guaranteed error bounds. Unfortunately, construction of abstractions for large-scale CPSs in a monolithic manner suffers severely from the curse of dimensionality. To relieve this issue, one promising solution is to consider the large-scale CPSs as an interconnected system composed of several smaller subsystems, and provide a compositional framework for the construction of abstractions for the given system using abstractions of smaller subsystems. This chapter is concerned with providing different compositional approaches for the construction of infinite abstractions for large-scale discrete-time stochastic control systems.

3.1.1 Related Literature

3.1.1.1 Infinite Abstraction Techniques

In the past few years, there have been some results on the construction of infinite abstractions for stochastic systems. Existing results include infinite approximation techniques for jump-diffusion systems [JP09], and infinite-horizon properties over discrete-time stochastic models with continuous-state spaces [TA11]. Compositional construction of infinite abstractions is discussed in [ZRME17] using small-gain type conditions. An (in)finite abstraction technique for synthesis of stochastic control systems is recently discussed in [NSZ19].

3.1.1.2 Temporal Logic Verification and Synthesis

There have been also several results on the verification and synthesis of stochastic systems over high-level properties expressed as LTL formulae such as safety, reachability or more complex properties denoted by omega-regular languages. In this respect, a policy refinement of general Markov decision processes via approximate similarity relations is initially proposed in [HSA17], and then generalized to synthesize policies for robust satisfaction of specifications in [HS18, HS19]. Formal controller synthesis of stochastic systems via control barrier certificates for LTL properties over finite traces is presented in [JSZ19]. Maximally safe Markov policies of abstract finite-space models to design suboptimal policies for original continuous-space systems are proposed in [Sou14, Chapter 5]. An optimal control synthesis approach is proposed in [TMKA17] in which the probability of a given event is either maximized or minimized over a controlled discrete-time Markov process model.

A new approach for the automated synthesis of safe and robust PID controllers for stochastic hybrid systems is proposed in [SPB+17]. An automated synthesis of digital controllers with formal safety guarantees for systems with nonlinear dynamics, noisy output measurements, and stochastic disturbances is recently presented in [SSP+19]. Optimal control policies satisfying temporal logic specifications for a team of robots moving in a stochastic environment are proposed in [CDB12, DCB17]. A general framework to synthesize controllers satisfying signal temporal logic specifications for piecewise affine systems subject to stochastic uncertainties is provided in [MSH+17]. These specifications are encoded as chance constraints and a method for designing model predictive controllers under such constraints is proposed in [FMPS17, FMPS19]. An application of these logic specifications in controlling wastewater systems is studied in [FSMOM18, FSMOM17]. An efficient and safe exploration algorithm for Markov decision processes with unknown transition models is developed in [BMAS19].

A reinforcement learning framework for the controller synthesis of unknown MDPs satisfying omega-regular objectives is proposed in [HPS+19a]. Measurability and safety verification of stochastic hybrid systems are discussed in [FHH+11]. A framework for analyzing probabilistic safety and reachability problems for discrete-time stochastic hybrid systems is proposed in [KDS+11, DKS+13]. A controller design scheme for stochastic hybrid systems satisfying specifications described by a finite automata is provided in [KSL13]. A probabilistic approach for the control of stochastic systems subject to LTL formula over a set of linear predicates in the state of the system is presented in [LAB09]. Computational methods for stochastic control systems with metric interval temporal logic specifications are proposed in [FT15]. A strategy synthesis for stochastic games with multiple long-run objectives is presented in [BKTW15]. A controller synthesis framework in turn-based stochastic games with both a qualitative LTL constraint and a quantitative discounted-sum objective is studied in [WT16]. A temporal logic control for stochastic linear systems using an abstraction refinement of probabilistic games is discussed in [SKC+17].

3.1.2 Contributions

In this chapter, we provide three different compositional methodologies (i.e., classic small-gain, max small-gain, and dissipativity approaches) for the construction of infinite abstractions for networks of stochastic control systems. The proposed techniques leverage sufficient small-gain and dissipativity type conditions to establish the compositionality results which rely on relations between subsystems and their infinite abstractions described by the existence of stochastic simulation functions. This type of relations enables us to compute the probabilistic error between the interconnection of concrete subsystems and that of their infinite abstractions. As a consequence, one can utilize the proposed results here to solve particularly safety/reachability problems over abstract interconnected systems and then carry the results back over concrete interconnected ones.

In the first part of this chapter, we leverage sufficient classic small-gain type conditions for the compositional quantification of the probabilistic distance between the interconnection of stochastic control subsystems and that of their infinite abstractions. We also provide a framework for the construction of infinite abstractions for the class of linear stochastic systems. Moreover, we consider a finite-horizon invariant specification and show how a synthesized policy for the abstract system can be refined to a policy for the original system while providing a guarantee on the probability of the satisfaction. It should be noted that we do not put any restriction on the sources of uncertainties in the concrete and abstract systems. Thus our results are more general than the ones obtained by [Zam14, ZRME17], where the noises in the concrete and abstract systems are assumed to be the same. This means the abstraction has access to the noise of the concrete system which is a strong assumption. We demonstrate the effectiveness of the proposed results by constructing an infinite abstraction (totally 4 dimensions) of an interconnection of four discrete-time linear stochastic control subsystems (together 100 dimensions) in a compositional fashion.

In the second part of the chapter, we propose a max small-gain condition and show that it is more general than the classic one since it does not require any linear growth on the gains of the subsystems which is the case in the classic small-gain approach. We also show that the approximation error provided by the max small-gain is completely independent of the size of the network, and is computed only based on the maximum error of subsystems instead of being a linear combination of them which is the case in the classic small-gain approach. Accordingly, the overall error computed by the max small-gain does not change as the number of subsystems grows. We also extend our proposed construction scheme (in the first part) from linear systems to a particular class of *nonlinear* stochastic systems whose nonlinearities satisfy a slope restriction.

In the last part of the chapter, we provide a compositional approach using an interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions. We show that the proposed compositionality conditions can enjoy the structure of interconnection topology and be potentially satisfied regardless of the number or gains of subsystems. We also provide a construction framework for the same nonlinear class of stochastic systems. Finally, we extend our specification from the finite-horizon invariant to a fragment of linear temporal logic known as syntactically co-safe linear temporal logic (scLTL) [KV01]. In particular, given such a co-safe LTL specification over the concrete system, we construct an epsilon-perturbed specification over the abstract system whose probability of satisfaction gives a lower bound for the probability of satisfaction in the concrete domain. We demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 3 dimensions) of an interconnection of three discrete-time nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion such that the compositionality condition does not require any constraint on the number or gains of subsystems. We also employ the constructed abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe LTL specification.

3.2 Classic Small-Gain Approach

In this section, we provide a compositional framework for the construction of infinite abstractions via a classic small-gain approach for dt-SCS defined in Definition 2.3.1. Our abstraction scheme is based on a notion of so-called stochastic simulation functions, using which one can quantify the distance between original interconnected stochastic control systems and that of their abstractions in a probabilistic setting. Accordingly, the infinite abstraction, which is itself a discrete-time stochastic control system with a lower dimension, performs as a substitute in the controller design process. In particular, one can utilize the proposed results here to solve particularly safety/reachability problems over the abstract interconnected systems and then carry the results back over the concrete interconnected ones.

We derive sufficient classic small-gain type conditions for the compositional quantification of the probabilistic distance between the interconnection of stochastic control subsystems and that of their abstractions. We then focus on the class of discrete-time linear stochastic control systems with independent noises in the abstract and concrete subsystems. For this class of systems, we propose a computational scheme to construct infinite abstractions together with their corresponding stochastic simulation functions. Moreover, we consider a finite-horizon invariant specification and show how a synthesized policy for the abstract system can be refined back to a policy for the original system while providing a guarantee on the probability of satisfaction. We demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 4 dimensions) of an interconnection of four discrete-time linear stochastic control subsystems (together 100 dimensions) in a compositional fashion.

3.2.1 sum-Type Stochastic Pseudo-Simulation and Simulation Functions

In this subsection, we first introduce a notion of so-called sum-type stochastic pseudo-simulation functions (sum-type SPSF) for discrete-time stochastic control systems with both internal and external inputs and outputs and then define sum-type stochastic simulation functions (sum-type SSF) for systems with only external inputs and outputs. These two definitions will be employed to quantify the closeness of two interconnected dt-SCS.

Remark 3.2.1. *Simulation functions are* Lyapunov-like *functions defined over the Cartesian product of state spaces, which relate the state trajectory of the abstract system to the* *state trajectory of the original one such that the mismatch between two systems remains within some guaranteed error bounds.*

Definition 3.2.2. *Consider two dt-SCS* $\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2)$ *and* $\widehat{\Sigma} = (\hat{X}, \hat{U}, W, \hat{\varsigma}, \hat{f}, Y^1, Y^2, \hat{h}^1, \hat{h}^2)$ *with the same internal input, and internal and external output spaces. A function* $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *is called a* sum*-type stochastic pseudo-simulation function (*sum*-type SPSF) from* $\widehat{\Sigma}$ *to* Σ *if*

- $\exists \alpha \in \mathcal{K}_\infty$ *such that*

$$\forall x \in X, \forall \hat{x} \in \hat{X}, \forall i \in \{1,2\}, \quad \alpha(\|h^i(x) - \hat{h}^i(\hat{x})\|_2) \leq V(x, \hat{x}), \tag{3.2.1}$$

- $\forall x \in X, \hat{x} \in \hat{X}, \hat{\nu} \in \hat{U}$, *and* $\forall \hat{w} \in \hat{W}$, $\exists \nu \in U$ *such that* $\forall w \in W$

$$\begin{aligned}&\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\Big|\, x(k){=}x, \hat{x}(k){=}\hat{x}, \nu(k){=}\nu, \hat{\nu}(k){=}\hat{\nu}, w(k){=}w, \hat{w}(k){=}\hat{w}\Big] \\ &\quad - V(x, \hat{x}) \leq -\kappa(V(x, \hat{x})) + \rho_{\text{int}}(\|w - \hat{w}\|_2) + \rho_{\text{ext}}(\|\hat{\nu}\|_2) + \psi,\end{aligned} \tag{3.2.2}$$

  for some $\kappa \in \mathcal{K}_\infty$, $\rho_{\text{int}}, \rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, *and* $\psi \in \mathbb{R}_{\geq 0}$.

We utilize the notation $\widehat{\Sigma} \preceq^{\mathsf{sum}}_{SPSF} \Sigma$ if there exists a sum-type SPSF V from $\widehat{\Sigma}$ to Σ, in which the control system $\widehat{\Sigma}$ is considered as an abstraction of the concrete (original) system Σ.

Remark 3.2.3. *The second condition in Definition 3.2.2 implicitly implies the existence of a function* $\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w})$ *for the satisfaction of* (3.2.2). *This function is called the* interface function *and can be employed to refine a synthesized policy* $\hat{\nu}$ *for* $\widehat{\Sigma}$ *to a policy* ν *for* Σ.

In this section, we study interconnected discrete-time stochastic control systems without internal inputs and outputs, resulting from the interconnection of discrete-time stochastic control subsystems having both internal and external signals. Thus we modify the above definition for systems without internal inputs and outputs by eliminating all the terms related to $w, \hat{w}$.

Definition 3.2.4. *Consider two dt-SCS* $\Sigma = (X, U, \varsigma, f, Y, h)$ *and* $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{\varsigma}, \hat{f}, Y, \hat{h})$ *with the same output spaces. A function* $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *is called a* sum*-type stochastic simulation function (*sum*-type SSF) from* $\widehat{\Sigma}$ *to* Σ *if*

- $\exists \alpha \in \mathcal{K}_\infty$ *such that*

$$\forall x \in X, \forall \hat{x} \in \hat{X}, \quad \alpha(\|h(x) - \hat{h}(\hat{x})\|_2) \leq V(x, \hat{x}), \tag{3.2.3}$$

- $\forall x \in X, \hat{x} \in \hat{X}, \hat{\nu} \in \hat{U}$, $\exists \nu \in U$ *such that*

$$\begin{aligned}&\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\Big|\, x(k) = x, \hat{x}(k) = \hat{x}, \nu(k) = \nu, \hat{\nu}(k) = \hat{\nu}\Big] - V(x, \hat{x}) \\ &\quad \leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{\nu}\|_2) + \psi,\end{aligned} \tag{3.2.4}$$

  for some $\kappa \in \mathcal{K}_\infty$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, *and* $\psi \in \mathbb{R}_{\geq 0}$.

We call $\widehat{\Sigma}$ an abstraction of Σ, and denote by $\widehat{\Sigma} \preceq^{\mathsf{sum}}_{SSF} \Sigma$ if there exists a sum-type SSF V from $\widehat{\Sigma}$ to Σ.

Remark 3.2.5. *Note that conditions* (3.2.1), (3.2.2), (3.2.3), *and* (3.2.4) *roughly speaking guarantee that if the concrete system and its abstraction start from two close initial conditions, then their outputs remain close (in terms of the expectation) after one step. This type of conditions is closely related to the ones in the notions of (bi)simulation relations [Tab09].*

In order to show the usefulness of the sum-type SSF in comparing output trajectories of two dt-SCS in a probabilistic setting, we need the following technical lemma borrowed from [Kus67, Theorem 3, pp. 86] with some slight modifications for the finite-time horizon, and also [Kus67, Theorem 12, pp. 71] for the infinite-time horizon.

Lemma 3.2.6. *Let $\Sigma = (X, \varsigma, f, Y, h)$ be a dt-SCS with the transition map $f : X \times \mathcal{V}_\varsigma \to X$.*
i) Finite-time horizon: Assume there exist $V : X \to \mathbb{R}_{\geq 0}$ and constants $0 < \hat{\kappa} < 1$ and $\hat{\psi} \in \mathbb{R}_{\geq 0}$ such that

$$\mathbb{E}\Big[V(x(k+1)) \,\big|\, x(k) = x\Big] \leq \hat{\kappa} V(x) + \hat{\psi}.$$

Then for any random variable a as the initial state of the dt-SCS, the following inequity holds:

$$\mathbb{P}\left\{\sup_{0 \leq k \leq T_d} V(x) \geq \varepsilon \,\big|\, a\right\} \leq \hat{\delta},$$

$$\hat{\delta} := \begin{cases} 1 - (1 - \frac{V(a)}{\varepsilon})(1 - \frac{\hat{\psi}}{\varepsilon})^{T_d}, & \text{if } \varepsilon \geq \frac{\hat{\psi}}{\hat{\kappa}}, \\ (\frac{V(a)}{\varepsilon})(1 - \hat{\kappa})^{T_d} + (\frac{\hat{\psi}}{\hat{\kappa}\varepsilon})(1 - (1 - \hat{\kappa})^{T_d}), & \text{if } \varepsilon < \frac{\hat{\psi}}{\hat{\kappa}}. \end{cases}$$

ii) Infinite-time horizon: Assume there exists a nonnegative $V : X \to \mathbb{R}_{\geq 0}$ such that

$$\mathbb{E}\Big[V(x(k+1)) \,\big|\, x(k) = x\Big] - V(x) \leq 0.$$

Function V satisfying the above inequality is called nonnegative supermartingale. Then for any random variable a as the initial state of the dt-SCS, the following inequity holds:

$$\mathbb{P}\left\{\sup_{0 \leq k < \infty} V(x) \geq \varepsilon \,\big|\, a\right\} \leq \frac{V(a)}{\varepsilon}.$$

Now by employing Lemma 3.2.6, we provide one of the main results of this section.

Theorem 3.2.7. *Let $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{\varsigma}, \hat{f}, Y, \hat{h})$ be two dt-SCS with the same output spaces. Suppose V is a sum-type SSF from $\widehat{\Sigma}$ to Σ, and there exists a constant $0 < \hat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}_\infty$ in* (3.2.4) *satisfies $\kappa(r) \geq \hat{\kappa} r$ $\forall r \in \mathbb{R}_{\geq 0}$. For any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the* *two dt-SCS, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds:*

$$\mathbb{P}\left\{\sup_{0\le k\le T_d} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \ge \varepsilon \,\middle|\, a, \hat{a}\right\} \le \hat{\delta}, \tag{3.2.5}$$

$$\hat{\delta} := \begin{cases} 1 - \left(1 - \frac{V(a,\hat{a})}{\alpha(\varepsilon)}\right)\left(1 - \frac{\hat{\psi}}{\alpha(\varepsilon)}\right)^{T_d}, & \text{if } \alpha(\varepsilon) \ge \frac{\hat{\psi}}{\hat{\kappa}}, \\ \left(\frac{V(a,\hat{a})}{\alpha(\varepsilon)}\right)(1-\hat{\kappa})^{T_d} + \left(\frac{\hat{\psi}}{\hat{\kappa}\alpha(\varepsilon)}\right)\left(1 - (1-\hat{\kappa})^{T_d}\right), & \text{if } \alpha(\varepsilon) < \frac{\hat{\psi}}{\hat{\kappa}}, \end{cases}$$

provided that there exists a constant $\hat{\psi} \ge 0$ satisfying $\hat{\psi} \ge \rho_{\text{ext}}(\|\hat{\nu}\|_\infty) + \psi$.

Proof. Since V is a sum-type SSF from $\widehat{\Sigma}$ to Σ, we have

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{0\le k\le T_d} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \ge \varepsilon \,\middle|\, a, \hat{a}\right\} \\
&= \mathbb{P}\left\{\sup_{0\le k\le T_d} \alpha\left(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2\right) \ge \alpha(\varepsilon) \,\middle|\, a, \hat{a}\right\} \\
&\le \mathbb{P}\left\{\sup_{0\le k\le T_d} V\left(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k)\right) \ge \alpha(\varepsilon) \,\middle|\, a, \hat{a}\right\}.
\end{aligned} \tag{3.2.6}$$

The equality holds due to α being a $\mathcal{K}_\infty$ function. The inequality is also true due to the condition (3.2.3) on the sum-type SSF V. The results follow by applying the first part of Lemma 3.2.6 to (3.2.6) with some slight modification and utilizing the inequality (3.2.4). □

Remark 3.2.8. *Note that since every infinity norm is upper bounded by an Euclidean norm, one can readily show that the provided results in Theorem 3.2.7 are also valid in the case of having the infinity norm (cf. Section 3.3).*

The results shown in Theorem 3.2.7 provide a closeness of output behaviours of two systems in the finite-time horizon. We can extend the result to an infinite-time horizon using the second part of Lemma 3.2.6 given that $\hat{\psi} = 0$ as stated in the following corollary.

Corollary 3.2.9. *Let Σ and $\widehat{\Sigma}$ be two dt-SCS with the same output spaces. Suppose V is a sum-type SSF from $\widehat{\Sigma}$ to Σ such that $\rho_{\text{ext}}(\cdot) \equiv 0$ and $\psi = 0$. For any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ preserving Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, there exists $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds:*

$$\mathbb{P}\left\{\sup_{0\le k<\infty} \|y_{a\nu}(k) - \hat{y}_{\hat{a}0}(k)\|_2 \ge \varepsilon \,\middle|\, a, \hat{a}\right\} \le \frac{V(a,\hat{a})}{\alpha(\varepsilon)}.$$

Proof. Since V is a sum-type SSF from $\widehat{\Sigma}$ to Σ with $\rho_{\text{ext}}(\cdot) \equiv 0$ and $\psi = 0$, for any $x(k) \in X$ and $\hat{x}(k) \in \hat{X}$ and any $\hat{\nu}(k) \in \hat{U}$, there exists $\nu(k) \in U$ such that

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\big|\, x(k), \hat{x}(k), \nu(k), \hat{\nu}(k)\Big] - V((x(k), \hat{x}(k)) \\ \leq -\kappa(V(x(k), \hat{x}(k)),$$

showing that $V\left(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k)\right)$ is a nonnegative supermartingale [Kus67, Chapter 1] for any initial conditions a and $\hat{a}$ and inputs $\nu, \hat{\nu}$. Following the same reasoning as in the proof of Theorem 3.2.7, we have

$$\mathbb{P}\left\{\sup_{0\leq k<\infty} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \geq \varepsilon \,\big|\, a, \hat{a}\right\} \\ = \mathbb{P}\left\{\sup_{0\leq k<\infty} \alpha\big(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2\big) \geq \alpha(\varepsilon) \,\big|\, a, \hat{a}\right\} \\ \leq \mathbb{P}\left\{\sup_{0\leq k<\infty} V(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k)) \geq \alpha(\varepsilon) \,\big|\, a, \hat{a}\right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon)},$$

where the last inequality is due to the nonnegative supermartingale property as presented in the second part of Lemma 3.2.6. □

Remark 3.2.10. *Note that $\psi = 0$ is possible potentially if concrete and abstract systems are both continuous-space but possibly with different dimensions and share the same multiplicative noise. Depending on the dynamic, function $\rho_{\text{ext}}(\cdot)$ can be identically zero (cf. Case study 3.2.4).*

The sum-type SSF defined before can be employed to guarantee an upper bound on the probability of the maximum difference in output trajectories. In particular, we consider a finite-horizon invariant specification and show how a synthesized policy for the abstract system can be refined to a policy for the original one while providing a guarantee on the probability of satisfaction. This idea can be utilized in conjunction with the stochastic safety/reachability analysis of systems, which is discussed next.

Suppose V is a sum-type SSF from $\widehat{\Sigma}$ to Σ. Then for any input strategy $\hat{\nu}$ of the system $\widehat{\Sigma}$, there exists an input strategy ν of Σ such that the following probability is bounded:

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \geq \varepsilon \,\big|\, a, \hat{a}\right\} \leq \hat{\delta},$$

with $\hat{\delta}$ being defined in Theorem 3.2.7 based on ε and T_d. Given the unsafe set $\mathcal{A}_1$ for Σ, we can construct another set $\mathcal{A}_2$, which is the ε neighborhood of $\mathcal{A}_1$, i.e.,

$$\mathcal{A}_2 = \{y' \,|\, \exists y \in \mathcal{A}_1, \|y' - y\|_2 \leq \varepsilon\}.$$

Now, we can provide the following corollary.

Corollary 3.2.11. *Suppose V is a* sum*-type SSF from $\widehat{\Sigma}$ to Σ. For any input $\hat{\nu}(\cdot)$ there exists $\nu(\cdot)$ such that the following inequality holds:*

$$\mathbb{P}\{\exists k \leq T_d,\ y_{a\nu}(k) \in \mathcal{A}_1\} \leq \mathbb{P}\{\exists k \leq T_d,\ \hat{y}_{\hat{a}\hat{\nu}}(k) \in \mathcal{A}_2\} + \hat{\delta}.$$

Proof. Denote the events $\mathcal{E}_1 := \{\exists k \leq T_d,\ y_{a\nu}(k) \in \mathcal{A}_1\}$ and $\mathcal{E}_2 := \{\exists k \leq T_d,\ \hat{y}_{\hat{a}\hat{\nu}}(k) \in \mathcal{A}_2\}$. Then we have

$$\mathbb{P}\{\mathcal{E}_1\} = \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} + \mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\} \leq \mathbb{P}\{\mathcal{E}_2\} + \mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\},$$

where $\bar{\mathcal{E}}_2$ is the complement of $\mathcal{E}_2$. Notice that the term $\mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\}$ is bounded by $\hat{\delta}$ due to the above results, which concludes the proof. □

3.2.2 Compositionality Results

In this subsection, we analyze networks of control systems and show how to construct their abstractions together with the corresponding sum-type SSF by employing sum-type SPSF of subsystems. We consider here Σ as the original dt-SCS and $\widehat{\Sigma}$ as its infinite abstraction with (potentially) a lower dimension.

3.2.2.1 Interconnected Stochastic Control Systems

Consider a complex stochastic control system Σ composed of $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i as

$$\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i^1, Y_i^2, h_i^1, h_i^2), \quad i \in \{1, \cdots, N\}, \tag{3.2.7}$$

with partitioned internal inputs and outputs as

$$\begin{aligned} w_i &= \left[w_{i1}; \ldots; w_{i(i-1)}; w_{i(i+1)}; \ldots; w_{iN}\right], \\ y_i^2 &= \left[y_{i1}^2; \ldots; y_{i(i-1)}^2; y_{i(i+1)}^2; \ldots; y_{iN}^2\right], \end{aligned} \tag{3.2.8}$$

and also its internal output function

$$h_i^2 = \left[h_{i1}^2; \ldots; h_{i(i-1)}^2; h_{i(i+1)}^2; \ldots; h_{iN}^2\right]. \tag{3.2.9}$$

In particular, we assume that the dimension of w_{ij} is equal to the dimension of y_{ji}^2. If there is no connection from stochastic control subsystem Σ_i to Σ_j, then we assume that the connecting output function is identically zero for all arguments, i.e., $h_{ij}^2 \equiv 0$. Now, we define the *interconnected stochastic control systems* as the following.

Definition 3.2.12. *Consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i^1, Y_i^2, h_i^1, h_i^2), i \in \{1, \cdots, N\}$, with the input-output configuration as in* (3.2.8) *and* (3.2.9)*. The interconnection of Σ_i for any $i \in \{1, \cdots, N\}$, is the interconnected stochastic control system $\Sigma = (X, U, \varsigma, f, Y, h)$, denoted by $\mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$, such that $X :=$*

$\prod_{i=1}^{N} X_i$, $U := \prod_{i=1}^{N} U_i$, $f := \prod_{i=1}^{N} f_i$, $Y := \prod_{i=1}^{N} Y_i^1$, and $h = \prod_{i=1}^{N} h_i^1$, subjected to the following constraint:

$$\forall i,j \in \{1,\cdots,N\}, i \neq j: \qquad w_{ij} = y_{ji}^2, \quad Y_{ji}^2 \subseteq W_{ij}. \tag{3.2.10}$$

An example of the interconnection of two concrete control subsystems Σ_1 and Σ_2 is illustrated in Figure 3.1.

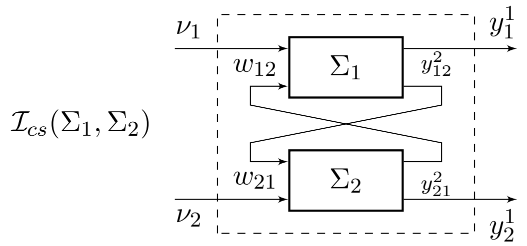

Figure 3.1: Interconnection of two concrete stochastic subsystems Σ_1 and Σ_2.

3.2.2.2 Compositional Abstractions of Interconnected Control Systems

We assume that we are given N stochastic control subsystems as in (3.2.7) together with their corresponding abstractions $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, W_i, \hat{\varsigma}_i, \hat{f}_i, Y_i^1, Y_i^2, \hat{h}_i^1, \hat{h}_i^2)$ with a sum-type SPSF V_i from $\widehat{\Sigma}_i$ to Σ_i. To prove the main compositionality result of the section, we raise the following classic small-gain assumption.

Assumption 3.2.13. *For any $i,j \in \{1,\cdots,N\}$, $i \neq j$, there exist $\mathcal{K}_\infty$ functions $\hat{\gamma}_i$ and constants $\hat{\lambda}_i \in \mathbb{R}_{>0}$ and $\hat{\delta}_{ij} \in \mathbb{R}_{\geq 0}$ such that for any $s \in \mathbb{R}_{\geq 0}$*

$$\kappa_i(s) \geq \hat{\lambda}_i \hat{\gamma}_i(s), \tag{3.2.11}$$

$$h_{ji}^2 \equiv 0 \implies \hat{\delta}_{ij} = 0, \tag{3.2.12}$$

$$h_{ji}^2 \not\equiv 0 \implies \rho_{\text{int}i}((N-1)\alpha_j^{-1}(s)) \leq \hat{\delta}_{ij}\hat{\gamma}_j(s), \tag{3.2.13}$$

where α_j, κ_i, and $\rho_{\text{int}i}$ represent the corresponding $\mathcal{K}_\infty$ functions of V_i appearing in Definition 3.2.2. Prior to presenting the next theorem, we define $\hat{\Lambda} := diag(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$, $\hat{\Delta} := \{\hat{\delta}_{ij}\}$, where $\hat{\delta}_{ii} = 0$ $\forall i \in \{1,\cdots,N\}$, and $\hat{\Gamma}(s) := [\hat{\gamma}_1(s_1); \ldots; \hat{\gamma}_N(s_N)]$, where $s = [s_1; \ldots; s_N]$. In the next theorem, we leverage the classic small-gain Assumption 3.2.13 to quantify the error between the interconnection of stochastic control subsystems and that of their infinite abstractions in a compositional way.

Theorem 3.2.14. *Consider the interconnected stochastic control system $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i. Suppose that each stochastic control subsystem Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding sum-type SPSF V_i. If Assumption 3.2.13 holds and there exists a vector $\mu \in \mathbb{R}_{>0}^N$ such that the inequality*

$$\mu^T(-\hat{\Lambda} + \hat{\Delta}) < 0 \tag{3.2.14}$$

is also met, then

$$V(x,\hat{x}) := \sum_{i=1}^{N} \mu_i V_i(x_i,\hat{x}_i)$$

is a sum*-type SSF function from* $\widehat{\Sigma} = \mathcal{I}_{cs}(\widehat{\Sigma}_1,\ldots,\widehat{\Sigma}_N)$ *to* $\Sigma = \mathcal{I}_{cs}(\Sigma_1,\ldots,\Sigma_N)$*.*

Proof. We first show that (3.2.3) in Definition 3.2.4 holds. For any $x := [x_1;\ldots;x_N]$ and $\hat{x} := [\hat{x}_1;\ldots;\hat{x}_N]$, one acquires

$$\begin{aligned}\|h(x)-\hat{h}(\hat{x})\|_2 &\le \sum_{i=1}^{N} \|h_i^1(x_i)-\hat{h}_i^1(\hat{x}_i)\|_2\\ &\le \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i,\hat{x}_i)) \le \hat{\beta}(V(x,\hat{x})),\end{aligned}$$

with function $\hat{\beta}:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ defined for all $s\in\mathbb{R}_{\ge 0}$ as

$$\hat{\beta}(s) := \max\Big\{\textstyle\sum_{i=1}^{N}\alpha_i^{-1}(s_i)\,\big|\, s_i\ge 0,\ \sum_{i=1}^{N}\mu_i s_i = s\Big\}.$$

It is not hard to verify that function $\hat{\beta}(\cdot)$ defined above is a $\mathcal{K}_\infty$ function. By taking the $\mathcal{K}_\infty$ function $\alpha(s) := \hat{\beta}^{-1}(s)$, $\forall s\in\mathbb{R}_{\ge 0}$, one obtains

$$\alpha(\|h(x)-\hat{h}(\hat{x})\|_2) \le V(x,\hat{x}),$$

satisfying inequality (3.2.3). Now we show that (3.2.4) holds, as well. Consider any $x = [x_1;\ldots;x_N]$, $\hat{x} = [\hat{x}_1;\ldots;\hat{x}_N]$, and $\hat{\nu} = [\hat{\nu}_1;\ldots;\hat{\nu}_N]$. By applying the following inequality

$$\rho_{\text{int}i}(s_1+\cdots+s_{N-1}) \le \sum_{i=1}^{N-1}\rho_{\text{int}i}((N-1)s_i), \tag{3.2.15}$$

which is valid for any $\rho_{\text{int}i}\in\mathcal{K}_\infty\cup\{0\}$, and any $s_i\in\mathbb{R}_{\ge 0}$, $i\in\{1,\cdots,N\}$, one can obtain the chain of inequalities in (3.2.17). By defining

$$\begin{aligned}\kappa(s) &:= \min\Big\{-\mu^T(-\hat{\Lambda}+\hat{\Delta})\hat{\Gamma}(V(x,\hat{x}))\,\big|\,\mu^T V(x,\hat{x}) = s\Big\},\\ \rho_{\text{ext}}(s) &:= \max\Big\{\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(s_i)\,\big|\, s_i\ge 0, \|[s_1;\ldots;s_N]\|_2 = s\Big\},\\ \psi &:= \sum_{i=1}^{N}\mu_i\psi_i,\end{aligned} \tag{3.2.16}$$

where $V(x,\hat{x}) = [V_1(x_1,\hat{x}_1);\ldots;V_N(x_N,\hat{x}_N)]$, the condition (3.2.4) is also satisfied. Then V is a sum-type SSF function from $\widehat{\Sigma}$ to Σ, which completes the proof. □

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^{N}\mu_i V_i(x_i(k+1),\hat{x}_i(k+1))\,\big|\,x,\hat{x},\hat{\nu}\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\hat{x}_i)\\
&=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i(x_i(k+1),\hat{x}_i(k+1))\,\big|\,x,\hat{x},\hat{\nu}\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\hat{x}_i)\\
&=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i(x_i(k+1),\hat{x}_i(k+1))\,\big|\,x_i,\hat{x}_i,\hat{\nu}_i\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\hat{x}_i)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\rho_{\mathrm{int}i}(\|w_i-\hat{w}_i\|_2)+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\rho_{\mathrm{int}i}(\sum_{j=1,i\neq j}^{N}\|w_{ij}-\hat{w}_{ij}\|_2)+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&=\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\rho_{\mathrm{int}i}(\sum_{j=1,i\neq j}^{N}\|y_{ji}^2-\hat{y}_{ji}^2\|_2)+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{j=1,i\neq j}^{N}\rho_{\mathrm{int}i}((N-1)\|y_{ji}^2-\hat{y}_{ji}^2\|_2)+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&=\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{j=1,i\neq j}^{N}\rho_{\mathrm{int}i}((N-1)\|h_j^2(x_j)-\hat{h}_j^2(\hat{x}_j)\|_2)+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{j=1,i\neq j}^{N}\rho_{\mathrm{int}i}((N-1)\alpha_j^{-1}(V_j(x_j,\hat{x}_j)))+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\hat{\lambda}_i\hat{\gamma}_i(V_i(x_i,\hat{x}_i))+\sum_{j=1,i\neq j}^{N}\hat{\delta}_{ij}\hat{\gamma}_j(V_j(x_j,\hat{x}_j))+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\big)\\
&=\mu^\top(-\hat{\Lambda}+\hat{\Delta})\hat{\Gamma}(V_1(x_1,\hat{x}_1);\ldots;V_N(x_N,\hat{x}_N))+\sum_{i=1}^{N}\mu_i\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i.
\end{aligned}
\tag{3.2.17}
$$

Remark 3.2.15. *A vector $\mu\in\mathbb{R}^N_{>0}$ satisfying $\mu^T(-\hat{\Lambda}+\hat{\Delta})<0$ exists if and only if the spectral radius of $\hat{\Lambda}^{-1}\hat{\Delta}$ is strictly less than one [DIW11]. In this case if $\hat{\Delta}$ is irreducible, μ can be chosen as a left eigenvector of $-\hat{\Lambda}+\hat{\Delta}$ corresponding to the largest eigenvalue, which is real and negative by the Perron-Frobenius theorem [Axe94].*

Remark 3.2.16. *If $\rho_{\mathrm{int}i}$ satisfies the triangle inequality $\rho_{\mathrm{int}i}(a+b)\leq\rho_{\mathrm{int}i}(a)+\rho_{\mathrm{int}i}(b)$ for all nonnegative values of a and b, the inequality 3.2.15 can be reduced to the following* *less conservative inequality:*

$$\rho_{\text{int}i}(s_1 + \cdots + s_{N-1}) \leq \sum_{i=1}^{N-1} \rho_{\text{int}i}(s_i),$$

and consequently, the condition 3.2.13 reduces to

$$h^2_{ji} \not\equiv 0 \implies \rho_{\text{int}i}(\alpha_j^{-1}(s)) \leq \hat{\delta}_{ij}\hat{\gamma}_j(s).$$

3.2.3 Construction of sum-type SPSF

3.2.3.1 Discrete-Time Linear Stochastic Control Systems

In this subsection, we focus on a class of discrete-time linear stochastic control systems defined as

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + B\nu(k) + Dw(k) + R\varsigma(k), \\ y^1(k) = C^1 x(k), \\ y^2(k) = C^2 x(k), \end{cases} \tag{3.2.18}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions (i.e., mean zero and covariance matrix identity). We use the tuple $\Sigma = (A, B, C^1, C^2, D, R)$ to refer to the class of systems in (3.2.18). Here, we provide conditions under which a candidate V is a sum-type SPSF facilitating the construction of an infinite abstraction $\widehat{\Sigma}$.

Assumption 3.2.17. *Assume that there exist a matrix K and a positive-definite matrix $\tilde{M}$ such that the matrix inequalities, $\forall i \in \{1, 2\}$,*

$$C^{iT}C^i \preceq \tilde{M}, \tag{3.2.19}$$

$$(1+\pi)(A+BK)^T\tilde{M}(A+BK) - \tilde{M} \preceq -\hat{\kappa}\tilde{M}, \tag{3.2.20}$$

hold for some positive constants π and $0 < \hat{\kappa} < 1$.

We employ the following quadratic function

$$V(x, \hat{x}) = (x - \tilde{P}\hat{x})^T\tilde{M}(x - \tilde{P}\hat{x}), \tag{3.2.21}$$

where $\tilde{P} \in \mathbb{R}^{n \times \hat{n}}$ is a matrix of an appropriate dimension. Assume that the equalities, $\forall i \in \{1, 2\}$,

$$A\tilde{P} = \tilde{P}\hat{A} - BQ \tag{3.2.22}$$

$$D = \tilde{P}\hat{D} - BS \tag{3.2.23}$$

$$C^i\tilde{P} = \hat{C}^i, \tag{3.2.24}$$

hold for some matrices Q and S of appropriate dimensions and potentially with the lowest possible $\hat{n}$. In the next theorem, we show that under the aforementioned conditions V in (3.2.21) is a sum-type SPSF from $\widehat{\Sigma}$ to Σ.

Theorem 3.2.18. *Let* $\Sigma = (A, B, C^1, C^2, D, R)$ *and* $\widehat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}^1, \hat{C}^2, \hat{D}, \hat{R})$ *be two discrete-time linear stochastic control subsystems with two* independent *additive noises. Suppose that there exist matrices* $\tilde{M}$*,* K*,* $\tilde{P}$*,* Q*, and* S *satisfying* (3.2.19)*,* (3.2.20)*,* (3.2.22)*,* (3.2.23)*, and* (3.2.24)*. Then* V *defined in* (3.2.21) *is a* sum*-type SPSF from* $\widehat{\Sigma}$ *to* Σ*.*

Proof. Here we show that $\forall x \in X$, $\forall \hat{x} \in \hat{X}$, $\forall \hat{\nu} \in \hat{U}$, $\forall \hat{w} \in \hat{W}$, $\exists \nu \in U$, $\forall w \in W$, such that V satisfies $\|C^i x - \hat{C}^i \hat{x}\|_2^2 \leq V(x, \hat{x}), i \in \{1, 2\}$, and

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\big|\, x(k) = x, \hat{x}(k) = \hat{x}, \hat{\nu}(k) = \hat{\nu}, w(k) = w, \hat{w}(k) = \hat{w}\Big] - V(x, \hat{x})$$
$$\leq -\hat{\kappa}(V(x, \hat{x})) + (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{\tilde{M}}D\|_2^2\|w - \hat{w}\|_2^2$$
$$+ (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|_2^2 + \mathrm{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big). \tag{3.2.25}$$

According to (3.2.24), we have $\|C^i x - \hat{C}^i \hat{x}\|_2^2 = (x - \tilde{P}\hat{x})^T C^{iT} C^i (x - \tilde{P}\hat{x})$. By applying (3.2.19), it can be easily verified that $\|C^i x - \hat{C}^i \hat{x}\|_2^2 \leq V(x, \hat{x})$ holds $\forall x \in X$, $\forall \hat{x} \in \hat{X}$. Now, we show the inequality (3.2.25). Given any x, $\hat{x}$, $\hat{\nu}$, and $\hat{w}$, we choose ν via the following *linear interface function*:

$$\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w}) := K(x - \tilde{P}\hat{x}) + Q\hat{x} + \tilde{R}\hat{\nu} + S\hat{w}, \tag{3.2.26}$$

for some matrix $\tilde{R}$ of an appropriate dimension. By employing equations (3.2.22), (3.2.23), and the definition of the interface function in (3.2.26), we simplify

$$Ax + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w}) + Dw - \tilde{P}(\hat{A}\hat{x} + \hat{B}\hat{\nu} + \hat{D}\hat{w}) + \big(R\varsigma - \tilde{P}\hat{R}\hat{\varsigma}\big)$$

to $(A + BK)(x - \tilde{P}\hat{x}) + D(w - \hat{w}) + (B\tilde{R} - \tilde{P}\hat{B})\hat{\nu} + \big(R\varsigma - \tilde{P}\hat{R}\hat{\varsigma}\big)$. One obtains

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\big|\, x(k) = x, \hat{x}(k) = \hat{x}, \hat{\nu}(k) = \hat{\nu}, w(k) = w, \hat{w}(k) = \hat{w}\Big] - V(x, \hat{x})$$
$$= (x - \tilde{P}\hat{x})^T\Big[(A + BK)^T\tilde{M}(A + BK) - \tilde{M}\Big](x - \tilde{P}\hat{x}) + \|\sqrt{\tilde{M}}D(w - \hat{w})\|_2^2$$
$$+ \Big[2(x - \tilde{P}\hat{x})^T(A + BK)^T\Big]\tilde{M}\Big[D(w - \hat{w})\Big] + \Big[2(w - \hat{w})^T D^T\Big]\tilde{M}\Big[(B\tilde{R} - \tilde{P}\hat{B})\hat{\nu}\Big]$$
$$+ \Big[2(x - \tilde{P}\hat{x})^T(A + BK)^T\Big]\tilde{M}\Big[(B\tilde{R} - \tilde{P}\hat{B})\hat{\nu}\Big] + \|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\hat{\nu}\|_2^2$$
$$+ \mathrm{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big).$$

Using Young's inequality [You12] as $ab \leq \frac{\pi}{2}a^2 + \frac{1}{2\pi}b^2$, for any $a, b \geq 0$ and any $\pi > 0$, and by employing Cauchy-Schwarz inequality and (3.2.20), one obtains the following upper bound:

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\big|\, x(k) = x, \hat{x}(k) = \hat{x}, \hat{\nu}(k) = \hat{\nu}, w(k) = w, \hat{w}(k) = \hat{w}\Big] - V(x, \hat{x})$$
$$\leq -\hat{\kappa}(V(x, \hat{x})) + (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{\tilde{M}}D\|_2^2\|w - \hat{w}\|_2^2$$
$$+ (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|_2^2 + \mathrm{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big).$$

Hence the proposed V in (3.2.21) is a sum-type SPSF from $\widehat{\Sigma}$ to Σ, which completes the proof. Note that the $\mathcal{K}_\infty$ functions κ, α, ρ_{int}, and ρ_{ext}, in Definition 3.2.2 associated with the sum-type SPSF in (3.2.21) are $\alpha(s) := s^2$, $\kappa(s) := \hat{\kappa}s$, and $\rho_{\text{int}}(s) := (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{\tilde{M}}D\|_2^2 s^2$, $\rho_{\text{ext}}(s) := (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2 s^2$, $\forall s \in \mathbb{R}_{\geq 0}$. Moreover, the positive constant ψ in (3.2.2) is $\psi = \text{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)$. □

Remark 3.2.19. *One can readily verify from the result of Theorem 3.2.18 that choosing $\hat{R}$ equal to zero results in a smaller constant ψ and, hence, a more closeness of subsystems and their abstractions. Observe that this is not the case when one assumes the noises of the concrete subsystem and its abstraction are the same as in [Zam14, ZRME17].*

Remark 3.2.20. *Note that the results in Theorem 3.2.18 do not impose any condition on matrix $\hat{B}$ and, therefore, it can be chosen arbitrarily. As an example, one can choose $\hat{B} = \mathbb{I}_{\hat{n}}$ which makes the abstract system $\widehat{\Sigma}$ fully actuated and consequently the synthesis problem over it much easier.*

Remark 3.2.21. *Since Theorem 3.2.18 does not impose any condition on matrix $\tilde{R}$, we choose $\tilde{R}$ to minimize function ρ_{ext} for V as suggested in [GP09]. The following choice for $\tilde{R}$*

$$\tilde{R} = (B^T\tilde{M}B)^{-1}B^T\tilde{M}\tilde{P}\hat{B}. \tag{3.2.27}$$

minimizes ρ_{ext}.

3.2.4 Case Study

Here, we demonstrate the effectiveness of the proposed results for an interconnected system consisting of four discrete-time linear stochastic control subsystems, i.e., $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$. The interconnection scheme of Σ with four external inputs and two outputs is illustrated in Figure 3.2. As seen, the internal output of Σ_1 (resp. Σ_2) is connected to the internal input of Σ_4 (resp. Σ_3) and the internal output of Σ_3 (resp. Σ_4) is connected to the internal input of Σ_1 (resp. Σ_2).

The system matrices are given by

$$A_i = \mathbb{I}_{25},\ B_i = \mathbb{I}_{25},\ C_i^{1T} = 0.1\mathbb{1}_{25},\ R_i = 0.01\mathbb{1}_{25},$$

for $i \in \{1, 2, 3, 4\}$. The internal input and output matrices are also given by:

$$\begin{aligned} &C_{14}^{2T} = C_{23}^{2T} = C_{31}^{2T} = C_{42}^{2T} = 0.1\mathbb{1}25, \\ &D_{13} = D_{24} = D_{32} = D_{41} = 0.1\mathbb{1}_{25}. \end{aligned}$$

In order to construct an infinite abstraction for $\mathcal{I}_{cs}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, we construct an infinite abstraction $\widehat{\Sigma}_i$ of each individual subsystem Σ_i, $i \in \{1, 2, 3, 4\}$. We first fix $\hat{\kappa}$

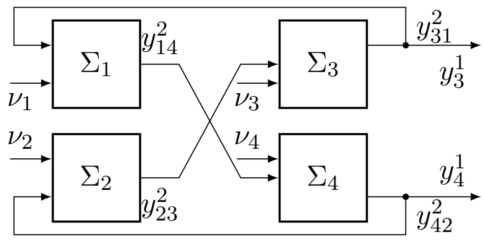

Figure 3.2: The interconnected system $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$.

and π for each subsystem, and then determine the matrices $\tilde{M}$ and K such that (3.2.19) and (3.2.20) hold for $i \in \{1,2,3,4\}$:

$$\tilde{M}_i = \mathbb{I}_{25},\ K_i = -0.95\mathbb{I}_{25},\ \hat{\kappa}_i = 0.98,\ \pi_i = 0.99.$$

We continue with determining other matrices such that (3.2.22), (3.2.23), and (3.2.24) hold:

$$\tilde{P}_i = \mathbb{1}_{25},\ Q_i = \mathbb{1}_{25},\ S_i = -0.003\mathbb{1}_{25},$$

for $i \in \{1,2,3,4\}$. Accordingly, the matrices of abstract subsystems are computed as:

$$\hat{A}_i = 2,\ \hat{C}_i = 2.5,\ \hat{D}_i = 0.096,$$

for $i \in \{1,2,3,4\}$. Note that here $\hat{R}_i$, $i \in \{1,2,3,4\}$, are considered zero in order to reduce the constant ψ_i for each V_i as discussed in Remark 3.2.19. Moreover, $\hat{B}_i$ is chosen 1 and we compute $\tilde{R}_i$, $i \in \{1,2,3,4\}$, using (3.2.27) as $\tilde{R}_i = \mathbb{1}_{25}$. The interface function for $i \in \{1,2,3,4\}$ follows by (3.2.26) as:

$$\nu_i = -0.95\mathbb{I}_{25}(x_i - \mathbb{1}_{25}\hat{x}_i) + \mathbb{1}_{25}\hat{x}_i + \mathbb{1}_{25}\hat{\nu}_i - 0.003\mathbb{1}_{25}\hat{w}_i.$$

Hence, Theorem 3.2.18 holds and $V_i(x_i, \hat{x}_i) = (x_i - \mathbb{1}_{25}\hat{x}_i)^T \tilde{M}_i (x_i - \mathbb{1}_{25}\hat{x}_i)$ is a sum-type SPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying conditions (3.2.1) and (3.2.2) with $\alpha_i(s) = s^2$, $\kappa_i(s) = 0.98s$, $\rho_{\text{ext}i}(s) = 0$, $\rho_{\text{int}i}(s) = 0.88s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 0.0025$, for $i \in \{1,2,3,4\}$. We now proceed with Theorem 3.2.14 to construct a sum-type SSF from $\widehat{\Sigma}$ to Σ. Assumption 3.2.13 holds with $\hat{\gamma}_i(s) = s$ and

$$\hat{\Delta} = \begin{bmatrix} 0 & 0 & 0.88 & 0 \\ 0 & 0 & 0 & 0.88 \\ 0 & 0.88 & 0 & 0 \\ 0.88 & 0 & 0 & 0 \end{bmatrix}, \qquad \hat{\Lambda} = \begin{bmatrix} 0.98 & 0 & 0 & 0 \\ 0 & 0.98 & 0 & 0 \\ 0 & 0 & 0.98 & 0 \\ 0 & 0 & 0 & 0.98 \end{bmatrix}.$$

Additionally, one can readily verify that a vector $\mu \in \mathbb{R}^4_{>0}$ exists here since the spectral radius of $\hat{\Lambda}^{-1}\hat{\Delta}$ is strictly less than one [DIW11]. By choosing vector μ as $\mu = \mathbb{1}_4$, the function

$$V(x, \hat{x}) = V_1(x_1, \hat{x}_1) + V_2(x_2, \hat{x}_2) + V_3(x_3, \hat{x}_3) + V_4(x_4, \hat{x}_4),$$

is a sum-type SSF from $\mathcal{I}_{cs}(\widehat{\Sigma}_1, \widehat{\Sigma}_2, \widehat{\Sigma}_3, \widehat{\Sigma}_4)$ to $\mathcal{I}_{cs}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ satisfying conditions (3.2.3) and (3.2.4) with $\alpha(s) = s^2$, $\kappa(s) = 0.1s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 0.01$. If the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are started from zero, one can readily verify that the norm of the error between outputs of Σ and of $\widehat{\Sigma}$ will not exceed 1 with a probability at least 90% computed by the sum-type SSF V using inequality (3.2.5) for $T_d = 10$, i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 1,\ \forall k \in [0, 10]) \geq 0.9.$$

3.3 max Small-Gain Approach

In this section, we propose a compositional methodology for the construction of infinite abstractions based on max small-gain conditions. We show that the new compositional framework is more general than the classic one proposed in the previous section since the provided max small-gain condition does not require a linear growth on the gains of the subsystems which is the case in the classic version. Moreover, we show that the provided approximation error via the max small-gain does not change as the number of subsystems grows since the proposed overall error (i.e., ψ) is completely independent of the size of the network (i.e., N), and is computed only based on the maximum error of subsystems (i.e., ψ_i) instead of being a linear combination of them which is the case in the classic small-gain approach.

3.3.1 max-Type Stochastic Pseudo-Simulation and Simulation Functions

Here, for dt-SCS with both internal and external inputs and outputs, we first introduce the notion of max-type stochastic pseudo-simulation functions (max-type SPSF). We then define the notion of max-type stochastic simulation functions (max-type SSF) for dt-SCS without internal signals. Although the former definition is employed to quantify the closeness of two dt-SCS, the latter is specifically utilized for the interconnected dt-SCS.

Definition 3.3.1. *Consider two dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, W, \varsigma, \hat{f}, Y^1, Y^2, \hat{h}^1, \hat{h}^2)$, with the same internal input, and internal and external output spaces. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a* max*-type stochastic pseudo-simulation function (*max*-type SPSF) from $\widehat{\Sigma}$ to Σ if there exist functions $\alpha, \kappa \in \mathcal{K}_\infty$, with $\kappa < \mathcal{I}_d$, $\rho_{\text{int}}, \rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and a constant $\psi \in \mathbb{R}_{\geq 0}$, such that*

- $$\forall x \in X, \forall \hat{x} \in \hat{X}, \forall i \in \{1, 2\}, \quad \alpha(\|h^i(x) - \hat{h}^i(\hat{x})\|) \leq V(x, \hat{x}), \tag{3.3.1}$$
- *and for all $x \in X$, $\hat{x} \in \hat{X}$, $\hat{\nu} \in \hat{U}$ there exists $\nu \in U$ such that $\forall \hat{w} \in \hat{W}$, $\forall w \in W$,*

$$\begin{aligned}\mathbb{E}\Big[V(f(x, \nu, w, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \nu, \hat{\nu}, w, \hat{w}\Big] \\ \leq \max\Big\{\kappa(V(x, \hat{x})), \rho_{\text{int}}(\|w - \hat{w}\|), \rho_{\text{ext}}(\|\hat{\nu}\|), \psi\Big\}.\end{aligned} \tag{3.3.2}$$

We denote $\widehat{\Sigma} \preceq^{\mathsf{max}}_{SPSF} \Sigma$ if there exists a max-type SPSF V from $\widehat{\Sigma}$ to Σ, and call the control system $\widehat{\Sigma}$ an abstraction of the concrete (original) system Σ.

Remark 3.3.2. *As a comparison, the notion of* max*-type SPSF here is equivalent to the* sum*-type defined in Definition 3.2.2 such that the existence of one implies that of the other one. However, the upper bound in* (3.3.2) *is in the* max *form, whereas the one in* (3.2.2) *is in the additive form.*

Definition 3.3.1 can also be stated for systems without internal signals as the following definition.

Definition 3.3.3. *Consider two dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, Y, \hat{h})$ without internal and external signals. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a* max*-type stochastic simulation function (*max*-type SSF) from $\widehat{\Sigma}$ to Σ if*

- *there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad \forall x \in X, \hat{x} \in \hat{X}, \tag{3.3.3}$$

- *and for all $x \in X$, $\hat{x} \in \hat{X}$, $\hat{\nu} \in \hat{U}$, there exists $\nu \in U$ such that*

$$\mathbb{E}\Big[V(f(x, \nu, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \varsigma)) \,\big|\, x, \hat{x}, \nu, \hat{\nu}\Big] \leq \max\Big\{\kappa(V(x, \hat{x})), \rho_{\mathrm{ext}}(\|\hat{\nu}\|), \psi\Big\}, \tag{3.3.4}$$

  for some $\kappa \in \mathcal{K}_\infty$ with $\kappa < \mathcal{I}_d$, $\rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.

We call $\widehat{\Sigma}$ an abstraction of Σ, and denote by $\widehat{\Sigma} \preceq^{\mathsf{max}}_{SSF} \Sigma$ if there exists a max-type SSF V from $\widehat{\Sigma}$ to Σ.

Now one can utilize Theorem 3.2.7 and show how the max-type SSF can be employed to compare output trajectories of two interconnected dt-SCS (without internal signals) in a probabilistic sense. Note that this theorem holds for the setting here since the max form of SSF here implies the additive form proposed in (3.2.2).

3.3.2 Compositionality Results

In this subsection, we analyze networks of stochastic control subsystems and discuss how to construct their infinite abstractions together with the max-type SSF based on corresponding max-type SPSF of their subsystems. Suppose we are given N concrete stochastic control subsystems (3.2.7) with their input-output configuration similar to (3.2.8) and (3.2.9), where their corresponding infinite abstractions are

$$\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, W_i, \varsigma_i, \hat{f}_i, Y_i^1, Y_i^2, \hat{h}_i^1, \hat{h}_i^2).$$

Moreover, we assume there exists a max-type SPSF V_i from $\widehat{\Sigma}_i$ to Σ_i with the corresponding functions and constants denoted by α_i, κ_i, $\rho_{\mathrm{int}i}$, $\rho_{\mathrm{ext}i}$, and ψ_i as in Definition 3.3.1. Now we raise the following max small-gain assumption that is essential for proposing the compositionality result in this section.

Assumption 3.3.4. *Assume that* $\mathcal{K}_\infty$ *functions* κ_{ij} *defined as*

$$\kappa_{ij}(s) := \begin{cases} \kappa_i(s), & \text{if } i = j, \\ \rho_{\text{int}i}(\alpha_j^{-1}(s)), & \text{if } i \neq j, \end{cases}$$

satisfy

$$\kappa_{i_1 i_2} \circ \kappa_{i_2 i_3} \circ \cdots \circ \kappa_{i_{r-1} i_r} \circ \kappa_{i_r i_1} < \mathcal{I}_d \tag{3.3.5}$$

for all sequences $(i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$ *and* $r \in \{1, \ldots, N\}$.

Remark 3.3.5. *Note that the* max *small-gain condition* (3.3.5) *is a standard one in studying the stability of large-scale interconnected systems via ISS Lyapunov functions [DRW07, DRW10]. This condition is automatically satisfied if each* κ_{ii} *is less than identity* ($\kappa_{ii} < \mathcal{I}_d, \forall i \in \{1, \ldots, N\}$). *Although this condition should be satisfied for all possible sequences* $(i_1, \ldots, i_r) \in \{1, \ldots, N\}^r, r \in \{1, \ldots, N\}$, *it allows some subsystems to compensate the undesirable effects of other subsystems in the interconnected network such that it is satisfied.*

The max *small-gain condition* (3.3.5) *implies the existence of* $\mathcal{K}_\infty$ *functions* $\sigma_i > 0$ *[Rüf10, Theorem 5.5] satisfying*

$$\max_{i,j} \left\{ \sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j \right\} < \mathcal{I}_d, \qquad i, j = \{1, \ldots, N\}. \tag{3.3.6}$$

In the next theorem, we show that if Assumption 3.3.4 holds and $\max_i \sigma_i^{-1}$ is concave (in order to employ Jensen's inequality), then we can compute the mismatch between the interconnection of stochastic control subsystems and that of their infinite abstractions in a compositional fashion.

Theorem 3.3.6. *Consider the interconnected dt-SCS* $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *stochastic control subsystems* Σ_i. *Suppose that each* Σ_i *admits an infinite abstraction* $\widehat{\Sigma}_i$ *together with a corresponding* max*-type SPSF* V_i. *If Assumption 3.3.4 holds and* $\max_i \sigma_i^{-1}$ *for* σ_i *as in* (3.3.6) *is concave, then a function* $V(x, \hat{x})$ *defined as*

$$V(x, \hat{x}) := \max_i \left\{ \sigma_i^{-1}(V_i(x_i, \hat{x}_i)) \right\}, \tag{3.3.7}$$

is a max*-type SSF from* $\widehat{\Sigma} = \mathcal{I}_{cs}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ *to* $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$.

Proof. We first show that for some $\mathcal{K}_\infty$ function α, SSF V in (3.3.7) satisfies the inequality (3.3.3). For any $x = [x_1; \ldots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, one gets

$$\begin{aligned} \|h(x) - \hat{h}(\hat{x})\| &= \max_i \left\{ \|h_i^1(x_i) - \hat{h}_i^1(\hat{x}_i)\| \right\} \leq \max_i \left\{ \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \right\} \\ &\leq \hat{\beta}\ (\max_i \left\{ \sigma_i^{-1}(V_i(x_i, \hat{x}_i)) \right\}) = \hat{\beta}(V(x, \hat{x})), \end{aligned}$$

where $\hat{\beta}(s) = \max_i \left\{ \alpha_i^{-1} \circ \sigma_i(s) \right\}$ for all $s \in \mathbb{R}_{\geq 0}$, which is a $\mathcal{K}_\infty$ function and (3.3.3) holds with $\alpha = \hat{\beta}^{-1}$.

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,\varsigma),\hat{f}(\hat{x},\hat{\nu},\varsigma))\,\big|\,x,\hat{x},\hat{\nu}\Big]\\
&\quad=\mathbb{E}\Big[\max_i\Big\{\sigma_i^{-1}(V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i)))\Big\}\,\big|\,x,\hat{x},\hat{\nu}\Big]\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i))\,\big|\,x,\hat{x},\hat{\nu}\Big])\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i))\,\big|\,x_i,\hat{x}_i,\hat{\nu}_i\Big])\Big\}\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\text{int}i}(\|w_i-\hat{w}_i\|),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\text{int}i}(\max_{j,j\neq i}\{\|w_{ij}-\hat{w}_{ij}\|\}),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\text{int}i}(\max_{j,j\neq i}\{\|y_{ji}^2-\hat{y}_{ji}^2\|\}),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\text{int}i}(\max_{j,j\neq i}\{\|h_j^2(x_j)-\hat{h}_j^2(\hat{x}_j)\|\}),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\text{int}i}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j(x_j,\hat{x}_j))\}),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}(V_j(x_j,\hat{x}_j)),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j(x_j,\hat{x}_j)),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad\leq\max_{i,j,l}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_l^{-1}(V_l(x_l,\hat{x}_l)),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j(V(x,\hat{x})),\rho_{\text{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max\Big\{\kappa(V(x,\hat{x})),\rho_{\text{ext}}(\|\hat{\nu}\|),\psi\Big\}.
\end{aligned}
\tag{3.3.9}
$$

We continue with showing (3.3.4), as well. Let $\kappa(s)=\max_{i,j}\{\sigma_i^{-1}\circ\kappa_{ij}\circ\sigma_j(s)\}$. It follows from (3.3.6) that $\kappa<\mathcal{I}_d$. Since $\max_i\sigma_i^{-1}$ is concave, one can readily get the chain of inequalities in (3.3.9) using Jensen's inequality, and by defining ρ_{ext}, and ψ as

$$
\begin{aligned}
\rho_{\text{ext}}(s) &:= \begin{cases}\max_i\{\sigma_i^{-1}\circ\rho_{\text{ext}i}(s_i)\},\\ \text{s.t.}\quad s_i\geq 0,\ \|[s_1;\ldots;s_N]\|=s,\end{cases}\\
\psi &:= \max_i \sigma_i^{-1}(\psi_i).
\end{aligned}
\tag{3.3.8}
$$

Note that κ and ρ_{ext} in (3.3.9) belong to $\mathcal{K}_\infty$ and $\mathcal{K}_\infty\cup\{0\}$, respectively, due to their definition provided above. Hence, V is a max-type SSF from $\widehat{\Sigma}$ to Σ which completes the proof. □

Remark 3.3.7. *As seen, the proposed overall error (i.e., ψ) in* (3.3.8) *is completely independent of the size of the network (i.e., N), and is computed only based on the* *maximum error of subsystems (i.e., ψ_i) instead of being a linear combination of them which is the case in* (3.2.16)*. Accordingly, the provided approximation error in* (3.2.5) *via the proposed* max *small-gain approach does not change as the number of subsystems grows.*

We emphasize that the proposed max small-gain condition (3.3.5) is more general than the *classic* one provided in Assumption 3.2.13 since it does not require linear growth on the gains of subsystems which is the case in Assumption 3.2.13. We provide the following example for a detailed comparison.

Example 3.3.8. *Consider the following system:*

$$\Sigma : \begin{cases} x_1(k+1) &= a_1 x_1(k) + b_1\sqrt{|x_2(k)|} + \varsigma_1(k), \\ x_2(k+1) &= a_2 x_2(k) + b_2 g(x_1(k)) \ + \varsigma_2(k), \end{cases}$$

where $0 < a_1 < 1$, $0 < a_2 < 1$, $b_1, b_2 \in \mathbb{R}$, and the function g satisfies the following quadratic Lipschitz assumption: there exists an $\mathscr{L} \in \mathbb{R}_{>0}$ such that: $|g(x) - g(x')| \leq \mathscr{L}|x - x'|^2$ for all $x, x' \in \mathbb{R}$. One can readily verify that functions $V_1(x_1, \hat{x}_1) = |x_1 - \hat{x}_1|$ and $V_2(x_2, \hat{x}_2) = |x_2 - \hat{x}_2|$ are sum*-type SPSF from subsystems x_1 and x_2 to themselves, respectively. Here, one cannot come up with gain functions that globally satisfy Assumption 3.2.13. In particular, this assumption requires existence of $\mathcal{K}_\infty$ functions being upper bounded by linear ones and lower bounded by quadratic ones which is impossible to satisfy globally. On the other hand, the proposed small-gain condition* (3.3.5) *is still applicable here showing that $V(x, \hat{x}) := \max\{\sigma_1^{-1} \circ V_1(x_1, \hat{x}_1), \sigma_2^{-1} \circ V_2(x_2, \hat{x}_2)\}$ is a* max*-type SSF from Σ to itself, for some appropriate $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ (with concave $\max_1 \sigma_1^{-1}$, $\max_2 \sigma_2^{-1}$) satisfying* (3.3.6) *which is guaranteed to exist if $|b_1|\sqrt{|b_2|\mathscr{L}} < 1$ and $|b_2|(b_1\mathscr{L})^2 < 1$. Therefore the* max *small-gain condition* (3.3.5) *is much more general than the* classic *one proposed in Assumption 3.2.13.*

Now in the next subsection, we extend our proposed construction scheme (in the previous section) from linear systems to a particular class of *nonlinear* stochastic systems whose nonlinearities satisfy a slope restriction. We impose conditions on the dt-SCS Σ enabling us to find a max-type SPSF from its infinite abstraction $\widehat{\Sigma}$ to Σ. The required conditions are presented via some matrix inequalities.

3.3.3 Construction of max-type SPSF

3.3.3.1 Stochastic Control Systems with Slope Restrictions on Nonlinearity

Here, we focus on a specific class of discrete-time nonlinear stochastic control systems Σ and a quadratic max-type SPSF V in the form of (3.2.21). The class of nonlinear systems is given by

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + E\varphi(Fx(k)) + B\nu(k) + Dw(k) + R\varsigma(k), \\ y^1(k) = C^1 x(k), \\ y^2(k) = C^2 x(k), \end{cases} \tag{3.3.10}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions, and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies

$$0 \leq \frac{\varphi(c) - \varphi(d)}{c - d} \leq b, \quad \forall c, d \in \mathbb{R}, c \neq d, \tag{3.3.11}$$

for some $b \in \mathbb{R}_{>0} \cup \{\infty\}$.

We use the tuple $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$ to refer to the class of nonlinear systems of the form (3.3.10).

Remark 3.3.9. *If E is a zero matrix or φ in* (3.3.10) *is linear including the zero function (i.e., $\varphi \equiv 0$), one can remove or push the term $E\varphi(Fx)$ to Ax, and consequently the nonlinear tuple reduces to the linear one $\Sigma = (A, B, C^1, C^2, D, R)$. Then, every time we mention the tuple $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$, it implicitly implies that φ is nonlinear and E is nonzero.*

Remark 3.3.10. *Although the lower bound in* (3.3.11) *is zero, one can also assume* (3.3.11) *with some nonlinear functions φ with a nonzero lower bound, e.g., $a \in \mathbb{R}$. In this case, one can make a change of coordinate and define a new function $\tilde{\varphi}(r) := \varphi(r) - ar$ which satisfies* (3.3.11) *with $\tilde{a} = 0$ and $\tilde{b} = b - a$, and rewrite* (3.3.10) *as*

$$\Sigma : \begin{cases} x(k+1) = \tilde{A}x(k) + E\tilde{\varphi}(Fx(k)) + B\nu(k) + Dw(k) + R\varsigma(k), \\ y^1(k) = C^1 x(k), \\ y^2(k) = C^2 x(k), \end{cases}$$

where $\tilde{A} = A + aEF$.

Remark 3.3.11. *We restrict ourselves here to systems with a single nonlinearity as in* (3.3.10) *for the sake of simple presentation. However, it would be straightforward to show similar results for systems with multiple nonlinearities as*

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{\bar{M}} E_i \varphi_i(F_i x(k)) + B\nu(k) + Dw(k) + R\varsigma(k), \\ y^1(k) = C^1 x(k), \\ y^2(k) = C^2 x(k), \end{cases}$$

where $\varphi_i : \mathbb{R} \to \mathbb{R}$ satisfies (3.3.11) *for some $b_i \in \mathbb{R}_{>0} \cup \{\infty\}$, for any $i \in \{1, \ldots, \bar{M}\}$.*

In order to show that V in (3.2.21) is a max-type SPSF from $\widehat{\Sigma}$ to Σ, we require the following key assumption on Σ.

Assumption 3.3.12. *Assume that for some constant $0 < \hat{\kappa} < 1$, there exist matrices $\tilde{M} \succ 0$, K, and L_1 of appropriate dimensions such that the matrix inequality* (3.3.12) *holds. Note that the left-hand side matrix in* (3.3.12) *is symmetric, as well.*

Remark 3.3.13. *Note that for any linear system $\Sigma = (A, B, C^1, C^2, D, R)$, stabilizability of the pair (A, B) is sufficient to satisfy Assumption 3.3.12 in where matrices E, F, and L_1 are identically zero.*

$$
\begin{bmatrix} (1+2/\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}(BL_1+E) \\ * & (1+2/\pi)(B\tilde{R}-\tilde{P}\hat{B})^T\tilde{M}(B\tilde{R}-\tilde{P}\hat{B}) \end{bmatrix}
\preceq \begin{bmatrix} \hat{\kappa}\tilde{M} & -F^T \\ -F & \frac{2}{\hat{b}} \end{bmatrix} \tag{3.3.12}
$$

Now, we provide one of the main results of this section showing conditions under which V in (3.2.21) is a max-type SPSF from $\widehat{\Sigma}$ to Σ.

Theorem 3.3.14. *Let Σ and $\widehat{\Sigma}$ be two stochastic control subsystems. Suppose Assumption 3.3.12 holds and there exist matrices $\tilde{P}$, Q, S, and L_2 of appropriate dimensions such that one has, $\forall i \in \{1,2\}$,*

$$
\begin{aligned}
A\tilde{P} &= \tilde{P}\hat{A} - BQ, && (3.3.13a)\\
E &= \tilde{P}\hat{E} - B(L_1 - L_2), && (3.3.13b)\\
D &= \tilde{P}\hat{D} - \hat{B}S, && (3.3.13c)\\
R &= \tilde{P}\hat{R}, && (3.3.13d)\\
\hat{F} &= F\tilde{P}, && (3.3.13e)\\
\hat{C}^i &= C^i\tilde{P}. && (3.3.13f)
\end{aligned}
$$

Then function V defined in (3.2.21) *is a max-type SPSF from $\widehat{\Sigma}$ to Σ.*

Proof. Here we first show that $\forall x, \forall \hat{x}, \forall \hat{\nu}, \exists \nu, \forall w$, and $\forall \hat{w}$, V satisfies $\frac{\lambda_{\min}(M)}{\lambda_{\max}(C^{iT}C^i)}\|C^ix - \hat{C}^i\hat{x}\|^2 \leq V(x,\hat{x})$ and then

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma)) \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\leq \max\Big\{(1-(1-\tilde{\pi})\hat{\kappa})(V(x,\hat{x})), (1+\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2,\\
&(1+1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(m(1+3\pi))\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\Big\}.
\end{aligned}
$$

According to (3.3.13f), we have $\|C^ix - \hat{C}^i\hat{x}\|^2 \leq n\lambda_{\max}(C^{iT}C^i)\|x - \tilde{P}\hat{x}\|^2$, and similarly $\lambda_{\min}(\tilde{M})\|x - \tilde{P}\hat{x}\|^2 \leq (x-\tilde{P}\hat{x})^T\tilde{M}(x-\tilde{P}\hat{x})$. Then one can readily verify that $\frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\|C^ix - \hat{C}^i\hat{x}\|^2 \leq V(x,\hat{x})$ holds $\forall x, \forall \hat{x}$, implying that the inequality (3.3.1) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\, s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (3.3.2) holds, as well. Given any x, $\hat{x}$, and $\hat{\nu}$, we choose ν via the following interface function:

$$
\nu = \nu_{\hat{\nu}}(x,\hat{x},\hat{\nu}) := K(x - \tilde{P}\hat{x}) + Q\hat{x} + \tilde{R}\hat{\nu} + S\hat{w} + L_1\varphi(Fx) - L_2\varphi(F\tilde{P}\hat{x}), \tag{3.3.14}
$$

for some matrix $\tilde{R}$ of an appropriate dimension. By employing the equations (3.3.13a)-(3.3.13e), and also the definition of the interface function in (3.3.14), we simplify

$$Ax + E\varphi(Fx) + B\nu(x, \hat{x}, \hat{\nu}) + Dw \\ - \tilde{P}(\hat{A}\hat{x} + \hat{E}\varphi(\hat{F}\hat{x}) + \hat{B}\hat{\nu} + \hat{D}\hat{w}) + (R\varsigma - \tilde{P}\hat{R}\varsigma)$$

to

$$(A + BK)(x - \tilde{P}\hat{x}) + D(w - \hat{w}) + (B\tilde{R} - \tilde{P}\hat{B})\hat{\nu} \\ + (BL_1 + E)(\varphi(Fx) - \varphi(F\tilde{P}\hat{x})). \tag{3.3.15}$$

From the slope restriction (3.3.11), one obtains

$$\varphi(Fx) - \varphi(F\tilde{P}\hat{x}) = \underline{\delta}(Fx - F\tilde{P}\hat{x}) = \underline{\delta}F(x - \tilde{P}\hat{x}), \tag{3.3.16}$$

where $\underline{\delta}$ is a function of x and $\hat{x}$ and takes values in the interval $[0, b]$. Using (3.3.16), the expression in (3.3.15) reduces to

$$((A + BK) + \underline{\delta}(BL_1 + E)F)(x - \tilde{P}\hat{x}) + D(w - \hat{w}) + (B\tilde{R} - \tilde{P}\hat{B})\hat{\nu}.$$

Using Young's inequality [You12] as $cd \leq \frac{\pi}{2}c^2 + \frac{1}{2\pi}d^2$, for any $c, d \geq 0$ and any $\pi > 0$, by employing Cauchy-Schwarz inequality, the matrix inequality (3.3.12), and by defining $\mathcal{Z} = \begin{bmatrix} x - \tilde{P}\hat{x} \\ \underline{\delta}F(x - \tilde{P}\hat{x}) \end{bmatrix}$, one can obtain the chain of inequalities in (3.3.17) in order to get an upper bound. Hence the proposed V in (3.2.21) is a max-type SPSF from $\widehat{\Sigma}$ to Σ, which completes the proof. Note that the last inequality in (3.3.17) is derived by applying Theorem 1 in [SGZ18]. The functions $\alpha, \kappa \in \mathcal{K}_\infty$, and $\rho_{\text{int}}, \rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ in Definition 3.3.1 associated with V in (3.2.21) are defined as $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\, s^2$, $\kappa(s) := (1 - (1 - \tilde{\pi})\tilde{\kappa})\, s$, $\rho_{\text{int}}(s) := (1 + \tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1 + 2\pi + 1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\, s^2$, $\rho_{\text{ext}}(s) := (1 + 1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(m(1 + 3\pi))\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2\, s^2$, $\forall s \in \mathbb{R}_{\geq 0}$ where $\tilde{\kappa} = 1 - \hat{\kappa}$, $0 < \tilde{\pi} < 1$, and $\tilde{\delta} > 0$. Moreover, the positive constant ψ in (3.3.2) is equal to zero. □

3.4 Dissipativity Approach

In this section, we provide a compositional approach for the construction of infinite abstractions from dt-SCS using an interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions. We show that the proposed compositionality conditions can enjoy the structure of the interconnection topology and be potentially satisfied regardless of the number or gains of subsystems. We also provide an abstract-construction framework for the same nonlinear class of stochastic systems in (3.3.10). Finally, we extend our specification from the finite-horizon invariant to a fragment of linear temporal logic known as syntactically co-safe linear temporal logic (scLTL) [KV01]. In particular, given such a co-safe LTL specification over the concrete

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad= (x-\tilde{P}\hat{x})^T\Big[((A+BK)+\underline{\delta}(BL_1+E)F)^T\tilde{M}((A+BK)+\underline{\delta}(BL_1+E)F)\Big](x-\tilde{P}\hat{x})\\
&\qquad+2\Big[(x-\tilde{P}\hat{x})^T((A+BK)+\underline{\delta}(BL_1+E)F)^T\Big]\tilde{M}\Big[D(w-\hat{w})\Big]+2\Big[(x-\tilde{P}\hat{x})^T((A+BK)\\
&\qquad+\underline{\delta}(BL_1+E)F)^T\Big]\tilde{M}\Big[(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}\Big]+2\Big[(w-\hat{w})^TD^T\Big]\tilde{M}\Big[(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}\Big]\\
&\qquad+\hat{\nu}^T(B\tilde{R}-\tilde{P}\hat{B})^T\tilde{M}(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}+(w-\hat{w})^TD^T\tilde{M}D(w-\hat{w})\\
&\quad\leq \mathcal{Z}^T\begin{bmatrix}(1+2/\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}(BL_1+E)\\ * & (1+2/\pi)(B\tilde{R}-\tilde{P}\hat{B})^T\tilde{M}(B\tilde{R}-\tilde{P}\hat{B})\end{bmatrix}\mathcal{Z}\\
&\qquad+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2+m(1+3\pi)\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\\
&\quad\leq \mathcal{Z}^T\begin{bmatrix}\hat{\kappa}\tilde{M} & -F^T\\ -F & \frac{2}{b}\end{bmatrix}\mathcal{Z}+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2\\
&\qquad+m(1+3\pi)\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\\
&\quad=\hat{\kappa}V(x,\hat{x})-2\underline{\delta}(1-\underline{\delta}/b)(x-\tilde{P}\hat{x})^TF^TF(x-\tilde{P}\hat{x})+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2\\
&\qquad+m(1+3\pi)\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\\
&\quad\leq\hat{\kappa}V(x,\hat{x})+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2+m(1+3\pi)\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\\
&\quad\leq\max\Big\{(1-(1-\tilde{\pi})\hat{\kappa})(V(x,\hat{x})),(1+\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2,\\
&\qquad(1+1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(m(1+3\pi))\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|^2\Big\}.
\end{aligned}
\tag{3.3.17}
$$

system, we construct an epsilon-perturbed specification over the abstract system whose probability of satisfaction gives a lower bound for the probability of satisfaction in the concrete domain. We demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 3 dimensions) of an interconnection of three discrete-time nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion such that the compositionality condition does not require any constraint on the number or gains of the subsystems. We employ the constructed abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe LTL specification. It should be also noted that we again do not put any restriction on the sources of uncertainties in the concrete and abstract systems. Then the noises in the concrete and abstract systems are independent from each other.

3.4.1 Stochastic Storage Functions

In this subsection, we introduce a notion of so-called stochastic storage functions (SStF) for the discrete-time stochastic control systems with both internal and external inputs and outputs which is adapted from the notion of storage functions from dissipativity theory [AMP16].

Definition 3.4.1. *Consider two dt-SCS* $\Sigma = (X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2)$ *and* $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{\varsigma}, \hat{f}, Y^1, \hat{Y}^2, \hat{h}^1, \hat{h}^2)$ *with the same external output spaces. A function* $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *is called a stochastic storage function (SStF) from* $\widehat{\Sigma}$ *to* Σ *if there exist* $\alpha, \kappa \in \mathcal{K}_\infty$*,* $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$*, some matrices* $G, \hat{G}, H$ *of appropriate dimensions, and some symmetric matrix* $\bar{X}$ *of an appropriate dimension with conformal block partitions* $\bar{X}^{ij}$*,* $i, j \in \{1, 2\}$*, such that for any* $x \in X$ *and* $\hat{x} \in \hat{X}$*, one has*

-
$$\alpha(\|h^1(x) - \hat{h}^1(\hat{x})\|_2) \leq V(x, \hat{x}), \tag{3.4.1}$$

- *and* $\forall x \in X$ $\forall \hat{x} \in \hat{X}$ $\forall \hat{\nu} \in \hat{U}$ $\exists \nu \in U$ *such that* $\forall \hat{w} \in \hat{W}$ $\forall w \in W$ *one obtains*
$$\begin{aligned}&\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\big|\, x(k)=x, \hat{x}(k)=\hat{x}, \nu(k)=\nu, \hat{\nu}(k)=\hat{\nu}, w(k)=w, \hat{w}(k)=\hat{w}\Big]\\ &\quad - V(x, \hat{x}) \leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{\nu}\|_2) + \psi\\ &\quad + \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h^2(x) - H\hat{h}^2(\hat{x})\end{bmatrix}^T \overbrace{\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}}^{\bar{X}:=} \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h^2(x) - H\hat{h}^2(\hat{x})\end{bmatrix},\end{aligned} \tag{3.4.2}$$

  for some $\psi \in \mathbb{R}_{\geq 0}$*.*

We use the notation $\widehat{\Sigma} \preceq_{SStF} \Sigma$ if there exists an SStF V from $\widehat{\Sigma}$ to Σ, in which $\widehat{\Sigma}$ is considered as an abstraction of the concrete system Σ.

Remark 3.4.2. *The last term in the inequality* (3.4.2) *is interpreted in dissipativity theory as the* supply rate *[AMP16]. Here we choose this function to be quadratic which results in tractable compositional conditions in the form of linear matrix (in)equalities (cf.* (3.4.3)*).*

For the dt-SCS without internal signals (including interconnected dt-SCS), the above notion reduces to the sum-type SSF as in Definition 3.2.4. Now one can utilize the results of Theorem 3.2.7 and show how the sum-type SSF can be employed to compare output trajectories of two interconnected dt-SCS (without internal signals) in a probabilistic sense.

3.4.2 Compositionality Results

In this subsection, we first provide a formal definition of an interconnection between discrete-time stochastic control subsystems.

Definition 3.4.3. *Consider* $N \in \mathbb{N}_{\geq 1}$ *stochastic control subsystems* $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$*,* $\forall i \in \{1, \dots, N\}$*, and a static matrix* M *of an appropriate dimension defining the coupling of these subsystems. The interconnection of* Σ_i *for any* $i \in \{1, \dots, N\}$*,* *is the interconnected stochastic control system* $\Sigma = (X, U, \varsigma, f, Y, h)$, *denoted by* $\mathcal{I}_{cd}(\Sigma_1, \ldots, \Sigma_N)$, *such that* $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, $f := \prod_{i=1}^N f_i$, $Y := \prod_{i=1}^N Y_i^1$, *and* $h = \prod_{i=1}^N h_i^1$, *with the internal variables constrained by:*

$$[w_1; \ldots; w_N] = M[h_1^2(x_1); \ldots; h_N^2(x_N)].$$

Assume that we are given N stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$ together with their corresponding abstractions $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \hat{\varsigma}_i, \hat{f}_i, Y_i^1, \hat{Y}_i^2, \hat{h}_i^1, \hat{h}_i^2)$ with the SStF V_i from $\widehat{\Sigma}_i$ to Σ_i. We use α_i, κ_i, $\rho_{\text{ext}i}$, H_i, G_i, $\hat{G}_i$, $\bar{X}_i$, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$ to denote the corresponding functions, matrices, and their conformal block partitions appearing in Definition 3.4.1. In the next theorem, as one of the main results of the section, we quantify the error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional way.

Theorem 3.4.4. *Consider the interconnected stochastic control system* $\Sigma = \mathcal{I}_{cd}(\Sigma_1, \ldots, \Sigma_N)$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *stochastic control subsystems* Σ_i *and the coupling matrix* M. *Suppose stochastic control subsystems* $\widehat{\Sigma}_i$ *are abstractions of* Σ_i *with the corresponding SStF* V_i. *If there exist* $\mu_i > 0$, $i \in \{1, \ldots, N\}$, *and the matrix* $\hat{M}$ *of an appropriate dimension such that the matrix (in)equalities*

$$\begin{bmatrix} GM \\ \mathbb{I}_{\tilde{q}} \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} GM \\ \mathbb{I}_{\tilde{q}} \end{bmatrix} \preceq 0, \tag{3.4.3}$$

$$GMH = \hat{G}\hat{M}, \tag{3.4.4}$$

are satisfied, where $\tilde{q} = \sum_{i=1}^N q_i^2$ *and* q_i^2 *are dimensions of internal outputs of subsystems* Σ_i, *and*

$$G := \mathsf{diag}(G_1, \ldots, G_N),\ \hat{G} := \mathsf{diag}(\hat{G}_1, \ldots, \hat{G}_N),\ H := \mathsf{diag}(H_1, \ldots, H_N), \tag{3.4.5}$$

$$\bar{X}_{cmp} := \begin{bmatrix} \mu_1 \bar{X}_1^{11} & & & \mu_1 \bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{11} & & & \mu_N \bar{X}_N^{12} \\ \mu_1 \bar{X}_1^{21} & & & \mu_1 \bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{21} & & & \mu_N \bar{X}_N^{22} \end{bmatrix}, \tag{3.4.6}$$

then

$$V(x, \hat{x}) := \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i), \tag{3.4.7}$$

is a sum*-type SSF from the interconnected control system* $\widehat{\Sigma} = \mathcal{I}_{cd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$, *with the coupling matrix* $\hat{M}$, *to* Σ.

Note that the matrix $\bar{X}_{cmp}$ in (3.4.6) has zero matrices in all its empty entries.

Proof. We first show that the inequality (3.2.3) holds for some $\mathcal{K}_\infty$ function α. For any $x = [x_1; \ldots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, one gets:

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\|_2 &= \|[h_1^1(x_1); \ldots; h_N^1(x_N)] - [\hat{h}_1^1(\hat{x}_1); \ldots; \hat{h}_N^1(\hat{x}_N)]\|_2 \\ &\le \sum_{i=1}^{N} \|h_i^1(x_i) - \hat{h}_i^1(\hat{x}_i)\|_2 \le \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \le \hat{\beta}(V(x, \hat{x})),\end{aligned}$$

with the function $\hat{\beta} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ defined for all $r \in \mathbb{R}_{\ge 0}$ as

$$\hat{\beta}(r) := \max\Big\{\textstyle\sum_{i=1}^{N} \alpha_i^{-1}(s_i) \,\big|\, s_i \ge 0,\ \sum_{i=1}^{N} \mu_i s_i = r\Big\}.$$

It is not hard to verify that the function $\hat{\beta}(\cdot)$ defined above is a $\mathcal{K}_\infty$ function. By taking the $\mathcal{K}_\infty$ function $\alpha(r) := \hat{\beta}^{-1}(r)$, $\forall r \in \mathbb{R}_{\ge 0}$, one obtains

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|_2) \le V(x, \hat{x}),$$

satisfying the inequality (3.2.3). Now we prove that the function V in (3.4.7) satisfies the inequality (3.2.4), as well. Consider any $x = [x_1; \ldots; x_N] \in X$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, and $\hat{\nu} = [\hat{\nu}_1; \ldots; \hat{\nu}_N] \in \hat{U}$. For any $i \in \{1, \ldots, N\}$, there exists $\nu_i \in U_i$, consequently, a vector $\nu = [\nu_1; \ldots; \nu_N] \in U$, satisfying (3.4.2) for each pair of subsystems Σ_i and $\widehat{\Sigma}_i$ with the internal inputs given by $[w_1; \ldots; w_N] = M[h_1^2(x_1); \ldots; h_N^2(x_N)]$ and $[\hat{w}_1; \ldots; \hat{w}_N] = \hat{M}[\hat{h}_1^2(\hat{x}_1); \ldots; \hat{h}_N^2(\hat{x}_N)]$. Then we have the chain of inequalities in (3.4.9) using conditions (3.4.3) and (3.4.4), and by defining $\kappa(\cdot), \rho_{\text{ext}}(\cdot)$, and ψ as

$$\begin{aligned}\kappa(r) &:= \min\Big\{\sum_{i=1}^{N} \mu_i \kappa_i(s_i) \,\big|\, s_i \ge 0,\ \sum_{i=1}^{N} \mu_i s_i = r\Big\}, \\ \rho_{\text{ext}}(r) &:= \max\Big\{\sum_{i=1}^{N} \mu_i \rho_{\text{ext}i}(s_i) \,\big|\, s_i \ge 0, \|[s_1; \ldots; s_N]\|_2 = r\Big\}, \\ \psi &:= \sum_{i=1}^{N} \mu_i \psi_i. \end{aligned} \tag{3.4.8}$$

Note that κ and ρ_{ext} in (3.4.9) belong to $\mathcal{K}_\infty$ and $\mathcal{K}_\infty \cup \{0\}$, respectively, because of their definitions provided above. Hence, we conclude that V is a sum-type SSF from $\widehat{\Sigma}$ to Σ. □

Remark 3.4.5. *Linear matrix inequality (LMI)* (3.4.3) *with* $G = \mathbb{I}$ *is similar to the LMI studied by [AMP16, Chapter 2] as a compositional stability condition based on the dissipativity theory. As discussed by [AMP16], the LMI holds independently of the number of subsystems in many physical applications with specific interconnection structures including communication networks, flexible joint robots, power generators, and so on. We refer the interested readers to [AMP16] for more details on the satisfaction of this type of LMI.*

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^{N}\mu_i V_i(x_i(k+1),\hat{x}_i(k+1))\,\Big|\,x(k)=x,\hat{x}(k)=\hat{x},\hat{\nu}(k)=\hat{\nu}\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\hat{x}_i)\\
&=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i(x_i(k+1),\hat{x}_i(k+1))\,\Big|\,x_i(k)=x_i,\hat{x}_i(k)=\hat{x}_i,\hat{\nu}_i(k)=\hat{\nu}_i\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\hat{x}_i)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i\\
&\quad+\begin{bmatrix}G_iw_i-\hat{G}_i\hat{w}_i\\ h_i^2(x_i)-H_i\hat{h}_i^2(\hat{x}_i)\end{bmatrix}^T\begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}\begin{bmatrix}G_iw_i-\hat{G}_i\hat{w}_i\\ h_i^2(x_i)-H_i\hat{h}_i^2(\hat{x}_i)\end{bmatrix}\big)\\
&=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix}G_1w_1-\hat{G}_1\hat{w}_1\\ \vdots\\ G_Nw_N-\hat{G}_N\hat{w}_N\\ h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}G_1w_1-\hat{G}_1\hat{w}_1\\ \vdots\\ G_Nw_N-\hat{G}_N\hat{w}_N\\ h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix}GM\begin{bmatrix}h_1^2(x_1)\\ \vdots\\ h_N^2(x_N)\end{bmatrix}-\hat{G}\hat{M}\begin{bmatrix}\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ \hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\ h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}GM\begin{bmatrix}h_1^2(x_1)\\ \vdots\\ h_N^2(x_N)\end{bmatrix}-\hat{G}\hat{M}\begin{bmatrix}\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ \hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\ h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix}h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\begin{bmatrix}GM\\ \mathbb{I}_{\tilde{q}}\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}GM\\ \mathbb{I}_{\tilde{q}}\end{bmatrix}\begin{bmatrix}h_1^2(x_1)-H_1\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-H_N\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&\leq\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\leq-\kappa\left(V\left(x,\hat{x}\right)\right)+\rho_{\mathrm{ext}}(\|\hat{\nu}\|_2)+\psi. \qquad (3.4.9)
\end{aligned}
$$

$$\begin{bmatrix} \bar{\mathcal{A}}^T \tilde{M} \bar{\mathcal{A}} & \bar{\mathcal{A}}^T \tilde{M} Z & \bar{\mathcal{A}}^T \tilde{M} \bar{\mathcal{B}} & \bar{\mathcal{A}}^T \tilde{M} \bar{\mathcal{R}} \\ * & Z^T \tilde{M} Z & Z^T \tilde{M} \bar{\mathcal{B}} & Z^T \tilde{M} \bar{\mathcal{R}} \\ * & * & \bar{\mathcal{B}}^T \tilde{M} \bar{\mathcal{B}} & \bar{\mathcal{B}}^T \tilde{M} \bar{\mathcal{R}} \\ * & * & * & \bar{\mathcal{R}}^T \tilde{M} \bar{\mathcal{R}} \end{bmatrix}$$
$$\preceq \begin{bmatrix} \hat{\kappa} \tilde{M} + C^{2T} \bar{X}^{22} C^2 & C^{2T} \bar{X}^{21} & -F^T & 0 \\ \bar{X}^{12} C^2 & \bar{X}^{11} & 0 & 0 \\ -F & 0 & \frac{2}{b} & 0 \\ 0 & 0 & 0 & \tilde{k} \bar{\mathcal{R}}^T \tilde{M} \bar{\mathcal{R}} \end{bmatrix} \tag{3.4.11}$$

Remark 3.4.6. *One can relax condition* (3.4.4) *and employ the linear least square approach instead of solving the equality exactly. In this case, an additional error resulting from the least square approach is added to ψ in* (3.4.8).

3.4.3 Construction of SStF

3.4.3.1 Stochastic Control Systems with Slope Restrictions on Nonlinearity

In this subsection, we focus on the nonlinear class of discrete-time stochastic control systems defined in (3.3.10) together with *quadratic* stochastic storage functions V in the form of (3.2.21), and provide an approach on the construction of their abstractions. In order to show that V in (3.2.21) is an SStF from $\widehat{\Sigma}$ to Σ, we require the following assumption on Σ.

Assumption 3.4.7. *Let $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$. Assume that for some constants $0 < \hat{\kappa} < 1$ and $\tilde{k} > 0$, there exist matrices $\tilde{M} \succ 0$, K, L_1, Z, G, $\bar{X}^{11}$, $\bar{X}^{12}$, $\bar{X}^{21}$, and $\bar{X}^{22}$ of appropriate dimensions such that the matrix equality*

$$D = ZG, \tag{3.4.10}$$

and the inequality (3.4.11) *hold with $\bar{\mathcal{A}} = (A + BK), \bar{\mathcal{B}} = (BL_1 + E), \bar{\mathcal{R}} = (B\tilde{R} - \tilde{P}\hat{B})$.*

Remark 3.4.8. *Note that for any linear system $\Sigma = (A, B, C^1, C^2, D, R)$, stabilizability of the pair (A, B) is sufficient to satisfy Assumption 3.4.7 in where matrices E, F, and L_1 are identically zero [AM07, Chapter 4].*

Now, we provide one of the main results of this section showing under which conditions V in (3.2.21) is an SStF from $\widehat{\Sigma}$ to Σ.

Theorem 3.4.9. *Let $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$ and $\widehat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}^1, \hat{C}^2, \hat{D}, \hat{E}, \hat{F}, \hat{R}, , \varphi)$ be two stochastic control subsystems with the same external output space. Suppose Assumption 3.4.7 holds and there exist matrices $\tilde{P}$, Q, H, L_2, and $\hat{G}$ of appropriate* *dimensions such that*

$$A\tilde{P} = \tilde{P}\hat{A} - BQ, \tag{3.4.12a}$$
$$C^1\tilde{P} = \hat{C}^1, \tag{3.4.12b}$$
$$\bar{X}^{12}C^2\tilde{P} = \bar{X}^{12}H\hat{C}^2, \tag{3.4.12c}$$
$$\bar{X}^{22}C^2\tilde{P} = \bar{X}^{22}H\hat{C}^2, \tag{3.4.12d}$$
$$\hat{F} = F\tilde{P}, \tag{3.4.12e}$$
$$E = \tilde{P}\hat{E} - B(L_1 - L_2), \tag{3.4.12f}$$
$$\tilde{P}\hat{D} = Z\hat{G}, \tag{3.4.12g}$$

hold. Then, function V *defined in* (3.2.21) *is an SStF from* $\widehat{\Sigma}$ *to* Σ.

