Compositional abstraction for interconnected systems over Riemannian manifolds: A dissipativity approach

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***Abstract*— In this work, we derive sufficient conditions under which compositional abstractions of interconnected systems evolving on Riemannian manifolds can be constructed using the interconnection topology and joint *differential* dissipativity-type properties of subsystems and their abstractions. This allows for a much broader variety of systems than the ones considered in the existing works defined over Euclidean spaces. In the proposed framework, the abstraction, itself a control system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process. We provide an example to illustrate the effectiveness of the proposed differential dissipativity-type compositional reasoning for interconnected control systems.**

I. Introduction

Control and analysis of large-scale interconnected systems has recently attracted significant attention because they appear in many modern applications such as transportation systems, power networks, and air traffic control. For those large-scale interconnected systems, controller design to achieve some complex specifications in a reliable and cost effective way is a challenging task. One direction which has been explored to overcome this challenge is to use a simpler (e.g. lower dimension) (in)finite approximation (referred to as *abstraction*) of the given system as a replacement in the controller design process. This allows for a design of a controller for the abstraction, which can be refined to the one for the original complex system. The error between the outputs of the original system and its abstraction can be quantified a priori.

Many large-scale complex systems can be regarded as interconnected systems consisting of smaller components. Rather than treating the interconnected system in a monolithic manner, an approach which severely restricts the capability of existing techniques to deal with many subsystems, one can employ a "divide and conquer" strategy wherein an abstraction of the original network can be provided by constructing abstractions of the subsystems and their interconnections. This is referred to as