Proof. We first show that $\forall x$, $\forall \hat{x}$, $\forall \hat{\nu}$, $\exists \nu$, $\forall \hat{w}$, and $\forall w$, V satisfies $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}\|C^1x - \hat{C}^1\hat{x}\|_2^2 \leq V(x,\hat{x})$ and then

$$\begin{aligned}\mathbb{E}\Big[&V(x(k+1),\hat{x}(k+1)\,\big|\,x(k)=x,\hat{x}(k)=\hat{x},\hat{\nu}(k)=\hat{\nu},w(k)=w,\hat{w}(k)=\hat{w}\Big] - V(x,\hat{x})\\ &\leq -(1-\hat{\kappa})(V(x,\hat{x})) + \tilde{k}\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|_2^2 + \mathrm{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)\\ &\quad + \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h^2(x) - H\hat{h}^2(\hat{x})\end{bmatrix}^T \begin{bmatrix}\bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix} \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h^2(x) - H\hat{h}^2(\hat{x})\end{bmatrix}.\end{aligned}$$

According to (3.4.12b), we have $\|C^1x - \hat{C}^1\hat{x}\|_2^2 = (x - \tilde{P}\hat{x})^TC^{1T}C^1(x - \tilde{P}\hat{x})$. Since $\lambda_{\min}(C^{1T}C^1)\|x - \tilde{P}\hat{x}\|_2^2 \leq (x - \tilde{P}\hat{x})^TC^{1T}C^1(x - \tilde{P}\hat{x}) \leq \lambda_{\max}(C^{1T}C^1)\|x - \tilde{P}\hat{x}\|_2^2$, and similarly, $\lambda_{\min}(\tilde{M})\|x - \tilde{P}\hat{x}\|_2^2 \leq (x - \tilde{P}\hat{x})^T\tilde{M}(x - \tilde{P}\hat{x}) \leq \lambda_{\max}(\tilde{M})\|x - \tilde{P}\hat{x}\|_2^2$, it can be readily verified that $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}\|C^1x - \hat{C}^1\hat{x}\|_2^2 \leq V(x,\hat{x})$ holds $\forall x$, $\forall \hat{x}$, implying that the inequality (3.4.1) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (3.4.2) holds, as well. Given any x, $\hat{x}$, and $\hat{\nu}$, we choose ν via the following *interface* function:

$$\nu = \nu_{\hat{\nu}}(x,\hat{x},\hat{\nu}) := K(x - \tilde{P}\hat{x}) + Q\hat{x} + \tilde{R}\hat{\nu} + L_1\varphi(Fx) - L_2\varphi(F\tilde{P}\hat{x}), \tag{3.4.13}$$

for some matrix $\tilde{R}$ of an appropriate dimension. By employing the equations (3.4.10), (3.4.12a), (3.4.12e), (3.4.12f) and also the definition of the interface function in (3.4.13), we simplify

$$\begin{aligned}&Ax + E\varphi(Fx) + B\nu_{\hat{\nu}}(x,\hat{x},\hat{\nu}) + Dw\\ &\qquad - \tilde{P}(\hat{A}\hat{x} + \hat{E}\varphi(\hat{F}\hat{x}) + \hat{B}\hat{\nu} + \hat{D}\hat{w}) + (R\varsigma - \tilde{P}\hat{R}\hat{\varsigma})\end{aligned}$$

to

$$\begin{aligned}&(A + BK)(x - \tilde{P}\hat{x}) + Z(Gw - \hat{G}\hat{w}) + (B\tilde{R} - \tilde{P}\hat{B})\hat{\nu}\\ &\qquad + (BL_1 + E)(\varphi(Fx) - \varphi(F\tilde{P}\hat{x})) + (R\varsigma - \tilde{P}\hat{R}\hat{\varsigma}).\end{aligned} \tag{3.4.14}$$

From the slope restriction (3.3.11), one obtains

$$\varphi(Fx) - \varphi(F\tilde{P}\hat{x}) = \underline{\delta}(Fx - F\tilde{P}\hat{x}) = \underline{\delta}F(x - \tilde{P}\hat{x}), \tag{3.4.15}$$

where $\underline{\delta}$ is a function of x and $\hat{x}$ and takes values in the interval $[0, b]$. Using (3.4.15), the expression in (3.4.14) reduces to

$$((A + BK) + \underline{\delta}(BL_1 + E)F)(x - \tilde{P}\hat{x}) + Z(Gw - \hat{G}\hat{w}) \\ + (B\tilde{R} - \tilde{P}\hat{B})\hat{\nu} + (R\varsigma - \tilde{P}\hat{R}\hat{\varsigma}).$$

Using Cauchy- Schwarz inequality, (3.4.11), (3.4.12c), and (3.4.12d), one can obtain the chain of inequalities in (3.4.16) in order to reach an upper bound. Hence, the proposed V in (3.2.21) is an SStF from $\widehat{\Sigma}$ to Σ, which completes the proof. □

Note that conditions (3.4.12) hold as long as the geometric conditions V-18 to V-23 in [ZA18] hold. The functions $\alpha \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and the matrix $\bar{X}$ in Definition 3.4.1 associated with the SStF in (3.2.21) are $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}s^2$, $\kappa(s) := (1 - \hat{\kappa})s$, $\rho_{\text{ext}}(s) := \tilde{\kappa}\|\sqrt{\tilde{M}}(B\tilde{R} - \tilde{P}\hat{B})\|_2^2 s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, where $\tilde{R}$ is a matrix of an appropriate dimension employed in the interface map (3.4.13), and $\bar{X} = \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix}$. Moreover, the positive constant ψ in (3.4.2) is $\psi = \text{Tr}\big(R^T\tilde{M}R + \hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)$.

The relation (3.2.5) lower bounds the probability such that the Euclidean distance between any output trajectory of the abstract model and the corresponding one of the concrete model remains close and is different from the probabilistic version discussed for finite state, discrete-time labeled Markov chains by [DLT08a], which hinges on the absolute difference between transition probabilities over sets covering the state space. However, one can still employ the results in Theorem 3.2.7 and design controllers for abstractions and refine them to concrete systems while providing the probability of satisfaction over the concrete domain. In particular, we extend our specification from the finite-horizon invariant to a fragment of linear temporal logic known as syntactically co-safe linear temporal logic (scLTL) [KV01]. We discuss given such a co-safe LTL specification over the concrete system, how one can construct an epsilon-perturbed specification over the abstract system whose probability of satisfaction gives a lower bound for the probability of satisfaction in the concrete domain.

3.4.4 Probability of Satisfaction for Properties Expressed as scLTL

Consider a dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ and a measurable target set $\bar{\mathcal{T}} \subset Y$. We say that an output trajectory $\{y(k)\}_{k\geq 0}$ reaches a target set $\bar{\mathcal{T}}$ within the time interval $[0, T_d] \subset \mathbb{N}$, if there exists a $k \in [0, T_d]$ such that $y(k) \in \bar{\mathcal{T}}$. This bounded reaching of $\bar{\mathcal{T}}$ is denoted by $\lozenge^{\leq T_d}\{y \in \bar{\mathcal{T}}\}$ or briefly $\lozenge^{\leq T_d}\bar{\mathcal{T}}$. For $T_d \to \infty$, we denote the reachability property as $\lozenge\bar{\mathcal{T}}$, i.e., eventually $\bar{\mathcal{T}}$. For a dt-SCS Σ with policy $\bar{\rho}$, we want to compute the probability that an output trajectory reaches $\bar{\mathcal{T}}$ within the time horizon $T_d \in \mathbb{N}$, i.e., $\mathbb{P}(\lozenge^{\leq T_d}\bar{\mathcal{T}})$. The *reachability probability* is the probability that the target set $\bar{\mathcal{T}}$ is eventually reached and is denoted by $\mathbb{P}(\lozenge\bar{\mathcal{T}})$.

$$\begin{aligned}
&\mathbb{E}\Big[V(x(k+1),\hat{x}(k+1)\,|\,x(k)=x,\hat{x}(k)=\hat{x},\hat{\nu}(k)=\hat{\nu},w(k)=w,\hat{w}(k)=\hat{w}\Big]-V(x,\hat{x})\\
&=(x-\tilde{P}\hat{x})^T\Big[((A+BK)+\underline{\delta}(BL_1+E)F)^T\tilde{M}((A+BK)+\underline{\delta}(BL_1+E)F)\Big](x-\tilde{P}\hat{x})\\
&\quad+2\Big[(x-\tilde{P}\hat{x})^T((A+BK)+\underline{\delta}(BL_1+E)F)^T\Big]\tilde{M}\Big[Z(Gw-\hat{G}\hat{w})\Big]\\
&\quad+2\Big[(x-\tilde{P}\hat{x})^T((A+BK)+\underline{\delta}(BL_1+E)F)^T\Big]\tilde{M}\Big[(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}\Big]\\
&\quad+2\Big[(Gw-\hat{G}\hat{w})^TZ^T\Big]\tilde{M}\Big[(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}\Big]+\hat{\nu}^T(B\tilde{R}-\tilde{P}\hat{B})^T\tilde{M}(B\tilde{R}-\tilde{P}\hat{B})\hat{\nu}\\
&\quad+(Gw-\hat{G}\hat{w})^TZ^T\tilde{M}Z(Gw-\hat{G}\hat{w})+\mathrm{Tr}\big(R^T\tilde{M}R+\hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)-V(x,\hat{x})\\
&=\begin{bmatrix}x-\tilde{P}\hat{x}\\Gw-\hat{G}\hat{w}\\\underline{\delta}F(x-\tilde{P}\hat{x})\\\hat{\nu}\end{bmatrix}^T\begin{bmatrix}\bar{\mathcal{A}}^T\tilde{M}\bar{\mathcal{A}} & \bar{\mathcal{A}}^T\tilde{M}Z & \bar{\mathcal{A}}^T\tilde{M}\bar{\mathcal{B}} & \bar{\mathcal{A}}^T\tilde{M}\bar{\mathcal{R}}\\ * & Z^T\tilde{M}Z & Z^T\tilde{M}\bar{\mathcal{B}} & Z^T\tilde{M}\bar{\mathcal{R}}\\ * & * & \bar{\mathcal{B}}^T\tilde{M}\bar{\mathcal{B}} & \bar{\mathcal{B}}^T\tilde{M}\bar{\mathcal{R}}\\ * & * & * & \bar{\mathcal{R}}^T\tilde{M}\bar{\mathcal{R}}\end{bmatrix}\begin{bmatrix}x-\tilde{P}\hat{x}\\Gw-\hat{G}\hat{w}\\\underline{\delta}F(x-\tilde{P}\hat{x})\\\hat{\nu}\end{bmatrix}\\
&\quad+\mathrm{Tr}\big(R^T\tilde{M}R+\hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)-V(x,\hat{x})\\
&\le\begin{bmatrix}x-\tilde{P}\hat{x}\\Gw-\hat{G}\hat{w}\\\underline{\delta}F(x-\tilde{P}\hat{x})\\\hat{\nu}\end{bmatrix}^T\begin{bmatrix}\hat{\kappa}\tilde{M}+C^{2T}\bar{X}^{22}C^2 & C^{2T}\bar{X}^{21} & -F^T & 0\\ \bar{X}^{12}C^2 & \bar{X}^{11} & 0 & 0\\ -F & 0 & \frac{2}{b} & 0\\ 0 & 0 & 0 & \tilde{k}\bar{\mathcal{R}}^T\tilde{M}\bar{\mathcal{R}}\end{bmatrix}\begin{bmatrix}x-\tilde{P}\hat{x}\\Gw-\hat{G}\hat{w}\\\underline{\delta}F(x-\tilde{P}\hat{x})\\\hat{\nu}\end{bmatrix}\\
&\quad+\mathrm{Tr}\big(R^T\tilde{M}R+\hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)-V(x,\hat{x})\\
&=-(1-\hat{\kappa})(V(x,\hat{x}))-2\underline{\delta}(1-\frac{\delta}{b})(x-\tilde{P}\hat{x})^TF^TF(x-\tilde{P}\hat{x})+\tilde{k}\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\nu\|_2^2\\
&\quad+\begin{bmatrix}Gw-\hat{G}\hat{w}\\C^2x-H\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\\bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix}Gw-\hat{G}\hat{w}\\C^2x-H\hat{C}^2\hat{x}\end{bmatrix}+\mathrm{Tr}\big(R^T\tilde{M}R+\hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big)\\
&\le-(1-\hat{\kappa})(V(x,\hat{x}))+\tilde{k}\|\sqrt{\tilde{M}}(B\tilde{R}-\tilde{P}\hat{B})\|_2^2\|\hat{\nu}\|_2^2\\
&\quad+\begin{bmatrix}Gw-\hat{G}\hat{w}\\C^2x-H\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\\bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix}Gw-\hat{G}\hat{w}\\C^2x-H\hat{C}^2\hat{x}\end{bmatrix}+\mathrm{Tr}\big(R^T\tilde{M}R+\hat{R}^T\tilde{P}^T\tilde{M}\tilde{P}\hat{R}\big).
\end{aligned}\tag{3.4.16}$$

More complex properties can be described using the temporal logic. Consider a set of atomic propositions AP and the alphabet $\Sigma_{\mathsf{a}} := 2^{AP}$. Let $\omega = \omega(0), \omega(1), \omega(2), \ldots \in \Sigma_{\mathsf{a}}^{\mathbb{N}}$ be an infinite word, that is, a string composed of letters from Σ_{a}. Of interest are atomic propositions that are relevant to the dt-SCS via a measurable labeling function L from the output space to the alphabet as $\mathsf{L} : Y \to \Sigma_{\mathsf{a}}$. Output trajectories $\{y(k)\}_{k\geq 0} \in Y^{\mathbb{N}}$ can be readily mapped to the set of infinite words $\Sigma_{\mathsf{a}}^{\mathbb{N}}$, as

$$\omega = \mathsf{L}(\{y(k)\}_{k\geq 0}) := \{\omega \in \Sigma_{\mathsf{a}}^{\mathbb{N}} \,|\, \omega(k) = \mathsf{L}(y(k))\}.$$

Consider LTL properties with syntax [BK08]

$$\phi ::= \text{true} \,|\, p \,|\, \neg\phi \,|\, \phi_1 \wedge \phi_2 \,|\, \bigcirc\phi \,|\, \phi_1 \,\mathsf{U}\, \phi_2.$$

Let $\omega_k = \omega(k), \omega(k+1), \omega(k+2), \ldots$ be a subsequence (postfix) of ω, then the satisfaction relation between ω and a property ϕ, expressed in LTL, is denoted by $\omega \vDash \phi$ (or equivalently $\omega_0 \vDash \phi$). The semantics of the satisfaction relation are defined recursively over ω_k and the syntax of the LTL formula ϕ. An atomic proposition $p \in AP$ is satisfied by ω_k, i.e., $\omega_k \vDash p$, iff $p \in \omega(k)$. Furthermore, $\omega_k \vDash \neg\phi$ if $\omega_k \nvDash \phi$ and we say that $\omega_k \vDash \phi_1 \wedge \phi_2$ if $\omega_k \vDash \phi_1$ and $\omega_k \vDash \phi_2$. The next operator $\omega_k \vDash \bigcirc\phi$ holds if the property holds at the next time instance $\omega_{k+1} \vDash \phi$. We denote by $\bigcirc^j$, $j \in \mathbb{N}$, j times composition of the next operator. With a slight abuse of the notation, one has $\bigcirc^0\phi = \phi$ for any property ϕ. The temporal until operator $\omega_k \vDash \phi_1 \,\mathsf{U}\, \phi_2$ holds if $\exists i \in \mathbb{N} : \omega_{k+i} \vDash \phi_2$, and $\forall j \in \mathbb{N} : 0 \leq j < i, \omega_{k+j} \vDash \phi_1$. Based on these semantics, operators such as disjunction ($\vee$) can also be defined through the negation and conjunction: $\omega_k \vDash \phi_1 \vee \phi_2 \;\Leftrightarrow\; \omega_k \vDash \neg(\neg\phi_1 \wedge \neg\phi_2)$.

Remark 3.4.10. *Note that in this subsection, the satisfaction relation $\vDash$ changes by varying the labeling functions L. In the following, we employ subscript for $\models$ to show its dependency on the labeling functions.*

We are interested in a fragment of LTL properties known as syntactically co-safe linear temporal logic (scLTL) [KV01]. This fragment is defined in the following definition.

Definition 3.4.11. *An scLTL over a set of atomic propositions AP has syntax*

$$\phi ::= \text{true} \,|\, p \,|\, \neg p \,|\, \phi_1 \wedge \phi_2 \,|\, \phi_1 \vee \phi_2 \,|\, \bigcirc\phi \,|\, \phi_1 \,\mathsf{U}\, \phi_2 \,|\, \Diamond\phi,$$

with $p \in AP$.

Even though scLTL formulas are defined over infinite words (as in LTL formulae), their satisfaction is guaranteed in the finite time [KV01]. Any infinite word $\omega \in \Sigma_{\mathsf{a}}^{\mathbb{N}}$ satisfying an scLTL formula ϕ has a finite word $\omega_f \in \Sigma_{\mathsf{a}}^n$, $n \in \mathbb{N}$, as its prefix such that all infinite words with prefix ω_f also satisfy the formula ϕ. We denote the language of such finite prefixes associated with an scLTL formula ϕ by $\mathcal{L}_f(\phi)$.

In the remainder, we consider scLTL properties since their verification can be performed via a reachability property over a finite state automaton [KV01, BYG17]. For this purpose, we introduce a class of models known as deterministic finite-state automata (DFA).

Definition 3.4.12. *A DFA is a tuple $\mathcal{A}_\phi = (Q_\ell, q_0, \Sigma_{\mathsf{a}}, F_{\mathsf{a}}, \mathsf{t})$, where Q_ℓ is a finite set of locations, $q_0 \in Q_\ell$ is the initial location, Σ_{a} is a finite set (a.k.a. alphabet), $F_{\mathsf{a}} \subseteq Q_\ell$ is a set of accept locations, and $\mathsf{t} : Q_\ell \times \Sigma_{\mathsf{a}} \to Q_\ell$ is a transition function.*

A finite word composed of letters of the alphabet, i.e., $\omega_f = (\omega_f(0), \ldots, \omega_f(n)) \in \Sigma_{\mathsf{a}}^{n+1}$, is accepted by a DFA $\mathcal{A}_\phi$ if there exists a finite run $q = (q(0), \ldots, q(n+1)) \in Q_\ell^{n+2}$ such that $q(0) = q_0$, $q(i+1) = \mathsf{t}(q(i), \omega_f(i))$ for all $0 \leq i \leq n$, and $q(n+1) \in F_{\mathsf{a}}$. The accepted language of $\mathcal{A}_\phi$, denoted $\mathcal{L}(\mathcal{A}_\phi)$, is the set of all words accepted by $\mathcal{A}_\phi$. For every scLTL property ϕ, cf. Definition 3.4.11, there exists a DFA $\mathcal{A}_\phi$ such that

$$\mathcal{L}_f(\phi) = \mathcal{L}(\mathcal{A}_\phi).$$

As a result, the satisfaction of the property ϕ now becomes equivalent to the reaching to the accept locations in the DFA. We use the DFA $\mathcal{A}_\phi$ to specify properties of dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ as follows. Recall that $\mathsf{L} : Y \to \Sigma_{\mathsf{a}}$ is a given measurable function. To each output $y \in Y$, it assigns the letter $\mathsf{L}(y) \in \Sigma_{\mathsf{a}}$. Given a policy $\bar{\rho}$, we can define the probability that an output trajectory of Σ satisfies an scLTL property ϕ over the time horizon $[0, T_d]$, i.e., $\mathbb{P}(\omega_f \in \mathcal{L}(\mathcal{A}_\phi) \text{ s.t. } |\omega_f| \leq T_d + 1)$, with $|\omega_f|$ denoting the length of ω_f [DLT08a].

The following example provides an automaton associated with a reach-avoid specification.

Example 3.4.13. *Consider two measurable sets $\bar{\mathcal{S}}, \bar{\mathcal{T}} \subset Y$ as the safe and target sets, respectively. We present the DFA for the specification $(\bar{\mathcal{S}} \mathsf{U} \bar{\mathcal{T}})$, which requires the output trajectories to reach the target set $\bar{\mathcal{T}}$ while remaining in the safe set $\bar{\mathcal{S}}$. Note that we do not assume these two sets being disjoint. Consider the set of atomic propositions $AP = \{\bar{\mathcal{S}}, \bar{\mathcal{T}}\}$ and the alphabet $\Sigma_{\mathsf{a}} = \{\emptyset, \{\bar{\mathcal{S}}\}, \{\bar{\mathcal{T}}\}, \{\bar{\mathcal{S}}, \bar{\mathcal{T}}\}\}$. Define the labeling function as*

$$\mathsf{L}(y) = \begin{cases} \{\bar{\mathcal{S}}\} =: a, & \text{if } y \in \bar{\mathcal{S}} \backslash \bar{\mathcal{T}}, \\ \{\bar{\mathcal{T}}\} =: b, & \text{if } y \in \bar{\mathcal{T}}, \\ \emptyset =: c, & \text{if } y \notin \bar{\mathcal{S}} \cup \bar{\mathcal{T}}. \end{cases}$$

As can be seen from the above definition of the labeling function L, it induces a partition over the output space Y as

$$\mathsf{L}^{-1}(a) = \bar{\mathcal{S}} \backslash \bar{\mathcal{T}}, \quad \mathsf{L}^{-1}(b) = \bar{\mathcal{T}}, \quad \mathsf{L}^{-1}(c) = Y \backslash (\bar{\mathcal{S}} \cup \bar{\mathcal{T}}).$$

Note that we have indicated the elements of Σ_{a} with lower-case letters for the ease of notation. The specification $(\bar{\mathcal{S}} \mathsf{U} \bar{\mathcal{T}})$ can be equivalently written as $(a \mathsf{U} b)$ with the associated DFA depicted in Figure 3.3. This DFA has the set of locations $Q_\ell = \{q_0, q_1, q_2, q_3\}$, the initial location q_0, and accepting location $F_{\mathsf{a}} = \{q_2\}$. Thus output trajectories of a dt-SCS Σ satisfy the specification $(a \mathsf{U} b)$ if and only if their associated words are accepted by this DFA.

In the rest of this section, we focus on the computation of the probability of $\omega \in \mathcal{L}(\mathcal{A}_\phi)$ over bounded intervals. In other words, we fix a time horizon T_d and compute $\mathbb{P}(\omega(0)\omega(1)\dots\omega(T_d) \in \mathcal{L}(\mathcal{A}_\phi))$. Suppose Σ and $\widehat{\Sigma}$ are two dt-SCS for which the results of Theorem 3.2.7 hold. Consider a labeling function L defined on their output space and an scLTL specification ϕ with DFA $\mathcal{A}_\phi$. In the following, we show how to construct a DFA $\mathcal{A}_{\hat{\phi}}$ of another specification $\hat{\phi}$ and a new labeling function L^ε such that the satisfaction probability of $\hat{\phi}$ by output trajectories of $\widehat{\Sigma}$ and labeling function L^ε give a lower bound on the satisfaction probability of ϕ by output trajectories of Σ and the labeling function L.

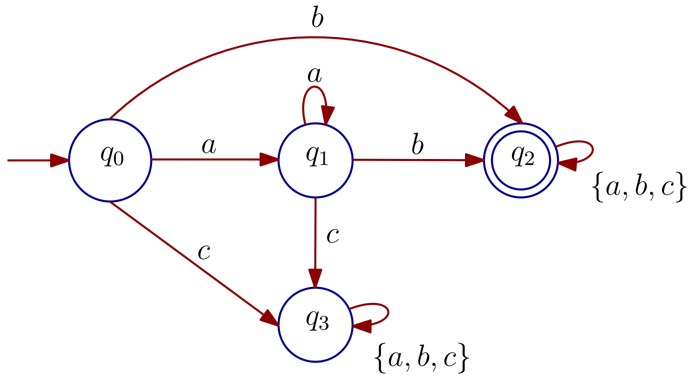

Figure 3.3: DFA $\mathcal{A}_\phi$ of the reach-avoid specification $(a \cup b)$.

Consider the labeling function $\mathsf{L} : Y \to \Sigma_\mathsf{a}$. The new labeling function $\mathsf{L}^\varepsilon : Y \to \bar{\Sigma}_\mathsf{a}$ is constructed using the ε-perturbation of subsets of Y. Define for any Borel measurable set $\mathsf{A} \subset Y$, its ε-perturbed version A^ε as the largest measurable set satisfying

$$\mathsf{A}^\varepsilon \subseteq \{y \in \mathsf{A} \,\big|\, \|\bar{y} - y\| \geq \varepsilon \text{ for all } \bar{y} \in Y \backslash \mathsf{A}\}.$$

Remark that the set A^ε is just the largest measurable set contained in the ε-deflated version of A and without loss of generality we assume it is nonempty. Then $\mathsf{L}^\varepsilon(y) = \mathsf{L}(y)$ for any $y \in \cup_{a \in \Sigma_\mathsf{a}} [\mathsf{L}^{-1}(a)]^\varepsilon$, otherwise $\mathsf{L}^\varepsilon(y) = \phi_\circ$.

Consider the DFA $\mathcal{A}_\phi = (Q_\ell, q_0, \Sigma_\mathsf{a}, F_\mathsf{a}, \mathsf{t})$. The new DFA

$$\mathcal{A}_{\hat{\phi}} = (\bar{Q}_\ell, q_0, \bar{\Sigma}_\mathsf{a}, F_\mathsf{a}, \bar{\mathsf{t}}) \tag{3.4.17}$$

will be constructed by adding one absorbing location q_{abs} and one letter $\phi_\circ$ as $\bar{Q}_\ell := Q_\ell \cup \{q_{\mathsf{abs}}\}$ and $\bar{\Sigma}_\mathsf{a} := \Sigma_\mathsf{a} \cup \{\phi_\circ\}$. The initial and accept locations are the same with $\mathcal{A}_\phi$. The transition relation is defined, $\forall q \in \bar{Q}_\ell, \forall a \in \bar{\Sigma}_\mathsf{a}$, as

$$\bar{\mathsf{t}}(q, a) := \begin{cases} \mathsf{t}(q, a), & \text{if } q \in Q_\ell, a \in \Sigma_\mathsf{a}, \\ q_{\mathsf{abs}}, & \text{if } a = \phi_\circ, q \in \bar{Q}_\ell, \\ q_{\mathsf{abs}}, & \text{if } q = q_{\mathsf{abs}}, a \in \bar{\Sigma}_\mathsf{a}. \end{cases}$$

In other words, we add an absorbing state q_{abs} and all the states will jump to this absorbing state with the label $\phi_\circ$. As an example, the modified DFA of the reach-avoid specification in Figure 3.3 is plotted in Figure 3.4.

In the next lemma, we employ the new labeling function to relate the satisfaction of specifications by output trajectories of two dt-SCS.

Lemma 3.4.14. *Suppose two observed sequences of output trajectories for two dt-SCS Σ and $\widehat{\Sigma}$ satisfy the inequality*

$$\sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\|_2 < \varepsilon,$$

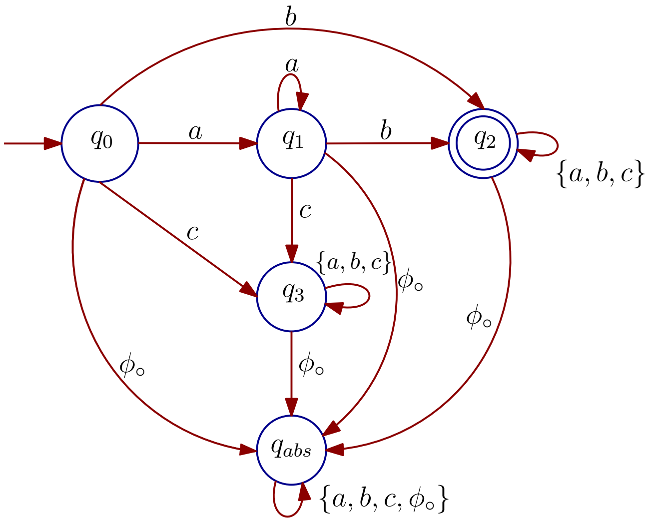

Figure 3.4: Modified DFA $\mathcal{A}_{\hat{\phi}}$ of the specification $(a \cup b)$.

for some time bound T_d and $\varepsilon > 0$. Then $y(\cdot) \vDash_{\mathsf{L}} \phi$ if $\hat{y}(\cdot) \vDash_{\mathsf{L}^\varepsilon} \hat{\phi}$ over the time interval $[0, T_d]$ with labeling functions L and L^ε, and the modified specification $\hat{\phi}$ defined in (3.4.17)*.*

Proof. Suppose $\hat{y}(\cdot) \vDash_{\mathsf{L}^\varepsilon} \hat{\phi}$ over the time interval $[0, T_d]$. According to the construction of DFA $\mathcal{A}_{\hat{\phi}}$, q_{abs} is an absorbing state and not an accepting state, thus $\mathsf{L}^\varepsilon(\hat{y}(k)) \neq \phi_\circ$, $\forall k \in [0, T_d]$. Then $\mathsf{L}^\varepsilon(\hat{y}(k)) \in \Sigma_{\mathsf{a}}$, $\forall k \in [0, T_d]$. Assume $\mathsf{L}^\varepsilon(\hat{y}(k)) = a$ then $\hat{y}(k) \in [\mathsf{L}^{-1}(a)]^\varepsilon$. Since we know that

$$\sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\|_2 < \varepsilon,$$

then according to the definition of ε-perturbed sets, $y(k) \in \mathsf{L}^{-1}(a)$ which gives $\mathsf{L}(y(k)) = a$. Thus $\mathsf{L}(y(\cdot)) = \mathsf{L}^\varepsilon(\hat{y}(\cdot))$ and having $\hat{y}(\cdot) \vDash_{\mathsf{L}^\varepsilon} \hat{\phi}$ guarantees $y(\cdot) \vDash_{\mathsf{L}} \phi$ due to the particular construction of $\hat{\phi}$. □

Next theorem presents the core result of this subsection.

Theorem 3.4.15. *Suppose Σ and $\widehat{\Sigma}$ are two dt-SCS for which the inequality* (3.2.5) *holds with the pair $(\varepsilon, \hat{\delta})$ and any time bound T_d. Suppose a specification ϕ and a labeling function L are defined for Σ. The following inequality holds for the labeling function L^ε on $\widehat{\Sigma}$ and the modified specification $\hat{\phi}$:*

$$\mathbb{P}(\hat{y}(\cdot) \vDash_{\mathsf{L}^\varepsilon} \hat{\phi}) - \hat{\delta} \leq \mathbb{P}(y(\cdot) \vDash_{\mathsf{L}} \phi), \tag{3.4.18}$$

where the satisfaction is over the time interval $[0, T_d]$.

Proof. According to Lemma 3.4.14, $y(\cdot) \nvDash_{\mathsf{L}} \phi$ results in $\hat{y}(\cdot) \nvDash_{\mathsf{L}^\varepsilon} \hat{\phi}$ over the time interval $[0, T_d]$ or

$$\sup_{0 \leq k \leq T_d} \|y(k) - \hat{y}(k)\|_2 \geq \varepsilon.$$

Then

$$\begin{aligned}
&\mathbb{P}(y(\cdot) \nvDash_{\mathsf{L}} \phi) \leq \mathbb{P}(\hat{y}(\cdot) \nvDash_{\mathsf{L}^{\varepsilon}} \hat{\phi}) + \overbrace{\mathbb{P}(\sup_{0\leq k\leq T_d} \|y(k) - \hat{y}(k)\|_2 \geq \varepsilon)}^{\leq \hat{\delta}},\\
&\Rightarrow 1 - \mathbb{P}(y(\cdot) \vDash_{\mathsf{L}} \phi) \leq 1 - \mathbb{P}(\hat{y}(\cdot) \vDash_{\mathsf{L}^{\varepsilon}} \hat{\phi}) + \hat{\delta},\\
&\Rightarrow \mathbb{P}(\hat{y}(\cdot) \vDash_{\mathsf{L}^{\varepsilon}} \hat{\phi}) - \hat{\delta} \leq \mathbb{P}(y(\cdot) \vDash_{\mathsf{L}} \phi),
\end{aligned}$$

which completes the proof. □

In order to get an upper bound for $\mathbb{P}(y(\cdot) \vDash_{\mathsf{L}} \phi)$, we need to define for any Borel measurable set $\mathsf{A} \subset Y$, its $(-\varepsilon)$-perturbed version $\mathsf{A}^{-\varepsilon}$ as the smallest measurable set satisfying

$$\mathsf{A}^{-\varepsilon} \supseteq \{y \in Y \,\big|\, \exists \bar{y} \in \mathsf{A} \text{ with } \|\bar{y} - y\| < \varepsilon\}.$$

Remark that the set $\mathsf{A}^{-\varepsilon}$ is just the smallest measurable set containing the ε-inflated version of A.

A new labeling map $\mathsf{L}^{-\varepsilon} : Y \to 2^{\Sigma_{\mathsf{a}}}$ is constructed using the $(-\varepsilon)-$perturbation of subsets of Y as

$$\mathsf{L}^{-\varepsilon}(y) := \left\{a \in \Sigma_{\mathsf{a}} \,\big|\, y \in [\mathsf{L}^{-1}(a)]^{-\varepsilon}\right\}. \tag{3.4.19}$$

Theorem 3.4.16. *Suppose Σ and $\widehat{\Sigma}$ are two dt-SCS for which the inequality* (3.2.5) *holds with the pair $(\varepsilon, \hat{\delta})$ and any time bound T_d. Suppose a specification ϕ and a labeling function L are defined for Σ. The following inequality holds for the labeling function $\mathsf{L}^{-\varepsilon}$ defined in* (3.4.19) *on $\widehat{\Sigma}$:*

$$\mathbb{P}(y(\cdot) \vDash_{\mathsf{L}} \phi) \leq \mathbb{P}(\hat{y}(\cdot) \vDash_{\mathsf{L}^{-\varepsilon}} \phi) + \hat{\delta}, \tag{3.4.20}$$

where the satisfaction is over the time interval $[0, T_d]$, and the probability in the right-hand side is computed for having $\hat{y}(\cdot) \vDash_{\mathsf{L}^{-\varepsilon}} \phi$ for any choice of non-determinism introduced by the labeling map $\mathsf{L}^{-\varepsilon}$.

The proof is similar to that of Theorem 3.4.15, and is omitted here.

In contrast with the inequality (3.4.18), the specification ϕ is the same in both sides of (3.4.20). The non-determinism originating from $\mathsf{L}^{-\varepsilon}$ in the right-hand side of (3.4.20) can be pushed to the DFA representation of ϕ, by constructing a finite automaton that is non-deterministic.

In the next subsection, we demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 3 dimensions) of an interconnected system consisting of three nonlinear stochastic control subsystems (together 222 dimensions) in a compositional fashion. We employ the constructed abstraction as a substitute to synthesize a controller enforcing a syntactically co-safe linear temporal logic specification.

3.4.5 Case Study

Consider a discrete-time nonlinear stochastic control system Σ satisfying

$$\Sigma : \begin{cases} x(k+1) = \bar{G}x(k) + \varphi(x(k)) + \nu(k) + R\varsigma(k), \\ y(k) = Cx(k), \end{cases} \tag{3.4.21}$$

for some matrix $\bar{G} = (\mathbb{I}_n - \tau\tilde{L}) \in \mathbb{R}^{n\times n}$ where $\tilde{L}$ is the Laplacian matrix of an undirected graph with sampling time $0 < \tau < 1/\bar{\Delta}$, where $\bar{\Delta}$ is the maximum degree of the graph [GR01]. Moreover, $R = \mathsf{diag}(0.007\mathbb{1}_{n_1}\ldots, 0.007\mathbb{1}_{n_N})$, $\varsigma(k) = [\varsigma_1(k);\ldots;\varsigma_N(k)]$, $\varphi(x) = [\mathbb{1}_{n_1}\varphi_1(F_1x_1(k));\ldots;\mathbb{1}_{n_N}\varphi_N(F_Nx_N(k))]$, where $n = \sum_{i=1}^{N} n_i$, $\varphi_i(x) = sin(x)$, and $F_i = [1;0;\cdots;0]^T \in \mathbb{R}^{n_i}$ $\forall i \in \{1,\ldots,N\}$, and C has the block diagonal structure as $C = \mathsf{diag}(C_1^1,\ldots,C_N^1)$, where $C_i^1 \in \mathbb{R}^{q_i^1\times n_i}, \forall i \in \{1,\ldots,N\}$. We partition x as $x = [x_1;\ldots;x_N]$ and ν as $\nu = [\nu_1;\ldots;\nu_N]$, where $x_i, \nu_i \in \mathbb{R}^{n_i}$. Now, by introducing $\Sigma = (\mathbb{I}_{n_i}, \mathbb{I}_{n_i}, C_i^1, \mathbb{I}_{n_i}, \mathbb{I}_{n_i}, \mathbb{1}_{n_i}, F_i, 0.007\mathbb{1}_{n_i}, \varphi_i)$ satisfying

$$\Sigma : \begin{cases} x_i(k+1) = x_i(k) + \mathbb{1}_{n_i}\varphi_i(F_ix_i(k)) + \nu_i(k) + w_i(k) + 0.007\mathbb{1}_{n_i}\varsigma_i(k), \\ y_i^1(k) = C_i^1x_i(k), \\ y_i^2(k) = x_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}_{cd}(\Sigma_1,\ldots,\Sigma_N)$, where the coupling matrix M is given by $M = -\tau\tilde{L}$. Our goal is to aggregate each x_i into a scalar-valued $\hat{x}_i$, governed by $\widehat{\Sigma}_i = (0.5, 1, \hat{C}_i^1, 1, 1, 0.1, 1, 0, \varphi_i)$ which satisfies

$$\widehat{\Sigma}_i : \begin{cases} \hat{x}_i(k+1) = 0.5\hat{x}_i(k) + 0.1\varphi_i(\hat{x}_i(k)) + \hat{\nu}_i(k) + \hat{w}_i(k), \\ \hat{y}_i^1(k) = \hat{C}_i^1\hat{x}_i(k), \\ \hat{y}_i^2(k) = \hat{x}_i(k), \end{cases}$$

where $\hat{C}_i^1 = C_i^1\mathbb{1}_{n_i}$. Note that here $\hat{R}_i$, $\forall i \in \{1,\ldots,N\}$, are considered zero in order to reduce constants ψ_i for each V_i as discussed in Remark 3.2.19. One can readily verify that, for any $i \in \{1,\ldots,N\}$, conditions (3.4.10) and (3.4.11) are satisfied with $\tilde{M}_i = \mathbb{I}_{n_i}$, $\hat{\kappa}_i = 0.95$, $\tilde{\kappa}_i = 1, b_i = 1$, $K_i = (\lambda_i' - 1)\mathbb{I}_{n_i}$, $\lambda_i' = 0.5$, $Z_i = G_i = \mathbb{I}_{n_i}$, $L_{1i} = -\mathbb{1}_{n_i}$, $\tilde{R} = \mathbb{1}_{n_i}$, $\bar{X}^{11} = \mathbb{I}_{n_i}$, $\bar{X}^{22} = \mathbf{0}_{n_i}$, and $\bar{X}^{12} = \bar{X}^{21} = \lambda_i'\mathbb{I}_{n_i}$. Moreover, for any $i \in \{1,\ldots,N\}$, $\tilde{P}_i = \mathbb{1}_{n_i}$ satisfies conditions (3.4.12) with $Q_i = -0.5\mathbb{1}_{n_i}$, $L_{2i} = -0.1\mathbb{1}_{n_i}$, and $H_i = \hat{G}_i = \mathbb{1}_{n_i}$. Hence, the function $V_i(x_i,\hat{x}_i) = (x_i - \mathbb{1}_{n_i}\hat{x}_i)^T(x_i - \mathbb{1}_{n_i}\hat{x}_i)$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (3.4.1) with $\alpha_i(s) = \frac{1}{\lambda_{\max}(C_{1i}^TC_{1i})}s^2$ and the condition (3.4.2) with $\kappa_i(s) := 0.05s$, $\rho_{\mathrm{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, $G_i = \mathbb{I}_{n_i}$, $\hat{H}_i = \mathbb{1}_{n_i}$, and

$$\bar{X}_i = \begin{bmatrix} \mathbb{I}_{n_i} & \lambda_i'\mathbb{I}_{n_i} \\ \lambda_i'\mathbb{I}_{n_i} & \mathbf{0}_{n_i} \end{bmatrix}, \tag{3.4.22}$$

where the input ν_i is given via the interface function in (3.4.13) as

$$\nu_i = (\lambda_i' - 1)(x_i - \mathbb{1}_{n_i}\hat{x}_i) - 0.5\mathbb{1}_{n_i}\hat{x}_i + \mathbb{1}_{n_i}\hat{\nu}_i - \mathbb{1}_{n_i}\varphi_i(F_ix_i) + 0.1\mathbb{1}_{n_i}\varphi_i(F_i\mathbb{1}_{n_i}\hat{x}_i).$$

Now, we look at $\widehat{\Sigma} = \mathcal{I}_{cd}(\widehat{\Sigma}_1,\ldots,\widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (3.4.4) as follows:

$$-\tau\tilde{L}\ \mathsf{diag}(\mathbb{1}_{n_1},\ldots,\mathbb{1}_{n_N}) = \mathsf{diag}(\mathbb{1}_{n_1},\ldots,\mathbb{1}_{n_N})\hat{M}. \tag{3.4.23}$$

Note that the existence of $\hat{M}$ satisfying (3.4.23) for the graph Laplacian $\tau\tilde{L}$ means that the N subgraphs form an *equitable partition* of the full graph [GR01]. Although this restricts the choice of a partition in general, for the complete graph any partition is equitable.

Choosing $\mu_1 = \cdots = \mu_N = 1$ and using $\bar{X}_i$ in (3.4.22), matrix $\bar{X}_{cmp}$ in (3.4.6) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} \mathbb{I}_n & \lambda'\mathbb{I}_n \\ \lambda'\mathbb{I}_n & \mathbf{0}_n \end{bmatrix},$$

where $\lambda' = \lambda'_1 = \cdots = \lambda'_N = 0.5$, and the condition (3.4.3) reduces to

$$\begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix} = \tau^2\tilde{L}^T\tilde{L} - \lambda'\tau\tilde{L} - \lambda'\tau\tilde{L}^T = \tau\tilde{L}(\tau\tilde{L} - 2\lambda'\mathbb{I}_n) \preceq 0,$$

without requiring any restrictions on the number or gains of the subsystems with $\tau = 0.9/(n-1)$. In order to show the above inequality, we used $\tilde{L} = \tilde{L}^T \succeq 0$ which is always true for Laplacian matrices of undirected graphs. Now, one can readily verify that $V(x,\hat{x}) = \sum_{i=1}^{n}(x_i - \mathbb{1}_{n_i}\hat{x}_i)^T(x_i - \mathbb{1}_{n_i}\hat{x}_i)$ is a sum-type SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (3.2.3) and (3.2.4).

For the sake of simulation, we assume $\tilde{L}$ is the Laplacian matrix of a complete graph as

$$\tilde{L} = \begin{bmatrix} n-1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}_{n\times n}. \tag{3.4.24}$$

We fix $N = 3$, $n = 222$, $n_i = 74$, and $C_i^1 = [1;0;\ldots;0]^T$, $i \in \{1,2,3\}$. By using the inequality (3.2.5) and starting the interconnected systems Σ and $\widehat{\Sigma}$ from initial states $-13\mathbb{1}_{222}$ and $-13\mathbb{1}_3$, respectively, we guarantee that the distance between outputs of Σ and $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with a probability at least 90%, i.e.,

$$\mathbb{P}\left(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 1,\ \forall k \in [0,10]\right) \geq 0.9.$$

Let us now synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ to enforce a specification, defined by the following scLTL formula (cf. Definition 3.4.11):

$$\varpi = \bigwedge_{j=0}^{T_d} \bigcirc^j \left(\bar{\mathcal{S}} \wedge \left(\bigwedge_{i=1}^{3}(\neg\bar{\mathcal{A}}_i)\right)\right) \wedge \Diamond\bar{\mathcal{T}}_1 \wedge \Diamond\bar{\mathcal{T}}_2, \tag{3.4.25}$$

which requires that any output trajectory y of the closed-loop system evolves inside the set $\bar{\mathcal{S}}$, avoids sets $\bar{\mathcal{A}}_i$, $i \in \{1,2,3\}$, indicated with blue boxes in Figure 3.5, over the bounded time interval $[0, T_d]$, and visits each $\bar{\mathcal{T}}_i$, $i \in \{1,2\}$, indicated with red boxes in

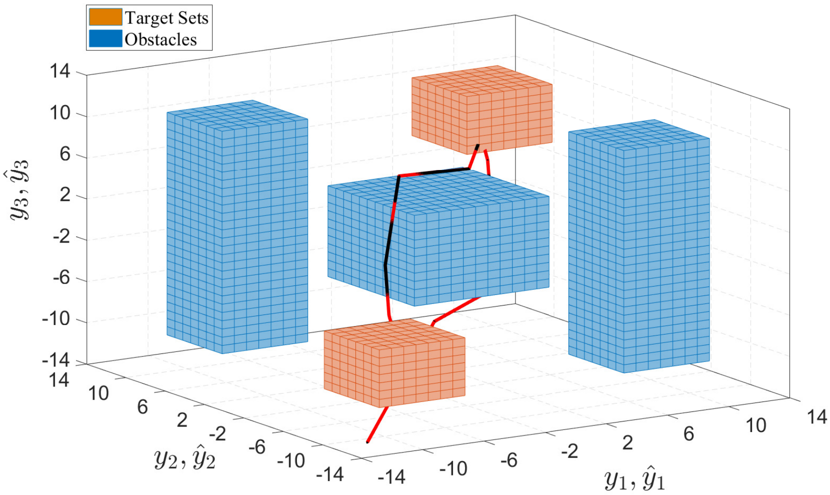

Figure 3.5: The specification with closed-loop output trajectories of Σ (black one) and $\widehat{\Sigma}$ (red one). The sets $\bar{\mathcal{S}}$, $\bar{\mathcal{A}}_i$, $i \in \{1,2,3\}$, and $\bar{\mathcal{T}}_i$, $i \in \{1,2\}$ are given by: $\bar{\mathcal{S}} = [-14,14]^3$, $\bar{O}_1 = [-10,-6] \times [6,10] \times [10,10]$, $\bar{O}_2 = [-5,5]^3$, and $\bar{O}_3 = [6,10] \times [-10,-6] \times [10,10]$, $\bar{\mathcal{T}}_1 = [-10,-6] \times [-10,-6] \times [-10,-6]$ and $\bar{\mathcal{T}}_2 = [6,10] \times [6,10] \times [6,10]$.

Figure 3.5. We want to satisfy ϖ over the bounded time interval $[0,10]$, i.e., $T_d = 10$. We use SCOTS [RZ16] to synthesize a controller for $\widehat{\Sigma}$ to enforce (3.4.25). In the synthesis process, we restrict the abstract inputs $\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3$ to $[-4,4]$. We also set the initial states of Σ to $x_i = \tilde{P}_i\hat{x}_i$, so that $V_i(x_i,\hat{x}_i) = 0$. A realization of closed-loop output trajectories of Σ and $\widehat{\Sigma}$ is illustrated in Figure 3.5. Also, several realizations of the norm of the error between outputs of Σ and $\widehat{\Sigma}$ are illustrated in Figure 3.6. In order to have some more practical analysis on the provided probabilistic bound, we also run Monte Carlo simulation of 10000 runs. In this case, one can statistically guarantee that the distance between outputs of Σ and $\widehat{\Sigma}$ is always less than or equal to 0.05 with the same probability, (i.e., at least 90%). This issue is expected and the reason is due to the conservatism nature of Lyapunov-like techniques (simulation functions), but with the gain of having a formal guarantee on the output trajectories rather than an empirical one. Note that it would not have been possible to synthesize a controller using SCOTS for the original 222-dimensional system Σ, without the 3-dimensional intermediate approximation $\widehat{\Sigma}$. Moreover, we have intentionally dropped the noise of the abstraction and employed SCOTS here to show that if the concrete system possesses some stability property and the noises of two systems are additive and independent, it is actually better to construct and employ the non-stochastic abstraction since the non-stochastic abstraction is closer that the stochastic version (as discussed in Remark 3.2.19).

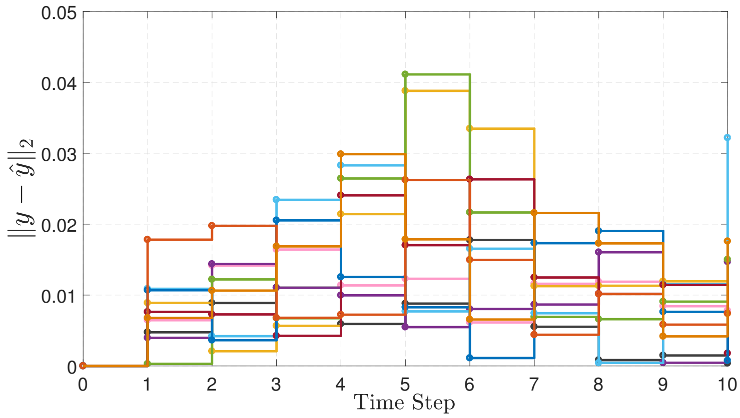

Figure 3.6: A few realizations of the norm of the error between outputs of Σ and of $\widehat{\Sigma}$, e.g., $\|y - \hat{y}\|_2$, for $T_d = 10$.

3.5 Summary

In this chapter, we have proposed compositional infinite abstractions (model order reductions) of original systems with three different compositional techniques including classic small-gain, max small-gain and dissipativity approaches. We showed that the proposed max small-gain approach is more general than the classic one since it does not require any linear growth on the gains of subsystems which is the case in the classic small-gain. We also showed that the provided approximation error via the max small-gain does not change as the number of subsystems grows since the proposed overall error is completely independent of the size of the network, and is computed only based on the maximum error of subsystems instead of being a linear combination of them which is the case in classic small-gain and dissipativity approaches. On the other hand, we discussed that the proposed dissipativity technique is less conservative than the classic (or max) small-gain approach in the sense that the provided dissipativity-type compositional condition can enjoy the structure of the interconnection topology and be potentially fulfilled independently of the number or gains of subsystems.

We have also extended our proposed construction scheme from linear to a particular class of *nonlinear* stochastic systems whose nonlinearities satisfy a slope restriction. Moreover, we extended our specification from a finite-horizon invariant to a fragment of linear temporal logic known as syntactically co-safe linear temporal logic (scLTL). We proved that given such a co-safe LTL specification over the concrete system, how one can construct an epsilon-perturbed specification over the abstract system whose probability of satisfaction gives a lower bound for the probability of satisfaction in the concrete domain. It should be noted that we did not put any restriction on the sources of uncertainties in concrete and abstract systems meaning that the noise of the abstraction could be completely independent of that of the concrete system. We showed that our results are more general than the ones available in the literature (e.g., [Zam14, ZRME17]), where the noises in concrete and abstract systems are assumed to be the same.

4 Finite Abstractions (Finite Markov Decision Processes)

4.1 Introduction

Construction of finite abstractions was introduced in recent years as a promising method to reduce the complexity of controller synthesis problems in particular for enforcing complex logical properties. In the second phase of the abstract procedure, one can construct finite abstractions (a.k.a. finite Markov decision processes (MDPs)) as approximate descriptions of (reduced-order) systems in which each discrete state corresponds to a collection of continuous states of (reduced-order) systems. Since final abstractions are finite, algorithmic machineries from Computer Science [BK08] are applicable to synthesize controllers over concrete systems enforcing complex logic properties. This chapter is concerned with providing compositional approaches for the construction of *finite* abstractions for large-scale discrete-time stochastic systems. We also propose a compositional technique for the construction of both infinite and finite abstractions in a unified framework via notions of approximate probabilistic relations. We show that the unified compositional framework is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions.

4.1.1 Related Literature

4.1.1.1 Finite Abstraction Techniques

There have been several results, proposed in the past few years, on the construction of finite abstractions for stochastic systems. Finite abstractions are initially employed in [APLS08] for the formal synthesis of discrete-time stochastic systems. An adaptive and sequential gridding scheme is proposed in [SA11, SA13a] that abstracts the system to a finite-state Markov chain. The main goal of the proposed algorithm is to make the discretization approach applicable to systems with larger dimensions. The approach generally relies on continuity of the stochastic kernel associated to the system and the error is a linear function of discretization parameters.

An approximation algorithm is proposed in [SA12a] with an error that depends on higher orders of discretization parameters. The continuity assumption is further relaxed in [SA12b, SA14b] by quantifying the discretization error for systems that have both deterministic and stochastic dynamics. Extension of the techniques to the formal abstraction-based policy synthesis is discussed in [TMKA13, Sou14]. Finite bisimilar abstractions for incrementally stable stochastic switched systems, randomly switched stochastic systems, and for incrementally stable stochastic control systems without discrete dynamics are respectively discussed in [ZAG15], [ZA14], and [ZMEM+14].

The use of abstraction techniques for modeling population of dynamical systems is studied in [SA15a, KES+13]. Construction of a finite stochastic dynamical model as the aggregation of temperature dynamics of a collection of thermostatically controlled loads is presented in [SA13b, SA15a] for discrete-time dynamics and in [SGE+14] for continuous-time dynamics. An abstraction framework for mapping a discrete-time stochastic system to an interval-valued Markov chain and mapping a switched discrete-time stochastic system to a bounded-parameter Markov decision process is proposed in [LAB15]. A method to generate finite Markovian abstractions for discrete-time linear stochastic systems evolving in full dimensional polytopes is provided in [LAB12]. An efficient abstraction framework for formal analysis and control synthesis of discrete-time stochastic hybrid systems with linear dynamics is developed in [CLL+19]. Safety verification of continuous-space Markov processes with jumps is studied in [SMA16] using discrete abstractions.

4.1.1.2 Compositional Techniques

In order to make the approaches provided by finite abstractions applicable to networks of interacting systems, compositional techniques are proposed in the past few years. Compositional construction of finite abstractions for discrete-time stochastic control systems is presented in [SAM15, SAM17] using dynamic Bayesian networks. A compositional strategy synthesis for stochastic games with multiple objectives is provided in [BKW18]. Compositional probabilistic verification via an assume-guarantee framework based on multi-objective probabilistic model checking is investigated in [KNPQ13] for finite systems.

4.1.1.3 Stochastic Similarity Relations

Stochastic similarity relations are employed to relate the probabilistic behavior of concrete models to that of their abstractions. These similarities can be presented in the context of stochastic simulation, bisimulation, exact, and approximate relations. Similarity relations over finite-state stochastic systems have been studied, either via exact notions of probabilistic (bi)simulation relations [LS91], [SL95] or approximate versions [DLT08b], [DAK12]. Similarity relations for models with general, uncountable state spaces have been also proposed in the literature. These relations either depend on stability requirements on model outputs via martingale theory or contractivity analysis [JP09], [ZMEM+14] or enforce structural abstractions of a model [DGJP04] by exploiting continuity conditions on its probability laws [Aba13], [AKNP14]. A new notion of approximate similarity relation is recently proposed in [HSA17, HS18] that takes into account both the deviation in the stochastic evolution and in outputs of the two systems.

4.1.1.4 Control Barrier Certificates

In order to deal with the computational complexity arising with the construction of finite abstractions, there have also been discretization-free approaches based on control barrier certificates, proposed in recent years. Discretization-free approaches based on barrier certificates for stochastic hybrid systems are initially proposed in [PJP07], and then extended for the probabilistic safety verification and finite-time regional verification in [HCL+17], and [ST12], respectively. Temporal logic verification of stochastic systems via control barrier certificates and its extension for formal synthesis are respectively presented in [JSZ18] and [JSZ19]. Recently, verification and control for a finite-time safety of stochastic systems via barrier functions are discussed in [SDC19].

4.1.1.5 Stability Verification of Large-Scale Systems

There have been also some results in the context of the stability verification of large-scale *non-stochastic* systems via finite-step Lyapunov-type functions. Nonconservative small-gain conditions based on finite-step Lyapunov functions are originally introduced in [AP98]. Nonconservative dissipativity and small-gain conditions for stability analysis of interconnected systems are proposed in [GL12, NR14]. Stability analysis of large-scale discrete-time systems via finite-step storage functions is discussed in [GL15]. Moreover, nonconservative small-gain conditions for closed sets using finite-step ISS Lyapunov functions are presented in [NGG+18]. Recently, compositional construction of finite abstractions via relaxed small-gain conditions for discrete-time non-stochastic systems is discussed in [NSWZ18]. The proposed results in [NSWZ18] employ finite-step ISS Lyapunov functions and their compositionality framework is only applicable to non-stochastic systems.

4.1.1.6 Learning Techniques

Reinforcement learning (RL) [SB18] is an approach to sequential decision making in which agents rely on reward signals to choose actions aimed at achieving prescribed objectives. Model-free RL [SLW+06] refers to a class of techniques that are asymptotically space-efficient because they do not construct a full model of the environment. These techniques include classic algorithms like TD(λ) [Sut88] and Q-learning [Wat89] as well as their extensions to deep neural networks such as deep deterministic policy gradient (DDPG) [LHP+15] and neural-fitted Q-iterations [Rie05]. Model-free reinforcement learning has achieved performance comparable to that of human experts in video and board games [Tes95, MKS+15, SHM+16]. This success has motivated extensions of RL to the control of safety-critical systems [LHP+15, LFDA16] in spite of a lack of theoretical convergence guarantees of RL for general continuous-state spaces [DSL+17].

4.1.2 Contributions

In the first part of this chapter, we provide $\max$ small-gain type conditions for the compositional quantification of the probabilistic distance between the interconnection of stochastic control subsystems and that of their finite abstractions. We show that if original systems are incrementally input-to-state stable, one can construct finite MDPs for the general setting of nonlinear stochastic control systems. We also show that for the same nonlinear class of stochastic control systems proposed in the previous chapter, the aforementioned incrementally ISS property can be readily verified by some easier to check matrix inequalities. We demonstrate the effectiveness of the proposed results by applying our approaches to a fully connected network of 20 nonlinear subsystems (totally 100 dimensions). We construct finite MDPs from their *reduced-order versions* proposed in the previous chapter (together 20 dimensions) with guaranteed error bounds on their output trajectories. We also apply the proposed results to the temperature regulation in a circular building and construct compositionally a finite abstraction of a network containing 1000 rooms. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies regulating the temperature in each room for a bounded time horizon.

We then extend our results to stochastic switched systems whose switch signals accept dwell-time with multiple Lyapunov functions. We show that under standard assumptions ensuring the incremental input-to-state stability of switched systems (i.e., existence of common incremental ISS Lyapunov functions, or multiple incremental ISS Lyapunov functions with dwell-time), one can construct finite MDPs for the general setting of nonlinear stochastic switched systems. To demonstrate the effectiveness of our proposed results, we first apply our approaches to a road traffic network in a circular cascade ring composed of 200 cells, and construct compositionally a finite MDP of the network. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies keeping the density of the traffic lower than 20 vehicles per cell. We benchmark our proposed results against the ones available in the literature. We also provide some discussions on the memory usage and computation time in the construction of finite MDPs for this case study in both monolithic and compositional manners, and compare the results in a table for different ranges of the state discretization parameter. We show that the proposed compositional approach in this work remarkably reduces the curse of dimensionality problem in constructing finite MDPs. We then apply our proposed techniques to a *fully interconnected* network of 500 nonlinear subsystems (totally 1000 dimensions), and construct their finite MDPs with guaranteed error bounds. We provide simulation results for this case study to have a more practical analysis on the proposed probabilistic bounds.

In the second part of the chapter, we first propose the dissipativity approach as our compositional framework for the construction of finite MDPs from stochastic control systems and provide the corresponding results. We utilize the incremental passivability property of original systems and propose an approach to construct finite MDPs for the general setting of nonlinear stochastic control systems. We apply our proposed results to the temperature regulation in a network of 200 rooms such that the compositionality condition does not require any constraint on the number or gains of the subsystems. We also illustrate the effectiveness of our results on an example of fully interconnected network. We benchmark our results against the compositional abstraction technique of [SAM15] which is based on construction of finite MDPs via dynamic Bayesian networks.

We then generalize our results to stochastic switched systems with multiple supply rates and multiple storage functions accepting dwell-time. We also enlarge the class of systems for the construction of finite MDPs by adding time-varying nonlinearities to the dynamics satisfying an incremental quadratic inequality, whereas the provided results in the first part of the chapter only handle the class of nonlinearities satisfying slope restrictions. We also relax one of the compositionality conditions proposed for stochastic control systems in the previous part that was implicit, without providing a direct method for satisfying it. We relax this condition at the cost of incurring an additional error term, but benefiting from choosing quantization parameters of internal input sets freely. We apply our proposed techniques to a *fully interconnected network* of 100 *nonlinear* subsystems (totally 200 dimensions), and also a *road traffic network* in a circular cascade ring composed of 50 cells.

In the third and forth parts of the chapter, we propose relaxed versions of max small-gain and dissipativity approaches and provide a less conservative framework in the sense that the stabilizability of individual subsystems for establishing the compositional results, required in all previous parts, is not here necessarily required. We also provide the probabilistic closeness guarantee between interconnected stochastic autonomous systems and that of their finite Markov chains (MCs) for the whole state trajectory. We quantify that if the state discretization parameter is small enough, the sampled MC will be close enough to the original system for the all time instances. We apply our proposed results to different case studies including three networks with unstabilizable subsystems, and a fully interconnected network of 500 nonlinear subsystems.

We then propose a compositional approach for the construction of (in)finite abstractions using notions of approximate probabilistic relations. The abstraction framework is based on δ-lifted relations, using which one can quantify the distance in probability between the interconnected original systems and that of their abstractions. This new approximate relation unifies compositionality results in the literature by incorporating the dependencies between state transitions explicitly and by allowing abstract models to have either finite or infinite state spaces. In particular, we focus on the nonlinear class of stochastic control systems satisfying slope restrictions and construct their abstractions using both the model order reduction and the space discretization in a unified framework. We show that the unified compositional scheme is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions.

Finally, we propose a novel reinforcement learning framework to synthesize policies for unknown, continuous-space MDPs. This scheme enables one to apply model-free, off-the-shelf reinforcement learning algorithms for finite MDPs to compute optimal strategies for the corresponding continuous-space MDPs without explicitly constructing the finite-state abstraction. The proposed approach is based on abstracting the process with a finite MDP with *unknown* transition probabilities, synthesizing strategies over the abstract MDP, and then mapping the results back over the concrete continuous-space MDP with bounded approximation guarantees. The system properties of interest belong to the co-safe LTL, and the synthesis requirement is to maximize the probability of satisfaction within a given bounded time horizon. A key contribution here is to leverage the classical convergence results for the reinforcement learning on finite MDPs and provide control strategies maximizing the probability of satisfaction over the unknown, continuous-space MDPs by providing probabilistic closeness guarantees.

4.2 max Small-Gain Approach

In this section, we provide a compositional methodology for the construction of *finite* abstractions for the both stochastic control and switched systems. The proposed technique leverages sufficient max small-gain type conditions to establish the compositionality results which rely on relations between subsystems and their finite abstractions described by the existence of max-type stochastic simulation functions.

4.2.1 Stochastic Control Systems

We first consider the stochastic control systems defined in (2.3.1) and the max-type SPSF and SSF in Definitions 3.3.1, 3.3.3. We present a computational scheme to construct finite MDPs together with their corresponding max-type SPSF for concrete models or their reduced-order versions. We then show that if original systems are incrementally input-to-state stable, one can construct finite MDPs for the general setting of nonlinear stochastic control systems. We also show that for the same nonlinear class of stochastic control systems defined in (3.3.10), the aforementioned incrementally ISS property can be readily verified by some easier to check matrix inequalities. We demonstrate the effectiveness of the proposed results by applying our approaches to a fully connected network of 20 nonlinear subsystems (totally 100 dimensions). We construct finite MDPs from their *reduced-order versions* (together 20 dimensions) proposed in Section 3.3 of the previous chapter with guaranteed error bounds on their output trajectories.

We also apply the proposed results to a temperature regulation in a circular building and construct compositionally a finite abstraction of a network containing 1000 rooms. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies regulating the temperature in each room for a bounded time horizon. Note that we provide the compositional frameworks for infinite and finite abstractions in the previous and this chapters separately since one may be interested in employing one of the proposed results. In addition, if construction of infinite abstractions provided in the previous section is not possible for some given dynamics, one can readily utilize the proposed results for finite abstractions (without performing the model order reduction) which is always possible as in this chapter.

In the next subsection, we show how to construct finite Markov decision processes (MDPs) from concrete models (or their reduced-order versions) as finite abstractions of original systems.

4.2.1.1 Finite Abstractions of dt-SCS

In this subsection, we approximate a dt-SCS Σ with a *finite* $\widehat{\Sigma}$ using Algorithm 1. This algorithm adapted from [SAM15] with some modifications presents this approximation. To construct such a finite approximation, we assume the state and input sets of the dt-SCS Σ are restricted to compact subsets over which we are interested to perform synthesis. The rest of the state sets can be considered as single absorbing states in both Σ and $\widehat{\Sigma}$. In order to make the notation easier, we assume this procedure is already applied to the system and eliminate the absorbing states from the presentation.

Algorithm 1 first constructs a finite partition from the state set X and input sets U, W. Then representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$, and $\bar{w}_i \in \mathsf{W}_i$ are selected as abstract states and inputs. Transition probabilities in the finite MDP $\widehat{\Sigma}$ are also computed according to (4.2.1). The output maps $\hat{h}^1, \hat{h}^2$ are the same as h^1, h^2 with their domain restricted to finite state set $\hat{X}$ (cf. Step 7) and the output sets $\hat{Y}^1, \hat{Y}^2$ are the image of $\hat{X}$ under h^1, h^2 (cf. Step 6).

Algorithm 1 Abstraction of dt-SCS Σ by a finite MDP $\widehat{\Sigma}$

Require: Input dt-SCS $\Sigma = \big(X, W, U, T_{\mathsf{x}}, Y^1, Y^2, h^1, h^2\big)$

1: Select finite partitions of sets X, U, W as $X = \cup_{i=1}^{n_x} \mathsf{X}_i$, $U = \cup_{i=1}^{n_\nu} \mathsf{U}_i$, $W = \cup_{i=1}^{n_w} \mathsf{W}_i$

2: For each $\mathsf{X}_i, \mathsf{U}_i$, and W_i, select single representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$, $\bar{w}_i \in \mathsf{W}_i$

3: Define $\hat{X} := \{\bar{x}_i, i = 1, ..., n_x\}$ as the finite state set of MDP $\widehat{\Sigma}$ with external and internal input sets $\hat{U} := \{\bar{\nu}_i, i = 1, ..., n_\nu\}$ $\hat{W} := \{\bar{w}_i, i = 1, ..., n_w\}$

4: Define the map $\Xi : X \to 2^X$ that assigns to any $x \in X$, the corresponding partition set it belongs to, i.e., $\Xi(x) = \mathsf{X}_i$ if $x \in \mathsf{X}_i$ for some $i = 1, 2, \ldots, n_x$

5: Compute the discrete transition probability matrix $\hat{T}_{\mathsf{x}}$ for $\widehat{\Sigma}$ as

$$\hat{T}_{\mathsf{x}}(x' \,|\, x, \nu, w) = T_{\mathsf{x}}(\Xi(x') \,|\, x, \nu, w), \tag{4.2.1}$$

for all $x, x' \in \hat{X}, \nu \in \hat{U}, w \in \hat{W}$

6: Define output spaces $\hat{Y}^1 := h^1(\hat{X}), \hat{Y}^2 := h^2(\hat{X})$

7: Define output maps $\hat{h}^1 := h^1|_{\hat{X}}$ and $\hat{h}^2 := h^2|_{\hat{X}}$

Ensure: Output finite MDP

$$\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{T}_{\mathsf{x}}, \hat{Y}^1, \hat{Y}^2, \hat{h}^1, \hat{h}^2) \tag{4.2.2}$$

In the following theorem, we give a dynamical representation of the finite MDP.

Theorem 4.2.1. *Given a dt-SCS* $\Sigma = \big(X, U, W, \varsigma, f, Y^1, Y^2, h^1, h^2\big)$*, the finite MDP* $\widehat{\Sigma}$ *constructed in Algorithm 1 can be represented as*

$$\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}^1, \hat{Y}^2, \hat{h}^1, \hat{h}^2), \tag{4.2.3}$$

where $\hat{f} : \hat{X} \times \hat{U} \times \hat{W} \times \mathcal{V}_\varsigma \to \hat{X}$ *is defined as*

$$\hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma) = \Pi_x(f(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)), \tag{4.2.4}$$

and $\Pi_x : X \to \hat{X}$ *is the map that assigns to any* $x \in X$*, the representative point* $\bar{x} \in \hat{X}$ *of the corresponding partition set containing* x*. The initial state of* $\widehat{\Sigma}$ *is also selected according to* $\hat{x}_0 := \Pi_x(x_0)$ *with* x_0 *being the initial state of* Σ*.*

Proof. It is sufficient to show that (4.2.1) holds for the dynamical representation of $\widehat{\Sigma}$ in (4.2.3) and that of Σ. For any $x, x' \in \hat{X}$, $\nu \in \hat{U}$ and $w \in \hat{W}$,

$$\begin{aligned}\hat{T}_{\mathsf{x}}(x' \,|\, x, \nu, w) &= \mathbb{P}(x' = \hat{f}(x, \nu, w, \varsigma)) \\ &= \mathbb{P}(x' = \Pi_x(f(x, \nu, w, \varsigma))) = \mathbb{P}(f(x, \nu, w, \varsigma) \in \Xi(x')),\end{aligned}$$

where $\Xi(x')$ is the partition set with x' as its representative point as defined in Step 4 of Algorithm 1. Using the probability measure $\vartheta(\cdot)$ of random variable ς, we can write

$$\hat{T}_{\mathsf{x}}(x' \,|\, x, \nu, w) = \int_{\Xi(x')} f(x, \nu, w, \varsigma) d\vartheta(\varsigma) = T_{\mathsf{x}}(\Xi(x') \,|\, x, \nu, w),$$

which completes the proof. □

Dynamical representation provided by Theorem 4.2.1 uses the map $\Pi_x : X \to \hat{X}$ that assigns to any $x \in X$, the representative point $\bar{x} \in \hat{X}$ of the corresponding partition set containing x. This map satisfies the inequality

$$\|\Pi_x(x) - x\| \le \bar{\delta}, \quad \forall x \in X, \tag{4.2.5}$$

where $\bar{\delta} := \sup\{\|x - x'\|,\ x, x' \in \mathsf{X}_i,\ i = 1, 2, \ldots, n_x\}$ is the discretization parameter. We use this inequality in the next subsection for the compositional construction of finite MDPs.

Remark 4.2.2. *Note that the proposed bound in* (4.2.5) *is valid for any type of norms provided that the state discretization parameter $\bar{\delta}$ is defined based on the corresponding norm.*

Remark 4.2.3. *We started from the concrete continuous-space dt-SCS as in* (2.3.1)*, constructed its representation as a continuous-space MDP as presented in* (2.4.1)*, then employed Algorithm 1 to construct a finite MDP* (4.2.2) *from the continuous-space MDP* (2.4.1)*, and finally transformed it back to a finite-space dt-SCS as in* (4.2.3) *as our final abstract model (which is more common to be presented in this form for the control community).*

Remark 4.2.4. *Note that we do not have any requirements for discretizing state, external, and internal input sets. However, the size of the state discretization parameter $\bar{\delta}$ appears in the formulated error as in* (4.2.13) *and* (4.2.19)*: one can decrease the error by reducing the state discretization parameter. We also do not have any constraint on the shape of the partition elements in general in constructing finite MDPs. For the sake of an easy implementation, one can consider partition sets as hyper-intervals and the center of them as representative points.*

In the next subsection, we provide an approach for the *compositional synthesis* of interconnected dt-SCS.

4.2.1.2 Compositionality Results

Here, we consider $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$ as the original subsystems (or their reduced-order versions constructed in the previous section) and $\widehat{\Sigma}_i$ as their finite abstractions as constructed in Algorithm 1 given by the tuple

$$\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{h}_i^1, \hat{h}_i^2),$$

with the input-output configuration similar to (3.2.8) and (3.2.9), where $\hat{W}_i \subseteq W_i$, $\hat{Y}_i^1 \subseteq Y_i^1$, and $\hat{Y}_i^2 \subseteq Y_i^2$. Moreover, we assume there exists the max-type SPSF V_i from $\widehat{\Sigma}_i$ to Σ_i with the corresponding functions and constants denoted by α_i, κ_i, $\rho_{\text{int}i}$, $\rho_{\text{ext}i}$, and ψ_i. In order to provide the compositionality result of this section for interconnected finite systems, we first define the abstraction map $\Pi_{w_{ji}}$ on W_{ji} that assigns to any $w_{ji} \in W_{ji}$, a representative point $\bar{w}_{ji} \in \hat{W}_{ji}$ of the corresponding partition set containing w_{ji}. The mentioned map satisfies

$$\|\Pi_{w_{ji}}(w_{ji}) - w_{ji}\| \leq \bar{\mu}_{ji}, \quad \forall w_{ji} \in W_{ji}, \tag{4.2.6}$$

where $\bar{\mu}_{ji}$ is an *internal input* discretization parameter defined similar to $\bar{\delta}$ in (4.2.5).

Remark 4.2.5. *Note that the condition* (4.2.6) *helps us to choose quantization parameters of internal input sets freely at the cost of incurring an additional error term for the overall network (i.e, ψ) which is formulated based on $\bar{\mu}_{ji}$ in* (4.2.10)*. Moreover, the state discretization parameter $\bar{\delta}$ appears in the formulated error for each subsystem (i.e, ψ_i) as in* (4.2.13) *and* (4.2.19)*. These two errors together affect the probabilistic closeness guarantee provided in Theorem 3.2.7.*

Now we define a notion of the interconnection applicable to finite MDPs.

Definition 4.2.6. *Consider $N \in \mathbb{N}_{\geq 1}$ finite stochastic control subsystems $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{h}_i^1, \hat{h}_i^2)$, $i \in \{1, \ldots, N\}$. The interconnection of $\widehat{\Sigma}_i$ is the finite interconnected stochastic control system $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$, denoted by $\widehat{\mathcal{I}}_{cs}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$, such that $\hat{X} := \prod_{i=1}^N \hat{X}_i$, $\hat{U} := \prod_{i=1}^N \hat{U}_i$, $\hat{f} := \prod_{i=1}^N \hat{f}_i$, $\hat{Y} := \prod_{i=1}^N \hat{Y}_i^1$, and $\hat{h} = \prod_{i=1}^N \hat{h}_i^1$, subject to the following constraint:*

$$\forall i, j \in \{1, \ldots, N\}, i \neq j: \qquad \hat{w}_{ji} = \Pi_{w_{ji}}(\hat{y}_{ij}^2), \quad \Pi_{w_{ji}}(\hat{Y}_{ij}^2) \subseteq \hat{W}_{ji}.$$

Now we raise the following max small-gain assumption similar to Assumption 3.3.4.

Assumption 4.2.7. *Assume that there exist $\mathcal{K}_\infty$ functions $\tilde{\delta}_f, \bar{\lambda}$ such that $(\bar{\lambda} - \mathcal{I}_d) \in \mathcal{K}_\infty$ and $\mathcal{K}_\infty$ functions κ_{ij} defined as*

$$\kappa_{ij}(s) := \begin{cases} \kappa_i(s), & \text{if } i = j, \\ (\mathcal{I}_d + \tilde{\delta}_f) \circ \rho_{\text{int}i} \circ \bar{\lambda} \circ \alpha_j^{-1}(s), & \text{if } i \neq j, \end{cases}$$

satisfy

$$\kappa_{i_1 i_2} \circ \kappa_{i_2 i_3} \circ \cdots \circ \kappa_{i_{r-1} i_r} \circ \kappa_{i_r i_1} < \mathcal{I}_d \tag{4.2.7}$$

for all sequences $(i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$ *and* $r \in \{1, \ldots, N\}$.

Similar to (3.3.6), *the small-gain condition* (4.2.7) *implies the existence of* $\mathcal{K}_\infty$ *functions* $\sigma_i > 0$ *[Rüf10, Theorem 5.5], satisfying*

$$\max_{i,j} \left\{ \sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j \right\} < \mathcal{I}_d, \qquad i, j = \{1, \ldots, N\}. \tag{4.2.8}$$

In the next theorem, we leverage the max small-gain Assumption 4.2.7 together with the concavity assumption of $\max_i \sigma_i^{-1}$ to quantify the error between the interconnection of stochastic control subsystems and that of their finite abstractions in a compositional manner.

Theorem 4.2.8. *Consider the interconnected dt-SCS* $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *stochastic control subsystems* Σ_i. *Suppose that each* Σ_i *admits a finite abstraction* $\widehat{\Sigma}_i$ *together with a* max*-type SPSF* V_i. *If Assumption 4.2.7 holds and* $\max_i \sigma_i^{-1}$ *for* σ_i *as in* (4.2.8) *is concave, then the function* $V(x, \hat{x})$ *defined as*

$$V(x, \hat{x}) := \max_i \left\{ \sigma_i^{-1}(V_i(x_i, \hat{x}_i)) \right\}, \tag{4.2.9}$$

is a max*-type SSF from* $\widehat{\Sigma} = \widehat{\mathcal{I}}_{cs}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ *to* $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$.

Proof. We first show that the max-type SSF V in (4.2.9) satisfies the inequality (3.3.3) for some $\mathcal{K}_\infty$ function α. For any $x = [x_1; \ldots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, one gets

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\| &= \max_i \left\{ \|h_i^1(x_i) - \hat{h}_i^1(\hat{x}_i)\| \right\} \leq \max_i \left\{ \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \right\} \\ &\leq \hat{\beta}\ (\max_i \left\{ \sigma_i^{-1}(V_i(x_i, \hat{x}_i)) \right\}) = \hat{\beta}(V(x, \hat{x})),\end{aligned}$$

where $\hat{\beta}(s) = \max_i \left\{ \alpha_i^{-1} \circ \sigma_i(s) \right\}$ for all $s \in \mathbb{R}_{\geq 0}$, which is a $\mathcal{K}_\infty$ function and (3.3.3) holds with $\alpha = \hat{\beta}^{-1}$.

We continue with showing (3.3.4). Let $\kappa(s) = \max_{i,j}\{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j(s)\}$. It follows from (4.2.8) that $\kappa < \mathcal{I}_d$. Since $\max_i \sigma_i^{-1}$ is concave, one can readily get the chain of inequalities in (4.2.12) using Jensen's inequality, the inequality (4.2.6), and by defining $\rho_{\text{ext}}(\cdot)$, and ψ as

$$\begin{aligned}\rho_{\text{ext}}(s) &:= \begin{cases} \max_i\{\sigma_i^{-1} \circ \rho_{\text{ext}i}(s_i)\} \\ \text{s.t.} \quad s_i \geq 0, \ \|[s_1; \ldots; s_N]\| = s \end{cases} \\ \psi &:= \max_i \sigma_i^{-1}(\Lambda_i),\end{aligned} \tag{4.2.10}$$

where $\Lambda_i := (\mathcal{I}_d + \tilde{\delta}_f^{-1}) \circ (\rho_{\text{int}i} \circ \bar{\lambda} \circ (\bar{\lambda} - \mathcal{I}_d)^{-1}(\max_{j, j \neq i}\{\bar{\mu}_{ji}\}) + \psi_i)$. Hence, V is a max-type SSF from $\widehat{\Sigma}$ to Σ which completes the proof. □

Remark 4.2.9. *Note that to show Theorem 4.2.8, we have employed the following inequalities:*

$$\begin{cases} \rho_{\text{int}}(a+b) \le \rho_{\text{int}} \circ \bar{\lambda}(a) + \rho_{\text{int}} \circ \bar{\lambda} \circ (\bar{\lambda} - \mathcal{I}_d)^{-1}(b), \\ a+b \le \max\{(\mathcal{I}_d + \tilde{\delta}_f)(a), (\mathcal{I}_d + \tilde{\delta}_f^{-1})(b)\}, \end{cases}$$

for any $a, b \in \mathbb{R}_{\ge 0}$, where $\rho_{\text{int}}, \tilde{\delta}_f, \bar{\lambda}, (\bar{\lambda} - \mathcal{I}_d) \in \mathcal{K}_\infty$.

Remark 4.2.10. *If $\rho_{\text{int}i}, \forall i \in \{1, \ldots, N\}$ are linear, κ_{ij} and Λ_i reduce to, respectively, $\kappa_{ij} = (\mathcal{I}_d + \tilde{\delta}_f) \circ \rho_{\text{int}i} \circ \alpha_j^{-1}(s)$, and $\Lambda_i := (\mathcal{I}_d + \tilde{\delta}_f^{-1}) \circ (\rho_{\text{int}i} \circ (\max_{j, j \neq i}\{\bar{\mu}_{ji}\}) + \psi_i), \forall i \in \{1, \ldots, N\}, j \neq i$.*

Figure 4.1 schematically shows the results of Theorem 4.2.8.

4.2.1.3 Construction of max-type SPSF

4.2.1.3.1 General Setting of Nonlinear Stochastic Control Systems

In this subsection, we assume that the output map $h^i, i \in \{1,2\}$, satisfies the following general Lipschitz assumption: there exists an $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\|h^i(x) - h^i(x')\| \le \tilde{\alpha}(\|x - x'\|)$ for all $x, x' \in X$, and $i \in \{1,2\}$. Note that this assumption on h^i is not restrictive provided that h^i is continuous and one works on a compact subset of X. We impose conditions on the infinite dt-SCS Σ enabling us to find a max-type SPSF from its finite abstraction $\widehat{\Sigma}$, constructed as in the previous subsection, to Σ. The existence of the max-type SPSF is established under the assumption that the original model (or its reduced-order version) is *incrementally input-to-state stable* (δ-ISS) as in Definition 2.7.1. Moreover, we need to raise the following assumption.

Assumption 4.2.11. *Assume that there exists a function $\gamma \in \mathcal{K}_\infty$ such that*

$$V(x, x') - V(x, x'') \le \gamma(\|x' - x''\|), \quad \forall x, x', x'' \in X. \tag{4.2.11}$$

Remark 4.2.12. *As shown in [ZMEM+14] and by employing the mean value theorem, the inequality* (4.2.11) *is always satisfied for any differentiable function V restricted to a compact subset of $X \times X$. Note that if one chooses $V = ((x-x')^T \tilde{M}(x-x'))^{\frac{1}{2}}, \forall x, x' \in X$, then $\gamma(s) = \sqrt{\lambda_{\max}(\tilde{M})}s, \forall s \in \mathbb{R}_{\ge 0}$.*

Now we show that under this mild condition, the function V is indeed a max-type SPSF from $\widehat{\Sigma}$ to Σ.

Theorem 4.2.13. *Let Σ be an incrementally input-to-state stable dt-SCS via a function V as in Definition 2.7.1 and $\widehat{\Sigma}$ be its finite MDP constructed as in Algorithm 1. If Assumption 4.2.11 holds, then V is a max-type SPSF from $\widehat{\Sigma}$ to Σ.*

Proof. Given the Lipschitz assumption on h^i, since Σ is incrementally input-to-state stable, and from (2.7.1), $\forall x \in X$ and $\forall \hat{x} \in \hat{X}$, we get

$$\|h^i(x) - \hat{h}^i(\hat{x})\| \le \tilde{\alpha}(\|x - \hat{x}\|) \le \hat{\beta}(V(x, \hat{x})),$$

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,\varsigma),\hat{f}(\hat{x},\hat{\nu},\varsigma))\,\big|\,x,\hat{x},\hat{\nu}\Big]\\
&=\mathbb{E}\Big[\max_i\Big\{\sigma_i^{-1}(V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i)))\Big\}\,\big|\,x,\hat{x},\hat{\nu}\Big]\\
&\leq\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i))\,\big|\,x,\hat{x},\hat{\nu}\Big])\Big\}\\
&=\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(f_i(x_i,\nu_i,w_i,\varsigma_i),\hat{f}_i(\hat{x}_i,\hat{\nu}_i,\hat{w}_i,\varsigma_i))\,\big|\,x_i,\hat{x}_i,\hat{\nu}_i\Big])\Big\}\\
&\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\|w_i-\hat{w}_i\|),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|w_{ij}-\hat{w}_{ij}\|\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|y_{ji}^2-\hat{y}_{ji}^2+\hat{y}_{ji}^2-\Pi_{w_{ji}}(\hat{y}_{ji}^2)\|\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|y_{ji}^2-\hat{y}_{ji}^2\|+\|\hat{y}_{ji}-\Pi_{w_{ji}}(\hat{y}_{ji})\|\}),\\
&\qquad\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j(x_j,\hat{x}_j))+\bar{\mu}_{ji}\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}\circ\bar{\lambda}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j(x_j,\hat{x}_j))\})\\
&\qquad+\rho_{\mathrm{int}i}\circ\bar{\lambda}\circ(\bar{\lambda}-\mathcal{I}_d)^{-1}(\max_{j,j\neq i}\{\bar{\mu}_{ji}\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),(\mathcal{I}_d+\tilde{\delta}_f)\circ\rho_{\mathrm{int}i}\circ\bar{\lambda}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j(x_j,\hat{x}_j))\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\Lambda_i\})\Big\}\\
&=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}(V_j(x_j,\hat{x}_j)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\Lambda_i\})\Big\}\\
&=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j(x_j,\hat{x}_j)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\Lambda_i\})\Big\}\\
&\leq\max_{i,j,l}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_l^{-1}(V_l(x_l,\hat{x}_l)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\Lambda_i\})\Big\}\\
&=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j(V(x,\hat{x})),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\Lambda_i\})\Big\}\\
&=\max\Big\{\kappa(V(x,\hat{x})),\rho_{\mathrm{ext}}(\|\hat{\nu}\|),\psi\Big\}. \qquad (4.2.12)
\end{aligned}
$$

where $\hat{\beta}=\tilde{\alpha}\circ\underline{\alpha}^{-1}$, which satisfies (3.3.1) with $\alpha(s):=\hat{\beta}^{-1}(s)\ \forall s\in\mathbb{R}_{\geq 0}$. Now by taking the conditional expectation from (4.2.11), $\forall x\in X,\forall\hat{x}\in\hat{X},\forall\hat{\nu}\in\hat{U},\forall w\in W,\forall\hat{w}\in\hat{W}$,

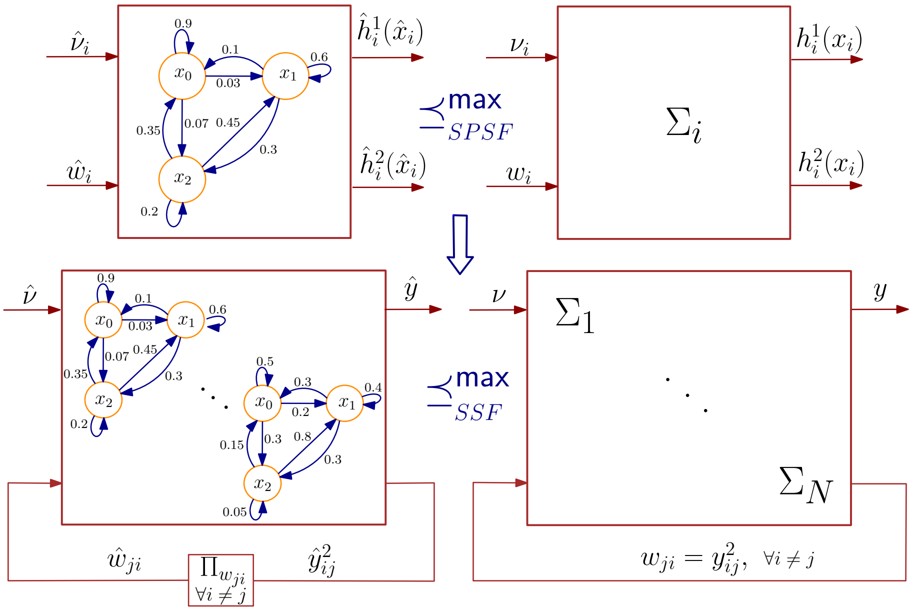

Figure 4.1: Compositionality results for constructing interconnected finite systems provided that the condition (4.2.7) is satisfied.

we have

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad-\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),f(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad\leq \mathbb{E}\Big[\gamma(\|\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma)-f(\hat{x},\hat{\nu},\hat{w},\varsigma)\|)\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big],
\end{aligned}$$

where $\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma)=\Pi_x(f(\hat{x},\hat{\nu},\hat{w},\varsigma))$. Using (4.2.5), the above inequality reduces to

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad-\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),f(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\leq\gamma(\bar{\delta}).
\end{aligned}$$

Employing (2.7.2), we get

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),f(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad\leq V(x,\hat{x})-\bar{\kappa}(V(x,\hat{x}))+\bar{\rho}_{\rm int}(\|w-\hat{w}\|).
\end{aligned}$$

It follows that $\forall x\in X,\forall\hat{x}\in\hat{X},\forall\hat{u}\in\hat{U}$, and $\forall w\in W,\forall\hat{w}\in\hat{W}$,

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]-V(x,\hat{x})\\
&\quad\leq-\bar{\kappa}(V(x,\hat{x}))+\bar{\rho}_{\rm int}(\|w-\hat{w}\|)+\gamma(\bar{\delta}).
\end{aligned}$$

$$\begin{bmatrix}(1+2/\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}E\\ * & (1+2/\pi)E^T\tilde{M}E\end{bmatrix} \preceq \begin{bmatrix}\hat{\kappa}\tilde{M} & -F^T\\ -F & \frac{2}{b}\end{bmatrix} \quad (4.2.15)$$

Using the previous inequality and by employing the similar argument as the one in [SGZ18, Theorem 1], one obtains

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,\hat{\nu},w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\ &\quad\leq \max\Big\{\tilde{\kappa}_f(V(x,\hat{x})),\tilde{\rho}_{\text{int}}(\|w-\hat{w}\|),\tilde{\gamma}(\bar{\delta})\Big\},\end{aligned} \quad (4.2.13)$$

where $\tilde{\kappa}_f = \mathcal{I}_d - (\mathcal{I}_d - \tilde{\pi}_f)\circ\underline{\kappa}$, $\tilde{\rho}_{\text{int}} = (\mathcal{I}_d + \tilde{\delta}_f)\circ\underline{\kappa}^{-1}\circ\tilde{\pi}_f^{-1}\circ\bar{\lambda}\circ\bar{\rho}_{\text{int}}$, $\tilde{\gamma} = (\mathcal{I}_d + \tilde{\delta}_f^{-1})\circ\underline{\kappa}^{-1}\circ\tilde{\pi}_f^{-1}\circ\bar{\lambda}\circ(\bar{\lambda}-\mathcal{I}_d)^{-1}\circ\gamma$ where $\tilde{\delta}_f,\tilde{\pi}_f,\bar{\lambda},\underline{\kappa}$ are some arbitrarily chosen $\mathcal{K}_\infty$ functions with $\mathcal{I}_d - \tilde{\pi}_f\in\mathcal{K}_\infty$, $\bar{\lambda}-\mathcal{I}_d\in\mathcal{K}_\infty$, $\mathcal{I}_d-\underline{\kappa}\in\mathcal{K}_\infty$, and $\underline{\kappa}\leq\bar{\kappa}$. Then the inequality (3.3.2) is satisfied with $\nu=\hat{\nu}$, $\kappa=\tilde{\kappa}_f$, $\rho_{int}=\tilde{\rho}_{int}$, and $\rho_{ext}\equiv 0$, and $\psi=\tilde{\gamma}(\bar{\delta})$. Hence, V is a max-type SPSF from $\widehat{\Sigma}$ to Σ. □

Now we provide similar results as in this subsection but tailored to the nonlinear class of stochastic control systems (3.3.10). We show that inequalities (2.7.1) and (2.7.2) for a candidate quadratic function V boil down to some matrix inequalities.

4.2.1.3.2 Stochastic Control Systems with Slope Restrictions on Nonlinearity

Here, we focus on Σ in (3.3.10) and propose an approach to construct its finite abstraction $\widehat{\Sigma}$ via a candidate quadratic function V as

$$V(x,\hat{x}) = (x-\hat{x})^T\tilde{M}(x-\hat{x}), \quad (4.2.14)$$

where $\tilde{M}$ is a positive-definite matrix of an appropriate dimension. In order to show that V in (4.2.14) is a max-type SPSF from $\widehat{\Sigma}$ to Σ, we require the following assumption on Σ.

Assumption 4.2.14. *Assume that for some constant $0<\hat{\kappa}<1$ and $\pi>0$, there exist matrices $\tilde{M}\succ 0$, and K of appropriate dimensions such that the inequality (4.2.15) holds.*

Now we provide another main result of this section showing under which conditions V in (4.2.14) is a max-type SPSF from $\widehat{\Sigma}$ to Σ.

Theorem 4.2.15. *Assume the system Σ satisfies Assumption 4.2.14 and $\hat{C}^i = C^i, i\in\{1,2\}$. Let $\widehat{\Sigma}$ be its finite abstraction with the state discretization parameter $\bar{\delta}$. Then function V defined in (4.2.14) is a max-type SPSF from $\widehat{\Sigma}$ to Σ.*

Proof. We first show that $\forall x$, $\forall \hat{x}$, $\forall \hat{\nu}$, $\exists \nu$, $\forall w$, and $\forall \hat{w}$, V satisfies $\frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\|C^i x - \hat{C}^i\hat{x}\|^2 \leq V(x,\hat{x})$ and then

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x=x(k),\hat{x}=\hat{x}(k),\hat{\nu}=\hat{\nu}(k),w=w(k),\hat{w}=\hat{w}(k)\Big]\\&\quad\leq \max\Big\{(1-(1-\tilde{\pi})\tilde{\kappa})(V(x,\hat{x})),(1+\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2,\\&\quad(1+1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(n(1+3\pi)\lambda_{\max}(\tilde{M}))\,\bar{\delta}^2\Big\}.\end{aligned}$$

Since $\hat{C}^i = C^i$, we have $\|C^i x - \hat{C}^i\hat{x}\|^2 \leq n\lambda_{\max}(C^{iT}C^i)\|x-\hat{x}\|^2$, and similarly $\lambda_{\min}(\tilde{M})\|x-\hat{x}\|^2 \leq (x-\hat{x})^T\tilde{M}(x-\hat{x})$. One can readily verify that $\frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\|C^i x-\hat{C}^i\hat{x}\|^2 \leq V(x,\hat{x})$ holds $\forall x$, $\forall \hat{x}$, implying that the inequality (3.3.1) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\,s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (3.3.2) holds, as well. Given any x, $\hat{x}$, and $\hat{\nu}$, we choose ν via the following interface function:

$$\nu = \nu_{\hat{\nu}}(x,\hat{x},\hat{\nu}) := K(x-\hat{x}) + \hat{\nu}. \tag{4.2.16}$$

By employing the definition of the interface function, we simplify

$$\begin{aligned}&Ax + E\varphi(Fx) + B\nu(x,\hat{x},\hat{\nu}) + Dw + R\varsigma\\&\qquad - \Pi_x(A\hat{x} + E\varphi(F\hat{x}) + B\hat{\nu} + D\hat{w} + R\varsigma)\end{aligned}$$

to

$$(A+BK)(x-\hat{x}) + D(w-\hat{w}) + E(\varphi(Fx)-\varphi(F\hat{x})) + \bar{N}, \tag{4.2.17}$$

where $\bar{N} = A\hat{x}+E\varphi(F\hat{x})+B\hat{\nu}+D\hat{w}+R\varsigma-\Pi_x(A\hat{x}+E\varphi(F\hat{x})+B\hat{\nu}+D\hat{w}+R\varsigma)$. From the slope restriction (3.3.11), one obtains

$$\varphi(Fx)-\varphi(F\hat{x}) = \underline{\delta}(Fx-F\hat{x}) = \underline{\delta}F(x-\hat{x}), \tag{4.2.18}$$

where $\underline{\delta}$ is a function of x and $\hat{x}$ and takes values in the interval $[0,b]$. Using (4.2.18), the expression in (4.2.17) reduces to

$$((A+BK)+\underline{\delta}EF)(x-\hat{x}) + D(w-\hat{w}) + \bar{N}.$$

Using Young's inequality [You12] as $cd \leq \frac{\pi}{2}c^2 + \frac{1}{2\pi}d^2$, for any $c,d \geq 0$ and any $\pi > 0$, by employing Cauchy-Schwarz inequality and (4.2.15), and since

$$\begin{cases}\|\bar{N}\| \leq \bar{\delta},\\ \bar{N}^T\tilde{M}\bar{N} \leq n\lambda_{\max}(M)\bar{\delta}^2,\end{cases}$$

one can obtain the chain of inequalities in (4.2.19). Hence, the proposed V in (4.2.14) is a max-type SPSF from $\widehat{\Sigma}$ to Σ, which completes the proof. Note that the last inequality in (4.2.19) is derived by applying Theorem 1 in [SGZ18]. The functions $\alpha,\kappa \in \mathcal{K}_\infty$, and $\rho_{\text{int}}, \rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ in Definition 3.3.1 associated with V in (4.2.14) are defined as $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{n\lambda_{\max}(C^{iT}C^i)}\,s^2$, $\kappa(s) := (1-(1-\tilde{\pi})\tilde{\kappa})\,s$, $\rho_{\text{int}}(s) := (1+\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\,s^2$, $\rho_{\text{ext}}(s) := 0$, $\forall s \in \mathbb{R}_{\geq 0}$ where $\tilde{\kappa} = 1-\hat{\kappa}$, $0 < \tilde{\pi} < 1$, and $\tilde{\delta} > 0$. Moreover, the positive constant ψ in (3.3.2) is $\psi = (1+1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(n(1+3\pi)\lambda_{\max}(\tilde{M}))\,\bar{\delta}^2$. □

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad=(x-\hat{x})^T\Big[((A+BK)+\underline{\delta}EF)^T\tilde{M}((A+BK)+\underline{\delta}EF)\Big](x-\hat{x})+2\Big[(x-\hat{x})^T((A+BK)\\
&\qquad+\underline{\delta}EF)^T\Big]\tilde{M}\Big[D(w-\hat{w})\Big]+2\Big[(x-\hat{x})^T((A+BK)+\underline{\delta}EF)^T\Big]\tilde{M}\mathbb{E}\Big[\bar{N}\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\qquad+2\Big[(w-\hat{w})^TD^T\Big]\tilde{M}\mathbb{E}\Big[\bar{N}\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]+(w-\hat{w})^TD^T\tilde{M}D(w-\hat{w})\\
&\qquad+\mathbb{E}\Big[\bar{N}^T\tilde{M}\bar{N}\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\
&\quad\le\begin{bmatrix}x-\hat{x}\\ \underline{\delta}F(x-\hat{x})\end{bmatrix}^T\begin{bmatrix}(1+2/\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}E\\ * & (1+2/\pi)E^T\tilde{M}E\end{bmatrix}\begin{bmatrix}x-\hat{x}\\ \underline{\delta}F(x-\hat{x})\end{bmatrix}\\
&\qquad+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2+n(1+3\pi)\lambda_{\max}(\tilde{M})\,\bar{\delta}^2\\
&\quad\le\begin{bmatrix}x-\hat{x}\\ \underline{\delta}F(x-\hat{x})\end{bmatrix}^T\begin{bmatrix}\hat{\kappa}\tilde{M} & -F^T\\ -F & \frac{2}{b}\end{bmatrix}\begin{bmatrix}x-\hat{x}\\ \underline{\delta}F(x-\hat{x})\end{bmatrix}+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2\\
&\qquad+n(1+3\pi)\lambda_{\max}(\tilde{M})\,\bar{\delta}^2\\
&\quad=\hat{\kappa}V(x,\hat{x})-2\underline{\delta}(1-\underline{\delta}/b)(x-\hat{x})^TF^TF(x-\hat{x})+\bar{p}(1+2\pi+1/\pi)\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2\\
&\qquad+n(1+3\pi)\lambda_{\max}(\tilde{M})\,\bar{\delta}^2\\
&\quad\le\hat{\kappa}V(x,\hat{x})+(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2+n(1+3\pi)\lambda_{\max}(\tilde{M})\,\bar{\delta}^2\\
&\quad\le\max\Big\{(1-(1-\tilde{\pi})\tilde{\kappa})(V(x,\hat{x})),(1+\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(\bar{p}(1+2\pi+1/\pi))\|\sqrt{\tilde{M}}D\|_2^2\|w-\hat{w}\|^2,\\
&\qquad(1+1/\tilde{\delta})(\frac{1}{\tilde{\kappa}\tilde{\pi}})(n(1+3\pi)\lambda_{\max}(\tilde{M}))\,\bar{\delta}^2\Big\}. \qquad (4.2.19)
\end{aligned}
$$

Next proposition establishes a so-called transitivity property for the computation of error bounds proposed in Theorem 3.2.7. This result is important especially when one first constructs a reduced-order model (an infinite abstraction) of an original stochastic system and then uses it to construct a finite MDP. The next proposition can provide the overall error bound in this two-step abstraction scheme. We refer the interested readers to Case study 4.2.1.4.1 for an application of this proposition.

Proposition 4.2.16. *Suppose Σ_1, Σ_2, and Σ_3 are three dt-SCS without internal signals. For any external input trajectories ν_1, ν_2, and ν_3 and for any random variables a_1, a_2, and a_3 as the initial states of the three dt-SCS, if*

$$
\mathbb{P}\left\{\sup_{0\le k\le T_d}\|y_{1a_1\nu_1}(k)-y_{2a_2\nu_2}(k)\|\ge\varepsilon_1\,\big|\,a_1,a_2\right\}\le\hat{\delta}_1,
$$

$$
\mathbb{P}\left\{\sup_{0\le k\le T_d}\|y_{2a_2\nu_2}(k)-y_{3a_3\nu_3}(k)\|\ge\varepsilon_2\,\big|\,a_2,a_3\right\}\le\hat{\delta}_2,
$$

for some $\varepsilon_1, \varepsilon_2 > 0$ *and* $\hat{\delta}_1, \hat{\delta}_2 \in]0\ 1[$*, then the probabilistic mismatch between output trajectories of* Σ_1 *and* Σ_3 *is quantified as*

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{1a_1\nu_1}(k) - y_{3a_3\nu_3}(k)\| \geq \varepsilon_1 + \varepsilon_2 \,\middle|\, a_1, a_2, a_3\right\} \leq \hat{\delta}_1 + \hat{\delta}_2.$$

Proof. By defining

$$\begin{aligned}
\mathcal{A} &= \{\|y_{1a_1\nu_1}(k) - y_{2a_2\nu_2}(k)\| < \varepsilon_1 \,|\, a_1, a_2, a_3\},\\
\mathcal{B} &= \{\|y_{2a_2\nu_2}(k) - y_{3a_3\nu_3}(k)\| < \varepsilon_2 \,|\, a_1, a_2, a_3\},\\
\mathcal{C} &= \{\|y_{1a_1\nu_1}(k) - y_{3a_3\nu_3}(k)\| < \varepsilon_1 + \varepsilon_2 \,|\, a_1, a_2, a_3\},
\end{aligned}$$

we have $\mathbb{P}\{\bar{\mathcal{A}}\} \leq \hat{\delta}_1$ and $\mathbb{P}\{\bar{\mathcal{B}}\} \leq \hat{\delta}_2$, where $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are the complement of $\mathcal{A}$ and $\mathcal{B}$, respectively. Since $\mathbb{P}\{\mathcal{A} \cap \mathcal{B}\} \leq \mathbb{P}\{\mathcal{C}\}$, we have

$$\mathbb{P}\{\bar{\mathcal{C}}\} \leq \mathbb{P}\{\bar{\mathcal{A}} \cup \bar{\mathcal{B}}\} \leq \mathbb{P}\{\bar{\mathcal{A}}\} + \mathbb{P}\{\bar{\mathcal{B}}\} \leq \hat{\delta}_1 + \hat{\delta}_2.$$

Then

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{1a_1\nu_1}(k) - y_{3a_3\nu_3}(k)\| \geq \varepsilon_1 + \varepsilon_2 \,\middle|\, a_1, a_2, a_3\right\} \leq \hat{\delta}_1 + \hat{\delta}_2.$$

□

4.2.1.4 Case Studies

Here we first apply our provided techniques to a fully interconnected network of 20 nonlinear subsystems (totally 100 dimensions) as depicted in Figure 4.2 right, and construct finite MDPs from their *reduced-order versions* (together 20 dimensions) with guaranteed probabilistic error bounds on their output trajectories. We then apply our proposed approaches to a temperature regulation in a circular building (cf. Figure 4.2 left) and construct compositionally a finite abstraction of the network containing 1000 rooms. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies regulating the temperature in each room for a bounded time horizon.

4.2.1.4.1 Nonlinear Fully Interconnected Network

In order to show the applicability of our approach to strongly interconnected networks with nonlinear dynamics, we consider nonlinear dt-SCS defined in (3.4.21). We assume $\tilde{L}$ is the Laplacian matrix of a complete graph as in (3.4.24) and $\tau = 0.001$. Moreover, $R = \mathsf{diag}(\mathbb{1}_{n_1}, \ldots, \mathbb{1}_{n_N})$, $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_N(k)]$, $\varphi(x) = [\mathbb{1}_{n_1}\varphi_1(F_1x_1(k)); \ldots; \mathbb{1}_{n_N}\varphi_N(F_Nx_N(k))]$ where $n = \sum_{i=1}^{N} n_i$, $\varphi_i(x) = sin(x)$, and $F_i^T = \begin{bmatrix}0.1 & 0 & \cdots & 0\end{bmatrix}^T \in \mathbb{R}^{n_i}$ $\forall i \in \{1, \ldots, N\}$. We partition $x(k)$ as $x(k) = [x_1(k); \ldots; x_N(k)]$ and $\nu(k)$ as $\nu(k) = [\nu_1(k); \ldots; \nu_N(k)]$, where $x_i(k), \nu_i(k) \in \mathbb{R}^{n_i}$. Now, we introduce Σ_i as

$$\Sigma_i : \begin{cases} x_i(k+1) = A_ix_i(k) + \mathbb{1}_{n_i}\varphi_i(F_ix_i(k)) + \nu_i(k) + D_iw_i(k) + \mathbb{1}_{n_i}\varsigma_i(k),\\ y_i^1(k) = x_i(k),\\ y_i^2(k) = x_i(k), \end{cases}$$

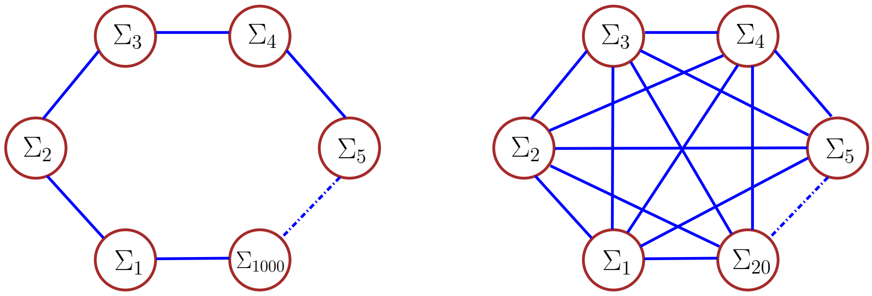

Figure 4.2: Left: A circular building in a network of 1000 rooms. Right: A fully interconnected network of 20 nonlinear components (totally 100 dimensions).

where $A_i = (\mathbb{I}_{n_i} - \tau \tilde{L}_i)$ with $\tilde{L}_i$ as in (3.4.24), $w_i(k) = [y_{1i}; \ldots; y_{(i-1)i}; y_{(i+1)i}; \ldots; y_{Ni}]$, $i \in \{1, \ldots, N\}$, and

$$D_i = -\tau \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & \cdots & -1 & -1 \end{bmatrix}_{n_i \times (n-n_i)}, \qquad \forall i \in \{1, \ldots, N\}.$$

We fix $N = 20$, $n = 100$, $n_i = 5$, $\forall i \in \{1, \ldots, N\}$. Then one can readily verify that $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \ldots, \Sigma_N)$. Our goal is to first aggregate each x_i into a scalar-valued $\hat{x}_{\mathsf{r}i}$ (index r signifies the reduced-order version of the original model), governed by $\widehat{\Sigma}_{\mathsf{r}i}$, which satisfies:

$$\widehat{\Sigma}_{\mathsf{r}i} : \begin{cases} \hat{x}_{\mathsf{r}i}(k+1) = 0.5\hat{x}_{\mathsf{r}i}(k) + 0.1\varphi_i(0.1\hat{x}_{\mathsf{r}i}(k)) + \hat{\nu}_{\mathsf{r}i}(k) + \hat{D}_i \hat{w}_{\mathsf{r}i}(k) + \varsigma_i(k), \\ \hat{y}^1_{\mathsf{r}i}(k) = \hat{C}^1_i \hat{x}_{\mathsf{r}i}(k), \\ \hat{y}^2_{\mathsf{r}i}(k) = \hat{C}^2_i \hat{x}_{\mathsf{r}i}(k), \end{cases}$$

where $\hat{D}_i = 0.001\mathbb{1}^T_{95}$, $\hat{C}^{\bar{i}}_i = \mathbb{1}_5, \bar{i} \in \{1, 2\}$, and $\hat{w}_{\mathsf{r}i}(k) \in \mathbb{R}^{95}$. One can readily verify that, for any $i \in \{1, \ldots, N\}$, the condition (3.3.12) is satisfied with $\tilde{M}_i = \mathbb{I}_5$, $\hat{\kappa}_i = 0.003$, $\pi_i = 1$, $\tilde{P}_i = \mathbb{1}_5$, $L_{1i} = -\mathbb{1}_5$, $\tilde{R}_i = \mathbb{1}_5$, $b_i = 1$, and K_i as a 5×5 matrix with diagonal elements -0.9, and off-diagonals -0.001. Moreover, for any $i \in \{1, \ldots, N\}$, conditions (3.3.13) are satisfied by $L_{2i} = -0.1\mathbb{1}_5$, $Q_i = -0.4\mathbb{1}_5$, and $S_i = \mathbf{0}_{5\times 95}$. We fix the max-type SPSF as in (3.2.21). By taking $\tilde{\pi}_i = 0.99$, $\tilde{\kappa}_i = 0.99$ and $\tilde{\delta}_i = 0.1$, $\forall i \in \{1, \ldots, N\}$, one can verify that $V_i(x_i, \hat{x}_{\mathsf{r}i}) = (x_i - \mathbb{1}_5\hat{x}_{\mathsf{r}i})^T \mathbb{I}_5 (x_i - \mathbb{1}_5\hat{x}_{\mathsf{r}i})$ is a max-type SPSF from $\widehat{\Sigma}_{\mathsf{r}i}$ to Σ_i satisfying condition (3.3.1) with $\alpha_i(s) = 1/5s^2$ and the condition (3.3.2) with $\kappa_i(s) = 0.99s$, $\rho_{\mathrm{int}i}(s) = 0.2s^2$, $\rho_{\mathrm{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 0$, where the input ν_i is given via the interface function in (3.3.14) as

$$\nu_i = -K_i(x_i - \mathbb{1}_5\hat{x}_{\mathsf{r}i}) - 0.4\mathbb{1}_5\hat{x}_{\mathsf{r}i} + \mathbb{1}_5\hat{\nu}_{\mathsf{r}i} - \mathbb{1}_5\varphi_i(F_i x_i) + 0.1\mathbb{1}_5\varphi_i(F_i\mathbb{1}_5\hat{x}_{\mathsf{r}i}).$$

By taking $\sigma_i(s) = s$ $\forall i \in \{1, \ldots, N\}$, one can readily verify that the max small-gain condition (3.3.5) and as a result condition (3.3.6) are satisfied. Hence, $V(x, \hat{x}_{\mathsf{r}}) = \max_i (x_i - \mathbb{1}_5\hat{x}_{\mathsf{r}i})^T \mathbb{I}_5 (x_i - \mathbb{1}_5\hat{x}_{\mathsf{r}i})$ is a max-type SSF from $\widehat{\Sigma}_{\mathsf{r}} = \mathcal{I}_{cs}(\widehat{\Sigma}_{\mathsf{r}1}, \ldots, \widehat{\Sigma}_{\mathsf{r}N})$ to Σ

satisfying conditions (3.3.3) and (3.3.4) with $\alpha(s) = 1/25s^2$, $\kappa(s) = 0.99\,s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 0$.

Now we proceed with finding a max-type SPSF from the finite MDP $\widehat{\Sigma}_i$ to the reduced-order model $\widehat{\Sigma}_{\mathsf{r}i}$. One can readily verify that, for any $i \in \{1, \dots, N\}$, the condition (4.2.15) is satisfied with $\tilde{M}_i = 1$, $\hat{\kappa}_i = 0.009$, $\pi_i = 1$, $K_i = -0.49$, and $b_i = 1$. By taking $\tilde{\pi}_i = 0.99$, $\tilde{\kappa}_i = 0.99$ and $\tilde{\delta}_i = 0.9$ $\forall i \in \{1, \dots, N\}$, the function $V_i(\hat{x}_{\mathsf{r}i}, \hat{x}_i) = (\hat{x}_{\mathsf{r}i} - \hat{x}_i)^2$ is a max-type SPSF from $\widehat{\Sigma}_i$ to $\widehat{\Sigma}_{\mathsf{r}i}$ satisfying the condition (3.3.1) with $\alpha_i(s) = 1/5s^2$ and the condition (3.3.2) with $\kappa_i(s) = 0.99s$, $\rho_{\text{int}i}(s) = 0.26s^2$, $\rho_{\text{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 8.42\bar{\delta}^2$, where the input ν_i is given via the interface function in (4.2.16) as

$$\hat{\nu}_{\mathsf{r}i} = -\,0.49(\hat{x}_{\mathsf{r}i} - \hat{x}_i) + \hat{\nu}_i.$$

By taking $\sigma_i(s) = s$ $\forall i \in \{1, \dots, N\}$, one can readily verify that the small-gain condition (4.2.7) and as a result the condition (4.2.8) are satisfied. Hence, $V(\hat{x}_{\mathsf{r}}, \hat{x}) = \max_i (\hat{x}_{\mathsf{r}i} - \hat{x}_i)^2$ is a max-type SSF from $\widehat{\Sigma} = \widehat{\mathcal{I}}_{cs}(\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_N)$ to $\widehat{\Sigma}_{\mathsf{r}}$ satisfying conditions (3.3.3) and (3.3.4) with $\alpha(s) = 1/25s^2$, $\kappa(s) = 0.99\,s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 8.42\bar{\delta}^2$.

By taking the state set discretization parameter $\bar{\delta} = 0.001$, and starting the initial states of the interconnected systems Σ from $\mathbf{0}_{100}$, $\widehat{\Sigma}_{\mathsf{r}}$ and $\widehat{\Sigma}$ from $\mathbf{0}_{20}$, and using Theorem 3.2.7 and Proposition 4.2.16, we guarantee that the mismatch between outputs of Σ and $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.5, (\varepsilon_1 = \varepsilon_2 = 0.25)$, during the time horizon $T_d = 100$ with the probability at least 92%, , i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 0.5,\ \forall k \in [0, 100]) \geq 0.92\,.$$

Note that for the construction of finite abstractions, we have selected the center of partition sets as representative points. Moreover, we assume $\hat{Y}_{ij}^2 = \hat{W}_{ji}$, i.e., the overall error in (4.2.10) reduces to $\psi = \max_i \sigma_i^{-1}(\psi_i)$.

In Figure 4.3 which is in the logarithmic scale, we have fixed $\bar{\delta} = 0.001$ and plotted the error (the upper bound of the probability in (3.2.5)) as a function of the number of subsystems N and the confidence bound ε. As seen, ψ in (3.2.5) is independent of the size of the network, and is computed only based on the *maximum* of ψ_i of subsystems instead of being a *linear combination* of them which is the case in the classic small-gain approach. Hence, by increasing the number of subsystems, the error does not change.

4.2.1.4.2 Room Temperature Network

Consider a network of $n \geq 3$ rooms each equipped with a heater and connected circularly (cf. Figure 4.2 left). The model of this case study is adapted from [MGW17] by including the stochasticity in the model as the additive noise. The evolution of temperatures $\tilde{T}$ can be described by the interconnected linear dt-SCS

$$\Sigma : \begin{cases} \tilde{T}(k+1) = \bar{A}\tilde{T}(k) + \bar{\theta}\tilde{T}_h\nu(k) + \beta\tilde{T}_E + \varsigma(k), \\ y(k) = \tilde{T}(k), \end{cases} \tag{4.2.20}$$

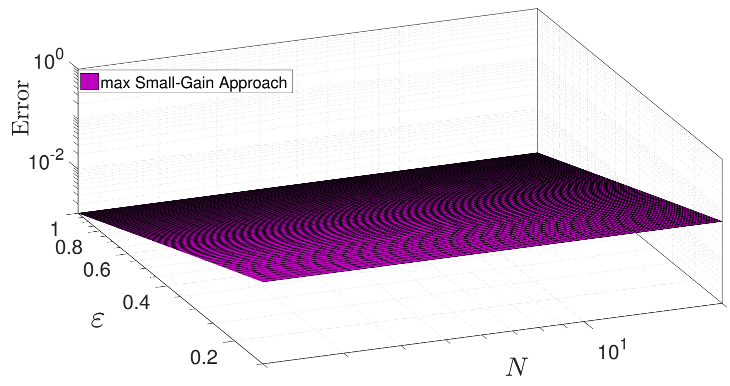

Figure 4.3: Fully interconnected network: Error bound in (3.2.5) provided by our approach based on max small-gain conditions. Plot is in the logarithmic scale for a fixed $\bar{\delta} = 0.001$ and $T_d = 100$. By increasing the number of subsystems, the error provided in (3.2.5) does not change since the overall ψ is independent of the size of the network (i.e., N), and is computed only based on the maximum ψ_i of subsystems instead of being a linear combination of them which is the case in the classic small-gain approach.

where $\bar{A}$ is a matrix with diagonal elements $\bar{a}_{ii} = (1 - 2\eta - \beta - \bar{\theta}\nu_i(k))$, $i \in \{1, \dots, n\}$, off-diagonal elements $\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = \bar{a}_{1,n} = \bar{a}_{n,1} = \eta$, $i \in \{1, \dots, n-1\}$, and all other elements are identically zero. Parameters η, β, and $\bar{\theta}$ are conduction factors, respectively, between rooms $i \pm 1$ and the room i, between the external environment and the room i, and between the heater and the room i. Moreover, $\tilde{T}(k) = [\tilde{T}_1(k); \dots; \tilde{T}_n(k)]$, $\nu(k) = [\nu_1(k); \dots; \nu_n(k)]$, $\varsigma(k) = [\varsigma_1(k); \dots; \varsigma_n(k)]$, $\tilde{T}_E = [\tilde{T}_{e1}; \dots; \tilde{T}_{en}]$, where $\tilde{T}_i(k)$ and $\nu_i(k)$ are taking values in sets $[19, 21]$ and $[0, 0.6]$, respectively, for all $i \in \{1, \dots, n\}$. Outside temperatures are the same for all rooms: $\tilde{T}_{ei} = -1\,°C$, $\forall i \in \{1, \dots, n\}$, and the heater temperature $\tilde{T}_h = 50\,°C$. Let us consider the individual rooms as Σ_i described as

$$\Sigma_i : \begin{cases} \tilde{T}_i(k+1) = A_i \tilde{T}_i(k) + \bar{\theta}\tilde{T}_h \nu_i(k) + D_i w_i(k) + \beta \tilde{T}_{ei} + \varsigma_i(k), \\ y_i^1(k) = \tilde{T}_i(k), \\ y_i^2(k) = \tilde{T}_i(k), \end{cases} \tag{4.2.21}$$

where $A_i = \bar{a}_{ii}$, $i \in \{1, \dots, n\}$. One can readily verify that $\Sigma = \mathcal{I}_{cs}(\Sigma_1, \dots, \Sigma_N)$ where $D_i = [\eta; \eta]^T$, and $w_i(k) = [y_{i-1}^2(k); y_{i+1}^2(k)]$ (with $y_0^2 = y_n^2$ and $y_{n+1}^2 = y_1^2$). Note that since the dynamics of each room is scaler (no need to reduce the order), our objective here is just to construct the finite abstraction of each room. First, we fix a max-type SPSF as in (4.2.14). Since the dynamics of the system is linear, the condition (4.2.15) reduces to

$$(1 + 2/\pi_i)(A_i + B_i K_i)^T \tilde{M}_i (A_i + B_i K_i) \preceq \hat{\kappa}_i \tilde{M}_i,$$

which is nothing more than the stabilizability of the temperature dynamic in the room i. One can verify that this condition is satisfied with $\tilde{M}_i = 1$, $K_i = 0$, $\pi_i = 1$, $\hat{\kappa}_i = 0.48$

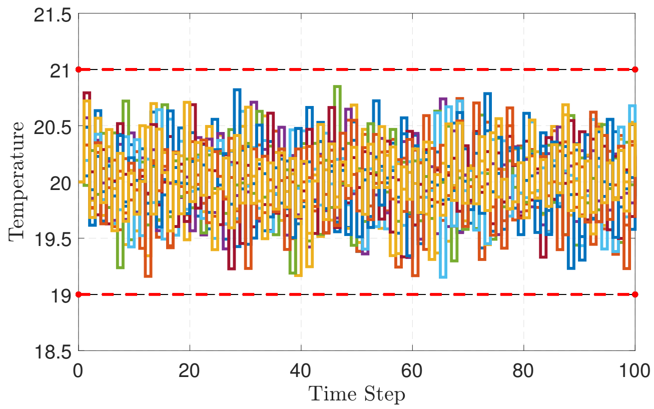

Figure 4.4: Closed loop state trajectories of a representative room with different noise realizations in a network of 1000 rooms.

$\forall i \in \{1,\ldots,n\}$, and $\eta = 0.1, \beta = 0.4, \bar{\theta} = 0.5$. Then function $V_i(\tilde{T}_i, \hat{\tilde{T}}_i) = (\tilde{T}_i - \hat{\tilde{T}}_i)^2$ is a max-type SPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (3.3.1) with $\alpha_i(s) = s^2$ and the condition (3.3.2) with $\kappa_i(s) = 0.99s$, $\rho_{\text{int}i}(s) = 0.91s^2$, $\rho_{\text{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 7.6\,\bar{\delta}_i^2$.

Now we check small-gain condition (4.2.7) that is required for the compositionality result. By taking $\sigma_i(s) = s$, $\forall i \in \{1,\ldots,n\}$, condition (4.2.7) and as a result condition (4.2.8) are always satisfied without any restriction on the number of rooms. Hence, $V(\tilde{T}, \hat{\tilde{T}}) = \max_i(\tilde{T}_i - \hat{\tilde{T}}_i)^2$ is a max-type SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (3.3.3) and (3.3.4) with $\alpha(s) = s^2$, $\kappa(s) = 0.99\,s$, $\rho_{\text{ext}}(s) = 0$, and $\psi = 7.6\,\bar{\delta}^2$.

We fix $n = 1000$ and set the state discretization parameter $\bar{\delta} = 0.005$. The initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are selected as $20\mathbb{1}_{1000}$. Using Theorem 3.2.7, we guarantee that the distance between outputs of Σ and $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.5$ during the time horizon $T_d = 100$ with probability at least 98%, i.e.,

$$\mathbb{P}(\|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 0.5,\ \forall k \in [0, 100]) \geq 0.98\,. \tag{4.2.22}$$

Let us now synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ such that the controller maintains the temperature of any room in the comfort zone $[19, 21]$. We design a local controller for the abstract subsystem $\widehat{\Sigma}_i$, and then refine it back to the subsystem Σ_i using the interface function. We employ the tool FAUST^2 [SGA15] to synthesize controllers for Σ_i by taking the external input discretization parameter as 0.04 and standard deviation of the noise as 0.21, $\forall i \in \{1,\ldots,n\}$. Closed-loop state trajectories of a representative room with different noise realizations are illustrated in Figure 4.4.

Similarly, we have fixed $\bar{\delta} = 0.005$ and plotted in Figure 4.5 the error between the finite MDP $\widehat{\Sigma}$ and the concrete model Σ as a function of the number of subsystems N and the confidence bound ε. As seen, by increasing the number of subsystems, the error does not change since ψ in (3.2.5) is independent of the size of the network.

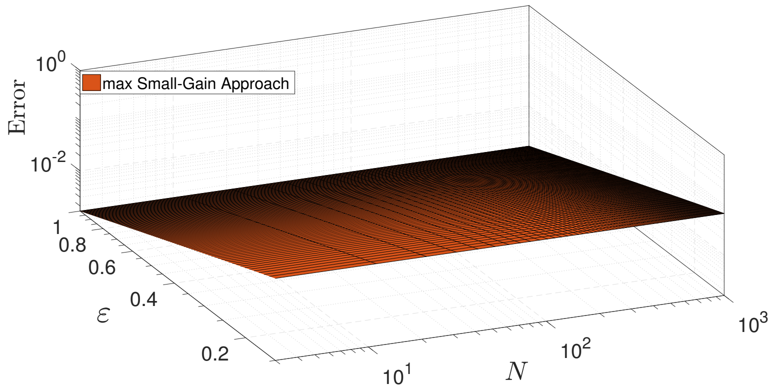

Figure 4.5: Temperature control: Error bound in (3.2.5) provided by our approach based on max small-gain conditions. Plot is in the logarithmic scale for a fixed $\bar{\delta} = 0.005$, and $T_d = 100$. By increasing the number of subsystems, the error provided in (3.2.5) does not change since the overall ψ is independent of the size of the network (i.e., N), and is computed only based on the maximum ψ_i of subsystems instead of being a linear combination of them which is the case in the classic small-gain approach.

4.2.2 Stochastic Switched Systems

In this section, we extend the results of the precious section to stochastic switched systems whose switch signals accept dwell-time with multiple Lyapunov functions. We show that under standard assumptions ensuring the incremental input-to-state stability of switched systems (i.e., existence of common incremental ISS Lyapunov functions, or multiple incremental ISS Lyapunov functions with dwell-time), one can construct finite MDPs for the general setting of nonlinear stochastic switched systems. To demonstrate the effectiveness of our proposed results, we first apply our approaches to a road traffic network in a circular cascade ring composed of 200 cells, and construct compositionally a finite MDP of the network. We employ the constructed finite abstractions as substitutes to compositionally synthesize policies keeping the density of the traffic lower than 20 vehicles per cell. We benchmark our proposed results against the ones available in the literature. We also provide some discussions on the memory usage and computation time in the construction of finite MDPs for this case study in both monolithic and compositional manners, and compare the results in a table for different ranges of the state discretization parameter. We show that the proposed compositional approach in this work remarkably reduces the curse of dimensionality problem in constructing finite MDPs. We then apply our proposed techniques to a *fully interconnected* network of 500 nonlinear subsystems (totally 1000 dimensions), and construct their finite MDPs with guaranteed error bounds. We provide simulation results for this case study to have more practical analysis on the proposed probabilistic bounds.

We should emphasize that extending the previous results from control systems to switched ones is very challenging. We first need to provide an augmented framework for presenting each switched system with different modes with a single system covering all modes (called global MDP in this chapter) whose external output trajectories are exactly the same as those of original switched systems. We then continue with the new global MDP to construct its finite abstraction and provide a probabilistic closeness between the two systems. Moreover, the definition of simulation functions for switched systems needs to be developed in order to encode the effect of discrete switching signals instead of continuous inputs. We also need to define the dwell-time condition for switched systems accepting multiple δ-ISS Lyapunov functions, and provide corresponding results for this class of systems.

Consider the stochastic switched systems defined in Definition 2.6.1. We assume that the signal $\mathbf{p}$ satisfies a *dwell-time* condition [Mor96] as defined in the next definition.

Definition 4.2.17. *Consider a switching signal* $\mathbf{p} : \mathbb{N} \rightarrow P$ *and define its switching time instants as*

$$\mathfrak{S}_{\mathbf{p}} := \{\mathfrak{s}_k \colon k \in \mathbb{N}_{\geq 1}\}.$$

Then $\mathbf{p} : \mathbb{N} \rightarrow P$ *has* dwell-time $k_d \in \mathbb{N}$ *[Mor96] if elements of* $\mathfrak{S}_{\mathbf{p}}$ *ordered as* $\mathfrak{s}_1 \leq \mathfrak{s}_2 \leq \mathfrak{s}_3 \leq \ldots$ *satisfy* $\mathfrak{s}_1 \geq k_d$ *and* $\mathfrak{s}_{k+1} - \mathfrak{s}_k \geq k_d, \forall k \in \mathbb{N}_{\geq 1}$.

Remark 4.2.18. *Note that the dwell-time in our setting is deterministic and always respected by the controller designed using the finite MDP. More precisely, switching signals in this work are control inputs and the main goal is to synthesize them with a specific dwell-time such that outputs of original systems satisfy some high-level specifications such as safety, reachability, etc. (cf. Case study 4.2.2.6.1). In existing works with a stochastic dwell-time (e.g., [BDS05], [XLSM13]), switching signals are not control inputs and are randomly changing in an adversarial manner.*

For any $p \in P$, we use Σ_p to refer to the system (2.6.2) with the constant switching signal $\mathbf{p}(k) = p$ for all $k \in \mathbb{N}$. We assume that the output map $h^i, i \in \{1, 2\}$, satisfies the following general assumption: there exists an $\mathscr{L} \in \mathcal{K}_\infty$ such that $\|h^i(x) - h^i(x')\| \leq \mathscr{L}(\|x - x'\|)$ for all $x, x' \in X$.

Remark 4.2.19. *Note that our assumption on* $h^i, i \in \{1, 2\}$, *with* $\mathscr{L} \in \mathcal{K}_\infty$ *is more general than the standard Lipschitz condition in which* $\mathscr{L}$ *is a linear function (i.e.,* $\mathscr{L}(\alpha) = L\alpha$, *for some nonnegative* L*). Moreover, this assumption on* $h^i, i \in \{1, 2\}$ *is not restrictive provided that* $h^i, i \in \{1, 2\}$ *are continuous and one works on a compact subset of* X*. More precisely, all uniformly continuous functions automatically satisfy this assumption [Ran03].*

Given the dt-SS in (2.6.1), we are interested in *Markov policies* similar to Definition 2.5.1 but for switched systems defined as follows.

Definition 4.2.20. *A Markov policy for the dt-SS* Σ *in* (2.6.1) *is a sequence* $\bar{\rho} = (\bar{\rho}_0, \bar{\rho}_1, \bar{\rho}_2, \ldots)$ *of universally measurable stochastic kernels* $\bar{\rho}_n$ *[BS96], each defined on* $P = \{1, \ldots, m\}$*, given* $X \times W$*. The class of all such Markov policies is denoted by* $\bar{\Pi}_M$*.*

Since we are interested in studying interconnected dt-SS without internal signals, the interconnected dt-SS is indicated by the simplified tuple $(X, P, \mathcal{P}, \varsigma, \mathbf{F}, Y, h)$ with $f_p : X \times \mathcal{V}_\varsigma \rightarrow X, \forall p \in P$.

4.2.2.1 Global Markov Decision Processes

In this subsection, we consider $\Sigma_p, \forall p \in P$, as *local* MDPs and introduce the notion of *global* Markov decision process as in the next definition. Note that this notion is adapted from the definition of labeled transition systems defined in [BK08] and modified to capture the stochastic nature of the system. This provides an alternative description of switched systems enabling us to represent a switched system and its finite MDP in a common framework.

Definition 4.2.21. *Given a dt-SS* $\Sigma = (X, P, \mathcal{P}, W, \varsigma, \mathbf{F}, Y^1, Y^2, h^1, h^2)$*, we define the associated global MDP* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$*, where:*

- $\mathbb{X} = X \times P \times \{0, \ldots, k_d - 1\}$ *is the set of states. A state* $(x, p, l) \in \mathbb{X}$ *means that the current state of* Σ *is* x*, the current value of the switching signal is* p*, and the time elapsed since the latest switching time instant saturated by* k_d *is* l*;*
- $\mathbb{U} = P$ *is the set of* external *inputs;*
- $\mathbb{W} = W$ *is the set of* internal *inputs;*
- ς *is a sequence of i.i.d. random variables;*
- $\mathbb{F} : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \times \mathcal{V}_\varsigma \to \mathbb{X}$ *is the one-step transition function given by* $(x', p', l') = \mathbb{F}((x, p, l), \nu, w, \varsigma)$ *if and only if* $x' = f_p(x, w, \varsigma)$*,* $\nu = p$ *and the following scenarios hold:*
  - $l < k_d - 1, p' = p$*, and* $l' = l + 1$*: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell-time;*
  - $l = k_d - 1, p' = p$*, and* $l' = k_d - 1$*: switching is allowed but no switch occurs;*
  - $l = k_d - 1, p' \neq p$*, and* $l' = 0$*: switching is allowed and a switch occurs;*
- $\mathbb{Y}^1 = Y^1$ *is the* external *output set;*
- $\mathbb{Y}^2 = Y^2$ *is the* internal *output set;*
- $\mathbb{H}^1 : \mathbb{X} \to \mathbb{Y}^1$ *is the* external *output map defined as* $\mathbb{H}^1(x, p, l) = h^1(x)$*;*
- $\mathbb{H}^2 : \mathbb{X} \to \mathbb{Y}^2$ *is the* internal *output map defined as* $\mathbb{H}^2(x, p, l) = h^2(x)$*.*

We associate respectively to $\mathbb{U}$ and $\mathbb{W}$ the sets $\mathcal{U}$ and $\mathcal{W}$ to be collections of sequences $\{\nu(k) : \Omega \to \mathbb{U},\ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to \mathbb{W},\ k \in \mathbb{N}\}$, in which $\nu(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. We also denote the initial conditions of p and l by p_0 and $l_0 = 0$.

Remark 4.2.22. *Note that in the global MDP* $\mathbb{G}(\Sigma)$ *in Definition 4.2.21, we added two additional variables* p *and* l *to the state tuple of the system* Σ*, in which* l *is a counter that depending on its value allows or prevents the system from switching, and* p *acts as a memory to record the input.*

Proposition 4.2.23. *Global MDP $\mathbb{G}(\Sigma)$ in Definition 4.2.21 is itself an MDP and the output trajectory of Σ defined in* (2.6.2) *can be uniquely mapped to an output trajectory of $\mathbb{G}(\Sigma)$ and vice versa.*

Proof. In order to show that the global MDP $\mathbb{G}(\Sigma)$ in Definition 4.2.21 is itself an MDP, we need to elaborate on this issue that $\mathbb{X}$ is itself a Borel space. Since X defined in (2.6.1) is a Borel space, one can readily verify that its Cartesian product by other discrete spaces as $\mathbb{X} = X \times P \times \{0, \dots, k_d - 1\}$ is also a Borel space [APLS08]. Then the global MDP $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$ can be *equivalently* represented as an MDP

$$\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \mathbb{T}_{\mathsf{x}}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2),$$

where the map $\mathbb{T}_{\mathsf{x}} : \mathcal{B}(\mathbb{X}) \times \mathbb{X} \times \mathbb{U} \times \mathbb{W} \rightarrow [0, 1]$, is a conditional stochastic kernel that assigns to any $x \in \mathbb{X}$, $\nu \in \mathbb{U}$, and $w \in \mathbb{W}$ a probability measure $\mathbb{T}_{\mathsf{x}}(\cdot \,|\, x, \nu, w)$ on the measurable space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ so that for any set $\mathcal{A} \in \mathcal{B}(\mathbb{X})$,

$$\mathbb{P}(x(k+1) \in \mathcal{A} \,|\, x(k), \nu(k), w(k)) = \int_{\mathcal{A}} \mathbb{T}_{\mathsf{x}}(d(k+1) \,|\, x(k), \nu(k), w(k)).$$

Moreover,

$$(p', l') := \begin{cases} (p, l+1), & \text{if} \quad l < k_d - 1, \\ (p, k_d - 1), & \text{if} \quad l = k_d - 1, \\ (\neq p, 0), & \text{if} \quad l = k_d - 1, \end{cases}$$

or equivalently,

$$\nu := \begin{cases} \text{no switch}, & \text{if} \quad l < k_d - 1, \\ \{1, 2, \dots, m\}, & \text{if} \quad l = k_d - 1. \end{cases}$$

Then the global MDP $\mathbb{G}(\Sigma)$ in Definition 4.2.21 is itself an MDP. Now we elaborate on the fact that output trajectories of Σ defined in (2.6.2) and of $\mathbb{G}(\Sigma)$ are equivalent. Given an initial state x_0, a switching signal $\mathbf{p} : \mathbb{N} \rightarrow P$, an internal input $w(\cdot)$, and a realization of the noise $\varsigma(\cdot)$, one can uniquely map the output trajectory of Σ to an output trajectory of $\mathbb{G}(\Sigma)$. Moreover, if we pick $p_0 \in P$ as the initial mode of the system and $l_0 = 0$, the output trajectory of $\mathbb{G}(\Sigma)$ can be uniquely projected to an output trajectory of Σ. Then one can uniquely map the output trajectory of Σ to an output trajectory of $\mathbb{G}(\Sigma)$ and vice versa, for the same initial conditions. □

4.2.2.2 Finite Global MDPs

Here, we first formally define the finite abstraction of global MDPs as in the following definition.

Definition 4.2.24. *Given a global MDP $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$ associated with Σ as in Definition 4.2.21, one can construct its finite abstraction as a finite global MDP $\widehat{\mathbb{G}}(\hat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}^1, \hat{\mathbb{Y}}^2, \hat{\mathbb{H}}^1, \hat{\mathbb{H}}^2)$, where:*

- $\hat{\mathbb{X}} = \hat{X} \times P \times \{0, \dots, k_d - 1\}$ *is the set of states;*
- $\hat{\mathbb{U}} = \mathbb{U} = P$ *is the set of* external *inputs that remains the same as in the global MDP;*
- $\hat{\mathbb{W}} = \hat{W}$ *is the set of* internal *inputs;*
- ς *is a sequence of i.i.d. random variables;*
- $\hat{\mathbb{F}} : \hat{\mathbb{X}} \times \hat{\mathbb{U}} \times \hat{\mathbb{W}} \times \mathcal{V}_\varsigma \to \hat{\mathbb{X}}$ *is the one-step transition function given by* $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x}, p, l), \hat{\nu}, \hat{w}, \varsigma)$ *if and only if* $\hat{x}' = \hat{f}_p(\hat{x}, \hat{w}, \varsigma)$ *as defined similar to* (4.2.4), $\hat{\nu} = p$ *and the following scenarios hold:*
  - $l < k_d - 1$, $p' = p$, *and* $l' = l + 1$;
  - $l = k_d - 1$, $p' = p$, *and* $l' = k_d - 1$;
  - $l = k_d - 1$, $p' \neq p$, *and* $l' = 0$;
- $\hat{\mathbb{Y}}^1 = \{\mathbb{H}^1(\hat{x}, p, l) \,|\, (\hat{x}, p, l) \in \hat{\mathbb{X}}\}$ *is the* external *output set;*
- $\hat{\mathbb{Y}}^2 = \{\mathbb{H}^2(\hat{x}, p, l) \,|\, (\hat{x}, p, l) \in \hat{\mathbb{X}}\}$ *is the* internal *output set;*
- $\hat{\mathbb{H}}^1 : \hat{\mathbb{X}} \to \hat{\mathbb{Y}}^1$ *is the* external *output map defined as* $\hat{\mathbb{H}}^1(\hat{x}, p, l) = \mathbb{H}^1(\hat{x}, p, l) = h^1(\hat{x})$;
- $\hat{\mathbb{H}}^2 : \hat{\mathbb{X}} \to \hat{\mathbb{Y}}^2$ *is the* internal *output map defined as* $\hat{\mathbb{H}}^2(\hat{x}, p, l) = \mathbb{H}^2(\hat{x}, p, l) = h^2(\hat{x})$.

In the next subsection, in order to provide an approach for *compositional synthesis* of interconnected dt-SS, we define notions of augmented stochastic pseudo-simulation and simulation functions. These two notions are employed to quantify the probabilistic error between the global MDP and its finite abstraction and also their interconnection without internal signals, respectively.

4.2.2.3 aug-Type Stochastic Pseudo-Simulation and Simulation Functions

Here we first introduce a notion of augmented stochastic pseudo-simulation functions (aug-type SPSF) for dt-SS with internal inputs and outputs. We then define a notion of augmented stochastic simulation functions (aug-type SSF) for switched systems without internal signals. We employ these definitions mainly to quantify the closeness of the global MDP and its finite abstraction.

Definition 4.2.25. *Consider two global MDPs* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$ *and* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}^1, \hat{\mathbb{Y}}^2, \hat{\mathbb{H}}^1, \hat{\mathbb{H}}^2)$. *A function* $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ *is called an augmented stochastic pseudo-simulation function (*aug*-type SPSF) from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$ *if there exist* $\alpha \in \mathcal{K}_\infty$, $0 < \kappa < 1$, $\rho_{\text{int}} \in \mathcal{K}_\infty \cup \{0\}$, *and a constant* $\psi \in \mathbb{R}_{\geq 0}$ *such that*

- $\forall (x, p, l) \in \mathbb{X}, \forall (\hat{x}, p, l) \in \hat{\mathbb{X}}, \forall i \in \{1, 2\}$,

$$\alpha(\|\mathbb{H}^i(x, p, l) - \hat{\mathbb{H}}^i(\hat{x}, p, l)\|) \leq V((x, p, l), (\hat{x}, p, l)), \tag{4.2.23}$$

- $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$, $\forall \hat{\nu} \in \hat{\mathbb{U}}, \forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$,

$$\begin{aligned}\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\,\big|\,x,\hat{x},p,l,w,\hat{w}\Big]\\ \leq \max\Big\{\kappa V((x,p,l),(\hat{x},p,l)), \rho_{\mathrm{int}}(\|w-\hat{w}\|), \psi\Big\},\end{aligned} \tag{4.2.24}$$

where the expectation operator $\mathbb{E}$ *is with respect to* ς *under the one-step transition of both global MDPs with* $\nu = \hat{\nu}$, *i.e.,* $(x',p',l') = \mathbb{F}((x,p,l),\hat{\nu},w,\varsigma)$ *and* $(\hat{x}',p',l') = \hat{\mathbb{F}}((\hat{x},p,l),\hat{\nu},\hat{w},\varsigma)$.

If there exists an aug-type SPSF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq^{\mathsf{aug}}_{SPSF} \mathbb{G}(\Sigma)$, and the system $\widehat{\mathbb{G}}(\widehat{\Sigma})$ is called an abstraction of the concrete (original) global MDP $\mathbb{G}(\Sigma)$.

Now, we modify the above notion for global MDPs without internal signals by eliminating all the terms related to $w, \hat{w}$ which will be employed later for relating interconnected global MDPs.

Definition 4.2.26. *Consider two global MDPs* $\mathbb{G}(\Sigma) = (\mathbb{X},\mathbb{U},\varsigma,\mathbb{F},\mathbb{Y},\mathbb{H})$ *and* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}},\hat{\mathbb{U}},\varsigma,\hat{\mathbb{F}},\hat{\mathbb{Y}},\hat{\mathbb{H}})$ *without internal signals. A function* $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ *is called an augmented stochastic simulation function (*aug*-type SSF) from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$ *if*

- *there exists* $\alpha \in \mathcal{K}_\infty$ *such that* $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$,

$$\alpha(\|\mathbb{H}(x,p,l) - \hat{\mathbb{H}}(\hat{x},p,l)\|) \leq V((x,p,l),(\hat{x},p,l)), \tag{4.2.25}$$

- $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$, $\forall \hat{\nu} \in \hat{\mathbb{U}}$,

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\,\big|\,x,\hat{x},p,l\Big] \leq \max\Big\{\kappa V((x,p,l),(\hat{x},p,l)), \psi\Big\}, \tag{4.2.26}$$

for some $0 < \kappa < 1$, *and* $\psi \in \mathbb{R}_{\geq 0}$, *where the expectation operator* $\mathbb{E}$ *is with respect to* ς *under the one-step transition of both global MDPs with* $\nu = \hat{\nu}$, *i.e.,* $(x',p',l') = \mathbb{F}((x,p,l),\hat{\nu},\varsigma)$ *and* $(\hat{x}',p',l') = \hat{\mathbb{F}}((\hat{x},p,l),\hat{\nu},\varsigma)$.

If there exists an aug-type SSF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq^{\mathsf{aug}}_{SSF} \mathbb{G}(\Sigma)$, and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ is called an abstraction of $\mathbb{G}(\Sigma)$.

In order to show the usefulness of the aug-type SSF in comparing output trajectories of two global MDPs (without internal inputs and outputs) in a probabilistic setting, we need the following technical lemma borrowed from [Kus67, Theorem 3, pp. 86] with some slight modifications adapted to stochastic switched systems.

Lemma 4.2.27. *Let* $\mathbb{G}(\Sigma) = (\mathbb{X},\mathbb{U},\varsigma,\mathbb{F},\mathbb{Y},\mathbb{H})$ *be a global MDP with the transition map* $\mathbb{F} : \mathbb{X} \times \mathbb{U} \times \mathcal{V}_\varsigma \to \mathbb{X}$. *Assume there exist* $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ *and constants* $0 < \kappa < 1$, *and* $\psi \in \mathbb{R}_{\geq 0}$ *such that*

$$\mathbb{E}\Big[V(x',p',l')\,\big|\,x,p,l\Big] \leq \kappa V(x,p,l) + \psi,$$

where $(x', p', l') = \mathbb{F}((x, p, l), p, \varsigma)$*. Then for any random variable* a *as the initial state of the underlying dt-SS, any initial mode* p_0*, and* $l_0 = 0$ *as the initial counter, the following inequity holds:*

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} V(x(k), p(k), l(k)) \geq \varepsilon \,\middle|\, a, p_0\right\} \leq \hat{\delta},$$

$$\hat{\delta} := \begin{cases} 1 - (1 - \frac{V(a,p_0,l_0)}{\varepsilon})(1 - \frac{\psi}{\varepsilon})^{T_d}, & \text{if } \varepsilon \geq \frac{\psi}{\kappa}, \\ (\frac{V(a,p_0,l_0)}{\varepsilon})(1-\kappa)^{T_d} + (\frac{\psi}{\kappa\varepsilon})(1 - (1-\kappa)^{T_d}), & \text{if } \varepsilon < \frac{\psi}{\kappa}. \end{cases}$$

Now by employing Lemma 4.2.27, we provide one of the results of the section.

Theorem 4.2.28. *Let* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ *and* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$ *be two global MDPs without internal inputs. Suppose* V *is an* aug*-type SSF from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$*. For any random variables* a *and* $\hat{a}$ *as the initial states of the two dt-SS, any initial mode* p_0*, and for any external input trajectory* $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ *that preserves Markov property for the closed-loop* $\widehat{\mathbb{G}}(\widehat{\Sigma})$*, the following inequality holds:*

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,\middle|\, a, \hat{a}, p_0\right\} \tag{4.2.27}$$
$$\leq \begin{cases} 1 - (1 - \frac{V((a,p_0,l_0),(\hat{a},p_0,l_0))}{\alpha(\varepsilon)})(1 - \frac{\psi}{\alpha(\varepsilon)})^{T_d}, & \text{if } \alpha(\varepsilon) \geq \frac{\psi}{\kappa}, \\ \frac{V((a,p_0,l_0),(\hat{a},p_0,l_0))}{\alpha(\varepsilon)}(1-\kappa)^{T_d} + \frac{\psi}{\kappa\alpha(\varepsilon)}(1 - (1-\kappa)^{T_d}), & \text{if } \alpha(\varepsilon) < \frac{\psi}{\kappa}. \end{cases}$$

Proof. For any $(x, p, l) \in \mathbb{X}$, and $(\hat{x}, p, l) \in \hat{\mathbb{X}}$, one gets

$$\|\mathbb{H}(x, p, l) - \hat{\mathbb{H}}(\hat{x}, p, l)\| = \|h(x) - \hat{h}(\hat{x})\| = \|y - \hat{y}\|.$$

Since V is an aug-type SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{0\leq k\leq T_d} \|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,\Big|\, a, \hat{a}, p_0\Big\} \\
&= \mathbb{P}\Big\{\sup_{0\leq k\leq T_d} \alpha\left(\|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|\right) \geq \alpha(\varepsilon) \,\Big|\, a, \hat{a}, p_0\Big\} \\
&\leq \mathbb{P}\Big\{\sup_{0\leq k\leq T_d} V((x_{a\hat{\nu}}(k), p(k), l(k), (\hat{x}_{\hat{a}\hat{\nu}}(k), p(k), l(k))) \geq \alpha(\varepsilon) \,\Big|\, a, \hat{a}, p_0\Big\}.
\end{aligned} \tag{4.2.28}$$

The equality holds due to α being a $\mathcal{K}_\infty$ function, and also the condition (4.2.25) on the aug-type SSF V. By applying Lemma 4.2.27 to (4.2.28), utilizing the inequality (4.2.26), and since

$$\max\Big\{\kappa V((x, p, l), (\hat{x}, p, l)), \psi\Big\} \leq \kappa V((x, p, l), (\hat{x}, p, l)) + \psi,$$

one can readily acquire the results in (4.2.27).

□

4.2.2.4 Compositionality Results

In this subsection, we analyze networks of stochastic switched subsystems by driving a max small-gain condition and discuss how to construct their *finite* global MDPs together with an aug-type SSF based on the aug-type SPSF of their subsystems.

4.2.2.4.1 Concrete Interconnected Stochastic Switched Systems

Suppose we are given N *concrete* stochastic switched subsystems $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, \mathbf{F}_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$, $i \in \{1, \ldots, N\}$ with their *equivalent* global MDPs $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, \mathbb{H}_i^1, \mathbb{H}_i^2)$, in which their internal inputs and outputs are partitioned as in (3.2.8) and (3.2.9). Now, we are ready to define the *interconnection* of concrete dt-SS Σ_i.

Definition 4.2.29. *Consider* $N \in \mathbb{N}_{\geq 1}$ *dt-SS* $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, \mathbf{F}_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$, *with the input-output configuration as in* (3.2.8) *and* (3.2.9). *The interconnection of* Σ_i, $\forall i \in \{1, \ldots, N\}$, *is the* concrete *interconnected dt-SS* $\Sigma = (X, P, \mathcal{P}, \varsigma, \mathbf{F}, Y, h)$, *denoted by* $\mathcal{I}_{ss}(\Sigma_1, \ldots, \Sigma_N)$, *such that* $X := \prod_{i=1}^N X_i$, $P := \prod_{i=1}^N P_i$, $\mathcal{P} := \prod_{i=1}^N \mathcal{P}_i$, $\mathbf{F} := \prod_{i=1}^N \mathbf{F}_i$, $Y := \prod_{i=1}^N Y_i^1$, *and* $h = \prod_{i=1}^N h_i^1$, *subjected to the following constraint:*

$$\forall i, j \in \{1, \ldots, N\}, i \neq j: \quad w_{ji} = y_{ij}^2, \quad Y_{ij}^2 \subseteq W_{ji}.$$

Similarly, given global MDPs $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, \mathbb{H}_i^1, \mathbb{H}_i^2)$, $i \in \{1, \ldots, N\}$, one can also define the *interconnection* of concrete global MDPs $\mathbb{G}(\Sigma_i)$ as $\mathcal{I}_{ss}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$.

Now assume that any *concrete* global MDP $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, \mathbb{H}_i^1, \mathbb{H}_i^2)$, $i \in \{1, \ldots, N\}$, admits an *abstract* global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma}_i) = (\hat{\mathbb{X}}_i, \hat{\mathbb{U}}_i, \hat{\mathbb{W}}_i, \varsigma_i, \hat{\mathbb{F}}_i, \hat{\mathbb{Y}}_i^1, \hat{\mathbb{Y}}_i^2, \hat{\mathbb{H}}_i^1, \hat{\mathbb{H}}_i^2)$ together with an aug-type SPSF V_i from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ with the corresponding functions and constants denoted by $\alpha_i, \rho_{\text{int}i}, \kappa_i$ and ψ_i as in Definition 4.2.25.

4.2.2.4.2 Compositional Abstractions of Interconnected Switched Systems

Now, we define a notion of the *interconnection* of *abstract* global MDPs $\widehat{\mathbb{G}}(\widehat{\Sigma}_i) = (\hat{\mathbb{X}}_i, \hat{\mathbb{U}}_i, \hat{\mathbb{W}}_i, \varsigma_i, \hat{\mathbb{F}}_i, \hat{\mathbb{Y}}_i^1, \hat{\mathbb{Y}}_i^2, \hat{\mathbb{H}}_i^1, \hat{\mathbb{H}}_i^2)$.

Definition 4.2.30. *Consider* $N \in \mathbb{N}_{\geq 1}$ abstract *global MDPs* $\widehat{\mathbb{G}}(\widehat{\Sigma}_i) = (\hat{\mathbb{X}}_i, \hat{\mathbb{U}}_i, \hat{\mathbb{W}}_i, \varsigma_i, \hat{\mathbb{F}}_i, , \hat{\mathbb{Y}}_i^1, \hat{\mathbb{Y}}_i^2, \hat{\mathbb{H}}_i^1, \hat{\mathbb{H}}_i^2)$, *with the input-output configuration similar to* (3.2.8) *and* (3.2.9). *The interconnection of* $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$, $\forall i \in \{1, \ldots, N\}$, *is the interconnected* abstract *global MDP* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$, *denoted by* $\hat{\mathcal{I}}_{ss}(\widehat{\mathbb{G}}(\widehat{\Sigma}_1), \ldots, \widehat{\mathbb{G}}(\widehat{\Sigma}_N))$, *such that* $\hat{\mathbb{X}} := \prod_{i=1}^N \hat{\mathbb{X}}_i$, $\hat{\mathbb{U}} := \prod_{i=1}^N \hat{\mathbb{U}}_i$, $\hat{\mathbb{Y}} := \prod_{i=1}^N \hat{\mathbb{Y}}_i^1$, $\hat{\mathbb{H}} = \prod_{i=1}^N \hat{\mathbb{H}}_i^1$, *and the map* $\hat{\mathbb{F}} = \prod_{i=1}^N \hat{\mathbb{F}}_i$ *is the transition function given by* $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x}, p, l), \hat{\nu}, \hat{w}, \varsigma)$ *if and only if* $\hat{x}' = \hat{f}_p(\hat{x}, \hat{w}, \varsigma)$ *as defined similar to* (4.2.4), $\hat{\nu} = p$ *and the following scenarios hold for any* $i \in \{1, \ldots, N\}$:

- $l_i < k_{d_i} - 1$, $p'_i = p_i$, *and* $l'_i = l_i + 1$;
- $l_i = k_{d_i} - 1$, $p'_i = p_i$, *and* $l'_i = k_{d_i} - 1$;

- $l_i = k_{d_i} - 1$, $p'_i \neq p_i$, *and* $l'_i = 0$;

where $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N], \hat{\nu} = [\hat{\nu}_1; \ldots; \hat{\nu}_N], p = [p_1; \ldots; p_N], l = [l_1; \ldots; l_N]$, *and subject to the following constraint:*

$$\forall i, j \in \{1, \ldots, N\}, i \neq j : \qquad \hat{w}_{ji} = \Pi_{w_{ji}}(\hat{y}^2_{ij}), \quad \Pi_{w_{ji}}(\hat{\mathbb{Y}}^2_{ij}) \subseteq \hat{\mathbb{W}}_{ji}.$$

Now we leverage the max small-gain Assumption 4.2.7 to quantify the error between the interconnection of concrete global MDPs and that of their finite abstractions in a compositional manner.

Theorem 4.2.31. *Consider the interconnected global MDP* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *global MDPs* $\mathbb{G}(\Sigma_i)$*. Suppose that each* $\mathbb{G}(\Sigma_i)$ *admits a finite abstraction* $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ *together with an* aug*-type SPSF* V_i*. If Assumption 4.2.7 holds, then the function* $V((x,p,l),(\hat{x},p,l))$ *defined as*

$$V((x,p,l),(\hat{x},p,l)) := \max_i \Big\{\sigma_i^{-1}(V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)))\Big\}, \tag{4.2.29}$$

for σ_i *as in* (4.2.8)*, is an* aug*-type SSF function from* $\widehat{\mathcal{I}}_{ss}(\widehat{\mathbb{G}}(\widehat{\Sigma}_1), \ldots, \widehat{\mathbb{G}}(\widehat{\Sigma}_N))$ *to* $\mathcal{I}_{ss}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$ *provided that* $\max_i \sigma_i^{-1}$ *is concave.*

Proof. We first show that the aug-type SSF V in (4.2.29) satisfies the inequality (4.2.25) for some $\mathcal{K}_\infty$ function α. For any $(x,p,l) \in \mathbb{X}$, and $(\hat{x},p,l) \in \hat{\mathbb{X}}$, one gets

$$\begin{aligned}
\|\mathbb{H}(x,p,l) - \hat{\mathbb{H}}(\hat{x},p,l)\| &= \max_i \Big\{\|\mathbb{H}^1_i(x_i,p_i,l_i) - \hat{\mathbb{H}}^1_i(\hat{x}_i,p_i,l_i)\|\Big\}\\
&\leq \max_i \Big\{\alpha_i^{-1}(V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)))\Big\}\\
&\leq \hat{\beta}\ (\max_i \Big\{\sigma_i^{-1}(V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)))\Big\})\\
&= \hat{\beta}(V((x,p,l),(\hat{x},p,l))),
\end{aligned}$$

where $\hat{\beta}(s) = \max_i \Big\{\alpha_i^{-1} \circ \sigma_i(s)\Big\}$ for all $s \in \mathbb{R}_{\geq 0}$, which is a $\mathcal{K}_\infty$ function and (4.2.25) holds with $\alpha = \hat{\beta}^{-1}$. We continue with showing that the inequality (4.2.26) holds, as well. Let $\kappa(s) = \max_{i,j}\{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j(s)\}$. It follows from (4.2.8) that $\kappa < \mathcal{I}_d$. Since $\max_i \sigma_i^{-1}$ is concave, one can readily get the chain of inequalities in (4.2.31) using Jensen's inequality, the inequality (4.2.6), and by defining ψ as

$$\psi := \max_i \sigma_i^{-1}(\Lambda_i), \tag{4.2.30}$$

where $\Lambda_i := (\mathcal{I}_d + \tilde{\delta}_f^{-1}) \circ (\rho_{\text{inti}} \circ \bar{\lambda} \circ (\bar{\lambda} - \mathcal{I}_d)^{-1}(\max_{j,j\neq i}\{\bar{\mu}_{ji}\}) + \psi_i)$. Hence V is an aug-type SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, which completes the proof. □

Figure 4.6 schematically shows the results of Theorem 4.2.31.

$$
\begin{aligned}
&\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\,\big|\,x,\hat{x},p,l\Big] = \mathbb{E}\Big[\max_i\Big\{\sigma_i^{-1}(V_i((x_i',p_i',l_i'),(\hat{x}_i',p_i',l_i')))\Big\}\,\big|\,x,\hat{x},p,l\Big]\\
&\quad\le \max_i\Big\{\sigma_i^{-1}\big(\mathbb{E}\Big[V_i((x_i',p_i',l_i'),(\hat{x}_i',p_i',l_i'))\,\big|\,x,\hat{x},p,l\Big]\big)\Big\}\\
&\quad= \max_i\Big\{\sigma_i^{-1}\big(\mathbb{E}\Big[V_i((x_i',p_i',l_i'),(\hat{x}_i',p_i',l_i'))\,\big|\,x_i,\hat{x}_i,p_i,l_i\Big]\big)\Big\}\\
&\quad\le \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}(\|w_i-\hat{w}_i\|),\psi_i\})\Big\}\\
&\quad= \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|w_{ij}-\hat{w}_{ij}\|\}),\psi_i\})\Big\}\\
&\quad= \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|y_{ji}^2-\hat{y}_{ji}^2+\hat{y}_{ji}^2-\Pi_{w_{ji}}(\hat{y}_{ji}^2)\|\}),\psi_i\})\Big\}\\
&\quad\le \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|\mathbb{H}_j^2(x_j,p_j,l_j)-\hat{\mathbb{H}}_j^2(\hat{x}_j,p_j,l_j)\|\\
&\qquad\qquad+\|\hat{y}_{ji}^2-\Pi_{w_{ji}}(\hat{y}_{ji}^2)\|\}),\psi_i\})\Big\}\\
&\quad\le \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j((x_j,p_j,l_j),(\hat{x}_j,p_j,l_j)))\\
&\qquad\qquad+\bar{\mu}_{ji}\}),\psi_i\})\Big\}\\
&\quad\le \max_i\Big\{\sigma_i^{-1}(\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),\rho_{\mathrm{int}i}\circ\bar{\lambda}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j((x_j,p_j,l_j),(\hat{x}_j,p_j,l_j)))\})\\
&\qquad\qquad+\rho_{\mathrm{int}i}\circ\bar{\lambda}\circ(\bar{\lambda}-\mathcal{I}_d)^{-1}(\max_{j,j\neq i}\{\bar{\mu}_{ji}\}),\psi_i\})\Big\}\\
&\quad\le \max_i\Big\{\sigma_i^{-1}\max\{\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)),(\mathcal{I}_d+\tilde{\delta}_f)\circ\rho_{\mathrm{int}i}\circ\bar{\lambda}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j((x_j,p_j,l_j),\\
&\qquad\qquad(\hat{x}_j,p_j,l_j)))\}),\Lambda_i\})\Big\}\\
&\quad= \max_{i,j}\Big\{\sigma_i^{-1}\max\{\kappa_{ij}(V_j((x_j,p_j,l_j),(\hat{x}_j,p_j,l_j)),\Lambda_i\})\Big\}\\
&\quad= \max_{i,j}\Big\{\sigma_i^{-1}\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j((x_j,p_j,l_j),(\hat{x}_j,p_j,l_j))),\Lambda_i\})\Big\}\\
&\quad\le \max_{i,j,\bar{j}}\Big\{\sigma_i^{-1}\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_{\bar{j}}^{-1}(V_{\bar{j}}((x_{\bar{j}},p_{\bar{j}},l_{\bar{j}}),(\hat{x}_{\bar{j}},p_{\bar{j}},l_{\bar{j}}))),\Lambda_i\})\Big\}\\
&\quad= \max_{i,j}\Big\{\sigma_i^{-1}\max\{\kappa_{ij}\circ\sigma_j(V((x,p,l),(\hat{x},p,l))),\Lambda_i\})\Big\}\\
&\quad= \max\Big\{\kappa V((x,p,l),(\hat{x},p,l)),\psi\Big\}. \qquad (4.2.31)
\end{aligned}
$$

4.2.2.5 Construction of aug-type SPSF

In this subsection, we impose conditions on the *concrete* dt-SS Σ enabling us to find an aug-type SPSF from the finite abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$. The required conditions are first presented in the *general setting* of nonlinear stochastic switched systems in the

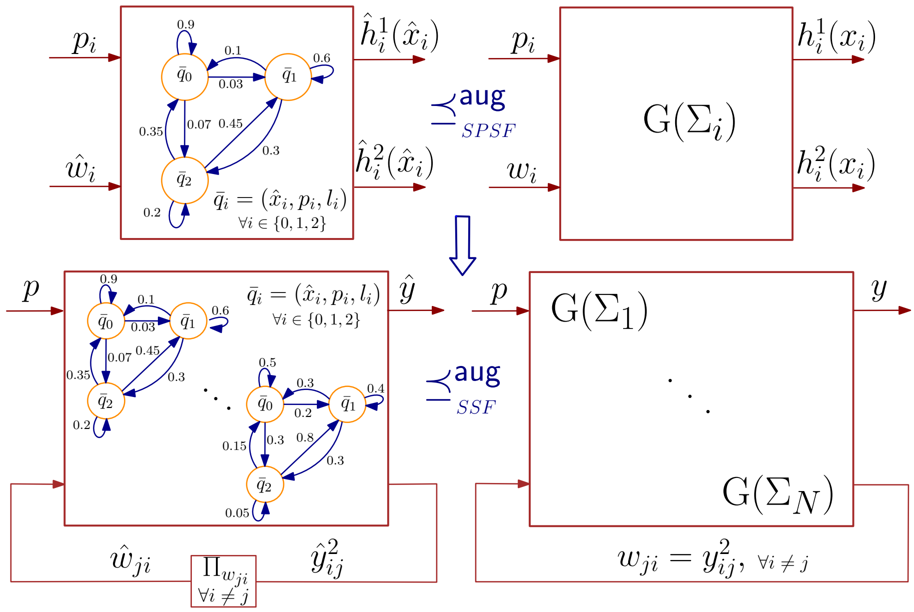

Figure 4.6: Compositionality results for constructing the interconnection of *finite* global MDPs provided that the condition (4.2.7) is satisfied.

next subsection and then represented via some matrix inequality for a nonlinear class of stochastic switched systems similar to (3.3.10).

4.2.2.5.1 General Setting of Nonlinear Stochastic Switched Systems

The aug-type SPSF from the finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ is established under the assumption that original discrete-time stochastic switched subsystems $\Sigma_p, \forall p \in P$, are *incrementally input-to-state stable* (δ-ISS) similar to Definition 2.7.1 but adapted to stochastic switched systems.

Definition 4.2.32. *A dt-SS Σ_p is called* incrementally input-to-state stable *(δ-ISS) if there exists a function $V_p : X \times X \to \mathbb{R}_{\geq 0}$ such that $\forall x, x' \in X$, $\forall w, w' \in W$, the following two inequalities hold:*

$$\underline{\alpha}_p(\|x - x'\|) \leq V_p(x, x') \leq \overline{\alpha}_p(\|x - x'\|), \qquad (4.2.32)$$

and

$$\mathbb{E}\Big[V_p(f_p(x, w, \varsigma), f_p(x', w', \varsigma)) \,\Big|\, x, x', w, w'\Big] \leq \bar{\kappa}_p V_p(x, x') + \bar{\rho}_{\mathrm{int}p}(\|w - w'\|), \qquad (4.2.33)$$

for some $\underline{\alpha}_p, \overline{\alpha}_p \in \mathcal{K}_\infty$, $0 < \bar{\kappa}_p < 1$, and $\bar{\rho}_{\mathrm{int}p} \in \mathcal{K}_\infty \cup \{0\}$.

In order to construct an aug-type SPSF from the finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, we need to raise the following assumptions. These assumptions are essential to show the main result of this section in Theorem 4.2.36.

Assumption 4.2.33. *There exists* $\tilde{\mu} \geq 1$ *such that*

$$\forall x, x' \in X,\ \forall p, p' \in P, \quad V_p(x, x') \leq \tilde{\mu} V_{p'}(x, x'). \tag{4.2.34}$$

Remark 4.2.34. *Assumption 4.2.33 is a standard one in switched systems accepting* multiple *Lyapunov functions with* dwell-time *similar to the one appeared in [Lib03, equation (3.6)]. Note that if the function* V_p *is quadratic in the form of* (4.2.40)*, there always exists* $\tilde{\mu} \geq 1$ *satisfying Assumption 4.2.33 as* $\tilde{\mu} = \max(\frac{\lambda_{\max}(\tilde{M}_p)}{\lambda_{\min}(\tilde{M}_{p'})}, \frac{\lambda_{\max}(\tilde{M}_{p'})}{\lambda_{\min}(\tilde{M}_p)}), \forall p, p' \in P$ *(cf. Case study 4.2.2.6.5). If there exists a common Lyapunov function between all modes, then* $\tilde{\mu} = 1$ *and* $V((x,p,l),(\hat{x},p,l)) = V(x,\hat{x})$ *(cf. Case study 4.2.2.6.1).*

Assumption 4.2.35. *Assume that* $\forall p \in P$*, there exists a function* $\gamma_p \in \mathcal{K}_\infty$ *such that*

$$V_p(x, x') - V_p(x, x'') \leq \gamma_p(\|x' - x''\|), \quad \forall x, x', x'' \in X. \tag{4.2.35}$$

Under Definition 4.2.32 and Assumptions 4.2.33 and 4.2.35, the next theorem shows a relation between $\mathbb{G}(\Sigma)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ via establishing an aug-type SPSF between them.

Theorem 4.2.36. *Let* $\Sigma = (X, P, \mathcal{P}, W, \varsigma, \mathbf{F}, Y^1, Y^2, h^1, h^2)$ *be a switched system with its equivalent global MDP* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$*. Consider the abstract global MDP* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}^1, \hat{\mathbb{Y}}^2, \hat{\mathbb{H}}^1, \hat{\mathbb{H}}^2)$ *constructed as in Definition 4.2.24. For any* $p \in P$*, let* Σ_p *be an incrementally input-to-state stable dt-SS via a function* V_p *as in Definition 4.2.32, and Assumptions 4.2.33 and 4.2.35 hold. Let* $\bar{\epsilon} > 1$*. If* $\forall p \in P$*,* $k_d \geq \bar{\epsilon}\frac{\ln(\tilde{\mu})}{\ln(1/\bar{\kappa}_p)} + 1$*, then*

$$V((x,p,l),(\hat{x},p,l)) = \frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}} V_p(x,\hat{x}), \tag{4.2.36}$$

is an aug*-type SPSF from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$*.*

Proof. Given the general assumption on h^i, since Σ_p is incrementally input-to-state stable, and from (4.2.32), $\forall (x,p,l) \in \mathbb{X}$ and $\forall (\hat{x},p,l) \in \hat{\mathbb{X}}$, we get

$$\begin{aligned}\|\mathbb{H}^i(x,p,l) - \hat{\mathbb{H}}^i(\hat{x},p,l)\| &= \|h^i(x) - \hat{h}^i(\hat{x})\| \leq \mathscr{L}(\|x - \hat{x}\|)\\ &\leq \mathscr{L} \circ \underline{\alpha}_p^{-1}(V(x,\hat{x})) = \mathscr{L} \circ \underline{\alpha}_p^{-1}(\bar{\kappa}_p^{l/\bar{\epsilon}} V((x,p,l),(\hat{x},p,l))).\end{aligned}$$

Since $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}} > 1$, one can conclude that the inequality (4.2.23) holds with $\alpha(s) = \min_p\{(\mathscr{L} \circ \underline{\alpha}_p^{-1}(s))^{-1}\}$, $\forall s \in \mathbb{R}_{\geq 0}$. Now we show that the inequality (4.2.24) holds, as well. By taking the conditional expectation from (4.2.35), $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall p \in P, \forall w \in W, \forall \hat{w} \in$

$\hat{W}$, we have

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\&\quad-\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),f_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\&\quad\leq\mathbb{E}\Big[\gamma(\|\hat{f}_p(\hat{x},\hat{w},\varsigma)-f_p(\hat{x},\hat{w},\varsigma)\|)\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big],\end{aligned}$$

where $\hat{f}_p(\hat{x},\hat{\nu},\varsigma)=\Pi_x(f_p(\hat{x},\hat{w},\varsigma))$. Using (4.2.5), the above inequality reduces to

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\&\quad-\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),f_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\leq\gamma_p(\bar{\delta}).\end{aligned}$$

Employing (4.2.33), we get

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},w,\hat{w}\Big]\\&\quad\leq\bar{\kappa}_pV_p(x,\hat{x})+\bar{\rho}_{\mathrm{int}p}(\|w-\hat{w}\|)+\gamma_p(\bar{\delta}).\end{aligned}\tag{4.2.37}$$

Now, in order to show that the function V in (4.2.36) satisfies (4.2.24), we should consider the different scenarios as in Definition 4.2.24. For the first scenario ($l<k_d-1,\|f_p(\hat{x},\hat{w},\varsigma)-\hat{f}_p(\hat{x},\hat{w},\varsigma)\|\leq\bar{\delta}$, $p'=p$, and $l'=l+1$), using (4.2.37) we have:

$$\begin{aligned}&\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\,\Big|\,x,\hat{x},p,l,w,\hat{w}\Big]\\&=\frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[V_{p'}(x',\hat{x}')\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\&=\frac{1}{\bar{\kappa}_p^{(l+1)/\bar{\epsilon}}}\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]\\&\leq\frac{1}{\bar{\kappa}_p^{(l+1)/\bar{\epsilon}}}\big(\bar{\kappa}_pV_p(x,\hat{x})+\bar{\rho}_{\mathrm{int}p}(\|w-\hat{w}\|)+\gamma_p(\bar{\delta})\big)\\&=\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\frac{1}{\bar{\kappa}_p^{(l+1)/\bar{\epsilon}}}\big(\bar{\rho}_{\mathrm{int}p}(\|w-\hat{w}\|)+\gamma_p(\bar{\delta})\big)\\&\leq\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\big(\bar{\rho}_{\mathrm{int}p}(\|w-\hat{w}\|)+\gamma_p(\bar{\delta})\big);\end{aligned}$$

Note that the last inequality here holds since $l<k_d-1$, and consequently, $l+1<k_d$.

For the second scenario ($l = k_d - 1, \|f_p(\hat{x}, \hat{w}, \varsigma) - \hat{f}_p(\hat{x}, \hat{w}, \varsigma)\| \le \bar{\delta}$, $p' = p$, and $l' = k_d - 1$), we have:

$$\begin{aligned}
&\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\Big|\, x, \hat{x}, p, l, w, \hat{w}\Big] \\
&= \frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}} \mathbb{E}\Big[V_{p'}(x', \hat{x}') \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&= \frac{1}{\bar{\kappa}_{p}^{l/\bar{\epsilon}}} \mathbb{E}\Big[V_{p}(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\le \frac{1}{\bar{\kappa}_{p}^{l/\bar{\epsilon}}} \big(\bar{\kappa}_p V_p(x, \hat{x}) + \bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big) \\
&= \bar{\kappa}_p V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\bar{\kappa}_{p}^{k_d/\bar{\epsilon}}} \big(\bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big) \\
&\le \bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\bar{\kappa}_{p}^{k_d/\bar{\epsilon}}} \big(\bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big);
\end{aligned}$$

Note that the last inequality holds since $\bar{\epsilon} > 1$, and consequently, $0 < \frac{\bar{\epsilon}-1}{\bar{\epsilon}} < 1$.

For the last scenario ($l = k_d - 1, \|f_p(\hat{x}, \hat{w}, \varsigma) - \hat{f}_p(\hat{x}, \hat{w}, \varsigma)\| \le \bar{\delta}, p' \neq p$, and $l' = 0$), using Assumption 4.2.33 we have:

$$\begin{aligned}
&\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\Big|\, x, \hat{x}, p, l, w, \hat{w}\Big] \\
&= \frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}} \mathbb{E}\Big[V_{p'}(x', \hat{x}') \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\le \tilde{\mu}\, \mathbb{E}\Big[V_{p}(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&= \tilde{\mu} \bar{\kappa}_p^{(k_d-1)/\bar{\epsilon}} \frac{1}{\bar{\kappa}_{p}^{l/\bar{\epsilon}}} \mathbb{E}\Big[V_{p}(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\le \tilde{\mu} \bar{\kappa}_p^{(k_d-1)/\bar{\epsilon}} \frac{1}{\bar{\kappa}_{p}^{l/\bar{\epsilon}}} \big(\bar{\kappa}_p V_p(x, \hat{x}) + \bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big) \\
&\le \tilde{\mu} \bar{\kappa}_p^{(k_d-1)/\bar{\epsilon}} \bar{\kappa}_p V((x, p, l), (\hat{x}, p, l)) + \tilde{\mu} \big(\bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big) \\
&\le \bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\bar{\kappa}_{p}^{k_d/\bar{\epsilon}}} \big(\bar{\rho}_{\text{int}p}(\|w - \hat{w}\|) + \gamma_p(\bar{\delta})\big);
\end{aligned}$$

Note that $\forall p \in P$, $\tilde{\mu} \bar{\kappa}_p^{(k_d-1)/\bar{\epsilon}} \le 1$ since $\forall p \in P$, $k_d \ge \bar{\epsilon} \frac{\ln(\tilde{\mu})}{\ln(1/\bar{\kappa}_p)} + 1$. By employing a similar argument as the one in [SGZ18, Theorem 1], and by defining $\tilde{\kappa} = \max_p \{\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}\}$, $\tilde{\rho}_{\text{int}}(s) = \max_p \{\frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}} \bar{\rho}_{\text{int}p}(s)\}, \forall s \in \mathbb{R}_{\ge 0}$, and $\bar{\gamma}(\bar{\delta}) = \max_p \{\frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}} \bar{\gamma}_p(\bar{\delta})\}$, the following inequality

$$\begin{aligned}
&\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\Big|\, x, \hat{x}, p, l, w, \hat{w}\Big] \\
&\quad \le \max\Big\{\tilde{\kappa} V((x, p, l), (\hat{x}, p, l)), \tilde{\rho}_{\text{int}}(\|w - \hat{w}\|), \tilde{\gamma}\Big\}
\end{aligned}$$

holds for the all scenarios, where $\tilde{\kappa} = (1-(1-\tilde{\pi})(1-\bar{\kappa})$, $\tilde{\rho}_{\text{int}} = (\mathcal{I}_d+\tilde{\delta}_f)\circ(\frac{1}{(1-\bar{\kappa})\tilde{\pi}}\bar{\lambda}\circ\bar{\rho}_{\text{int}})$, $\tilde{\gamma} = (\mathcal{I}_d+\tilde{\delta}_f^{-1})\circ(\frac{1}{(1-\bar{\kappa})\tilde{\pi}}\circ\bar{\lambda}\circ(\bar{\lambda}-\mathcal{I}_d)^{-1}\circ\bar{\gamma})$ where $\tilde{\delta}_f, \bar{\lambda}$, are some arbitrarily chosen $\mathcal{K}_\infty$ functions with $\bar{\lambda}-\mathcal{I}_d \in \mathcal{K}_\infty$, and $0<\tilde{\pi}<1$, $1-\bar{\kappa}>0$. Then the inequality (4.2.24) is satisfied with $\nu=\hat{\nu}$, $\kappa=\tilde{\kappa}$, $\rho_{int}=\tilde{\rho}_{int}$, and $\psi=\tilde{\gamma}(\bar{\delta})$. Hence V is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ which completes the proof. □

Remark 4.2.37. *Note that if there exists a common Lyapunov function* $V: X\times X\to\mathbb{R}_{\geq 0}$ *between all switching modes (i.e.,* $p=p', \forall p,p'\in P$*) satisfying Definition 4.2.32 and Assumptions 4.2.33 and 4.2.35, then* $V((x,p,l),(\hat{x},p,l)) = V(x,\hat{x})$ *and Definitions 4.2.25 and 4.2.26 reduce to Definitions 3.3.1 and 3.3.3 (cf. Case study 4.2.2.6.1). Accordingly, the functions and constants* $\alpha, \bar{\kappa}, \bar{\rho}_{\text{int}}$ *and* $\bar{\gamma}$ *reduce to* $\alpha(s) = (\mathscr{L}_p\circ\underline{\alpha}_p^{-1}(s))^{-1}$, $\bar{\rho}_{\text{int}}(s) = \bar{\rho}_{\text{int}p}(s), \forall s\in\mathbb{R}_{\geq 0}$, *and* $\bar{\kappa}=\bar{\kappa}_p$, $\bar{\gamma}(\bar{\delta}) = \bar{\gamma}_p(\bar{\delta})$.

Now we provide similar results in the next subsection but tailored to a particular class of *nonlinear* stochastic switched systems.

4.2.2.5.2 Stochastic Switched Systems with Slope Restrictions on Nonlinearity

Here we focus on a specific class of discrete-time *nonlinear* stochastic switched systems similar to (3.3.10) together with *quadratic* functions V_p, and provide an approach on the construction of their finite global MDPs. The class of nonlinear switched systems is given by

$$\Sigma : \begin{cases} x(k+1) = A_{\mathbf{p}(k)}x(k) + E_{\mathbf{p}(k)}\varphi_{\mathbf{p}(k)}(F_{\mathbf{p}(k)}x(k)) + B_{\mathbf{p}(k)} + D_{\mathbf{p}(k)}w(k) + R_{\mathbf{p}(k)}\varsigma(k), \\ y^1(k) = C^1x(k), \\ y^2(k) = C^2x(k), \end{cases} \tag{4.2.38}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions, and $\varphi_p:\mathbb{R}\to\mathbb{R}$ satisfies

$$0 \leq \frac{\varphi_p(c)-\varphi_p(d)}{c-d} \leq b_p, \quad \forall c,d\in\mathbb{R}, c\neq d, \tag{4.2.39}$$

for some $b_p\in\mathbb{R}_{>0}\cup\{\infty\}$.

We use the tuple

$$\Sigma = (A,B,C^1,C^2,D,E,F,R,\varphi),$$

to refer to the class of nonlinear switched systems of the form (4.2.38), where $A = \{A_1,\ldots,A_m\}, B=\{B_1,\ldots,B_m\}, D=\{D_1,\ldots,D_m\}, E=\{E_1,\ldots,E_m\}, F=\{F_1,\ldots, F_m\}, R=\{R_1,\ldots,R_m\}, \varphi=\{\varphi_1,\ldots,\varphi_m\}$, for the finite set of $P=\{1,\ldots,m\}$.

We employ a quadratic function of the form

$$V_p(x,\hat{x}) = (x-\hat{x})^T\tilde{M}_p(x-\hat{x}), \quad \forall p\in P, \tag{4.2.40}$$

where $\tilde{M}_p \succ 0$ is a positive-definite matrix of an appropriate dimension. In order to show that a nominated V employing V_p in (4.2.40) is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ associated with Σ, we raise the following assumption on Σ.

Assumption 4.2.38. *Assume that there exist constants $0 < \bar{\kappa}_p < 1$, $\pi_p \in \mathbb{R}_{>0}$, and matrix $\tilde{M}_p \succ 0$ such that the following inequality holds:*

$$\begin{bmatrix} (1+2\pi_p)A_p^T\tilde{M}_pA_p & A_p^T\tilde{M}_pE_p \\ E_p^T\tilde{M}_pA_p & (1+2\pi_p)E_p^T\tilde{M}_pE_p \end{bmatrix} \preceq \begin{bmatrix} \bar{\kappa}_p\tilde{M}_p & -F_p^T \\ -F_p & 2/b_p \end{bmatrix}. \tag{4.2.41}$$

Remark 4.2.39. *Note that for any* linear *system $\Sigma = (A, B, C^1, C^2, D, R)$ with matrices E_p and F_p being identically zero, matrices A_p being Hurwitz is sufficient to satisfy Assumption 4.2.38.*

Now we provide another main result of this section showing under which conditions a nominated V using V_p in (4.2.40) is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.

Theorem 4.2.40. *Consider the global MDP $\mathbb{G}(\Sigma)$ associated with $\Sigma = (A, B, C^1, C^2, D, , E, F, R, \varphi)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ as its* finite *abstraction with the* state *discretization parameter $\bar{\delta}$. Let $\bar{\epsilon} > 1$ and $\hat{C}^i = C^i, i \in \{1,2\}$. If Assumption 4.2.38 holds, and $\forall p \in P$, $k_d \geq \bar{\epsilon}\frac{\ln(\tilde{\mu})}{\ln(1/\bar{\kappa}_p)} + 1$, then*

$$V((x,p,l),(\hat{x},p,l)) = \frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}}V_p(x,\hat{x}), \tag{4.2.42}$$

with V_p in (4.2.40) *is an* aug*-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.*

Proof. Since $\hat{C}^i = C^i$, we have $\|\mathbb{H}^i(x,p,l) - \hat{\mathbb{H}}^i(\hat{x},p,l)\| = \|C^ix - \hat{C}^i\hat{x}\|^2 \leq n\lambda_{\max}(C^{iT}C^i)\|x - \hat{x}\|^2$, and similarly $\lambda_{\min}(\tilde{M}_p)\|x-\hat{x}\|^2 \leq (x-\hat{x})^T\tilde{M}_p(x-\hat{x})$. One can readily verify that $\frac{\lambda_{\min}(\tilde{M}_p)}{n\lambda_{\max}(C^{iT}C^i)}\|C^ix - \hat{C}^i\hat{x}\|^2 \leq V_p(x,\hat{x})$ holds $\forall x, \forall \hat{x}$, and consequently, $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}}\frac{\lambda_{\min}(\tilde{M}_p)}{n\lambda_{\max}(C^{iT}C^i)}\|C^ix - \hat{C}^i\hat{x}\|^2 \leq V((x,p,l),(\hat{x},p,l))$, $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$. Since $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}} > 1$, one can conclude that the inequality (4.2.23) holds with $\alpha(s) = \min_p\{\frac{\lambda_{\min}(\tilde{M}_p)}{n\lambda_{\max}(C^{iT}C^i)}\}\, s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (4.2.24) holds, as well. We simplify

$$\begin{aligned} &A_px + E_p\varphi_p(F_px) + B_p + D_pw + R_p\varsigma \\ &\qquad - \Pi_x(A_p\hat{x} + E_p\varphi_p(F_p\hat{x}) + B_p + D_p\hat{w} + R_p\varsigma) \end{aligned}$$

to

$$A_p(x-\hat{x}) + D_p(w-\hat{w}) + E_p(\varphi_p(F_px) - \varphi_p(F_p\hat{x})) + \bar{N}_p, \tag{4.2.43}$$

where $\bar{N}_p = A_p\hat{x} + E_p\varphi_p(F_p\hat{x}) + B_p + D_p\hat{w} + R_p\varsigma - \Pi_x(A_p\hat{x} + E_p\varphi_p(F_p\hat{x}) + B_p + D_p\hat{w} + R_p\varsigma)$. From the slope restriction (4.2.39), one obtains

$$\varphi_p(F_px) - \varphi_p(F_p\hat{x}) = \underline{\delta}_p(F_px - F_p\hat{x}) = \underline{\delta}_pF_p(x-\hat{x}), \tag{4.2.44}$$

where $\underline{\delta}_p$ is a constant and depending on x and $\hat{x}$ takes values in the interval $[0, b_p]$. Using (4.2.44), the expression in (4.2.43) reduces to

$$(A_p + \underline{\delta}_p E_p F_p)(x - \hat{x}) + D_p(w - \hat{w}) + \bar{N}_p.$$

Using Young's inequality [You12] as $cd \leq \frac{\pi}{2}c^2 + \frac{1}{2\pi}d^2$, for any $c, d \geq 0$ and any $\pi > 0$, by employing Cauchy-Schwarz inequality and (4.2.41), and since

$$\|\bar{N}_p\| \ \leq \ \bar{\delta}, \quad \bar{N}_p^T \tilde{M}_p \bar{N}_p \leq n\lambda_{\max}(\tilde{M}_p)\bar{\delta}^2,$$

one can obtain the chain of inequalities in (4.2.46) including the different scenarios as in Definition 4.2.24. By employing the similar argument as the one in [SGZ18, Theorem 1], and by defining $\bar{\kappa} = \max_p\{\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}\}$, $\bar{\rho}_{\text{int}}(s) = \max_p\{\frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\}s^2, \forall s \in \mathbb{R}_{\geq 0}$, and $\bar{\gamma} = \max_p\{\frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}n(1 + 3/\pi_p)\lambda_{\max}(\tilde{M}_p)\}\bar{\delta}^2$, the following inequality

$$\begin{aligned}&\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\Big|\, x, \hat{x}, p, l, w, \hat{w}\Big] \\ &\leq \max\Big\{\tilde{\kappa}V((x, p, l), (\hat{x}, p, l)), \tilde{\rho}_{\text{int}}(\|w - \hat{w}\|), \tilde{\gamma}\Big\} \qquad (4.2.45)\end{aligned}$$

holds for all the scenarios, where $\tilde{\kappa} = (1-(1-\tilde{\pi})(1-\bar{\kappa})$, $\tilde{\rho}_{\text{int}} = \frac{(1+\tilde{\delta}_c)}{(1-\tilde{\kappa})\tilde{\pi}}\bar{\rho}_{\text{int}}$, $\tilde{\gamma} = \frac{(1+1/\tilde{\delta}_c)}{(1-\tilde{\kappa})\tilde{\pi}}\bar{\gamma}$, where $\tilde{\pi}, \tilde{\delta}_c$, can be arbitrarily chosen such that $0 < \tilde{\pi} < 1$, $\tilde{\delta}_c > 0$, $1 - \tilde{\kappa} > 0$. Then the inequality (4.2.24) is satisfied with $\nu = \hat{\nu}$, $\kappa = \tilde{\kappa}$, $\rho_{int} = \tilde{\rho}_{int}$, and $\psi = \tilde{\gamma}$. Hence V defined in (4.2.42) is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, which completes the proof. □

Remark 4.2.41. *If $\forall p \in P$, there exists a common $V : X \times X \to \mathbb{R}_{\geq 0}$ satisfies Assumption 4.2.38, then $V, \alpha, \bar{\kappa}, \bar{\rho}_{\text{int}}$ and $\bar{\gamma}$ reduce to the functions $V((x, p, l), (\hat{x}, p, l)) = V(x, \hat{x}), \alpha(s) = \frac{\lambda_{\min}(\tilde{M}_p)}{n\lambda_{\max}(C^TC)}s^2, \bar{\rho}_{\text{int}}(s) = \bar{p}(1 + \pi_p + 2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2 s^2, \forall s \in \mathbb{R}_{\geq 0}$, and constants $\bar{\kappa} = \bar{\kappa}_p$, $\bar{\gamma} = n(1 + 3/\pi_p)\lambda_{\max}(\tilde{M}_p)\bar{\delta}^2$.*

Remark that $\tilde{\mu}$ used in Theorem 4.2.40 is the one appearing in Assumption 4.2.33. Given the quadratic form of V_p in (4.2.40), $\forall p \in P$, we can always choose $\tilde{\mu} \geq 1$ satisfying Assumption 4.2.33 as discussed in Remark 4.2.34.

4.2.2.6 Case Studies

In this subsection, to demonstrate the effectiveness of our proposed results, we first apply our approaches to a road traffic network in a circular cascade ring composed of 200 identical cells, each of which has the length of 500 meters with 1 entry and 1 way out, and construct compositionally a *finite* MDP of the network. We employ the constructed finite abstraction as a substitute to compositionally synthesize policies keeping the density of the traffic lower than 20 vehicles per cell. Finally, to show the applicability of our results to switched systems accepting *Multiple* Lyapunov functions with *dwell-time*, we apply our proposed techniques to a *fully interconnected* network of 500 *nonlinear* subsystems (totally 1000 dimensions) and construct their *finite* MDPs with guaranteed error bounds on their probabilistic output trajectories.

- First Scenario ($l < k_d - 1, \|f(\hat{x},\hat{w},\varsigma) - \hat{f}_p(\hat{x},\hat{w},\varsigma)\| \leq \bar{\delta}, p' = p, l' = l+1$):

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l')) \,\big|\, x,\hat{x},p,l,w,\hat{w}\Big] = \frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[V_{p'}(x',\hat{x}') \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$= \frac{1}{\bar{\kappa}_p^{(1+l)/\bar{\epsilon}}}\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma)) \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$= \frac{1}{\bar{\kappa}_p^{(1+l)/\bar{\epsilon}}}\big((x-\hat{x})^T\Big[(A_p+\underline{\delta}_pE_pF_p)^T\tilde{M}_p(A_p+\underline{\delta}_pE_pF_p)\Big](x-\hat{x})+(w-\hat{w})^TD_p^T\tilde{M}_pD_p(w-\hat{w})$$
$$+ 2\Big[(x-\hat{x})^T(A_p+\underline{\delta}_pE_pF_p)^T\Big]\tilde{M}_p\Big[D_p(w-\hat{w})\Big]+2\Big[(w-\hat{w})^TD_p^T\Big]\tilde{M}_p\mathbb{E}\Big[\bar{N}_p \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$+ 2\Big[(x-\hat{x})^T(A_p+\underline{\delta}_pE_pF_p)^T\Big]\tilde{M}_p\mathbb{E}\Big[\bar{N}_p \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big] + \mathbb{E}\Big[\bar{N}_p^T\tilde{M}_p\bar{N}_p \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]\big)$$
$$\leq \frac{1}{\bar{\kappa}_p^{(1+l)/\bar{\epsilon}}}\big(\begin{bmatrix} x-\hat{x} \\ \underline{\delta}_pF_p(x-\hat{x})\end{bmatrix}^T\begin{bmatrix}(1+2\pi_p)A_p^T\tilde{M}_pA_p & A_p^T\tilde{M}_pE_p \\ * & (1+2\pi_p)E_p^T\tilde{M}_pE_p\end{bmatrix}\begin{bmatrix} x-\hat{x} \\ \underline{\delta}_pF_p(x-\hat{x})\end{bmatrix}$$
$$+ \bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2 + n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big)$$
$$\leq \frac{1}{\bar{\kappa}_p^{(1+l)/\bar{\epsilon}}}\big(\begin{bmatrix} x-\hat{x} \\ \underline{\delta}_pF_p(x-\hat{x})\end{bmatrix}^T\begin{bmatrix}\bar{\kappa}_p\tilde{M}_p & -F_p^T \\ -F_p & \frac{2}{b_p}\end{bmatrix}\begin{bmatrix} x-\hat{x} \\ \underline{\delta}_pF_p(x-\hat{x})\end{bmatrix}$$
$$+ \bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2 + n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big)$$
$$= \frac{1}{\bar{\kappa}_p^{(1+l)/\bar{\epsilon}}}\big(\bar{\kappa}_p(V_p(x,\hat{x})) - 2\underline{\delta}_p(1-\frac{\bar{\delta}}{b_p})(x-\hat{x})^TF_p^TF_p(x-\hat{x})$$
$$+ \bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2 + n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big)$$
$$\leq \bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l)) + \frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\big(\bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2$$
$$+ n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big);$$

- Second Scenario ($l = k_d - 1, \|f(\hat{x},\hat{w},\varsigma) - \hat{f}_p(\hat{x},\hat{w},\varsigma)\| \leq \bar{\delta}, p' = p, l' = k_d - 1$):

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l')) \,\big|\, x,\hat{x},p,l,w,\hat{w}\Big] = \frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[V_{p'}(x',\hat{x}') \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$= \frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}}\mathbb{E}\Big[V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\varsigma)) \,\big|\, x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$\leq \bar{\kappa}_pV((x,p,l),(\hat{x},p,l)) + \frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\big(\bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2$$
$$+ n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big)$$
$$\leq \bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l)) + \frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\big(\bar{p}(1+\pi_p+2/\pi_p)\|\sqrt{\tilde{M}_p}D_p\|_2^2\|w-\hat{w}\|^2$$
$$+ n(1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2\big);$$

- **Last Scenario** $(l = k_d - 1, \|f(\hat{x}, \hat{w}, \varsigma) - \hat{f}_p(\hat{x}, \hat{w}, \varsigma)\| \leq \bar{\delta}, p' \neq p, l' = 0)$:

$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\big|\, x, \hat{x}, p, l, w, \hat{w}\Big] = \frac{1}{\bar{\kappa}_{p'}^{l'/\bar{\epsilon}}} \mathbb{E}\Big[V_{p'}(x', \hat{x}') \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$
$$= \tilde{\mu}\; \mathbb{E}\Big[V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \varsigma)) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$
$$\leq \tilde{\mu}\bar{\kappa}_p^{(k_d-1)/\bar{\epsilon}} \bar{\kappa}_p V((x, p, l), (\hat{x}, p, l)) + \tilde{\mu}\big(\bar{p}(1 + \pi_p + 2/\pi_p)\|\sqrt{\tilde{M}_p} D_p\|_2^2 \|w - \hat{w}\|^2$$
$$+ n(1 + 3/\pi_p)\lambda_{\max}(\tilde{M}_p)\, \bar{\delta}^2\big)$$
$$\leq \bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\bar{\kappa}_p^{k_d/\bar{\epsilon}}}\big(\bar{p}(1 + \pi_p + 2/\pi_p)\|\sqrt{\tilde{M}_p} D_p\|_2^2 \|w - \hat{w}\|^2$$
$$+ n(1 + 3/\pi_p)\lambda_{\max}(\tilde{M}_p)\, \bar{\delta}^2\big). \tag{4.2.46}$$

4.2.2.6.1 Road Traffic Network

In this subsection, we apply our results to a road traffic network in a circular cascade ring which is composed of 200 identical cells, each of which has the length of 500 meters with 1 entry and 1 way out, as schematically depicted in Figure 4.7. The model of this case study is borrowed from [LCGG13] by including the stochasticity in the model as the additive noise. The entry is controlled by a traffic light, that enables (green light)

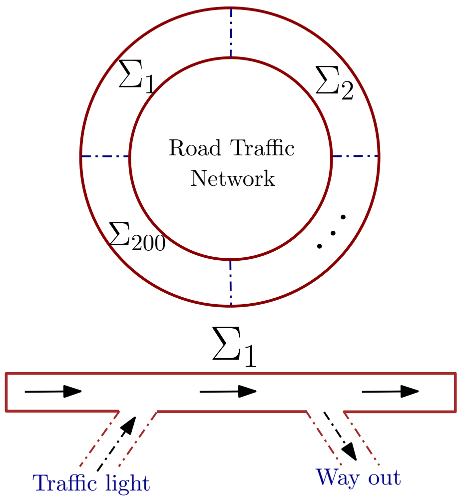

Figure 4.7: Model of a road traffic network in a circular cascade ring composed of 200 identical cells, each of which has the length of 500 meters with 1 entry and 1 way out.

or not (red light) the vehicles to pass. In this model the length of a cell is in kilometers (0.5 [km]), and the flow speed of vehicles is 100 kilometers per hour ([km/h]). Moreover, during the sampling time interval $\tau = 6.48$ seconds, it is assumed that 8 vehicles pass the entry controlled by the green light, and one quarter of vehicles goes out on the exit of each cell (the ratio denoted $\tilde{q}$). We want to observe the density of the traffic x_i, given in vehicles per cell, for each cell i of the road. The set of modes is $P_i = \{1, 2\}, i \in \{1, \ldots, n\}$ such that

- mode 1 means the traffic light is red;
- mode 2 means the traffic light is green.

Note that here we only have traffic signals on the on-ramps. The dynamic of the interconnected system is described by:

$$\Sigma : \begin{cases} x(k+1) = A\,x(k) + B_{\mathbf{p}(k)} + \varsigma(k), \\ y(k) = x(k), \end{cases} \tag{4.2.47}$$

where A is a matrix with diagonal elements $a_{ii} = (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})$, $i \in \{1, \ldots, n\}$, off-diagonal elements $a_{i+1,i} = \frac{\tau v_i}{\tilde{l}_i}$, $i \in \{1, \ldots, n-1\}$, $a_{1,n} = \frac{\tau v_n}{\tilde{l}_n}$, and all other elements are identically zero. Moreover, $B_p = [b_{1p_1}; \ldots; b_{np_n}]$, $x(k) = [x_1(k); \ldots; x_n(k)]$, $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_n(k)]$, and

$$b_{ip_i} = \begin{cases} 0, & \text{if} \quad p_i = 1, \\ 8, & \text{if} \quad p_i = 2. \end{cases}$$

Furthermore, the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions (i.e., mean zero and covariance matrix identity). Now by introducing the individual cells Σ_i described as

$$\Sigma_i : \begin{cases} x_i(k+1) = (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})\,x_i(k) + D_i w_i(k) + b_{i\mathbf{p}_i(k)} + \varsigma_i(k), \\ y_i^1(k) = x_i(k), \\ y_i^2(k) = x_i(k), \end{cases} \tag{4.2.48}$$

where $D_i = \frac{\tau v_{i-1}}{\tilde{l}_{i-1}}$ (with $v_0 = v_n$, $\tilde{l}_0 = \tilde{l}_n$) and $w_i(k) = y_{i-1}^2(k)$ (with $y_0^2 = y_n^2$), one can readily verify that $\Sigma = \mathcal{I}_{ss}(\Sigma_1, \ldots, \Sigma_N)$, equivalently $\Sigma = \mathcal{I}_{ss}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$. Note that we consider sets $X_i = W_i = [0 \ \ 20]$, $\forall i \in \{1, \ldots, n\}$. Since the dynamic of the system is linear, condition (4.2.41) reduces to

$$(1 + 2\pi_i)A_i^T \tilde{M}_i A_i \preceq \bar{\kappa}_i \tilde{M}_i,$$

which is nothing more than the stability of each cell i. Note that in this example $V_p = V_{p'}, \forall p, p' \in P$ (i.e., the common Lyapunov function). Then one can readily verify that this condition is satisfied with $\tilde{M}_i = 1$, $\pi_i = 0.85$, $\bar{\kappa}_i = 0.41$ $\forall i \in \{1, \ldots, n\}$, and the function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ satisfying the condition (4.2.23) with $\alpha_i(s) = s^2$ and the condition (4.2.24) with $\kappa_i = 0.99$, $\rho_{\text{int}i}(s) = 0.72s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 84.96\,\bar{\delta}_i^2$.

Now we check the small-gain condition (4.2.7) that is required for the compositionality result. By taking $\sigma_i(s) = s$, $\forall i \in \{1, \ldots, n\}$, the condition (4.2.7) and as a result the condition (4.2.8) are always satisfied without any restriction on the number of cells. Hence, $V(x,\hat{x}) = \max_i (x_i - \hat{x}_i)^2$ is an aug-type SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ satisfying conditions (4.2.25) and (4.2.26) with $\alpha(s) = s^2$, $\kappa = 0.99$, and $\psi = 84.96\,\bar{\delta}^2$.

We take the state and internal input discretization parameters as 0.02. Then we have $n_{x_i} = n_{w_i} = 1000$. By taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $10\mathbb{1}_{200}$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 15$ with the probability at least 88%, i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 1,\ \forall k \in [0, 15]) \geq 0.88.$$

4.2.2.6.2 Compositional Controller Synthesis

Let us now synthesize a controller for Σ via the abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma})$ such that the *safety* controller maintains the density of the traffic lower than 20 vehicles per cell. The idea here is to first design a local controller for the abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$, and then refine it back to the system Σ_i. Consequently, the controller for the interconnected system Σ would be a vector such that each of its components is the controller for systems Σ_i. We employ here the software tool FAUST^2 [SGA15] by doing some modifications to accept internal inputs as disturbances, and synthesize a controller for Σ by choosing the standard deviation of the noise $\bar{\sigma}_i = 0.83$, $\forall i \in \{1, \ldots, n\}$. An optimal switch for a representative cell in a network of $N = 200$ cells is plotted in Figure 4.8 top. The optimal switch here is sub-optimal for each subsystem and is obtained by assuming that other subsystems do not violate the safety specification. An optimal switch w.r.t. time for a representative cell with different noise realizations is also illustrated in Figure 4.8 middle, with 10 different realizations. Moreover, closed-loop state trajectories of the representative cell with different noise realizations are illustrated in Figure 4.8 bottom, with 10 realizations.

4.2.2.6.3 Memory Usage and Computation Time

Now we discuss the memory usage and computation time of constructing finite MDPs in both monolithic and compositional manners. The monolithic finite MDP would be a matrix with the dimension of $(n_{x_i}^N \times 2^N) \times n_{x_i}^N$ with $n_{x_i} = 1000$ and $N = 200$. By allocating 8 bytes for each entry of the matrix to be stored as a double-precision floating point, one needs a memory of $\frac{8\times 1000^{200}\times 2^{200}\times 1000^{200}}{10^9} \approx 10^{1252}$ GB for building the finite MDP in the monolithic manner which is impossible in practice. Now we proceed with the compositional construction of finite MDPs proposed in this thesis. The constructed MDP for each subsystem here is a matrix with the dimension of $(n_{x_i} \times 2 \times n_{w_i}) \times n_{x_i}$ with $n_{x_i} = n_{w_i} = 1000$. This has the memory usage of $\frac{8\times 1000\times 2\times 1000\times 1000}{10^9} = 16$ GB. We can compute such a finite MDP with the software tool FAUST^2, which takes 112 seconds on a machine with Windows operating system (Intel i7@3.6GHz CPU and 16 GB of RAM).

A comparison on the required memory for the construction of finite MDPs between the monolithic and compositional manners for different state discretization parameters is provided in Table 4.1. As seen, in order to provide even a very weak closeness guarantee

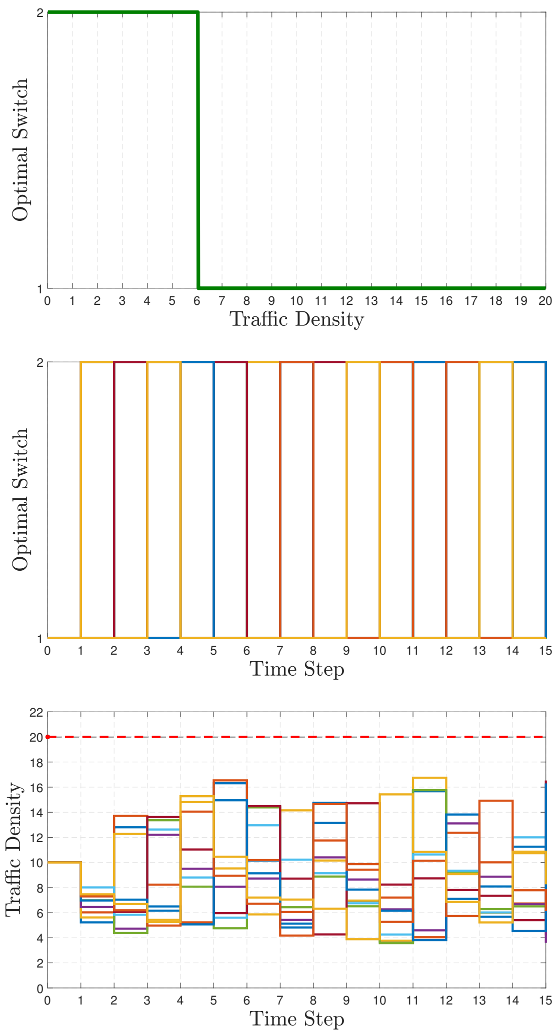

Figure 4.8: Top: An optimal switch for a representative cell in a network of 200 cells. Middle: An optimal switch w.r.t. time for a representative cell with different noise realizations. Bottom: Closed-loop state trajectories of a representative cell with different noise realizations.

of 2% between trajectories of Σ and of $\widehat{\Sigma}$, the required memory in the monolithic fashion is 10^{972} GB which is still impossible in practice. This implementation clearly shows that the proposed compositional approach in this work significantly mitigates the curse of dimensionality problem in constructing finite MDPs monolithically. In particular, in order to quantify the probabilistic closeness between two networks Σ and $\widehat{\Sigma}$ via the inequality (4.2.27) as provided in Table 4.1, one needs to only build finite MDPs of individual subsystems (i.e., $\widehat{\Sigma}_i$), construct an aug-type SPSF between each Σ_i and $\widehat{\Sigma}_i$, and then employ the proposed compositionality results of the section to build an aug-type SSF between Σ and $\widehat{\Sigma}$.

Table 4.1: Required memory for the construction of finite MDPs in both monolithic and compositional manners for different state discretization parameters.

$\bar{\delta}$	Closeness	$\widehat{\Sigma}_i$ (GB)	$\widehat{\Sigma}$ (GB)
0.01	97%	128	10^{1372}
0.02	88%	16	10^{1252}
0.03	75%	4.72	10^{1181}
0.04	60%	2	10^{1131}
0.05	44%	1.02	10^{1092}
0.06	30%	0.59	10^{1061}
0.07	19%	0.37	10^{1033}
0.08	11%	0.25	10^{1011}
0.09	5%	0.17	10^{990}
0.1	2%	0.12	10^{972}

4.2.2.6.4 Comparisons with DBN Approach of [SAM17]

We compare the probabilistic closeness guarantees provided by the results of this section with that of [SAM17]. Note that our results are based on the max small-gain approach while [SAM17] employs dynamic Bayesian network (DBN) to capture the dependencies between subsystems. The comparison is shown in Figures 4.9-4.11 in the logarithmic scale. In Figure 4.9, we have fixed $\varepsilon = 1$, $\bar{\sigma}_i = 0.83$, $T_d = 15$, and plotted the error as a function of the state discretization parameter $\bar{\delta}$ and the number of subsystems N. As seen, by increasing the number of subsystems, our error provided in (4.2.27) does not change since the overall ψ is independent of the size of the network (i.e., N), and is computed only based on the maximum ψ_i of subsystems instead of being a linear combination of them which is the case in [SAM17]. In Figure 4.10, we have fixed $N = 200$, $\varepsilon = 1$, $T_d = 15$, and plotted the error as a function of $\bar{\delta}$ and the standard deviation of the noise $\bar{\sigma}$. Our error in (4.2.27) is independent of $\bar{\sigma}$ while the error in [SAM17] grows when $\bar{\sigma}$ goes to zero. In Figure 4.11, we have fixed $N = 200$, $\bar{\sigma}_i = 0.83$, $T_d = 15$, and plotted the error as a function of $\bar{\delta}$ and ε. The error in [SAM17] is independent of ε while our error increases when ε goes to zero.

In conclusion, the proposed approach in [SAM17] is more general than our setting here. It does not require original systems to be incrementally input-to-state stable (δ-ISS) and only the Lipschitz continuity of the associated stochastic kernels is enough for validity of the results. The refinement does not require running the abstract systems and obtaining the input according to an interface function. On the other hand, the abstraction error in [SAM17] depends on the number of subsystems and also the Lipschitz constants of the stochastic kernels associated with the system. Thus, our approach outperforms the results in [SAM17] for large-scale stochastic systems with a small standard deviation of the noise as long as the imposed assumptions are satisfied.

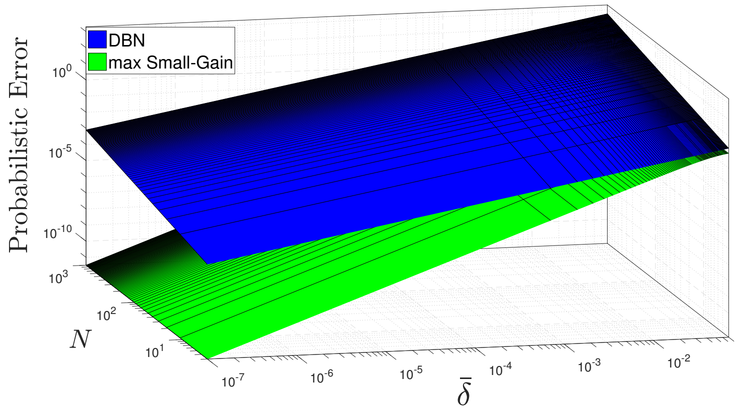

Figure 4.9: Comparison of the probabilistic error bound in (4.2.27) provided by our approach based on the max small-gain with that of [SAM17] based on DBN. Plots are in the logarithmic scale for a fixed $\varepsilon = 1$, $\bar{\sigma}_i = 0.83$, and $T_d = 15$.

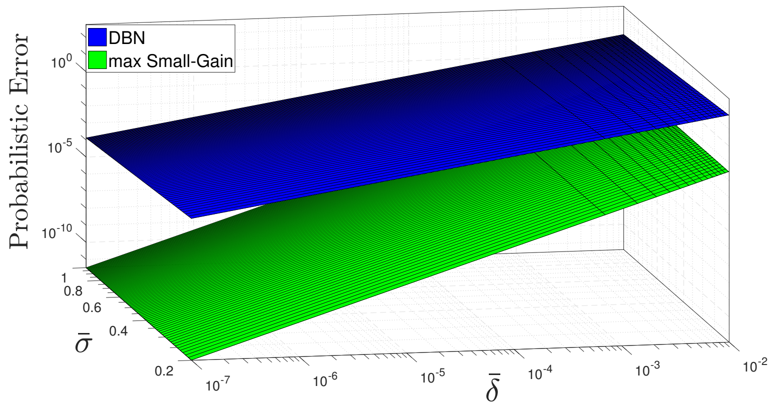

Figure 4.10: Comparison of the probabilistic error bound in (4.2.27) provided by our approach based on max the small-gain with that of [SAM17] based on DBN. Plots are in the logarithmic scale for a fixed $N = 200$, $\varepsilon = 1$, and $T_d = 15$.

4.2.2.6.5 Switched Systems Accepting Multiple Lyapunov Functions with Dwell-Time

In order to show the applicability of our results to switched systems accepting *multiple* Lyapunov functions with *dwell-time*, we apply our proposed techniques to a *fully interconnected* network of 500 *nonlinear* subsystems in the form of (4.2.38) (totally 1000 dimensions), as illustrated in Figure 4.12. The model of the system does not have a common Lyapunov function because it exhibits unstable behaviors for different switching signals [Lib03] (i.e., if one periodically switches between different modes, the trajectory

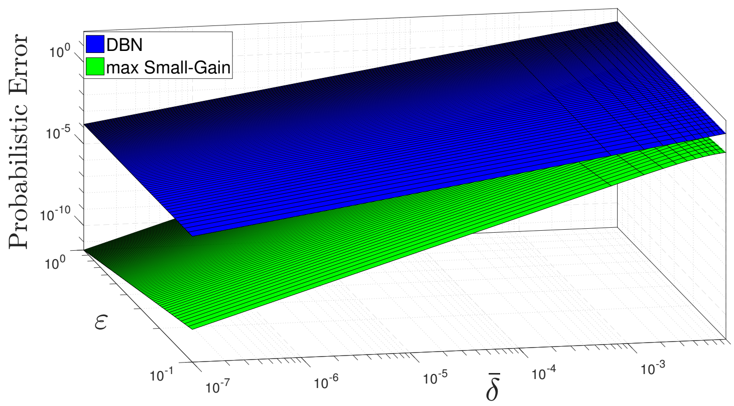

Figure 4.11: Comparison of the probabilistic error bound in (4.2.27) provided by our approach based on the max small-gain with that of [SAM17] based on DBN. Plots are in the logarithmic scale for a fixed $N = 200$, $\bar{\sigma}_i = 0.83$, and $T_d = 15$.

goes to infinity). The dynamic of the interconnected system is described by:

$$\Sigma : \begin{cases} x(k+1) = A_{\mathbf{p}(k)}x(k) + B_{\mathbf{p}(k)} + \varphi(x(k)) + R\varsigma(k), \\ y(k) = x(k), \end{cases} \tag{4.2.49}$$

where

$$A_{\mathbf{p}(k)} = \begin{bmatrix} \bar{A}_{pi} & \tilde{A} & \cdots & \cdots & \tilde{A} \\ \tilde{A} & \bar{A}_{pi} & \tilde{A} & \cdots & \tilde{A} \\ \tilde{A} & \tilde{A} & \bar{A}_{pi} & \cdots & \tilde{A} \\ \vdots & & \ddots & \ddots & \vdots \\ \tilde{A} & \cdots & \cdots & \tilde{A} & \bar{A}_{pi} \end{bmatrix}_{n\times n},$$

$$\tilde{A} = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \qquad \bar{A}_{p_i} = \begin{cases} \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix}, & \text{if} \quad p_i = 1, \\ \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix}, & \text{if} \quad p_i = 2. \end{cases}$$

Moreover, we choose $R = \mathsf{diag}(\mathbb{1}_2, \ldots, \mathbb{1}_2)$, $\varphi(x) = [0.1\mathbb{1}_2\varphi_1(0.1\mathbb{1}_2^T x_1(k)); \ldots; 0.1\mathbb{1}_2\varphi_N (0.1\mathbb{1}_2^T x_N(k))]$, and $\varphi_i(x) = sin(x)$, $\forall i \in \{1, \ldots, N\}$. Note that functions φ_i satisfy the condition (4.2.39) with $b_{p_i} = 1$. We fix here $N = 500$. Furthermore, $B_p = [b_{1p_1}; \ldots; b_{Np_N}]$ such that

$$b_{ip_i} = \begin{cases} \begin{bmatrix} -0.9 \\ 0.5 \end{bmatrix}, & \text{if} \quad p_i = 1, \\ \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix}, & \text{if} \quad p_i = 2. \end{cases}$$

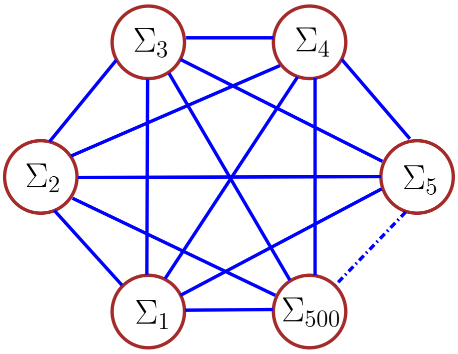

Figure 4.12: A *fully interconnected* network of 500 *nonlinear* components (totally 1000 dimensions).

We partition $x(k)$ as $x(k) = [x_1(k); \ldots; x_N(k)]$ and $\varsigma(k)$ as $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_N(k)]$, where $x_i(k), \varsigma_i(k) \in \mathbb{R}^2$. Now, by introducing the individual subsystems Σ_i described as

$$\Sigma_i : \begin{cases} x_i(k+1) = \bar{A}_{\mathbf{p}_i(k)} x_i(k) + b_{i\mathbf{p}_i(k)} + D_i w_i(k) + 0.1\mathbb{1}_2 \varphi_i(0.1\mathbb{1}_2^T x_i(k)) + \mathbb{1}_2 \varsigma_i(k), \\ y_i^1(k) = x_i(k), \\ y_i^2(k) = x_i(k), \end{cases} \tag{4.2.50}$$

where

$$D_i = [\tilde{A}; \ldots; \tilde{A}]^T_{2\times(n-2)},$$
$$w_i(k) = [y_{i1}^2; \ldots; y_{i(i-1)}^2; y_{i(i+1)}^2; \ldots; y_{iN}^2], \ i \in \{1, \ldots, N\},$$

one can readily verify that $\Sigma = \mathcal{I}_{ss}(\Sigma_1, \ldots, \Sigma_N)$, equivalently $\Sigma = \mathcal{I}_{ss}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$. One can also verify that, $\forall i \in \{1, \ldots, N\}$, the condition (4.2.41) is satisfied with

$$\text{for } p_i = 1: \ \tilde{M}_{p_i} = \begin{bmatrix} 1.311 & 0.001 \\ 0.001 & 0.492 \end{bmatrix}, \bar{\kappa}_{p_i} = 0.7, \pi_{p_i} = 0.5,$$
$$\text{for } p_i = 2: \ \tilde{M}_{p_i} = \begin{bmatrix} 0.4 & 0.01 \\ 0.01 & 1.49 \end{bmatrix}, \bar{\kappa}_{p_i} = 0.7, \pi_{p_i} = 0.4.$$

By taking $\bar{\epsilon} = 1.75$ and choosing $\tilde{\mu} = 3.27$, one can get the dwell-time $k_d = 7$. Hence, $V_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = \frac{1}{\bar{\kappa}_{p_i}^{-l/1.75}}(x_i - \hat{x}_i)^T \tilde{M}_{ip_i}(x_i - \hat{x}_i)$ is an aug-type SPSF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ satisfying the condition (4.2.23) with $\alpha_i(s) = 0.2s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and the condition (4.2.24) with $\kappa_i = 0.99$, $\rho_{\text{int}i}(s) = 0.19s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 2266\,\bar{\delta}_i^2$.

Now we check small-gain condition (4.2.7) that is required for the compositionality result. By taking $\sigma_i(s) = s$, $\forall i \in \{1, \ldots, N\}$, the condition (4.2.7) and as a result the condition (4.2.8) are satisfied. Hence, $V((x, p, l), (\hat{x}, p, l)) = \max_i\{\frac{1}{\bar{\kappa}_{p_i}^{-l/1.75}}(x_i - \hat{x}_i)^T \tilde{M}_{ip_i}(x_i - \hat{x}_i)\}$ is an aug-type SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ satisfying conditions (4.2.25) and (4.2.26) with $\alpha(s) = 0.2s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, $\kappa = 0.99$, and $\psi = 2266\,\bar{\delta}^2$.

By taking the state discretization parameter $\bar{\delta}_i = 0.001$, and choosing the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $\mathbb{1}_{1000}$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with the probability at least 90%, i.e.,

$$\mathbb{P}(\|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 1, \ \forall k \in [0, 10]) \geq 0.9.$$

4.2.2.6.6 Analysis on Probabilistic Closeness Guarantee

In order to have a practical analysis on the proposed probabilistic closeness guarantee, we plotted in Figure 4.13 the probabilistic error bound provided in (4.2.27) in terms of the state discretization parameter $\bar{\delta}$ and the confidence bound ε. As seen, the probabilistic closeness guarantee is improved by either decreasing $\bar{\delta}$ or increasing ε. Note that the constant ψ in (4.2.27) is formulated based on the state discretization parameter $\bar{\delta}$ as in (4.2.46). It is worth mentioning that there are some other parameters in (4.2.27) such as $\mathcal{K}_\infty$ function α, and the value of SSF V at initial conditions $a, \hat{a}, p_0, l_0$ which can also improve the proposed bound for given values of T_d and initial conditions of the system.

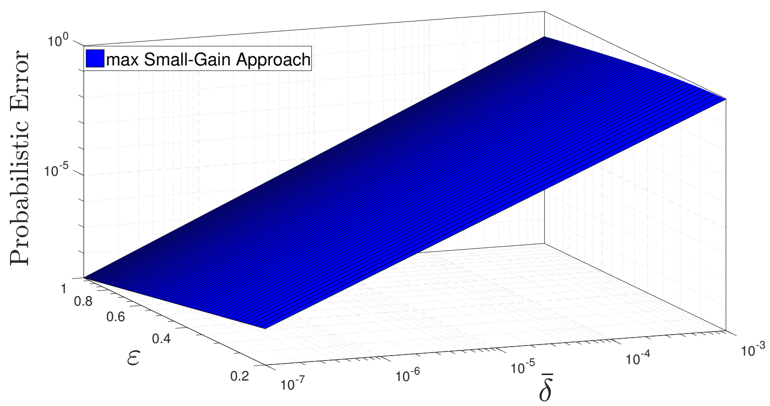

Figure 4.13: Probabilistic error bound proposed in (4.2.27) based on $\bar{\delta}$ and ε. Plot is in the logarithmic scale for $T_d = 10$. The probabilistic closeness guarantee is improved by either decreasing the state discretization parameter $\bar{\delta}$ or increasing the confidence bound ε.

4.3 Dissipativity Approach

In this section, we provide a compositional methodology based on the dissipativity approach for the construction of *finite* abstractions for the both stochastic control and switched systems. The proposed technique leverages the interconnection structure and joint dissipativity-type properties of subsystems and their abstractions characterized via stochastic storage functions. The provided compositionality conditions can enjoy the structure of the interconnection topology and be potentially satisfied independently of the number or gains of subsystems.

4.3.1 Stochastic Control Systems

We first consider the stochastic control systems defined in (2.3.1) and the SStF and sum-type SSF in Definitions 3.4.1, and 3.2.4 for quantifying the probabilistic error between two dt-SCS (with both internal and external signals) and two interconnected dt-SCS (without internal signals), respectively. We rewrite Theorem 3.4.4 as the compositional results of this section to establish a sum-type SSF between Σ and its finite MDP $\widehat{\Sigma}$.

Theorem 4.3.1. *Consider the interconnected stochastic control system $\Sigma = \mathcal{I}_{cd}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i and the coupling matrix M. Suppose that each stochastic control subsystem Σ_i admits its finite abstraction $\widehat{\Sigma}_i$ with the corresponding SStF V_i. Suppose conditions* (3.4.3)*, and* (3.4.4) *are satisfied. Then the weighted sum* (3.4.7) *is a* sum*-type SSF from the interconnected finite MDP $\widehat{\Sigma} = \mathcal{I}_{cd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$, with coupling matrix $\hat{M}$, to $\Sigma = \mathcal{I}_{cd}(\Sigma_1, \ldots, \Sigma_N)$ if $\mu_i > 0$, $i \in \{1, \ldots, N\}$, and the following inclusion holds:*

$$\hat{M} \prod_{i=1}^{N} \hat{Y}_i^2 \subseteq \prod_{i=1}^{N} \hat{W}_i. \tag{4.3.1}$$

The proof is similar to that of Theorem 3.4.4 and is omitted here.

Remark 4.3.2. *Note that the condition* (4.3.1) *is not restrictive since $\hat{W}_i$ and $\hat{Y}_i^2$ are internal input and output sets of abstract subsystems $\widehat{\Sigma}_i$, which are finite. Thus one can readily choose internal input sets $\hat{W}_i$ such that $\prod_{i=1}^{n} \hat{W}_i := \hat{M} \prod_{i=1}^{n} \hat{Y}_i^2$ which implicitly implies a condition on the granularity of the discretization for sets W_i and Y_i^2. In other words, the condition* (4.3.1) *is required for just having a well-posed interconnection.*

4.3.1.1 Construction of SStF

In this subsection, we impose conditions on the infinite dt-SCS Σ enabling us to find an SStF from its finite abstraction $\widehat{\Sigma}$ to Σ. The required conditions are first presented in a general setting of nonlinear stochastic control systems in the next subsection and then represented via some matrix inequality for linear stochastic control systems in Subsection 4.3.1.1.2.

4.3.1.1.1 General Setting of Nonlinear Stochastic Control Systems

The stochastic storage function from the finite MDP $\widehat{\Sigma}$ to Σ is established under the assumption that the original discrete-time stochastic control system Σ is incrementally passivable as in Definition 2.8.1.

Remark 4.3.3. *Note that Definition 2.8.1 implies that V is an SStF from system Σ equipped with the state feedback controller $\bar{H}$ to itself. This type of property is closely related to the notion of so-called incremental stabilizability [Ang02, PTS09].*

In Subsection 4.3.1.1.2, we show that inequalities (2.8.1), (2.8.2) for a candidate quadratic function V and linear stochastic control systems boil down to some matrix inequality. Under Definition 2.8.1, the next theorem shows a relation between Σ and $\widehat{\Sigma}$, constructed as in Algorithm 1, via establishing an SStF between them.

Theorem 4.3.4. *Let Σ be an incrementally passivable dt-SCS via a function V as in Definition 2.8.1 and $\widehat{\Sigma}$ be its finite MDP as in Algorithm 1. Assume that there exists a function $\gamma \in \mathcal{K}_\infty$ such that the condition* (4.2.11) *is satisfied. Then V is an SStF from $\widehat{\Sigma}$ to Σ.*

Proof. Since the system Σ is incrementally passivable, from (2.8.1), and since $h^1 = \hat{h}^1$, $\forall x \in X$ and $\forall \hat{x} \in \hat{X}$, we have

$$\underline{\alpha}(\|h^1(x) - h^1(\hat{x})\|_2) = \underline{\alpha}(\|h^1(x) - \hat{h}^1(\hat{x})\|_2) \leq V(x, \hat{x}),$$

satisfying (3.4.1) with $\alpha(s) := \underline{\alpha}(s)\ \forall s \in \mathbb{R}_{\geq 0}$. Now by taking the conditional expectation from (4.2.11), $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall \hat{\nu} \in \hat{U}, \forall w \in W, \forall \hat{w} \in \hat{W}$, we have

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\quad - \mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), f(\hat{x}, \bar{H}(\hat{x}) + \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\quad \leq \mathbb{E}\Big[\gamma(\|\hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma) - f(\hat{x}, \bar{H}(\hat{x}) + \hat{\nu}, \hat{w}, \varsigma)\|_2) \,\Big|\, \hat{x}, \hat{x}, \hat{\nu}, w, \hat{w}\Big],
\end{aligned}$$

where $\hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma) = \Pi_x(f(\hat{x}, \bar{H}(\hat{x}) + \hat{\nu}, \hat{w}, \varsigma))$. Using (4.2.5), the above inequality reduces to

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\
&\quad - \mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), f(\hat{x}, \bar{H}(\hat{x}) + \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \leq \gamma(\bar{\delta}).
\end{aligned}$$

Employing (2.8.2) and since $h^2 = \hat{h}^2$, we get

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), f(\hat{x}, \bar{H}(\hat{x}) + \hat{\nu}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] - V(x, \hat{x}) \\
&\quad \leq -\hat{\kappa}(V(x, \hat{x})) + \begin{bmatrix} w - \hat{w} \\ h^2(x) - \hat{h}^2(\hat{x}) \end{bmatrix}^T \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix} \begin{bmatrix} w - \hat{w} \\ h^2(x) - \hat{h}^2(\hat{x}) \end{bmatrix}.
\end{aligned}$$

It follows that $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall \hat{u} \in U$, and $\forall w \in W, \forall \hat{w} \in \hat{W}$,

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x, \bar{H}(x) + \hat{\nu}, w, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma))) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] - V(x, \hat{x}) \\
&\quad \leq -\hat{\kappa}(V(x, \hat{x})) + \gamma(\bar{\delta}) + \begin{bmatrix} w - \hat{w} \\ h^2(x) - \hat{h}^2(\hat{x}) \end{bmatrix}^T \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix} \begin{bmatrix} w - \hat{w} \\ h^2(x) - \hat{h}^2(\hat{x}) \end{bmatrix},
\end{aligned}$$

satisfying (3.4.2) with $\psi = \gamma(\bar{\delta})$, $\nu = \bar{H}(x) + \hat{\nu}$, $\kappa = \hat{\kappa}$, $\rho_{ext} \equiv 0$, and G, $\hat{G}$, H are identity matrices of appropriate dimensions. Hence, V is an SStF from $\widehat{\Sigma}$ to Σ which completes the proof. $\square$

Now we provide similar results as in this subsection but tailored to linear stochastic control systems.

4.3.1.1.2 Discrete-Time Linear Stochastic Control Systems

In this subsection, we focus on the linear class of discrete-time stochastic control systems Σ as defined in (3.2.18) and *quadratic* functions V in (4.2.14). In order to show that V in (4.2.14) is an SStF from $\widehat{\Sigma}$ to Σ, we require the following key assumption on Σ.

Assumption 4.3.5. *Let $\Sigma = (A, B, C^1, C^2, D, R)$. Assume that for some constants $0 < \hat{\kappa} < 1$ and $\pi > 0$, there exist matrices $\tilde{M} \succ 0$, K, $\bar{X}^{11}$, $\bar{X}^{12}$, $\bar{X}^{21}$, and $\bar{X}^{22}$ of appropriate dimensions such that the following matrix inequality holds:*

$$\begin{bmatrix}(1+\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}D\\ D^T\tilde{M}(A+BK) & (1+\pi)D^T\tilde{M}D\end{bmatrix} \preceq \begin{bmatrix}\hat{\kappa}\tilde{M}+C^{2T}\bar{X}^{22}C^2 & C^{2T}\bar{X}^{21}\\ \bar{X}^{12}C^2 & \bar{X}^{11}\end{bmatrix}. \tag{4.3.2}$$

Now we provide another main result of this section showing that under some conditions V in (4.2.14) is an SStF from $\widehat{\Sigma}$ to Σ.

Theorem 4.3.6. *Let $\Sigma = (A, B, C^1, C^2, D, R)$ and $\widehat{\Sigma}$ be its finite MDP with the discretization parameter $\bar{\delta}$, and $\hat{Y}_1 \subseteq Y_1$. Suppose Assumption 4.3.5 holds, and $C^1 = \hat{C}^1$, $C^2 = \hat{C}^2$. Then the function V defined in (4.2.14) is an SStF from $\widehat{\Sigma}$ to Σ.*

Proof. Here we show that $\forall x, \forall \hat{x}, \forall \hat{\nu}, \exists \nu, \forall \hat{w}, \forall w$, V satisfies $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}\|C^1x - \hat{C}^1\hat{x}\|_2^2 \leq V(x,\hat{x})$ and

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{\nu},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] - V(x,\hat{x})\\ &\leq -(1-\hat{\kappa})(V(x,\hat{x})) + (1+2/\pi)\lambda_{\max}(\tilde{M})\bar{\delta}^2\\ &+\begin{bmatrix}w-\hat{w}\\ h^2(x)-\hat{h}^2(\hat{x})\end{bmatrix}^T\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix}w-\hat{w}\\ h^2(x)-\hat{h}^2(\hat{x})\end{bmatrix}.\end{aligned}$$

Since $C^1 = \hat{C}^1$, we have $\|C^1x - \hat{C}^1\hat{x}\|_2^2 = (x-\hat{x})^TC^{1T}C^1(x-\hat{x})$. Since $\lambda_{\min}(C^{1T}C^1)\|x-\hat{x}\|_2^2 \leq (x-\hat{x})^TC^{1T}C^1(x-\hat{x}) \leq \lambda_{\max}(C^{1T}C^1)\|x-\hat{x}\|_2^2$ and similarly $\lambda_{\min}(\tilde{M})\|x-\hat{x}\|_2^2 \leq (x-\hat{x})^T\tilde{M}(x-\hat{x}) \leq \lambda_{\max}(\tilde{M})\|x-\hat{x}\|_2^2$, it can be readily verified that $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}\|C^1x - \hat{C}^1\hat{x}\|_2^2 \leq V(x,\hat{x})$ holds $\forall x$, $\forall \hat{x}$, implying that the inequality (3.4.1) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C^{1T}C^1)}s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (3.4.2) holds, as well. Given any x, $\hat{x}$, and $\hat{\nu}$, we choose ν via the *interface* function proposed in (4.2.16). Then we simplify

$$Ax + B\nu_{\hat{\nu}}(x,\hat{x},\hat{\nu}) + Dw + R\varsigma - \Pi_x(A\hat{x} + B\hat{\nu} + D\hat{w} + R\varsigma)$$

to

$$(A+BK)(x-\hat{x}) + D(w-\hat{w}) + \bar{N},$$

where $\bar{N} = A\hat{x} + B\hat{\nu} + D\hat{w} + R\varsigma - \Pi_x(A\hat{x} + B\hat{\nu} + D\hat{w} + R\varsigma)$. Using Young's inequality [You12] as $ab \leq \frac{\pi}{2}a^2 + \frac{1}{2\pi}b^2$, for any $a, b \geq 0$ and any $\pi > 0$, by employing Cauchy-Schwarz inequality, $C^2 = \hat{C}^2$, and since

$$\|\bar{N}\|_2 \leq \bar{\delta}, \quad \bar{N}^T\tilde{M}\bar{N} \leq \lambda_{\max}(\tilde{M})\bar{\delta}^2,$$

$$
\begin{aligned}
&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat f(\hat x,\hat\nu,\hat w,\varsigma))\,\Big|\,x,\hat x,\hat\nu,w,\hat w\Big]-V(x,\hat x)\\
&\quad=(x-\hat x)^T(A+BK)^T\tilde M(A+BK)(x-\hat x)+2(x-\hat x)^T(A+BK)^T\tilde M D(w-\hat w)\\
&\qquad+(w-\hat w)^TD^T\tilde M D(w-\hat w)+2(x-\hat x)^T(A+BK)^T\tilde M\mathbb{E}\Big[\bar N\,\Big|\,x,\hat x,\hat\nu,w,\hat w\Big]\\
&\qquad+2(w-\hat w)^TD^T\tilde M\mathbb{E}\Big[\bar N\,\Big|\,x,\hat x,\hat\nu,w,\hat w\Big]+\mathbb{E}\Big[\bar N^T\tilde M\bar N\,\Big|\,x,\hat x,\hat\nu,w,\hat w\Big]-V(x,\hat x)\\
&\quad\leq\begin{bmatrix}x-\hat x\\ w-\hat w\end{bmatrix}^T\begin{bmatrix}(1+\pi)(A+BK)^T\tilde M(A+BK) & (A+BK)^T\tilde M D\\ D^T\tilde M(A+BK) & (1+\pi)D^T\tilde M D\end{bmatrix}\begin{bmatrix}x-\hat x\\ w-\hat w\end{bmatrix}\\
&\qquad+(1+2/\pi)\lambda_{\max}(\tilde M)\bar\delta^2-V(x,\hat x)\\
&\quad\leq\begin{bmatrix}x-\hat x\\ w-\hat w\end{bmatrix}^T\begin{bmatrix}\hat\kappa\tilde M+C^{2T}\bar X^{22}C^2 & C^{2T}\bar X^{21}\\ \bar X^{12}C^2 & \bar X^{11}\end{bmatrix}\begin{bmatrix}x-\hat x\\ w-\hat w\end{bmatrix}+(1+2/\pi)\lambda_{\max}(\tilde M)\bar\delta^2-V(x,\hat x)\\
&\quad=-(1-\hat\kappa)(V(x,\hat x))+\begin{bmatrix}w-\hat w\\ C^2x-\hat C^2\hat x\end{bmatrix}^T\begin{bmatrix}\bar X^{11} & \bar X^{12}\\ \bar X^{21} & \bar X^{22}\end{bmatrix}\begin{bmatrix}w-\hat w\\ C^2x-\hat C^2\hat x\end{bmatrix}+(1+2/\pi)\lambda_{\max}(\tilde M)\bar\delta^2.
\end{aligned}
\tag{4.3.3}
$$

one can obtain the chain of inequalities in (4.3.3). Then the inequality (3.4.2) is also satisfied. Hence the proposed V in (4.2.14) is an SStF from $\widehat\Sigma$ to Σ, which completes the proof. Note that functions $\alpha\in\mathcal{K}_\infty$, $\kappa\in\mathcal{K}$, $\rho_{\text{ext}}\in\mathcal{K}_\infty\cup\{0\}$, and the matrix $\bar X$ in Definition 3.4.1 associated with V in (4.2.14) are defined as $\alpha(s)=\frac{\lambda_{\min}(\tilde M)}{\lambda_{\max}(C^{1T}C^1)}s^2$, $\kappa(s):=(1-\hat\kappa)s$, $\rho_{\text{ext}}(s):=0$, $\forall s\in\mathbb{R}_{\geq0}$, and $\bar X=\begin{bmatrix}\bar X^{11} & \bar X^{12}\\ \bar X^{21} & \bar X^{22}\end{bmatrix}$. Moreover, the positive constant ψ in (3.4.2) is $\psi=(1+2/\pi)\lambda_{\max}(\tilde M)\bar\delta^2$. □

4.3.1.2 Case Studies

To demonstrate the effectiveness of our proposed approaches, we first apply our results to the temperature regulation in a circular building containing 200 rooms by constructing compositionally a finite abstraction of the network. Then, to show its applicability to strongly connected networks, the results are illustrated on a network with a fully-connected interconnection graph.

4.3.1.2.1 Room Temperature Network

In this subsection, we apply our results to the temperature regulation of $n\geq3$ rooms with the interconnected network as defined in (4.2.20). By introducing Σ_i described as (4.2.21), one can readily verify that $\Sigma=\mathcal{I}_{cd}(\Sigma_1,\ldots,\Sigma_N)$ where the coupling matrix M is with elements $m_{i,i+1}=m_{i+1,i}=m_{1,n}=m_{n,1}=1$, $i\in\{1,\ldots,n-1\}$, and all other elements identically zero. One can also verify that, $\forall i\in\{1,\ldots,n\}$, the condition (4.3.2) is satisfied with $\tilde M_i=1$, $K_i=0$, $\bar X_i^{11}=\eta^2(1+\pi_i)$, $\bar X_i^{22}=-3.38\eta(1+\pi_i)$,

$\bar{X}_i^{12} = \bar{X}_i^{21} = \eta\lambda'_i$, where $\lambda'_i = 1 - 2\eta - \beta - \bar{\theta}\nu_i(k)$, and selecting some appropriate values for $\eta, \beta, \bar{\theta}, \hat{\kappa}_i, \pi_i, \forall i \in \{1, \ldots, n\}$. Hence, the function $V_i(\tilde{T}_i, \hat{\tilde{T}}_i) = (\tilde{T}_i - \hat{\tilde{T}}_i)^2$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (3.4.1) with $\alpha_i(s) = s^2$ and the condition (3.4.2) with $\kappa_i(s) := (1 - \hat{\kappa}_i)s$, $\rho_{\text{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, $\psi_i = (1 + 2/\pi_i)\bar{\delta}_i^2$, $G_i = \hat{G}_i = H_i = 1$, and

$$\bar{X}_i = \begin{bmatrix} \eta^2(1+\pi_i) & \eta\lambda'_i \\ \eta\lambda'_i & -3.38\eta(1+\pi_i) \end{bmatrix}, \tag{4.3.4}$$

where the input ν_i is given via the interface function in (4.2.16) as $\nu_i = \hat{\nu}_i$. Now we look at $\widehat{\Sigma} = \mathcal{I}_{cd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (3.4.4) as $\hat{M} = M$. Choosing $\mu_1 = \cdots = \mu_N = 1$ and using $\bar{X}_i$ in (4.3.4), the matrix $\bar{X}_{cmp}$ in (3.4.6) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} \eta^2(1+\pi)\mathbb{I}_n & \eta\lambda'\mathbb{I}_n \\ \eta\lambda'\mathbb{I}_n & -3.38\eta(1+\pi)\mathbb{I}_n \end{bmatrix},$$

where $\lambda' = \lambda'_1 = \cdots = \lambda'_N$, $\pi = \pi_1 = \cdots = \pi_N$, and accordingly the condition (3.4.3) reduces to

$$\begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} = \eta^2(1+\pi)M^T M + \eta\lambda' M + \eta\lambda' M^T - 3.38\eta(1+\pi)\mathbb{I}_n \preceq 0,$$

without requiring any restrictions on the number or gains of the subsystems. In order to satisfy the above inequality, we used $M = M^T$, and $4\eta^2(1+\pi) + 4\eta\lambda' - 3.38\eta(1+\pi) \preceq 0$ employing Gershgorin circle theorem [Bel65] which can be satisfied for appropriate values of η, π and λ'. By choosing finite internal input sets $\hat{W}_i$ of $\widehat{\Sigma}$ such that $\prod_{i=1}^n \hat{W}_i = \hat{M} \prod_{i=1}^n \hat{X}_i$, the condition (4.3.1) is also satisfied. Now, one can verify that $V(\tilde{T}, \hat{\tilde{T}}) = \sum_{i=1}^n (\tilde{T}_i - \hat{\tilde{T}}_i)^2$ is a sum-type SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (3.2.3) and (3.2.4) with $\alpha(s) = s^2$, $\kappa(s) := (1 - \hat{\kappa})s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = n(1 + 2/\pi)\bar{\delta}^2$.

To demonstrate the effectiveness of the proposed approach, we first fix $n = 15$. By taking the state discretization parameter $\bar{\delta}_i = 0.005$, and constants $\hat{\kappa}_i = 0.99, \pi_i = 0.05, \forall i \in \{1, \ldots, n\}, \eta = 0.1, \beta = 0.022, \bar{\theta} = 0.05$, one can readily verify that conditions (3.4.3) and (4.3.2) are satisfied. Accordingly, by taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $20\mathbf{1}_{15}$, we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.63$ during the time horizon $T_d = 10$ with the probability at least 90%, i.e.,

$$\mathbb{P}(\|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 0.63,\ \forall k \in [0, 10]) \geq 0.9\,.$$

Let us now synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ such that the controller maintains the temperature of any room in the safe set [19,21]. We employ here the software tool FAUST^2 [SGA15] to synthesize a controller for Σ by taking the external input discretization parameter as 0.04, and the standard deviation of the noise $\bar{\sigma}_i = 0.28$, $\forall i \in \{1, \ldots, n\}$. Closed-loop state trajectories of the representative room with different noise realizations are illustrated in Figure 4.14. Policy ν and the associated safety probability for a representative room in the network are respectively plotted in

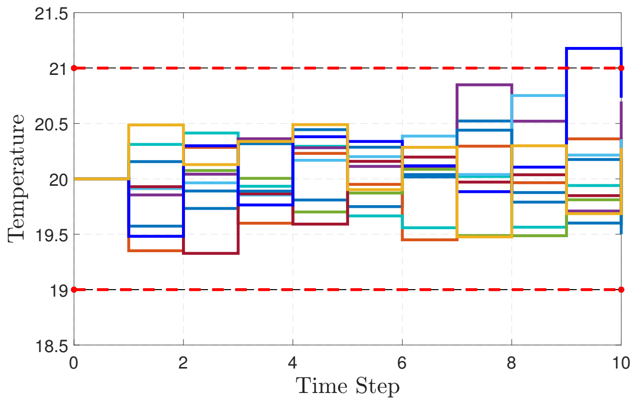

Figure 4.14: Closed-loop trajectories of a representative room with different noise realizations in a network of 15 rooms.

Figures 4.15 and 4.16 as a function of the initial temperature of the room. Policy ν is *locally sub-optimal* for each subsystem and is obtained by assuming that other subsystems do not violate the safety specification. The synthesized policy ν is smoothly decreasing from the maximum input 0.6 to the minimum 0 as temperature increases. The maximum safety probability is around the center of the interval $[19, 21]$, and its minimums are at the two boundaries. Note that the oscillations appeared in Figures 4.15 and 4.16 are due to the state and input discretization parameters. We now compare the guarantees provided by our approach and by [SAM15]. Note that our result is based on the dissipativity approach while [SAM15] uses dynamic Bayesian network (DBN) to capture the dependencies between subsystems. The comparison is shown in Figures 4.17 and 4.18 in the logarithmic scale. In Figure 4.17, we have fixed $\varepsilon = 0.2$ (cf. (3.2.5)) and plotted the error as a function of the discretization parameter $\bar{\delta}$ and the standard deviation of the noise $\bar{\sigma}$. Our error of (3.2.5) is independent of $\bar{\sigma}$ while the error of [SAM15] converges to infinity when $\bar{\sigma}$ goes to zero. Thus our approach outperforms [SAM15] for a smaller standard deviation of the noise. In Figure 4.18, we have fixed $\bar{\sigma} = 0.28$ and plotted the error as a function of the discretization parameter $\bar{\delta}$ and the confidence bound ε. The error in [SAM15] is independent of ε while our error increases when ε goes to zero. Thus there is a trade-off between ε and $\bar{\delta}$ to get better bounds in comparison with [SAM15].

In order to show the scalability of our approach, we increase the number of rooms to $n = 200$. If we take the state discretization parameter $\bar{\delta}_i = 0.005$, and constants $\hat{\kappa}_i = 0.99, \pi_i = 0.98, \forall i \in \{1, \ldots, n\}, \eta = 0.1, \beta = 0.4, \bar{\theta} = 0.5$, conditions (3.4.3) and (4.3.2) are readily met. Moreover, if the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are started from $20\mathbf{1}_{200}$, one can readily verify that the norm of the error between outputs of Σ and of $\widehat{\Sigma}$ will not exceed 0.63 with the probability at least 90% for $T_d = 10$. Similarly, we synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ by taking the external input discretization parameter as 0.04, and $\bar{\sigma}_i = 0.21, \forall i \in \{1, \ldots, n\}$. Closed-loop state

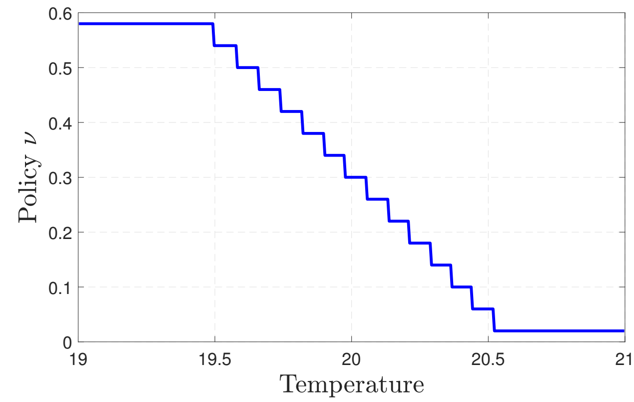

Figure 4.15: Policy ν for a representative room in a network of 15 rooms.

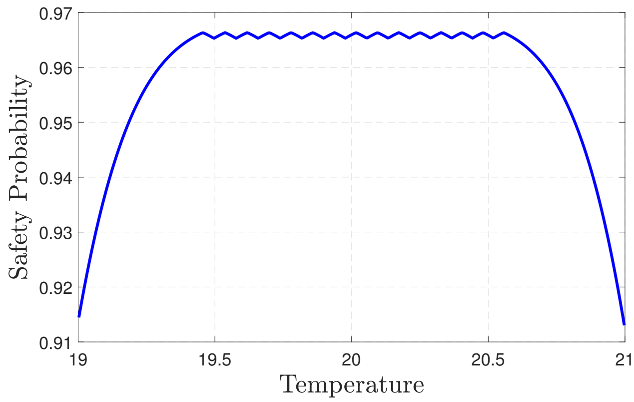

Figure 4.16: Closed-loop safety probability of a representative room with the time horizon $T_d = 10$ in a network of 15 rooms.

trajectories of the representative room with different noise realizations are illustrated in Figure 4.19.

4.3.1.2.2 Comparisons with Small-Gain Approach in Subsection 4.2.2

Since the road traffic network in (4.2.20) admits a common Lyapunov function, the results proposed in Subsection 4.2.2 recover the ones here (as discussed in Remark 4.2.37) by considering switching signals as discrete inputs. Then we make a comparison between the both proposed results. The comparison is shown in Figure 4.20 in the logarithmic scale. We have fixed $\varepsilon = 1$, $T_d = 15$, and plotted the error as a function of $\bar{\delta}$ and the number of subsystems N. By increasing the number of subsystems, the probabilistic error bound does not change since the overall ψ is independent of N, and is computed only based on the maximum of ψ_i of subsystems instead of being a linear combination of them which is the case here. Nevertheless, for networks with small number of subsystems,

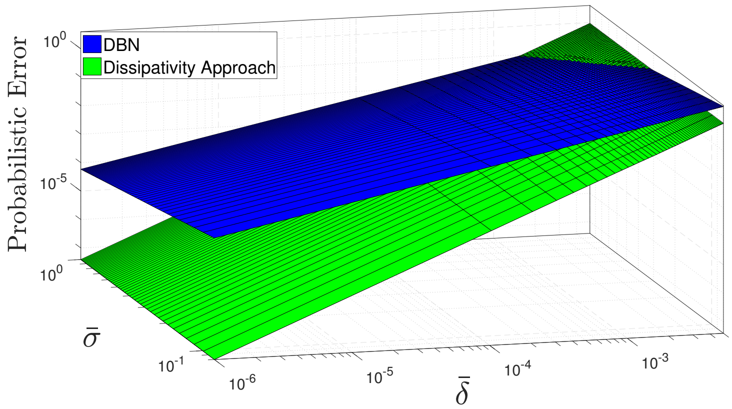

Figure 4.17: Comparison of the error bound provided by the approach of this section based on the dissipativity with that of [SAM15] based on DBN. Plots are in the logarithmic scale for a fixed $\varepsilon = 0.2$ (cf. (3.2.5)).

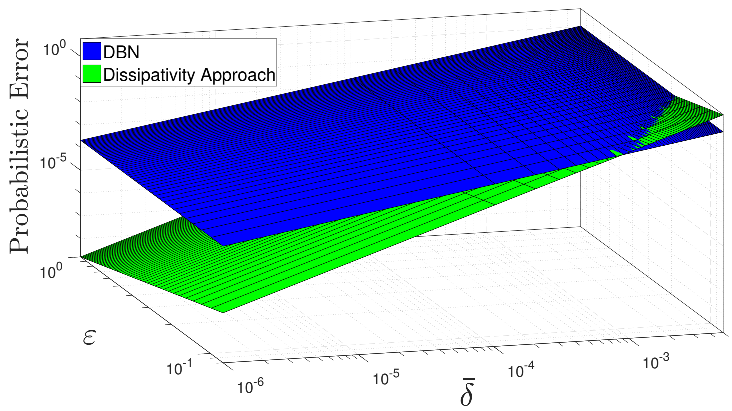

Figure 4.18: Comparison of the error bound provided by the approach of this section based on the dissipativity with that of [SAM15] based on DBN. Plots are in the logarithmic scale for a fixed noise standard deviation $\bar{\sigma} = 0.28$.

the proposed errors here are better than the ones provided in Subsection 4.2.2. This issue is expected and the reason is due to the conservatism nature of the approach that we employed in Subsection 4.2.2 ([SGZ18, Theorem 1]) to transfer the additive form of pseudo-simulation functions to a max form (cf. (4.2.45)), but with the gain of providing an overall error for the network only based on the maximum error of subsystems instead of a linear combination of them.

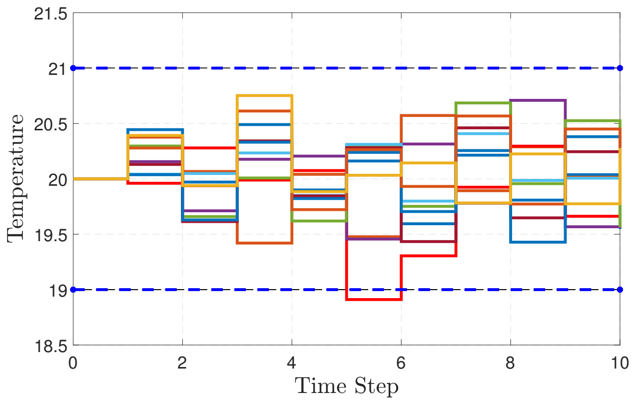

Figure 4.19: Closed-loop trajectories of a representative room with different noise realizations in a network of 200 rooms.

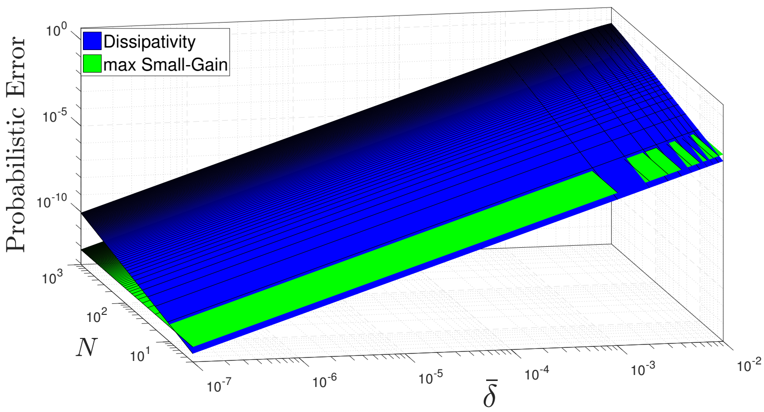

Figure 4.20: Comparison of the probabilistic error bound provided by this section based on the dissipativity approach with that of Subsection 4.2.2 based on the max small-gain. Plots are in the logarithmic scale for a fixed $\varepsilon = 1$, and $T_d = 15$.

4.3.1.2.3 Fully Interconnected Network

In order to show the applicability of our approach to strongly connected networks, we consider the following interconnected *linear* dt-SCS

$$\Sigma : \begin{cases} x(k+1) = \bar{G}x(k) + \nu(k) + \varsigma(k), \\ y(k) = x(k), \end{cases}$$

with the matrix $\bar{G} = (\mathbb{I}_n - \tau\tilde{L}) \in \mathbb{R}^{n\times n}$ where $\tilde{L}$ is the Laplacian matrix of an undirected graph, and $0 < \tau < 1/\bar{\Delta}$ with $\bar{\Delta}$ being the maximum degree of the graph [GR01]. We expand the state $x(k) = [x_1(k); \ldots ; x_n(k)]$, the external input $\nu(k) = [\nu_1(k); \ldots ; \nu_n(k)]$, and the noise $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_n(k)]$. Now by defining Σ_i as

$$\Sigma_i : \begin{cases} x_i(k+1) = x_i(k) + \nu_i(k) + w_i(k) + \varsigma_i(k), \\ y_i^1(k) = x_i(k), \\ y_i^2(k) = x_i(k), \end{cases}$$

one can verify that $\Sigma = \mathcal{I}_{cd}(\Sigma_1, \ldots, \Sigma_N)$ where the coupling matrix M is given by $M = -\tau \tilde{L}$. One can also verify that, $\forall i \in \{1, \ldots, n\}$, the condition (4.3.2) is satisfied with $\tilde{M}_i = 1$, $K_i = -0.2$, $\bar{X}^{11} = (1 + \pi_i)$, $\bar{X}^{22} = 0$, $\bar{X}^{12} = \bar{X}^{21} = \lambda'_i$, where $\lambda'_i = 1 + K_i$, and $\hat{\kappa}_i = 0.99$, $\pi_i = 0.55, \forall i \in \{1, \ldots, n\}$. Hence the function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (3.4.1) with $\alpha_i(s) = s^2$ and the condition (3.4.2) with $\kappa_i(s) := (1 - \hat{\kappa}_i)s$, $\rho_{\text{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, $\psi_i = (1 + 2/\pi_i)\bar{\delta}_i^2$, and $G_i = \hat{G}_i = H_i = 1$. Now, we look at $\widehat{\Sigma} = \mathcal{I}_{cd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (3.4.4) by $\hat{M} = M$. Choosing $\mu_1 = \cdots = \mu_N = 1$, the matrix $\bar{X}_{cmp}$ in (3.4.6) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} (1+\pi)\mathbb{I}_n & \lambda'\mathbb{I}_n \\ \lambda'\mathbb{I}_n & 0 \end{bmatrix},$$

where $\lambda' = \lambda'_1 = \cdots = \lambda'_N$, $\pi = \pi_1 = \cdots = \pi_N$, and the condition (3.4.3) reduces to

$$\begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix} = (1+\pi)\tau^2 \tilde{L}^T \tilde{L} - \lambda'\tau\tilde{L} - \lambda'\tau\tilde{L}^T = \tau\tilde{L}((1+\pi)\tau\tilde{L} - 2\lambda'\mathbb{I}_n) \preceq 0,$$

which is always satisfied without requiring any restrictions on the number or gains of the subsystems. In order to show the above inequality, we used $\tilde{L} = \tilde{L}^T \succeq 0$ which is always true for Laplacian matrices of undirected graphs. By choosing finite internal input sets $\hat{W}_i$ of $\widehat{\Sigma}$ such that $\prod_{i=1}^n \hat{W}_i = \hat{M} \prod_{i=1}^n \hat{X}_i$, the condition (4.3.1) is also satisfied. Now, one can verify that $V(x, \hat{x}) = \sum_{i=1}^n (x_i - \hat{x}_i)^2$ is a sum-type SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (3.2.3) and (3.2.4) with $\alpha(s) = s^2$, $\kappa(s) := (1 - \hat{\kappa})s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = n(1 + 2/\pi)\bar{\delta}^2$.

To illustrate the results, we assume $\tilde{L}$ is the Laplacian matrix of a complete graph as in (3.4.24) and $\tau = 0.1$. We fix $n = 150$, and the state discretization parameter $\bar{\delta}_i = 0.005, \forall i \in \{1, \ldots, n\}$. By using the sum-type SSF V and the inequality (3.2.5), and taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $20\mathbf{1}_{150}$, we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.63$ during the time horizon $T_d = 10$ with the probability at least 90%.

4.3.2 Stochastic Switched Systems

In this subsection, we extend the results of the precious section to stochastic switched systems whose switch signals accept dwell-time with multiple supply rates and multiple storage functions. The proposed compositionality conditions here can enjoy the structure of the interconnection topology and be potentially fulfilled independently of the number or gains of the subsystems. We show that if a switched system is *incremental passive* (i.e., existence of a common incremental storage function, or *multiple* incremental storage functions with *dwell-time*), one can construct finite MDPs of concrete models for the general setting of nonlinear stochastic switched systems such that each switching mode has its independent supply rate.

We also enlarge the class of systems for the construction of finite MDPs by adding time-varying nonlinearities to the dynamics satisfying an incremental quadratic inequality, whereas the provided results in the previous sections only handle the class of nonlinearities satisfying slope restrictions. We show that for this class of nonlinear switched systems, the aforementioned incremental passivity property can be readily verified by some easier to check matrix inequalities. Moreover, we generalize the results of Section 4.2.2 by allowing the noises of abstractions be completely independent of those of concrete systems. We also relax the compositionality condition (4.3.1) that was implicit, without providing a direct method for satisfying it. We relax this condition at the cost of incurring an additional error term, but benefiting from choosing quantization parameters of internal input sets freely. Finally we apply our proposed techniques to a *fully interconnected network* of 100 *nonlinear* subsystems (totally 200 dimensions), and also the *road traffic network* in a circular cascade ring composed of 50 cells.

In this section, we assume that $f_p, \forall p \in P$, satisfies the following Lipschitz assumption: there exists an $\mathscr{L}_p \in \mathbb{R}_{\geq 0}$ such that $\|f_p(x,w,\varsigma)-f_p(x,w,\hat\varsigma)\|_2 \leq \mathscr{L}_p\|\varsigma-\hat\varsigma\|_2$ for all $x \in X$, $w \in W$, $\varsigma \in \mathcal{V}_\varsigma$, $\hat\varsigma \in \mathcal{V}_{\hat\varsigma}$.

4.3.2.1 aug-Type Stochastic Storage and Pseudo-Storage Functions

We first introduce a notion of augmented stochastic storage functions (aug-type SStF) for dt-SS with internal inputs and outputs. We then define a notion of augmented stochastic pseudo-storage functions (aug-type SPStF) for switched systems without internal signals. We employ these definitions mainly to quantify the closeness of interconnected global MDPs and their finite abstractions.

Definition 4.3.7. *Consider two global MDPs $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$ and $\widehat{\mathbb{G}}(\widehat\Sigma) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \hat\varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}^1, \hat{\mathbb{Y}}^2, \hat{\mathbb{H}}^1, \hat{\mathbb{H}}^2)$ with internal inputs and outputs. A function $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ is called an augmented stochastic storage function (aug-type SStF) from $\widehat{\mathbb{G}}(\widehat\Sigma)$ to $\mathbb{G}(\Sigma)$ if there exist $\alpha \in \mathcal{K}_\infty$, $0 < \kappa < 1$, $\psi \in \mathbb{R}_{\geq 0}$, and a symmetric matrix $\bar X$ with conformal block partitions $\bar X^{ij}$, $i, j \in \{1, 2\}$, such that*

- $\forall (x,p,l) \in \mathbb{X}, \forall (\hat x,p,l) \in \hat{\mathbb{X}}$,

$$\alpha(\|\mathbb{H}^1(x,p,l) - \hat{\mathbb{H}}^1(\hat x,p,l)\|_2) \leq V((x,p,l),(\hat x,p,l)), \tag{4.3.5}$$

- $\forall (x,p,l) \in \mathbb{X}, \forall (\hat x,p,l) \in \hat{\mathbb{X}}$, $\forall \hat\nu \in \hat{\mathbb{U}}, \forall w \in \mathbb{W}, \forall \hat w \in \hat{\mathbb{W}}$,

$$\begin{aligned}&\mathbb{E}\Big[V((x',p',l'),(\hat x',p',l')) \,\big|\, x, \hat x, p, l, w, \hat w\Big]\\ &\quad \leq \kappa V((x,p,l),(\hat x,p,l)) + \bar z^T \bar X \bar z + \psi,\end{aligned} \tag{4.3.6}$$

where

$$\bar{z} = \begin{bmatrix} w - \hat{w} \\ \mathbb{H}_2(x,p,l) - \hat{\mathbb{H}}_2(\hat{x},p,l) \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix},$$

and the expectation operator $\mathbb{E}$ *is with respect to* ς *under the one-step transition of both global MDPs with* $\nu = \hat{\nu}$*, i.e.,* $(x', p', l') = \mathbb{F}((x,p,l), \hat{\nu}, w, \varsigma)$ *and* $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x},p,l), \hat{\nu}, \hat{w}, \hat{\varsigma})$.

If there exists an aug-type SStF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq^{\mathsf{aug}}_{SStF} \mathbb{G}(\Sigma)$, and the system $\widehat{\mathbb{G}}(\widehat{\Sigma})$ is called an abstraction of the concrete (original) global MDP $\mathbb{G}(\Sigma)$.

Now we modify the above notion for global MDPs without internal inputs and outputs by eliminating all the terms related to $w, \hat{w}$ which is employed in Theorem 4.2.28 for relating interconnected systems.

Definition 4.3.8. *Consider two global MDPs* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ *and* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\varsigma}, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$ *without internal inputs and outputs. A function* $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ *is called an augmented stochastic pseudo-storage function (*aug*-type SPStF) from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$ *if*

- *there exists* $\alpha \in \mathcal{K}_\infty$ *such that* $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$,

$$\alpha(\|\mathbb{H}(x,p,l) - \hat{\mathbb{H}}(\hat{x},p,l)\|_2) \leq V((x,p,l),(\hat{x},p,l)), \tag{4.3.7}$$

- $\forall (x,p,l) \in \mathbb{X}, \forall (\hat{x},p,l) \in \hat{\mathbb{X}}$, $\forall \hat{\nu} \in \hat{\mathbb{U}}$,

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l')) \,\big|\, x, \hat{x}, p, l\Big] \leq \kappa V((x,p,l),(\hat{x},p,l)) + \psi, \tag{4.3.8}$$

for some $0 < \kappa < 1$*, and* $\psi \in \mathbb{R}_{\geq 0}$*, where the expectation operator* $\mathbb{E}$ *is with respect to* ς *under the one-step transition of both global MDPs with* $\nu = \hat{\nu}$*, i.e.,* $(x', p', l') = \mathbb{F}((x,p,l), \hat{\nu}, \varsigma)$ *and* $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x},p,l), \hat{\nu}, \hat{\varsigma})$.

If there exists an aug-type SPStF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq^{\mathsf{aug}}_{SPStF} \mathbb{G}(\Sigma)$, and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ is called an abstraction of $\mathbb{G}(\Sigma)$. Now one can utilize Theorem 4.2.28 and compare output trajectories of two global MDPs (without internal inputs and outputs) in a probabilistic setting.

4.3.2.2 Compositionality Results

4.3.2.2.1 Interconnected Stochastic Switched Systems

Suppose we are given N *concrete* stochastic switched subsystems, $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, \mathbf{F}_i, Y_i^1, Y_i^2, h_i^1, h_i^2), i \in \{1, \ldots, N\}$, with its equivalent global MDP $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, \mathbb{H}_i^1, \mathbb{H}_i^2)$. Now we provide a formal definition of the interconnection of concrete dt-SS $\Sigma_i, \forall i \in \{1, \ldots, N\}$.

Definition 4.3.9. *Consider $N \in \mathbb{N}_{\geq 1}$ dt-SS $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, \mathbf{F}_i, Y_i^1, Y_i^2, h_i^1, h_i^2)$, and a matrix M defining the coupling between these subsystems. The interconnection of Σ_i, $\forall i \in \{1, \ldots, N\}$, is the concrete interconnected dt-SS $\Sigma = (X, P, \mathcal{P}, \varsigma, \mathbf{F}, Y, h)$, denoted by $\mathcal{I}_{sd}(\Sigma_1, \ldots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $P := \prod_{i=1}^N P_i$, $\mathcal{P} := \prod_{i=1}^N \mathcal{P}_i$, $\mathbf{F} := \prod_{i=1}^N \mathbf{F}_i$, $Y := \prod_{i=1}^N Y_i^1$, and $h = \prod_{i=1}^N h_i^1$, with the internal inputs constrained according to*

$$[w_1; \ldots; w_N] = M\big[h_1^2(x_1); \ldots; h_N^2(x_N)\big]. \tag{4.3.9}$$

Similarly, given global MDPs $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, \mathbb{H}_i^1, \mathbb{H}_i^2), i \in \{1, \ldots, N\}$, one can also define the interconnection of $\mathbb{G}(\Sigma_i)$ as $\mathcal{I}_{sd}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$.

4.3.2.2.2 Compositional Abstractions of Interconnected Switched Systems

In order to provide compositionality results of the section, we utilize an abstraction map Π_w on W (defined similar to (4.2.6)) that assigns to any $w \in W$ a representative point $\bar{w} \in \hat{W}$ of the corresponding partition set containing w. Now we define a notion of the interconnection of *abstract* global MDPs $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$.

Definition 4.3.10. *Consider $N \in \mathbb{N}_{\geq 1}$* abstract *global MDPs $\widehat{\mathbb{G}}(\widehat{\Sigma}_i) = (\hat{\mathbb{X}}_i, \hat{\mathbb{U}}_i, \hat{\mathbb{W}}_i, \hat{\varsigma}_i, \hat{\mathbb{F}}_i, , \hat{\mathbb{Y}}_i^1, \hat{\mathbb{Y}}_i^2, \hat{\mathbb{H}}_i^1, \hat{\mathbb{H}}_i^2)$. The interconnection of $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$, $\forall i \in \{1, \ldots, N\}$, is the interconnected* abstract *global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\varsigma}, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$, denoted by $\hat{\mathcal{I}}_{sd}(\widehat{\mathbb{G}}(\widehat{\Sigma}_1), \ldots, \widehat{\mathbb{G}}(\widehat{\Sigma}_N))$, such that $\hat{\mathbb{X}} := \prod_{i=1}^N \hat{\mathbb{X}}_i$, $\hat{\mathbb{U}} := \prod_{i=1}^N \hat{\mathbb{U}}_i$, $\hat{\mathbb{Y}} := \prod_{i=1}^N \hat{\mathbb{Y}}_i^1$, $\hat{\mathbb{H}} = \prod_{i=1}^N \hat{\mathbb{H}}_i^1$, and the map $\hat{\mathbb{F}} = \prod_{i=1}^N \hat{\mathbb{F}}_i$ is the transition function given by $(\hat{x}', p', l') = \hat{\mathbb{F}}\left((\hat{x}, p, l), \hat{\nu}, \hat{w}, \hat{\varsigma}\right)$ if and only if $\hat{x}' = \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})$ as defined similar to* (4.2.4)*, $\hat{\nu} = p$ and the following scenarios hold for any $i \in \{1, \ldots, N\}$:*

- *$l_i < k_{d_i} - 1$, $p_i' = p_i$, and $l_i' = l_i + 1$;*
- *$l_i = k_{d_i} - 1$, $p_i' = p_i$, and $l_i' = k_{d_i} - 1$;*
- *$l_i = k_{d_i} - 1$, $p_i' \neq p_i$, and $l_i' = 0$;*

where $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N], \hat{\nu} = [\hat{\nu}_1; \ldots; \hat{\nu}_N], p = [p_1; \ldots; p_N], l = [l_1; \ldots; l_N]$, and subject to the following constraint:

$$\begin{aligned} [\hat{w}_1; \ldots; \hat{w}_N] &= \Pi_w(\hat{M}\Big[\hat{h}_1^2(\hat{x}_1); \ldots; \hat{h}_N^2(\hat{x}_N)\Big]), \\ \Pi_w(\hat{M}\prod_{i=1}^N \hat{Y}_i^2) &\subseteq \prod_{i=1}^N \hat{W}_i, \end{aligned} \tag{4.3.10}$$

where $\hat{M}$ is the coupling matrix between subsystems $\widehat{\Sigma}_i, \forall i \in \{1, \ldots, N\}$.

Remark 4.3.11. *Note that the proposed condition* (4.3.10) *is less conservative than the compositionality condition* (4.3.1) *presented in Theorem 4.3.1. In particular, the proposed condition in* (4.3.1) *is an implicit one meaning that there is no direct way to* *satisfy it. Moreover, our compositionality framework here allows to choose quantization parameters of internal input sets freely such that one can reduce the cardinality of the internal input sets of finite abstractions. Although the compositionality condition* (4.3.1) *presented in Theorem 4.3.1 is relaxed here to* (4.3.10)*, our proposed compositionality approach suffers from an additional error formulated in* (4.3.15) *based on* $\bar{\boldsymbol{\mu}}$*.*

In the next theorem, we provide sufficient conditions to quantify the error between the interconnection of global MDPs and that of their finite abstractions in a compositional manner.

Theorem 4.3.12. *Consider the interconnected global MDP* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *global MDPs* $\mathbb{G}(\Sigma_i)$*. Suppose that each* $\mathbb{G}(\Sigma_i)$ *admits a finite global MDP* $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ *together with an* aug*-type SStF* V_i*. Then the function* $V((x,p,l),(\hat{x},p,l))$ *defined as*

$$V((x,p,l),(\hat{x},p,l)) := \sum_{i=1}^{N} \mu_i V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)), \tag{4.3.11}$$

is an aug*-type SPStF function from* $\widehat{\mathcal{I}}_{sd}(\widehat{\mathbb{G}}(\widehat{\Sigma}_1),\ldots,\widehat{\mathbb{G}}(\widehat{\Sigma}_N))$ *with the coupling matrix* $\hat{M}$*, to* $\mathcal{I}_{sd}(\mathbb{G}(\Sigma_1),\ldots,\mathbb{G}(\Sigma_N))$*, if* $\mu_i > 0$*,* $i \in \{1,\ldots,N\}$*, and there exists* $0 < \underline{\mu} < 1$ *such that* $(1+\underline{\mu})\max_i(\kappa_i) < 1$*, and for all* $x_i \in X_i$*,* $\hat{x}_i \in \hat{X}_i$*,* $i \in \{1,\ldots,N\}$*:*

$$\|\mathbb{H}_i^2(x_i,p_i,l_i) - \hat{\mathbb{H}}_i^2(\hat{x}_i,p_i,l_i)\|_2^2 \leq \frac{\mu_i\kappa_i}{\underline{\mu}} V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)), \tag{4.3.12}$$

and

$$M = \hat{M}, \tag{4.3.13}$$

$$\begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \preceq 0, \tag{4.3.14}$$

where $\bar{X}_{cmp}$ *is as in* (3.4.6)*.*

Proof. We first show that the aug-type SPStF V in (4.3.11) satisfies the inequality (4.3.7) for some $\mathcal{K}_\infty$ function α. For any $(x,p,l) \in \mathbb{X}$, and $(\hat{x},p,l) \in \hat{\mathbb{X}}$, one gets

$$\begin{aligned}\|\mathbb{H}(x,p,l) - \hat{\mathbb{H}}(\hat{x},p,l)\|_2 &= \|[\mathbb{H}_1^1(x_1,p_1,l_1);\ldots;\mathbb{H}_N^1(x_N,p_N,l_N)] \\ &\quad - [\hat{\mathbb{H}}_1^1(\hat{x}_1,p_1,l_1);\ldots;\hat{\mathbb{H}}_N^1(\hat{x}_N,p_N,l_N)]\|_2 \\ &\leq \sum_{i=1}^{N} \|\mathbb{H}_i^1(x_i,p_i,l_i) - \hat{\mathbb{H}}_i^1(\hat{x}_i,p_i,l_i)\|_2 \\ &\leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))) \\ &\leq \hat{\beta}(V((x,p,l),(\hat{x},p,l))),\end{aligned}$$

with the function $\hat{\beta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined for all $s \in \mathbb{R}_{\geq 0}$ as

$$\hat{\beta}(s) := \max\Big\{\sum_{i=1}^{N} \alpha_i^{-1}(s_i) \,\big|\, s_i \geq 0,\ \sum_{i=1}^{N} \mu_i s_i = s\Big\}.$$

It is not hard to verify that the function $\hat{\beta}(\cdot)$ defined above is a $\mathcal{K}_\infty$ function. By taking the $\mathcal{K}_\infty$ function $\alpha(s) := \hat{\beta}^{-1}(s)$, $\forall s \in \mathbb{R}_{\geq 0}$, one can satisfy the inequality (4.3.7). We continue with showing that the inequality (4.3.8) holds, as well. By defining $[\bar{w}_1; \dots; \bar{w}_N] = \hat{M}[\hat{h}_1^2(\hat{x}_1); \dots; \hat{h}_N^2(\hat{x}_N)]$, we have the chain of inequalities in (4.3.16) using conditions (4.3.12), (4.3.13), (4.3.14) and by defining κ, ψ as

$$\kappa := \max\Big\{\sum_{i=1}^{N}(1+\underline{\mu})\mu_i\kappa_i s_i \,\big|\, s_i \geq 0,\ \sum_{i=1}^{N}\mu_i s_i = 1, (1+\underline{\mu})\max_i(\kappa_i) < 1\Big\},$$
$$\psi := \begin{cases} \sum_{i=1}^{N}\mu_i\psi_i + \frac{\|\bar{\boldsymbol{\mu}}\|_2^2}{\underline{\mu}^2}\lambda_{\max}(\underline{P}) & \text{if } \bar{X}_{cmp} \leq 0,\\ \sum_{i=1}^{N}\mu_i\psi_i + \|\bar{\boldsymbol{\mu}}\|_2^2(\frac{1}{\underline{\mu}^2}\lambda_{\max}(\underline{P}) + \rho(\bar{X}_{cmp})) & \text{if } \bar{X}_{cmp} > 0, \end{cases} \tag{4.3.15}$$

where $\underline{P} = \bar{X}_{cmp}^T \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp}$, $\bar{\boldsymbol{\mu}} = [\bar{\mu}_1; \dots; \bar{\mu}_N]$, and ρ is the *spectral radius*. Hence, V is an aug-type SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ which completes the proof. □

Figure 4.21 schematically illustrates the result of this theorem.

4.3.2.3 Construction of aug-Type SStF

In this subsection, we impose conditions on the concrete dt-SS Σ enabling us to find an aug-type SStF from the finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$. The required conditions are first presented in the *general setting* of nonlinear stochastic switched systems in the next subsection and then represented via some matrix inequality for a particular class of *nonlinear* stochastic switched systems whose nonlinearities satisfy an incremental quadratic inequality in Subsection 4.3.2.3.2.

4.3.2.3.1 General Setting of Nonlinear Stochastic Switched Systems

The aug-type SStF from the finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ is established under the assumption that original discrete-time stochastic switched subsystems $\Sigma_p, \forall p \in P$, are *incremental passive* as in the following definition. Note that this definition is similar to Definition 2.8.1 but adapted for switched systems.

Definition 4.3.13. *A dt-SS Σ_p is called* incremental passive *if there exists a storage function $V_p : X \times X \to \mathbb{R}_{\geq 0}$ such that $\forall x, x' \in X$, $\forall w, w' \in W$, $\forall p \in P$, the following two inequalities hold:*

$$\underline{\alpha}_p(\|h^1(x) - h^1(x')\|_2) \leq V_p(x, x'), \tag{4.3.17}$$

$$
\begin{aligned}
&\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\,\big|\,x,\hat{x},p,l\Big]=\mathbb{E}\Big[\sum_{i=1}^{N}\mu_iV_i((x_i',p_i',l_i'),(\hat{x}_i',p_i',l_i'))\,\big|\,x,\hat{x},p,l\Big]\\
&=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i((x_i',p_i',l_i'),(\hat{x}_i',p_i',l_i'))\,\big|\,x_i,\hat{x}_i,p_i,l_i\Big]\\
&\leq\sum_{i=1}^{N}\mu_i\big(\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))+\psi_i\\
&\quad+\begin{bmatrix} w_i-\hat{w}_i\\ \mathbb{H}_i^2(x_i,p_i,l_i)-\hat{\mathbb{H}}_i^2(\hat{x}_i,p_i,l_i)\end{bmatrix}^T\begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}\begin{bmatrix} w_i-\hat{w}_i\\ \mathbb{H}_i^2(x_i,p_i,l_i)-\hat{\mathbb{H}}_i^2(\hat{x}_i,p_i,l_i)\end{bmatrix}\big)\\
&=\sum_{i=1}^{N}\mu_i\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix} w_1-\hat{w}_1\\ \vdots\\ w_N-\hat{w}_N\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\begin{bmatrix}\mu_1\bar{X}_1^{11} & & & \mu_1\bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{11} & & & \mu_N\bar{X}_N^{12}\\ \mu_1\bar{X}_1^{21} & & & \mu_1\bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{21} & & & \mu_N\bar{X}_N^{22}\end{bmatrix}\begin{bmatrix} w_1-\hat{w}_1\\ \vdots\\ w_N-\hat{w}_N\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&=\sum_{i=1}^{N}\mu_i\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix} w_1-\bar{w}_1+\bar{w}_1-\hat{w}_1\\ \vdots\\ w_N-\bar{w}_N+\bar{w}_N-\hat{w}_N\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix} w_1-\bar{w}_1+\bar{w}_1-\hat{w}_1\\ \vdots\\ w_N-\bar{w}_N+\bar{w}_N-\hat{w}_N\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&=\sum_{i=1}^{N}\mu_i\kappa_iV_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix} M\begin{bmatrix}h_1^2(x_1)\\ \vdots\\ h_N^2(x_N)\end{bmatrix}-\hat{M}\begin{bmatrix}\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ \hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix} M\begin{bmatrix}h_1^2(x_1)\\ \vdots\\ h_N^2(x_N)\end{bmatrix}-\hat{M}\begin{bmatrix}\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ \hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\
&\quad+\begin{bmatrix}\bar{w}_1-\hat{w}_1\\ \vdots\\ \bar{w}_N-\hat{w}_N\\ \mathbf{0}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{w}_1-\hat{w}_1\\ \vdots\\ \bar{w}_N-\hat{w}_N\\ \mathbf{0}_N\end{bmatrix}+2\begin{bmatrix} M\begin{bmatrix}h_1^2(x_1)\\ \vdots\\ h_N^2(x_N)\end{bmatrix}-\hat{M}\begin{bmatrix}\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ \hat{h}_N^2(\hat{x}_N)\end{bmatrix}\\ h_1^2(x_1)-\hat{h}_1^2(\hat{x}_1)\\ \vdots\\ h_N^2(x_N)-\hat{h}_N^2(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{w}_1-\hat{w}_1\\ \vdots\\ \bar{w}_N-\hat{w}_N\\ \mathbf{0}_N\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \mu_i \kappa_i V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)) + \sum_{i=1}^{N} \mu_i \psi_i \\
&+ \begin{bmatrix} h_1^2(x_1) - \hat{h}_1^2(\hat{x}_1) \\ \vdots \\ h_N^2(x_N) - \hat{h}_N^2(\hat{x}_N) \end{bmatrix}^T \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} h_1^2(x_1) - \hat{h}_1^2(\hat{x}_1) \\ \vdots \\ h_N^2(x_N) - \hat{h}_N^2(\hat{x}_N) \end{bmatrix} \\
&+ \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix} + 2 \begin{bmatrix} h_1^2(x_1) - \hat{h}_1^2(\hat{x}_1) \\ \vdots \\ h_N^2(x_N) - \hat{h}_N^2(\hat{x}_N) \end{bmatrix}^T \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix} \\
&\leq \sum_{i=1}^{N} \mu_i \kappa_i V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)) + \sum_{i=1}^{N} \mu_i \psi_i + \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix} \\
&+ \underline{\mu}^2 \begin{bmatrix} h_1^2(x_1) - \hat{h}_1^2(\hat{x}_1) \\ \vdots \\ h_N^2(x_N) - \hat{h}_N^2(\hat{x}_N) \end{bmatrix}^T \begin{bmatrix} h_1^2(x_1) - \hat{h}_1^2(\hat{x}_1) \\ \vdots \\ h_N^2(x_N) - \hat{h}_N^2(\hat{x}_N) \end{bmatrix} \\
&+ \frac{1}{\underline{\mu}^2} \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix}^T \bar{X}_{cmp}^T \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} \bar{w}_1 - \hat{w}_1 \\ \vdots \\ \bar{w}_N - \hat{w}_N \\ \mathbf{0}_N \end{bmatrix} \\
&\leq \sum_{i=1}^{N} \mu_i \kappa_i V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)) + \sum_{i=1}^{N} \mu_i \psi_i + \|\bar{\boldsymbol{\mu}}\|_2^2 \sigma_{\max}\big(\bar{X}_{cmp}\big) \\
&+ \underline{\mu} \sum_{i=1}^{N} \mu_i \kappa_i V_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)) + \frac{1}{\underline{\mu}^2} \|\bar{\boldsymbol{\mu}}\|_2^2 \lambda_{\max}\big(\bar{X}_{cmp}^T \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp}\big) \\
&\leq \kappa V((x,p,l),(\hat{x},p,l)) + \psi. \qquad (4.3.16)
\end{aligned}
$$

---

and

$$
\begin{aligned}
&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(x',w',\varsigma)) \,\Big|\, x,x',w,w'\Big] \\
&\quad \leq \bar{\kappa}_p V_p(x,x') + \begin{bmatrix} w - w' \\ h^2(x) - h^2(x') \end{bmatrix}^T \begin{bmatrix} S_p^{11} & S_p^{12} \\ S_p^{21} & S_p^{22} \end{bmatrix} \begin{bmatrix} w - w' \\ h^2(x) - h^2(x') \end{bmatrix}, \qquad (4.3.18)
\end{aligned}
$$

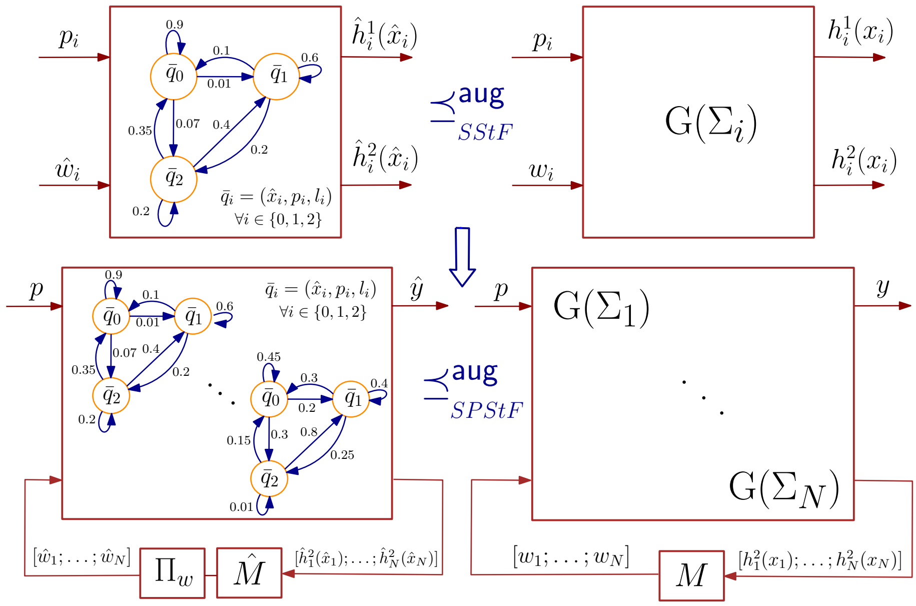

Figure 4.21: Compositionality results provided that conditions (4.3.12), (4.3.13), and (4.3.14) are satisfied.

for some $\underline{\alpha}_p, \in \mathcal{K}_\infty$, $0 < \bar{\kappa}_p < 1$, *and matrices* S_p^{11}, S_p^{12}, S_p^{21}, *and* S_p^{22} *of appropriate dimensions.*

In order to construct an aug-type SStF from the finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, we need to raise the following assumption.

Assumption 4.3.14. *Assume that for constants* $\bar{\kappa}_p$, $p \in \{1, \ldots, m\}$ *as appeared in Definition 4.3.13,* $\bar{\epsilon} > 1$, *and* $\forall l \in \{0, \ldots, k_d - 2\}$, *where* $k_d \geq \bar{\epsilon}\frac{\ln(\tilde{\mu})}{\ln(1/\max_p\{\bar{\kappa}_p\})} + 1$, *there exist matrices* $\bar{X}^{11}$, $\bar{X}^{12}$, $\bar{X}^{21}$, *and* $\bar{X}^{22}$ *of appropriate dimensions such that the following inequality holds:*

$$\frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}}\sum_{p=1}^{m}\begin{bmatrix} S_p^{11} & S_p^{12} \\ S_p^{21} & S_p^{22}\end{bmatrix} \leq \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}.$$

Under Definition 4.3.13 and Assumptions 4.2.33, 4.2.35 and 4.3.14, the next theorem shows a relation between $\mathbb{G}(\Sigma)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ via establishing an aug-type SStF between them.

Theorem 4.3.15. *Let* $\Sigma = (X, P, \mathcal{P}, W, \varsigma, \mathbf{F}, Y^1, Y^2, h^1, h^2)$ *be a switched system with its equivalent global MDP* $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}^1, \mathbb{Y}^2, \mathbb{H}^1, \mathbb{H}^2)$. *Consider the abstract global MDP* $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \hat{\varsigma}, \hat{\mathbb{F}}, \hat{\mathbb{Y}}^1, \hat{\mathbb{Y}}^2, \hat{\mathbb{H}}^1, \hat{\mathbb{H}}^2)$ *constructed as in Definition 4.2.24. For any* $p \in \{1, \ldots, m\}$, *let* Σ_p *be an* incrementally passive *dt-SS via a function* V_p *as* *in Definition 4.3.13. If Assumptions 4.2.33, 4.2.35 and 4.3.14 hold, then*

$$V((x,p,l),(\hat{x},p,l)) = \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m} V_p(x,\hat{x}), \tag{4.3.19}$$

is an aug*-type SStF from* $\widehat{\mathbb{G}}(\widehat{\Sigma})$ *to* $\mathbb{G}(\Sigma)$.

Proof. Since Σ_p is *incrementally passive*, using (4.3.17), $\forall (x,p,l) \in \mathbb{X}$ and $\forall (\hat{x},p,l) \in \hat{\mathbb{X}}$, and since $\hat{h}^1(\hat{x}) = h^1(\hat{x})$, we get

$$\begin{aligned}\|\mathbb{H}^1(x,p,l) - \hat{\mathbb{H}}^1(\hat{x},p,l)\|_2 &= \|h^1(x) - \hat{h}^1(\hat{x})\|_2 = \|h^1(x) - h^1(\hat{x})\|_2 \\ &\leq \underline{\alpha}_p^{-1}(V(x,\hat{x})) = \underline{\alpha}_p^{-1}(\bar{\kappa}_p^{l/\bar{\epsilon}} V((x,p,l),(\hat{x},p,l))).\end{aligned}$$

Since $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}} > 1$, one can conclude that the inequality (4.3.5) holds with $\alpha(s) = \min_p\{\underline{\alpha}_p(s)\}$, $\forall s \in \mathbb{R}_{\geq 0}$. Now we show that the inequality (4.3.6) holds, as well. By taking the conditional expectation from (4.2.35), $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall p \in P, \forall w \in W, \forall \hat{w} \in \hat{W}$, we have

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), \hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] - \mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\quad\leq \mathbb{E}\Big[\gamma(\|\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}) - f_p(\hat{x},\hat{w},\hat{\varsigma})\|_2)\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big],\end{aligned}$$

where $\hat{f}_p(\hat{x},\hat{\nu},\hat{\varsigma}) = \Pi_x(f_p(\hat{x},\hat{w},\hat{\varsigma}))$. Using (4.2.5), the above inequality reduces to

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), \hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\quad - \mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \leq \gamma_p(\bar{\delta}).\end{aligned} \tag{4.3.20}$$

Employing (4.2.35), one has

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\quad\leq \mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\qquad + \mathbb{E}\Big[\gamma_p(\|f_p(\hat{x},\hat{w},\hat{\varsigma}) - f_p(\hat{x},\hat{w},\varsigma)\|_2)\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big].\end{aligned} \tag{4.3.21}$$

Then using (4.3.21), one can rewrite (4.3.20) as

$$\begin{aligned}&\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), \hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\quad\leq \mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \\ &\qquad + \mathbb{E}\Big[\gamma_p(\|f_p(\hat{x},\hat{w},\hat{\varsigma}) - f_p(\hat{x},\hat{w},\varsigma)\|_2)\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] + \gamma_p(\bar{\delta}).\end{aligned} \tag{4.3.22}$$

Employing (4.3.18) and since $\hat{h}^2(\hat{x}) = h^2(\hat{x})$, we get

$$\mathbb{E}\Big[V_p(f_p(x,w,\varsigma), f_p(\hat{x},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big] \leq \bar{\kappa}_p V_p(x,\hat{x}) + z^T S_p z,$$

where

$$z = \begin{bmatrix} w - \hat{w} \\ h^2(x) - \hat{h}^2(\hat{x}) \end{bmatrix}, \quad S_p = \begin{bmatrix} S_p^{11} & S_p^{12} \\ S_p^{21} & S_p^{22} \end{bmatrix}.$$

Then one has

$$\begin{aligned} &\mathbb{E}\Big[V_p(f_p(x, w, \varsigma), f_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \\ &\quad \leq \bar{\kappa}_p V_p(x, \hat{x}) + z^T S_p z + \gamma_p(\bar{\delta}) + \mathbb{E}\Big[\gamma_p(\|f_p(\hat{x}, \hat{w}, \hat{\varsigma}) - f_p(\hat{x}, \hat{w}, \varsigma)\|_2) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]. \end{aligned}$$

Given the Lipschitz assumption on f_p, one can conclude that

$$\mathbb{E}\Big[V_p(f_p(x, w, \varsigma), f_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] \leq \bar{\kappa}_p V_p(x, \hat{x}) + z^T S_p z + \gamma_p(\bar{\delta}) + \underline{\Lambda}_p, \tag{4.3.23}$$

where

$$\underline{\Lambda}_p = \mathbb{E}\Big[\gamma_p(\mathscr{L}_p \|\hat{\varsigma} - \varsigma\|_2) \,\Big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big].$$

Now employing (4.3.23) and Assumptions 4.2.33 (required for the last scenario), and 4.3.14, one can obtain the chain of inequalities in (4.3.24) including the three different scenarios as discussed in Definition 4.2.24. By defining $\kappa = \max_p\{\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}\}$, and $\psi = \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}} \sum_{p=1}^{m} \gamma_p(\bar{\delta})$, one can conclude that V defined in (4.3.19) is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, which completes the proof. Note that the last inequality in the first scenario holds since $l < k_d - 1$, and consequently, $l + 1 < k_d$. In addition, the last inequality of the second scenario holds since $\bar{\epsilon} > 1$, and consequently, $0 < \frac{\bar{\epsilon}-1}{\bar{\epsilon}} < 1$. Finally in the last scenario, $\tilde{\mu}\max_p\{\bar{\kappa}_p\}^{(k_d-1)/\bar{\epsilon}} \leq 1$ since $k_d \geq \bar{\epsilon}\frac{\ln(\tilde{\mu})}{\ln(1/\max_p\{\bar{\kappa}_p\})} + 1$. Hence, the last inequality of the last scenario also holds. □

Remark 4.3.16. *Note that if there exists a common storage function* $V : X \times X \to \mathbb{R}_{\geq 0}$ *between all switching modes* $p \in P$ *satisfying Definition 4.3.13 and Assumptions 4.2.33, 4.2.35 and 4.3.14, and there exists a common supply rate satisfying Definition 4.3.13, then* $V((x, p, l), (\hat{x}, p, l)) = V(x, \hat{x})$ *and Definitions 4.3.7 and 4.3.8 reduce to, respectively, Definitions 3.4.1 and 3.2.4 (cf. Case study 4.3.2.4.2).*

Now we provide similar results as this subsection but tailored to a particular class of *nonlinear* stochastic switched systems whose nonlinearities satisfy an incremental quadratic inequality.

4.3.2.3.2 Switched Systems with Incremental Quadratic Constraint on Nonlinearity

Here, we enlarge the nonlinear class of discrete-time stochastic switched systems Σ proposed in (4.2.38) by adding time-varying nonlinearities to the dynamics satisfying an

- First Scenario $(l < k_d - 1, \|f(\hat{x}, \hat{w}, \hat{\varsigma}) - \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})\|_2 \leq \bar{\delta}, p' = p, l' = l + 1)$:

$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\big|\, x, \hat{x}, p, l, w, \hat{w}\Big] = \frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p'=1}^{m} V_{p'}(x', \hat{x}') \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$= \frac{1}{\max_p\{\bar{\kappa}_p\}^{(l+1)/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{(l+1)/\bar{\epsilon}}}(\sum_{p=1}^{m}(\bar{\kappa}_p V_p(x, \hat{x}) + z^T S_p z + \gamma_p(\bar{\delta}) + \underline{\Lambda}_p))$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{(l+1)/\bar{\epsilon}}}(\max_p\{\bar{\kappa}_p\}\sum_{p=1}^{m} V_p(x, \hat{x}) + \sum_{p=1}^{m}(z^T S_p z + \gamma_p(\bar{\delta}) + \underline{\Lambda}_p))$$

$$= \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\max_p\{\bar{\kappa}_p\}^{(l+1)/\bar{\epsilon}}}\sum_{p=1}^{m} z^T S_p z$$

$$+ \frac{1}{\max_p\{\bar{\kappa}_p\}^{(l+1)/\bar{\epsilon}}}\sum_{p=1}^{m}(\gamma_p(\bar{\delta}) + \underline{\Lambda}_p)$$

$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \bar{z}^T \bar{X} \bar{z} + \frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}(\gamma_p(\bar{\delta}) + \underline{\Lambda}_p);$$

- Second Scenario $(l = k_d - 1, \|f(\hat{x}, \hat{w}, \hat{\varsigma}) - \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})\|_2 \leq \bar{\delta}, p' = p, l' = k_d - 1)$:

$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\big|\, x, \hat{x}, p, l, w, \hat{w}\Big] = \frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p'=1}^{m} V_{p'}(x', \hat{x}') \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$= \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}(\max_p\{\bar{\kappa}_p\}\sum_{p=1}^{m} V_p(x, \hat{x}) + \sum_{p=1}^{m}(z^T S_p z + \gamma_p(\bar{\delta}) + \underline{\Lambda}_p))$$

$$= \max_p\{\bar{\kappa}_p\} V((x, p, l), (\hat{x}, p, l)) + \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m} z^T S_p z + \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m}(\gamma_p(\bar{\delta}) + \underline{\Lambda}_p)$$

$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l)) + \bar{z}^T \bar{X} \bar{z} + \frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}(\gamma_p(\bar{\delta}) + \underline{\Lambda}_p);$$

- Last Scenario $(l = k_d - 1, \|f(\hat{x}, \hat{w}, \hat{\varsigma}) - \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})\|_2 \leq \bar{\delta}, p' \neq p, l' = 0)$:

$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\big|\, x, \hat{x}, p, l, w, \hat{w}\Big] = \frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p'=1}^{m} V_{p'}(x', \hat{x}') \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$\leq \tilde{\mu}\mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$= \tilde{\mu}\max_p\{\bar{\kappa}_p\}^{(k_d-1)/\bar{\epsilon}}\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$
$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}(\max_p\{\bar{\kappa}_p\}\sum_{p=1}^{m} V_p(x,\hat{x})+\sum_{p=1}^{m}(z^T S_p z+\gamma_p(\bar{\delta})+\underline{\Lambda}_p))$$
$$=\max_p\{\bar{\kappa}_p\}V((x,p,l),(\hat{x},p,l))+\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}(\sum_{p=1}^{m} z^T S_p z+\sum_{p=1}^{m}(\gamma_p(\bar{\delta})+\underline{\Lambda}_p))$$
$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\bar{z}^T\bar{X}\bar{z}+\frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}(\gamma_p(\bar{\delta})+\underline{\Lambda}_p). \quad (4.3.24)$$

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incremental quadratic inequality, and provide an approach on the construction of an aug-type SStF. The time-varying nonlinearity is the one considered in [AC11], which satisfies an incremental quadratic inequality: for all $\tilde{Q}_p \in \tilde{\mathcal{Q}}_p$, where $\tilde{\mathcal{Q}}_p$ is the set of symmetric matrices referred to incremental multiplier matrices, the following incremental quadratic constraint holds for all $k \in \mathbb{N}$, and $d_1, d_2 \in \mathbb{R}$:

$$\begin{bmatrix} d_2-d_1 \\ \varphi_p(k,d_2)-\varphi_p(k,d_1)\end{bmatrix}^T \tilde{Q}_p \begin{bmatrix} d_2-d_1 \\ \varphi_p(k,d_2)-\varphi_p(k,d_1)\end{bmatrix} \geq 0. \quad (4.3.25)$$

To facilitate subsequent analysis, we write the matrix $\tilde{Q}_p$ in the following conformal partitioned form:

$$\tilde{Q}_p = \begin{bmatrix} \tilde{Q}_{11_p} & \tilde{Q}_{12_p} \\ \tilde{Q}_{12_p}^T & \tilde{Q}_{22_p}\end{bmatrix}.$$

Remark 4.3.17. *As discussed in [AC11], the time-varying nonlinearity proposed in* (4.3.25) *is more general that the one presented in* (4.2.39). *For instance, one can readily recover the slope restriction in* (4.2.39) *for* $\varphi_p(k,x) = sin(x), \forall k \in \mathbb{N}$, *by considering* $\tilde{Q}_{11_p} = 1, \tilde{Q}_{12_p} = 0, \tilde{Q}_{22_p} = -1$.

In order to show that a nominated V employing V_p in (4.2.40) is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, we raise the following assumption.

Assumption 4.3.18. *Assume that for some constants* $0 < \bar{\kappa}_p < 1$, *and* $\pi_p \in \mathbb{R}_{>0}$, *there exist matrices* $\tilde{M}_p \succ 0$, S_p^{11}, S_p^{12}, S_p^{21}, *and* S_p^{22} *of appropriate dimensions such that the inequality* (4.3.26) *holds.*

Remark 4.3.19. *Note that for any linear system* $\Sigma = (A, B, C^1, C^2, D, R)$, *the stability of matrices* A_p *is sufficient to satisfy Assumption 4.3.18, where matrices* E_p *and* F_p *are identically zero.*

$$\begin{bmatrix}(1+\pi_p)A_p^T\tilde{M}_pA_p & A_p^T\tilde{M}_pD_p & A_p^T\tilde{M}_pE_p\\ D_p^T\tilde{M}_p^TA_p & (1+\pi_p)D_p^T\tilde{M}_pD_p & D_p^T\tilde{M}_pE_p\\ E_p^T\tilde{M}_p^TA_p & E_p^T\tilde{M}_p^TD_p & (1+\pi_p)E_p^T\tilde{M}_pE_p\end{bmatrix}$$
$$\preceq\begin{bmatrix}\bar{\kappa}_p\tilde{M}_p+C^{2T}S_p^{22}C^2-F_p^T\tilde{Q}_{11_p}F_p & C^{2T}S_p^{21} & -F_p^T\tilde{Q}_{12_p}\\ S_p^{12}C^2 & S_p^{11} & 0\\ -\tilde{Q}_{12_p}^TF_p & 0 & -\tilde{Q}_{22_p}\end{bmatrix}. \tag{4.3.26}$$

Now we provide another main result of the section showing that under which conditions a nominated V using V_p in (4.2.40) is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.

Theorem 4.3.20. *Consider the global MDP $\mathbb{G}(\Sigma)$ associated with $\Sigma=(A,B,C^1,C^2,D,E,F,R,\varphi)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ as its finite abstraction with the state discretization parameter $\bar{\delta}$. Let $\bar{\epsilon}>1$ and $k_d\geq\bar{\epsilon}\frac{\ln(\tilde{\mu})}{\ln(1/\bar{\kappa}_p)}+1$, $\forall p\in P$. If Assumptions 4.3.18, and 4.3.14 (with $\bar{\kappa}_p$ as appeared in (4.3.26)) hold, then*

$$V((x,p,l),(\hat{x},p,l))=\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m}V_p(x,\hat{x}), \tag{4.3.27}$$

with V_p nominated in (4.2.40), is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.

Proof. Since $\hat{C}^1=C^1$, we have $\|\mathbb{H}^1(x,p,l)-\hat{\mathbb{H}}^1(\hat{x},p,l)\|_2=\|C^1x-\hat{C}^1\hat{x}\|_2^2=(x-\hat{x})^TC^{1T}C^1(x-\hat{x})$. Since $\lambda_{\min}(C^{1T}C^1)\|x-\hat{x}\|_2^2\leq(x-\hat{x})^TC^{1T}C^1(x-\hat{x})\leq\lambda_{\max}(C^{1T}C^1)\|x-\hat{x}\|_2^2$ and similarly $\lambda_{\min}(\tilde{M}_p)\|x-\hat{x}\|_2^2\leq(x-\hat{x})^T\tilde{M}_p(x-\hat{x})\leq\lambda_{\max}(\tilde{M}_p)\|x-\hat{x}\|_2^2$, it can be readily verified that $\frac{\lambda_{\min}(\tilde{M}_p)}{\lambda_{\max}(C^{1T}C^1)}\|C^1x-\hat{C}^1\hat{x}\|_2^2\leq V_p(x,\hat{x})$ holds $\forall x$, $\forall\hat{x}$, and consequently, $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}}\frac{\lambda_{\min}(\tilde{M}_p)}{\lambda_{\max}(C^{1T}C^1)}\|C^1x-\hat{C}^1\hat{x}\|_2^2\leq V((x,p,l),(\hat{x},p,l))$, $\forall(x,p,l)\in\mathbb{X},\forall(\hat{x},p,l)\in\hat{\mathbb{X}}$. Since $\frac{1}{\bar{\kappa}_p^{l/\bar{\epsilon}}}>1$, one can conclude that the inequality (4.3.5) holds with $\alpha(s)=\min_p\{\frac{\lambda_{\min}(\tilde{M}_p)}{\lambda_{\max}(C^{1T}C^1)}\}\,s^2$ for any $s\in\mathbb{R}_{\geq0}$. We proceed with showing that the inequality (4.3.6) holds, as well. We simplify

$$\begin{aligned}&A_px+E_p\varphi_p(k,F_px)+B_p+D_pw+R_p\varsigma\\&\qquad-\Pi_x(A_p\hat{x}+E_p\varphi_p(k,F_p\hat{x})+B_p+D_p\hat{w}+R_p\hat{\varsigma})\end{aligned}$$

to

$$A_p(x-\hat{x})+D_p(w-\hat{w})+E_p(\varphi_p(k,F_px)-\varphi_p(k,F_p\hat{x}))+R_p(\varsigma-\hat{\varsigma})+\bar{N}_p, \tag{4.3.28}$$

where $\bar{N}_p=A_p\hat{x}+E_p\varphi_p(k,F_p\hat{x})+B_p+D_p\hat{w}+R_p\hat{\varsigma}-\Pi_x(A_p\hat{x}+E_p\varphi_p(k,F_p\hat{x})+B_p+D_p\hat{w}+R_p\hat{\varsigma})$. By defining $\bar{\varphi}_p=\varphi_p(k,F_px)-\varphi_p(k,F_p\hat{x})$, and employing the fact that $\forall x\in X,\forall\hat{x}\in\hat{X}$, [AC11],

$$\begin{bmatrix}x-\hat{x}\\ \bar{\varphi}_p\end{bmatrix}^T\begin{bmatrix}F_p & 0\\ 0 & \mathbb{I}\end{bmatrix}^T\tilde{Q}_p\begin{bmatrix}F_p & 0\\ 0 & \mathbb{I}\end{bmatrix}\begin{bmatrix}x-\hat{x}\\ \bar{\varphi}_p\end{bmatrix}\geq0,$$

using Young's inequality [You12] as $cd \leq \frac{\pi}{2}c^2 + \frac{1}{2\pi}d^2$, for any $c, d \geq 0$ and any $\pi > 0$, employing Cauchy-Schwarz inequality, the matrix inequality (4.3.26), and since

$$\|\bar{N}_p\|_2 \ \leq \ \bar{\delta}, \quad \bar{N}_p^T \tilde{M}_p \bar{N}_p \leq \lambda_{\max}(\tilde{M}_p)\bar{\delta}^2,$$

one can obtain the chain of inequalities in (4.3.29) including the three different scenarios as discussed in Definition 4.2.24. By defining $\kappa = \max_p\{\bar{\kappa}_p^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}\}$, and $\psi = \frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}} \sum_{p=1}^m ((1{+}4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2{+}(2{+}\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p))$, one can conclude that V defined in (4.3.27) is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, which completes the proof. Note that in the first scenario of chain of inequalities (4.3.29), we utilize $\mathcal{J}_1$ and $\mathcal{J}_2$ to show respectively the left and right-hand sides of the matrix inequality (4.3.26). □

Remark 4.3.21. *If $\forall p \in P$, there exists a common $V : X \times X \to \mathbb{R}_{\geq 0}$ satisfying Assumptions 4.3.14, and 4.3.18, and there is a common supply rate satisfying Assumption 4.3.18, then V, α, κ, and ψ in Theorem 4.3.20 reduce to $V((x,p,l),(\hat{x},p,l)) = V(x,\hat{x}), \alpha(s) = \frac{\lambda_{\min}(\tilde{M}_p)}{\lambda_{\max}(C^{1T}C^1)}s^2$, $\kappa = \bar{\kappa}_p$, and $\psi = (1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\bar{\delta}^2 + (2+\pi_p)\,Tr(R_p^T\tilde{M}_pR_p)$.*

Remark 4.3.22. *Note that if the noises in the concrete and abstract systems are assumed to be the same, the constant ψ in (4.3.6) reduces to $\psi = \frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}} \sum_{p=1}^m (1 + 3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2$.*

4.3.2.4 Case Studies

To show the applicability of our results to stochastic switched systems with multiple supply rates and multiple storage functions accepting the dwell-time, we first apply our proposed techniques to a *fully interconnected network* of 100 *nonlinear* subsystems (totally 200 dimensions), and construct their finite MDPs with guaranteed error bounds on their probabilistic output trajectories. We then apply our approaches to the *road traffic network* in a circular cascade ring composed of 50 cells, and construct compositionally a finite MDP of the network such that the compositionality condition does not require any constraint on the number or gains of subsystems. We employ the constructed abstraction as a substitute to compositionally synthesize policies keeping the density of the traffic lower than 20 vehicles per cell.

4.3.2.4.1 Switched Network with Multiple Supply Rates and Multiple Storage Functions Accepting Dwell-Time

We first apply our proposed techniques to a fully interconnected network of 100 nonlinear subsystems (totally 200 dimensions) and construct their finite MDPs with guaranteed error bounds on their probabilistic output trajectories. Note that the model of the system does not have a common storage function because it exhibits unstable behaviors for different switching signals [Lib03] (i.e., if one periodically switches between different modes, the trajectory goes to infinity). We assume that there is no common supply

- **First Scenario** $(l < k_d - 1, \|f(\hat{x}, \hat{w}, \hat{\varsigma}) - \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})\|_2 \leq \bar{\delta}, p' = p, l' = l+1)$:

$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,\big|\, x, \hat{x}, p, l, w, \hat{w}\Big] = \frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}} \mathbb{E}\Big[\sum_{p'=1}^{m} V_{p'}(x', \hat{x}') \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$= \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \mathbb{E}\Big[\sum_{p=1}^{m} V_p(f_p(x, w, \varsigma), \hat{f}_p(\hat{x}, \hat{w}, \hat{\varsigma})) \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$

$$= \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \Big(\sum_{p=1}^{m} \big((x-\hat{x})^T A_p^T \tilde{M}_p A_p (x-\hat{x}) + \bar{\varphi}_p^T E_p^T \tilde{M}_p E_p \bar{\varphi}_p$$
$$+ (w-\hat{w})^T D_p^T \tilde{M}_p D_p (w-\hat{w}) + \mathbb{E}\Big[\bar{N}_p^T \tilde{M}_p \bar{N}_p \big| x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] + \mathbb{E}\Big[\varsigma^T R_p^T \tilde{M}_p R_p \varsigma \big| x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$
$$+ \mathbb{E}\Big[\hat{\varsigma}^T R_p^T \tilde{M}_p R_p \hat{\varsigma} \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] + 2\mathbb{E}\Big[\hat{\varsigma}^T R_p^T \tilde{M}_p \bar{N}_p \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]$$
$$+ 2(x-\hat{x})^T A_p^T \tilde{M}_p D_p (w-\hat{w}) + 2\bar{\varphi}_p^T E_p^T \tilde{M}_p D_p (w-\hat{w})$$
$$+ 2(w-\hat{w})^T D_p^T \tilde{M}_p \mathbb{E}\Big[\bar{N}_p \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] + 2(x-\hat{x})^T A_p^T \tilde{M}_p E_p \bar{\varphi}_p$$
$$+ 2(x-\hat{x})^T A_p^T \tilde{M}_p \mathbb{E}\Big[\bar{N}_p \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big] + 2\bar{\varphi}_p^T E_p^T \tilde{M}_p \mathbb{E}\Big[\bar{N}_p \,\big|\, x, \hat{x}, \hat{\nu}, w, \hat{w}\Big]\big)\Big)$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \Big(\sum_{p=1}^{m} \Big(\begin{bmatrix} x-\hat{x} \\ w-\hat{w} \\ \bar{\varphi}_p \end{bmatrix}^T \mathcal{J}_1 \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \\ \bar{\varphi}_p \end{bmatrix} + (1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\bar{\delta}^2$$
$$+ (2+\pi_p)\mathrm{Tr}(R_p^T \tilde{M}_p R_p)\Big)\Big)$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \Big(\sum_{p=1}^{m} \Big(\begin{bmatrix} x-\hat{x} \\ w-\hat{w} \\ \bar{\varphi}_p \end{bmatrix}^T \mathcal{J}_2 \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \\ \bar{\varphi}_p \end{bmatrix} + (1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2$$
$$+ (2+\pi_p)\mathrm{Tr}(R_p^T \tilde{M}_p R_p)\Big)\Big)$$

$$= \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \Big(\sum_{p=1}^{m} \bar{\kappa}_p V_p(x, \hat{x}) - \sum_{p=1}^{m} \begin{bmatrix} x-\hat{x} \\ \bar{\varphi}_p \end{bmatrix}^T \begin{bmatrix} F_p & 0 \\ 0 & \mathbb{I} \end{bmatrix}^T \tilde{Q}_p \begin{bmatrix} F_p & 0 \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} x-\hat{x} \\ \bar{\varphi}_p \end{bmatrix}$$
$$+ \sum_{p=1}^{m} ((1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2 + (2+\pi_p)\mathrm{Tr}(R_p^T \tilde{M}_p R_p))$$
$$+ \sum_{p=1}^{m} \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \end{bmatrix}^T \begin{bmatrix} C^{2T} S_p^{22} C^2 & C^{2T} S_p^{21} \\ S_p^{12} C^2 & S_p^{11} \end{bmatrix} \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \end{bmatrix}\Big)$$

$$\leq \frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}} \Big(\max_p\{\bar{\kappa}_p\} \sum_{p=1}^{m} V_p(x, \hat{x}) + \sum_{p=1}^{m} \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \end{bmatrix}^T \begin{bmatrix} C^{2T} S_p^{22} C^2 & C^{2T} S_p^{21} \\ S_p^{12} C^2 & S_p^{11} \end{bmatrix} \begin{bmatrix} x-\hat{x} \\ w-\hat{w} \end{bmatrix}$$
$$+ \sum_{p=1}^{m} ((1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2 + (2+\pi_p)\mathrm{Tr}(R_p^T \tilde{M}_p R_p))\Big)$$

$$= \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}} V((x, p, l), (\hat{x}, p, l))$$

$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}}\sum_{p=1}^{m}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix} S_p^{11} & S_p^{12}\\ S_p^{21} & S_p^{22}\end{bmatrix}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}$$

$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{(1+l)/\bar{\epsilon}}}\sum_{p=1}^{m}((1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p))$$

$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}$$

$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}((1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p));$$

- Second Scenario $(l=k_d-1,\|f(\hat{x},\hat{w},\hat{\varsigma})-\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma})\|_2\leq\bar{\delta},p'=p,l'=k_d-1)$:

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\big|x,\hat{x},p,l,w,\hat{w}\Big]=\frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p'=1}^{m}V_{p'}(x',\hat{x}')\big|x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$=\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m}V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$\leq \max_p\{\bar{\kappa}_p\}V((x,p,l),(\hat{x},p,l))+\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix} S_p^{11} & S_p^{12}\\ S_p^{21} & S_p^{22}\end{bmatrix}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}$$

$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\sum_{p=1}^{m}((1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p))$$

$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix} w-\hat{w}\\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}$$

$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}((1+3/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p));$$

- Last Scenario $(l=k_d-1,\|f(\hat{x},\hat{w},\hat{\varsigma})-\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma})\|_2\leq\bar{\delta},p'\neq p,l'=0)$:

$$\mathbb{E}\Big[V((x',p',l'),(\hat{x}',p',l'))\big|x,\hat{x},p,l,w,\hat{w}\Big]=\frac{1}{\max_p\{\bar{\kappa}_{p'}\}^{l'/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p'=1}^{m}V_{p'}(x',\hat{x}')\big|x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$\leq\tilde{\mu}\;\mathbb{E}\Big[\sum_{p=1}^{m}V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$=\tilde{\mu}\max_p\{\bar{\kappa}_p\}^{(k_d-1)/\bar{\epsilon}}\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m}V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$\leq\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}\mathbb{E}\Big[\sum_{p=1}^{m}V_p(f_p(x,w,\varsigma),\hat{f}_p(\hat{x},\hat{w},\hat{\varsigma}))\,\big|\,x,\hat{x},\hat{\nu},w,\hat{w}\Big]$$

$$\leq \max_p\{\bar{\kappa}_p\}V((x,p,l),(\hat{x},p,l))+\frac{1}{\max_p\{\bar{\kappa}_p\}^{l/\bar{\epsilon}}}(\sum_{p=1}^{m}\begin{bmatrix} x-\hat{x}\\ w-\hat{w}\end{bmatrix}^T\begin{bmatrix} S^{11} & S^{12}\\ S^{21} & S^{22}\end{bmatrix}\begin{bmatrix} x-\hat{x}\\ w-\hat{w}\end{bmatrix}$$

$$+\sum_{p=1}^{m}((1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p)))$$
$$\leq \max_p\{\bar{\kappa}_p\}^{\frac{\bar{\epsilon}-1}{\bar{\epsilon}}}V((x,p,l),(\hat{x},p,l))+\begin{bmatrix} w-\hat{w} \\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}^T\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix} w-\hat{w} \\ C^2x-\hat{C}^2\hat{x}\end{bmatrix}$$
$$+\frac{1}{\max_p\{\bar{\kappa}_p\}^{k_d/\bar{\epsilon}}}\sum_{p=1}^{m}((1+4/\pi_p)\lambda_{\max}(\tilde{M}_p)\,\bar{\delta}^2+(2+\pi_p)\mathrm{Tr}(R_p^T\tilde{M}_pR_p)). \quad (4.3.29)$$

---

rate satisfying the condition (4.3.26). The dynamic of the interconnected system is as in (4.2.49) (but with a time-varying nonlinearity) with

$$\tilde{A}=\begin{bmatrix}0.0012 & 0\\ 0 & 0.0012\end{bmatrix},$$

and $R=\mathsf{diag}(0.001\mathbb{1}_2,\ldots,0.001\mathbb{1}_2)$, $\varphi(k,x(k))=[0.1\mathbb{1}_2\varphi_1(k,0.1\mathbb{1}_2^Tx_1(k));\ldots;0.1\mathbb{1}_2\varphi_N(k,0.1\mathbb{1}_2^Tx_N(k))]$, and $\varphi_p(k,x)=sin(x),\forall k\in\mathbb{N}$. Note that nonlinear functions φ_i satisfy the incremental quadratic constraint (4.3.25) with

$$Q_p=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \forall p\in P.$$

Furthermore, $B_p=[b_{1p_1};\ldots;b_{Np_N}]$ such that

$$b_{ip_i}=\begin{cases}\begin{bmatrix}-0.9\\ 0.5\end{bmatrix}, & \text{if}\quad p_i=1,\\ \begin{bmatrix}0.1\\ -0.3\end{bmatrix}, & \text{if}\quad p_i=2.\end{cases}$$

We partition $x(k)$ as $x(k)=[x_1(k);\ldots;x_N(k)]$ and $\varsigma(k)$ as $\varsigma(k)=[\varsigma_1(k);\ldots;\varsigma_N(k)]$, where $x_i(k),\varsigma_i(k)\in\mathbb{R}^2$. Now by introducing the individual subsystems Σ_i as in (4.2.50) (with $D_i=\mathbb{I}_2$), one can readily verify that $\Sigma=\mathcal{I}_{sd}(\Sigma_1,\ldots,\Sigma_N)$, equivalently $\mathbb{G}(\Sigma)=\mathcal{I}_{sd}(\mathbb{G}(\Sigma_1),\ldots,\mathbb{G}(\Sigma_N))$, where the coupling matrix M is

$$M=0.0012\begin{bmatrix}\mathbf{0}_2 & \mathbb{I}_2 & \cdots & \cdots & \mathbb{I}_2\\ \mathbb{I}_2 & \mathbf{0}_2 & \mathbb{I}_2 & \cdots & \mathbb{I}_2\\ \mathbb{I}_2 & \mathbb{I}_2 & \mathbf{0}_2 & \cdots & \mathbb{I}_2\\ \vdots & & \ddots & \ddots & \vdots\\ \mathbb{I}_2 & \cdots & \cdots & \mathbb{I}_2 & \mathbf{0}_2\end{bmatrix}_{200\times 200}.$$

One can also verify that, $\forall i\in\{1,\ldots,N\}$, the condition (4.3.26) is satisfied with

$$\text{for}\, p_i=1\colon\ \tilde{M}_{p_i}=\begin{bmatrix}1.311 & 0.001\\ 0.001 & 0.492\end{bmatrix},\ \bar{\kappa}_{p_i}=0.7,\ \pi_{p_i}=0.5,$$
$$\text{for}\, p_i=2\colon\ \tilde{M}_{p_i}=\begin{bmatrix}0.4 & 0.01\\ 0.01 & 1.49\end{bmatrix},\ \bar{\kappa}_{p_i}=0.7,\ \pi_{p_i}=0.4,$$

and

$$S_1^{11} = \begin{bmatrix} 2.4799 & -0.1017 \\ -0.1017 & 1.4646 \end{bmatrix}, \quad S_1^{12} = \begin{bmatrix} 0.1389 & 0.1744 \\ 0.1744 & -0.0461 \end{bmatrix},$$
$$S_1^{21} = (S_1^{12})^T, \quad S_1^{22} = \begin{bmatrix} -0.1464 & 0.0654 \\ 0.0654 & -0.2233 \end{bmatrix},$$
$$S_2^{11} = \begin{bmatrix} 3.1467 & -0.7962 \\ -0.7962 & 4.1511 \end{bmatrix}, \quad S_2^{12} = \begin{bmatrix} -0.6347 & -0.1068 \\ -0.1068 & -0.5404 \end{bmatrix},$$
$$S_2^{21} = (S_2^{12})^T, \quad S_2^{22} = \begin{bmatrix} -0.0318 & -0.0101 \\ -0.0101 & -0.0202 \end{bmatrix}.$$

By taking $\bar{\epsilon} = 1.75$, and $\tilde{\mu} = 3.27$, one can get the dwell-time $k_d = 7$. Then Assumption 4.3.14 is also satisfied with

$$\bar{X}^{11} = \begin{bmatrix} 19.4343 & -3.0642 \\ -3.0642 & 19.4581 \end{bmatrix}, \quad \bar{X}^{12} = \begin{bmatrix} -1.2599 & 0.1942 \\ 0.1942 & -1.4565 \end{bmatrix},$$
$$X^{21} = (\bar{X}^{21})^T, \quad \bar{X}^{22} = \begin{bmatrix} -0.8721 & -0.0480 \\ -0.0480 & -0.8474 \end{bmatrix}.$$

Hence, $V_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = \frac{1}{\max_{p_i}\{\bar{\kappa}_{p_i}\}^{l/1.75}} \sum_{p_i=1}^{2} (x_i - \hat{x}_i)^T \tilde{M}_{ip_i}(x_i - \hat{x}_i)$ is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ satisfying the condition (4.3.5) with $\alpha_i(s) = 0.39s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and the condition (4.3.6) with $\kappa_i = 0.85$, and $\psi_i = 117.41\, \bar{\delta}_i^2 + (3.7 \times 10^{-5})$.

Now we look at $\widehat{\Sigma} = \mathcal{I}_{sd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (4.3.13) as $\hat{M} = M$. By taking $\mu_1 = \cdots = \mu_N = 1$, the condition (4.3.14) is also satisfied. Hence, $V((x,p,l),(\hat{x},p,l)) = \sum_{i=1}^{100} (\frac{1}{\max_{p_i}\{\bar{\kappa}_{p_i}\}^{l/1.75}} \sum_{p_i=1}^{2} (x_i - \hat{x}_i)^T \tilde{M}_{ip_i}(x_i - \hat{x}_i))$ is an aug-type SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ satisfying conditions (4.3.7) and (4.3.8) with $\alpha(s) = 0.39s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, $\kappa = 0.85$, and $\psi = \sum_{i=1}^{100} \psi_i = 1.17 \times 10^4\, \bar{\delta}^2 + (3.7 \times 10^{-3})$.

By taking the state discretization parameter $\bar{\delta} = \bar{\delta}_i = 0.0003, \forall i \in \{1, \ldots, N\}$, and taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $\mathbb{1}_{200}$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with the probability at least 88%, i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 1, \ \forall k \in [0, 10]) \geq 0.88.$$

Note that for the construction of finite abstractions, we selected the center of partition sets as representative points. Moreover, we assume a well-defined interconnection of abstractions (i.e., $\hat{M} \prod_{i=1}^{N} \hat{Y}_i^2 = \prod_{i=1}^{N} \hat{W}_i$). Then satisfying the compositionality condition (4.3.12) is no more needed, and accordingly, the overall error formulated in (4.3.15) is reduced to $\psi = \sum_{i=1}^{N} \mu_i \psi_i$.

4.3.2.4.2 Road Traffic Network

In this subsection, we apply our results to the road traffic network in a circular cascade ring which is composed of 50 identical cells, each of which has the length of 500 meters with 1 entry and 1 way out. We compositionally construct a finite MDP of the network such that the compositionality condition does not require any constraint on the number or gains of subsystems. We then employ the constructed finite abstraction as a substitute to compositionally synthesize policies keeping the density of the traffic lower than 20 vehicles per cell.

The dynamic of the interconnected system is as in (4.2.47). Now by introducing the individual cells Σ_i described as in (4.2.48) with $D_i = \frac{\tau v_{i-1}}{\tilde{l}_{i-1}}$ (with $v_0 = v_n$, $\tilde{l}_0 = \tilde{l}_n$), one can readily verify that $\Sigma = \mathcal{I}_{sd}(\Sigma_1, \ldots, \Sigma_N)$, equivalently $\mathbb{G}(\Sigma) = \mathcal{I}_{sd}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$, where the coupling matrix M is with elements $m_{i+1,i} = 1, i \in \{1, \ldots, n-1\}$, $m_{1,n} = 1$, and all other elements are identically zero. Note that here $V_p = V_{p'}, \forall p, p' \in P$ (i.e., a common storage function). Moreover, we assume that the noises of the concrete and abstract systems are the same in order to reduce the error as discussed in Remark 4.3.22. Then one can readily verify that the condition (4.3.26) (applied to linear systems with $E_p = F_p = 0, \forall p \in P$, and $S_p^{ij} = \bar{X}^{ij}$, $i, j \in \{1, 2\}$) is satisfied with $\tilde{M}_i = 1$, $\pi_i = 1.48$, $\bar{\kappa}_i = 0.99$, $\forall i \in \{1, \ldots, n\}$, and

$$\bar{X}_i = \begin{bmatrix} (\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i) & (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})\frac{\tau v_i}{\tilde{l}_i} \\ (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})\frac{\tau v_i}{\tilde{l}_i} & -1.9(\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i) \end{bmatrix}. \tag{4.3.30}$$

Then the function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an aug-type SStF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ satisfying the condition (4.3.5) with $\alpha_i(s) = s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and the condition (4.3.6) with $\kappa_i = 0.99$, and $\psi_i = 2.34\,\bar{\delta}_i^2$.

Now we look at $\widehat{\Sigma} = \mathcal{I}_{sd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (4.3.13) as $\hat{M} = M$. By taking $\mu_1 = \cdots = \mu_N = 1$, and using $\bar{X}_i$ as in (4.3.30), the condition (4.3.14) is satisfied as

$$\begin{aligned} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} &= (\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i)M^TM + (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})\frac{\tau v_i}{\tilde{l}_i}(M^T + M) - 1.9(\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i)\mathbb{I}_n \\ &= (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})\,\frac{\tau v_i}{\tilde{l}_i}(M^T + M) - 0.9(\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i)\mathbb{I}_n \leq 0, \end{aligned}$$

without requiring any restrictions on the number or gains of the subsystems. Note that M^TM is an identity matrix and $M^T + M$ is a matrix with $\bar{m}_{i,i+1} = \bar{m}_{i+1,i} = \bar{m}_{1,n} = \bar{m}_{n,1} = 1$, $i \in \{1, \ldots, n-1\}$, and all other elements are identically zero. In order to show the above inequality, we used, $i \in \{1, \ldots, n\}$,

$$2(1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})(\frac{\tau v_i}{\tilde{l}_i}) - 0.9(\frac{\tau v_i}{\tilde{l}_i})^2(1+\pi_i) \preceq 0,$$

employing Gershgorin circle theorem [Bel65]. Hence, $V(x, \hat{x}) = \sum_{i=1}^{50}(x_i - \hat{x}_i)^2$ is an aug-type SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ satisfying conditions (4.3.7) and (4.3.8) with $\alpha(s) = s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, $\kappa = 0.99$, and $\psi = \sum_{i=1}^{50} \psi_i = 117\,\bar{\delta}^2$.

By taking $\bar{\delta} = \bar{\delta}_i = 0.02, \forall i \in \{1, \ldots, N\}$, and choosing the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $10\mathbb{1}_{50}$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with the probability at least 90%, i.e.,

$$\mathbb{P}(\|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 1,\ \forall k \in [0, 10]) \geq 0.9.$$

Let us now synthesize a controller for Σ via the abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma})$ using the software tool FAUST^2 [SGA15] such that the *safety* controller maintains the density of the traffic lower than 20 vehicles per cell. We fix the standard deviation of the noise as $\bar{\sigma}_i = 0.83$, $\forall i \in \{1, \ldots, n\}$. An optimal switch for a representative cell in a network of 50 cells is plotted in Figure 4.22 top. An optimal switch w.r.t. time for a representative cell with different noise realizations is also illustrated in Figure 4.22 middle, with 10 realizations. Moreover, closed-loop state trajectories of the representative cell with different noise realizations are illustrated in Figure 4.22 bottom.

4.3.2.4.3 Analysis on Probabilistic Closeness Guarantee

In order to have more practical analysis on the proposed probabilistic closeness guarantee, we plotted the probabilistic error bound provided in (4.2.27) in terms of the state discretization parameter $\bar{\delta}$ and the confidence bound ε in Figure 4.23. As seen, the probabilistic closeness guarantee is improved by either decreasing $\bar{\delta}$ or increasing ε. Note that the constant ψ in (4.2.27) is formulated based on the state discretization parameter $\bar{\delta}$.

4.4 Relaxed max Small-Gain Approach

In this section, we propose a relaxed version of max small-gain conditions for the construction of finite MDPs for networks of not necessarily stabilizable stochastic systems. The proposed framework relies on a relation between the original system and its finite abstraction employing a new notion of so-called max-type *finite-step* stochastic simulation functions. In comparison with the notions of max-type SSF (proposed in the previous sections) in which stability or stabilizability of each subsystem is required, a max-type finite-step simulation function needs to decay only after some finite numbers of steps instead of at each time step. This relaxation results in a *less* conservative version of small-gain conditions, using which one can compositionally construct finite MDPs such that the stabilizability of each subsystem is not necessarily required.

4.4.1 Stochastic Control Systems

We first focus on stochastic *control* systems as in Definition 2.3.1 and propose a compositional scheme based on relaxed max small-gain conditions. In order to make the notation easier, we assume that the internal and external output maps are identity. In the following subsection, we define $\mathcal{M}$-sampled systems, based on which one can employ max-type *finite-step* stochastic simulation functions to quantify the mismatch between the interconnected dt-SCS and that of their finite abstractions.

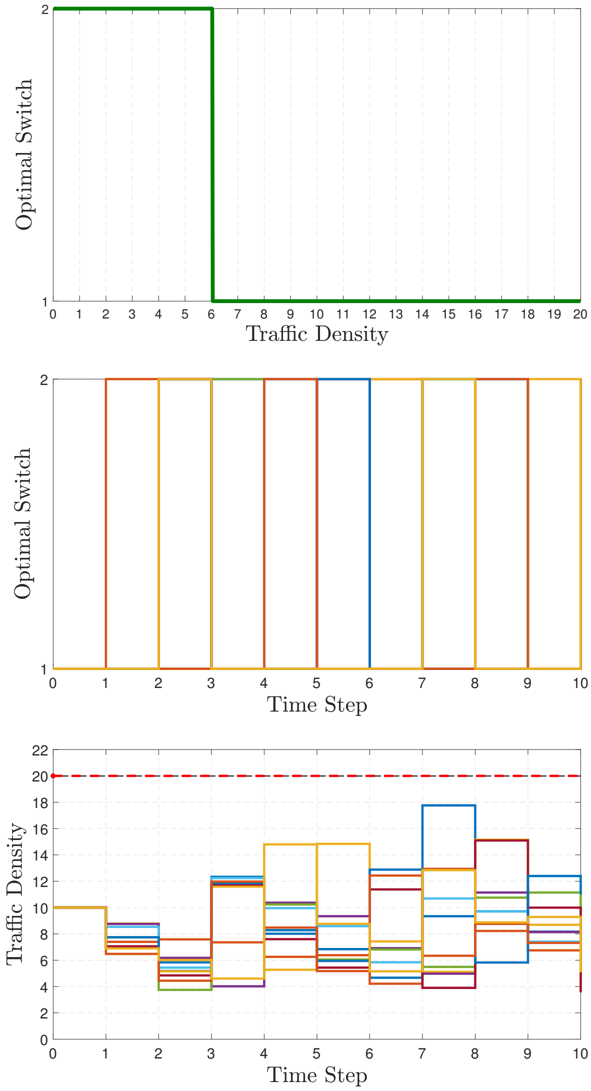

Figure 4.22: Top: An optimal switch for a representative cell in a network of 50 cells. Middle: An optimal switch w.r.t. time for a representative cell with different noise realizations. Bottom: Closed-loop state trajectories of a representative cell with different noise realizations.

4.4.1.1 $\mathcal{M}$-Sampled Systems

The existing methodologies for compositional (in)finite abstractions of interconnected discrete-time stochastic control systems proposed in the previous sections rely on the assumption of each subsystem to be individually stabilizable. This assumption does not hold in general even if the interconnected system is stabilizable. The main idea behind the *relaxed* max small-gain approach proposed in this section is as follows. We show that the individual stabilizability requirement can be relaxed by incorporating the stabilizing

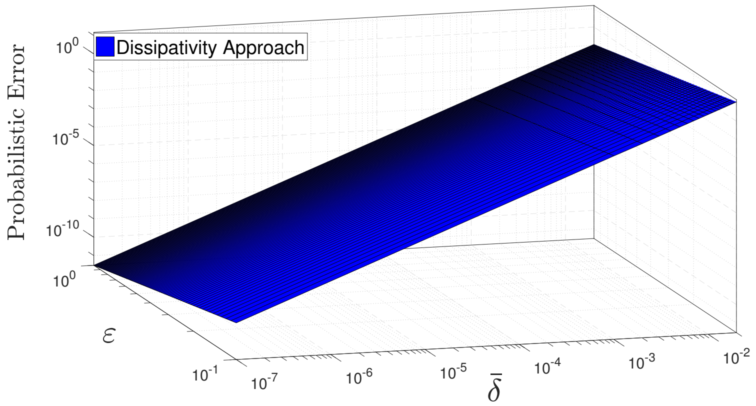

Figure 4.23: Probabilistic error bound proposed in (4.2.27) based on $\bar{\delta}$ and ε. Plot is in the logarithmic scale for $T_d = 10$. The probabilistic closeness guarantee is improved by either decreasing the state discretization parameter $\bar{\delta}$ or increasing the confidence bound ε.

effect of neighboring subsystems in a local unstabilizable subsystem. Once the stabilizing effect is appeared, we construct abstractions of subsystems and employ small-gain theory to provide compositionality results. Our approach here relies on looking at the solution process of the system in future time instances while incorporating the interconnection of subsystems. The following example illustrates this idea.

Example 4.4.1. *Consider two linear dt-SCS* Σ_1, Σ_2 *with dynamics*

$$\Sigma_i : \begin{cases} x_1(k+1) = 1.01x_1(k) + 0.4w_1(k) + \varsigma_1(k), \\ x_2(k+1) = 0.55x_2(k) - 0.2w_2(k) + \varsigma_2(k), \end{cases} \tag{4.4.1}$$

that are connected with the constraint $w_i = x_{3-i}$*, for* $i = \{1, 2\}$*. For simplicity, these two dt-SCS do not have external inputs, i.e.,* $\nu_i \equiv 0$ *for* $i = \{1, 2\}$*. Note that the first subsystem is not stable thus not stabilizable as well. Therefore the proposed results in Section 4.2 are not applicable to this network. By looking at the solution process two steps ahead and considering the interconnection, one can write*

$$\Sigma_{auxi} : \begin{cases} x_1(k+2) = 0.94x_1(k) + 0.62\mathsf{w}_1(k) + 0.4\varsigma_2(k) + 1.01\varsigma_1(k) + \varsigma_1(k+1), \\ x_2(k+2) = 0.22x_2(k) - 0.31\mathsf{w}_2(k) - 0.2\varsigma_1(k) + 0.55\varsigma_2(k) + \varsigma_2(k+1), \end{cases} \tag{4.4.2}$$

which we denote them by $\Sigma_{aux1}, \Sigma_{aux2}$ *in which* $\mathsf{w}_i = x_{3-i}$*, for* $i = \{1, 2\}$*. These two subsystems in* (4.4.2) *are now stable. This motivates us to construct abstractions of original subsystems* (4.4.1) *based on auxiliary subsystems* (4.4.2)*.*

Remark 4.4.2. *Note that after interconnecting the subsystems with each other and propagating the dynamics in the next* $\mathcal{M}$*-steps, the interconnection topology may change (cf. Case study 4.4.1.6). Then the* internal input *of the auxiliary system (i.e.,* w*) may be different from the original one (i.e.,* w*).*

The main contribution of this section is to provide a general methodology for the compositional synthesis of the interconnected dt-SCS with not necessarily stabilizable subsystems, by looking at the solution process $\mathcal{M}$-*step* ahead. For this, we raise the following assumption on the input signal.

Assumption 4.4.3. *The control input is nonzero only at time instances* $\{(k + \mathcal{M} - 1),\ k = j\mathcal{M}, j \in \mathbb{N}\}$.

Remark 4.4.4. *Note that in order to provide a* fully decentralized *controller synthesis framework, each subsystem in our setting must depend only on its own external input. In particular, after interconnecting subsystems with each other based on their interconnection topology and coming up with an* $\mathcal{M}$*-sampled system with all subsystems stabilizable, some subsystems may depend on external inputs of other subsystems. Then Assumption 4.4.3 here helps us in decomposing the network after* $\mathcal{M}$ *transitions such that each subsystem of the* $\mathcal{M}$*-sampled model is described only based on its own external input. This is essential in our proposed setting to have a* fully decentralized *controller synthesis.*

Remark 4.4.5. *Assumption 4.4.3 restricts external inputs to take values only at particular time instances, and consequently, reduces the times at which a policy can be applied. In addition, the proposed* $\mathcal{M}$*-sampled systems may increase the interconnectivity of the network's structure (less sparsity) and then increase the computational effort. These issues are conservatism aspects of our proposed approach in this section but with the gain of providing a compositional framework for the construction of finite MDPs for networks of* not necessarily stabilizable *stochastic subsystems (cf. Case study 4.4.1.6).*

Next lemma shows how dynamics of $\mathcal{M}$-sampled systems, call auxiliary system Σ_{aux}, can be acquired.

Lemma 4.4.6. *Suppose we are given* N *dt-SCS* Σ_i *defined by*

$$\Sigma_i : \begin{cases} x_i(k+1) = f_i(x_i(k), \nu_i(k), w_i(k), \varsigma_i(k)), \\ x_i(\cdot) \in X_i, \nu_i(\cdot) \in U_i, w_i(\cdot) \in W_i, k \in \mathbb{N}, \end{cases} \tag{4.4.3}$$

which are connected in a network with constraints $w_i = [x_1; \ldots; x_{i-1}; x_{i+1}; \ldots; x_N], \forall i \in \{1, \cdots, N\}$. *Under Assumption 4.4.3, the* $\mathcal{M}$*-sampled systems* Σ_{auxi}*, which are the solutions of* Σ_i *at time instances* $k = j\mathcal{M}, j \in \mathbb{N}$*, have the dynamics*

$$\Sigma_{auxi} : \begin{cases} x_i(k+\mathcal{M}) = \tilde{f}_i(x_i(k), \nu_i(k+\mathcal{M}-1), \mathsf{w}_i(k), \tilde{\varsigma}_i(k)), \\ x_i(\cdot) \in X_i, \nu_i(\cdot) \in U_i, \mathsf{w}_i(\cdot) \in \tilde{W}_i, k = j\mathcal{M}, j \in \mathbb{N}, \end{cases} \tag{4.4.4}$$

where $\tilde{\varsigma}_i(k)$ *is a vector containing noise terms as follows:*

$$\begin{aligned} &\tilde{\varsigma}_i(k) = [\bar{\varsigma}_1(k); \ldots; \bar{\varsigma}_i^*(k); \ldots; \bar{\varsigma}_N(k)], \quad \bar{\varsigma}_i^*(k) = [\varsigma_i(k); \ldots; \varsigma_i(k+\mathcal{M}-1)], \\ &\bar{\varsigma}_j(k) = [\varsigma_j(k); \ldots; \varsigma_j(k+\mathcal{M}-2)], \quad \forall j \in \{1, \ldots N\}, j \neq i. \end{aligned} \tag{4.4.5}$$

Note that some of the noise terms in $\tilde{\varsigma}_i(k)$ may be eliminated depending on the interconnection graph, but all the terms are present for a *fully interconnected* network. Proof of Lemma 4.4.6 is based on the recursive application of the vector field f_i and utilizing Assumption 4.4.3. Computation of vector field $\tilde{f}_i$ is illustrated in the next example on a network of two linear dt-SCS.

Example 4.4.7. *Consider two linear dt-SCS Σ_i with dynamics*

$$\Sigma_i : \begin{cases} x_1(k+1) = A_1 x_1(k) + B_1 \nu_1(k) + D_1 w_1(k) + R_1 \varsigma_1(k), \\ x_2(k+1) = A_2 x_2(k) + B_2 \nu_2(k) + D_2 w_2(k) + R_2 \varsigma_2(k), \end{cases} \tag{4.4.6}$$

connected with constraints $w_i = x_{3-i}, i \in \{1,2\}$. Matrices A_i, B_i, D_i, R_i, $i \in \{1,2\}$, have appropriate dimensions. We can rewrite the given dynamics as

$$x(k+1) = \bar{A}x(k) + \bar{B}\nu(k) + \bar{D}w(k) + \bar{R}\varsigma(k),$$

with $x = [x_1; x_2], \nu = [\nu_1; \nu_2], w = [w_1; w_2], \varsigma = [\varsigma_1; \varsigma_2]$, where

$$\bar{A} = \mathsf{diag}(A_1, A_2), \bar{B} = \mathsf{diag}(B_1, B_2), \bar{D} = \mathsf{diag}(D_1, D_2), \bar{R} = \mathsf{diag}(R_1, R_2).$$

By applying the interconnection constraints $w = [w_1; w_2] = [x_2; x_1] = \mathcal{C}[x_1; x_2]$ with $\mathcal{C} = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}$, we have

$$x(k+1) = (\bar{A} + \bar{D}\mathcal{C})x(k) + \bar{B}\nu(k) + \bar{R}\varsigma(k).$$

Now by looking at the solutions $\mathcal{M}$ steps ahead, one gets

$$\begin{aligned} x(k+\mathcal{M}) =& (\bar{A} + \bar{D}\mathcal{C})^{\mathcal{M}} x(k) + \sum_{n=0}^{\mathcal{M}-1} (\bar{A} + \bar{D}\mathcal{C})^n \bar{B}\nu(k+\mathcal{M}-n-1) \\ &+ \sum_{n=0}^{\mathcal{M}-1} (\bar{A} + \bar{D}\mathcal{C})^n \bar{R}\varsigma(k+\mathcal{M}-n-1). \end{aligned}$$

After applying Assumption 4.4.3 and by partitioning $(\bar{A} + \bar{D}\mathcal{C})^{\mathcal{M}}$ as

$$(\bar{A} + \bar{D}\mathcal{C})^{\mathcal{M}} = \left[\begin{array}{c|c} \tilde{A}_1 & \tilde{D}_1 \\ \hline \tilde{A}_2 & \tilde{D}_2 \end{array}\right],$$

one can decompose the network and obtain the auxiliary subsystems proposed in (4.4.4) *as follows:*

$$\Sigma_{\mathsf{aux}i} : \begin{cases} x_1(k+\mathcal{M}) = \tilde{A}_1 x_1(k) + B_1 \nu_1(k+\mathcal{M}-1) + \tilde{D}_1 \mathsf{w}_1(k) + \tilde{R}_1 \tilde{\varsigma}_1(k), \\ x_2(k+\mathcal{M}) = \tilde{A}_2 x_2(k) + B_2 \nu_2(k+\mathcal{M}-1) + \tilde{D}_2 \mathsf{w}_2(k) + \tilde{R}_2 \tilde{\varsigma}_2(k), \end{cases}$$

where $\mathsf{w}_i = x_{3-i}$, for $i = \{1,2\}$, are the new internal inputs, $\tilde{\varsigma}_1(k), \tilde{\varsigma}_2(k)$ are defined as in (4.4.5) *with $N = 2$, and $\tilde{R}_1$, $\tilde{R}_2$ are matrices of appropriate dimensions which can be computed based on the matrices in* (4.4.6). *As seen, $\tilde{A}_1$ and $\tilde{A}_2$ now depend also on D_1, D_2, which may make the pairs $(\tilde{A}_1, B_1)$ and $(\tilde{A}_2, B_2)$ stabilizable.*

Remark 4.4.8. *The main idea behind the proposed approach is that we first look at the solutions of the unstabilizable subsystems, during which we connect subsystems with each other based on their interconnection networks. We go ahead until all subsystems are stabilizable (if possible). Once the stabilizing effect is evident, we decompose the network such that each subsystem is only in terms of its own state, and external input. In contrast to given original systems, the interconnection topology in $\mathcal{M}$-sampled systems may change meaning that the internal input of auxiliary systems may be different from the original ones. Furthermore, the external input of auxiliary systems after doing the $\mathcal{M}$-step analysis is given at instants $k + \mathcal{M} - 1$, $k = j\mathcal{M}$, $j \in \mathbb{N}$. Finally, the noise in auxiliary systems is now a sequence of noises of other subsystems in different time steps depending on the type of the interconnection.*

Remark 4.4.9. *If after interconnecting subsystems to each other and looking ahead in times at the solutions of unstable subsystems the stability effect is not present in finite time steps, we cannot employ the proposed relaxed small-gain condition to provide compositionality results. In particular, in order to establish finite-step stochastic pseudo-simulation functions from $\widehat{\Sigma}_i$ to Σ_i for the general setting of nonlinear stochastic systems, the auxiliary system $\Sigma_{\mathsf{aux}i}$ should be* incrementally input-to-state stable*. To the best of our knowledge, it is not possible in general to provide some conditions on original systems based on which one can guarantee the stabilizability of subsystems after $\mathcal{M}$ transitions or provide an upper bound for $\mathcal{M}$. In fact, such $\mathcal{M}$ depends not only on the subsystem dynamics but also on the interconnection topology.*

4.4.1.2 max-Type Finite-Step Stochastic Pseudo-Simulation and Simulation Functions

In this subsection, we introduce the notion of max-type *finite-step* stochastic pseudo-simulation functions (max-type FPSF) for dt-SCS with both internal and external signals. We also define the notion of max-type *finite-step* stochastic simulation functions (max-type FSF) for dt-SCS without internal signals. We then quantify the closeness of two interconnected dt-SCS based on the max-type FSF. We employ here the notion of max-type finite-step simulation functions inspired by the notion of finite-step Lyapunov functions [GGLW14].

Definition 4.4.10. *Consider dt-SCS Σ_i and $\widehat{\Sigma}_i$, where $\hat{W}_i \subseteq W_i$ and $\hat{X}_i \subseteq X_i$. A function $V_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ is called a* max*-type* finite-step *stochastic pseudo-simulation function (*max*-type FPSF) from $\widehat{\Sigma}_i$ to Σ_i if there exist $\mathcal{M} \in \mathbb{N}_{\geq 1}$, $\alpha_i, \kappa_i \in \mathcal{K}_\infty$, with $\kappa_i < \mathcal{I}_d$, $\rho_{\mathrm{int}i}, \rho_{\mathrm{ext}i} \in \mathcal{K}_\infty \cup \{0\}$, and a constant $\psi_i \in \mathbb{R}_{\geq 0}$, such that for all $k = j\mathcal{M}, j \in \mathbb{N}$, $x_i := x_i(k) \in X_i, \hat{x}_i := \hat{x}_i(k) \in \hat{X}_i$,*

-
$$\alpha_i(\|x_i - \hat{x}_i\|) \leq V_i(x_i, \hat{x}_i), \tag{4.4.7}$$

- *and for any* $\hat{\nu}_i := \hat{\nu}_i(k+\mathcal{M}-1) \in \hat{U}_i$*, there exists* $\nu_i := \nu_i(k+\mathcal{M}-1) \in U_i$ *such that for any* $\mathsf{w}_i := \mathsf{w}_i(k) \in \tilde{W}_i$ *and* $\hat{\mathsf{w}}_i := \hat{\mathsf{w}}_i(k) \in \hat{W}_i$*,*

$$\begin{aligned}&\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}), \hat{x}_i(k+\mathcal{M})) \,\big|\, x_i, \hat{x}_i, \nu_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big] \\ &\quad \leq \max\Big\{\kappa_i(V_i(x_i,\hat{x}_i)), \rho_{\mathrm{int}i}(\|\mathsf{w}_i - \hat{\mathsf{w}}_i\|), \rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|), \psi_i\Big\}. \end{aligned} \tag{4.4.8}$$

We denote by $\widehat{\Sigma}_i \preceq^{\mathsf{max}}_{FPSF} \Sigma_i$ if there exists a max-type FPSF V_i from $\widehat{\Sigma}_i$ to Σ_i. We drop the term *finite-step* for the case $\mathcal{M}=1$, and instead call it the max-type SPSF as defined in Definition 3.3.1.

Remark 4.4.11. *Note that* κ_i *defined in* (4.4.8) *depends on* $\mathcal{M}$ *and is required to be less than* $\mathcal{I}_d$*. Then the* max*-type FPSF* V_i *here is* less *conservative than the* max*-type SPSF in Definition 3.3.1. In other words, the condition* (4.4.8) *may not be satisfied for* $\mathcal{M}=1$ *but may hold for some* $\mathcal{M} \in \mathbb{N}_{>1}$ *which is the case in this section. Such an implicit dependency on* $\mathcal{M}$ *increases the class of systems for which the condition* (4.4.8) *is satisfiable. This relaxation allows some of the individual subsystems to be even unstabilizable.*

Definition 4.4.10 can also be stated for systems without internal inputs by eliminating all the terms related to $\mathsf{w}, \hat{\mathsf{w}}$, as the next definition.

Definition 4.4.12. *Consider two dt-SCS* Σ *and* $\widehat{\Sigma}$ *without internal inputs, where* $\hat{X} \subseteq X$*. A function* $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *is called a* max*-type* finite-step *stochastic simulation function (*max*-type FSF) from* $\widehat{\Sigma}$ *to* Σ *if there exist* $\mathcal{M} \in \mathbb{N}_{\geq 1}$*, and* $\alpha \in \mathcal{K}_\infty$ *such that*

- $\forall x(k) := x \in X, \hat{x}(k) := \hat{x} \in \hat{X}$,

$$\alpha(\|x - \hat{x}\|) \leq V(x, \hat{x}), \tag{4.4.9}$$

- *and* $\forall x(k) := x \in X$, $\forall \hat{x}(k) := \hat{x} \in \hat{X}$, $\forall \hat{\nu}(k+\mathcal{M}-1) := \hat{\nu} \in \hat{U}$, $\exists \nu(k+\mathcal{M}-1) := \nu \in U$ *such that*

$$\mathbb{E}\Big[V(x(k+\mathcal{M}), \hat{x}(k+\mathcal{M})) \,\big|\, x, \hat{x}, \nu, \hat{\nu}\Big] \leq \max\Big\{\kappa(V(x,\hat{x})), \rho_{\mathrm{ext}}(\|\hat{\nu}\|), \psi\Big\}, \tag{4.4.10}$$

for some $\kappa \in \mathcal{K}_\infty$ *with* $\kappa < \mathcal{I}_d$*,* $\rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$*,* $\psi \in \mathbb{R}_{\geq 0}$*, and* $k = j\mathcal{M}, j \in \mathbb{N}$*.*

We call $\widehat{\Sigma}$ an abstraction of Σ, and denote by $\widehat{\Sigma} \preceq^{\mathsf{max}}_{FSF} \Sigma$ if there exists a max-type FSF V from $\widehat{\Sigma}$ to Σ.

We rewrite Theorem 3.2.7 for the $\mathcal{M}$-sampled systems, and show how the max-type FSF can be employed to compare state trajectories of two dt-SCS (without internal inputs) in a probabilistic setting.

Theorem 4.4.13. *Let Σ and $\widehat{\Sigma}$ be two dt-SCS without internal inputs, where $\hat{X} \subseteq X$. Suppose V is a* max*-type FSF from $\widehat{\Sigma}$ to Σ at times $k = j\mathcal{M}, j \in \mathbb{N}$, and there exists a constant $0 < \hat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}_\infty$ in (4.4.10) satisfies $\kappa(r) \geq \hat{\kappa} r$, $\forall r \in \mathbb{R}_{\geq 0}$. For any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, and for any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds:*

$$\mathbb{P}\left\{\sup_{k=j\mathcal{M},\, 0\leq j\leq T_d} \|x_{a\nu}(k) - \hat{x}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,\Big|\, a, \hat{a}\right\} \tag{4.4.11}$$
$$\leq \begin{cases} 1 - (1 - \frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1 - \frac{\hat{\psi}}{\alpha(\varepsilon)})^{T_d}, & \text{if } \alpha(\varepsilon) \geq \frac{\hat{\psi}}{\hat{\kappa}}, \\ (\frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1 - \hat{\kappa})^{T_d} + (\frac{\hat{\psi}}{\hat{\kappa}\alpha(\varepsilon)})(1 - (1 - \hat{\kappa})^{T_d}), & \text{if } \alpha(\varepsilon) < \frac{\hat{\psi}}{\hat{\kappa}}, \end{cases}$$

where the constant $\hat{\psi} \geq 0$ satisfies $\hat{\psi} \geq \rho_{\text{ext}}(\|\hat{\nu}\|_\infty) + \psi$.

The proof is similar to that of Theorem 3.2.7 and is omitted here.

Remark 4.4.14. *Note that the results shown in Theorem 4.4.13 provide a closeness of state trajectories of two interconnected dt-SCS only at times $k = j\mathcal{M}$, $0 \leq j \leq T_d$, for some $\mathcal{M} \in \mathbb{N}_{\geq 1}$. This guarantee will be generalized in Section 4.4.2 by providing the closeness guarantee for all time instances.*

4.4.1.3 Finite Abstractions of Auxiliary Systems

In this subsection, we modify Algorithm 1 and approximate an dt-SCS Σ_{aux} with a finite $\widehat{\Sigma}_{\mathsf{aux}}$. Algorithm 2 presents this approximation.

Given a dt-SCS $\Sigma_{\mathsf{aux}} = (X, U, \tilde{W}, \varsigma, f)$, a finite MDP $\widehat{\Sigma}_{\mathsf{aux}} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f})$ can be constructed based on Algorithm 2, where $\hat{f} : \hat{X} \times \hat{U} \times \hat{W} \times V_\varsigma \to \hat{X}$ is defined as

$$\hat{f}(\hat{x}(k), \hat{\nu}(k+M-1), \hat{\mathsf{w}}(k), \tilde{\varsigma}(k)) = \Pi_x(\tilde{f}(\hat{x}(k), \hat{\nu}(k+M-1), \hat{\mathsf{w}}(k), \tilde{\varsigma}(k))), \tag{4.4.12}$$

and $\Pi_x : X \to \hat{X}$ is the map that assigns to any $x \in X$, the representative point $\bar{x} \in \hat{X}$ of the corresponding partition set containing x. The initial state of $\widehat{\Sigma}_{\mathsf{aux}}$ is also selected according to $\hat{x}_0 := \Pi_x(x_0)$ with x_0 being the initial state of Σ_{aux}. Dynamical representation provided by (4.4.12) uses the map $\Pi_x : X \to \hat{X}$ that satisfies the inequality (4.2.5).

4.4.1.4 Compositionality Results

In this subsection, we assume that we are given a complex stochastic control system Σ composed of $N \in \mathbb{N}_{\geq 1}$ discrete-time stochastic control subsystems Σ_i as in (4.4.3), where their internal inputs w_i are partitioned as in (3.2.8). Now we define the *interconnected* stochastic control systems.

Definition 4.4.15. *Suppose we are given $N \in \mathbb{N}_{\geq 1}$ discrete-time stochastic control subsystems Σ_i, $i \in \{1, \ldots, N\}$, with the internal input configuration as in* (3.2.8). *The*

Algorithm 2 Abstraction of dt-SCS Σ_{aux} by a finite MDP $\widehat{\Sigma}_{\mathsf{aux}}$

Require: Input dt-SCS $\Sigma_{\mathsf{aux}} = (X, U, \tilde{W}, T_{\mathsf{x}})$
1: Select finite partitions of sets $X, U, \tilde{W}$ as $X = \cup_{i=1}^{n_x} \mathsf{X}_i$, $U = \cup_{i=1}^{n_\nu} \mathsf{U}_i$, $\tilde{W} = \cup_{i=1}^{n_w} \tilde{\mathsf{W}}_\mathsf{i}$
2: For each $\mathsf{X}_i, \mathsf{U}_i$, and $\tilde{\mathsf{W}}_\mathsf{i}$, select single representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$, $\bar{\mathsf{w}}_\mathsf{i} \in \tilde{\mathsf{W}}_\mathsf{i}$
3: Define $\hat{X} := \{\bar{x}_i, i = 1, ..., n_x\}$ as the finite state set of MDP $\widehat{\Sigma}_{\mathsf{aux}}$ with external and internal input sets $\hat{U} := \{\bar{\nu}_i, i = 1, ..., n_\nu\}$ $\hat{W} := \{\bar{\mathsf{w}}_\mathsf{i}, i = 1, ..., n_w\}$
4: Define the map $\Xi : X \to 2^X$ that assigns to any $x \in X$, the corresponding partition set it belongs to, i.e., $\Xi(x) = \mathsf{X}_i$ if $x \in \mathsf{X}_i$ for some $i = 1, 2, \dots, n_x$
5: Compute the discrete transition probability matrix $\hat{T}_{\mathsf{x}}$ for $\widehat{\Sigma}_{\mathsf{aux}}$ as:

$$\hat{T}_{\mathsf{x}}(x' \,|\, x, \nu, \mathsf{w}) = T_{\mathsf{x}}(\Xi(x') \,|\, x, \nu, \mathsf{w}),$$

for all $x := x(k), x' := x(k + \mathcal{M}) \in \hat{X}, \nu := \nu(k + \mathcal{M} - 1) \in \hat{U}, \mathsf{w} := \mathsf{w}(k) \in \hat{W}$, $k = j\mathcal{M}, j \in \mathbb{N}$,

Ensure: Output finite MDP $\widehat{\Sigma}_{\mathsf{aux}} = (\hat{X}, \hat{U}, \hat{W}, \hat{T}_{\mathsf{x}})$

interconnection of Σ_i for any $i \in \{1, \dots, N\}$, denoted by $\mathcal{I}_{fs}(\Sigma_1, \dots, \Sigma_N)$, is the interconnected stochastic control system Σ, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, and the function $f := \prod_{i=1}^N f_i$, subjected to the following constraint:

$$\forall i, j \in \{1, \dots, N\}, i \neq j: \quad w_{ij} = x_j, \quad X_j \subseteq W_{ij}. \tag{4.4.13}$$

In the next theorem, we leverage max small-gain Assumption 3.3.4 together with the concavity assumption of $\max_i \sigma_i^{-1}$ to show the main compositionality result of the section.

Theorem 4.4.16. *Suppose we are given the interconnected dt-SCS $\Sigma = \mathcal{I}_{fs}(\Sigma_1, \dots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i. Let each Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding* max*-type FPSF V_i. If Assumption 3.3.4 holds and also*

$$\forall i, j \in \{1, \dots, N\}, i \neq j: \qquad \hat{X}_j \subseteq \hat{W}_{ij},$$

then the function $V(x, \hat{x})$ defined as

$$V(x, \hat{x}) := \max_i \Big\{ \sigma_i^{-1}(V_i(x_i, \hat{x}_i)) \Big\}, \tag{4.4.14}$$

for σ_i as in (3.3.6), is a max*-type FSF function from $\widehat{\Sigma} = \mathcal{I}_{fs}(\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_N)$ to $\Sigma = \mathcal{I}_{fs}(\Sigma_1, \dots, \Sigma_N)$ at times $k = j\mathcal{M}, j \in \mathbb{N}$ provided that $\max_i \sigma_i^{-1}$ is concave.*

Proof. We first show that for some $\mathcal{K}_\infty$ function α, the max-type FSF V in (4.4.14) satisfies the inequality (4.4.9). For any $x = [x_1; \dots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \dots; \hat{x}_N] \in \hat{X}$, one gets

$$\|x-\hat{x}\| = \max_i \left\{\|x_i-\hat{x}_i\|\right\} \le \max_i \left\{\alpha_i^{-1}(V_i(x_i,\hat{x}_i))\right\}$$
$$\le \hat{\beta}\ (\max_i \left\{\sigma_i^{-1}(V_i(x_i,\hat{x}_i))\right\}) = \hat{\beta}(V(x,\hat{x}))$$

where $\hat{\beta}(s) = \max_i \left\{\alpha_i^{-1} \circ \sigma_i(s)\right\}$ for all $s \in \mathbb{R}_{\ge 0}$, which is a $\mathcal{K}_\infty$ function and thus (4.4.9) holds with $\alpha = \hat{\beta}^{-1}$.

We proceed with showing (4.4.10). Let $\kappa(s) = \max_{i,j}\{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j(s)\}$. It follows from (3.3.6) that $\kappa < \mathrm{id}$. Since $\max_i \sigma_i^{-1}$ is concave and by using Jensen's inequality, one can readily acquire the chain of inequalities in (4.4.15) where $\rho_{\mathrm{ext}}(\cdot)$ and ψ are defined as

$$\rho_{\mathrm{ext}}(s) := \begin{cases} \max_i\{\sigma_i^{-1} \circ \rho_{\mathrm{ext}i}(s_i)\}, \\ \text{s.t.} \quad s_i \ge 0,\ \|[s_1;\ldots;s_N]\| = s, \end{cases}$$
$$\psi := \max_i \sigma_i^{-1}(\psi_i).$$

Since κ and ρ_{ext} in (4.4.15) are $\mathcal{K}_\infty$ and $\mathcal{K}_\infty \cup \{0\}$, respectively, V is a max-type FSF from $\widehat{\Sigma}$ to Σ which completes the proof. □

4.4.1.5 Construction of max-Type FPSF

4.4.1.5.1 Discrete-Time Linear Stochastic Control Systems

In this subsection, we focus on the linear class of dt-SCS. Suppose we are given a network composed of N linear discrete-time stochastic control subsystems as follows:

$$\Sigma_i : x_i(k+1) = A_i x_i(k) + D_i w_i(k) + B_i \nu_i(k) + R_i \varsigma_i(k), \tag{4.4.16}$$

where the additive noise $\varsigma_i(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. Suppose w_i is partitioned as (3.2.8), and $\mathcal{M} \in \mathbb{N}_{\ge 1}$ be given. By employing the interconnection constraint (4.4.13) and Assumption 4.4.3, the dynamic of the sampled system at $\mathcal{M}$-*step* forward can be written as

$$\Sigma_{\mathsf{aux}i} : x_i(k+\mathcal{M}) = \tilde{A}_i x_i(k) + B_i \nu_i(k+\mathcal{M}-1) + \tilde{D}_i \mathsf{w}_i(k) + \tilde{R}_i \tilde{\varsigma}_i(k),$$

where $\tilde{\varsigma}_i(k)$ for the fully interconnected network is obtained as in (4.4.5). Although the pairs (A_i, B_i) may not be necessarily stabilizable, we assume that the pairs $(\tilde{A}_i, B_i)$ after $\mathcal{M}$-*step* are stabilizable as discussed in Example 4.4.1. Therefore, we can construct the finite MDP as presented in Algorithm 2 from the new auxiliary system. To do so, we candidate the following function

$$V_i(x_i,\hat{x}_i) = ((x_i-\hat{x}_i)^T \tilde{M}_i (x_i-\hat{x}_i))^{\frac{1}{2}}, \tag{4.4.17}$$

where $\tilde{M}_i$ is a positive-definite matrix of an appropriate dimension. In order to show that V_i in (4.4.17) is a max-type FPSF from $\widehat{\Sigma}_i$ to Σ_i, we require the following assumption on $\Sigma_{\mathsf{aux}i}$.

$$
\begin{aligned}
&\mathbb{E}\Big[V(x(k+\mathcal{M}),\hat{x}(k+\mathcal{M}))\,\big|\,x(k),\hat{x}(k),\nu(k+\mathcal{M}-1),\hat{\nu}(k+\mathcal{M}-1)\Big]\\
&\quad=\mathbb{E}\Big[\max_i\Big\{\sigma_i^{-1}V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\Big\}\,\big|\,x(k),\hat{x}(k),\nu(k+\mathcal{M}-1),\hat{\nu}(k+\mathcal{M}-1)\Big]\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\big|\,x(k),\hat{x}(k),\nu(k+\mathcal{M}-1),\hat{\nu}(k+\mathcal{M}-1)\Big])\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\big|\,x_i=x_i(k),\hat{x}_i=\hat{x}_i(k),\nu_i=\nu_i(k+\mathcal{M}-1),\\
&\qquad\qquad\hat{\nu}_i=\hat{\nu}_i(k+\mathcal{M}-1)\Big])\Big\}\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\|\mathsf{w}_i-\hat{\mathsf{w}}_i\|),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|\mathsf{w}_{ij}-\hat{\mathsf{w}}_{ij}\|\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\|x_j-\hat{x}_j\|\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad\leq\max_i\Big\{\sigma_i^{-1}(\max\{\kappa_i(V_i(x_i,\hat{x}_i)),\rho_{\mathrm{int}i}(\max_{j,j\neq i}\{\alpha_j^{-1}(V_j(x_j,\hat{x}_j))\}),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}(V_j(x_j,\hat{x}_j)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j(x_j,\hat{x}_j)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad\leq\max_{i,j,l}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j\circ\sigma_l^{-1}(V_l(x_l,\hat{x}_l)),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max_{i,j}\Big\{\sigma_i^{-1}(\max\{\kappa_{ij}\circ\sigma_j(V(x,\hat{x})),\rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|),\psi_i\})\Big\}\\
&\quad=\max\Big\{\kappa(V(x,\hat{x})),\rho_{\mathrm{ext}}(\|\hat{\nu}\|),\psi\Big\}.
\end{aligned}
\tag{4.4.15}
$$

Assumption 4.4.17. *Assume that there exist matrices $\tilde{M}_i \succ 0$, and K_i of appropriate dimensions such that the matrix inequality*

$$
(1+2\pi_i)(\tilde{A}_i+B_iK_i)^T\tilde{M}_i(\tilde{A}_i+B_iK_i)\preceq\hat{\kappa}_i\tilde{M}_i, \tag{4.4.18}
$$

holds for some constants $0<\hat{\kappa}_i<1$ and $\pi_i>0$.

Now we raise the main result of this subsection.

Theorem 4.4.18. *Assume the system $\Sigma_{\mathsf{aux}i}$ satisfies Assumption 4.4.17. Let $\widehat{\Sigma}_{\mathsf{aux}i}$ be its finite abstraction as constructed in Algorithm 2 with the state discretization parameter $\bar{\delta}_i$. Then the function V_i defined in (4.4.17) is a* max*-type FPSF from $\widehat{\Sigma}_i$ to Σ_i.*

Proof. We first show that $\forall x_i(k)$, $\forall \hat{x}_i(k)$, $\forall \hat{\nu}_i(k+\mathcal{M}-1)$, $\exists \nu_i(k+\mathcal{M}-1)$, $\forall \hat{\mathsf{w}}_i(k)$, $\forall \mathsf{w}_i(k)$, such that V_i satisfies $\sqrt{\lambda_{\min}(\tilde{M}_i)}\|x_i(k)-\hat{x}_i(k)\| \leq V_i(x_i(k),\hat{x}_i(k))$ and then

$$\begin{aligned}&\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\big|x_i=x_i(k),\hat{x}_i=\hat{x}_i(k),\nu_i=\nu_i(k+\mathcal{M}-1),\\&\quad\hat{\nu}_i=\hat{\nu}_i(k+\mathcal{M}-1),\mathsf{w}_i=\mathsf{w}_i(k),\hat{\mathsf{w}}_i=\hat{\mathsf{w}}_i(k)\Big]\\&\leq\max\Big\{(1-(1-\tilde{\pi}_i)\tilde{\kappa}_i)(V_i(x_i,\hat{x}_i)),(1+\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(\bar{p}_i(1+\pi_i+1/\pi_i))^{\frac{1}{2}}\\&\quad\|\sqrt{\tilde{M}_i}\tilde{D}_i\|_2\|\mathsf{w}_i-\hat{\mathsf{w}}_i\|,(1+1/\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(n_i(1+2/\pi_i)\lambda_{\max}(\tilde{M}_i))^{\frac{1}{2}}\,\bar{\delta}_i\Big\}.\end{aligned}$$

Since $\sqrt{\lambda_{\min}(\tilde{M}_i)}\|x_i-\hat{x}_i\| \leq ((x_i-\hat{x}_i)^T\tilde{M}_i(x_i-\hat{x}_i))^{\frac{1}{2}}$, one can readily verify that $\sqrt{\lambda_{\min}(\tilde{M}_i)}\|x_i-\hat{x}_i\| \leq V_i(x_i,\hat{x}_i)$ $\forall x_i$, $\forall \hat{x}_i$. Then the inequality (4.4.7) holds with $\alpha_i(s)=\sqrt{\lambda_{\min}(\tilde{M}_i)}\,s$ for any $s\in\mathbb{R}_{\geq 0}$. We proceed with showing the inequality (4.4.8). Given any $x_i(k)$, $\hat{x}_i(k)$, and $\hat{\nu}_i(k+\mathcal{M}-1)$, we choose $\nu_i(k+\mathcal{M}-1)$ via the following *interface* function:

$$\nu_i(k+\mathcal{M}-1)=K_i(x_i(k)-\hat{x}_i(k))+\hat{\nu}_i(k+\mathcal{M}-1), \tag{4.4.19}$$

and simplify

$$\begin{aligned}&\tilde{A}_ix_i(k)+B_i\nu_i(k+\mathcal{M}-1)+\tilde{D}_i\mathsf{w}_i(k)+\tilde{R}_i\varsigma_i(k)\\&\qquad-\Pi_{x_i}(\tilde{A}_i\hat{x}_i(k)+B_i\hat{\nu}_i(k+\mathcal{M}-1)+\tilde{D}_i\hat{\mathsf{w}}_i(k)+\tilde{R}_i\varsigma_i(k))\end{aligned}$$

to

$$(\tilde{A}_i+B_iK_i)(x_i(k)-\hat{x}_i(k))+\tilde{D}_i(\mathsf{w}_i(k)-\hat{\mathsf{w}}_i(k))+\bar{N}_i,$$

where $\bar{N}_i=\tilde{A}_i\hat{x}_i(k)+B_i\hat{\nu}_i(k+\mathcal{M}-1)+\tilde{D}_i\hat{\mathsf{w}}_i(k)+\tilde{R}_i\varsigma_i(k)-\Pi_{x_i}(\tilde{A}_i\hat{x}_i(k)+B_i\hat{\nu}_i(k+\mathcal{M}-1)+\tilde{D}_i\hat{\mathsf{w}}_i(k)+\tilde{R}_i\varsigma_i(k))$. By employing Cauchy-Schwarz inequality, Young's inequality, Assumption 4.4.17, and since

$$\begin{cases}\|\bar{N}_i\| \leq \bar{\delta}_i,\\ \bar{N}_i^T\tilde{M}_i\bar{N}_i\leq n_i\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2,\end{cases}$$

one can obtain the chain of inequalities in (4.4.20). Hence, the proposed V_i in (4.4.17) is a max-type FPSF from $\widehat{\Sigma}_i$ to Σ_i, which completes the proof. Note that the last inequality in (4.4.20) is derived by applying Theorem 1 in [SGZ18]. The functions $\alpha_i,\kappa_i\in\mathcal{K}_\infty$, and $\rho_{\text{int}i}$, $\rho_{\text{ext}i}\in\mathcal{K}_\infty\cup\{0\}$ in Definition 4.4.10 associated with V_i in (4.4.17) are defined as $\alpha_i(s)=\sqrt{\lambda_{\min}(\tilde{M}_i)}\,s$, $\kappa_i(s):=(1-(1-\tilde{\pi}_i)\tilde{\kappa}_i)\,s$, $\rho_{\text{int}i}(s):=(1+\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(\bar{p}_i(1+\pi_i+1/\pi_i))^{\frac{1}{2}}\|\sqrt{\tilde{M}_i}\tilde{D}_i\|_2\,s$, $\rho_{\text{ext}i}(s):=0$, $\forall s\in\mathbb{R}_{\geq 0}$ where $\tilde{\kappa}_i=1-\sqrt{\kappa_i}$, $0<\tilde{\pi}_i<1$, and $\tilde{\delta}_i>0$. Moreover, the positive constant ψ_i in (4.4.8) is $\psi_i=(1+1/\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(n_i(1+2/\pi_i)\lambda_{\max}(\tilde{M}_i))^{\frac{1}{2}}\,\bar{\delta}_i$. □

$$\begin{aligned}
&\mathbb{E}\Big[V_i(x_i(k{+}\mathcal{M}), \hat{x}_i(k{+}\mathcal{M})) \,\big|\, x_i{=}x_i(k), \hat{x}_i{=}\hat{x}_i(k), \nu_i{=}\nu_i(k{+}\mathcal{M}{-}1), \hat{\nu}_i{=}\hat{\nu}_i(k{+}\mathcal{M}{-}1),\\
&\quad \mathsf{w}_i = \mathsf{w}_i(k), \hat{\mathsf{w}}_i = \hat{\mathsf{w}}_i(k)\Big]\\
&\quad = \big((x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}_i(\tilde{A}_i + B_iK_i)(x_i - \hat{x}_i)\\
&\qquad + 2(x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}_i\tilde{D}_i(\mathsf{w}_i - \hat{\mathsf{w}}_i) + (\mathsf{w}_i - \hat{\mathsf{w}}_i)^T\tilde{D}_i^T\tilde{M}_i\tilde{D}_i(\mathsf{w}_i - \hat{\mathsf{w}}_i)\\
&\qquad + 2(x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\big|\, x_i, \hat{x}_i, \nu_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big]\\
&\qquad + 2(\mathsf{w}_i - \hat{\mathsf{w}}_i)^T\tilde{D}_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\big|\, x_i, \hat{x}_i, \nu_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big] + \mathbb{E}\Big[\bar{N}_i^T\tilde{M}_i\bar{N}_i \,\big|\, x_i, \hat{x}_i, \nu_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big]\big)^{\frac{1}{2}}\\
&\quad \leq \sqrt{\hat{\kappa}_i}(V_i(x_i, \hat{x}_i)) + (\bar{p}_i(1{+}\pi_i{+}1/\pi_i))^{\frac{1}{2}}\|\sqrt{\tilde{M}_i}\tilde{D}_i\|_2\|\mathsf{w}_i{-}\hat{\mathsf{w}}_i\| + (n_i(1{+}2/\pi_i)\lambda_{\max}(\tilde{M}_i))^{\frac{1}{2}}\,\bar{\delta}_i\\
&\quad \leq \max\Big\{(1{-}(1{-}\tilde{\pi}_i)\tilde{\kappa}_i)(V_i(x_i, \hat{x}_i)), (1 + \tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(\bar{p}_i(1{+}\pi_i{+}1/\pi_i))^{\frac{1}{2}}\|\sqrt{\tilde{M}_i}\tilde{D}_i\|_2\|\mathsf{w}_i{-}\hat{\mathsf{w}}_i\|,\\
&\qquad (1 + 1/\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(n_i(1 + 2/\pi_i)\lambda_{\max}(\tilde{M}_i))^{\frac{1}{2}}\,\bar{\delta}_i\Big\}. \qquad (4.4.20)
\end{aligned}$$

4.4.1.6 Case Study

In this subsection, we demonstrate the effectiveness of the proposed results by considering an interconnected system composed of four discrete-time linear stochastic control subsystems, i.e., $\Sigma = \mathcal{I}_{fs}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, such that one of them is not stabilizable. The discrete-time linear stochastic control subsystems are given by

$$\Sigma_i : \begin{cases} x_1(k+1) = 1.001x_1(k) + 0.4w_1(k) + \varsigma_1(k),\\ x_2(k+1) = -0.95x_2(k) + \nu_2(k) - 0.08w_2(k) + \varsigma_2(k),\\ x_3(k+1) = -0.94x_3(k) + \nu_3(k) - 0.05w_3(k) + \varsigma_3(k),\\ x_4(k+1) = 0.6x_4(k) + \nu_4(k) + 0.9w_4(k) + \varsigma_4(k), \end{cases} \qquad (4.4.21)$$

where

$$w_1 = x_2 + x_3, w_2 = x_1 + x_3, w_3 = x_2, w_4 = x_3.$$

As seen, the first subsystem is not stabilizable. Then we proceed with looking at the solution of Σ_i two steps ahead, i.e., $\mathcal{M} = 2$,

$$\Sigma_{\mathsf{aux}i} : \begin{cases} x_1(k+2) = 0.97x_1(k) + \tilde{D}_1\mathsf{w}_1(k) + \tilde{R}_1\tilde{\varsigma}_1(k),\\ x_2(k+2) = 0.8745x_2(k) + \nu_2(k+1) + \tilde{D}_2\mathsf{w}_2(k) + \tilde{R}_2\tilde{\varsigma}_2(k),\\ x_3(k+2) = 0.8876x_3(k) + \nu_3(k+1) + \tilde{D}_3\mathsf{w}_3(k) + \tilde{R}_3\tilde{\varsigma}_3(k),\\ x_4(k+2) = 0.36x_4(k) + \nu_4(k+1) + \tilde{D}_4\mathsf{w}_4(k) + \tilde{R}_4\tilde{\varsigma}_4(k), \end{cases} \qquad (4.4.22)$$

where

$$\tilde{D}_1 = [-0.0004; -0.0076]^T, \quad \tilde{D}_2 = [-0.0041; 0.1192]^T,$$
$$\tilde{D}_3 = [0.004; 0.0945]^T, \quad \tilde{D}_4 = [-0.045; -0.306]^T,$$
$$\mathsf{w}_1 = [x_2; x_3], \quad \mathsf{w}_2 = [x_1; x_3], \quad \mathsf{w}_3 = [x_1; x_2], \mathsf{w}_4 = [x_2; x_3],$$
$$\tilde{\varsigma}_1(k) = [\varsigma_3(k); \varsigma_2(k); \varsigma_1(k); \varsigma_1(k+1)], \quad \tilde{\varsigma}_3(k) = [\varsigma_2(k); \varsigma_3(k); \varsigma_3(k+1)],$$
$$\tilde{\varsigma}_2(k) = [\varsigma_3(k); \varsigma_1(k); \varsigma_2(k); \varsigma_2(k+1)], \quad \tilde{\varsigma}_4(k) = [\varsigma_3(k); \varsigma_4(k); \varsigma_4(k+1)].$$

Moreover, $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}; \tilde{R}_{i4}]^T, \forall i \in \{1, 2\}$, where

$$\tilde{R}_{11} = 0.4, \quad \tilde{R}_{12} = 0.4, \quad \tilde{R}_{13} = 1.001, \quad \tilde{R}_{14} = 1,$$
$$\tilde{R}_{21} = -0.08, \quad \tilde{R}_{22} = -0.08, \quad \tilde{R}_{23} = -0.95, \quad \tilde{R}_{24} = 1.$$

and $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}]^T, \forall i \in \{3, 4\}$, where

$$\tilde{R}_{31} = -0.05, \quad \tilde{R}_{32} = -0.941, \quad \tilde{R}_{33} = 1, \quad \tilde{R}_{41} = 0.9, \quad \tilde{R}_{42} = 0.6, \quad \tilde{R}_{43} = 1.$$

One can readily see that $\tilde{A}_1$ is stable. Now, we proceed with constructing the finite MDP from the $\mathcal{M}$-sampled system as acquired in (4.4.22). We fix the max-type FPSF as (4.2.14). One can readily verify that the condition (4.4.18) is satisfied with

$$\hat{\kappa}_1 = 0.9597, \quad \hat{\kappa}_2 = 0.588, \quad \hat{\kappa}_3 = 0.7115, \quad \hat{\kappa}_4 = 0.337,$$
$$K_2 = -0.1745, \quad K_3 = -0.1176, \quad K_4 = 0,$$
$$\pi_1 = 0.01, \quad \pi_2 = 0.1, \quad \pi_3 = 0.1, \quad \pi_4 = 0.8, \quad \tilde{M}_i = 1, \quad \forall i \in \{1, 2, 3, 4\}.$$

Then function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is a max-type FPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (4.4.7) with $\alpha_i(s) = s^2, \forall i \in \{1, 2, 3, 4\}$, $\forall s \in \mathbb{R}_{\geq 0}$, and the condition (4.4.8) with

$$\kappa_i(s) = 0.99s, \quad \rho_{\text{ext}i}(s) = 0, \forall i \in \{1, 2, 3, 4\}, \quad \rho_{\text{int}1}(s) = 0.8802s^2,$$
$$\rho_{\text{int}2}(s) = 0.8517s^2, \quad \rho_{\text{int}3}(s) = 0.8344s^2, \quad \rho_{\text{int}4}(s) = 0.9779s^2, \forall s \in \mathbb{R}_{\geq 0},$$
$$\psi_1 = 7409\,\bar{\delta}^2, \quad \psi_2 = 555\,\bar{\delta}^2, \quad \psi_3 = 433\,\bar{\delta}^2, \quad \psi_4 = 57.48\,\bar{\delta}^2.$$

Now we check the max small-gain condition (3.3.5) that is required for the compositionality result. By taking $\sigma_i(s) = s \ \forall i \in \{1, 2, 3, 4\}$, one can readily verify that the max small-gain condition (3.3.5) and as a result the condition (3.3.6) are satisfied. Hence, $V(x, \hat{x}) = \max_i (x_i - \hat{x}_i)^2$ is a max-type FSF from $\widehat{\Sigma}$ to Σ satisfying conditions (4.4.9) and (4.4.10) with $\alpha(s) = s^2$, $\kappa(s) = 0.99\,s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 7409\,\bar{\delta}^2$.

By taking the state discretization parameter $\bar{\delta} = 0.001$, and starting the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ from $\mathbb{1}_4$ and employing Theorem 4.4.13, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ at times $k = 2j, j = \{0, \ldots, 30\}$ with the probability at least 90%, i.e.,

$$\mathbb{P}(\|x_{a\nu}(k) - \hat{x}_{\hat{a}\hat{\nu}}(k)\| \leq 1, \ \forall k = 2j, j = \{0, \ldots, 30\}) \geq 0.9.$$

4.4.2 Stochastic Autonomous Systems

In this subsection, we focus on discrete-time stochastic autonomous systems (dt-SAS) (i.e., the dt-SCS in (4.4.3) without external inputs) and extend the proposed compositional framework of the previous section. In particular, although the provided results in the previous section do not ask the individual subsystems to be stable, our probabilistic closeness guarantee presented in this section is *more general* than the one provided in (4.4.11) since we propose the closeness guarantee for the whole state trajectory, while (4.4.11) quantifies the error only at some specific steps without providing the closeness for all time steps.

Similar to the previous section, we consider the internal and external output maps as identity. Furthermore, we employ in this section Definitions 4.4.10, and 4.4.12 but with $\rho_{\text{ext}}(\cdot) \equiv 0$. In the next subsection, we leverage the results of Theorem 4.4.13 and provide the closeness guarantee for all time instances.

4.4.2.1 Closeness Guarantee for All Time Instances

Suppose we are given an interconnected network composed of N stochastic subsystems Σ_i as in (4.4.3) where $\nu_i \equiv 0$, and with the interconnection constraint $w_{ij} = x_j, \forall i, j \in \{1, \ldots, N\}, i \neq j$. The $\mathcal{M}$-sampled systems for all time instances contain $\mathcal{M}$ different dynamics starting from initial values $\{x_i(0), x_i(1), \ldots, x_i(\mathcal{M}-1)\}$ as follows:

$$\begin{aligned}
x_i(k+\mathcal{M}) &= \tilde{f}_i(x_i(k), \mathsf{w}_i(k), \tilde{\varsigma}_i(k)),\\
x_i(k+\mathcal{M}+1) &= \tilde{f}_i(x_i(k+1), \mathsf{w}_i(k+1), \tilde{\varsigma}_i(k+1)),\\
&\vdots\\
x_i(k+2\mathcal{M}-1) &= \tilde{f}_i(x_i(k+\mathcal{M}-1), \mathsf{w}_i(k+\mathcal{M}-1), \tilde{\varsigma}_i(k+\mathcal{M}-1)),
\end{aligned} \tag{4.4.23}$$

where $k = j\mathcal{M}, j \in \mathbb{N}$. In order to show the closeness of two interconnected dt-SAS for all time instants using Theorem 4.4.13, we require the following assumption.

Assumption 4.4.19. *Assume that there exist $\mathcal{K}_\infty$ and concave function $\tilde{\eta}$ and a constant $\tilde{\theta} \geq 0$ such that for all $x(k) := x \in X$ and $\hat{x}(k) := \hat{x} \in \hat{X}$,*

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,\Big|\, x, \hat{x}\Big] \leq \tilde{\eta}(V(x, \hat{x})) + \tilde{\theta}.$$

Remark 4.4.20. *Note that Assumption 4.4.19 is a standard one employed in the definition of the* sum*-type SSF ($\mathcal{M} = 1$) similar to the one appeared in* (3.2.4)*. Remark that Assumption 4.4.19 is less restrictive than the condition* (4.4.10) *since we do not require $\tilde{\eta} < \mathcal{I}_d$ (cf. Case study 4.4.2.3).*

Next theorem shows the closeness of two interconnected dt-SAS for all time instants.

Theorem 4.4.21. *Let Σ and $\widehat{\Sigma}$ be two dt-SAS without internal inputs (i.e., $\rho_{\text{int}}(\cdot) \equiv 0$), where $\hat{X} \subseteq X$. Suppose V is a* max*-type FSF from $\widehat{\Sigma}$ to Σ at times $k = j\mathcal{M}, j \in \mathbb{N}$, and Assumption 4.4.19 holds. For any random variables a and $\hat{a}$ as the initial states of the* *two dt-SS, the closeness of the two interconnected systems for all time instants within the time horizon $T_d\mathcal{M}$ can be acquired as*

$$\mathbb{P}\left\{\sup_{k\in\mathbb{N},\,0\leq k\leq T_d\mathcal{M}}\|x_a(k)-\hat{x}_{\hat{a}}(k)\|\geq\varepsilon\,\middle|\,a,\hat{a}\right\} \tag{4.4.24}$$
$$\leq\begin{cases}\bar{\eta}_0+\bar{\beta}_0V(a,\hat{a}), & \text{if } \mathcal{M}=1,\\ \sum_{i=0}^{\mathcal{M}-1}(\bar{\eta}_i+\bar{\beta}_i\tilde{\eta}^i(V(a,\hat{a}))+\bar{\beta}_i\sum_{n=0}^{i-1}\tilde{\eta}^n(\tilde{\theta})), & \text{if } \mathcal{M}\geq 2,\end{cases}$$

where $\tilde{\eta}^0=\mathcal{I}_d$, and $\bar{\eta}_i,\bar{\beta}_i$ are some constants coming from the right-hand side of (4.4.11) in the form of $\bar{\eta}_i+\bar{\beta}_iV(a,\hat{a})$, with

$$\begin{cases}\bar{\eta}_i=1-(1-\frac{\psi_i}{\alpha_i(\varepsilon)})^{T_d},\ \bar{\beta}_i=\frac{1}{\alpha_i(\varepsilon)}(1-\frac{\psi_i}{\alpha_i(\varepsilon)})^{T_d}, & \text{if } \alpha_i\left(\varepsilon\right)\geq\frac{\psi_i}{\hat{\kappa}_i},\\ \bar{\eta}_i=\frac{\psi_i}{\hat{\kappa}_i\alpha_i(\varepsilon)}(1-(1-\hat{\kappa}_i)^{T_d}),\ \bar{\beta}_i=\frac{1}{\alpha_i(\varepsilon)}(1-\hat{\kappa}_i)^{T_d}, & \text{if } \alpha_i\left(\varepsilon\right)<\frac{\psi_i}{\hat{\kappa}_i}.\end{cases}$$

Proof. We write our closeness guarantee proposed in (4.4.11) (i.e., at times $k=j\mathcal{M},j\in\mathbb{N}$, $\mathcal{M}\in\mathbb{N}_{\geq 1}$) for $\mathcal{M}$-sampled systems with $\mathcal{M}$ different dynamics as appeared in (4.4.23) starting from initial values $\{x_i(0),x_i(1),\ldots,x_i(\mathcal{M}-1)\}$ as follows:

$$\mathbb{P}\left\{\sup\nolimits_{k=j\mathcal{M},\,0\leq j\leq T_d}\|x_a(k)-\hat{x}_{\hat{a}}(k)\|\geq\varepsilon\,\middle|\,a,\hat{a}\right\}\leq\bar{h}_0(a,\hat{a}),$$
$$\mathbb{P}\left\{\sup\nolimits_{k=j\mathcal{M}+1,\,0\leq j\leq T_d}\|x_{x_1}(k)-\hat{x}_{\hat{x}_1}(k)\|\geq\varepsilon\,\middle|\,x_1,\hat{x}_1\right\}\leq\bar{h}_1(x_1,\hat{x}_1),$$
$$\vdots$$
$$\mathbb{P}\left\{\sup\nolimits_{k=j\mathcal{M}+\mathcal{M}-1,\,0\leq j\leq T_d}\|x_{x_{\mathcal{M}-1}}(k)-\hat{x}_{\hat{x}_{\mathcal{M}-1}}(k)\|\geq\varepsilon\,\middle|\,x_{\mathcal{M}-1},\hat{x}_{\mathcal{M}-1}\right\}\leq\bar{h}_{\mathcal{M}-1}(x_{\mathcal{M}-1},\hat{x}_{\mathcal{M}-1}).$$

Now one can write the chain of inequalities in (4.4.25) which completes the proof. The first inequality is based on Boole's inequality,

$$\mathbb{P}\{A_1\cup A_2\,|\,a_1,a_2\}\leq\mathbb{P}\{A_1\,|\,a_1,a_2\}+\mathbb{P}\{A_2\,|\,a_1,a_2\}.$$

The last inequality is valid due to

$$\mathbb{E}\Big[V(x_i,\hat{x}_i)\,\Big|\,a,\hat{a}\Big]\leq\tilde{\eta}^i(V(a,\hat{a}))+\sum_{n=0}^{i-1}\tilde{\eta}^n(\tilde{\theta}),$$

which can be proved inductively. It holds for $i=1$ using $\tilde{\eta}^0=\mathcal{I}_d$, and Assumption 4.4.19. For $(i+1)$ we have

$$\begin{aligned}\mathbb{E}\Big[V(x_{i+1},\hat{x}_{i+1})\,\Big|\,a,\hat{a}\Big]&=\mathbb{E}\Big[\mathbb{E}\left[V(x_{i+1},\hat{x}_{i+1})\,\middle|\,x_i,\hat{x}_i\right]\,\Big|\,a,\hat{a}\Big]\\ &\leq\mathbb{E}\Big[\tilde{\eta}(V(x_i,\hat{x}_i))+\tilde{\theta}\,\Big|\,a,\hat{a}\Big]\leq\tilde{\eta}(\mathbb{E}\Big[V(x_i,\hat{x}_i)\,\Big|\,a,\hat{a}\Big])+\tilde{\theta}\\ &\leq\tilde{\eta}(\tilde{\eta}^i(V(a,\hat{a}))+\sum_{n=0}^{i-1}\tilde{\eta}^n(\tilde{\theta}))+\tilde{\theta}\leq\tilde{\eta}^{i+1}(V(a,\hat{a}))+\sum_{n=0}^{i-1}\tilde{\eta}^{n+1}(\tilde{\theta})+\tilde{\theta}\\ &\leq\tilde{\eta}^{i+1}(V(a,\hat{a}))+\sum_{n=0}^{i}\tilde{\eta}^n(\tilde{\theta}),\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{k\in\mathbb{N},\,0\le k\le T_d\mathcal{M}}\|x_a(k)-\hat{x}_{\hat{a}}(k)\|\ge\varepsilon\,\Big|\,a,\hat{a}\Big\}\\
&\le\sum_{i=0}^{\mathcal{M}-1}\mathbb{P}\Big\{\sup_{k=j\mathcal{M}+i,\,0\le j\le T_d}\|x_{x_i}(k)-\hat{x}_{\hat{x}_i}(k)\|\ge\varepsilon\,\Big|\,a,\hat{a}\Big\}\\
&\le\bar{h}_0(a,\hat{a})+\sum_{i=1}^{\mathcal{M}-1}\mathbb{E}\Big[\bar{h}_i(x_i,\hat{x}_i)\,\Big|\,a,\hat{a}\Big]=\sum_{i=0}^{\mathcal{M}-1}\mathbb{E}\Big[\bar{\eta}_i+\bar{\beta}_iV(x_i,\hat{x}_i)\,\Big|\,a,\hat{a}\Big]\\
&=\sum_{i=0}^{\mathcal{M}-1}(\bar{\eta}_i+\bar{\beta}_i\mathbb{E}\Big[V(x_i,\hat{x}_i)\,\Big|\,a,\hat{a}\Big])\le\sum_{i=0}^{\mathcal{M}-1}(\bar{\eta}_i+\bar{\beta}_i\tilde{\eta}^i(V(a,\hat{a}))+\bar{\beta}_i\sum_{n=0}^{i-1}\tilde{\eta}^n(\tilde{\theta})).
\end{aligned}\tag{4.4.25}$$

where we have respectively used the law of total expectation, Assumption 4.4.19, Jensen's inequality for the concave function $\tilde{\eta}$, assumption of the induction step for i, and finally using the fact that $\tilde{\eta}$ is subadditive. Note that $\tilde{\eta}$ is indeed subadditive because it is *concave* and $\tilde{\eta}:[0,\infty]\rightarrow[0,\infty]$ with $\tilde{\eta}(0)=0$ [Sch96, Chapter 12]. □

4.4.2.2 Stochastic Autonomous Systems with Incremental Quadratic Constraint on Nonlinearity

In this subsection, we impose conditions on the infinite dt-SAS Σ_i in order to find a max-type SPSF (i.e., $\mathcal{M}=1$) from $\widehat{\Sigma}_i$ to Σ_i for the nonlinear class of stochastic systems with an incremental quadratic constraint on the nonlinearity. The class of nonlinear stochastic autonomous systems, considered here, is given by

$$x_i(k+1)=A_ix_i(k)+E_i\varphi_i(k,F_ix_i(k))+D_iw_i(k)+R_i\varsigma_i(k),\tag{4.4.26}$$

where the additive noise $\varsigma_i(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. Moreover, the time-varying nonlinearity satisfies the incremental quadratic inequality in (4.3.25) for $\varphi_i,\forall i\in\{1,\ldots,N\}$. We use the tuple $\Sigma_i=(A_i,D_i,E_i,F_i,R_i,\varphi_i)$, to refer to the class of nonlinear stochastic systems of the form (4.4.26).

Now we provide a condition under which a candidate V_i in the quadratic form of (4.2.14) is a max-type SPSF from $\widehat{\Sigma}_i$ to Σ_i.

Assumption 4.4.22. *Assume that for some constants* $0<\hat{\kappa}_i<1$*, and* $\pi_i>0$*, there exists a matrix* $\tilde{M}_i$ *of an appropriate dimension such that the following inequality holds:*

$$\begin{bmatrix}(1+2/\pi_i)A_i^T\tilde{M}_iA_i & A_i^T\tilde{M}_iE_i\\ E_i^T\tilde{M}_iA_i & (1+2/\pi_i)E_i^T\tilde{M}_iE_i\end{bmatrix}\preceq\begin{bmatrix}\hat{\kappa}_i\tilde{M}_i-F_i^T\tilde{Q}_{11i}F_i & -F_i^T\tilde{Q}_{12i}\\ -\tilde{Q}_{12i}^TF_i & -\tilde{Q}_{22i}\end{bmatrix}.\tag{4.4.27}$$

Now we raise the main result of this subsection.

Theorem 4.4.23. *Assume $\Sigma_i = (A_i, D_i, E_i, F_i, R_i, \varphi_i)$ satisfies Assumption 4.4.22. Let $\widehat{\Sigma}_i$ be its finite MC as described in Algorithm 1 (but for stochastic autonomous systems) with the state discretization parameter $\bar{\delta}_i$. Then the function V_i defined in* (4.2.14) *is a* max*-type SPSF (with $\mathcal{M} = 1$) from $\widehat{\Sigma}_i$ to Σ_i.*

Proof. Since $\lambda_{\min}(\tilde{M}_i)\|x_i - \hat{x}_i\|^2 \leq (x_i - \hat{x}_i)^T \tilde{M}_i (x_i - \hat{x}_i)$, it can be readily verified that $\lambda_{\min}(\tilde{M}_i)\|x_i - \hat{x}_i\|^2 \leq V_i(x_i, \hat{x}_i)$ holds $\forall x_i$, $\forall \hat{x}_i$, implying that the inequality (3.3.1) holds with $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the the inequality (3.3.2) holds, as well. Given any $x_i := x_i(k)$, and $\hat{x}_i := \hat{x}_i(k)$, we simplify

$$A_i x_i + D_i w_i + E_i \varphi_i(k, F_i x_i) + R_i \varsigma_i \\ - \Pi_{x_i}(A_i \hat{x}_i + D_i \hat{w}_i + E_i \varphi_i(k, F_i \hat{x}_i) + R_i \varsigma_i)$$

to

$$A_i(x_i - \hat{x}_i) + D_i(w_i - \hat{w}_i) + E_i(\varphi_i(k, F_i x_i) - \varphi_i(k, F_i \hat{x}_i)) + \bar{N}_i,$$

where $\bar{N}_i = A_i \hat{x}_i + D_i \hat{w}_i + E_i \varphi_i(k, F_i \hat{x}_i) + R_i \varsigma_i - \Pi_{x_i}(A_i \hat{x}_i + D_i \hat{w}_i + E_i \varphi_i(k, F_i \hat{x}_i) + R_i \varsigma_i)$. By defining $\bar{\varphi}_i = \varphi_i(k, F_i x_i) - \varphi_i(k, F_i \hat{x}_i)$, and employing the fact that $\forall x_i \in X_i, \forall \hat{x}_i \in \hat{X}_i$, [AC11],

$$\begin{bmatrix} x_i - \hat{x}_i \\ \bar{\varphi}_i \end{bmatrix}^T \begin{bmatrix} F_i & 0 \\ 0 & \mathbb{I} \end{bmatrix}^T \tilde{Q}_i \begin{bmatrix} F_i & 0 \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} x_i - \hat{x}_i \\ \bar{\varphi}_i \end{bmatrix} \geq 0,$$

using Young's inequality, Cauchy-Schwarz inequality, the matrix inequality (4.4.27), and since

$$\|\bar{N}_i\| \leq \bar{\delta}_i, \quad \bar{N}_i^T \tilde{M}_i \bar{N}_i \leq \lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2,$$

one can obtain the chain of inequalities in (4.4.28). Hence, the proposed V_i in (4.2.14) is a max-type SPSF from $\widehat{\Sigma}_i$ to Σ_i, which completes the proof. Note that functions $\alpha_i, \kappa_i \in \mathcal{K}_\infty$, and $\rho_{\text{int}i} \in \mathcal{K}_\infty \cup \{0\}$, are defined as $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)s^2$, $\kappa_i(s) := (1 - (1 - \tilde{\pi}_i)\tilde{\kappa}_i)\, s$, $\rho_{\text{int}i}(s) := (1 + \tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i \tilde{\pi}_i})(\bar{p}_i(1 + 2\pi_i + 1/\pi_i))\|\sqrt{\tilde{M}_i} D_i\|_2^2\, s^2$, $\forall s \in \mathbb{R}_{\geq 0}$ where $\tilde{\kappa}_i = 1 - \hat{\kappa}_i$, $0 < \tilde{\pi}_i < 1$, and $\tilde{\delta}_i > 0$. Moreover, the positive constant ψ_i is $\psi_i = (1 + 1/\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i \tilde{\pi}_i})(n_i(1 + 3\pi_i)\lambda_{\max}(\tilde{M}_i))\, \bar{\delta}_i^2$. □

4.4.2.3 Case Study

In this subsection, we demonstrate the effectiveness of the proposed results by considering an interconnected system composed of four discrete-time linear stochastic subsystems, i.e., $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, such that one of them is not stable. The stochastic subsystems are given by

$$\Sigma_i : \begin{cases} x_1(k+1) = 1.001 x_1(k) + 0.5 w_1(k) + \varsigma_1(k), \\ x_2(k+1) = -0.95 x_2(k) - 0.08 w_2(k) + \varsigma_2(k), \\ x_3(k+1) = -0.94 x_3(k) - 0.05 w_3(k) + \varsigma_3(k), \\ x_4(k+1) = 0.6 x_4(k) + 0.9 w_4(k) + \varsigma_4(k), \end{cases} \tag{4.4.29}$$

$$
\begin{aligned}
&\mathbb{E}\Big[V_i(x_i(k+1),\hat{x}_i(k+1))\,\Big|\,x_i=x_i(k),\hat{x}_i=\hat{x}_i(k),w_i=w_i(k),\hat{w}_i=\hat{w}_i(k)\Big]\\
&=(x_i-\hat{x}_i)^TA_i^T\tilde{M}_iA_i(x_i-\hat{x}_i)+\bar{\varphi}_i^TE_i^T\tilde{M}_iE_i\bar{\varphi}_i+(w_i-\hat{w}_i)^TD_i^T\tilde{M}_iD_i(w_i-\hat{w}_i)\\
&\quad+\mathbb{E}\Big[\bar{N}_i^T\tilde{M}_i\bar{N}_i\,\big|\,x,\hat{x}_i,w_i,\hat{w}_i\Big]+2(x_i-\hat{x}_i)^TA_i^T\tilde{M}_iD_i(w_i-\hat{w}_i)+2\bar{\varphi}_i^TE_i^T\tilde{M}_iD_i(w_i-\hat{w}_i)\\
&\quad+2(w_i-\hat{w}_i)^TD_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i\,\big|\,x_i,\hat{x}_i,w_i,\hat{w}_i\Big]+2(x_i-\hat{x}_i)^TA_i^T\tilde{M}_iE_i\bar{\varphi}_i\\
&\quad+2(x_i-\hat{x}_i)^TA_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i\,\big|\,x_i,\hat{x}_i,w_i,\hat{w}_i\Big]+2\bar{\varphi}_i^TE_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i\,\big|\,x_i,\hat{x}_i,w_i,\hat{w}_i\Big]\\
&\le\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}^T\begin{bmatrix}(1+2/\pi_i)A_i^T\tilde{M}_iA_i & A_i^T\tilde{M}_iE_i\\ E_i^T\tilde{M}_iA_i & (1+2/\pi_i)E_i^T\tilde{M}_iE_i\end{bmatrix}\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}\\
&\quad+\bar{p}_i(1+2\pi_i+1/\pi_i)\|\sqrt{\tilde{M}_i}D_i\|_2^2\|w_i-\hat{w}_i\|^2+n_i(1+3\pi_i)\lambda_{\max}(\tilde{M}_i)\,\bar{\delta}_i^2\\
&\le\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}^T\begin{bmatrix}\hat{\kappa}_i\tilde{M}_i-F_i^T\tilde{Q}_{11i}F_i & -F_i^T\tilde{Q}_{12i}\\ -\tilde{Q}_{12i}^TF_i & -\tilde{Q}_{22i}\end{bmatrix}\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}\\
&\quad+\bar{p}_i(1+2\pi_i+1/\pi_i)\|\sqrt{\tilde{M}_i}D_i\|_2^2\|w_i-\hat{w}_i\|^2+n_i(1+3\pi_i)\lambda_{\max}(\tilde{M}_i)\,\bar{\delta}_i^2\\
&=\hat{\kappa}_iV_i(x_i,\hat{x}_i)-\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}^T\begin{bmatrix}F_i & 0\\ 0 & \mathbb{I}\end{bmatrix}^T\tilde{Q}_i\begin{bmatrix}F_i & 0\\ 0 & \mathbb{I}\end{bmatrix}\begin{bmatrix}x_i-\hat{x}_i\\ \bar{\varphi}_i\end{bmatrix}\\
&\quad+\bar{p}_i(1+2\pi_i+1/\pi_i)\|\sqrt{\tilde{M}_i}D_i\|_2^2\|w_i-\hat{w}_i\|^2+n_i(1+3\pi_i)\lambda_{\max}(\tilde{M}_i)\,\bar{\delta}_i^2\\
&\le\hat{\kappa}_iV_i(x_i,\hat{x}_i)+\bar{p}_i(1+2\pi_i+1/\pi_i)\|\sqrt{\tilde{M}_i}D_i\|_2^2\|w_i-\hat{w}_i\|^2+n_i(1+3\pi_i)\lambda_{\max}(\tilde{M}_i)\,\bar{\delta}_i^2\\
&\le\max\Big\{(1-(1-\tilde{\pi}_i)\tilde{\kappa}_i)(V_i(x_i,\hat{x}_i)),(1+\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(\bar{p}_i(1+2\pi_i+1/\pi_i))\|\sqrt{\tilde{M}_i}D_i\|_2^2\|w_i-\hat{w}_i\|^2,\\
&\quad(1+1/\tilde{\delta}_i)(\frac{1}{\tilde{\kappa}_i\tilde{\pi}_i})(n_i(1+3\pi_i)\lambda_{\max}(\tilde{M}_i))\,\bar{\delta}_i^2\Big\}.
\end{aligned}
\tag{4.4.28}
$$

where

$$w_1=x_2+x_3,w_2=x_1+x_3,w_3=x_2,w_4=x_3,$$

with sets $X_i=[0\ \ 0.1]$, and $W_i=[0\ \ 0.2]$, $\forall i\in\{1,2,3,4\}$. As seen, the first subsystem is not stable. Then we proceed with looking at the solution of Σ_i two steps ahead, i.e., $\mathcal{M}=2$,

$$
\Sigma_{\mathsf{aux}i}:\begin{cases}x_1(k+2)=0.962x_1(k)+\tilde{D}_1\mathsf{w}_1(k)+\tilde{R}_1\tilde{\varsigma}_1(k),\\ x_2(k+2)=0.8665x_2(k)+\tilde{D}_2\mathsf{w}_2(k)+\tilde{R}_2\tilde{\varsigma}_2(k),\\ x_3(k+2)=0.8876x_3(k)+\tilde{D}_3\mathsf{w}_3(k)+\tilde{R}_3\tilde{\varsigma}_3(k),\\ x_4(k+2)=0.36x_4(k)+\tilde{D}_4\mathsf{w}_4(k)+\tilde{R}_4\tilde{\varsigma}_4(k),\end{cases}
\tag{4.4.30}
$$

where

$$\tilde{D}_1 = [0.0005; -0.0095]^T, \quad \tilde{D}_2 = [-0.0041; 0.1112]^T,$$
$$\tilde{D}_3 = [0.004; 0.0945]^T, \quad \tilde{D}_4 = [-0.045; -0.306]^T,$$
$$\mathsf{w}_1 = [x_2; x_3], \quad \mathsf{w}_2 = [x_1; x_3], \quad \mathsf{w}_3 = [x_1; x_2], \quad \mathsf{w}_4 = [x_2; x_3],$$
$$\tilde{\varsigma}_1(k) = [\varsigma_3(k); \varsigma_2(k); \varsigma_1(k); \varsigma_1(k+1)], \quad \tilde{\varsigma}_3(k) = [\varsigma_2(k); \varsigma_3(k); \varsigma_3(k+1)],$$
$$\tilde{\varsigma}_2(k) = [\varsigma_3(k); \varsigma_1(k); \varsigma_2(k); \varsigma_2(k+1)], \quad \tilde{\varsigma}_4(k) = [\varsigma_3(k); \varsigma_4(k); \varsigma_4(k+1)].$$

Moreover, $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}; \tilde{R}_{i4}]^T, \forall i \in \{1, 2\}$, where

$$\tilde{R}_{11} = 0.5, \quad \tilde{R}_{12} = 0.5, \quad \tilde{R}_{13} = 1.001, \quad \tilde{R}_{14} = 1,$$
$$\tilde{R}_{21} = -0.08, \quad \tilde{R}_{22} = -0.08, \quad \tilde{R}_{23} = -0.95, \quad \tilde{R}_{24} = 1.$$

and $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}]^T \, \forall i \in \{3, 4\}$, where

$$\tilde{R}_{31} = -0.05, \quad \tilde{R}_{32} = -0.94, \quad \tilde{R}_{33} = 1, \quad \tilde{R}_{41} = 0.9, \quad \tilde{R}_{42} = 0.6, \quad \tilde{R}_{43} = 1.$$

As seen, $\tilde{A}_1$ is now stable. Now we proceed with constructing the finite MC from the $\mathcal{M}$-sampled system as acquired in (4.4.30). We fix $V_i(x_i, \hat{x}_i) = \|x_i - \hat{x}_i\|$. One can readily verify that V_i is a max-type FPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (4.4.7) with $\alpha_i(s) = s, \forall i \in \{1, 2, 3, 4\} \; \forall s \in \mathbb{R}_{\geq 0}$, and the condition (4.4.8) with

$$\kappa_i(s) = 0.99s, \quad \forall i \in \{1, 2, 3, 4\},$$
$$\rho_{\text{int}1}(s) = 0.8838s, \quad \rho_{\text{int}2}(s) = 0.9676s, \quad \rho_{\text{int}3}(s) = 0.9936s, \quad \rho_{\text{int}4}(s) = 0.9659s,$$
$$\psi_1 = 36.47\,\bar{\delta}_1, \quad \psi_2 = 56.85\,\bar{\delta}_2, \quad \psi_3 = 60.61\,\bar{\delta}_3, \quad \psi_4 = 3.09\,\bar{\delta}_4.$$

Now we check the max small-gain condition (3.3.5) that is required for the compositionality result. By taking $\sigma_i(s) = s \; \forall i \in \{1, 2, 3, 4\}$, one can readily verify that the max small-gain condition (3.3.5) and as a result the condition (3.3.6) are satisfied. Hence, $V(x, \hat{x}) = \max_i \|x_i - \hat{x}_i\|$ is a max-type FSF from $\widehat{\Sigma}$ to Σ satisfying conditions (4.4.9) and (4.4.10) with $\alpha(s) = s$, $\kappa(s) = 0.99\, s$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 60.61\,\bar{\delta}$.

We take the state and internal input discretization parameters as 0.0001. Then we have $n_{x_i} = 1000$, and $n_{w_i} = 2000$. By starting the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ from $\mathbb{1}_4$, and by employing Theorem 4.4.13, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ at times $k = 2j, j = \{0, \ldots, 15\}$ with the probability at least 91%, i.e.,

$$\mathbb{P}\Big\{\|x_a(k) - \hat{x}_{\hat{a}}(k)\| \leq 1, \; \forall k = 2j, j = \{0, \ldots, 15\}\Big\} \geq 0.91.$$

Now we quantify the probabilistic closeness guarantee for the whole state trajectory as proposed in (4.4.24). Assumption 4.4.19 is satisfied for the original interconnected system with $\tilde{\eta}(s) = 1.001s, \forall s \in \mathbb{R}_{\geq 0}$ and $\tilde{\theta} = \bar{\delta}$. Then by employing Theorem 4.4.21, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ for the *whole state trajectory* within the time horizon $[0, 30]$ with the probability at least 88%, i.e.,

$$\mathbb{P}\Big\{\|x_a(k) - \hat{x}_{\hat{a}}(k)\| \leq 1, \; \forall k \in [0, 30]\Big\} \geq 0.88.$$

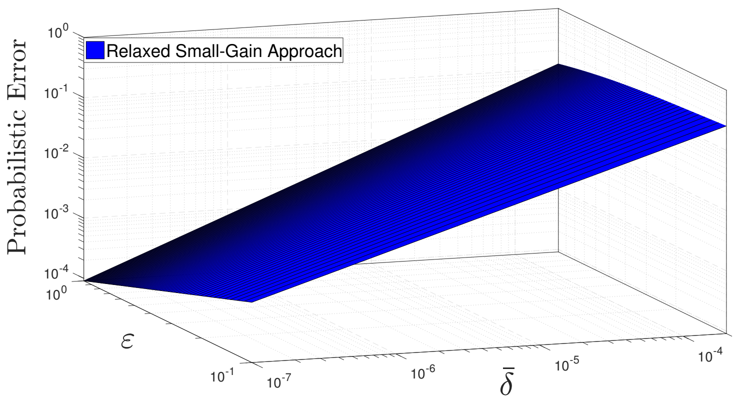

Figure 4.24: Error bound proposed in (4.4.24) for $\mathcal{M} = 2$. Plot is in the logarithmic scale for $T_d = 30$. The probabilistic closeness guarantee is improved by either decreasing the state discretization parameter $\bar{\delta}$ or increasing the confidence bound ε.

4.4.2.3.1 Analysis on Probabilistic Closeness Guarantee, Computation Time and Memory Usage

In order to have more practical analysis on the proposed closeness guarantee, we plotted the error bound provided in (4.4.24) for $\mathcal{M} = 2$ in terms of the state discretization parameter $\bar{\delta}$ and the confidence bound ε in Figure 4.24. As seen, the probabilistic closeness guarantee is improved by either decreasing $\bar{\delta}$ or increasing ε.

Now we provide some discussions on the computation time and memory usage in constructing the finite MC in both monolithic and compositional manners. To do so, we employ the software tool FAUST[2] on a machine with Windows operating system (Intel i7@3.6GHz CPU and 16 GB of RAM). The monolithic MC would be a matrix with the dimension of $n_{x_i}^4 \times n_{x_i}^4$ with $n_{x_i} = 1000$. By allocating 8 bytes for each entry of the matrix to be stored, one needs a memory of roughly $\frac{8\times 1000^4 \times 1000^4}{10^9} = 8 \times 10^{15}$ GB for building the finite MC in the monolithic manner which is impossible in practice. Now, we proceed with the compositional construction of the finite MC proposed in this section for each subsystem. The constructed MC for each subsystem here is a matrix with the dimension of $(n_{x_i} \times n_{w_i}) \times n_{x_i}$ (with $n_{x_i} = 1000$, $n_{w_i} = 2000$) with a memory usage of roughly $\frac{8\times 1000\times 2000\times 1000}{10^9} = 16$ GB for each MC and 64 GB for all 4 MCs, and the computation time of 112 seconds for constructing each MC. This implementation clearly shows that the proposed compositional approach in this section significantly mitigates the curse of dimensionality problem in constructing finite MCs monolithically.

4.5 Relaxed Dissipativity Approach

In this section, we develop a compositional approach for the construction of finite MDPs for networks of not necessarily passivable stochastic control systems using the dissipativity approach. In particular, the proposed compositional technique leverages the interconnection structure and joint dissipativity-type properties of subsystems and their abstractions characterized via a notion of *finite-step stochastic storage functions*. The provided compositionality conditions can enjoy the structure of the interconnection topology and be potentially satisfied *regardless* of the number or gains of subsystems. The finite-step stochastic storage functions of subsystems are utilized to establish a sum-*type finite-step stochastic simulation function* between the interconnection of concrete stochastic subsystems and that of their finite MDPs.

In order to make the notation easier, we assume that the internal and external output maps are identity. Similar to Example 4.4.1, we first raise the following example with an interconnection constraint based on the dissipativity approach to illustrate the idea.

Example 4.5.1. *Consider two linear dt-SCS* Σ_1, Σ_2 *as in* (4.4.1) *with the interconnection constraint* $[w_1; w_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} [x_1; x_2]$. *Note that the first subsystem is not stable thus not stabilizable as well. Therefore the proposed results in Section 4.3 are not applicable to this network. By looking at the solution process of the system two steps ahead and considering the interconnection, one can obtain*

$$\Sigma_{auxi}: \begin{cases} x_1(k+2) = 0.29x_1(k) + 0.38\mathsf{w}_1(k) + 0.4\varsigma_2(k) + 0.61\varsigma_1(k) + \varsigma_1(k+1), \\ x_2(k+2) = 0.04x_2(k) - 0.19\mathsf{w}_2(k) - 0.2\varsigma_1(k) + 0.35\varsigma_2(k) + \varsigma_2(k+1), \end{cases} \tag{4.5.1}$$

where $[\mathsf{w}_1; \mathsf{w}_2] = [x_2; x_1]$. *The two subsystems in* (4.5.1), *denoted by* $\Sigma_{aux1}, \Sigma_{aux2}$, *are now stable. This motivates us to construct abstractions of original subsystems* (4.4.1) *based on auxiliary subsystems* (4.5.1).

Now one can utilize Assumption 4.4.3 and Lemma 4.4.6 to come up with the dynamics of the $\mathcal{M}$-sampled systems, i.e., auxiliary systems $\Sigma_{\mathsf{aux}i}$, as in (4.4.4). Similar to Example 4.4.7, we illustrate the computation of the vector field $\tilde{f}_i$ on a network of two linear dt-SCS in the next example.

Example 4.5.2. *Consider linear dt-SCS* Σ_i, $i \in \{1, 2\}$ *in* (4.4.6) *with the interconnection constraint* $[w_1; w_2] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} [x_1; x_2]$. *Matrices* A_i, B_i, D_i, R_i, $i \in \{1, 2\}$, *have appropriate dimensions. We can rewrite the given dynamics as*

$$x(k+1) = \bar{A}x(k) + \bar{B}\nu(k) + \bar{D}w(k) + \bar{R}\varsigma(k),$$

with $x = [x_1; x_2], \nu = [\nu_1; \nu_2], w = [w_1; w_2], \varsigma = [\varsigma_1; \varsigma_2]$, *where*

$$\bar{A} = \mathsf{diag}(A_1, A_2), \bar{B} = \mathsf{diag}(B_1, B_2), \bar{D} = \mathsf{diag}(D_1, D_2), \bar{R} = \mathsf{diag}(R_1, R_2).$$

By applying the interconnection constraint $w = [w_1; w_2] = M[x_1; x_2]$ *with* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, *we have*

$$x(k+1) = (\bar{A} + \bar{D}M)x(k) + \bar{B}\nu(k) + \bar{R}\varsigma(k).$$

Now by looking at the solutions $\mathcal{M}$ steps ahead, one gets

$$x(k+\mathcal{M}) = (\bar{A}+\bar{D}M)^{\mathcal{M}}x(k) + \sum_{n=0}^{\mathcal{M}-1}(\bar{A}+\bar{D}M)^n\bar{B}\nu(k+\mathcal{M}-n-1) + \sum_{n=0}^{\mathcal{M}-1}(\bar{A}+\bar{D}M)^n\bar{R}\varsigma(k+\mathcal{M}-n-1).$$

After applying Assumption 4.4.3 and by partitioning $(\bar{A}+\bar{D}M)^{\mathcal{M}}$ as

$$(\bar{A}+\bar{D}M)^{\mathcal{M}} = \left[\begin{array}{c|c}\tilde{A}_1 & \tilde{D}_1\\ \hline \tilde{A}_2 & \tilde{D}_2\end{array}\right],$$

one can decompose the network and obtain the auxiliary subsystems proposed in (4.4.4) *as follows, $i \in \{1,2\}$:*

$$\Sigma_{auxi} : x_i(k+\mathcal{M}) = \tilde{A}_i x_i(k) + B_i\nu_i(k+\mathcal{M}-1) + \tilde{D}_i \mathsf{w}_i(k) + \tilde{R}_i\tilde{\varsigma}_i(k), \tag{4.5.2}$$

where $\mathsf{w}_1(k), \mathsf{w}_2(k)$ are the new internal inputs, $\tilde{\varsigma}_1(k), \tilde{\varsigma}_2(k)$ are defined as in (4.4.5) *with $N=2$, and $\tilde{R}_i$ is a matrix of an appropriate dimension which can be computed based on the matrices in* (4.4.6). *As seen, $\tilde{A}_1$ and $\tilde{A}_2$ now depend also on D_1, D_2 and the interconnection matrix M, which may result in the pairs $(\tilde{A}_1, B_1)$ and $(\tilde{A}_2, B_2)$ being stabilizable.*

Remark 4.5.3. *Note that in order to establish finite-step stochastic storage functions from $\widehat{\Sigma}_i$ to Σ_i for the general setting of nonlinear stochastic systems, the auxiliary system Σ_{auxi} should be* incrementally passivable. *This incremental passivability property is equivalent to the classical stability property for the class of linear stochastic systems. After interconnecting subsystems to each other and looking at the solutions in future time instances, checking the stability property for unstable subsystems is easy since it only needs matrix manipulations (as discussed in Example 4.5.2).*

4.5.1 Finite-Step Stochastic Storage and sum-Type Finite-Step Stochastic Simulation Functions

In this subsection, we first introduce the notion of finite-step stochastic storage functions (FStF) for dt-SCS with both internal and external signals, which is adapted from the notion of storage functions from the dissipativity theory. We then define the notion of sum-type finite-step stochastic simulation functions (sum-type FSF) for systems with only external signals. We utilize these two definitions to quantify the probabilistic closeness of two interconnected dt-SCS.

Definition 4.5.4. *Consider dt-SCS Σ_i and $\widehat{\Sigma}_i$ where $\hat{X}_i \subseteq X_i$. A function $V_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ is called a* finite-step *stochastic storage function (FStF) from $\widehat{\Sigma}_i$ to Σ_i if there exist $\mathcal{M} \in \mathbb{N}_{\geq 1}$, $\alpha_i, \kappa_i \in \mathcal{K}_\infty$, $\rho_{\text{ext}i} \in \mathcal{K}_\infty \cup \{0\}$, a constant $\psi_i \in \mathbb{R}_{\geq 0}$, and a symmetric matrix $\bar{X}_i$ with conformal block partitions $\bar{X}_i^{l\bar{l}}$, $l, \bar{l} \in \{1,2\}$, such that for all $k = j\mathcal{M}, j \in \mathbb{N}$, $x_i := x_i(k) \in X_i, \hat{x}_i := \hat{x}_i(k) \in \hat{X}_i$,*

- $$\alpha_i(\|x_i - \hat{x}_i\|_2) \leq V_i(x_i, \hat{x}_i), \tag{4.5.3}$$

- *and for any $\hat{\nu}_i := \hat{\nu}_i(k + \mathcal{M} - 1) \in \hat{U}_i$, there exists $\nu_i := \nu_i(k + \mathcal{M} - 1) \in U_i$ such that for any $\mathsf{w}_i := \mathsf{w}_i(k) \in \tilde{W}_i$ and $\hat{\mathsf{w}}_i := \hat{\mathsf{w}}_i(k) \in \hat{W}_i$, one obtains*

$$\begin{aligned}&\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}), \hat{x}_i(k+\mathcal{M}))\,\Big|\, x_i, \hat{x}_i, \nu_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big] - V_i(x_i, \hat{x}_i) \qquad (4.5.4)\\ &\leq -\kappa_i(V_i(x_i, \hat{x}_i)) + \rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|_2) + \psi_i + \begin{bmatrix}\mathsf{w}_i - \hat{\mathsf{w}}_i\\ x_i - \hat{x}_i\end{bmatrix}^T \underbrace{\begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}}_{\bar{X}_i :=} \begin{bmatrix}\mathsf{w}_i - \hat{\mathsf{w}}_i\\ x_i - \hat{x}_i\end{bmatrix}.\end{aligned}$$

If there exists an FStF V_i from $\widehat{\Sigma}_i$ to Σ_i, denoted by $\widehat{\Sigma}_i \preceq_{FStF} \Sigma_i$, the control system $\widehat{\Sigma}_i$ is called an abstraction of the concrete (original) system Σ_i.

Remark 4.5.5. *Note that for the sake of readability, we assume that Σ_i and $\widehat{\Sigma}_i$ both have the same dimension (without performing any model order reductions). But if this is not the case and they have different dimensionality, one can employ the techniques proposed in Section 3.4 to first reduce the dimension of concrete systems, and then apply the proposed results of this section.*

Definition 4.5.6. *Consider two dt-SCS Σ and $\widehat{\Sigma}$ without internal signals, where $\hat{X} \subseteq X$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a* sum*-type* finite-step *stochastic simulation function (*sum*-type FSF) from $\widehat{\Sigma}$ to Σ if there exist $\mathcal{M} \in \mathbb{N}_{\geq 1}$, and $\alpha \in \mathcal{K}_\infty$ such that*

- $$\forall x := x(k) \in X, \forall \hat{x} := \hat{x}(k) \in \hat{X}, \quad \alpha(\|x - \hat{x}\|_2) \leq V(x, \hat{x}), \tag{4.5.5}$$

- *and $\forall x := x(k) \in X$, $\forall \hat{x} := \hat{x}(k) \in \hat{X}$, $\forall \hat{\nu} := \hat{\nu}(k + \mathcal{M} - 1) \in \hat{U}$, $\exists \nu := \nu(k + \mathcal{M} - 1) \in U$ such that*

$$\begin{aligned}&\mathbb{E}\Big[V(x(k+\mathcal{M}), \hat{x}(k+\mathcal{M}))\,\Big|\, x, \hat{x}, \nu, \hat{\nu}\Big] - V(x, \hat{x})\\ &\leq -\kappa(V(x, \hat{x})) + \rho_{\mathrm{ext}}(\|\hat{\nu}\|_2) + \psi, \end{aligned} \tag{4.5.6}$$

  for some $\kappa \in \mathcal{K}$, $\rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$, $\psi \in \mathbb{R}_{\geq 0}$, and $k = j\mathcal{M}, j \in \mathbb{N}$.

If there exists a sum-type FSF V from $\widehat{\Sigma}$ to Σ, denoted by $\widehat{\Sigma} \preceq_{FSF}^{\mathsf{sum}} \Sigma$, $\widehat{\Sigma}$ is called an abstraction of Σ.

Now one can utilize Theorem 4.4.13 and compare state trajectories of two dt-SCS without internal inputs in a probabilistic setting at times $k = j\mathcal{M}$, $0 \leq j \leq T_d$, for some $\mathcal{M} \in \mathbb{N}_{\geq 1}$.

Remark 4.5.7. *Note that one can consider original stochastic systems as autonomous without external inputs and provide a closeness guarantee for all time instances similar to results of the previous section proposed in Theorem 4.4.21.*

4.5.2 Compositionality Results

We first provide a formal definition of *concrete* interconnected stochastic control subsystems.

Definition 4.5.8. *Consider* $N \in \mathbb{N}_{\geq 1}$ concrete *stochastic control subsystems* Σ_i, $i \in \{1,\dots,N\}$, *and a matrix* M *defining the coupling between them. The interconnection of* Σ_i, $\forall i \in \{1,\dots,N\}$, *is the* concrete *dt-SCS* Σ, *denoted by* $\mathcal{I}_{fd}(\Sigma_1,\dots,\Sigma_N)$, *such that* $X := \prod_{i=1}^{N} X_i$, $U := \prod_{i=1}^{N} U_i$, *and* $f := \prod_{i=1}^{N} f_i$, *with the internal inputs constrained according to*

$$[w_1;\dots;w_N] = M[x_1;\dots;x_N]. \tag{4.5.7}$$

We require the condition $M\prod_{i=1}^{N} X_i \subseteq \prod_{i=1}^{N} W_i$ *to have a well-posed interconnection.*

As mentioned in Remark 4.4.2, after interconnecting subsystems with each other and doing the $\mathcal{M}$-step analysis, the interconnection coupling matrix M may change. Then the interconnection constraint for auxiliary systems is defined as

$$[\mathsf{w}_1;\dots;\mathsf{w}_N] = M_a[x_1;\dots;x_N], \tag{4.5.8}$$

where M_a is an *auxiliary* coupling matrix.

We assume that we are given N concrete stochastic control subsystems Σ_i together with their corresponding abstractions $\widehat{\Sigma}_i$ with an FStF V_i from $\widehat{\Sigma}_i$ to Σ_i. We indicate by α_i, κ_i, $\rho_{\text{ext}i}$, $\bar{X}_i$, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$, the corresponding functions and the conformal block partitions appearing in Definition 4.5.4. In order to provide the compositionality results of the section, we define a notion of the interconnection for *abstract* stochastic control subsystems.

Definition 4.5.9. *Consider* $N \in \mathbb{N}_{\geq 1}$ abstract *stochastic control subsystems* $\widehat{\Sigma}_i$, $i \in \{1,\dots,N\}$, *and a matrix* $\hat{M}$ *defining the coupling between them. The interconnection of* $\widehat{\Sigma}_i$, $\forall i \in \{1,\dots,N\}$, *is the* abstract *dt-SCS* $\widehat{\Sigma}$, *denoted by* $\widehat{\mathcal{I}}_{fd}(\widehat{\Sigma}_1,\dots,\widehat{\Sigma}_N)$, *such that* $\hat{X} := \prod_{i=1}^{N} \hat{X}_i$, $\hat{U} := \prod_{i=1}^{N} \hat{U}_i$, *and* $\hat{f} := \prod_{i=1}^{N} \hat{f}_i$, *with the internal inputs constrained according to*

$$[\hat{w}_1;\dots;\hat{w}_N] = \Pi_w(\hat{M}[\hat{x}_1;\dots;\hat{x}_N]),$$

where Π_w *is the abstraction map defined similarly to the one in* (4.2.6)*. Accordingly, the interconnection constraint for abstractions of auxiliary subsystems is defined as*

$$[\hat{\mathsf{w}}_1;\dots;\hat{\mathsf{w}}_N] = \Pi_{\mathsf{w}}(\hat{M}_a[\hat{x}_1;\dots;\hat{x}_N]), \tag{4.5.9}$$

where $\hat{M}_a$ *is an* auxiliary *coupling matrix for abstractions.*

Remark 4.5.10. *Note that Definition 4.5.9 implicitly assumes that the following constraints are satisfied to have well-posed interconnections:*

$$\Pi_w(\hat{M}\prod_{i=1}^{N} \hat{X}_i) \subseteq \prod_{i=1}^{N} \hat{W}_i, \quad \Pi_{\mathsf{w}}(\hat{M}_a\prod_{i=1}^{N} \hat{X}_i) \subseteq \prod_{i=1}^{N} \hat{W}_i. \tag{4.5.10}$$

In the next theorem, as the compositionality results of the section, we provide sufficient conditions to have a sum-type FSF from the interconnection of abstractions $\widehat{\Sigma} = \widehat{\mathcal{I}}_{fd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ to that of concrete ones $\Sigma = \mathcal{I}_{fd}(\Sigma_1, \ldots, \Sigma_N)$. This theorem enables us to quantify the probabilistic error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional manner by leveraging Theorem 4.4.13.

Theorem 4.5.11. *Consider the interconnected stochastic auxiliary system $\Sigma_{\mathsf{aux}} = \mathcal{I}_{fd}(\Sigma_{\mathsf{aux}1}, \ldots, \Sigma_{\mathsf{aux}N})$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic auxiliary subsystems $\Sigma_{\mathsf{aux}i}$ and the* auxiliary *coupling matrix M_a. Suppose that each stochastic control subsystem Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding FStF V_i. Then the weighted sum*

$$V(x, \hat{x}) := \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i) \tag{4.5.11}$$

is a sum*-type FSF from the interconnected control system $\widehat{\Sigma} = \widehat{\mathcal{I}}_{fd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ to $\Sigma = \mathcal{I}_{fd}(\Sigma_1, \ldots, \Sigma_N)$ if $\mu_i > 0$, $i \in \{1, \ldots, N\}$, and there exists $0 < \underline{\mu} < 1$ such that $\forall x_i \in X_i$, $\forall \hat{x}_i \in \hat{X}_i$, $i \in \{1, \ldots, N\}$,*

$$\|x_i - \hat{x}_i\|_2^2 \leq \frac{\mu_i}{\underline{\mu}} \kappa_i(V_i(x_i, \hat{x}_i)), \tag{4.5.12}$$

and

$$M_a = \hat{M}_a, \tag{4.5.13}$$

$$\begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix} \preceq 0, \tag{4.5.14}$$

where $\bar{X}_{cmp}$ is as in (3.4.6).

Proof. We first show that the sum-type FSF V in (4.5.11) satisfies the inequality (4.5.5) for some $\mathcal{K}_\infty$ function α. For any $x = [x_1; \ldots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, one gets:

$$\|x - \hat{x}\|_2 \leq \sum_{i=1}^{N} \|x_i - \hat{x}_i\|_2 \leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \leq \hat{\beta}(V(x, \hat{x})),$$

with the function $\hat{\beta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined for all $s \in \mathbb{R}_{\geq 0}$ as

$$\hat{\beta}(s) := \max\Big\{\textstyle\sum_{i=1}^{N} \alpha_i^{-1}(s_i) \,\big|\, s_i \geq 0,\ \sum_{i=1}^{N} \mu_i s_i = s\Big\}.$$

It is not hard to verify that the function $\hat{\beta}(\cdot)$ defined above is a $\mathcal{K}_\infty$ function. By taking the $\mathcal{K}_\infty$ function $\alpha(r) := \hat{\beta}^{-1}(s)$, $\forall s \in \mathbb{R}_{\geq 0}$, one obtains

$$\alpha(\|x - \hat{x}\|_2) \leq V(x, \hat{x}),$$

satisfying the inequality (4.5.5). Now we prove that the sum-type FSF V in (4.5.11) satisfies the inequality (4.5.6), as well. Consider any $x = [x_1; \ldots; x_N] \in X$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \hat{X}$, and $\hat{\nu} = [\hat{\nu}_1; \ldots; \hat{\nu}_N] \in \hat{U}$. For any $i \in \{1, \ldots, N\}$, there exists $\nu_i \in U_i$, consequently, a vector $\nu = [\nu_1; \ldots; \nu_N] \in U$, satisfying (4.5.4) for each pair of subsystems Σ_i and $\widehat{\Sigma}_i$ with internal inputs given by $[\mathsf{w}_1; \ldots; \mathsf{w}_N] = M_a[x_1; \ldots; x_N]$ and $[\hat{\mathsf{w}}_1; \ldots; \hat{\mathsf{w}}_N] = \Pi_w(\hat{M}_a[\hat{x}_1; \ldots; \hat{x}_N])$. By defining $[\bar{\mathsf{w}}_1; \ldots; \bar{\mathsf{w}}_N] = \hat{M}_a[\hat{x}_1; \ldots; \hat{x}_N]$, we have the chain of inequalities in (4.5.16) using conditions (4.5.12), (4.5.13), (4.5.14) and by defining $\kappa(\cdot), \rho_{\text{ext}}(\cdot), \psi$ as

$$
\begin{aligned}
\kappa(s) &:= (1 - \underline{\mu}) \min \Big\{ \sum_{i=1}^{N} \mu_i \kappa_i(s_i) \,\Big|\, s_i \geq 0, \sum_{i=1}^{N} \mu_i s_i = s \Big\} \\
\rho_{\text{ext}}(s) &:= \max \Big\{ \sum_{i=1}^{N} \mu_i \rho_{\text{ext}i}(s_i) \,\Big|\, s_i \geq 0, \|[s_1; \ldots; s_N]\| = s \Big\}, \\
\psi &:= \begin{cases} \sum_{i=1}^{N} \mu_i \psi_i + \frac{\|\bar{\boldsymbol{\mu}}\|_2^2}{\underline{\mu}^2} \lambda_{\max}(\underline{P}), & \text{if } \bar{X}_{cmp} \leq 0, \\ \sum_{i=1}^{N} \mu_i \psi_i + \|\bar{\boldsymbol{\mu}}\|_2^2 (\frac{1}{\underline{\mu}^2} \lambda_{\max}(\underline{P}) + \rho(\bar{X}_{cmp})), & \text{if } \bar{X}_{cmp} > 0, \end{cases}
\end{aligned}
\tag{4.5.15}
$$

where $\underline{P} = \bar{X}_{cmp}^T \begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp}$, $\bar{\boldsymbol{\mu}} = [\bar{\mu}_1; \ldots; \bar{\mu}_N]$, and ρ is the *spectral radius*. Note that κ and ρ_{ext} in (4.5.16) belong to $\mathcal{K}_\infty$ and $\mathcal{K}_\infty \cup \{0\}$, respectively, due to their definition provided above. Hence, we conclude that V is a sum-type FSF from $\widehat{\Sigma}$ to Σ. □

Remark 4.5.12. *Condition* (4.5.12) *is satisfied if one can find* $\mu_i > 0$ *and* $0 < \underline{\mu} < 1$ *such that* $(\alpha_i^{-1}(s))^2 \leq \frac{\mu_i}{\underline{\mu}} \kappa_i(s), \forall s \in \mathbb{R}_{\geq 0}, i \in \{1, \ldots, N\}$. *Note that the previous inequality is always satisfied for linear systems and quadratic functions* $V_i(x_i, \hat{x}_i)$ *(cf. Case study 4.5.4.1).*

4.5.3 Construction of FStF

In this subsection, we first focus on the nonlinear class of discrete-time stochastic control systems Σ_i and *quadratic* functions V_i by providing an approach on the construction of their stochastic storage functions (with $\mathcal{M} = 1$). We then propose a technique to construct an FStF for a linear class of stochastic control systems.

4.5.3.1 Stochastic Control Systems with Slope Restrictions on Nonlinearity

The class of discrete-time nonlinear stochastic control systems, considered here, is given by

$$
x_i(k+1) = A_i x(k) + E_i \varphi_i(F_i x_i(k)) + B_i \nu_i(k) + D_i w_i(k) + R_i \varsigma_i(k), \tag{4.5.17}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[V(x(k+\mathcal{M}),\hat{x}(k+\mathcal{M}))\,\Big|\,x(k),\hat{x}(k),\nu(k+\mathcal{M}-1),\hat{\nu}(k+\mathcal{M}-1)\Big]-V(x,\hat{x})\\
&=\mathbb{E}\Big[\sum_{i=1}^{N}\mu_i\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\Big|\,x(k),\hat{x}(k),\nu(k+\mathcal{M}-1),\hat{\nu}(k+\mathcal{M}-1)\Big]\Big]\\
&\quad-\sum_{i=1}^{N}\mu_iV_i(x_i,\hat{x}_i)\\
&=\mathbb{E}\Big[\sum_{i=1}^{N}\mu_i\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\Big|\,x_i=x_i(k),\hat{x}_i=\hat{x}_i(k),\nu_i=\nu_i(k+\mathcal{M}-1),\\
&\qquad\hat{\nu}_i=\hat{\nu}_i(k+\mathcal{M}-1)\Big]\Big]-\sum_{i=1}^{N}\mu_iV_i(x_i,\hat{x}_i)\\
&\leq\sum_{i=1}^{N}\mu_i\big(-\kappa_i(V_i(x_i,\hat{x}_i))+\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\psi_i+\begin{bmatrix}\mathsf{w}_i-\hat{\mathsf{w}}_i\\x_i-\hat{x}_i\end{bmatrix}^T\begin{bmatrix}\bar{X}_i^{11}&\bar{X}_i^{12}\\\bar{X}_i^{21}&\bar{X}_i^{22}\end{bmatrix}\begin{bmatrix}\mathsf{w}_i-\hat{\mathsf{w}}_i\\x_i-\hat{x}_i\end{bmatrix}\big)\\
&=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix}\mathsf{w}_1-\hat{\mathsf{w}}_1\\\vdots\\\mathsf{w}_N-\hat{\mathsf{w}}_N\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}^T\begin{bmatrix}\mu_1\bar{X}_1^{11}&&&\mu_1\bar{X}_1^{12}&&\\&\ddots&&&\ddots&\\&&\mu_N\bar{X}_N^{11}&&&\mu_N\bar{X}_N^{12}\\\mu_1\bar{X}_1^{21}&&&\mu_1\bar{X}_1^{22}&&\\&\ddots&&&\ddots&\\&&\mu_N\bar{X}_N^{21}&&&\mu_N\bar{X}_N^{22}\end{bmatrix}\begin{bmatrix}\mathsf{w}_1-\hat{\mathsf{w}}_1\\\vdots\\\mathsf{w}_N-\hat{\mathsf{w}}_N\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}\\
&=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i\\
&\quad+\begin{bmatrix}\mathsf{w}_1-\bar{\mathsf{w}}_1+\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\\vdots\\\mathsf{w}_N-\bar{\mathsf{w}}_N+\bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\mathsf{w}_1-\bar{\mathsf{w}}_1+\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\\vdots\\\mathsf{w}_N-\bar{\mathsf{w}}_N+\bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}\\
&=\begin{bmatrix}M_a\begin{bmatrix}x_1\\\vdots\\x_N\end{bmatrix}-\hat{M}_a\begin{bmatrix}\hat{x}_1\\\vdots\\\hat{x}_N\end{bmatrix}\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}M_a\begin{bmatrix}x_1\\\vdots\\x_N\end{bmatrix}-\hat{M}_a\begin{bmatrix}\hat{x}_1\\\vdots\\\hat{x}_N\end{bmatrix}\\x_1-\hat{x}_1\\\vdots\\x_N-\hat{x}_N\end{bmatrix}+\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))
\end{aligned}
$$

$$+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i+\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}$$

$$+2\begin{bmatrix}M_a\begin{bmatrix}x_1\\ \vdots\\ x_N\end{bmatrix}-\hat{M}_a\begin{bmatrix}\hat{x}_1\\ \vdots\\ \hat{x}_N\end{bmatrix}\\ x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}$$

$$=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i+\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}$$

$$+\begin{bmatrix}x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}^T\begin{bmatrix}M_a\\ \mathbb{I}_n\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}M_a\\ \mathbb{I}_n\end{bmatrix}\begin{bmatrix}x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}+2\begin{bmatrix}x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}^T\begin{bmatrix}M_a\\ \mathbb{I}_n\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}$$

$$\leq\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i+\underline{\mu}^2\begin{bmatrix}x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}^T\begin{bmatrix}x_1-\hat{x}_1\\ \vdots\\ x_N-\hat{x}_N\end{bmatrix}$$

$$+\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}+\frac{1}{\underline{\mu}^2}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}^T\bar{X}_{cmp}^T\begin{bmatrix}M_a\\ \mathbb{I}_n\end{bmatrix}\begin{bmatrix}M_a\\ \mathbb{I}_n\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}\bar{\mathsf{w}}_1-\hat{\mathsf{w}}_1\\ \vdots\\ \bar{\mathsf{w}}_N-\hat{\mathsf{w}}_N\\ \mathbf{0}_N\end{bmatrix}$$

$$\leq\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{\text{ext}i}(\|\hat{\nu}_i\|_2)+\sum_{i=1}^{N}\mu_i\psi_i+\underline{\mu}\sum_{i=1}^{N}\mu_i\kappa_i(V_i(x_i,\hat{x}_i))$$

$$+ \frac{1}{\underline{\mu}^2}\|\bar{\boldsymbol{\mu}}\|_2^2\lambda_{\max}\big(\bar{X}_{cmp}^T \begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix} \begin{bmatrix} M_a \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp}\big) + \|\bar{\boldsymbol{\mu}}\|_2^2\sigma_{\max}\big(\bar{X}_{cmp}\big)$$
$$\leq -\kappa\left(V\left(x, \hat{x}\right)\right) + \rho_{\text{ext}}(\|\hat{\nu}\|_2) + \psi. \tag{4.5.16}$$

$$\begin{bmatrix} (1+\pi_i)(A_i+B_iK_i)^T\tilde{M}_i(A_i+B_iK_i) & (A_i+B_iK_i)^T\tilde{M}_iD_i & (A_i+B_iK_i)^T\tilde{M}_iE_i \\ * & (1+\pi_i)D_i^T\tilde{M}_iD_i & D_i^T\tilde{M}_iE_i \\ * & * & (1+\pi_i)E_i^T\tilde{M}_iE_i \end{bmatrix}$$
$$\preceq \begin{bmatrix} \hat{\kappa}_i\tilde{M}_i + \bar{X}_i^{22} & \bar{X}_i^{21} & -F_i^T \\ \bar{X}_i^{12} & \bar{X}_i^{11} & 0 \\ -F_i & 0 & 2/b_i \end{bmatrix} \tag{4.5.18}$$

---

where the additive noise $\varsigma_i(k)$ is a sequence of independent random vectors with multivariate standard normal distributions, and $\varphi_i : \mathbb{R} \to \mathbb{R}$ satisfies slope restrictions in (3.3.11). We provide conditions under which a candidate V_i in the quadratic form of (4.2.14) is an SStF from $\widehat{\Sigma}_i$ to Σ_i. To do so, we require the following assumption on Σ_i.

Assumption 4.5.13. *Assume that for some constants $0 < \hat{\kappa}_i < 1$, and $\pi_i > 0$, there exist matrices K_i, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$ of appropriate dimensions such that the inequality* (4.5.18) *holds.*

Now we propose the main result of this subsection.

Theorem 4.5.14. *Assume the system $\Sigma_i = (A_i, B_i, D_i, E_i, F_i, R_i, \varphi_i)$ satisfies Assumption 4.5.13. Let $\widehat{\Sigma}_i$ be its finite abstraction as constructed in Algorithm 1 with the state discretization parameter $\bar{\delta}_i$, and $\hat{X}_i \subseteq X_i$. Then function V_i defined in* (4.2.14) *is an SStF (with $\mathcal{M} = 1$) from $\widehat{\Sigma}_i$ to Σ_i.*

Proof. Since $\lambda_{\min}(\tilde{M}_i)\|x_i - \hat{x}_i\|_2^2 \leq (x_i - \hat{x}_i)^T\tilde{M}_i(x_i - \hat{x}_i)$, it can be readily verified that $\lambda_{\min}(\tilde{M}_i)\|x_i - \hat{x}_i\|_2^2 \leq V_i(x_i, \hat{x}_i)$ holds $\forall x_i$, $\forall \hat{x}_i$, implying that the inequality (4.5.3) holds with $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (4.5.4) holds, as well. Given any $x_i := x_i(k)$, $\hat{x}_i := \hat{x}_i(k)$, and $\hat{\nu}_i := \hat{\nu}_i(k)$, we choose $\nu_i := \nu_i(k)$ via the following *interface* function:

$$\nu_i = \nu_{\hat{\nu}_i}(x_i, \hat{x}_i, \hat{\nu}_i) := K_i(x_i - \hat{x}_i) + \hat{\nu}_i. \tag{4.5.19}$$

By employing the above definition of the interface function, we simplify

$$A_ix_i + B_i\nu_{\hat{\nu}_i}(x_i, \hat{x}_i, \hat{\nu}_i) + D_iw_i + E_i\varphi_i(F_ix_i) + R_i\varsigma_i$$
$$- \Pi_{x_i}(A_i\hat{x}_i + B_i\hat{\nu}_i + D_i\hat{w}_i + E_i\varphi_i(F_i\hat{x}_i) + R_i\varsigma_i)$$

to

$$(A_i + B_iK_i)(x_i - \hat{x}_i) + D_i(w_i - \hat{w}_i) + E_i(\varphi_i(F_ix_i) - \varphi_i(F_i\hat{x}_i)) + \bar{N}_i, \tag{4.5.20}$$

where $\bar{N}_i = A_i\hat{x}_i + B_i\hat{\nu}_i + D_i\hat{w}_i + E_i\varphi_i(F_i\hat{x}_i) + R_i\varsigma_i - \Pi_{x_i}(A_i\hat{x}_i + B_i\hat{\nu}_i + D_i\hat{w}_i + E_i\varphi_i(F_i\hat{x}_i) + R_i\varsigma_i)$. From the slope restriction (3.3.11), one obtains

$$\varphi_i(F_ix_i) - \varphi_i(F_i\hat{x}_i) = \underline{\delta}_iF_i(x_i - \hat{x}_i), \tag{4.5.21}$$

where $\underline{\delta}_i$ is a function of x_i and $\hat{x}_i$ and takes values in the interval $[0, b_i]$. Using (4.5.21), the expression in (4.5.20) reduces to

$$(A_i + B_iK_i)(x_i - \hat{x}_i) + \underline{\delta}_iE_iF_i(x_i - \hat{x}_i) + D_i(w_i - \hat{w}_i) + \bar{N}_i.$$

Using Cauchy-Schwarz inequality, Young's inequality, Assumption 4.5.13, and since

$$\begin{cases} \|\bar{N}_i\|_2 \ \leq \ \bar{\delta}_i, \\ \bar{N}_i^T\tilde{M}_i\bar{N}_i \leq \lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2, \end{cases}$$

one can obtain the chain of inequalities in (4.5.22). Hence, the proposed V_i in (4.2.14) is an SStF (with $\mathcal{M} = 1$) from $\widehat{\Sigma}_i$ to Σ_i which completes the proof. Note that functions $\alpha_i \in \mathcal{K}_\infty$, $\kappa_i \in \mathcal{K}$, $\rho_{\text{ext}i} \in \mathcal{K}_\infty \cup \{0\}$, and the matrix $\bar{X}_i$ in Definition 4.5.4 associated with V_i in (4.2.14) are defined as $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)s^2$, $\kappa_i(s) := (1 - \hat{\kappa}_i)s$, $\rho_{\text{ext}i}(s) := 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\bar{X}_i = \begin{bmatrix} \bar{X}_i^{11} & \bar{X}_i^{12} \\ \bar{X}_i^{21} & \bar{X}_i^{22} \end{bmatrix}$. Moreover, the positive constant ψ_i is $\psi_i = (1 + 3/\pi)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2$. Note that in the chain of inequalities (4.5.22), we defined $\bar{\mathcal{Z}} = \begin{bmatrix} x_i - \hat{x}_i \\ w_i - \hat{w}_i \\ \underline{\delta}_iF_i(x_i - \hat{x}_i) \end{bmatrix}$. □

4.5.3.2 Discrete-Time Linear Stochastic Control Systems

In this subsection, we focus on the class of linear dt-SCS and propose a technique to construct an FStF from $\widehat{\Sigma}_i$ to Σ_i. Suppose we are given a network composed of N linear stochastic control subsystems $\Sigma_i = (A_i, B_i, D_i, R_i)$, $i \in \{1, \ldots, N\}$. Let $\mathcal{M} \in \mathbb{N}_{\geq 1}$ be given. By employing the interconnection constraint (4.5.7) and Assumption 4.4.3, the dynamics of the auxiliary system $\Sigma_{\mathsf{aux}i}$, $i \in \{1, \ldots, N\}$, at $\mathcal{M}$-*step* forward can be obtained similar to (4.5.2) but for the N subsystems. Although the pairs (A_i, B_i) may not be necessarily stabilizable, we assume that the pairs $(\tilde{A}_i, B_i)$ after $\mathcal{M}$-*step* are stabilizable as discussed in Example 4.5.1. Therefore, we can construct finite MDPs as presented in Algorithm 2 from the new auxiliary system. To do so, we nominate the quadratic function (4.2.14). In order to show that V_i in (4.2.14) is an FStF from $\widehat{\Sigma}_i$ to Σ_i, we require the following assumption on $\Sigma_{\mathsf{aux}i}$.

Assumption 4.5.15. *Assume that for some constants* $0 < \hat{\kappa}_i < 1$ *and* $\pi_i > 0$*, there exist matrices* K_i, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, *and* $\bar{X}_i^{22}$ *of appropriate dimensions such that inequality* (4.5.23) *holds.*

$$
\begin{aligned}
&\mathbb{E}\Big[V_i(x_i(k+1),\hat{x}_i(k+1)) \,\Big|\, x_i=x_i(k),\hat{x}_i=\hat{x}_i(k),\nu_i=\nu_i(k),\hat{\nu}_i=\hat{\nu}_i(k),w_i=w_i(k),\hat{w}_i=\hat{w}_i(k)\Big]\\
&\quad - V_i(x_i,\hat{x}_i)\\
&\quad = (x_i-\hat{x}_i)^T\Big[(A_i+B_iK_i)^T\tilde{M}_i(A_i+B_iK_i)\Big](x_i-\hat{x}_i)+\underline{\delta}_i(x_i-\hat{x}_i)^TF_i^TE_i^T\tilde{M}_iE_iF_i(x_i-\hat{x}_i)\underline{\delta}_i\\
&\qquad + 2\Big[(x_i-\hat{x}_i)^T(A_i+B_iK_i)^T\Big]\tilde{M}_i\Big[\underline{\delta}_iE_iF_i(x_i-\hat{x}_i)\Big] + (w_i-\hat{w}_i)^TD_i^T\tilde{M}_iD_i(w_i-\hat{w}_i)\\
&\qquad + 2\Big[\underline{\delta}_i(x_i-\hat{x}_i)^TF_i^TE_i^T\Big]\tilde{M}_i\Big[D_i(w_i-\hat{w}_i)\Big]+2\Big[(x_i-\hat{x}_i)^T(A_i+B_iK_i)^T\Big]\tilde{M}_i\Big[D_i(w_i-\hat{w}_i)\Big]\\
&\qquad + 2\Big[(x_i-\hat{x}_i)^T(A_i+B_iK_i)^T\Big]\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\Big|\, x_i,\hat{x}_i,\hat{\nu}_i,w_i,\hat{w}_i\Big]+\mathbb{E}\Big[\bar{N}_i^T\tilde{M}_i\bar{N}_i\Big|x,\hat{x}_i,\hat{\nu}_i,w_i,\hat{w}_i\Big]\\
&\qquad + 2\Big[\underline{\delta}_i(x_i-\hat{x}_i)^TF_i^TE_i^T\Big]\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\Big|\, x_i,\hat{x}_i,\hat{\nu}_i,w_i,\hat{w}_i\Big]\\
&\qquad + 2(w_i-\hat{w}_i)^TD_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\Big|\, x_i,\hat{x}_i,\hat{\nu}_i,w_i,\hat{w}_i\Big] - V_i(x_i,\hat{x}_i)\\
&\quad \le \bar{\mathcal{Z}}^T\begin{bmatrix}(1+\pi_i)(A_i+B_iK_i)^T\tilde{M}_i(A_i+B_iK_i) & (A_i+B_iK_i)^T\tilde{M}_iD_i & (A_i+B_iK_i)^T\tilde{M}_iE_i\\ * & (1+\pi_i)D_i^T\tilde{M}_iD_i & D_i^T\tilde{M}_iE_i\\ * & * & (1+\pi_i)E_i^T\tilde{M}_iE_i\end{bmatrix}\bar{\mathcal{Z}}\\
&\qquad + (1+3/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2 - V_i(x_i,\hat{x}_i)\\
&\quad \le \bar{\mathcal{Z}}^T\begin{bmatrix}\hat{\kappa}_i\tilde{M}_i+\bar{X}_i^{22} & \bar{X}_i^{21} & -F_i^T\\ \bar{X}_i^{12} & \bar{X}_i^{11} & 0\\ -F_i & 0 & 2/b_i\end{bmatrix}\bar{\mathcal{Z}} + (1+3/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2 - V_i(x_i,\hat{x}_i)\\
&\quad = -(1-\hat{\kappa}_i)(V_i(x_i,\hat{x}_i)) - 2\underline{\delta}_i(1-\frac{\underline{\delta}_i}{b_i})(x_i-\hat{x}_i)^TF_i^TF_i(x_i-\hat{x}_i) + (1+3/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2\\
&\qquad + \begin{bmatrix}x_i-\hat{x}_i\\ w_i-\hat{w}_i\end{bmatrix}^T\begin{bmatrix}\bar{X}_i^{22} & \bar{X}_i^{21}\\ \bar{X}_i^{12} & \bar{X}_i^{11}\end{bmatrix}\begin{bmatrix}x_i-\hat{x}_i\\ w_i-\hat{w}_i\end{bmatrix}\\
&\quad \le -(1-\hat{\kappa}_i)(V_i(x_i,\hat{x}_i)) + \begin{bmatrix}w_i-\hat{w}_i\\ x_i-\hat{x}_i\end{bmatrix}^T\begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}\begin{bmatrix}w_i-\hat{w}_i\\ x_i-\hat{x}_i\end{bmatrix} + (1+3/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2.
\end{aligned}
\tag{4.5.22}
$$

$$
\begin{bmatrix}(1+\pi_i)(\tilde{A}_i+B_iK_i)^T\tilde{M}_i(\tilde{A}_i+B_iK_i) & (\tilde{A}_i+B_iK_i)^T\tilde{M}_i\tilde{D}_i\\ * & (1+\pi_i)\tilde{D}_i^T\tilde{M}_i\tilde{D}_i\end{bmatrix} \preceq \begin{bmatrix}\hat{\kappa}_i\tilde{M}_i+\bar{X}_i^{22} & \bar{X}_i^{21}\\ \bar{X}_i^{12} & \bar{X}_i^{11}\end{bmatrix}
\tag{4.5.23}
$$

Now we propose the main result of this subsection.

Theorem 4.5.16. *Assume the system Σ_{auxi} satisfies Assumption 4.5.15. Let $\widehat{\Sigma}_{auxi}$ be its finite abstraction as constructed in Algorithm 2 with the state discretization parameter $\bar{\delta}_i$. Then the function V_i proposed in* (4.2.14) *is an FStF from $\widehat{\Sigma}_i$ to Σ_i.*

Proof. We first show that $\forall x_i := x_i(k)$, $\forall \hat{x}_i := \hat{x}_i(k)$, $\forall \hat{\nu}_i := \hat{\nu}_i(k+\mathcal{M}-1)$, $\exists \nu_i := \nu_i(k+\mathcal{M}-1)$, $\forall \mathsf{w}_i := \mathsf{w}_i(k)$, $\forall \hat{\mathsf{w}}_i := \hat{\mathsf{w}}_i(k)$, such that V_i satisfies $\lambda_{\min}(\tilde{M}_i)\|x_i-\hat{x}_i\|_2^2 \leq V_i(x_i,\hat{x}_i)$ and then

$$\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}),\hat{x}_i(k+\mathcal{M}))\,\Big|\,x_i,\hat{x}_i,\nu_i,\hat{\nu}_i,\mathsf{w}_i,\hat{\mathsf{w}}_i,\Big] - V_i(x_i,\hat{x}_i)$$
$$\leq -(1-\hat{\kappa}_i)(V_i(x_i,\hat{x}_i)) + (1+2/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2 + \begin{bmatrix}\mathsf{w}_i-\hat{\mathsf{w}}_i\\ x_i-\hat{x}_i\end{bmatrix}^T \begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}\begin{bmatrix}\mathsf{w}_i-\hat{\mathsf{w}}_i\\ x_i-\hat{x}_i\end{bmatrix}.$$

Since $\lambda_{\min}(\tilde{M}_i)\|x_i-\hat{x}_i\|_2^2 \leq (x_i-\hat{x}_i)^T\tilde{M}_i(x_i-\hat{x}_i)$, one can readily verify that $\lambda_{\min}(\tilde{M}_i)\|x_i-\hat{x}_i\|_2^2 \leq V_i(x_i,\hat{x}_i)$ $\forall x_i$, $\forall \hat{x}_i$. Then the inequality (4.5.3) holds with $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)\,s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing the inequality (4.5.4). Given any $x_i(k)$, $\hat{x}_i(k)$, and $\hat{\nu}_i(k+\mathcal{M}-1)$, we choose $\nu_i(k+\mathcal{M}-1)$ via the interface function (4.4.19) and simplify

$$\tilde{A}_ix_i(k) + B_i\nu_i(k+\mathcal{M}-1) + \tilde{D}_i\mathsf{w}_i(k) + \tilde{R}_i\varsigma_i(k)$$
$$- \Pi_{x_i}(\tilde{A}_i\hat{x}_i(k) + B_i\hat{\nu}_i(k+\mathcal{M}-1) + \tilde{D}_i\hat{\mathsf{w}}_i(k) + \tilde{R}_i\tilde{\varsigma}_i(k))$$

to

$$(\tilde{A}_i + B_iK_i)(x_i(k)-\hat{x}_i(k)) + \tilde{D}_i(\mathsf{w}_i(k)-\hat{\mathsf{w}}_i(k)) + \bar{N}_i,$$

where $\bar{N}_i = \tilde{A}_i\hat{x}_i(k) + B_i\hat{\nu}_i(k+\mathcal{M}-1) + \tilde{D}_i\hat{\mathsf{w}}_i(k) + \tilde{R}_i\tilde{\varsigma}_i(k) - \Pi_{x_i}(\tilde{A}_i\hat{x}_i(k) + B_i\hat{\nu}_i(k+\mathcal{M}-1) + \tilde{D}_i\hat{\mathsf{w}}_i(k) + \tilde{R}_i\tilde{\varsigma}_i(k))$. By employing Cauchy-Schwarz inequality, Young's inequality, and Assumption 4.5.15, one can obtain the chain of inequalities in (4.5.24). Hence the proposed V_i in (4.2.14) is an FStF from $\widehat{\Sigma}_i$ to Σ_i, which completes the proof. Note that functions $\alpha_i \in \mathcal{K}_\infty$, $\kappa_i \in \mathcal{K}$, $\rho_{\text{ext}i} \in \mathcal{K}_\infty \cup \{0\}$, and the matrix $\bar{X}_i$ in Definition 4.5.4 associated with V_i in (4.2.14) are defined as $\alpha_i(s) = \lambda_{\min}(\tilde{M}_i)s^2$, $\kappa_i(s) := (1-\hat{\kappa}_i)s$, $\rho_{\text{ext}i}(s) := 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\bar{X}_i = \begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}$. Moreover, the positive constant ψ_i in (4.5.4) is $\psi_i = (1+2/\pi)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2$. □

4.5.4 Case Studies

In this subsection, to demonstrate the effectiveness of our proposed results, we first apply our approaches to an interconnected system composed of 4 subsystems such that 2 of them are not stabilizable. Then to show the applicability of our results to *nonlinear* systems having strongly connected networks, we apply our proposed techniques to a *fully interconnected* network of 500 nonlinear subsystems and construct their finite MDPs with guaranteed error bounds on their probabilistic output trajectories.

4.5.4.1 Network with Unstabilizable Subsystems

We demonstrate the effectiveness of the proposed results by considering an interconnected system composed of four linear stochastic control subsystems, i.e., $\Sigma = \mathcal{I}_{fd}(\Sigma_1, \Sigma_2,$

$$
\begin{aligned}
&\mathbb{E}\Big[V_i(x_i(k+\mathcal{M}), \hat{x}_i(k+\mathcal{M})) \,\big|\, x_i = x_i(k), \hat{x}_i = \hat{x}_i(k), \nu_i = \nu_i(k+\mathcal{M}-1), \hat{\nu}_i = \hat{\nu}_i(k+\mathcal{M}-1),\\
&\quad \mathsf{w}_i = \mathsf{w}_i(k), \hat{\mathsf{w}}_i = \hat{\mathsf{w}}_i(k)\Big] - V_i(x_i, \hat{x}_i)\\
&= (x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}_i(\tilde{A}_i + B_iK_i)(x_i - \hat{x}_i) + (\mathsf{w}_i - \hat{\mathsf{w}}_i)^T\tilde{D}_i^T\tilde{M}_i\tilde{D}_i(\mathsf{w}_i - \hat{\mathsf{w}}_i)\\
&\quad + 2(x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}_i\tilde{D}_i(\mathsf{w}_i - \hat{\mathsf{w}}_i) + 2(\mathsf{w}_i - \hat{\mathsf{w}}_i)^T\tilde{D}_i^T\tilde{M}_i\mathbb{E}\Big[\bar{N}_i \,\big|\, x_i, \hat{x}_i, \hat{\nu}_i, \mathsf{w}_i, \hat{w}_i\Big]\\
&\quad + 2_i(x_i - \hat{x}_i)^T(\tilde{A}_i + B_iK_i)^T\tilde{M}\mathbb{E}\Big[\bar{N}_i \,\big|\, x_i, \hat{x}_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big] + \mathbb{E}\Big[\bar{N}_i^T\tilde{M}\bar{N}_i \,\big|\, x_i, \hat{x}_i, \hat{\nu}_i, \mathsf{w}_i, \hat{\mathsf{w}}_i\Big]\\
&\quad - V_i(x_i, \hat{x}_i)\\
&\le \begin{bmatrix} x_i - \hat{x}_i \\ \mathsf{w}_i - \hat{\mathsf{w}}_i \end{bmatrix}^T \begin{bmatrix} (1+\pi_i)(\tilde{A}_i + B_iK_i)^T\tilde{M}_i(\tilde{A}_i + B_iK_i) & (\tilde{A}_i + B_iK_i)^T\tilde{M}_i\tilde{D}_i \\ * & (1+\pi_i)\tilde{D}_i^T\tilde{M}_i\tilde{D}_i \end{bmatrix} \begin{bmatrix} x_i - \hat{x}_i \\ \mathsf{w}_i - \hat{\mathsf{w}}_i \end{bmatrix}\\
&\quad + (1 + 2/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2 - V_i(x_i, \hat{x}_i)\\
&\le \begin{bmatrix} x_i - \hat{x}_i \\ \mathsf{w}_i - \hat{\mathsf{w}}_i \end{bmatrix}^T \begin{bmatrix} \hat{\kappa}_i\tilde{M}_i + \bar{X}_i^{22} & \bar{X}_i^{21} \\ \bar{X}_i^{12} & \bar{X}_i^{11} \end{bmatrix} \begin{bmatrix} x_i - \hat{x}_i \\ \mathsf{w}_i - \hat{\mathsf{w}}_i \end{bmatrix} + (1 + 2/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}^2 - V_i(x_i, \hat{x}_i)\\
&= -(1 - \hat{\kappa}_i)(V_i(x_i, \hat{x}_i)) + \begin{bmatrix} \mathsf{w}_i - \hat{\mathsf{w}}_i \\ x_i - \hat{x}_i \end{bmatrix}^T \begin{bmatrix} \bar{X}_i^{11} & \bar{X}_i^{12} \\ \bar{X}_i^{21} & \bar{X}_i^{22} \end{bmatrix} \begin{bmatrix} \mathsf{w}_i - \hat{\mathsf{w}}_i \\ x_i - \hat{x}_i \end{bmatrix} + (1 + 2/\pi_i)\lambda_{\max}(\tilde{M}_i)\bar{\delta}_i^2.
\end{aligned}
\tag{4.5.24}
$$

Σ_3, Σ_4), with the interconnection matrix

$$
M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.
$$

The linear stochastic control subsystems are given by

$$
\Sigma: \begin{cases} x_1(k+1) = 1.02x_1(k) - 0.07w_1(k) + \varsigma_1(k), \\ x_2(k+1) = 1.04x_2(k) - 0.06w_2(k) + \varsigma_2(k), \\ x_3(k+1) = 0.5x_3(k) + \nu_3(k) + 0.04w_3(k) + \varsigma_3(k), \\ x_4(k+1) = 0.6x_4(k) + \nu_4(k) + 0.05w_4(k) + \varsigma_4(k), \end{cases} \tag{4.5.25}
$$

with $X_i = [0 \;\; 0.5], W_i = [0 \;\; 1], \forall i \in \{1, \ldots, 4\}$ and $U_i = [0 \;\; 0.45], \forall i \in \{3, 4\}$. As seen, the first two subsystems are not stabilizable. Then we proceed with looking at the solution of Σ_i two steps ahead, i.e., $\mathcal{M} = 2$,

$$
\Sigma_{\mathsf{aux}}: \begin{cases} x_1(k+2) = 0.89x_1(k) + \mathsf{w}_1(k) + \tilde{R}_1\tilde{\varsigma}_1(k), \\ x_2(k+2) = 0.95x_2(k) + \mathsf{w}_2(k) + \tilde{R}_2\tilde{\varsigma}_2(k), \\ x_3(k+2) = 0.24x_3(k) + \nu_3(k+1) + \mathsf{w}_3(k) + \tilde{R}_3\tilde{\varsigma}_3(k), \\ x_4(k+2) = 0.35x_4(k) + \nu_4(k+1) + \mathsf{w}_4(k) + \tilde{R}_4\tilde{\varsigma}_4(k), \end{cases} \tag{4.5.26}
$$

where

$$\tilde{\varsigma}_1(k) = [\varsigma_3(k); \varsigma_1(k); \varsigma_1(k+1)], \quad \tilde{\varsigma}_3(k) = [\varsigma_1(k); \varsigma_2(k); \varsigma_3(k); \varsigma_3(k+1)],$$
$$\tilde{\varsigma}_2(k) = [\varsigma_4(k); \varsigma_2(k); \varsigma_2(k+1)], \quad \tilde{\varsigma}_4(k) = [\varsigma_1(k); \varsigma_2(k); \varsigma_4(k); \varsigma_4(k+1)].$$

Moreover, $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}]^T, \ \forall i \in \{1, 2\}$, where

$$\tilde{R}_{11} = 0.95, \quad \tilde{R}_{12} = -0.07, \quad \tilde{R}_{13} = 1,$$
$$\tilde{R}_{21} = 0.98, \quad \tilde{R}_{22} = -0.06, \quad \tilde{R}_{23} = 1,$$

and $\tilde{R}_i = [\tilde{R}_{i1}; \tilde{R}_{i2}; \tilde{R}_{i3}; \tilde{R}_{i4}]^T, \ \forall i \in \{3, 4\}$, where

$$\tilde{R}_{31} = 0.04, \quad \tilde{R}_{32} = 0.04, \quad \tilde{R}_{33} = 0.5, \quad \tilde{R}_{34} = 1,$$
$$\tilde{R}_{41} = 0.05, \quad \tilde{R}_{42} = 0.05, \quad \tilde{R}_{43} = 0.6, \quad \tilde{R}_{44} = 1.$$

In addition, the new interconnection matrix for the the *auxiliary* system is

$$M_a = \begin{bmatrix} 0 & -0.002 & -0.1 & 0 \\ -0.003 & 0 & 0 & -0.09 \\ 0.05 & 0.05 & 0 & -0.002 \\ 0.07 & 0.07 & -0.003 & 0 \end{bmatrix}. \tag{4.5.27}$$

One can readily see that the first two subsystems are now stable. Then, we proceed with constructing finite MDPs from auxiliary systems (4.5.26) as proposed in Algorithm 2. Based on the auxiliary coupling matrix M_a in (4.5.27), one has $\tilde{W}_1 = [-0.051 \ \ 0], \tilde{W}_2 = [-0.0465 \ \ 0], \tilde{W}_3 = [-0.001 \ \ 0.05], \tilde{W}_4 = [-0.0015 \ \ 0.07]$. By taking state, internal and external input discretization parameters as $\bar{\delta}_i = 0.004, \bar{\mu}_i = 0.0001, \forall i \in \{1, \dots, 4\}$, $\theta_i = 0.006, \forall i \in \{3, 4\}$, one has $n_{x_i} = 125, \forall i \in \{1, \dots, 4\}$, $n_{w_1} = 510, n_{w_2} = 465, n_{w_3} = 510, n_{w_4} = 715$, $n_{u_i} = 75, \forall i \in \{3, 4\}$. We consider here partition sets as intervals and the center of each interval as representative points. One can readily verify that the condition (4.5.23) is satisfied with

$$\hat{\kappa}_1 = 0.96, \hat{\kappa}_2 = 0.99, \hat{\kappa}_3 = 0.64, \hat{\kappa}_4 = 0.63, K_3 = K_4 = 0,$$
$$\pi_1 = 0.1, \pi_2 = 0.05, \pi_3 = \pi_4 = 0.99, \tilde{M}_i = 1, \forall i \in \{1, 2, 3, 4\},$$
$$\bar{X}_1^{11} = 1.1, \bar{X}_1^{12} = \bar{X}_1^{21} = 0.89, \bar{X}_1^{22} = -0.05,$$
$$\bar{X}_2^{11} = 1.05, \bar{X}_2^{12} = \bar{X}_2^{21} = 0.95, \bar{X}_2^{22} = -0.03,$$
$$\bar{X}_3^{11} = 1.99, \bar{X}_3^{12} = \bar{X}_3^{21} = 0.24, \bar{X}_3^{22} = -0.2,$$
$$\bar{X}_4^{11} = 1.99, \bar{X}_4^{12} = \bar{X}_4^{21} = 0.35, \bar{X}_4^{22} = -0.03.$$

Then, the function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an FStF from $\widehat{\Sigma}_i$ to Σ_i satisfying the condition (4.5.3) with $\alpha_i(s) = s^2, \forall s \in \mathbb{R}_{\geq 0}, \forall i \in \{1, 2, 3, 4\}$, and the condition (4.5.4) with

$$\kappa_1(s) = 0.03s, \kappa_2(s) = 0.0051s, \kappa_3(s) = 0.35s, \kappa_4(s) = 0.36s, \forall s \in \mathbb{R}_{\geq 0},$$
$$\rho_{\text{ext}i}(s) = 0, \forall i \in \{1, 2, 3, 4\}, \psi_1 = 21\,\bar{\delta}^2, \psi_2 = 41\,\bar{\delta}^2, \psi_3 = 3.02\,\bar{\delta}^2, \psi_4 = 3.02\,\bar{\delta}^2,$$

where the input ν_i is given via the interface function in (4.4.19) as $\nu_i = \hat{\nu}_i$. Now we look at $\widehat{\Sigma} = \widehat{\mathcal{I}}_{fd}(\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}_a$ satisfying the condition (4.5.13) as $\hat{M}_a = M_a$. Choosing $\mu_1 = \dots = \mu_4 = 1$, condition (4.5.14) is satisfied as

$$\begin{bmatrix} M_a \\ \mathbb{I}_4 \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M_a \\ \mathbb{I}_4 \end{bmatrix} = \begin{bmatrix} -0.03 & 0.01 & -0.07 & 0.02 \\ 0.01 & -0.01 & 0.01 & -0.06 \\ -0.07 & 0.01 & -0.18 & -0.001 \\ 0.15 & 0.06 & -0.007 & -0.02 \end{bmatrix} \preceq 0.$$

By selecting $\underline{\mu} = 0.005$, the condition (4.5.12) is also satisfied. Now, one can readily verify that $V(x, \hat{x}) = \sum_{i=1}^{4}(x_i - \hat{x}_i)^2$ is a sum-type FSF from $\widehat{\Sigma}$ to Σ satisfying conditions (4.5.5) and (4.5.6) with $\alpha(s) = s^2$, $\kappa(s) := 0.005s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and the overall error of the network formulated in (4.5.15) as $\psi = 68.04\bar{\delta}^2 + (1.6 \times 10^5)\bar{\boldsymbol{\mu}}^2$.

By starting the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ from $\mathbb{1}_4$ and employing Theorem 4.4.13, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.5$ at times $k = 2j, j = \{0, \dots, 7\}$ with the probability at least 90%, i.e.,

$$\mathbb{P}(\|x_{a\hat{\nu}}(k) - \hat{x}_{\hat{a}\hat{\nu}}(k)\|_2 \leq 0.5, \forall k = 2j, j = \{0, \dots, 7\}) \geq 0.9.$$

4.5.4.1.1 Discussions on Memory Usage and Computation Time for Constructing $\widehat{\Sigma}_{\mathsf{aux}i}$

Now we provide some discussions on the memory usage and computation time in constructing finite MDPs in both monolithic and compositional manners. The monolithic finite MDP constructed from the given system in (4.5.25) would be a matrix with the dimension of $(n_{x_i}^4 \times n_{u_i}^2) \times n_{x_i}^4$. By allocating 8 bytes for each entry of the matrix to be stored as a double-precision floating point, one needs a memory of roughly $\frac{8\times125^4\times75^2\times125^4}{10^9} \approx 2.6822\times10^{12}$ GB for building the finite MDP in the monolithic manner which is impossible in practice. Now, we proceed with the compositional construction of finite MDPs proposed in this work for each subsystem of the $\mathcal{M}$-sampled system in (4.5.26). The construction procedure is performed via the software tool FAUST^2 on a machine with Windows operating system (Intel i7@3.6GHz CPU and 16 GB of RAM). The constructed MDP for each subsystem here is a matrix with the dimension of $(n_{x_i} \times n_{w_i} \times n_{u_i}) \times n_{x_i}$. Then the memory usage and computation time for all subsystems are as follows:
$\widehat{\Sigma}_{\mathsf{aux}1}$: Memory usage: 0.0638 GB, computation time: 9 (s),
$\widehat{\Sigma}_{\mathsf{aux}2}$: Memory usage: 0.0581 GB, computation time: 7 (s),
$\widehat{\Sigma}_{\mathsf{aux}3}$: Memory usage: 4.7813 GB, computation time: 43 (s),
$\widehat{\Sigma}_{\mathsf{aux}4}$: Memory usage: 6.7031 GB, computation time: 65 (s).
A comparison on the required memory for the construction of finite MDPs between the monolithic and compositional manners for different ranges of the state discretization parameter is provided in Table 4.2. Note that the third column of the table is about the maximum required memory for the construction of $\widehat{\Sigma}_{\mathsf{aux}i}$ (which is corresponding to $\widehat{\Sigma}_{\mathsf{aux}4}$). As seen, in order to provide even a weak closeness guarantee of 18% between

Table 4.2: Required memory for the construction of finite MDPs in both monolithic and compositional manners for different ranges of the state discretization parameter.

$\bar{\delta}$	Closeness	Memory for $\widehat{\Sigma}_{\mathsf{aux}i}$ (GB)	Memory for $\widehat{\Sigma}$ (GB)
0.002	92%	44.6875	1.9073×10^{15}
0.004	90%	6.7031	2.6822×10^{12}
0.006	88%	1.6156	3.0289×10^{10}
0.008	85%	0.6816	1.6786×10^{9}
0.01	83%	0.3575	195312500
0.02	61%	0.0429	175780
0.04	18%	0.0049	123.8347

states of Σ and $\widehat{\Sigma}$, the required memory for the monolithic fashion is 123.8347 GB which is still too big. This implementation clearly shows that the proposed compositional approach in this work significantly mitigates the curse of dimensionality problem in constructing finite MDPs monolithically. In particular, in order to quantify the probabilistic closeness between states of two networks Σ and $\widehat{\Sigma}$ via Theorem 4.4.13 as provided in Table 4.2, one needs to only build finite MDPs of individual auxiliary subsystems (i.e., $\widehat{\Sigma}_{\mathsf{aux}i}$), construct an FStF between each Σ_i and $\widehat{\Sigma}_i$, and then employ the proposed compositionality results of the section to build a sum-type FSF between Σ and $\widehat{\Sigma}$.

4.5.4.1.2 Compositional Controller Synthesis

In order to study the level of conservatism originating from Assumption 4.4.3, we compositionally synthesize a safety controller for Σ_{aux} in (4.5.26). We also compositionally abstract the original system Σ using the approach in [SAM17] which is based on dynamic Bayesian network (DBN), and employ FAUST^2 [SGA15] to synthesize a controller. We then compare the probabilities of satisfying a safety specification obtained by using these two controllers.

Note that the approach of [SAM17] does not require original subsystems to be stabilizable and only the Lipschitz continuity of the associated stochastic kernels is enough for the validity of the results. However, their proposed closeness guarantee converges to infinity when the standard deviation $\bar{\sigma}$ goes to zero whereas our probabilistic error in (4.4.11) is independent of $\bar{\sigma}$. Thus our proposed closeness bound outperforms [SAM17] for a smaller standard deviation of the noise. A detailed comparison on this issue has been made in Section 4.3, Figure 4.17. Although the comparison there is done for 1-step models, the same reasoning is valid for the $\mathcal{M}$-step ones as well.

Let $X_i = [-2 \ \ 2], W_i = [-2 \ \ 2], \forall i \in \{1, \ldots, 4\}$, and $U_i = [0 \ \ 1], \forall i \in \{3, 4\}$. We take $\bar{\delta}_i = 0.005, \bar{\mu}_i = 0.01, \forall i \in \{1, \ldots, 4\}$, and $\theta_i = 0.01, \forall i \in \{3, 4\}$. The main goal is to compositionally synthesize a safety controller for Σ_{aux} and Σ such that the controller maintains states of the systems in the set $[-2 \ \ 1.5]$ for $T_d = 14$ time steps. In order to make a fair comparison and since $\mathcal{M} = 2$, this safety requirement is required for only *even* time instances.

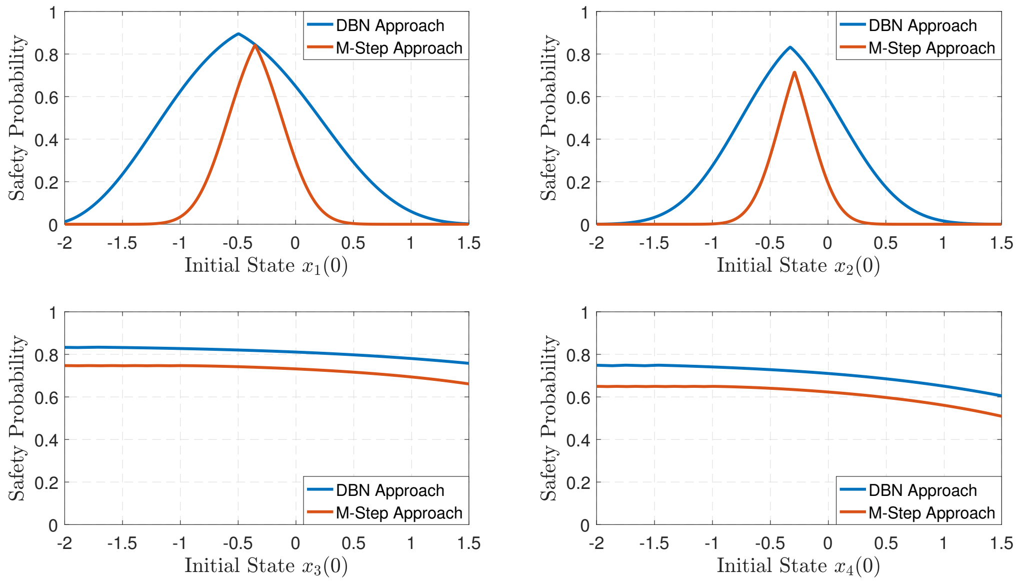

Figure 4.25: Comparison of safety probabilities by our approach and that of [SAM17] based on DBN. Plots are probabilities as a function of the initial state of one state variable while the other state variables have an initial value according to $x_1(0) = -0.35, x_2(0) = -0.285, x_3(0) = -1.705, x_4(0) = -1.745$. The time horizon is $T_d = 14$.

A comparison of safety probabilities for the $\mathcal{M}$-step and original subsystems is provided in Figure 4.25. We selected the initial conditions $x_1(0) = -0.35, x_2(0) = -0.285$, $x_3(0) = -1.705, x_4(0) = -1.745$. In each plot of Figure 4.25, we fixed three of these initial states and showed the probability as a function of the other state. We also fixed the standard deviation of the noise as $\bar{\sigma}_i = 0.1, \forall i \in \{1, 2\}, \bar{\sigma}_i = 0.6, \forall i \in \{3, 4\}$. As seen, safety probabilities using the DBN approach are better than those using the $\mathcal{M}$-step approach. This is mainly due the fact that the external inputs in the $\mathcal{M}$-step setting are allowed to take nonzero values only at particular time instances (here at $2j+1,\ j = \{0, \ldots, 6\}$), which makes the controller synthesis problem more conservative (as discussed in Remark 4.4.5).

We now plot one realization of input trajectories for the third and fourth subsystems in both $\mathcal{M}$-step and DBN approaches in Figure 4.26. As seen, the DBN approach allows taking nonzero input values at all time steps whereas the $\mathcal{M}$-step one only allows nonzero input values at $2j+1,\ j = \{0, \ldots, 6\}$.

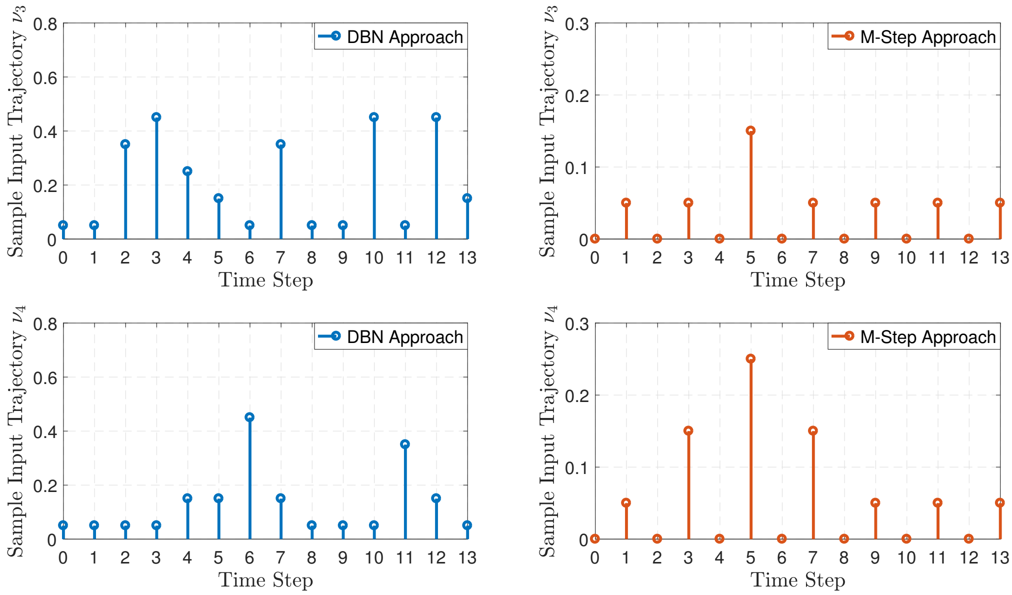

Figure 4.26: One realization of input trajectories ν_3, ν_4 via our approach and that of [SAM17] based on DBN. The DBN approach allows taking nonzero inputs at all time steps whereas the $\mathcal{M}$-step one allows this only at $2j+1,\ j = \{0, \dots, 6\}$.

4.5.4.2 Nonlinear Fully Interconnected Network

In order to show the applicability of our approach to *strongly interconnected* networks with *nonlinear* dynamics, we consider the nonlinear dt-SCS

$$\Sigma : x(k+1) = \bar{G}x(k) + \varphi(x(k)) + \nu(k) + \varsigma(k),$$

for some matrix $\bar{G} = (\mathbb{I}_n - \tau\tilde{L}) \in \mathbb{R}^{n\times n}$ where $\tilde{L}$ is the Laplacian matrix of an undirected graph with $0 < \tau < 1/\bar{\Delta}$, and $\bar{\Delta}$ is the maximum degree of the graph [GR01]. We assume $\tilde{L}$ is the Laplacian matrix of a *complete graph* as in (3.4.24). Moreover, $\varsigma(k) = [\varsigma_1(k); \dots; \varsigma_N(k)]$, $\varphi(x(k)) = [E_1\varphi_1(F_1x_1(k)); \dots; E_N\varphi_N(F_Nx_N(k))]$ where $\varphi_i(x) = sin(x)$, $\forall i \in \{1, \dots, N\}$. We partition $x(k)$ as $x(k) = [x_1(k); \dots; x_N(k)]$ and $\nu(k)$ as $\nu(k) = [\nu_1(k); \dots; \nu_N(k)]$. Now, by introducing Σ_i described as

$$\Sigma_i : x_i(k+1) = x_i(k) + E_i\varphi_i(F_ix_i(k)) + \nu_i(k) + w_i(k) + \varsigma_i(k),$$

one can readily verify that $\Sigma = \mathcal{I}_{fd}(\Sigma_1, \dots, \Sigma_N)$ where the coupling matrix M is given by $M = -\tau\tilde{L}$. Then one can verify that, $\forall i \in \{1, \dots, N\}$, the condition (4.5.18) is satisfied with $\tilde{M}_i = 1$, $K_i = -0.5$, $E_i = 0.1$, $F_i = 0.1$, $b_i = 1$, $\bar{X}^{11} = (1 + \pi_i)$, $\bar{X}^{22} = 0$, $\bar{X}^{12} = \bar{X}^{21} = \lambda_i'$, where $\lambda_i' = 1 + K_i$, $\hat{\kappa}_i = 0.99$, and $\pi_i = 1$, $\forall i \in \{1, \dots, N\}$. Hence, the function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i (with $\mathcal{M} = 1$) satisfying the condition (4.5.3) with $\alpha_i(s) = s^2$ and the condition (4.5.4) with $\kappa_i(s) := (1 - \hat{\kappa}_i)s$,

$\rho_{\text{iext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 4\bar{\delta}_i^2$. Now, we look at $\widehat{\Sigma} = \widehat{\mathcal{I}}_{fd}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying the condition (4.5.13) by $\hat{M} = M$. Choosing $\mu_1 = \cdots = \mu_N = 1$, the matrix $\bar{X}_{cmp}$ in (3.4.6) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} (1+\pi)\mathbb{I}_n & \lambda'\mathbb{I}_n \\ \lambda'\mathbb{I}_n & 0 \end{bmatrix},$$

where $\lambda' = \lambda'_1 = \cdots = \lambda'_n$, $\pi = \pi_1 = \cdots = \pi_n$, and the condition (4.5.14) reduces to

$$\begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} -\tau\tilde{L} \\ \mathbb{I}_n \end{bmatrix} = (1+\pi)\tau^2\tilde{L}^T\tilde{L} - \lambda'\tau\tilde{L} - \lambda'\tau\tilde{L}^T = \tau\tilde{L}((1+\pi)\tau\tilde{L} - 2\lambda'\mathbb{I}_n) \preceq 0,$$

which is always satisfied without requiring any restrictions on the number or gains of the subsystems with $\tau = 0.4/(n-1)$. In order to show the above inequality, we used $\tilde{L} = \tilde{L}^T \succeq 0$ which is always true for Laplacian matrices of undirected graphs. We fix here $n = 500$. Now one can readily verify that $V(x, \hat{x}) = \sum_{i=1}^{500}(x_i - \hat{x}_i)^2$ is an SStF $\widehat{\Sigma}$ to Σ satisfying conditions (4.5.5) and (4.5.6) with $\alpha(s) = s^2$, $\kappa(s) := (1-\hat{\kappa})s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 2000\bar{\delta}^2$.

By taking the state discretization parameter $\bar{\delta} = 0.005$, and selecting the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $\mathbb{1}_{500}$, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with the probability at least 88%.

4.6 Approximate Probabilistic Relations

In this section, we propose a compositional approach for constructing abstractions of Markov decision processes in (2.4.1) using approximate probabilistic relations. The abstraction framework is based on the notion of δ-lifted relations, using which one can quantify the distance in the probability between interconnected MDPs and that of their abstractions. This new approximate relation unifies compositionality results in the literature by incorporating the dependencies between state transitions explicitly and by allowing abstract models to have either finite or infinite state spaces. Accordingly, one can leverage the proposed results to perform analysis and synthesis over abstract models, and then carry the results back over concrete ones.

We provide conditions under which the proposed similarity relations between individual MDPs can be extended to relations between their respective interconnections. These conditions enable a compositional quantification of the probabilistic distance between interconnected MDPs and that of their abstractions. The proposed notion has the advantage of encoding prior knowledge on dependencies between uncertainties of the two models. Our compositional scheme allows constructing both infinite and finite abstractions in a unified framework. We benchmark our results against the compositional abstraction techniques proposed in Sections 3.4 and 4.3.1 which are based on the dissipativity approach and provide a compositional methodology for constructing both infinite abstractions (reduced-order models) and finite MDPs in two consecutive steps.

We show that the proposed unified approach here is less conservative than the two-step construction one proposed in Sections 3.4 and 4.3.1.

Similarities between two MDPs have been recently studied in [HSA17] using a notion of δ-lifted relations, but only for single MDPs. The result is generalized in [HSA18] to a larger class of temporal properties and in [HS18] to synthesize policies for the robust satisfaction of specifications. One of the main contributions of this section is to extend this notion such that it can be applied to networks of MDPs. In particular, we extend the notion of δ-lifted relations for networks of MDPs and show that under specific conditions systems can be composed while preserving the relation. This type of relations enables us to provide the probabilistic closeness guarantee between two interconnected MDPs (cf. Theorem 4.6.6). Furthermore, we provide an approach for the construction of finite MDPs in a unified framework for the nonlinear class of stochastic control systems (3.3.10), whereas the construction scheme in [HSA17] only handles the class of linear systems.

4.6.1 Approximate Probabilistic Relations based on Lifting

In this subsection, we first introduce the notion of δ-lifted relations over general state spaces. We then define (ϵ, δ)-approximate probabilistic relations based on lifting for MDPs with internal inputs and outputs. Finally, we define (ϵ, δ)-approximate relations for interconnected MDPs without internal signals resulting from the interconnection of MDPs having internal inputs and outputs.

We provide the notion of δ-lifted relation borrowed from [HSA17] as the following definition.

Definition 4.6.1. *Let $X, \hat{X}$ be two sets with associated measurable spaces $(X, \mathcal{B}(X))$ and $(\hat{X}, \mathcal{B}(\hat{X}))$. Consider a relation $\mathscr{R}_x \in \mathcal{B}(X \times \hat{X})$. We denote by $\bar{\mathscr{R}}_\delta \subseteq \mathcal{P}(X, \mathcal{B}(X)) \times \mathcal{P}(\hat{X}, \mathcal{B}(\hat{X}))$, the corresponding δ-lifted relation if there exists a probability space $(X \times \hat{X}, \mathcal{B}(X \times \hat{X}), \bar{\mathscr{L}})$ (equivalently, a lifting $\bar{\mathscr{L}}$) such that $(\Phi, \Theta) \in \bar{\mathscr{R}}_\delta$ if and only if*

- $\forall \mathcal{A} \in \mathcal{B}(X),\ \bar{\mathscr{L}}(\mathcal{A} \times \hat{X}) = \Phi(\mathcal{A})$,
- $\forall \hat{\mathcal{A}} \in \mathcal{B}(\hat{X}),\ \bar{\mathscr{L}}(X \times \hat{\mathcal{A}}) = \Theta(\hat{\mathcal{A}})$,
- *for the probability space $(X \times \hat{X}, \mathcal{B}(X \times \hat{X}), \bar{\mathscr{L}})$, it holds that $x \mathscr{R}_x \hat{x}$ with the probability at least $1 - \delta$, equivalently, $\bar{\mathscr{L}}(\mathscr{R}_x) \geq 1 - \delta$.*

For a given relation $\mathscr{R}_x \subseteq X \times \hat{X}$, the above definition specifies required properties for the lifting relation $\mathscr{R}_x$ to a relation $\bar{\mathscr{R}}_\delta$ that relates probability measures over X and $\hat{X}$.

We are interested in using *δ-lifted* relation for specifying similarities between an MDP and its abstraction. Therefore, internal inputs of the two MDPs should be in a relation denoted by $\mathscr{R}_w$. Next definition gives conditions for having a stochastic simulation relation between two MDPs.

Definition 4.6.2. *Consider two MDPs $\Sigma = (X, U, W, T_{\mathsf{x}}, Y^1, Y^2, h^1, h^2)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, , \hat{W}, \hat{T}_{\mathsf{x}}, Y^1, Y^2, \hat{h}^1, \hat{h}^2)$ with the same output spaces. Let $\pi : \mathcal{B}(\hat{X}) \rightarrow [0, 1]$ and $\hat{\pi}$:*

$\mathcal{B}(\hat{X}) \rightarrow [0,1]$ be initial probability distributions respectively for Σ and $\widehat{\Sigma}$. System $\widehat{\Sigma}$ is (ϵ, δ)-stochastically simulated by Σ, i.e., $\widehat{\Sigma} \preceq_{\epsilon}^{\delta} \Sigma$, if there exist relations $\mathscr{R}_x \subseteq X \times \hat{X}$ and $\mathscr{R}_w \subseteq W \times \hat{W}$ for which there exists a Borel measurable stochastic kernel $\bar{\mathscr{L}}_T(\cdot \mid x, \hat{x}, \hat{\nu}, w, \hat{w})$ on $X \times \hat{X}$ such that

- $\forall (x, \hat{x}) \in \mathscr{R}_x, \forall i \in \{1, 2\}, \quad \|h^i(x) - \hat{h}^i(\hat{x})\| \leq \epsilon,$
- *$\forall (x, \hat{x}) \in \mathscr{R}_x$, $\forall \hat{w} \in \hat{W}$, $\forall \hat{\nu} \in \hat{U}$, there exists $\nu \in U$ such that $\forall w \in W$ with $(w, \hat{w}) \in \mathscr{R}_w$,*
$$T_{\mathsf{x}}(\cdot \mid x, \nu, w) \ \bar{\mathscr{R}}_\delta \ \hat{T}_{\mathsf{x}}(\cdot \mid \hat{x}, \hat{\nu}, \hat{w})$$
with lifting $\bar{\mathscr{L}}_T(\cdot \mid x, \hat{x}, \hat{\nu}, w, \hat{w})$,
- $\pi \ \bar{\mathscr{R}}_\delta \ \hat{\pi}$.

The second condition of Definition 4.6.2 implicitly implies that there exists an interface function $\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w})$ such that state probability measures are in the lifted relation after one transition for any $(x, \hat{x}) \in \mathscr{R}_x$, $\hat{\nu} \in \hat{U}$, and $\hat{w} \in \hat{W}$. This function can be employed for refining a synthesized policy $\hat{\nu}$ for $\widehat{\Sigma}$ to a policy ν for Σ.

Remark 4.6.3. *Definition 4.6.2 extends the approximate probabilistic relation in [HSA17] by adding the relation $\mathscr{R}_w$ to capture the effect of internal inputs. The interface function $\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w})$ is also allowed to depend on the internal input of the abstract MDP $\widehat{\Sigma}$.*

Remark 4.6.4. *Note that Definition 4.6.2 generalizes the results of Section 3.4, that assumes independent noises in two similar MDPs, and of Section 4.3.1, that assumes shared noises, by making no particular assumption but requiring this dependency to be reflected in lifting $\bar{\mathscr{L}}_T$. We emphasize that this generalization is considered only for a concrete MDP and its abstraction. We still retain the assumption of independent uncertainties between MDPs in a network (cf. Definition 4.6.7 and Remark 4.6.8).*

Figure 4.27 illustrates ingredients of Definition 4.6.2. As seen, the relation $\mathscr{R}_w$ and the stochastic kernel $\bar{\mathscr{L}}_T$ capture the effect of internal inputs, and the relation of two noises, respectively. Moreover, the interface function $\nu_{\hat{\nu}}(x, \hat{x}, \hat{w}, \hat{\nu})$ is employed to refine a synthesized policy $\hat{\nu}$ for $\widehat{\Sigma}$ to a policy ν for Σ.

In this section, we are interested in networks of MDPs that are obtained from composing MDPs having both internal and external signals. The resulting interconnected MDP will have only *external* inputs and outputs and will be denoted by the tuple $\Sigma = (X, U, T_{\mathsf{x}}, Y, h)$ with the stochastic kernel $T_{\mathsf{x}} : \mathcal{B}(X) \times X \times U \rightarrow [0, 1]$.

Accordingly, Definition 4.6.2 can be applied to MDPs without internal inputs and outputs that may arise from composing MDPs via their internal signals. For such MDPs, we eliminate $\mathscr{R}_w$ and the interface function becomes independent of internal inputs, thus the definition reduces to the following definition.

Definition 4.6.5. *Consider two MDPs without internal signals $\Sigma = (X, U, T_{\mathsf{x}}, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{T}_{\mathsf{x}}, Y, \hat{h})$, that have the same output spaces. $\widehat{\Sigma}$ is (ϵ, δ)-stochastically simulated by Σ, i.e., $\widehat{\Sigma} \preceq_{\epsilon}^{\delta} \Sigma$, if there exists a relation $\mathscr{R}_x \subseteq X \times \hat{X}$ for which there exists a Borel measurable stochastic kernel $\bar{\mathscr{L}}_T(\cdot \mid x, \hat{x}, \hat{\nu})$ on $X \times \hat{X}$ such that*

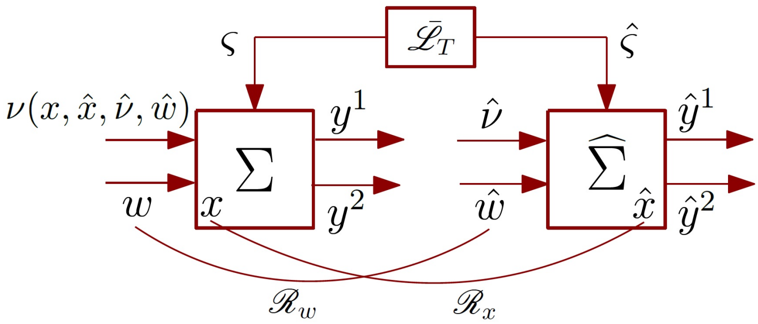

Figure 4.27: Notion of the *lifting* for specifying the similarity between an MDP and its abstraction. Relations $\mathscr{R}_x$ and $\mathscr{R}_w$ are the ones between states and internal inputs, respectively. $\bar{\mathscr{L}}_T$ specifies the relation of two noises, and the interface function $\nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w})$ is used for the refinement policy.

- $\forall (x, \hat{x}) \in \mathscr{R}_x, \ \|h(x) - \hat{h}(\hat{x})\| \leq \epsilon,$
- $\forall (x, \hat{x}) \in \mathscr{R}_x, \forall \hat{\nu} \in \hat{U}, \exists \nu \in U$ *such that* $T_{\mathsf{x}}(\cdot \,|\, x, \nu(x, \hat{x}, \hat{\nu})) \ \bar{\mathscr{R}}_\delta \ \hat{T}_{\mathsf{x}}(\cdot \,|\, \hat{x}, \hat{\nu})$ *with* $\bar{\mathscr{L}}_T(\cdot \,|\, x, \hat{x}, \hat{\nu}),$
- $\pi \ \bar{\mathscr{R}}_\delta \ \hat{\pi}.$

Definition 4.6.2 enables us to quantify the error in probability between a concrete system Σ and its abstraction $\widehat{\Sigma}$. In any (ϵ, δ)-approximate probabilistic relation, δ is used to quantify the distance in the probability between MDPs and ϵ for the closeness of output trajectories as stated in the next theorem.

Theorem 4.6.6. *If* $\widehat{\Sigma} \preceq_\epsilon^\delta \Sigma$ *and* $(w(k), \hat{w}(k)) \in \mathscr{R}_w$ *for all* $k \in \{0, 1, \dots, T_d\}$*, then for all policies on* $\widehat{\Sigma}$ *there exists a policy for* Σ *such that, for all measurable events* $\mathsf{A} \subset Y^{T_d+1}$*,*

$$\mathbb{P}\{\{\hat{y}(k)\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\} - \underline{\gamma} \leq \mathbb{P}\{\{y(k)\}_{0:T_d} \in \mathsf{A}\} \leq \mathbb{P}\{\{\hat{y}(k)\}_{0:T_d} \in \mathsf{A}^{\epsilon}\} + \underline{\gamma}, \tag{4.6.1}$$

with the constant $1 - \underline{\gamma} := (1 - \delta)^{T_d+1}$*, and with the* ϵ*-expansion and* ϵ*-contraction of* A *defined as*

$$\mathsf{A}^{\epsilon} := \{y(\cdot) \in Y^{T_d+1} \,|\, \exists \bar{y}(\cdot) \in \mathsf{A} \ \textit{with} \ \textit{max}_{k \leq T_d} \|\bar{y}(k) - y(k)\| \leq \epsilon\},$$
$$\mathsf{A}^{-\epsilon} := \{y(\cdot) \in \mathsf{A} \,|\, \bar{y}(\cdot) \in \mathsf{A} \ \textit{for all} \ \bar{y}(\cdot) \ \textit{with} \ \textit{max}_{k \leq T_d} \|\bar{y}(k) - y(k)\| \leq \epsilon\}.$$

Proof. The definition of the lifting implies that the initial states of two systems are in a relation with the probability at least $1 - \delta$. Moreover, if the two states are in the relation at the time k, they remain in the relation at time $k + 1$ with the probability at least $1 - \delta$. Then, we can write

$$\mathbb{P}\{(x(k), \hat{x}(k)) \in \mathscr{R}_x \text{ for all } k \in [0, T_d]\} \geq (1 - \delta)^{T_d+1}.$$

This can be proved by induction and conditioning the probability on the intermediate states.

Note that if $\{\hat{h}(\hat{x}(k))\}_{0:T_d} \in \mathsf{A}^{-\epsilon}$ and $(x(k),\hat{x}(k)) \in \mathscr{R}_x$ for all $k \in [0,T_d]$, then $\{y(k)\}_{0:T_d} \in \mathsf{A}$. As a consequence

$$\mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\} \wedge (x(k),\hat{x}(k)) \in \mathscr{R}_x \text{ for all } k \in [0,T_d]\} \leq \mathbb{P}\{\{h(x)\}_{0:T_d} \in \mathsf{A}\}.$$

Now by employing the union bounding argument, we have

$$\mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\} - (1-\delta)^{T_d+1} \leq \mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon} \wedge (x(k),\hat{x}(k)) \in \mathscr{R}_x,$$
$$\text{for all } k \in [0,T_d]\}.$$

Then

$$\begin{aligned}
&1 - \mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon} \wedge (x(k),\hat{x}(k)) \in \mathscr{R}_x \text{ for all } k \in [0,T_d]\}\\
&\leq (1 - \mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\}) + (1 - \mathbb{P}\{(x(k),\hat{x}(k)) \in \mathscr{R}_x \text{ for all } k \in [0,T_d]\})\\
&\leq (1 - \mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\}) + (1-(1-\delta)^{T_d+1}).
\end{aligned}$$

One can deduce that

$$\mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{-\epsilon}\} - (1-(1-\delta)^{T_d+1}) \leq \mathbb{P}\{\{h(x)\}_{0:T_d} \in \mathsf{A}\}.$$

Similarly, if $\{h(x(k))\}_{0:T_d} \in \mathsf{A}$ and $(x(k),\hat{x}(k)) \in \mathscr{R}_x$, then $\{\hat{h}(\hat{x}(k))\}_{0:T_d} \in \mathsf{A}^{\epsilon}$. Thus via similar arguments it holds that

$$\mathbb{P}\{\{h(x)\}_{0:T_d} \in \mathsf{A}\} \leq \mathbb{P}\{\{\hat{h}(\hat{x})\}_{0:T_d} \in \mathsf{A}^{\epsilon}\} + (1-(1-\delta)^{T_d+1}).$$

□

We employ this theorem to provide the probabilistic closeness guarantee between interconnected MDPs and that of their compositional abstractions. In the next subsection, we define the composition of MDPs via their internal inputs and outputs, and discuss how to relate them to a network of the interconnected abstraction based on their individual relations.

4.6.2 Compositionality Results

4.6.2.1 Interconnected MDPs

Let Σ be a network of $N \in \mathbb{N}_{\geq 1}$ MDPs

$$\Sigma_i = (X_i, W_i, U_i, T_{\mathsf{x}_i}, Y_i^1, Y_i^2, h_i^1, h_i^2), \quad i \in \{1,\ldots,N\}. \tag{4.6.2}$$

We partition internal inputs and outputs of Σ_i as in (3.2.8) and (3.2.9). Since internal inputs are employed for the interconnection by requiring $w_{ji} = y_{ij}^2$, this can be explicitly written using appropriate functions g_i defined as

$$w_i = \mathsf{g}_i(x_1,\ldots,x_N) := \Big[h_{1i}^2(x_1);\ldots;h_{(i-1)i}^2(x_{i-1});h_{(i+1)i}^2(x_{i+1});\ldots;h_{Ni}^2(x_N)\Big]. \tag{4.6.3}$$

Now, we define the *interconnected MDP* Σ as follows.

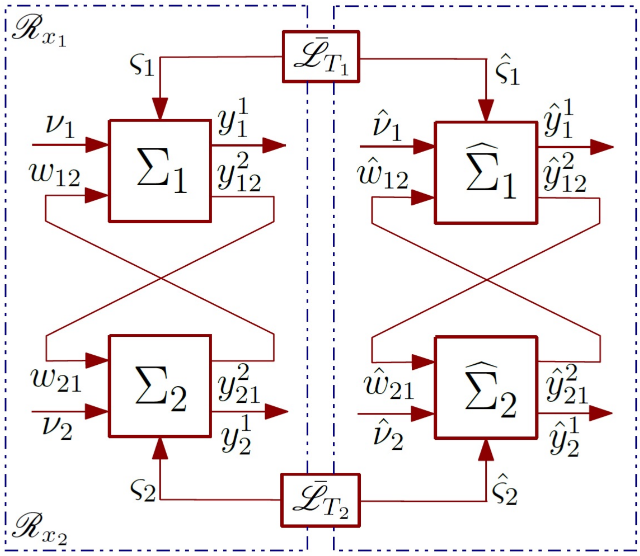

Figure 4.28: Interconnection of two MDPs Σ_1 and Σ_2 and that of their abstractions.

Definition 4.6.7. *Consider $N \in \mathbb{N}_{\geq 1}$ MDPs $\Sigma_i = (X_i, W_i, U_i, T_{\mathsf{x}_i}, Y_i^1, Y_i^2, h_i^1, h_i^2), i \in \{1, \dots, N\}$, with the input-output configuration as in* (3.2.8) *and* (3.2.9)*. The interconnection of Σ_i, $i \in \{1, \dots, N\}$, is an MDP $\Sigma = (X, U, T_{\mathsf{x}_i}, Y, h)$, denoted by $\mathcal{I}_{cl}(\Sigma_1, \dots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, $Y := \prod_{i=1}^N Y_i^1$, and $h = \prod_{i=1}^N h_i^1$, with the following constraints:*

$$\forall i, j \in \{1, \dots, N\},\ i \neq j: \qquad w_{ji} = y_{ij}^2, \quad Y_{ij}^2 \subseteq W_{ji}. \tag{4.6.4}$$

Moreover, one has the conditional stochastic kernel $T_{\mathsf{x}} := \prod_{i=1}^N T_{\mathsf{x}_i}$ and the initial probability distribution $\pi := \prod_{i=1}^N \pi_i$.

An example of the interconnection of two MDPs Σ_1 and Σ_2 and that of their abstractions is illustrated in Figure 4.28.

Remark 4.6.8. *Definition 4.6.7 assumes that uncertainties affecting individual MDPs in a network $\mathcal{I}_{cl}(\Sigma_1, \dots, \Sigma_N)$ are independent and, thus, constructs T_{x} and π by taking products of T_{x_i} and π_i, respectively. This definition can be generalized for dependent uncertainties by using their joint distribution in the construction of T_{x} and π, in the same manner as we discussed in Remark 4.6.4 for expressing dependent uncertainties in concrete and abstract MDPs.*

4.6.2.2 Compositional Abstractions of Interconnected MDPs

We assume that we are given N MDPs as in (4.6.2) together with their corresponding abstractions $\widehat{\Sigma}_i = (\hat{X}_i, \hat{W}_i, \hat{U}_i, \hat{T}_{\mathsf{x}_i}, Y_i^1, Y_i^2, \hat{h}_i^1, \hat{h}_i^2)$ such that $\widehat{\Sigma}_i \preceq_{\epsilon_i}^{\delta_i} \Sigma_i$ for some relation

$\mathscr{R}_{x_i}$ and constants ϵ_i, δ_i. The next theorem shows the main compositionality result of the section.

Theorem 4.6.9. *Consider the interconnected MDP $\Sigma = \mathcal{I}_{cl}(\Sigma_1, \dots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ MDPs Σ_i. Suppose $\widehat{\Sigma}_i$ is (ϵ_i, δ_i)-stochastically simulated by Σ_i with the corresponding relations $\mathscr{R}_{x_i}$ and $\mathscr{R}_{w_i}$ and lifting $\bar{\mathscr{L}_i}$. If*

$$\mathsf{g}_i(x)\mathscr{R}_{w_i}\hat{\mathsf{g}}_i(\hat{x}), \quad \forall (x, \hat{x}) \in \mathscr{R}_{x_i}, \tag{4.6.5}$$

with interconnection constraint maps $\mathsf{g}_i, \hat{\mathsf{g}}_i$ defined as in (4.6.3), then $\widehat{\Sigma} = \mathcal{I}_{cl}(\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_N)$ is (ϵ, δ)-stochastically simulated by $\Sigma = \mathcal{I}_{cl}(\Sigma_1, \dots, \Sigma_N)$ with a relation $\mathscr{R}_x$ defined as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \mathscr{R}_x \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{bmatrix} \Leftrightarrow \begin{cases} x_1 \mathscr{R}_{x_1} \hat{x}_1, \\ \quad\vdots \\ x_N \mathscr{R}_{x_N} \hat{x}_N, \end{cases}$$

and constants $\epsilon = \sum_{i=1}^{N} \epsilon_i$, and $\delta = 1 - \prod_{i=1}^{N}(1 - \delta_i)$. Lifting $\bar{\mathscr{L}}$ and the interface ν are obtained by taking products $\bar{\mathscr{L}} = \prod_{i=1}^{N} \bar{\mathscr{L}_i}$ and $\nu = \prod_{i=1}^{N} \nu_i$, and then substituting interconnection constraints (4.6.4)*.*

Proof. We first show that the first condition in Definition 4.6.5 holds. For any $x = [x_1; \dots; x_N] \in X$ and $\hat{x} = [\hat{x}_1; \dots; \hat{x}_N] \in \hat{X}$ with $x\mathscr{R}_x\hat{x}$, one gets:

$$\begin{aligned} \|h(x) - \hat{h}(\hat{x})\| &= \|[h_1^1(x_1); \dots; h_N^1(x_N)] - [\hat{h}_1^1(\hat{x}_1); \dots; \hat{h}_N^1(\hat{x}_N)]\| \\ &\leq \sum_{i=1}^{N} \|h_i^1(x_i) - \hat{h}_i^1(\hat{x}_i)\| \leq \sum_{i=1}^{N} \epsilon_i. \end{aligned}$$

As seen, the first condition in Definition 4.6.5 holds with $\epsilon = \sum_{i=1}^{N} \epsilon_i$. The second condition is also satisfied as follows. For any $(x, \hat{x}) \in \mathscr{R}_x$, and $\hat{\nu} \in \hat{U}$, we have:

$$\begin{aligned} \bar{\mathscr{L}}\Big\{x'\mathscr{R}_x\hat{x}' \,\Big|\, x, \hat{x}, \hat{\nu}\Big\} &= \bar{\mathscr{L}}\Big\{x_i'\mathscr{R}_{x_i}\hat{x}_i',\ i \in \{1, 2, \dots, N\} \,\Big|\, x, \hat{x}, \hat{\nu}\Big\} \\ &= \prod_{i=1}^{N} \bar{\mathscr{L}_i}\Big\{x_i'\mathscr{R}_{x_i}\hat{x}_i' \,\big|\, \mathsf{g}_i(x), \hat{\mathsf{g}}_i(x), \hat{\nu}_i\Big\} \geq \prod_{i=1}^{N}(1 - \delta_i). \end{aligned}$$

The second condition in Definition 4.6.5 also holds with $\delta = 1 - \prod_{i=1}^{N}(1 - \delta_i)$ which completes the proof. □

Remark 4.6.10. *Note that Theorem 4.6.9 requires $\mathsf{g}_i(x)\mathscr{R}_{w_i}\hat{\mathsf{g}}_i(\hat{x})$ for any $(x, \hat{x}) \in \mathscr{R}_x$. This condition puts a restriction on the structure of the network and how the dynamics of MDPs are coupled in the network (cf. Remark 4.6.3). It is similar to the condition imposed in the disturbance bisimulation relation defined in [MSSM19, MSSM17].*

We provide the following example to illustrate our compositionality results.

Example 4.6.11. *Assume that we are given two linear stochastic control systems as*

$$\Sigma_i : \begin{cases} x_i(k+1) = A_i x_i(k) + B_i \nu_i(k) + D_i w_i(k) + R_i \varsigma_i(k), \\ y_i^1(k) = x_i(k), \\ y_i^2(k) = x_i(k), \qquad i \in \{1,2\}, \end{cases} \tag{4.6.6}$$

where the additive noise $\varsigma_i(\cdot)$ is a sequence of independent random vectors with multivariate standard normal distributions for $i \in \{1,2\}$, and $R_i, i \in \{1,2\}$, are invertible. Let $\widehat{\Sigma}_i$ be the abstraction of MDP (4.6.6) *as*

$$\widehat{\Sigma}_i : \begin{cases} \hat{x}_i(k+1) = \hat{A}_i \hat{x}_i(k) + \hat{B}_i \hat{\nu}_i(k) + \hat{D}_i \hat{w}_i(k) + \hat{R}_i \hat{\varsigma}_i(k), \\ \hat{y}_i^1(k) = \hat{x}_i(k), \\ \hat{y}_i^2(k) = \hat{x}_i(k). \end{cases}$$

Transition kernels of Σ_i and $\widehat{\Sigma}_i$ can be written as

$$\begin{aligned} T_{\mathsf{x}_i}(\cdot \,|\, x_i, \nu_i, w_i) &= \mathcal{N}(\cdot \,|\, A_i x_i + B_i \nu_i + D_i w_i, R_i R_i^T), \\ \hat{T}_{\mathsf{x}_i}(\cdot \,|\, \hat{x}_i, \hat{\nu}_i, \hat{w}_i) &= \mathcal{N}(\cdot \,|\, \hat{A}_i \hat{x}_i + \hat{B}_i \hat{\nu}_i + \hat{D}_i \hat{w}_i, \hat{R}_i \hat{R}_i^T), \ \forall i \in \{1,2\}, \end{aligned}$$

where $\mathcal{N}(\cdot \,|\, \mu_c, \tilde{\Sigma})$ indicates a normal distribution with mean μ_c and covariance matrix $\tilde{\Sigma}$.

Independent uncertainties. *If $\varsigma_i(\cdot)$ and $\hat{\varsigma}_i(\cdot)$ in concrete and abstract systems are independent, a candidate for the lifted measure is*

$$\begin{aligned} \mathscr{L}_{T_i}(\cdot \,|\, x_i, \hat{x}_i, \hat{\nu}_i, w_i, \hat{w}_i) = \mathcal{N}(\cdot \,|\, & A_i x_i + B_i \nu_i + D_i w_i, R_i R_i^T) \\ & \times \mathcal{N}(\cdot \,|\, \hat{A}_i \hat{x}_i + \hat{B}_i \hat{\nu}_i + \hat{D}_i \hat{w}_i, \hat{R}_i \hat{R}_i^T). \end{aligned}$$

Now we connect two subsystems with each other based on the interconnection constraint (4.6.4) *which are $w_i = x_{3-i}$ and $\hat{w}_i = \hat{x}_{3-i}$ for $i \in \{1,2\}$. For any $x = [x_1; x_2] \in X, \hat{x} = [\hat{x}_1; \hat{x}_2] \in \hat{X}, \nu = [\nu_1; \nu_2] \in U, \hat{\nu} = [\hat{\nu}_1; \hat{\nu}_2] \in \hat{U}$, compositional transition kernels for interconnected MDPs are*

$$T_{\mathsf{x}}(\cdot \,|\, x, \nu) = \mathcal{N}(\cdot \,|\, Ax + B\nu, RR^T), \ \hat{T}_{\mathsf{x}}(\cdot \,|\, \hat{x}, \hat{\nu}) = \mathcal{N}(\cdot \,|\, \hat{A}\hat{x} + \hat{B}\hat{\nu}, \hat{R}\hat{R}^T),$$

where $\nu := \nu(x, \hat{x}, \hat{\nu})$ and

$$\begin{aligned} A &= \begin{bmatrix} A_1 & D_1 \\ D_2 & A_2 \end{bmatrix}, \quad B = \mathsf{diag}(B_1, B_2), \quad R = \mathsf{diag}(R_1, R_2), \\ \hat{A} &= \begin{bmatrix} \hat{A}_1 & \hat{D}_1 \\ \hat{D}_2 & \hat{A}_2 \end{bmatrix}, \quad \hat{B} = \mathsf{diag}(\hat{B}_1, \hat{B}_2), \quad \hat{R} = \mathsf{diag}(\hat{R}_1, \hat{R}_2). \end{aligned} \tag{4.6.7}$$

Then the candidate lifted measure for interconnected MDPs is

$$\bar{\mathscr{L}}_T(\cdot \,|\, x, \hat{x}, \hat{\nu}) = \mathcal{N}(\cdot \,|\, Ax + B\nu, RR^T)\mathcal{N}(\cdot \,|\, \hat{A}\hat{x} + \hat{B}\hat{\nu}, \hat{R}\hat{R}^T).$$

Note that after connecting subsystems with each other using the proposed interconnection constraint in (4.6.4)*, internal inputs will disappear.*

Dependent uncertainties. *Suppose Σ_i and $\widehat{\Sigma}_i$ share the same noise $\varsigma_i(\cdot) = \hat{\varsigma}_i(\cdot)$. In this case, the candidate lifted measure for $i \in \{1,2\}$ is obtained by*

$$\begin{aligned}\bar{\mathscr{L}}_{T_i}(dx_i' \times d\hat{x}_i' \,|\, x_i, \hat{x}_i, \hat{\nu}_i, w_i, \hat{w}_i) &= \mathcal{N}(dx_i' \,|\, A_i x_i + B_i \nu_i + D_i w_i, R_i R_i^T) \\ &\times \delta_d(d\hat{x}_i' \,|\, \hat{A}_i \hat{x}_i + \hat{B}_i \hat{\nu}_i + \hat{D}_i \hat{w}_i + \hat{R}_i R_i^{-1}(x_i' - A_i x_i - B_i \nu_i - D_i w_i)),\end{aligned}$$

where $\delta_d(\cdot \,|\, \mathsf{c})$ indicates Dirac delta distribution centered at c. Now we connect two subsystems with each other. For any $x = [x_1; x_2] \in X, \hat{x} = [\hat{x}_1; \hat{x}_2] \in \hat{X}, \nu = [\nu_1; \nu_2] \in U, \hat{\nu} = [\hat{\nu}_1; \hat{\nu}_2] \in \hat{U}$, the candidate lifted measure for interconnected MDPs is

$$\mathscr{L}_T(dx' \times d\hat{x}' \,|\, x, \hat{x}, \hat{\nu}) = \mathcal{N}(dx' \,|\, Ax + B\nu, RR^T) \times \delta_d(d\hat{x}' \,|\, A\hat{x} + B\hat{\nu} - \bar{A}x + \tilde{A}x' - \bar{B}\nu),$$

where $A, B, R, \hat{A}, \hat{B}$ are defined as in (4.6.7)*, and*

$$\bar{A} = \begin{bmatrix} \hat{R}_1 R_1^{-1} A_1 & \hat{R}_1 R_1^{-1} D_1 \\ \hat{R}_2 R_2^{-1} D_2 & \hat{R}_2 R_2^{-1} A_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \hat{R}_1 R_1^{-1} & 0 \\ 0 & \hat{R}_2 R_2^{-1} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \hat{R}_1 R_1^{-1} B_1 & 0 \\ 0 & \hat{R}_2 R_2^{-1} B_2 \end{bmatrix}.$$

In the next subsection, we focus on the nonlinear class of stochastic control systems in (3.3.10) and construct its infinite and finite abstractions in a unified framework. We provide explicit inequalities for establishing Theorem 4.6.9, which gives a probabilistic relation after composition and enables us to get guarantees of Theorem 4.6.6 on the closeness of the composed systems and that of their abstractions.

4.6.3 Stochastic Control Systems with Slope Restrictions on Nonlinearity

Here we focus on the nonlinear class of stochastic control systems in (3.3.10) where $\varsigma(\cdot) \sim \mathcal{N}(0, \mathbb{I}_n)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies slope restrictions (3.3.11). Existing compositional abstraction results for this class of models are based on either model order reductions or finite MDPs as proposed in the previous sections. Our proposed results here combine these two approaches in one unified framework. In other words, our abstract model here is obtained by discretizing the state space of a reduced-order version of the concrete model.

4.6.3.1 Construction of Finite MDPs

Consider a nonlinear system $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$ and its reduced-order version $\widehat{\Sigma}_{\mathsf{r}} = (\hat{A}_{\mathsf{r}}, \hat{B}_{\mathsf{r}}, \hat{C}_{\mathsf{r}}^1, \hat{C}_{\mathsf{r}}^2, \hat{D}_{\mathsf{r}}, \hat{E}_{\mathsf{r}}, \hat{F}_{\mathsf{r}}, \hat{R}_{\mathsf{r}}, \varphi)$. Note that the index r in the whole thesis signifies the reduced-order version of the original model. We discuss the construction of $\widehat{\Sigma}_{\mathsf{r}}$ from Σ in Theorem 4.6.12 of the next subsection. Construction of a finite MDP from $\widehat{\Sigma}_{\mathsf{r}}$ follows the approach of Algorithm 1. Denote the state and input spaces of $\widehat{\Sigma}_{\mathsf{r}}$ respectively by $\hat{X}_{\mathsf{r}}, \hat{U}_{\mathsf{r}}, \hat{W}_{\mathsf{r}}$. We construct a finite MDP by selecting partitions $\hat{X}_{\mathsf{r}} = \cup_i \mathsf{X}_i$, $\hat{U}_{\mathsf{r}} = \cup_i \mathsf{U}_i$, and $\hat{W}_{\mathsf{r}} = \cup_i \mathsf{W}_i$, and choosing representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$, and $\bar{w}_i \in \mathsf{W}_i$, as abstract states and inputs. The finite abstraction of Σ is an MDP $\widehat{\Sigma} = (\hat{X}, \hat{W}, \hat{U}, \hat{T}_{\mathsf{x}}, Y, \hat{h})$, where

$$\hat{X} = \{\bar{x}_i, i = 1, \ldots, n_x\}, \; \hat{U} = \{\bar{u}_i, i = 1, \ldots, n_u\}, \; \hat{W} = \{\bar{w}_i, i = 1, \ldots, n_w\}.$$

The transition probability matrix $\hat{T}_{\mathsf{x}}$ is constructed according to the dynamics $\hat{x}(k+1) = \hat{f}(\hat{x}(k), \hat{\nu}(k), \hat{w}(k), \varsigma(k))$ with

$$\hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma) := \Pi_x(\hat{A}_{\mathsf{r}}\hat{x} + \hat{E}_{\mathsf{r}}\varphi(\hat{F}_{\mathsf{r}}\hat{x}) + \hat{B}_{\mathsf{r}}\hat{\nu} + \hat{D}_{\mathsf{r}}\hat{w} + \hat{R}_{\mathsf{r}}\varsigma), \tag{4.6.8}$$

where $\Pi_x : \hat{X}_{\mathsf{r}} \to \hat{X}$ is the map that assigns to any $\hat{x}_{\mathsf{r}} \in \hat{X}_{\mathsf{r}}$, the representative point $\bar{x} \in \hat{X}$ of the corresponding partition set containing $\hat{x}_{\mathsf{r}}$. The initial state of $\widehat{\Sigma}$ is also selected according to $\hat{x}_0 := \Pi_x(\hat{x}_{\mathsf{r}}(0))$ with $\hat{x}_{\mathsf{r}}(0)$ being the initial state of $\widehat{\Sigma}_{\mathsf{r}}$. The abstraction map Π_x satisfies the inequality (4.2.5).

4.6.3.2 Establishing Probabilistic Relations

In this subsection, we provide conditions under which $\widehat{\Sigma}$ is (ϵ, δ)-stochastically simulated by Σ, i.e., $\widehat{\Sigma} \preceq_{\epsilon}^{\delta} \Sigma$, with relations $\mathscr{R}_x$ and $\mathscr{R}_w$. Here we candidate relations

$$\mathscr{R}_x = \Big\{(x, \hat{x}) \,\Big|\, (x - \tilde{P}\hat{x})^T \tilde{M}(x - \tilde{P}\hat{x}) \le \epsilon^2\Big\}, \tag{4.6.9a}$$

$$\mathscr{R}_w = \Big\{(w, \hat{w}) \,\Big|\, (w - \tilde{P}_w\hat{w})^T \tilde{M}_w(w - \tilde{P}_w\hat{w}) \le \epsilon_w^2\Big\}, \tag{4.6.9b}$$

where $\tilde{P} \in \mathbb{R}^{n\times\hat{n}}$ and $\tilde{P}_w \in \mathbb{R}^{m\times\hat{m}}$ are matrices of appropriate dimensions (potentially with the lowest $\hat{n}$ and $\hat{m}$), and $\tilde{M}, \tilde{M}_w$ are positive-definite matrices.

The next theorem gives conditions for having $\widehat{\Sigma} \preceq_{\epsilon}^{\delta} \Sigma$ with relations (4.6.9a) and (4.6.9b).

Theorem 4.6.12. *Let* $\Sigma = (A, B, C^1, C^2, D, E, F, R, \varphi)$ *and* $\widehat{\Sigma}_r = (\hat{A}_r, \hat{B}_r, \hat{C}_r^1, \hat{C}_r^2, \hat{D}_r, \hat{E}_r, \hat{F}_r, \hat{R}_r, \varphi)$ *be two nonlinear systems with the same additive noise. Suppose* $\widehat{\Sigma}$ *is a finite MDP constructed from* $\widehat{\Sigma}_r$ *according to Subsection 4.6.3.1. Then* $\widehat{\Sigma}$ *is* (ϵ, δ)*-stochastically simulated by* Σ *with relations* (4.6.9a)-(4.6.9b) *if there exist matrices* K, Q, S, L_1, L_2 *and* $\tilde{R}$ *such that,* $\forall i \in \{1, 2\}$,

$$\tilde{M} \succeq C^{iT}C^i, \tag{4.6.10a}$$

$$\hat{C}_r^i = C^i\tilde{P}, \tag{4.6.10b}$$

$$\hat{F}_r = F\tilde{P}, \tag{4.6.10c}$$

$$E = \tilde{P}\hat{E}_r - B(L_1 - L_2), \tag{4.6.10d}$$

$$A\tilde{P} = \tilde{P}\hat{A}_r - BQ, \tag{4.6.10e}$$

$$D\tilde{P}_w = \tilde{P}\hat{D}_r - BS, \tag{4.6.10f}$$

$$\mathbb{P}\{(\tilde{H}_1 + \tilde{P}\tilde{H}_2)^T\tilde{M}(\tilde{H}_1 + \tilde{P}\tilde{H}_2) \le \epsilon^2\} \succeq 1 - \delta, \tag{4.6.10g}$$

where

$$\tilde{H}_1 = ((A + BK) + \bar{\delta}(BL_1 + E)F)(x - \tilde{P}\hat{x}) + (B\tilde{R} - \tilde{P}\hat{B}_r)\hat{\nu} + D(w - \tilde{P}_w\hat{w}) + (R - \tilde{P}\hat{R}_r)\varsigma,$$

$$\tilde{H}_2 = \hat{A}_r\hat{x} + \hat{E}_r\varphi(\hat{F}_r\hat{x}) + \hat{B}_r\hat{\nu} + \hat{D}_r\hat{w} + \hat{R}_r\varsigma - \Pi_x(\hat{A}_r\hat{x} + \hat{E}_r\varphi(\hat{F}_r\hat{x}) + \hat{B}_r\hat{\nu} + \hat{D}_r\hat{w} + \hat{R}_r\varsigma).$$

Proof. First we show that the first condition in Definition 4.6.2 holds for all $(x, \hat{x}) \in \mathscr{R}_x$. According to (4.6.10a) and (4.6.10b), we have

$$\|C^i x - \hat{C}^i_{\mathsf{r}} \hat{x}\|^2 = (x - \tilde{P}\hat{x})^T C^{iT} C^i (x - \tilde{P}\hat{x}) \leq (x - \tilde{P}\hat{x})^T \tilde{M} (x - \tilde{P}\hat{x}) \leq \epsilon^2,$$

for any $(x, \hat{x}) \in \mathscr{R}_x$. Now we proceed with showing the second condition. This condition requires that $\forall (x, \hat{x}) \in \mathscr{R}_x, \forall (w, \hat{w}) \in \mathscr{R}_w, \forall \hat{\nu} \in \hat{U}$, the next states $(x', \hat{x}')$ should also be in the relation $\mathscr{R}_x$ with the probability at least $1 - \delta$:

$$\mathbb{P}\{(x' - \tilde{P}\hat{x}')^T \tilde{M} (x' - \tilde{P}\hat{x}') \leq \epsilon^2\} \geq 1 - \delta.$$

Given any x, $\hat{x}$, and $\hat{\nu}$, we choose ν via the following interface function:

$$\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w}) := K(x - \tilde{P}\hat{x}) + Q\hat{x} + \tilde{R}\hat{\nu} + S\hat{w} + L_1 \varphi(Fx) - L_2 \varphi(F\tilde{P}\hat{x}). \quad (4.6.11)$$

By substituting dynamics of Σ and $\widehat{\Sigma}$, employing (4.6.10c)-(4.6.10f), and the definition of the interface function in (4.6.11), we simplify

$$\begin{aligned} x' - P\hat{x}' = {} & Ax + E\varphi(Fx) + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w}) + Dw + R\varsigma \\ & - \tilde{P}(\hat{A}_{\mathsf{r}}\hat{x} + \hat{E}_{\mathsf{r}}\varphi(\hat{F}_{\mathsf{r}}x) + \hat{B}_{\mathsf{r}}\hat{\nu} + \hat{D}_{\mathsf{r}}\hat{w} + \hat{R}_{\mathsf{r}}\varsigma) + \tilde{P}\tilde{H}_2, \end{aligned}$$

to

$$\begin{aligned} & (A + BK)(x - \tilde{P}\hat{x}) + (B\tilde{R} - \tilde{P}\hat{B}_{\mathsf{r}})\hat{\nu} + D(w - \tilde{P}_w\hat{w}) \\ & \qquad + (BL_1 + E)(\varphi(Fx) - \varphi(FP\hat{x}_{\mathsf{r}})) + (R - \tilde{P}\hat{R}_{\mathsf{r}})\varsigma + \tilde{P}\tilde{H}_2, \quad (4.6.12) \end{aligned}$$

with $\tilde{H}_2 = \hat{A}_{\mathsf{r}}\hat{x} + \hat{E}_{\mathsf{r}}\varphi(\hat{F}_{\mathsf{r}}\hat{x}) + \hat{B}_{\mathsf{r}}\hat{\nu} + \hat{D}_{\mathsf{r}}\hat{w} + \hat{R}_{\mathsf{r}}\varsigma - \Pi_x(\hat{A}_{\mathsf{r}}\hat{x} + \hat{E}_{\mathsf{r}}\varphi(\hat{F}_{\mathsf{r}}\hat{x}) + \hat{B}_{\mathsf{r}}\hat{\nu} + \hat{D}_{\mathsf{r}}\hat{w} + \hat{R}_{\mathsf{r}}\varsigma)$. From the slope restriction (3.3.11), one obtains

$$\varphi(Fx) - \varphi(F\tilde{P}\hat{x}) = \underline{\delta}(Fx - F\tilde{P}\hat{x}) = \underline{\delta}F(x - \tilde{P}\hat{x}), \quad (4.6.13)$$

where $\underline{\delta}$ is a function of x and $\hat{x}$, and takes values in the interval $[0, b]$. Using (4.6.13), the expression in (4.6.12) reduces to

$$\begin{aligned} & ((A + BK) + \underline{\delta}(BL_1 + E)F)(x - \tilde{P}\hat{x}) + (B\tilde{R} - \tilde{P}\hat{B}_{\mathsf{r}})\hat{\nu} + D(w - \tilde{P}_w\hat{w}) \\ & \quad + (R - \tilde{P}\hat{R}_{\mathsf{r}})\varsigma + \tilde{P}\tilde{H}_2. \end{aligned}$$

This gives the condition (4.6.10g) for having the probabilistic relation, which completes the proof. □

Remark 4.6.13. *Note that the condition* (4.6.10g) *is a chance constraint. We satisfy this condition by selecting the constant* c_ς *such that* $\mathbb{P}\{\varsigma^T\varsigma \leq c_\varsigma^2\} \geq 1 - \delta$, *and requiring* $(\tilde{H}_1 + \tilde{P}\tilde{H}_2)^T \tilde{M} (\tilde{H}_1 + \tilde{P}\tilde{H}_2) \leq \epsilon^2$ *for any* ς *with* $\varsigma^T\varsigma \leq c_\varsigma^2$. *Since* $\varsigma \sim (0, \mathbb{I}_n)$, $\varsigma^T\varsigma$ *has a chi-square distribution with* 2 *degrees of freedom. Thus,* $c_\varsigma = \mathcal{X}_2^{-1}(1 - \delta)$ *with* $\mathcal{X}_2^{-1}$ *being chi-square inverse cumulative distribution function with* 2 *degrees of freedom.*

4.6.4 Case Study

In this section, we demonstrate the effectiveness of the proposed results on the network of four nonlinear stochastic control systems (totally 12 dimensions), i.e., $\Sigma = \mathcal{I}_{cl}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ as illustrated in Figure 3.2. We want to construct finite MDPs from their reduced-order versions (together 4 dimensions).

The matrices of the system are given as

$$A_i = \begin{bmatrix} 0.7882 & 0.3956 & 0.8333 \\ 0.7062 & 0.7454 & 0.9552 \\ 0.6220 & 0.3116 & 0.4409 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0.7555 & 0.1557 & 0.3487 \\ 0.1271 & 0.9836 & 0.2030 \\ 0.4735 & 0.4363 & 0.4493 \end{bmatrix}, \quad C_i^1 = 0.01\mathbb{1}_3^T,$$
$$E_i = \begin{bmatrix} 0.6482; & 0.6008; & 0.6209 \end{bmatrix}, \quad F_i = \begin{bmatrix} 0.5146; & 0.8756; & 0.2461 \end{bmatrix}^T,$$
$$R_i = \begin{bmatrix} 0.4974; & 0.3339; & 0.4527 \end{bmatrix}, \tag{4.6.14}$$

for $i \in \{1,2,3,4\}$. Internal input and output matrices are also given by

$$C_{14}^2 = C_{23}^2 = C_{31}^2 = C_{42}^2 = 0.01\mathbb{1}_3^T, \quad D_{13} = D_{24} = D_{32} = D_{41} = \begin{bmatrix} 0.074; 0.010; 0.086 \end{bmatrix}.$$

We consider $\varphi_i(x) = sin(x)$, $\forall i \in \{1,\dots,4\}$. Then functions φ_i satisfy the slope condition (3.3.11) with $b = 1$. In the following, we first construct the reduced-order version of the given dynamic by satisfying conditions (4.6.10a)-(4.6.10f). We then establish relations between subsystems by fulfilling the condition (4.6.10g). Afterwards, we satisfy the compositionality condition (4.6.5) to get a relation on the composed system, and finally, we utilize Theorem 4.6.6 to provide the probabilistic closeness guarantee between the interconnected model and its constructed finite MDP.

Conditions (4.6.10a)-(4.6.10f) are satisfied by, $\forall i \in \{1,2,3,4\}$,

$$\begin{aligned} Q_i &= \begin{bmatrix} -1.6568; -1.2280; 1.9276 \end{bmatrix}, \quad S_i = \begin{bmatrix} 0.0775; 0.0726; -0.1759 \end{bmatrix}, \\ \tilde{P}_i &= \begin{bmatrix} 0.5931; 0.3981; 0.5398 \end{bmatrix}, \quad L_{1i} = \begin{bmatrix} -0.6546; -0.4795; -0.2264 \end{bmatrix}, \\ L_{2i} &= \begin{bmatrix} -0.1713; -0.0777; -0.1044 \end{bmatrix}, \quad \tilde{P}_{wi} = 1, \quad \tilde{M}_i = \mathbb{I}_3. \end{aligned}$$

Accordingly, matrices of reduced-order systems can be obtained as, $\forall i \in \{1,2,3,4\}$, $\forall \bar{i} \in \{1,2\}$,

$$\hat{A}_{\mathsf{r}i} = 0.5127, \hat{E}_{\mathsf{r}i} = 0.3, \hat{F}_{\mathsf{r}i} = 0.7866, \hat{C}_{\mathsf{r}i}^{\bar{i}} = 0.0371, \hat{D}_{\mathsf{r}i} = 0.1403, \hat{R}_{\mathsf{r}i} = 0.8386.$$

Moreover, we compute $\tilde{R}_i = (B_i^T \tilde{M}_i B_i)^{-1} B_i^T \tilde{M}_i \tilde{P}_i \hat{B}_{\mathsf{r}i}$, $i \in \{1,2,3,4\}$, as discussed in Remark 3.2.21, to make the chance constraint (4.6.10g) less conservative. By taking $\hat{B}_{\mathsf{r}i} = 2$, we have $\tilde{R}_i = [1.1418; 0.5182; 0.6965]$. The interface functions for $i \in \{1,2,3,4\}$ are acquired by (4.6.11) as

$$\begin{aligned} \nu_i = &\begin{bmatrix} -0.6665 & -0.3652 & -0.9680 \\ -0.4372 & -0.5536 & -0.5781 \\ -0.4012 & -0.1004 & -0.2612 \end{bmatrix} (x_i - \tilde{P}_i \hat{x}_i) + Q_i \hat{x}_i + \tilde{R}_i \hat{\nu}_i + S_i \hat{w}_i \\ &+ L_{1i} \varphi_i(F_i x_i) - L_{2i} \varphi_i(F_i \tilde{P}_i \hat{x}_i). \end{aligned}$$

We proceed with showing that the condition (4.6.10g) holds as well, using Remark 4.6.13. This condition can be satisfied via the S-procedure [BV04], which enables us to reformulate (4.6.10g) as an existence of $\underline{\lambda} \geq 0$ such that the matrix inequality

$$\underline{\lambda}_i \begin{bmatrix} \tilde{F}_{1i} & \tilde{g}_{1i} \\ \tilde{g}_{1i}^T & \tilde{h}_{1i} \end{bmatrix} - \begin{bmatrix} \tilde{F}_{2i} & \tilde{g}_{2i} \\ \tilde{g}_{2i}^T & \tilde{h}_{2i} \end{bmatrix} \succeq 0, \tag{4.6.15}$$

holds. Here, $\tilde{F}_{1i}$, $\tilde{F}_{2i}$ are symmetric matrices, $\tilde{g}_{1i}$, $\tilde{g}_{2i}$ are vectors, and $\tilde{h}_{1i}$, $\tilde{h}_{2i}$ are real numbers. We first bound the external input of abstract systems as $\hat{\nu}_i^2 \leq c_{\hat{\nu}i}$ and select $c_{\varsigma i} = \mathcal{X}_2^{-1}(1-\delta_i)$, for all $i \in \{1,2,3,4\}$. Then matrices $\tilde{F}_{1i}$ and $\tilde{F}_{2i}$, $\forall i \in \{1,2,3,4\}$, can be constructed as

$$\tilde{F}_{1i} = \begin{bmatrix} \tilde{M}_i & \mathbf{0}_{3\times 3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ * & * & \tilde{M}_{wi} & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix},$$

$$\tilde{F}_{2i} = \begin{bmatrix} \tilde{F}_{11i} & \tilde{F}_{12i} & \tilde{F}_{13i} & \tilde{F}_{14i} & \tilde{F}_{15i} & \tilde{F}_{16i} \\ * & \tilde{F}_{22i} & \tilde{F}_{23i} & \tilde{F}_{24i} & \tilde{F}_{25i} & \tilde{F}_{26i} \\ * & * & \tilde{F}_{33i} & \tilde{F}_{34i} & \tilde{F}_{35i} & \tilde{F}_{36i} \\ * & * & * & \tilde{F}_{44i} & \tilde{F}_{45i} & \tilde{F}_{46i} \\ * & * & * & * & \tilde{F}_{55i} & \tilde{F}_{56i} \\ * & * & * & * & * & \tilde{F}_{66i} \end{bmatrix}, \tag{4.6.16}$$

where

$\tilde{F}_{11i} = (A_i + B_iK_i)^T\tilde{M}_i(A_i + B_iK_i), \tilde{F}_{12i} = (A_i + B_iK_i)^T\tilde{M}_i(B_iL_{1i} + E_i)F_i,$
$\tilde{F}_{13i} = (A_i + B_iK_i)^T\tilde{M}_iD_i, \tilde{F}_{14i} = (A_i + B_iK_i)^T\tilde{M}_i(B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i}),$
$\tilde{F}_{15i} = (A_i + B_iK_i)^T\tilde{M}_i\tilde{P}_i, \tilde{F}_{16i} = (A_i + B_iK_i)^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}),$
$\tilde{F}_{22i} = F_i^T(B_iL_{1i} + E_i)^T\tilde{M}(B_iL_{1i} + E_i)F_i, \tilde{F}_{23i} = F_i^T(B_iL_{1i} + E_i)^T\tilde{M}_iD_i,$
$\tilde{F}_{24i} = F_i^T(B_iL_{1i} + E_i)^T\tilde{M}_i(B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i}), \tilde{F}_{25i} = F_i^T(B_iL_{1i} + E_i)^T\tilde{M}_i\tilde{P}_i,$
$\tilde{F}_{26i} = F_i^T(B_iL_{1i} + E_i)^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}), \tilde{F}_{33i} = D_i^T\tilde{M}_iD_i, \tilde{F}_{34i} = D_i^TM_i(B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i}),$
$\tilde{F}_{35i} = D_i^TM_i\tilde{P}_i, \tilde{F}_{36i} = D_i^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}), \tilde{F}_{44i} = (B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i})^T\tilde{M}_i(B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i}),$
$\tilde{F}_{45i} = (B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i})^T\tilde{M}_i\tilde{P}_i, \tilde{F}_{46i} = (B_i\tilde{R}_i - \tilde{P}_i\hat{B}_{\mathrm{r}i})^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}), \tilde{F}_{55i} = \tilde{P}_i^T\tilde{M}_i\tilde{P}_i,$
$\tilde{F}_{56i} = \tilde{P}_i^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}), \tilde{F}_{66i} = (R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i})^T\tilde{M}_i(R_i - \tilde{P}_i\hat{R}_{\mathrm{r}i}).$

Moreover, vectors and real numbers of the inequality (4.6.15) are obtained as

$$\tilde{g}_{1i} = \tilde{g}_{2i} = \mathbf{0}_{10}, \quad \tilde{h}_{1i} = -(\epsilon_i^2 + \epsilon_{wi}^2 + c_{\hat{\nu}i} + c_{\varsigma i} + \bar{\delta}_i), \tilde{h}_{2i} = -\epsilon_i^2. \tag{4.6.17}$$

By taking $\epsilon_i = 1.25$, $\epsilon_{w_i} = 0.05$, $c_{\hat{\nu}_i} = 0.25$, $\delta_i = 0.001$, $\bar{\delta}_i = 0.1$, $\underline{\lambda}_i = 0.347$, for all $i \in \{1, 2, 3, 4\}$, one can readily verify that the matrix inequality (4.6.15) holds. Then $\widehat{\Sigma}_i$ is (ϵ_i, δ_i)-stochastically simulated by Σ_i with relations

$$\mathscr{R}_{xi} = \Big\{(x_i, \hat{x}_i) \,\big|\, (x_i - \tilde{P}_i\hat{x}_i)^T \tilde{M}_i (x_i - \tilde{P}_i\hat{x}_i) \le \epsilon_i^2\Big\},$$
$$\mathscr{R}_{wi} = \Big\{(w_i, \hat{w}_i) \,\big|\, (w_i - \hat{w}_i)^2 \le \epsilon_{wi}^2\Big\},$$

for $i \in \{1, 2, 3, 4\}$. We proceed with showing that the compositionality condition in (4.6.5) holds, as well. To do so, by employing the S-procedure, one should satisfy the matrix inequality in (4.6.15) with the following matrices:

$$\tilde{F}_{1i} = \begin{bmatrix} \tilde{M}_i & -\tilde{M}_i\tilde{P}_i \\ * & \tilde{P}_i^T\tilde{M}_i\tilde{P}_i \end{bmatrix}, \quad \tilde{F}_{2i} = \begin{bmatrix} C_{\mathsf{r}i}^{2T}\tilde{M}_{wi}C_{\mathsf{r}i}^2 & -C_{\mathsf{r}i}^{2T}\tilde{M}_{wi}\tilde{P}_{wi}\hat{C}_{\mathsf{r}i}^2 \\ * & \hat{C}_{\mathsf{r}i}^{2T}\tilde{P}_{wi}^T\tilde{M}_{wi}\tilde{P}_{wi}\hat{C}_{\mathsf{r}i}^2 \end{bmatrix},$$
$$\tilde{g}_{1i} = \tilde{g}_{2i} = \mathbf{0}_4, \quad \tilde{h}_{1i} = -\epsilon_i^2, \quad \tilde{h}_{2i} = -\epsilon_{wi}^2,$$

for $i \in \{1, 2, 3, 4\}$. This condition is satisfiable with $\underline{\lambda}_i = 0.001, \forall i \in \{1, 2, 3, 4\}$, thus $\widehat{\Sigma}$ is (ϵ, δ)-stochastically simulated by Σ with $\epsilon = 6$, and $\delta = 0.003$. According to (4.6.1), we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\epsilon = 6$ during the time horizon $T_d = 10$ with the probability at least 96% ($\underline{\gamma} = 0.04$).

4.6.4.1 Comparison with Sections 3.4 and 4.3.1

To demonstrate the effectiveness of the proposed approach, let us now compare the guarantees provided by the approach of this section and by Sections 3.4 and 4.3.1. Note that the proposed results here are based on the δ-lifted relation while Sections 3.4 and 4.3.1 employ the dissipativity approach to provide a compositional methodology for constructing both infinite abstractions (reduced-order models) and finite MDPs in two consecutive steps. Since we are not able to satisfy the proposed matrix inequalities in (3.4.11) and (4.5.18) for the given system in (4.6.14), we change the system dynamics to have a fair comparison. In other words, in order to show the conservatism nature of the existing techniques in Sections 3.4 and 4.3.1, we provide another example and compare our techniques with the existing ones in great detail.

The matrices of the new system are given by

$$A_i = \mathbb{I}_5, \; B_i = \mathbb{I}_5, \; C_i^1 = 0.05\mathbb{1}_5^T, \; R_i = \mathbb{1}_5,$$

for $i \in \{1, 2, 3, 4\}$, where matrices E_i, F_i are identically zero. The internal input and output matrices are also given by:

$$C_{14}^2 = C_{23}^2 = C_{31}^2 = C_{42}^2 = 0.05\mathbb{1}_5^T, \quad D_{13} = D_{24} = D_{32} = D_{41} = 0.1\mathbb{1}_5.$$

Conditions (4.6.10a),(4.6.10b),(4.6.10e),(4.6.10f) are satisfied by:

$$M_i = \mathbb{I}_5, \; P_{xi} = \mathbb{1}_5, \; P_{wi} = 1, \; Q_i = \mathbb{1}_5, \; S_i = 0.1\mathbb{1}_5,$$

for $i \in \{1,2,3,4\}$. Accordingly, the matrices of reduced-order systems are acquired as

$$\hat{A}_{\mathsf{r}i} = 2, \hat{C}^{\bar{i}}_{\mathsf{r}i} = 0.25, \hat{D}_{\mathsf{r}i} = 0.2, \hat{R}_{\mathsf{r}i} = 0.97, \quad \forall i \in \{1,2,3,4\}, \forall \bar{i} \in \{1,2\}.$$

Moreover by taking $\hat{B}_{\mathsf{r}i} = 1$, we compute $\tilde{R}_i$, $i \in \{1,2,3,4\}$, as $\tilde{R}_i = \mathbb{1}_5$. The interface function for $i \in \{1,2,3,4\}$ is computed as:

$$\nu_i = -0.95\mathbb{I}_5(x_i - \mathbb{1}_5\hat{x}_i) + \mathbb{1}_5\hat{x}_i + \mathbb{1}_5\hat{\nu}_i + 0.1\mathbb{1}_5\hat{\omega}_i.$$

We proceed with showing that the condition (4.6.10g) holds, as well. By taking

$$\epsilon_i = 5, \epsilon_{w_i} = 0.75, c_{\hat{\nu}_i} = 0.25, \delta_i = 0.001, \bar{\delta}_i = 0.1, \underline{\lambda}_i = 0.825, \quad \forall i \in \{1,2,3,4\},$$

and by employing the S-procedure, one can readily verify that the condition (4.6.10g) holds. Then $\widehat{\Sigma}_i$ is (ϵ_i, δ_i)-stochastically simulated by Σ_i, for $i \in \{1,2,3,4\}$. Additionally, by applying S-procedure, one can readily verify that $\widehat{\Sigma}$ is (ϵ, δ)-stochastically simulated by Σ with $\epsilon = 20$, and $\delta = 0.005$. According to (4.6.1), we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\epsilon = 20$ during the time horizon $T_d = 5$ with the probability at least 97% ($\underline{\gamma} = 0.03$).

Now we apply the proposed results in Sections 3.4 and 4.3.1 for the same matrices of the new system and also employ the same ϵ and the discretization parameter $\bar{\delta}$. By applying the proposed results in Section 3.4 to construct the infinite abstraction $\widehat{\Sigma}_{\mathsf{r}}$, one can guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}_{\mathsf{r}}$ will exceed $\epsilon_1 = 15$ during the time horizon $T_d = 5$ with the probability at most 87.94%, i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\mathsf{r}\hat{a}_{\mathsf{r}}\hat{\nu}_{\mathsf{r}}}(k)\| \geq 15,\ \forall k \in [0,5]) \leq 87.94\,.$$

After applying the proposed results in Section 4.3.1 to construct the finite abstraction $\widehat{\Sigma}$ from $\widehat{\Sigma}_{\mathsf{r}}$, one can guarantee that the distance between outputs of $\widehat{\Sigma}_{\mathsf{r}}$ and of $\widehat{\Sigma}$ will exceed $\epsilon_2 = 5$ during the time horizon $T_d = 5$ with the probability at most 0.0117%, i.e.,

$$\mathbb{P}(\|\hat{y}_{\mathsf{r}\hat{a}_{\mathsf{r}}\hat{\nu}_{\mathsf{r}}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq 5, \quad \forall k \in [0,5]) \leq 0.0117.$$

By employing Proposition 4.2.16, one can guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will exceed $\epsilon = 20$ during the time horizon $T_d = 5$ with the probability at most 0.8911%, i.e.,

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq 20, \quad \forall k \in [0,5]) \leq 0.8911.$$

This means the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\epsilon = 20$ during the time horizon $T_d = 5$ with the probability at least 0.1089%. As seen, the provided results in this section outperform the ones proposed in Sections 3.4 and 4.3.1. More precisely, since the proposed approach here is presented in a unified framework than a two-step abstraction scheme which is the case in Sections 3.4 and 4.3.1, one only needs to check the proposed conditions one time, and consequently, the proposed approach here is less conservative.

4.7 Model-Free Reinforcement Learning

In this section, we propose a novel reinforcement learning scheme to synthesize policies for *unknown*, continuous-space MDPs. This scheme enables one to apply model-free, off-the-shelf reinforcement learning algorithms for finite MDPs to compute optimal strategies for the corresponding continuous-space MDPs without explicitly constructing the finite-state abstraction provided in Algorithm 1. The proposed approach is based on abstracting the process with a finite MDP with *unknown* transition probabilities, synthesizing strategies over the abstract MDP, and then mapping the results back over the concrete continuous-space MDP with approximate optimality guarantees. The system properties of interest belong to the scLTL as discussed in Subsection 3.4.4, and the synthesis requirement is to maximize the probability of satisfaction within a given bounded time horizon. A key contribution of the section is to leverage the classical convergence results for the reinforcement learning on finite MDPs and provide control strategies maximizing the probability of satisfaction over unknown, continuous-space MDPs by providing probabilistic closeness guarantees.

Consider the discrete-time stochastic control system $\Sigma = (X, U, \varsigma, f)$ with the *finite* input space U. Note that since any input sequence will be implemented by a digital controller, without loss of generality and from now on in this section, we assume that the input space U is finite. In the following, we consider scLTL properties ϕ as in Subsection 3.4.4 since their verification can be performed via a reachability property over a deterministic finite-state automaton (DFA) $\mathcal{A}_\phi$ such that $\mathcal{L}_f(\phi) = \mathcal{L}(\mathcal{A}_\phi)$ [KV01]. Given a policy $\bar{\rho}$, we can define the probability that an output trajectory of Σ satisfies an scLTL property ϕ over the time horizon $[0, T_d]$, i.e., $\mathbb{P}(\omega_f \in \mathcal{L}(\mathcal{A}_\phi)$ s.t. $|\omega_f| \leq T_d + 1)$, with $|\omega_f|$ denoting the length of ω_f [DLT08a].

We should emphasize that there is no *closed-form solution* for computing optimal policies enforcing scLTL specifications over *continuous-space* MDPs. One can employ the approximation approaches, discussed before, to synthesize those policies which, however, suffer from the curse of dimensionality and require knowing precisely the probabilistic evolution of states in the models. Instead, we propose in this section an RL approach providing policies for unknown, continuous-space MDPs while providing *quantitative guarantees* on the satisfaction of properties.

4.7.1 Controller Synthesis for Unknown Continuous-Space MDPs

We are interested in automatically synthesizing controllers for unknown continuous-space MDPs whose requirements are provided as scLTL specifications. Given a discrete-time stochastic control system $\Sigma = (X, U, \varsigma, f)$, where f and distribution of ς are unknown, and given an scLTL formula ϕ, we wish to synthesize a Markov policy enforcing the property ϕ over Σ with the probability of satisfaction within a guaranteed threshold from the unknown optimal probability.

In order to provide any formal guarantee, we need to make further assumptions about the dt-SCS. In particular, we assume that the dynamical system in (2.3.3) is Lipschitz-continuous with a constant $\mathscr{H}$. We follow the results of [SA14a, SA15b] for the characterization of the Lipschitz constant. Consider the dynamical system in (2.3.3) where $\varsigma(\cdot)$ is i.i.d. with the known distribution $t_\varsigma(\cdot)$. Suppose that the vector field f is continuously differentiable and that the matrix $\frac{\partial f}{\partial \varsigma}$ is invertible. Then the *implicit function theorem* guarantees the existence and uniqueness of a function $\bar{g} : X \times X \times U \to \mathcal{V}_\varsigma$ such that $\varsigma(k) = \bar{g}(x(k+1), x(k), \nu(k))$. In this case, the conditional density function is:

$$t_x(x' \mid x, \nu) = \left|\det\left[\frac{\partial \bar{g}}{\partial x'}(x', x, \nu)\right]\right| t_\varsigma(\bar{g}(x', x, \nu)).$$

The Lipschitz constant $\mathscr{H}$ is specified by the dependence of the function $\bar{g}(x', x, \nu)$ on the variable x. As a special case, consider a nonlinear system with an additive noise

$$f(x, \nu, \varsigma) = f_a(x, \nu) + \varsigma.$$

Then the invertibility of $\frac{\partial f}{\partial \varsigma}$ is guaranteed and $\bar{g}(x', x, \nu) = x' - f_a(x, \nu)$. In this case, $\mathscr{H}$ is the product of the Lipschitz constant of $t_\varsigma(\cdot)$ and $f_a(\cdot)$.

The next example provides a systematic way of computing $\mathscr{H}$ for a class of linear MDPs.

Example 4.7.1. *Consider a dt-SCS Σ with linear dynamics $x(k+1) = Ax(k) + B\nu(k) + \varsigma(k)$, $A = [a_{ij}]$ where $\varsigma(k)$ are i.i.d. for $k = 0, 1, 2, \dots$ with normal distribution having the zero mean and the covariance matrix $\mathsf{diag}(\bar{\sigma}_1, \dots, \bar{\sigma}_n)$. Then one obtains $\mathscr{H} = \sum_{i,j} \frac{2|a_{ij}|}{\bar{\sigma}_i\sqrt{2\bar{\pi}}}$ with $\bar{\pi} = 3.14159$. Note that for the computation of the error, it is sufficient to know an upper bound on entries of the matrix A and a lower bound on the standard deviation of the noise $\bar{\sigma}$.*

An alternative way of computing the Lipschitz constant $\mathscr{H}$ is to estimate it from sample trajectories of Σ. This can be done by first constructing a non-parametric estimation of the conditional density function using techniques from [Sco92] and then compute the Lipschitz constant numerically using the derivative of the estimated conditional density function.

Now we have all required ingredients to state the main problem we solve in this section.

Problem 4.7.2. *Let ϕ be an scLTL formula and $\Sigma = (X, U, \varsigma, f)$ a continuous-space MDP, where f and distribution of ς are unknown, but the Lipschitz constant $\mathscr{H}$ is known. Synthesize a Markov policy that satisfies the property ϕ over Σ with the probability within a guaranteed threshold from the unknown optimal probability.*

Prior to proposing our solution to this problem, we first present the following theorem borrowed from [SA13a, Sou14] that shows the closeness between a continuous-space MDP Σ and its finite abstraction $\widehat{\Sigma}$ in a probabilistic setting. We will then exploit the result of this theorem in the next subsection to provide a reinforcement learning-based solution to Problem 4.7.2.

Theorem 4.7.3. *Let $\Sigma = (X, U, \varsigma, f)$ be a continuous-space MDP and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f})$ its finite abstraction as constructed in Algorithm 1. For a given scLTL specification ϕ,* *and for any policy $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$ (denoted by $\widehat{\Sigma}_{\hat{\nu}}$), the closeness between two systems can be acquired as*

$$|\mathbb{P}(\Sigma_{\hat{\nu}} \vDash \phi) - \mathbb{P}(\widehat{\Sigma}_{\hat{\nu}} \vDash \phi)| \leq \tilde{\varepsilon}, \quad \text{with } \tilde{\varepsilon} := T_d \bar{\delta} \mathscr{H} \underline{\mathscr{L}}, \tag{4.7.1}$$

where T_d is the finite time horizon, $\bar{\delta}$ is the state discretization parameter, $\mathscr{H}$ is the Lipschitz constant of the stochastic kernel, and $\underline{\mathscr{L}}$ is the Lebesgue measure of the specification set. Moreover, optimal probabilities of satisfying the specification over the two models are different with a distance of at most $2\tilde{\varepsilon}$:

$$\big| \max_{\nu \in \bar{\Pi}_M} \mathbb{P}(\Sigma_\nu \vDash \varphi) - \max_{\hat{\nu} \in \hat{\bar{\Pi}}_M} \mathbb{P}(\widehat{\Sigma}_{\hat{\nu}} \vDash \varphi) \big| \leq 2\tilde{\varepsilon}, \tag{4.7.2}$$

where $\bar{\Pi}_M$ and $\hat{\bar{\Pi}}_M$ are the set of Markov policies over Σ and $\widehat{\Sigma}$, respectively.

Remark 4.7.4. *Note that in order to employ Theorem 4.7.3, one can first a-priori fix the desired threshold $\tilde{\varepsilon}$ in (4.7.1). According to the values of $\mathscr{H}$, $\underline{\mathscr{L}}$, and T_d, one computes the required discretization parameter as $\bar{\delta} = \frac{\tilde{\varepsilon}}{T_d \mathscr{H} \underline{\mathscr{L}}}$. For instance in the case of a uniform quantizer, one can divide each dimension of the set X into intervals of size $\bar{\delta}/\sqrt{n}$ with n being the dimension of the set.*

4.7.2 Controller Synthesis via Reinforcement Learning

It follows from Theorem 4.7.3 that one can construct a finite abstraction $\widehat{\Sigma}$ from a given continuous-space dt-SCS Σ with known stochastic kernels such that the optimal probability of satisfaction of an scLTL specification ϕ for T_d steps in $\widehat{\Sigma}$ is no more than $2\tilde{\varepsilon}$-worse that the optimal strategy in Σ. Hence, given a dt-SCS Σ with known stochastic kernels, an scLTL property ϕ, and a time-horizon T_d, a $2\tilde{\varepsilon}$-optimal strategy to satisfy ϕ in T_d steps can be computed using a suitable finite MDP with $\bar{\delta}$ as the state discretization parameter. This problem can be solved using the dynamic programming over the product of $\widehat{\Sigma}$ and $\mathcal{A}_\phi$ by providing a scalar reward for all transitions once a final state of the DFA $\mathcal{A}_\phi$ is reached.

On the other hand, when the stochastic kernels are unknown, Theorem 4.7.3 still provides the correct probabilistic bound given a discretization parameter $\bar{\delta}$ if the Lipschitz constant $\mathscr{H}$ is known. This observation enables us to employ the reinforcement learning algorithm over the underlying discrete MDP without explicitly constructing the abstraction by restricting the observations of the reinforcement learner to the closest representative point in the set of partitions (cf. Algorithm 1). The model-free reinforcement learning can be used under such observations by using DFA $\mathcal{A}_\phi$ to provide scalar rewards by following an approach similar to the one presented in [HPS+19b] to combine the automaton and MDP. Observations of the MDP are used by an *interpreter* process to compute a run of the DFA. When the DFA reaches a final state, the interpreter gives the reinforcement learner a positive reward and the training episode terminates. Any RL algorithm that maximizes this probabilistic reward is guaranteed [HPS+19b] to converge to a policy that maximizes the probability of satisfaction of the scLTL objective.

It follows that any converging reinforcement learning algorithm [JJS94, BM00] over such finite observation space then converges to a 2ε-optimal strategy over the concrete dt-SCS Σ thanks to Theorem 4.7.3. We summarize the proposed solution in the following theorem.

Theorem 4.7.5. *Let ϕ be an scLTL formula, $\tilde{\varepsilon} > 0$, and $\Sigma = (X, U, \varsigma, f)$ a continuous-space MDP, where f and distribution of ς are unknown but the Lipschitz constant $\mathscr{H}$ as defined in Theorem 4.7.3 is known. For a discretization parameter $\bar{\delta}$ satisfying $T_d\bar{\delta}\mathscr{H}\underline{\mathscr{L}} \leq \tilde{\varepsilon}$, a convergent model-free reinforcement learning algorithm (e.g., Q-learning [BM00] or TD(λ) [JJS94]) over $\widehat{\Sigma}$ with a reward function guided by the DFA $\mathcal{A}_\phi$, converges to a $2\tilde{\varepsilon}$-optimal strategy over Σ.*

Before illustrating our results via some experiments, we elaborate on the dimension dependency in our proposed RL techniques compared to the abstraction-based ones. Assuming a uniform quantizer, the finite MDP constructed by Algorithm 1 is a matrix with a dimension of $(n_x \times n_\nu) \times n_x$. Computing this matrix is one of the bottlenecks in abstraction-based approaches since an n-dimensional integration has to be done numerically for each entries of this matrix. Moreover, n_x (i.e., the cardinality of the state set) grows exponentially with the dimension n. Once this matrix is computed, it is employed for the dynamic programming on a vector of size $(n_x \times n_\nu)$. This is a second bottleneck of the process. On the other hand, by employing the proposed RL approach, the curse of dimensionality reduces to only *learning* the vector of size $(n_x \times n_\nu)$ without having to compute the full matrix. Moreover, the abstraction-based techniques need to precisely know the probabilistic evolution of the states in the models, whereas in this section we only need to know the Lipschitz constant $\mathscr{H}$.

Concerning the trade-off between the iteration count, discretization size, and performance, we should mention that by decreasing the discretization parameter, the closeness error in Theorem 4.7.3 is reduced. On the other hand, one needs more training episodes as the size of the problem increases. Note that in our proposed setting, we do not need to compute transition probabilities $\hat{T}_{\mathsf{x}}$ in Algorithm 1, since we directly learn the value functions using RL.

4.7.3 Case Studies

Table 4.3 shows a comparison of Q-learning to the computed optimal probabilities. Two systems are analyzed. The first is the model of the room-temperature control system as

$$\Sigma : \tilde{T}(k+1) = (1 - 2\eta - \beta - \bar{\theta}\nu(k))\tilde{T}(k) + \bar{\theta}\tilde{T}_h\nu(k) + \beta\tilde{T}_e + 0.3162\varsigma(k),$$

where $\eta = 0$, $\beta = 0.022$, $\bar{\theta} = 0.05$, $\tilde{T}_e = -1\,°C$ and $\tilde{T}_h = 50\,°C$. Moreover, $T(k)$ and $\nu(k)$ are taking values in $[19, 21]$ and a finite input set $\{0.03, 0.09, 0.15, 0.21, 0.27, 0.33, 0.39, 0.45, 0.51, 0.57\}$, respectively. The objective of the controller is to keep the temperature between $19°C$ and $21°C$.

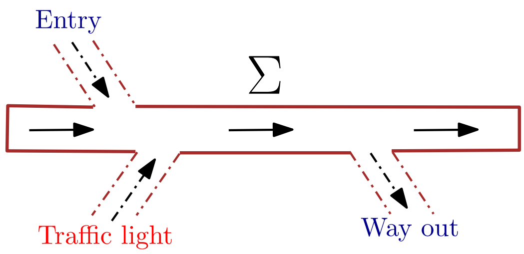

Figure 4.29: Diagram of the traffic cell model.

Table 4.3: Q-learning results.

$\bar{\delta}$	Room					Traffic				
	p_r	p_*	$\tilde{\varepsilon}$	p_l	p_u	p_r	p_*	$\tilde{\varepsilon}$	p_l	p_u
0.01	0.9698	0.9753	0.2468	0.7285	1.0	0.9856	0.9995	0.0160	0.9835	1.0
0.02	0.9745	0.9753	0.4936	0.4817	1.0	0.9975	0.9995	0.0319	0.9676	1.0
0.05	0.9543	0.9753	1.2339	0.0000	1.0	0.9993	0.9995	0.0798	0.9197	1.0
0.1	0.9779	0.9754	2.4678	0.0000	1.0	0.9999	0.9995	0.1596	0.8399	1.0
0.2	0.9732	0.9743	4.9357	0.0000	1.0	0.9999	0.9995	0.3193	0.6802	1.0

The second system is the model of the road traffic cell (Figure 4.29) with the following dynamics:

$$\Sigma : x(k+1) = (1 - \frac{\tau v}{\tilde{l}} - \tilde{q})x(k) + 6\nu(k) + 1.9494\varsigma(k) + 3,$$

where the length of a cell is 0.5 kilometers [km], and the flow speed of the vehicles is 100 kilometers per hour [km/h]. Moreover, during the sampling time interval $\tau = 6.48$ seconds, it is assumed that 6 vehicles pass the entry controlled by the traffic light, 3 vehicles go into the entry of the cell, and one quarter of vehicles goes out on the exit of the cell (the ratio denoted by $\tilde{q}$). The road has an input ramp regulated by a traffic light. The control strategy turns the light red and green trying to keep the density of the traffic fewer than 20 in the cell, while allowing as many cars as possible to enter the road.

For each model, five different discretization steps ($\bar{\delta}$) are considered and for each value of $\bar{\delta}$ the probabilities of satisfaction of the safety objectives are reported in the columns labeled p_r. These probabilities are the Q-values of the initial state of the finite-state MDP for the policy computed by Q-learning after 10^6 episodes. The objective is to keep the system safe for at least 10 steps. For the comparison, the optimal probability p_* for a time-dependent policy is reported assuming that we know the exact dynamics for these two examples. Note that we compute p_* using the dynamic programming over constructed finite MDPs as proposed in Algorithm 1. The optimal probability p_* reported in Table 4.3 corresponds to the same initial condition that is utilized in the learning process. The optimal probability for the original *continuous-space* MDP is always within an interval $[p_l, p_u]$ centered at p_* and with a radius $\tilde{\varepsilon}$ as reported in

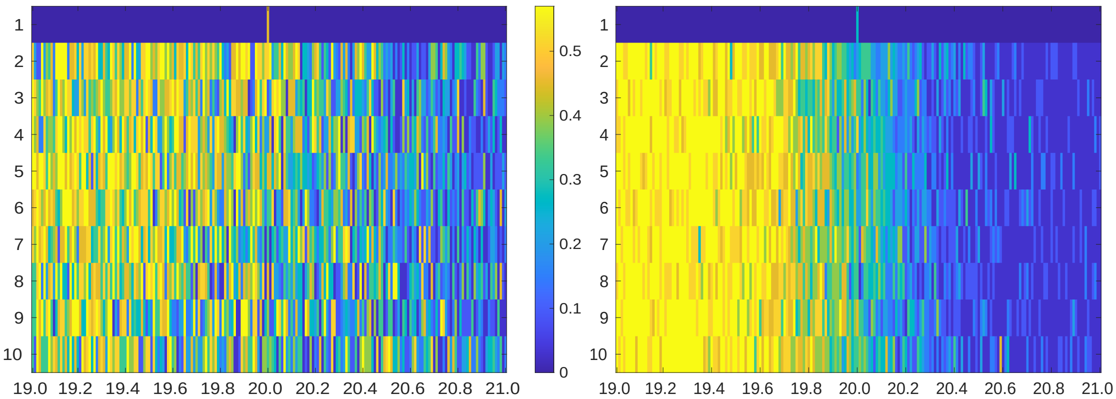

Figure 4.30: Room temperature control: A heat-map visualization of strategies learned via Reinforcement Learning after 10^5 episodes (left) and after $8 \cdot 10^6$ episodes (right). The X axis represents the room temperature in ℃, while the Y axis represents time steps $1 \leq k \leq 10$. The action suggested by the strategy is in the finite input set $\{0.03, 0.09, 0.15, 0.21, 0.27, 0.33, 0.39, 0.45, 0.51, 0.57\}$ and is color-coded according to the map shown in the middle: Bright yellow and deep blue represent maximum and minimum heats. In the first step, the strategies are only defined for the initial state; this causes the blue bands at the top.

Table 4.3. One can readily see from Table 4.3 that as the discretization parameter $\bar{\delta}$ decreases, the size of this interval shrinks, which implies that the optimal probability for the original *continuous-space* MDP converges to p_*. While finer abstractions give better theoretical guarantees, for a fixed number of episodes it is easier to learn good strategies for coarser abstractions. This is reflected in Table 4.3, where the values of p_r do not necessarily get better with smaller values of $\bar{\delta}$. However, by increasing the number of episodes, the strategies converge toward the optimal one, as illustrated in Figure 4.30, which visualizes room-temperature control strategies computed by the Q-learning after different numbers of episodes. Note that in Table 4.3, the error bound $\tilde{\varepsilon}$ exceeds one for $\bar{\delta} \geq 0.05$ in the room-temperate control example, which is not a useful probability bound for the *continuous-space* MDP. However, we prefer to report the corresponding values of p_r and p_* so that they can still be compared.

4.7.3.1 Autonomous Vehicle

The case studies discussed so far are the representative of what can be solved by discretization and tabular methods like Q-learning. Relaxing those constraints, we were able to apply a deep deterministic policy gradient (DDPG) [LHP+15] to a hybrid 7-dimensional *nonlinear* single track (ST) model of a BMW 320i car to synthesize a reach-avoid controller. The model is borrowed from [Alt19, Section 5.1] by including the stochasticity inside the dynamics as additive noises:

For $|x_4(k)| < 0.1$:

$$
\begin{aligned}
x_i(k+1) &= x_i(k) + \tau \bar{a}_i(k) + 0.5\varsigma_i(k), \quad i \in \{1,\dots,7\}\backslash\{3,4\},\\
x_3(k+1) &= x_3(k) + \tau \text{Sat}_1(\nu_1) + 0.5\varsigma_3(k),\\
x_4(k+1) &= x_4(k) + \tau \text{Sat}_2(\nu_2) + 0.5\varsigma_4(k),
\end{aligned}
$$

and for $|x_4(k)| \geq 0.1$:

$$
\begin{aligned}
x_i(k+1) &= x_i(k) + \tau \bar{b}_i(k) + 0.5\varsigma_i(k), \quad i \in \{1,\dots,7\}\backslash\{3,4\},\\
x_3(k+1) &= x_3(k) + \tau \text{Sat}_1(\nu_1) + 0.5\varsigma_3(k),\\
x_4(k+1) &= x_4(k) + \tau \text{Sat}_2(\nu_2) + 0.5\varsigma_4(k),
\end{aligned}
\tag{4.7.3}
$$

where,

$$
\begin{aligned}
&\bar{a}_1 = x_4\cos(x_5(k)), \quad \bar{a}_2 = x_4\sin(x_5(k)), \quad \bar{a}_5 = \frac{x_4}{l_{wb}}\tan(x_3(k)),\\
&\bar{a}_6 = \frac{\nu_2(k)}{l_{wb}}\tan(x_3(k)) + \frac{x_4}{l_{wb}\cos^2(x_3(k))}\nu_1(k), \quad a_7 = 0,\\
&\bar{b}_1 = x_4(k)\cos(x_5(k) + x_7(k)), \quad \bar{b}_2 = x_4(k)\sin(x_5(k) + x_7(k)), \quad \bar{b}_5 = x_6(k),\\
&\bar{b}_6 = \frac{\bar{\mu}_f \bar{m}}{I_z(l_r + l_f)}(l_f C_{S,f}(g l_r - \nu_2(k)h_{cg})x_3(k) + (l_r C_{S,r}(g l_f + \nu_2(k)h_{cg}) - l_f C_{S,f}(g l_r\\
&\qquad - \nu_2(k)h_{cg}))x_7(k) - (l_f^2 C_{S,f}(g l_r - \nu_2(k)h_{cg}) + l_r^2 C_{S,r}(g l_f + \nu_2(k)h_{cg}))\frac{x_6(k)}{x_4(k)}),\\
&\bar{b}_7 = \frac{\bar{\mu}_f}{x_4(k)(l_r + l_f)}(C_{S,f}(g l_r - \nu_2(k)h_{cg})x_3(k) + (C_{S,r}(g l_f + \nu_2(k)h_{cg}) + C_{S,f}(g l_r\\
&\qquad - \nu_2(k)h_{cg}))x_7(k) - (l_f C_{S,f}(g l_r - \nu_2(k)h_{cg}) - l_r C_{S,r}(g l_f + \nu_2(k)h_{cg}))\frac{x_6(k)}{x_4(k)}) - x_6(k).
\end{aligned}
$$

Moreover, $\text{Sat}_1(\cdot)$ and $\text{Sat}_2(\cdot)$ are input saturation functions introduced in [Alt19, Section 5.1], x_1 and x_2 are position coordinates, x_3 is the steering angle, x_4 is the heading velocity, x_5 is the yaw angle, x_6 is the yaw rate, and x_7 is the slip angle. Variables ν_1 and ν_2 are inputs and they control the steering angle and heading velocity, respectively.

The model takes into account the tire slip making it a good candidate for studies that consider planning of evasive maneuvers that are very close to physical limits. We consider an update period $\tau = 0.001$ seconds and the following parameters for a BMW 320i car: $l_{wb} = 2.5789$ as the wheelbase, $\bar{m} = 1093.3$ [kg] as the total mass of the vehicle, $\bar{\mu}_f = 1.0489$ as the friction coefficient, $l_f = 1.156$ [m] as the distance from the front axle to the center of gravity (CoG), $l_r = 1.422$ [m] as the distance from the rear axle to CoG, $h_{cg} = 0.574$ [m] as the hight of CoG, $I_z = 1791.6$ [kg m^2] as the moment of inertia for the entire mass around z axis, $C_{S,f} = 20.89$ [1/rad] as the front cornering stiffness coefficient, and $C_{S,r} = 20.89$ [1/rad] as the rear cornering stiffness coefficient.

We consider a bounded version of the state set $X := [0,84] \times [0,6] \times [-0.18, 0.18] \times [12,21] \times [-0.5,0.5] \times [-0.8,0.8] \times [-0.1,0.1]$, and a quantized version of the input set $U := [-0.4,0.4] \times [-4,4]$ with a very fine quantization parameter.

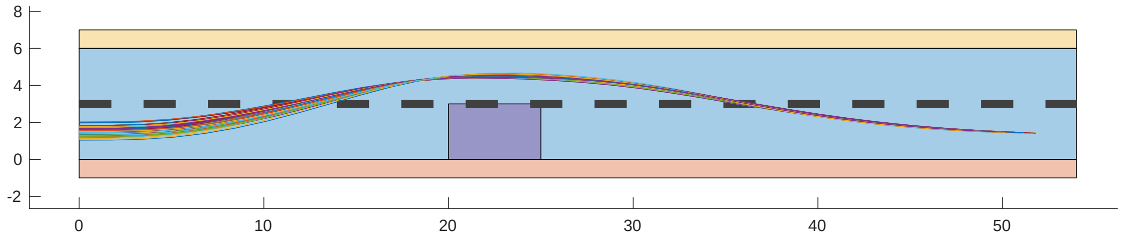

Figure 4.31: Trajectories of 100 simulations of the RL-synthesized controller for a 7-dimensional model of a BMW 320*i* car trained using DDPG. The road is 6 meter wide and 50 meter long, and the length of the car is 4.508 meters and its width is 1.610 meters.

We are interested in an autonomous operation of the vehicle on a highway. Consider a situation on a two-lane highway when an accident suddenly happens on the same lane on which our vehicle is traveling. The vehicle's controller should find a safe maneuver to avoid the crash with the next-appearing obstacle.

Figure 4.31 shows simulations from 100 samples with varying initial positions and initial heading velocities (16–18 m/s) for the learned controller. Though convergence guarantees are not available for DDPG and most RL algorithms with nonlinear function approximations, breakthroughs in this direction (e.g., SBEED by [DSL$^+$17]) will expand the applicability of our results to more complex safety-critical applications.

4.8 Summary

In this chapter, we have proposed compositional frameworks for the construction of finite MDPs as finite abstractions of given (reduced-order) systems. We showed that if the original system is incrementally input-to-state stable (or incrementally passivable in the dissipativity setting), one can construct finite MDPs of original systems for the general setting of nonlinear stochastic control systems. We have also proposed novel frameworks for the construction of finite MDPs for some particular classes of nonlinear stochastic systems whose nonlinearities satisfy a slope restriction or (in a more general form) an incremental quadratic inequality. We extended our results from control systems to switched ones whose switching signals accept dwell-time condition with multiple Lyapunov functions. Moreover, we proposed relaxed versions of small-gain and dissipativity approaches in which the stabilizability of individual subsystems for providing the compositionality results is not necessarily required.

We have also proposed a compositional technique for the construction of both infinite and finite abstractions in a unified framework via notions of approximate probabilistic relations. We showed that the unified compositional framework is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions. We finally proposed a novel model-free reinforcement learning framework to synthesize policies for unknown, continuous-space MDPs. We provided probabilistic closeness guarantees between unknown original models and that of their finite MDPs. We discussed that via the proposed model-free learning framework not only one can synthesize controllers for unknown stochastic systems, but also the curse of dimensionality problem is remarkably mitigated.

5 AMYTISS: Parallel Automated Controller Synthesis for Large-Scale Stochastic Systems

5.1 Introduction

To alleviate the computational complexity arising from the abstraction construction proposed in the previous chapter, one promising solution is to employ high-performance computing (HPC) platforms together with cloud-computing services to mitigate the state-explosion problem which is always the case in analyzing large-scale stochastic systems. In this chapter, we develop a software tool, called AMYTISS, in C++/OpenCL that provides scalable parallel algorithms to first construct finite MDPs from discrete-time stochastic control systems and then synthesize automatically their controllers satisfying complex logic properties including safety, reachability, and reach-avoid specifications. This tool significantly improves performances w.r.t. the computation time and memory usage by the parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators (HWAs). In other words, unlike all existing tools, AMYTISS offers highly scalable, distributed execution of parallel algorithms utilizing all available processing elements (PEs) in any heterogeneous computing platform. To the best of our knowledge, AMYTISS is the only tool of this kind for the stochastic systems that is able to utilize this type of compute units (CUs), simultaneously.

5.1.1 Related Literature

There exist several software tools on the verification and synthesis of stochastic systems with different classes of models. SReachTools [VGO19] performs the stochastic reachability analysis for linear, potentially time-varying, discrete-time stochastic systems. ProbReach [SZ15] is a tool for verifying the probabilistic reachability for stochastic hybrid systems. SReach [WZK+15] solves probabilistic bounded reachability problems for two classes of models: (i) nonlinear hybrid automata with the parametric uncertainty, and (ii) probabilistic hybrid automata with the additional randomness for both transition probabilities and variable resets. Modest Toolset [HH14] performs the modeling and analysis for hybrid, real-time, distributed and stochastic systems. Two competitions on tools for the formal verification and policy synthesis of stochastic models are organized with reports in [ABC+18, ABC+19].

Table 5.1: Comparison between AMYTISS, FAUST^2 and StocHy based on native features.

Aspect	**FAUST^2**	**StocHy**	**AMYTISS**
Platform	CPU	CPU	All platforms
Algorithms	Serial on HPC	Serial on HPC	Parallel on HPC
Model	Stochastic control systems: linear, bilinear	Stochastic hybrid systems: linear, bilinear	Stochastic control systems: nonlinear
Specification	Safety, reachability	Safety, reachability	Safety, reachability, reach-avoid
Stochasticity	Additive noise	Additive noise	Additive & multiplicative noises
Distribution	Normal, user-defined	Normal, user-defined	Normal, uniform, exponential, beta, user-defined
Disturbance	Not supported	Not supported	Supported

FAUST^2 [SGA15] generates formal abstractions for continuous-space discrete-time stochastic processes, and performs the verification and synthesis for safety and reachability specifications. However, FAUST^2 is originally implemented in MATLAB and suffers from the curse of dimensionality due to its lack of scalability for large models. StocHy [CDA19] provides the quantitative analysis of discrete-time stochastic hybrid systems such that it constructs finite abstractions, and performs the verification and synthesis for safety and reachability specifications.

AMYTISS differs from FAUST^2 and StocHy in two main directions. First, AMYTISS implements novel parallel algorithms and data structures targeting HPC platforms to reduce undesirable effects of the state-explosion problem. Accordingly, it is able to perform the parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators (HWAs). Whereas, FAUST^2 and StocHy can only run serially in one CPU, and consequently, it is limited to small systems. Additionally, AMYTISS can handle the abstraction construction and controller synthesis for two and a half player games (e.g., stochastic systems with bounded disturbances), whereas FAUST^2 and StocHy only handle one and a half player games (disturbance-free systems).

We compare AMYTISS with FAUST^2 and StocHy in Table 5.1 in detail in terms of different technical aspects. Although there have been some efforts in FAUST^2 and StocHy for parallel implementations, these are not compatible with HPC platforms. Specifically, FAUST^2 employs some parallelization techniques using parallel for-loops and sparse matrices inside Matlab, and StocHy uses Armadillo, a multi-threaded library for the scientific computing. However, these tools are not designed for the parallel computation on HPC platforms. Consequently, they can only utilize CPUs but they cannot run on GPUs or HWAs. In comparison, AMYTISS introduces novel parallel algorithms that support heterogeneous computing platforms combining CPUs, GPUs and HWAs.

Note that FAUST^2 and StocHy do not natively support reach-avoid specifications in the sense that users can explicitly provide some avoid sets. Implementing this type of properties requires some modifications inside those tools. In addition, we do not make a comparison here with SReachTools since it is mainly for the stochastic reachability analysis of linear, potentially time-varying, discrete-time stochastic systems, while AMYTISS is not limited to the reachability analysis and can handle nonlinear systems as well.

5.1.2 Contributions

In this chapter, we propose novel scalable parallel algorithms and efficient distributed data structures for constructing finite MDPs of large-scale discrete-time stochastic systems and automating the computation of their correct-by-construction controllers, given high-level specifications such as safety, reachability and reach-avoid. The main contributions and merits of this work are:

(1) We propose a novel data-parallel algorithm for constructing finite MDPs from discrete-time stochastic systems and storing them in efficient distributed data containers. The proposed algorithm handles large-scale systems.

(2) We propose parallel algorithms for synthesizing discrete controllers using the constructed MDPs to satisfy safety, reachability, or reach-avoid properties. More specifically, we introduce a novel parallel algorithm for the iterative computation of Bellman equation in the standard dynamic programming [Sou14].

(3) Unlike the existing tools in the literature, AMYTISS accepts bounded disturbances and natively supports both additive and multiplicative noises with different practical distributions including normal, uniform, exponential, and beta.

We apply the proposed implementations to real-world applications including room temperature and road traffic networks, and autonomous vehicles. This extends the applicability of formal method techniques to some safety-critical real-world applications with high dimensions. The results show remarkable reductions in the memory usage and computation time outperforming all existing tools in the literature.

We provide AMYTISS as an *open-source* tool. After compilation, AMYTISS is loaded via pFaces [KZ19] and launched for the parallel execution within available parallel computing resources. The source of AMYTISS and detailed instructions on its building and running can be found in:

https://github.com/mkhaled87/pFaces-AMYTISS

5.2 AMYTISS

In this chapter, we develop scalable parallel algorithms such that they support the parallel execution within CPUs, GPUs and hardware accelerators (HWAs). The results show that AMYTISS outperforms all existing tools. In this respect, we benchmark our tool against the most recent tools in the literature using several physical case studies including robot examples, and room temperature and road traffic networks. We also apply our algorithms to a 3-dimensional autonomous vehicle and a 7-dimensional *nonlinear* model of a BMW 320i car by synthesizing autonomous parking controllers.

Algorithm 3 Traditional serial algorithm for computing $\hat{T}_{\mathsf{x}}$

Require: $\hat{X}, \hat{U}, \hat{W}$, and a noise covariance matrix $\tilde{\Sigma} \in \mathbb{R}^{n \times n}$
Ensure: Transition probability matrix $\hat{T}_{\mathsf{x}}$ with the dimension of $(n_x \times n_\nu \times n_w, n_x)$
1: for all $\bar{x}_i \in \hat{X}$, s.t. $i \in \{1, ..., n_x\}$, do
2: for all $\bar{\nu}_j \in \hat{U}$, s.t. $j \in \{1, ..., n_\nu\}$, do
3: for all $\bar{w}_k \in \hat{W}$, s.t. $k \in \{1, ..., n_w\}$, do
4: Compute mean μ_c taking into account the given dynamic as

$$\mu_c = f(\bar{x}_i, \bar{\nu}_j, \bar{w}_k, 0)$$

5: for all $\bar{x}'_{\bar{l}} \in \hat{X}$, s.t. $\bar{l} \in \{1, ..., n_x\}$, do

$$\hat{T}_{\mathsf{x}}(\bar{x}'_{\bar{l}} | \bar{x}_i, \bar{\nu}_j, \bar{w}_k) := \int_{\Xi(x')} \text{PDF}(dx \mid \mu_c, \tilde{\Sigma}),$$

where PDF is the probability density function of the normal distribution.
6: end
7: end
8: end
9: end

We consider here stochastic control systems in Definition 2.3.1 in which W is a disturbance set. In the next subsection, we propose parallel algorithms for the construction of finite MDPs.

5.2.1 Parallel Construction of Finite MDPs

Here, we propose an approach to efficiently compute the transition probability matrix $\hat{T}_{\mathsf{x}}$ of the finite MDP $\widehat{\Sigma}$, which is essential for any controller synthesis procedure, as we discuss later in Section 5.2.2. Algorithm 3 presents the traditional serial algorithm for computing $\hat{T}_{\mathsf{x}}$. Note that if there are no disturbances in the given dynamics as presented in (2.3.3), one can still employ Algorithm 3 to compute the transition probability matrix but without step 3.

In subsections 5.2.1.1, 5.2.1.2, we address improvements of Algorithm 3. Each subsection targets one inefficient aspect of Algorithm 3 and discusses how to improve it. In subsection 5.2.1.3, we combine the proposed improvements and introduce a parallel algorithm for constructing $\hat{T}_{\mathsf{x}}$.

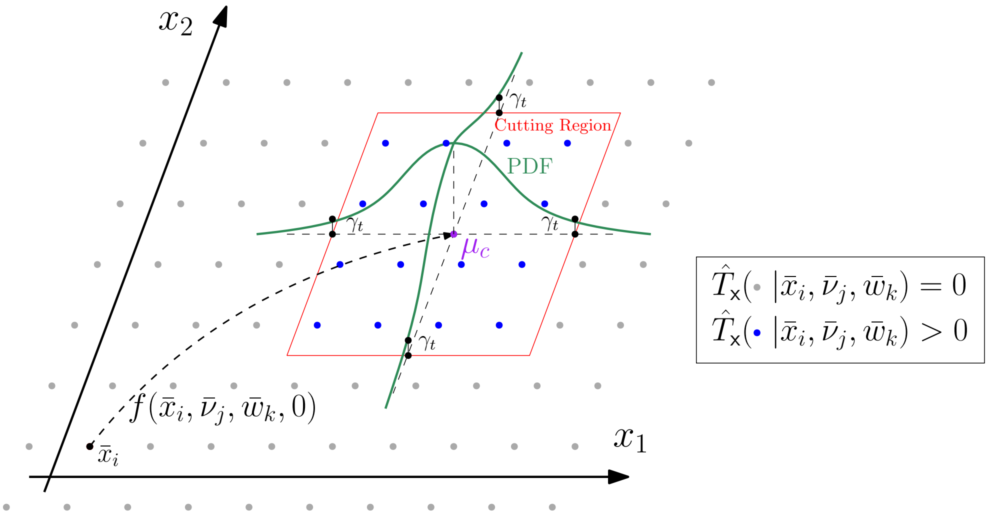

Figure 5.1: A 2-dimensional visualization of the cutting probability region (shown in red) with a cutting threshold of γ_t. The cutting region encloses representative post states (blue dots) that have nonzero probabilities in $\hat{T}_{\mathsf{x}}$. Other representative post states outside of the cutting region are considered to have zero probabilities in $\hat{T}_{\mathsf{x}}$.

5.2.1.1 Data-Parallel Threads for Computing $\hat{T}_{\mathsf{x}}$

The inner steps inside the nested for-loops 1, 2, and 3 in Algorithm 3 are computationally independent. More specifically, computations of μ_c, $\text{PDF}(x \,|\, \mu_c, \tilde{\Sigma})$, and $\hat{T}_{\mathsf{x}}$ all do not share date from one inner-loop to another. Hence, this is an embarrassingly data-parallel section of the algorithm. pFaces can be used to launch the necessary number of parallel threads on the employed hardware configuration (HWC) to improve the computation time of the algorithm. Each thread will eventually compute and store, independently, its corresponding values within $\hat{T}_{\mathsf{x}}$.

5.2.1.2 Less Memory for Post States in $\hat{T}_{\mathsf{x}}$

$\hat{T}_{\mathsf{x}}$ is a matrix with a dimension of $(n_x \times n_\nu \times n_w, n_x)$. The number of its columns is n_x as we need to compute and store the probability for each reachable partition element $\Xi(x'_{\bar{l}})$, corresponding to the representing post state $x'_{\bar{l}}$. Note that PDFs in this section follow Gaussian distributions.

For simplicity, we now focus on the computation done for a tuple $(\bar{x}_i, \bar{\nu}_j, \bar{w}_k)$. In many applications, when the PDF is decaying fast, only those partition elements near μ_c have relatively high probability values for being reached, starting from $\bar{x}_i$ and applying an input $\bar{\nu}_j$.

We set a cutting probability threshold $\gamma_t \in [0, 1]$ to control how much information for the partition elements around μ_c is stored. For a given mean value μ_c, a covariance matrix $\tilde{\Sigma}$ and a cutting probability threshold γ_t, $x \in X$ is called a PDF cutting point if $\gamma_t = \text{PDF}(x|\mu_c, \tilde{\Sigma})$. Since PDFs are symmetric, we have cutting points that form a hyper-rectangle in X, which we call it cutting region and denote it by $\hat{X}^{\tilde{\Sigma}}_{\gamma_t}$. This is visualized in Figure 5.1 for a 2-dimensional system.

For a tuple $(\bar{x}_i, \bar{\nu}_j, \bar{w}_k)$, $\hat{X}^{\tilde{\Sigma}}_{\gamma_t}$ is the set of representative points with probabilities of being reached greater than γ_t. Formally,

$$\hat{X}^{\tilde{\Sigma}}_{\gamma_t} = \{\bar{x} \in \hat{X} \,\big|\, \mathbb{P}\big(x(k+1) \in \Xi(\bar{x}) \,\big|\, x(k) = x_i, \nu(k) = \nu_j, w(k) = w_k\big) \geq \gamma_t\}.$$

Any partition element $\Xi(x'_l)$ with x'_l outside the cutting region is considered to have a zero probability of being reached. Such approximation allows controlling the sparsity of columns of $\hat{T}_{\mathsf{x}}$. The closer the value of γ_t to zero, the more accurate $\hat{T}_{\mathsf{x}}$ in representing the transitions of $\widehat{\Sigma}$. On the other hand, the closer the value of γ_t to one, less post state values need to be stored as columns in $\hat{T}_{\mathsf{x}}$. The number of probabilities to be stored for each tuple $(\bar{x}_i, \bar{\nu}_j, \bar{w}_k)$ is $|\hat{X}^{\tilde{\Sigma}}_{\gamma_t}|$. Figure 5.1 also visualizes how the proposed γ_t can help controlling the required memory for storing the transitions in $\hat{T}_{\mathsf{x}}$.

Note that since $\tilde{\Sigma}$ is fixed prior to running the algorithm, number of columns needed for a fixed γ_t can be identified before launching the computation. We can then accurately allocate a uniform fixed number of memory locations for any tuple $(\bar{x}_i, \bar{\nu}_j, \bar{w}_k)$ in $\hat{T}_{\mathsf{x}}$. Hence, there is no need for a dynamic sparse matrix data structure and $\hat{T}_{\mathsf{x}}$ is now a matrix with a dimension of $(n_x \times n_\nu \times n_w, |\hat{X}^{\tilde{\Sigma}}_{\gamma_t}|)$.

Remark 5.2.1. *Construction of $\hat{X}^{\tilde{\Sigma}}_{\gamma_t}$ is practically a simple process. We start by solving the equation $PDF(x^* \,\big|\, 0, \tilde{\Sigma}) = \gamma_t$ for $x^* \in \mathbb{R}^n_{>0}$ and computing the zero-mean cutting points at each dimension. Now since the PDF is symmetric, one obtains*

$$\hat{X}^{\tilde{\Sigma}}_{\gamma_t} = \{\bar{x} \in \hat{X} | \bar{x} \in [\![\mu_c - x^*, \mu_c + x^*]\!]\}.$$

Remark 5.2.2. *The reduction in the memory usage discussed in this subsection is tailored to Gaussian distributions for the sake of better presentation of the idea. Users interested in adding additional distributions to AMYTISS have the option of providing a subroutine that describes how other distributions should behave in terms of the required memory and with respect to the cutting threshold γ_t.*

5.2.1.3 A Parallel Algorithm for Constructing Finite MDP $\widehat{\Sigma}$

We present a novel parallel algorithm (Algorithm 4) to efficiently construct and store $\hat{T}_{\mathsf{x}}$ as a successor to Algorithm 3. We employ all the discussed enhancements in subsections 5.2.1.1, and 5.2.1.2, within the proposed algorithm. We do not parallelize the for-loop in Algorithm 4, Step 2, to avoid excessive parallelism. Note that, practically, for large-scale systems, $|\hat{X} \times \hat{U}|$ can reach up to billions. We are always interested in the number of parallel threads that can be scheduled reasonably to available HW computing units.

5.2.2 Parallel Synthesis of Controllers

In this subsection, we employ the dynamic programming to synthesize controllers for constructed finite MDPs $\widehat{\Sigma}$ satisfying safety, reachability, and reach-avoid properties [Sou14,

Algorithm 4 Proposed *parallel* algorithm for computing $\hat{T}_{\mathsf{x}}$

Require: $\hat{X}, \hat{U}, \hat{W}, \gamma_t$, and a noise covariance matrix $\tilde{\Sigma} \in \mathbb{R}^{n \times n}$
Ensure: Transition probability matrix $\hat{T}_{\mathsf{x}}$ with the dimension of $(n_x \times n_\nu \times 2, |\hat{X}_{\gamma_t}^{\tilde{\Sigma},W}|)$

```
for all (x̄, ν̄) ∈ X̂ × Û in parallel do
    for all w̄ ∈ Ŵ do
        Set μ_c = f(x̄, ν̄, w̄)
        Construct X̂_{γt}^{Σ̃} as described in Remark 5.2.1
        for all x* ∈ X̂_{γt}^{Σ̃} do
            Set T̂_x(x*|x̄, ν̄, w̄) := ∫_{Ξ(x*)} PDF(dx|μ_c, Σ̃)
        end
    end
end
```

SA13a]. We first present the traditional serial algorithm for the controller synthesis satisfying safety, reachability, and reach-avoid specifications as Algorithm 5. Note that if there are no disturbances in the given dynamics, Steps 16 and 17 of Algorithm 5 are to be excluded.

The serial algorithm does, repetitively, matrix multiplications in each loop that corresponds to different time instance of the bounded time T_d. We cannot parallelize the for-loop in Step 9 due to the data dependency, however, we can parallelize the contents of this loop by simply considering the standard parallel algorithms for the matrix multiplication.

Algorithm 6 is a parallelization of Algorithm 5. Step 10 in Algorithm 6 is the parallel implementation of the matrix multiplication in Algorithm 5, Step 10. Step 19 in Algorithm 6 selects and stores the inputs $\bar{\nu}$ that maximizes probabilities of enforcing the specifications.

A significant reduction in the computation of the intermediate matrix $\mathbb{V}_{int}$ is also introduced in Algorithm 6. In Algorithm 5, Step 10, the computation of $\mathbb{V}_{int}$ requires a matrix multiplication between T_{x} (dimension of $(n_x \times n_\nu \times n_w, n_x)$) and $\mathbb{V}_s(:,\cdot)$ (dimension of $(n_x, 1)$). On the other hand, in the parallel version in Algorithm 6, for each $\bar{w}$, the corresponding computation is done for $\mathbb{V}_{int}$ such that each element, i.e., $\mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w})$, requires only $|\hat{X}_{\gamma_t}^{\tilde{\Sigma}}|$ scalar multiplications. Here, we clearly utilize the technique discussed in Subsection 5.2.1.2 to consider only those post states in the cutting region $\hat{X}_{\gamma_t}^{\tilde{\Sigma}}$. Remember that other post states outside $\hat{X}_{\gamma_t}^{\tilde{\Sigma}}$ are considered to have the probability zero which means we can avoid their scalar multiplications.

5.2.3 On-the-Fly Construction of $\hat{T}_{\mathsf{x}}$

In AMYTISS, we also use another technique that further reduces the required memory for computing $\hat{T}_{\mathsf{x}}$. We refer to this approach as *on-the-fly abstractions* (OFA). In OFA

Algorithm 5 Traditional *serial* algorithm for controller synthesis satisfying safety, reachability and reach-avoid specifications

Require: $\hat{X}$, $\hat{U}$, $\hat{W}$, bounded time horizon T_d, $specs \in \{Safety, Reachability, ReachAvoid\}$, target set $\bar{\mathcal{T}}$ (in case $specs = Reachability, ReachAvoid$), and avoid set $\bar{\mathcal{A}}$ (in case $specs = ReachAvoid$)
Ensure: Optimal satisfaction probability $\mathbb{V}_s$ at time step $T_d = 1$, and optimal policy $\nu^\star$ corresponding to optimal satisfaction probability

1: Compute $\hat{T}_{\mathsf{x}}$ as presented in Algorithm 3
2: if $specs == Safety$ do
3: Set value function $\mathbb{V}_s := ones(n_x, T_d + 1)$
4: else
5: Compute a transition probability matrix $\hat{T}_{0\mathsf{x}}$ from $\hat{X} \backslash (\bar{\mathcal{T}} \cup \bar{\mathcal{A}})$ to $\bar{\mathcal{T}}$
6: Set $\hat{T}_{\mathsf{x}}$ to zero for any post-state in $(\bar{\mathcal{T}} \cup \bar{\mathcal{A}})$
7: Set value function $\mathbb{V}_s := zeros(n_x, T_d + 1)$
8: end
9: for $k = T_d : -1 : 1$ (*backward in time*) do
10: if $specs == Safety$ do
11: Set $V_{int} = \hat{T}_{\mathsf{x}} \mathbb{V}_s(:, k+1)$ $\{\mathbb{V}_{int}$ has dimension of $(n_x \times n_\nu \times n_w, 1)\}$
12: else
13: Set $\mathbb{V}_{int} = \hat{T}_{0\mathsf{x}} + \hat{T}_{\mathsf{x}} \mathbb{V}_s(:, k+1)$ $\{\mathbb{V}_{int}$ has dimension of $(n_x \times n_\nu \times n_w, 1)\}$
14: end
15: Reshape $\mathbb{V}_{int}$ to a matrix $\bar{\mathbb{V}}_{int}$ of dimension $(n_x \times n_\nu, n_w)$
16: Minimize $\bar{\mathbb{V}}_{int}$ with respect to disturbance set $\hat{W}$ as $\mathbb{V}_{min}$
17: Reshape $\mathbb{V}_{min}$ to a matrix $\bar{\mathbb{V}}_{min}$ of dimension (n_x, n_ν)
18: Maximize $\bar{\mathbb{V}}_{min}$ with respect to input set $\hat{U}$ as $\mathbb{V}_{max}$ of dimension $(n_x, 1)$
19: Update $\mathbb{V}_s(:, k) := \mathbb{V}_{max}$
20: end

version of Algorithm 6, we skip computing and storing the MDP $\hat{T}_{\mathsf{x}}$ and the matrix $\hat{T}_{0\mathsf{x}}$ (i.e., Steps 1 and 5). We instead compute the required entries of $\hat{T}_{\mathsf{x}}$ and $\hat{T}_{0\mathsf{x}}$ on-the-fly as they are needed (i.e., Steps 13 and 15). This reduces the required memory for $\hat{T}_{\mathsf{x}}$ and $\hat{T}_{0\mathsf{x}}$ but at the cost of the repeated computation of their entries in each time step from 1 to T_d. However, this gives the user an additional control over the trade-off between the computation time and memory usage.

5.2.4 Supporting Multiplicative Noises and Practical Distributions

AMYTISS natively supports multiplicative noises and practical distributions such as uniform, exponential, and beta distributions. The technique introduced in Subsection 5.2.1.2 for reducing the memory usage is also tuned for other distributions based on the support of their PDFs. Since AMYTISS is designed for extensibility, it allows also for customized distributions. Users need to specify their desired PDFs and hyper-rectangles

Algorithm 6 Proposed *parallel* algorithm for controller synthesis satisfying safety, reachability and reach-avoid specifications

Require: $\hat{X}$, $\hat{U}$, $\hat{W}$, bounded time horizon T_d, *specs* $\in \{Safety, Reachability, ReachAvoid\}$, target set $\bar{\mathcal{T}}$ (in case *specs* = *Reachability*, *ReachAvoid*), and avoid set $\bar{\mathcal{A}}$ (in case *specs* = *ReachAvoid*)
Ensure: Optimal satisfaction probability $\mathbb{V}_s$ at time step $T_d = 1$, and optimal policy $\nu^\star$ corresponding to optimal satisfaction probability

Compute $\hat{T}_\times$ in parallel as presented in Algorithm 4
if $specs == Safety$ do
    Set value function $\mathbb{V}_s := ones(n_x, T_d + 1)$
else
    Compute a transition probability matrix $\hat{T}_{0\times}$ from $\hat{X}\backslash(\bar{\mathcal{T}} \cup \bar{\mathcal{A}})$ to $\bar{\mathcal{T}}$
    Set $\hat{T}_\times$ to zero for any post-state in $(\bar{\mathcal{T}} \cup \bar{\mathcal{A}})$
    Set value function $\mathbb{V}_s := zeros(n_x, T_d + 1)$
end
for $k = T_d : -1 : 1$ (*backward in time*) do
  for all $(\bar{x}, \bar{\nu}) \in \hat{X} \times \hat{U}$ in parallel do
    for all $\bar{w} \in \hat{W}$
      Construct $\hat{X}^{\tilde{\Sigma}}_{\gamma_t}$ as discussed in Subsection 5.2.1.2
      Set $\mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w}) := \sum_{x^* \in \hat{X}^{\tilde{\Sigma}}_{\gamma_t}} \mathbb{V}_s(x^*, k+1) T_\times(x^* | \bar{x}, \bar{\nu}, \bar{w})$
      if *specs* == *ReachAvoid* and $\bar{x} \notin (\bar{\mathcal{T}} \cup \bar{\mathcal{A}})$ do
        Set $\mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w}) := \mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w}) + T_{0\times}(\bar{x}, \bar{\nu}, \bar{w})$
      end
    end
  end
  for all $\bar{x} \in \hat{X}$ in parallel do
    Set $\mathbb{V}_s(\bar{x}, k) := \max_{\bar{\nu} \in \hat{U}} \{ \min_{\bar{w} \in \hat{W}} \{ \mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w}) \} \}$
    Set $\nu^\star(\bar{x}, k) := \operatorname{argmax}_{\bar{\nu} \in \hat{U}} \{ \min_{\bar{w} \in \hat{W}} \{ \mathbb{V}_{int}(\bar{x}, \bar{\nu}, \bar{w}) \} \}$
  end
end

enclosing their supports so that AMYTISS can include them in the parallel computation of $\hat{T}_\times$. Further details on specifying customized distributions are provided in the README file.

AMYTISS also supports multiplicative noises as introduced in [LTS05]. Currently, the memory reduction technique of Subsection 5.2.1.2 is disabled when users provide systems with multiplicative noise. This means users should expect larger memory requirements for systems that have multiplicative noises. However, users can still benefit from the OFA version of Algorithm 6 to compensate for such increase in the memory requirement. We plan to include this feature for multiplicative noises in a future update of AMYTISS. We should mention that for the sake of better demonstration, previous sections considered the additive noise and the Gaussian normal distribution as a PDF to introduce the concepts of the idea.

5.2.5 AMYTISS by Running Example

AMYTISS is self-contained and requires only a modern C++ compiler. It supports the three major operating systems: Windows, Linux and Mac OS. We tested AMYTISS on Windows 10 x64, MacOS Mojave, Ubuntu 16.04, and Ubuntu 18.04, and found no major computation time differences.

Once compiled, utilizing it is a matter of providing text configuration files and launching the tool to operate on them. Please refer to the provided README file in the repository of AMYTISS for the general installation instruction.

For the sake of illustrating the proposed algorithms and the usage of AMYTISS, we first introduce a simple 2-dimensional example. Consider a robot described by the following difference equation:

$$\begin{cases} x_1(k+1) = x_1(k) + \tau\nu_1(k)\cos(\nu_2(k)) + w(k) + \varsigma_1(k) \\ x_2(k+1) = x_2(k) + \tau\nu_2(k)\sin(\nu_2(k)) + w(k) + \varsigma_2(k), \end{cases} \tag{5.2.1}$$

where $(x_1, x_2) \in X := [-10, 10]^2$ is a state vector representing a spacial coordinate, $(\nu_1, \nu_2) \in U := [-1, 1]^2$ is an input vector, $w \in W := [-1, 1]$ is a disturbance, $(\varsigma_1, \varsigma_2)$ is noises following a Gaussian distribution with the covariance matrix $\tilde{\Sigma} := \mathsf{diag}(0.75, 0.75)$, and $\tau := 10$ is a constant.

To construct MDPs approximating the system, we consider state quantization parameters of $(0.5, 0.5)$, input quantization parameters of $(0.1, 0.1)$, disturbance quantization parameters of 0.2, and a cutting probability level γ_t of 0.001. Using such quantization parameters, the number of state-input pairs $|\hat{X} \times \hat{U}|$ in $\hat{\Sigma}$ is 203401. We use $|\hat{X} \times \hat{U}|$ as an indicator to the size of the system.

System descriptions and controller synthesis requirements are provided to AMYTISS as text configuration files. The configuration files of this example are located in the directory %AMYTISS%/examples/ex_toy_XXXX, where %AMYTISS% is the installation directory of AMYTISS and XXXX should be replaced with the controller synthesis specification of interests and can be any of: safety, reachability, or reach-avoid. For a detailed description of the key-value pairs in each configuration file, refer to the README file in the repository of AMYTISS.

5.2.5.1 Synthesis for Safety Specifications

We synthesize a controller for the robot system in (5.2.1) to keep the state of the robot inside X within 8 time steps. The synthesized controller should enforce the safety specification in presence of the disturbance and the noise. The corresponding configuration file is located in file %AMYTISS%/examples/ex_toy_safety/toy2d.cfg, which describes the

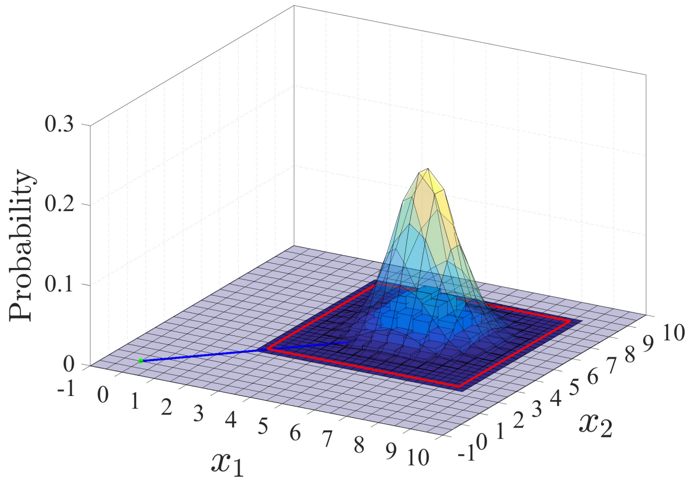

Figure 5.2: A visualization of the transitions for one source state $x := (0, 0)$ and an input $\nu = (0.7, 0.8)$ of the MDP approximating the robot example. The green point is the source state, the transparent bell-like shape is the PDF and the red rectangle is the cutting region. Probabilities of reaching partition elements inside cutting regions are shown as bars below the PDF.

system in (5.2.1) and the safety requirement. To launch AMYTISS and run it for synthesizing the safety controller of this example, navigate to the install directory %AMYTISS% and run the command:

```
$ pfaces -CGH -d 1 -k amytiss.cpu@./kernel-pack -cfg ./examples/ex_toy_safety/toy2d.cfg -p
```

where `pfaces` calls pFaces, `-CGH -d 1` asks pFaces to consider the first device from all CPU, GPU and HWA devices, `-k amytiss.cpu@./kernel-pack` asks pFaces to launch AMYTISS's kernel from its main source folder, `-cfg ./examples/ex_toy_safety/toy2d.cfg` asks pFaces to hand the configuration file to AMYTISS, and `-p` asks pFaces to collect profiling information. For more details about other arguments you may use, please refer to the manual of pFaces.

This launches AMYTISS to construct an MDP of the robot system and synthesize a safety controller for it. The results are stored in the output file specified in the configuration file. Using the provided MATLAB interface in AMYTISS, we visualize some transitions of the constructed MDP and show them in Figure 5.2. The used MATLAB script is located in %AMYTISS%/examples/ex_toy_safety/make_figs.m.

The output file contains also the control strategy which we use to simulate the closed-loop behavior of the system. Again, we rely on the provided MATLAB interface in AMYTISS to simulate the closed-loop behavior. The MATLAB script in %AMYTISS% /examples/ex_toy_safety/closedloop.m simulates the system with random choices on $\bar{w} \in \hat{W}$ and random values for the noise according to the provided covariance matrix. For

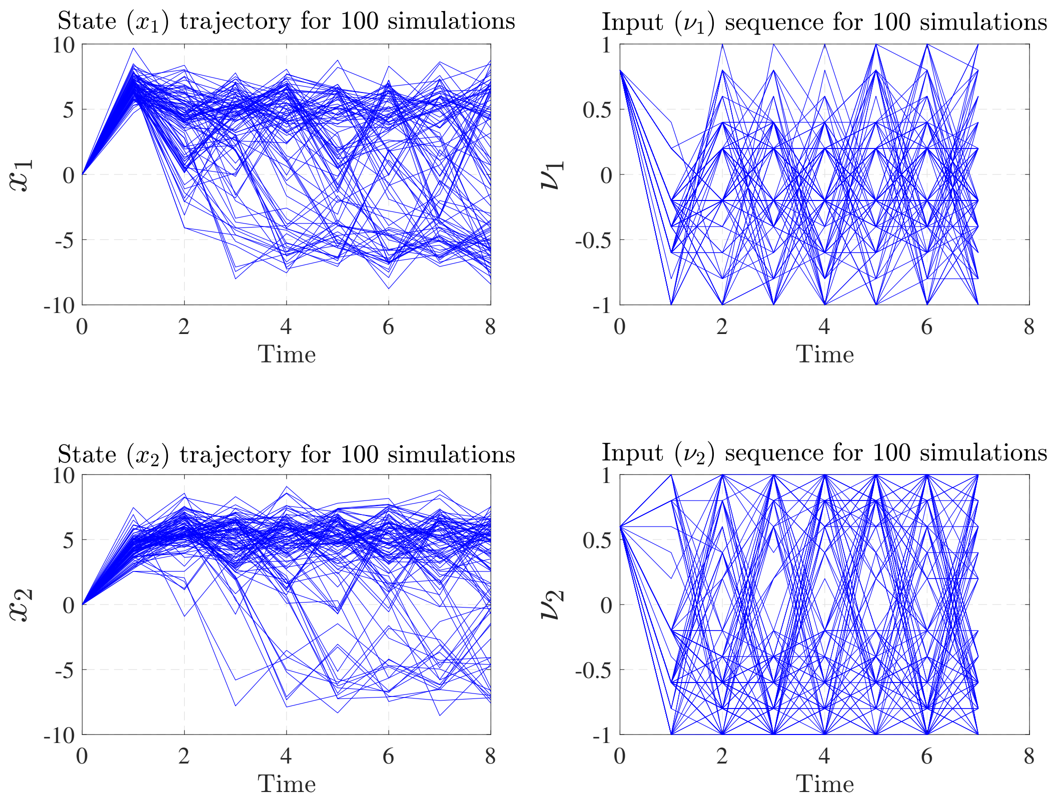

Figure 5.3: 100 simulations of the closed-loop behavior of the robot under a safety controller synthesized for maintaining the robot inside X. At left, we show the trajectory of each component of the state of the system at each time step. At right, we show the applied input at each time step. For the sake of readability, the input plot is shown as the piece-wise linear signal. The system is discrete-time and inputs are utilized only at update times.

each time step, the simulation queries the strategy from the output file and applies it to the system. We repeat the simulation 100 times. Figure 5.3 shows the closed-loop simulation results. Note that the input is always fixed at the time step $k = 0$. This is because we store only one input, which is the one maximizing the probability of satisfying the specification. After the time step $k = 0$, and because of noise/disturbance, the system lands in different states which requires applying different inputs to satisfy the specification.

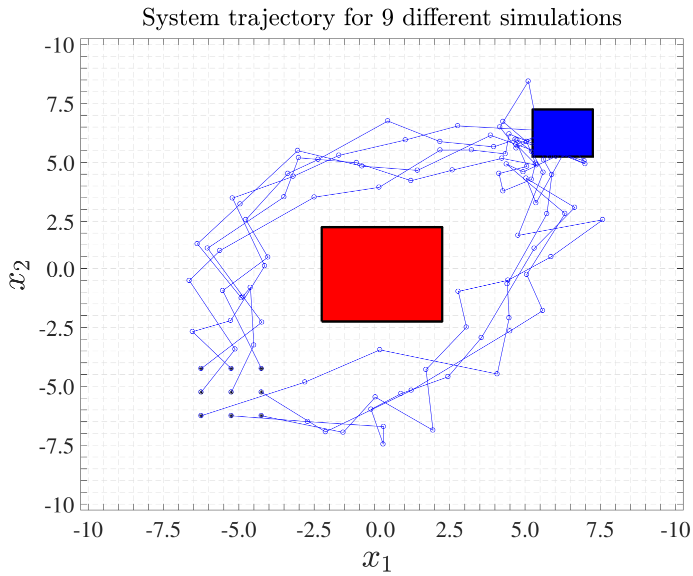

Figure 5.4: 9 simulations of the closed-loop behavior of the robot example under a controller synthesized for reaching a target set of states in X while avoiding another set of states. The 9 dots at the left bottom correspond to 9 initial states for 9 different simulation runs. The red and blue rectangles are avoid and target sets, respectively.

5.2.5.2 Synthesis for Reach-Avoid Specifications

We synthesize a controller for the robot system in (5.2.1) to reach the set $[5, 7]^2$ while avoiding the set $[-2, 2]^2$ within 16 time steps. To launch AMYTISS and run it for synthesizing the reachability controller of this example, navigate to the install directory %AMYTISS% and run the command:

```
$ pfaces -CGH -k amytiss.cpu@./kernel-pack -cfg ./examples/ex_toy_reachavoid/toy2d.cfg -d 1 -p
```

This launches AMYTISS to construct an MDP of the robot system and synthesize a reachability controller for it. A MATLAB script simulates the closed loop and it is located in %AMYTISS%/examples/ex_toy_reachavoid/closedloop.m. This runs 9 different simulations from 9 different initial states. Figure 5.4 shows the closed-loop simulation results.

5.2.6 Benchmarking and Case Studies

5.2.6.1 Controlling Computational Complexities

AMYTISS implements scalable parallel algorithms that run on top of pFaces. Hence, users can utilize the computing power in HPC platforms and cloud computing to scale the computation and control the computational complexities of their problems. We fix the system (i.e., the robot example) in hand and show how AMYTISS scales with respect to different computing platforms. Table 5.2 lists the HW configuration (HWC) we use

Table 5.2: Used HW configurations for benchmarking AMYTISS.

Id	Description	PEs	Frequency
CPU$_1$	Local machine: Intel Xeon E5-1620	8	3.6 GHz
CPU$_2$	Macbook Pro 15: Intel i9-8950HK	12	2.9 GHz
CPU$_3$	AWS instance c5.18xlarge: Intel Xeon Platinum 8000	72	3.6 GHz
GPU$_1$	Macbook Pro 15 laptop laptop: Intel UHD Graphics 630	23	0.35 GHz
GPU$_2$	Macbook Pro 15 laptop: AMD Radeon Pro Vega 20	1280	1.2 GHz
GPU$_3$	AWS p3.2xlarge instance: NVIDIA Tesla V100	5120	0.8 GHz

to benchmark AMYTISS. The devices range from local devices in laptops and desktop computers to advanced compute devices in Amazon AWS cloud computing services.

Table 5.4 shows the benchmarking results running AMYTISS with these HWCs for several case studies and makes comparisons between AMYTISS, FAUST^2, and StocHy. We employ a machine with Windows operating system (Intel i7@3.6GHz CPU and 16 GB of RAM) for FAUST^2, and StocHy. We should mention that FAUST^2 predefines a minimum number of representative points based on the desired abstraction error, and accordingly the computation time and memory usage reported in Table 5.4 are based on the minimum number of representative points. In addition, to have a fair comparison, we run all the case studies with additive noises since neither FAUST^2 nor StocHy support multiplicative noises.

For each HWC, we show the time in seconds to solve the problem. Clearly, employing HWCs with more PEs reduces the time to solve the problem. This is a strong indication for the scalability of the proposed algorithms. This also becomes very useful in real-time applications, where users can control the computation time of their problems by adding more resources. Since AMYTISS is the only tool that can utilize the reported HWCs, we do not compare with other similar tools.

To show the applicability of our results to large-scale stochastic systems, we apply our proposed techniques to several physical case studies. First, we synthesize a controller for 3- and 5-dimensional *room temperature networks* to keep the temperature of rooms in a comfort zone. Then we synthesize a controller for *road traffic networks* with 3 and 5 dimensions to maintain the density of the traffic below some level. We then consider 3- and 7-dimensional *nonlinear* models of autonomous vehicles and synthesize reach-avoid controllers to automatically park the vehicles. For each case study, we compare our tool with FAUST^2 and StocHy and report the technical details in Table 5.4.

5.2.6.2 Room Temperature Network

5-Dimensional System. We first apply our results to the temperature regulation of 5 rooms each equipped with a heater and connected on a circle. The evolution of temperatures $\tilde{T}_i$ can be described by individual rooms as

$$\begin{aligned}
\tilde{T}_i(k+1) &= a_{ii}\tilde{T}_i(k) + \bar{\theta}\tilde{T}_h\nu_i(k) + \eta w_i(k) + \beta\tilde{T}_{ei} + 0.01\varsigma_i(k), i \in \{1,3\},\\
\tilde{T}_i(k+1) &= b_{ii}\tilde{T}_i(k) + \eta w_i(k) + \beta\tilde{T}_{ei} + 0.01\varsigma_i(k), i \in \{2,4,5\},
\end{aligned}$$

where $a_{ii} = (1-2\eta-\beta-\bar{\theta}\nu_i(k))$, $b_{ii} = (1-2\eta-\beta)$, and $w_i(k) = \tilde{T}_{i-1}(k)+\tilde{T}_{i+1}(k)$ (with $\tilde{T}_0 = \tilde{T}_n$ and $\tilde{T}_{n+1} = \tilde{T}_1$). Furthermore, $\eta = 0.3$, $\beta = 0.022$, and $\bar{\theta} = 0.05$, $\tilde{T}_{ei} = -1\,°C$, $\tilde{T}_h = 50\,°C$, and $\tilde{T}_i(k)$ and $\nu_i(k)$ are taking values in sets $[19, 21]$ and $[0, 1]$, respectively, $\forall i \in \{1, \ldots, n\}$.

Let us now synthesize a controller for the 5-dimensional system via its finite abstraction $\widehat{\Sigma}$ such that the controller maintains the temperature of any room in the safe set $[19, 21]$ for at least 8 time steps.

We also applied our algorithms to a smaller version of this case study (3-dimensional system) with the results reported in Table 5.4.

5.2.6.3 Road Traffic Network

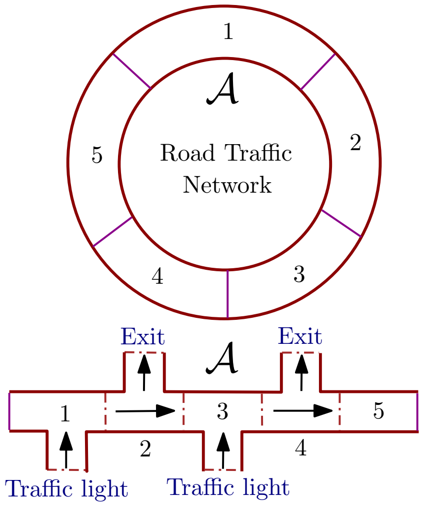

Figure 5.5: Model of a road traffic network composed of 5 cells of 500 meters with 2 entries and 2 ways out.

5-Dimensional System. Consider a road traffic network divided in 5 cells of 500 meters with 2 entries and 2 ways out, as schematically depicted in Figure 5.5. The model of this case study is borrowed from [LCGG13] by including stochasticity in the model as the additive noise.

The two entries are controlled by traffic lights, denoted by ν_1 and ν_3, that enable (green light) or not (red light) the vehicles to pass. In this model, the length of a cell is in kilometers [km] and the flow speed of the vehicles is 100 kilometers per hour [km/h]. Moreover, during the sampling time interval $\tau = 6.48$ seconds, it is assumed that 6 vehicles pass the entry controlled by the light ν_1, 8 vehicles pass the entry controlled by the light ν_3, and one quarter of vehicles that leave cells 1 and 3 goes out on the first exit (the ratio denoted by $\tilde{q}$). We want to observe the density of the traffic x_i, given in vehicles per cell, for each cell i of the road. The model of cells is described by:

$$x_1(k+1) = (1 - \frac{\tau v_1}{\tilde{l}_1})x_1(k) + \frac{\tau v_5}{\tilde{l}_5}w_1(k) + 6\nu_1(k) + 0.7\varsigma_1(k),$$
$$x_i(k+1) = (1 - \frac{\tau v_i}{\tilde{l}_i} - \tilde{q})x_i(k) + \frac{\tau v_{i-1}}{\tilde{l}_{i-1}}w_i(k) + 0.7\varsigma_i(k), \quad i \in \{2,4\},$$
$$x_3(k+1) = (1 - \frac{\tau v_3}{\tilde{l}_3})x_3(k) + \frac{\tau v_2}{\tilde{l}_2}w_3(k) + 8\nu_3(k) + 0.7\varsigma_3(k),$$
$$x_5(k+1) = (1 - \frac{\tau v_5}{\tilde{l}_5})x_5(k) + \frac{\tau v_4}{\tilde{l}_4}w_5(k) + 0.7\varsigma_5(k),$$

where $w_i(k) = x_{i-1}(k)$ (with $x_0 = x_5$), and $v_0 = v_5, \tilde{l}_0 = \tilde{l}_5$. We are interested first in constructing the finite MDP of the given 5-dimensional system and then synthesizing policies keeping the density of the traffic lower than 10 vehicles per cell.

For this case study, we have $X := [0,10]^5$ with quantization parameters of $(0.37, 0.37, 0.37, 0.37, 0.37)$, $U = [0,1]^2$ with quantization parameters of $(1,1)$, the noise covariance matrix $\Sigma := \mathsf{diag}(0.7, 0.7, 0.7, 0.7, 0.7)$, and a cutting probability level γ_t of $2e-2$.

We also applied our algorithms to the same case study but with 3-dimensions for the sake of benchmarking.

5.2.6.4 Autonomous Vehicle

7-Dimensional BMW 320i. Here, to show the applicability of our approaches to *nonlinear* models, we consider the 7-dimensional discrete-time nonlinear model of the BMW 320i car as presented in (4.7.3) with the sampling time $\tau = 0.1$ and the standard deviation of the noise $\bar{\sigma}_i = 0.2, \forall i \in \{1, ..., 7\}$. To construct a finite MDP $\widehat{\Sigma}$, we consider a bounded version of the state set $X := [-10.0, 10.0] \times [-10.0, 10.0] \times [-0.40, 0.40] \times [-2, 2] \times [-0.3, 0.3] \times [-0.4, 0.4] \times [-0.04, 0.04]$, a state discretization vector $(4.0; 4.0; 0.2; 1.0; 0.1; 0.2; 0.02)$, an input set $U := [-0.4, 0.4] \times [-4, 4]$, and an input discretization vector $[0.2; 2.0]$.

We are interested in an autonomous operation of the vehicle. The vehicle should park automatically in the parking lot located in the projected set $[-1.5, 0.0] \times [0.0, 1.5]$ within 32 time steps. The vehicle should avoid hitting a barrier represented by the set $[-1.5, 0.0] \times [-0.5, 0.0]$.

We also applied our algorithms to a 3-dimensional autonomous vehicle [RWR16, Section IX-A] for the sake of benchmarking.

5.2.6.5 Benchmark in StocHy

We benchmark our results against the ones provided by StocHy [CDA19]. We employ the same case study as in [CDA19, Case study 3] which starts from a 2-dimensional to a 12-dimensional continuous-space system with the same parameters.

To have a fair comparison, we utilize a machine with the same configuration as the one employed in [CDA19] (a laptop having an Intel Core i7 − 8550U CPU at 1.80GHz with 8 GB of RAM). We build a finite MDP for the given model and compare our computation time with the results provided by StocHy.

Table 5.3 shows the comparison between StocHy and AMYTISS. StocHy suffers significantly from the state-explosion problem as seen from its exponentially growing computation time. AMYTISS, on the other hand, outperforms StocHy and can handle bigger systems using the same hardware. This comparison shows speedups up to maximum 375 times for the 12-dimensional system. Note that we only reported up to 12-dimensions but AMYTISS can readily go beyond this limit for this example. For instance, AMYTISS manged to handle the 20-dimensional version of this system in 1572 seconds using an NVIDIA Tesla V100 GPU in Amazon AWS.

Table 5.3: Comparison between StocHy and AMYTISS for a continuous-space system with dimensions up to 12. The reported system is autonomous and, hence, $\hat{U}$ is singleton. $|\hat{X}|$ refers to the size of the system.

Dimension	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
$\lvert\hat{X}\rvert$	4	8	16	32	64	128	265	512	1024	2048	4096
Time (s) - StocHy	0.015	0.08	0.17	0.54	2.17	9.57	40.5	171.6	385.5	1708.2	11216
Time (s) - AMYTISS	0.02	0.92	0.20	0.47	1.02	1.95	3.52	6.32	10.72	17.12	29.95

Readers are highly advised to pay attention to the size of the system $|\hat{X} \times \hat{U}|$ (or $|\hat{X}|$ when $\hat{U}$ is singleton), not to its dimension. Actually, here, the 12-dimensional system, which has a size of 4096 state-input pairs is much smaller than the 2-dimensional illustrative example we introduced in Subsection 5.2.5, which has a size of 203401 state-input pairs. The current example has small size due to the very coarse quantization parameters and the tight bounds used to quantize X.

As seen in Table 5.4, AMYTISS clearly outperforms FAUST^2 and StocHy in all the case studies (with maximum speedups respectively up to 1680000 and 676000 times). Moreover, only AMYTISS can utilize the available HW resources to reduce the computation time. The OFA feature in AMYTISS reduces dramatically the required memory, while still solves the problems in reasonable amounts of time. FAUST^2 and StocHy fail to solve many of the problems since they lack the native support for nonlinear systems, they require of large amounts of memory, or they do not finish computing within 24 hours.

Table 5.4: Comparison between AMYTISS, FAUST^2 and StocHy based on their native features for several (physical) case studies. CSB refers to the continuous-space benchmark provided in [CDA19]. † refers to cases when we run AMYTISS with the OFA algorithm. N/M refers to the situation when there is not enough memory to run the case study. N/S refers to the lack of native support for nonlinear systems. (N x) refers to an N-times speedup. (N Kx) refers to an $(1000 \times N)$-times speedup. The presented speedup is the maximum speedup value across all reported devices. The required memory usage and computation time for FAUST^2 and StocHy are reported for just constructing finite MDPs. The reported times and memories are respectively in seconds and MB, unless other units are denoted.

				AMYTISS (time)							FAUST^2		StocHy		Speedup w.r.t	
Problem	**Spec.**	$\hat{X} \times \hat{U}$	T_d	**Mem.**	CPU_1	CPU_2	CPU_3	GPU_1	GPU_2	GPU_3	**Mem.**	**Time**	**Mem.**	**Time**	FAUST	StocHy
2-d StocHy CSB	Safety	4	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0001	≤ 1.0	0.002	8.5	0.015	**20 x**	**150 x**
3-d StocHy CSB	Safety	8	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0001	≤ 1.0	0.002	8.5	0.08	**20 x**	**800 x**
4-d StocHy CSB	Safety	16	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0002	≤ 1.0	0.01	8.5	0.17	**50 x**	**850 x**
5-d StocHy CSB	Safety	32	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0003	≤ 1.0	0.01	8.7	0.54	**33 x**	**1.8 Kx**
6-d StocHy CSB	Safety	64	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0006	4.251	1.2	9.6	2.17	**2.0 Kx**	**3.6 Kx**
7-d StocHy CSB	Safety	128	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0012	38.26	13	12.9	9.57	**10.8 Kx**	**7.9 Kx**
8-d StocHy CSB	Safety	256	6	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0026	344.3	104	26.6	40.5	**40 Kx**	**15.6 Kx**
9-d StocHy CSB	Safety	512	6	1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0057	3 GB	1126	80.7	171.6	**197 Kx**	**30.1 Kx**
10-d StocHy CSB	Safety	1024	6	4.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0122	N/M		297.5	385.5	N/A	**32 Kx**
11-d StocHy CSB	Safety	2048	6	16.0	1.0912	≤ 1.0	≤ 1.0	≤ 1.0	≤ 1.0	0.0284	N/M		1 GB	1708.2	N/A	**60 Kx**
12-d StocHy CSB	Safety	4096	6	64.0	4.3029	4.1969	≤ 1.0	≤ 1.0	≤ 1.0	0.0624	N/M		4 GB	11216	N/A	**179 Kx**
13-d StocHy CSB	Safety	8192	6	256.0	18.681	19.374	1.8515	1.6802	≤ 1.0	0.1277	N/M		N/A	$\geq$ 24h	N/A	**≥676 Kx**
14-d StocHy CSB	Safety	16384	6	1024.0	81.647	94.750	7.9987	7.3489	6.1632	0.2739	N/M		N/A	$\geq$ 24h	N/A	**≥320 Kx**
2-d Robot†	Safety	203401	8	≤ 1.0	8.5299	5.0991	0.7572	≤ 1.0	≤ 1.0	0.0154	N/A		N/A		N/A	N/A
2-d Robot	R.Avoid	741321	16	482.16	48.593	18.554	4.5127	2.5311	3.4353	0.3083	N/S		N/S		N/A	N/A
2-d Robot†	R.Avoid	741321	16	4.2484	132.10	41.865	11.745	5.3161	3.6264	0.1301	N/A		N/A		N/A	N/A
3-d Room Temp.	Safety	7776	8	6.4451	0.1072	0.0915	0.0120	≤ 1.0	≤ 1.0	0.0018	3.12	3027	N/M		**1680 Kx**	N/A
3-d Room Temp.†	Safety	7776	8	≤ 1.0	0.5701	0.3422	0.0627	≤ 1.0	≤ 1.0	0.0028	N/A		N/A		N/A	N/A
5-d Room Temp.	Safety	279936	8	3338.4	200.00	107.93	19.376	10.084	N/M	1.8663	2 GB	6822	N/M		**3790 x**	N/A
5-d Room Temp.†	Safety	279936	8	1.36	716.84	358.23	63.758	30.131	22.334	0.5639	N/A		N/A		N/A	N/A
3-d Road Traffic	Safety	2125764	16	1765.7	29.200	131.30	3.0508	5.7345	10.234	1.2895	N/M		N/M		N/A	N/A
3-d Road Traffic†	Safety	2125764	16	14.19	160.45	412.79	13.632	12.707	11.657	0.3062	N/A		N/A		N/A	N/A
5-d Road Traffic	Safety	68841472	7	8797.4	N/M	537.91	38.635	N/M	N/M	4.3935	N/M		N/M		N/A	N/A
5-d Road Traffic†	Safety	68841472	7	393.9	1148.5	1525.1	95.767	44.285	36.487	0.7397	N/A		N/A		N/A	N/A
3-d Vehicle	R.Avoid	1528065	32	1614.7	2.5h	1.1h	871.89	898.38	271.41	10.235	N/S		N/S		N/A	N/A
3-d Vehicle†	R.Avoid	1528065	32	11.17	2.8h	1.9h	879.78	903.2	613.55	107.68	N/A		N/A		N/A	N/A
7-d BMW 320i	R.Avoid	3937500	32	10169.4	N/M	$\geq$ 24h	21.5h	N/M	N/M	825.62	N/S		N/S		N/A	N/A
7-d BMW 320i†	R.Avoid	3937500	32	30.64	$\geq$ 24h	$\geq$ 24h	$\geq$ 24h	$\geq$ 24h	$\geq$ 24h	1251.7	N/A		N/A		N/A	N/A

5.3 Summary

In this chapter, we developed a software tool, called AMYTISS, in C++/OpenCL that provides scalable parallel algorithms for first constructing finite MDPs from discrete-time stochastic control systems and then synthesizing automatically their controllers satisfying complex logic properties including safety, reachability, and reach-avoid specifications. The software tool is developed based on theoretical results of the previous chapter (finite abstraction), and can utilize high-performance computing platforms and cloud-computing services to mitigate effects of the state-explosion problem, which is always present in analyzing large-scale stochastic systems. We showed that this tool significantly improves performances w.r.t. the computation time and memory usage by the parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators. We also showed that this tool outperforms all existing tools available in the literature.

6 Conclusions and Future Contributions

6.1 Conclusions

In this thesis, we proposed novel compositional techniques to analyze and control large-scale stochastic CPSs in an automated as well as formal fashion. In the first part of the thesis, we discussed compositional infinite abstractions (model order reductions) of original systems with three different compositional techniques including classic small-gain, max small-gain and dissipativity approaches. We showed that the proposed max small-gain approach is more general than the classic one since it does not require any linear growth on the gains of subsystems which is the case in the classic small-gain. We also proved that the provided approximation error via the max small-gain does not change as the number of subsystems grows. This issue is due to the fact that the proposed overall error is completely independent of the size of the network, and is computed only based on the maximum error of subsystems instead of being a linear combination of them which is the case in classic small-gain and dissipativity approaches. On the other hand, we discussed that the proposed dissipativity technique is less conservative than the classic (or max) small-gain approach in the sense that the provided dissipativity-type compositionality condition can enjoy the structure of the interconnection topology and be potentially fulfilled independently of the number or gains of subsystems.

In the second part of the thesis, we proposed compositional construction of finite MDPs as finite abstractions of given (reduced-order) systems with the same compositionality techniques. We showed that if the original system is incremental input-to-state stable (or incrementally passivable in the dissipativity setting), one can construct finite MDPs of original systems for the general setting of nonlinear stochastic control systems. We also proposed novel frameworks for the construction of finite MDPs for some particular classes of nonlinear stochastic systems whose nonlinearities satisfy a slope restriction or (in a more general form) an incremental quadratic inequality. We generalized our results from control systems to switched ones whose switching signals accept a dwell-time condition with multiple Lyapunov-like functions. Moreover, we proposed relaxed versions of small-gain and dissipativity approaches in which the stabilizability of individual subsystems for providing the compositionality results is not necessarily required. We then proposed a compositional technique for the construction of both infinite and finite abstractions in a unified framework via notions of approximate probabilistic relations. We showed that the unified compositional framework is less conservative than the two-step consecutive procedure that independently constructs infinite and finite abstractions. We finally proposed a novel model-free reinforcement learning scheme to synthesize policies for unknown, continuous-space MDPs. We provided approximate optimality guarantees between unknown original models and that of their finite MDPs. We discussed that via the proposed model-free learning framework not only one can synthesize controllers for unknown stochastic systems, but also the curse of dimensionality problem is remarkably mitigated.

In the last part of the thesis, we developed a software tool in C++/OpenCL, called AMYTISS, for designing correct-by-construction controllers of large-scale discrete-time stochastic systems. This software tool provides scalable parallel algorithms that allow to (i) construct finite MDPs from discrete-time stochastic control systems, and (ii) synthesize controllers automatically that satisfy complex logic properties including safety, reachability, and reach-avoid specifications. AMYTISS is developed based on theoretical results on constructing finite abstractions by employing high-performance computing platforms and cloud-computing services to alleviate effects of the state-explosion problem, which is always the case in analyzing large-scale stochastic systems. We showed that this tool significantly improves performances w.r.t. the computation time and memory usage by the parallel execution in different heterogeneous computing platforms including CPUs, GPUs and hardware accelerators. We also showed that this tool outperforms all existing tools available in the literature.

6.2 Recommendations for Future Research

In this section, we discuss some interesting topics that could be considered as potential future research lines.

- **Compositional controller synthesis.** In this thesis, we widely studied different compositional approaches for the construction of (in)finite abstractions for networks of stochastic control (switched) systems. One potential direction as a future work is to investigate the compositional controller synthesis for stochastic systems. In particular given a specification over the interconnected system, one can study the formal relation between the probability of satisfactions provided by local controllers for individual subsystems and that of their monolithic ones in the interconnected case.

- **Decomposition of more complex LTL properties.** In this thesis, we mainly considered our specifications as the safety. In particular, we considered the overall safety specification as a hyper-rectangle (a.k.a. hyper interval) and decomposed and projected it to different dimensions corresponding to subsystems. We first designed local controllers for abstractions $\widehat{\Sigma}_i$, and then refined them back to subsystems Σ_i using interface functions. Consequently, the controller for the interconnected system Σ is simply constructed by augmenting controllers of subsystems Σ_i. Another direction as the future research line is to consider more complex LTL properties including reachability, reach-avoid, etc., and study how to decompose these high-level specifications in order to provide a compositional synthesis framework for them.

- **Compositional barrier certificate.** In order to deal with the computational complexity arising with the construction of finite abstractions proposed in this thesis, there have also been discretization-free approaches based on control barrier certificates. One promising direction is to develop the barrier certificate approach to come up with a compositional approach for the temporal logic verification and synthesis of stochastic CPSs.

- max **dissipativity approaches.** In Sections 3.3 and 4.2, we proposed max small-gain approaches and showed that they are less conservative than the classic one provided in Section 3.2 since their approximation error does not change as the number of subsystems grows. One potential direction for an extension is to develop a compositional approach based on max dissipativity with the approximation error independent of the size of the network, and being only based on the maximum error of subsystems instead of a linear combination of them which is the case in the current dissipativity approach proposed in Sections 3.4 and 4.3.

- **Switched systems with unstable (unstabilizable) subsystems**. In Sections 4.2.2 and 4.3.2, we assumed that the given original switched subsystems are stable. It would be interesting if one can provide a compositional framework for stochastic switched systems accepting dwell-time and multiple Lyapunov functions but with some unstabilizable modes.

- **Constructing finite MDPs with discretization-free approaches.** In order to construct finite MDPs from original stochastic systems via Algorithm 1, we needed to discretize the state space of the system. This issue in general creates the state-explosion problem which is always present in analyzing large-scale stochastic systems. There are some discretization-free approaches for building symbolic models of original systems in the non-stochastic setting [ZAG15],[ZTA14],[ZG15]. It would be interesting if one can leverage the ideas there and provide a discretization-free framework for the construction of finite MDPs.

- **Compositional controller synthesis for unknown stochastic systems via reinforcement learning.** In Section 4.7, we proposed an approach for the controller synthesis of unknown continuous-space MDPs via the model-free reinforcement learning. One potential direction is to provide a compositional framework for the controller synthesis of unknown stochastic systems via the reinforcement learning.

- **Closeness guarantee for unknown stochastic systems via reinforcement learning for infinite-time horizon.** In Section 4.7, we proposed probabilistic closeness guarantees between unknown continuous-space original models and that of their finite MDPs for the finite-time horizon. It would be interesting if one can extend the results to an infinite-time horizon via the model-free reinforcement learning.

- **Extension of AMYTISS.** In our proposed software tool, AMYTISS, in Chapter 5, we assumed that our dynamics are *discrete-time* stochastic control systems. Providing a tool for large-scale *continuous-time* stochastic systems is an interesting direction as a future work.

Bibliography

[Aba13] A. Abate. Approximation metrics based on probabilistic bisimulations for general state-space Markov processes: A survey. *Electronic Notes in Theoretical Computer Science*, 297:3–25, 2013.

[ABC+18] A. Abate, H. Blom, N. Cauchi, S. Haesaert, A. Hartmanns, K. Lesser, M. Oishi, V. Sivaramakrishnan, S. Soudjani, C. I. Vasile, et al. ARCH-COMP18 category report: Stochastic modelling. In *ARCH@ ADHS*, pages 71–103, 2018.

[ABC+19] A. Abate, H. Blom, N. Cauchi, K. Degiorgio, M. Fränzle, E. M. Hahn, S. Haesaert, H. Ma, M. Oishi, C. Pilch, et al. ARCH-COMP19 category report: Stochastic modelling. *EPiC Series in Computing*, 61:62–102, 2019.

[AC11] B. Açıkmeşe and M. Corless. Observers for systems with nonlinearities satisfying incremental quadratic constraints. *Automatica*, 47(7):1339–1348, 2011.

[AKNP14] Al. Abate, M. Kwiatkowska, G. Norman, and D. Parker. Probabilistic model checking of labelled Markov processes via finite approximate bisimulations. In *Horizons of the Mind. A Tribute to Prakash Panangaden*, pages 40–58. 2014.

[Alt19] M. Althof. Commonroad: Vehicle models (version 2018a). Tech. rep. In *Technical University of Munich, 85748 Garching, Germany (October 2018)*, https://commonroad.in.tum.de. 2019.

[AM07] P. J. Antsaklis and A. N. Michel. *A linear systems primer*, volume 1. Birkhäuser Boston, 2007.

[AMP16] M. Arcak, C. Meissen, and A. Packard. *Networks of dissipative systems.* Springer, 2016.

[Ang02] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

[AP98] D. Aeyels and J. Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. *IEEE Transactions on automatic control*, 43(7):968–971, 1998.

[APLS08] A. Abate, M. Prandini, J. Lygeros, and S. Sastry. Probabilistic reachability and safety for controlled discrete-time stochastic hybrid systems. *Automatica*, 44(11):2724–2734, 2008.

[Axe94] O. Axelsson. Iterative solution methods. cambridge univ. *Press, Cambridge*, 1994.

[BDS05] S. Battilotti and A. De Santis. Dwell-time controllers for stochastic systems with switching Markov chain. *Automatica*, 41(6):923–934, 2005.

[Bel65] H. E. Bell. Gershgorin's theorem and the zeros of polynomials. *The American Mathematical Monthly*, 72(3):292–295, 1965.

[BK08] C. Baier and J.-P. Katoen. *Principles of model checking*. MIT press, 2008.

[BKTW15] N. Basset, M. Kwiatkowska, U. Topcu, and C. Wiltsche. Strategy synthesis for stochastic games with multiple long-run objectives. In *Proceedings of the International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 256–271, 2015.

[BKW18] N. Basset, M. Kwiatkowska, and C. Wiltsche. Compositional strategy synthesis for stochastic games with multiple objectives. *Information and Computation*, 261:536–587, 2018.

[BM00] V. S. Borkar and S. P. Meyn. The ODE method for convergence of stochastic approximation and reinforcement learning. *SIAM Journal on Control and Optimization*, 38(2):447–469, 2000.

[BMAS19] E. Bıyık, J. Margoliash, S. R. Alimo, and D. Sadigh. Efficient and safe exploration in deterministic Markov decision processes with unknown transition models. *arXiv:1904.01068*, 2019.

[BS96] D. P. Bertsekas and S. E. Shreve. *Stochastic optimal control: The discrete-time case*. Athena Scientific, 1996.

[BV04] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[BYG17] C. Belta, B. Yordanov, and E. Aydin Gol. *Formal methods for discrete-time dynamical systems*, volume 89 of *Studies in Systems, Decision and Control*. Springer, 2017.

[CDA19] N. Cauchi, K. Degiorgio, and A. Abate. StocHy: Automated verification and synthesis of stochastic processes. In *TACAS'19*, Lecture Notes in Computer Science, pages 247–264. 2019.

[CDB12] Y. Chen, K. Deng, and C. Belta. Multi-agent persistent monitoring in stochastic environments with temporal logic constraints. In *Proceedings of* *the 51st IEEE Conference on Decision and Control (CDC)*, pages 2801–2806, 2012.

[CLL+19] N. Cauchi, L. Laurenti, M. Lahijanian, A. Abate, M. Kwiatkowska, and L. Cardelli. Efficiency through uncertainty: scalable formal synthesis for stochastic hybrid systems. In *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, pages 240–251, 2019.

[DAK12] A. D'Innocenzo, A. Abate, and J.P. Katoen. Robust PCTL model checking. In *Proceedings of the 15th ACM international conference on Hybrid Systems: Computation and Control*, pages 275–286, 2012.

[DCB17] K. Deng, Y. Chen, and C. Belta. An approximate dynamic programming approach to multiagent persistent monitoring in stochastic environments with temporal logic constraints. *IEEE Transactions on Automatic Control*, 62(9):4549–4563, 2017.

[DGJP04] J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden. Metrics for labelled Markov processes. *Theoretical computer science*, 318(3):323–354, 2004.

[DIW11] S. Dashkovskiy, H. Ito, and F. Wirth. On a small gain theorem for ISS networks in dissipative Lyapunov form. *European Journal of Control*, 17(4):357–365, 2011.

[DKS+13] J. Ding, M. Kamgarpour, S. Summers, A. Abate, J. Lygeros, and C. Tomlin. A stochastic games framework for verification and control of discrete time stochastic hybrid systems. *Automatica*, 49(9):2665–2674, 2013.

[DLT08a] J. Desharnais, F. Laviolette, and M. Tracol. Approximate analysis of probabilistic processes: logic, simulation and games. In *Proceedings of the International Conference on Quantitative Evaluation of SysTems*, pages 264–273, 2008.

[DLT08b] J. Desharnais, F. Laviolette, and M. Tracol. Approximate analysis of probabilistic processes: Logic, simulation and games. In *Proceedings of the 5th international conference on quantitative evaluation of system*, pages 264–273, 2008.

[DRW07] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems (MCSS)*, 19(2):93–122, 2007.

[DRW10] S. N Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[DSL+17] Bo Dai, Albert Shaw, Lihong Li, Lin Xiao, Niao He, Zhen Liu, Jianshu Chen, and Le Song. Sbeed: Convergent reinforcement learning with nonlinear function approximation. *arXiv:1712.10285*, 2017.

[FHH+11] M. Fränzle, E. M. Hahn, H. Hermanns, N. Wolovick, and L. Zhang. Measurability and safety verification for stochastic hybrid systems. In *Proceedings of the 14th international conference on Hybrid systems: computation and control*, pages 43–52, 2011.

[FMPS17] S. S. Farahani, R. Majumdar, V. S. Prabhu, and S. Soudjani. Shrinking horizon model predictive control with chance-constrained signal temporal logic specifications. In *2017 American Control Conference (ACC)*, pages 1740–1746, 2017.

[FMPS19] S. S. Farahani, R. Majumdar, V. S. Prabhu, and S. Soudjani. Shrinking horizon model predictive control with signal temporal logic constraints under stochastic disturbances. *IEEE Transactions on Automatic Control*, 64(8):3324–3331, 2019.

[FSMOM17] S. S. Farahani, S. Soudjani, R. Majumdar, and C. Ocampo-Martinez. Robust model predictive control with signal temporal logic constraints for barcelona wastewater system. *IFAC-PapersOnLine*, 50(1):6594 – 6600, 2017. 20th IFAC World Congress.

[FSMOM18] S. S. Farahani, S. Soudjani, R. Majumdar, and C. Ocampo-Martinez. Formal controller synthesis for wastewater systems with signal temporal logic constraints: The barcelona case study. *Journal of Process Control*, 69:179 – 191, 2018.

[FT15] J. Fu and U. Topcu. Computational methods for stochastic control with metric interval temporal logic specifications. In *Proceedings of the 54th IEEE Conference on Decision and Control (CDC)*, pages 7440–7447, 2015.

[GGLW14] R. Geiselhart, R. H. Gielen, M. Lazar, and F. R. Wirth. An alternative converse Lyapunov theorem for discrete-time systems. *Systems & Control Letters*, 70:49–59, 2014.

[GL12] R. H. Gielen and M. Lazar. Non-conservative dissipativity and small-gain conditions for stability analysis of interconnected systems. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, pages 4187–4192, 2012.

[GL15] R. H. Gielen and M. Lazar. On stability analysis methods for large-scale discrete-time systems. *Automatica*, 55:66–72, 2015.

[GP09] A. Girard and G. J. Pappas. Hierarchical control system design using approximate simulation. *Automatica*, 45(2):566–571, 2009.

[GR01] C. Godsil and G. Royle. *Algebraic graph theory*. Graduate Texts in Mathematics. Springe, New York, 2001.

[HCL+17] C. Huang, X. Chen, W. Lin, Z. Yang, and X. Li. Probabilistic safety verification of stochastic hybrid systems using barrier certificates. *ACM Transactions on Embedded Computing Systems (TECS)*, 16(5s):186, 2017.

[HH14] A. Hartmanns and H. Hermanns. The modest toolset: An integrated environment for quantitative modelling and verification. In *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 593–598, 2014.

[HPS+19a] E. M. Hahn, M. Perez, S. Schewe, F. Somenzi, A. Trivedi, and D. Wojtczak. Omega-regular objectives in model-free reinforcement learning. In *Proceedings of the International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 395–412, 2019.

[HPS+19b] E. M. Hahn, M. Perez, S. Schewe, F. Somenzi, A. Trivedi, and D. Wojtczak. Omega-regular objectives in model-free reinforcement learning. In *Tools and Algorithms for the Construction and Analysis of Systems*, pages 395–412, 2019. LNCS 11427.

[HS18] S. Haesaert and S. Soudjani. Robust dynamic programming for temporal logic control of stochastic systems. *CoRR*, abs/1811.11445, 2018.

[HS19] S. Haesaert and S. Soudjani. Achievements in correct-by-design control for stochastic systems. In *Proceedings of the 5th International Workshop on Symbolic-Numeric Methods for Reasoning about CPS and IoT*, 2019.

[HSA17] S. Haesaert, S. Soudjani, and A. Abate. Verification of general Markov decision processes by approximate similarity relations and policy refinement. *SIAM Journal on Control and Optimization*, 55(4):2333–2367, 2017.

[HSA18] S. Haesaert, S. Soudjani, and A. Abate. Temporal logic control of general Markov decision processes by approximate policy refinement. *Proceedings of the 6th IFAC Conference on Analysis and Design of Hybrid Systems*, 51(16):73 – 78, 2018.

[JJS94] T. Jaakkola, M. I. Jordan, and S. P. Singh. Convergence of stochastic iterative dynamic programming algorithms. In *Advances in neural information processing systems*, pages 703–710, 1994.

[JP09] A. A. Julius and G. J. Pappas. Approximations of stochastic hybrid systems. *IEEE Transactions on Automatic Control*, 54(6):1193–1203, 2009.

[JSZ18] P. Jagtap, S. Soudjani, and M. Zamani. Temporal logic verification of stochastic systems using barrier certificates. In *Proceedings of the International Symposium on Automated Technology for Verification and Analysis*, pages 177–193, 2018.

[JSZ19] P. Jagtap, S. Soudjani, and M. Zamani. Formal synthesis of stochastic systems via control barrier certificates. *arXiv: 1905.04585*, 2019.

[Kal97] O. Kallenberg. *Foundations of modern probability.* Springer-Verlag, New York, 1997.

[KDS+11] M. Kamgarpour, J. Ding, S. Summers, A. Abate, J. Lygeros, and C. Tomlin. Discrete time stochastic hybrid dynamical games: Verification & controller synthesis. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 6122–6127, 2011.

[KES+13] M. Kamgarpour, C. Ellen, S. Soudjani, S. Gerwinn, J.L. Mathieu, N. Mullner, A. Abate, D.S. Callaway, M. Franzle, and J. Lygeros. Modeling options for demand side participation of thermostatically controlled loads. In *Proceedings of the Bulk Power System Dynamics and Control - IX Optimization, Security and Control of the Emerging Power Grid (IREP)*, pages 1–15, 2013.

[KNPQ13] M. Kwiatkowska, G. Norman, D. Parker, and H. Qu. Compositional probabilistic verification through multi-objective model checking. *Information and Computation*, 232:38–65, 2013.

[KSL13] M. Kamgarpour, S. Summers, and J. Lygeros. Control design for specifications on stochastic hybrid systems. In *Proceedings of the 16th international conference on Hybrid systems: computation and control*, pages 303–312, 2013.

[Kus67] H. J. Kushner. *Stochastic Stability and Control.* Mathematics in Science and Engineering. Elsevier Science, 1967.

[KV01] O. Kupferman and M. Y. Vardi. Model checking of safety properties. *Formal Methods in System Design*, 19(3):291–314, 2001.

[KZ19] M. Khaled and M. Zamani. pFaces: An acceleration ecosystem for symbolic control. In *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, pages 252–257, 2019.

[LAB09] M. Lahijanian, S. B. Andersson, and C. Belta. A probabilistic approach for control of a stochastic system from LTL specifications. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, pages 2236–2241, 2009.

[LAB12] M. Lahijanian, S. B. Andersson, and C. Belta. Approximate Markovian abstractions for linear stochastic systems. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, pages 5966–5971, 2012.

[LAB15] M. Lahijanian, S. B. Andersson, and C. Belta. Formal verification and synthesis for discrete-time stochastic systems. *IEEE Transactions on Automatic Control*, 60(8):2031–2045, 2015.

[LCGG13] E.l Le Corronc, A. Girard, and G. Goessler. Mode sequences as symbolic states in abstractions of incrementally stable switched systems. In *Proceedings of the 52th IEEE Conference on Decision and Control*, pages 3225–3230, 2013.

[LFDA16] S. Levine, C. Finn, T. Darrell, and P. Abbeel. End-to-end training of deep visuomotor policies. *J. Mach. Learn. Res.*, 17(1):1334–1373, 2016.

[LHP+15] T. P. Lillicrap, J. J. Hunt, A. Pritzel, N. Heess, T. Erez, Y. Tassa, D. Silver, and D. Wierstra. Continuous control with deep reinforcement learning. *CoRR*, abs/1509.02971, 2015.

[Lib03] D. Liberzon. *Switching in systems and control.* Springer Science & Business Media, 2003.

[LKSZ20a] A. Lavaei, M. Khaled., S. Soudjani, and M. Zamani. AMYTISS: A parallelized tool on automated controller synthesis for large-scale stochastic systems. In *Proceedings of the 23rd ACM International Conference on Hybrid Systems: Computation and Control, to apear*, 2020.

[LKSZ20b] A. Lavaei, M. Khaled, S. Soudjani, and M. Zamani. AMYTISS: Parallel automated controller synthesis for large-scale stochastic Systems. *Submitted for Publication*, 2020.

[LS91] K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and computation*, 94(1):1–28, 1991.

[LSMZ17] A. Lavaei, S. Soudjani, R. Majumdar, and M. Zamani. Compositional abstractions of interconnected discrete-time stochastic control systems. In *Proceedings of the 56th IEEE Conference on Decision and Control*, pages 3551–3556, 2017.

[LSS+20] A. Lavaei, F. Somenzi, S. Soudjani, A. Trivedi, and M. Zamani. Formal controller synthesis of unknown continuous-space MDPs via model-free reinforcement learning. In *Proceedings of the 11th ACM/IEEE Conference on Cyber-Physical Systems (ICCPS), to appear*, 2020.

[LSZ18a] A. Lavaei, S. Soudjani, and M. Zamani. Compositional synthesis of finite abstractions for continuous-space stochastic control systems: A small-gain approach. *Proceedings of the 6th IFAC Conference on Analysis and Design of Hybrid Systems*, 51(16):265–270, 2018.

[LSZ18b] A. Lavaei, S. Soudjani, and M. Zamani. Compositional synthesis of interconnected stochastic control systems based on finite MDPs. In *Proceedings of the 21st International Conference on Hybrid Systems: Computation and Control*, pages 273–274, 2018.

[LSZ18c] A. Lavaei, S. Soudjani, and M. Zamani. From dissipativity theory to compositional construction of finite Markov decision processes. In *Proceedings of the 21st ACM International Conference on Hybrid Systems: Computation and Control*, pages 21–30, 2018.

[LSZ19a] A. Lavaei, S. Soudjani, and M. Zamani. Approximate probabilistic relations for compositional synthesis of stochastic systems. In *Proceedings of the Numerical Software Verification*, pages 101–109, 2019. Lecture Notes in Computer Science 11652.

[LSZ19b] A. Lavaei, S. Soudjani, and M. Zamani. Compositional abstraction-based synthesis of general MDPs via approximate probabilistic relations. *Submitted for Publication, arXiv:1906.02930*, 2019.

[LSZ19c] A. Lavaei, S. Soudjani, and M. Zamani. Compositional construction of infinite abstractions for networks of stochastic control systems. *Automatica*, 107:125–137, 2019.

[LSZ19d] A. Lavaei, S. Soudjani, and M. Zamani. Compositional synthesis of not necessarily stabilizable stochastic systems via finite abstractions. In *Proceedings of the 18th European Control Conference*, pages 2802–2807, 2019.

[LSZ20a] A. Lavaei, S. Soudjani, and M. Zamani. Compositional abstraction-based synthesis for networks of stochastic switched systems. *Automatica*, 114, 2020.

[LSZ20b] A. Lavaei, S. Soudjani, and M. Zamani. Compositional abstraction of large-scale stochastic systems: A relaxed dissipativity approach. *Nonlinear Analysis: Hybrid Systems*, 36, 2020.

[LSZ20c] A. Lavaei, S. Soudjani, and M. Zamani. Compositional (in)finite abstractions for large-scale interconnected stochastic systems. *IEEE Transactions on Automatic Control, to appear as a full paper*, 2020.

[LTS05] W. Li, E. Todorov, and R. E. Skelton. Estimation and control of systems with multiplicative noise via linear matrix inequalities. In *Proceedings of the 2005, American Control Conference, 2005.*, pages 1811–1816, 2005.

[LZ19a] A. Lavaei and M. Zamani. Compositional construction of finite MDPs for large-scale stochastic switched systems: A dissipativity approach. *Proceedings of the 15th IFAC Symposium on Large Scale Complex Systems: Theory and Applications*, 52(3):31–36, 2019.

[LZ19b] A. Lavaei and M. Zamani. Compositional finite abstractions for large-scale stochastic switched systems. In *Proceedings of the 5th International Workshop on Symbolic-Numeric Methods for Reasoning about CPS and IoT (SNR)*, pages 3–5, 2019.

[LZ19c] A. Lavaei and M. Zamani. Compositional verification of large-scale stochastic systems via relaxed small-gain conditions. In *Proceedings of the 58th IEEE Conference on Decision and Control, to appear*, 2019.

[LZ20] A. Lavaei and M. Zamani. From dissipativity theory to compositional synthesis of large-scale stochastic switched systems. *Under Submission*, 2020.

[MGW17] P. J. Meyer, A. Girard, and E. Witrant. Compositional abstraction and safety synthesis using overlapping symbolic models. *IEEE Transactions on Automatic Control*, 63(6):1835–1841, 2017.

[MKS+15] V. Mnih, K. Kavukcouglu, D. Silver, A. A. Rusu, J. Veness, M. G. Bellemare, A. Graves, M. Riedmiller, A. K. Fidjeland, G. Ostrovski, S. Petersen, C. Beattie, A. Sadik, I. Antonoglou, H. King, D. Kumaran, D. Wierstra, S. Legg, and D. Hassabis. Human-level control through reinforcement learning. *Nature*, 518:529–533, 2015.

[Mor96] A. S. Morse. Supervisory control of families of linear set-point controllers-part i. exact matching. *IEEE transactions on Automatic Control*, 41(10):1413–1431, 1996.

[MSH+17] N. Mehr, D. Sadigh, R. Horowitz, S. S. Sastry, and S. A. Seshia. Stochastic predictive freeway ramp metering from signal temporal logic specifications. In *Proceedings of the American Control Conference (ACC)*, pages 4884–4889, 2017.

[MSSM17] K. Mallik, S. Soudjani, A.-K. Schmuck, and R. Majumdar. Compositional construction of finite state abstractions for stochastic control systems. In *Proceedings of the 56th IEEE International Conference on Decision and Control (CDC)*, pages 550–557, 2017.

[MSSM19] K. Mallik, A. Schmuck, S. Soudjani, and R. Majumdar. Compositional synthesis of finite-state abstractions. *IEEE Transactions on Automatic Control*, 64(6):2629–2636, June 2019.

[NGG+18] N. Noroozi, R. Geiselhart, L. Grüne, B. S. Rüffer, and F. R. Wirth. Non-conservative discrete-time ISS small-gain conditions for closed sets. *IEEE Transactions on Automatic Control*, 63(5):1231–1242, 2018.

[NR14] N. Noroozi and B. S. Rüffer. Non-conservative dissipativity and small-gain theory for ISS networks. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 3131–3136, 2014.

[NSWZ18] N. Noroozi, A. Swikir, F. R. Wirth, and M. Zamani. Compositional construction of abstractions via relaxed small-gain conditions part ii: discrete case. In *Proceedings of the European Control Conference (ECC)*, pages 1–4, 2018.

[NSZ19] A. Nejati, S. Soudjani, and M. Zamani. Abstraction-based synthesis of continuous-time stochastic control systems. In *Proceedings of the 18th European Control Conference*, pages 3212–3217, 2019.

[PJP07] S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8):1415–1428, 2007.

[Pnu77] A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th Annual Symposium on Foundations of Computer Science*, pages 46–57, 1977.

[PTS09] Q. C. Pham, N. Tabareau, and J. J. Slotine. A contraction theory approach to stochastic incremental stability. *IEEE Transactions on Automatic Control*, 54(4):816–820, 2009.

[Ran03] B. Randrianantoanina. On the structure of level sets of uniform and Lipschitz quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$. *ArXiv Mathematics e-prints*, January 2003.

[Rie05] M Riedmiller. Neural fitted Q iteration – First experiences with a data efficient neural reinforcement learning method. In *Machine Learning: ECML 2005*, pages 317–328. Springer, 2005.

[Rüf10] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. *Positivity*, 14(2):257–283, 2010.

[RWR16] G. Reissig, A. Weber, and M. Rungger. Feedback refinement relations for the synthesis of symbolic controllers. *IEEE Transactions on Automatic Control*, 62(4):1781–1796, 2016.

[RZ16] M. Rungger and M. Zamani. SCOTS: A tool for the synthesis of symbolic controllers. In *Proceedings of the 19th ACM International Conference on Hybrid Systems: Computation and Control*, pages 99–104, 2016.

[SA11] S. Soudjani and A. Abate. Adaptive gridding for abstraction and verification of stochastic hybrid systems. In *Proceedings of the 8th International Conference on Quantitative Evaluation of Systems*, pages 59–69, 2011.

[SA12a] S. Soudjani and A. Abate. Higher-order approximations for verification of stochastic hybrid systems. In *Automated Technology for Verification and Analysis*, volume 7561 of *Lecture Notes in Computer Science*, pages 416–434. 2012.

[SA12b] S. Soudjani and A. Abate. Probabilistic invariance of mixed deterministic-stochastic dynamical systems. In *ACM Proceedings of the 15th International Conference on Hybrid Systems: Computation and Control*, pages 207–216, 2012.

[SA13a] S. Soudjani and A. Abate. Adaptive and sequential gridding procedures for the abstraction and verification of stochastic processes. *SIAM Journal on Applied Dynamical Systems*, 12(2):921–956, 2013.

[SA13b] S. Soudjani and A. Abate. Aggregation of thermostatically controlled loads by formal abstractions. In *ECC 2013: European Control Conference*, pages 4232–4237, 2013.

[SA14a] S. Soudjani and A. Abate. Precise approximations of the probability distribution of a Markov process in time: An application to probabilistic invariance. In *Proceedings of the International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 547–561, 2014.

[SA14b] S. Soudjani and A. Abate. Probabilistic reach-avoid computation for partially degenerate stochastic processes. *IEEE Transactions on Automatic Control*, 59(2):528–534, 2014.

[SA15a] S. Soudjani and A. Abate. Aggregation and control of populations of thermostatically controlled loads by formal abstractions. *Control Systems Technology, IEEE Transactions on*, 23(3):975–990, 2015.

[SA15b] S. Soudjani and A. Abate. Quantitative approximation of the probability distribution of a Markov process by formal abstractions. *Logical Methods in Computer Science*, 11(3):1–29, 2015.

[SAM15] S. Soudjani, A. Abate, and R. Majumdar. Dynamic Bayesian networks as formal abstractions of structured stochastic processes. In *Proceedings of the 26th International Conference on Concurrency Theory*, pages 1–14, 2015.

[SAM17] S. Soudjani, A. Abate, and R. Majumdar. Dynamic Bayesian networks for formal verification of structured stochastic processes. *Acta Informatica*, 54(2):217–242, 2017.

[SB18] R. S. Sutton and A. G. Barto. *Reinforcement Learnging: An Introduction.* MIT Press, second edition, 2018.

[Sch96] E. Schechter. *Handbook of Analysis and its Foundations.* San Diego: Academic Press, 1996.

[Sco92] D. W. Scott. *Multivariate Density Estimation. Theory, Practice, and Visualization.* Wiley, 1992.

[SDC19] C. Santoyo, M. Dutreix, and S.l Coogan. Verification and control for finite-time safety of stochastic systems via barrier functions. In *Proceedings of the IEEE Conference on Control Technology and Applications*, pages 712–717, 2019.

[SGA15] S. Soudjani, C. Gevaerts, and A. Abate. FAUST^2: Formal abstractions of uncountable-state stochastic processes. In *TACAS'15*, volume 9035 of *Lecture Notes in Computer Science*, pages 272–286. 2015.

[SGE+14] S. Soudjani, S. Gerwinn, C. Ellen, M. Fränzle, and A. Abate. Formal synthesis and validation of inhomogeneous thermostatically controlled loads. In *Proceedings of the International Conference on Quantitative Evaluation of Systems*, pages 57–73, 2014.

[SGZ18] A. Swikir, A. Girard, and M. Zamani. From dissipativity theory to compositional synthesis of symbolic models. In *Proceedings of the 4th Indian Control Conference (ICC)*, pages 30–35, 2018.

[SHM+16] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. van den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, S. Dieleman, D. Grewe, J. Nham, N. Kalchbrenner, I. Sutskever, T. Lillicrap, , M. Leach, K. Kavukcuoglu, , T. Graepel, and D. Hassabis. Mastering the game of Go with deep neural networks and tree search. *Nature*, 529:484–489, January 2016.

[SKC+17] M. Svoreňová, J. Křetínskỳ, M. Chmelík, K. Chatterjee, I. Černá, and C. Belta. Temporal logic control for stochastic linear systems using abstraction refinement of probabilistic games. *Nonlinear Analysis: Hybrid Systems*, 23:230–253, 2017.

[SL95] R. Segala and N. Lynch. Probabilistic simulations for probabilistic processes. *Nordic Journal of Computing*, 2(2):250–273, 1995.

[SLW+06] A. L. Strehl, L. Li, E. Wiewiora, J. Langford, and M. L. Littman. PAC model-free reinforcement learning. In *Proceedings of the International Conference on Machine Learning, ICML*, pages 881–888, 2006.

[SMA16] S. Soudjani, R. Majumdar, and A. Abate. Safety verification of continuous-space pure jump Markov processes. In *Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*, pages 147–163. 2016.

[Sou14] S. Soudjani. *Formal Abstractions for Automated Verification and Synthesis of Stochastic Systems.* PhD thesis, Technische Universiteit Delft, The Netherlands, 2014.

[SPB+17] F. Shmarov, N. Paoletti, E. Bartocci, S. Lin, S. A. Smolka, and P. Zuliani. Automated synthesis of safe and robust PID controllers for stochastic hybrid systems. *arXiv:1707.05229*, 2017.

[SSP+19] F. Shmarov, S. Soudjani, N. Paoletti, E. Bartocci, S. Lin, S. A. Smolka, and P. Zuliani. Automated synthesis of safe digital controllers for sampled-data stochastic nonlinear systems. *arXiv:1901.03315*, 2019.

[ST12] J. Steinhardt and R. Tedrake. Finite-time regional verification of stochastic non-linear systems. *The International Journal of Robotics Research*, 31(7):901–923, 2012.

[Sut88] R. S. Sutton. Learning to predict by the methods of temporal differences. *Machine Learning*, 3(1):9–44, Aug 1988.

[SZ15] F. Shmarov and P. Zuliani. ProbReach: verified probabilistic delta-reachability for stochastic hybrid systems. In *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, pages 134–139, 2015.

[TA11] I. Tkachev and A. Abate. On infinite-horizon probabilistic properties and stochastic bisimulation functions. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 526–531, 2011.

[Tab09] P. Tabuada. *Verification and control of hybrid systems: a symbolic approach.* Springer Science & Business Media, 2009.

[Tes95] G. Tesauro. Temporal difference learning and TD-Gammon. *Commun. ACM*, 38(3):58–68, March 1995.

[TMKA13] I. Tkachev, A. Mereacre, Joost-Pieter Katoen, and A. Abate. Quantitative automata-based controller synthesis for non-autonomous stochastic hybrid systems. In *Proceedings of the 16th ACM International Conference on Hybrid Systems: Computation and Control*, pages 293–302, 2013.

[TMKA17] I. Tkachev, A. Mereacre, J.-P. Katoen, and A. Abate. Quantitative model-checking of controlled discrete-time Markov processes. *Information and Computation*, 253:1–35, 2017.

[TRK18] D. N. Tran, B. S. Rüffer, and C. M. Kellett. Convergence properties for discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 64(8):3415–3422, 2018.

[VGO19] A. P. Vinod, J. D. Gleason, and M. M. Oishi. SReachTools: A MATLAB stochastic reachability toolbox. In *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, pages 33–38, 2019.

[Wat89] C. J. C. H. Watkins. *Learning from delayed rewards.* PhD thesis, King's College, Cambridge, 1989.

[WT16] M. Wen and U. Topcu. Probably approximately correct learning in stochastic games with temporal logic specifications. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 3630–3636, 2016.

[WZK+15] Q. Wang, P. Zuliani, S. Kong, S. Gao, and E. M. Clarke. SReach: A probabilistic bounded delta-reachability analyzer for stochastic hybrid systems. In *Proceedings of the International Conference on Computational Methods in Systems Biology*, pages 15–27, 2015.

[XLSM13] J. Xiong, J. Lam, Z. Shu, and X. Mao. Stability analysis of continuous-time switched systems with a random switching signal. *IEEE Transactions on Automatic Control*, 59(1):180–186, 2013.

[You12] W. H. Young. On classes of summable functions and their Fourier series. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 87(594):225–229, 1912.

[ZA14] M. Zamani and A. Abate. Approximately bisimilar symbolic models for randomly switched stochastic systems. *Systems & Control Letters*, 69:38–46, 2014.

[ZA18] M. Zamani and M. Arcak. Compositional abstraction for networks of control systems: A dissipativity approach. *IEEE Transactions on Control of Network Systems*, 5(3):1003–1015, 2018.

[ZAG15] M. Zamani, A. Abate, and A. Girard. Symbolic models for stochastic switched systems: A discretization and a discretization-free approach. *Automatica*, 55:183–196, 2015.

[Zam14] M. Zamani. Compositional approximations of interconnected stochastic hybrid systems. In *Proceedings of the 53rd IEEE Conference on Decision and Control (CDC)*, pages 3395–3400, 2014.

[ZG15] M. Zamani and A. Girard. Bisimilar symbolic models for stochastic switched systems: A discretization-free approach. In *Proceedings of the American Control Conference (ACC)*, pages 1923–1928, 2015.

[ZMEM+14] M. Zamani, P. Mohajerin Esfahani, R. Majumdar, A. Abate, and J. Lygeros. Symbolic control of stochastic systems via approximately bisimilar finite abstractions. *IEEE Transactions on Automatic Control*, 59(12):3135–3150, 2014.

[ZRME17] M. Zamani, M. Rungger, and P. Mohajerin Esfahani. Approximations of stochastic hybrid systems: A compositional approach. *IEEE Transactions on Automatic Control*, 62(6):2838–2853, 2017.

[ZTA14] M. Zamani, I. Tkachev, and A. Abate. Bisimilar symbolic models for stochastic control systems without state-space discretization. In *Proceedings of the 17th international conference on Hybrid systems: computation and control*, pages 41–50, 2014.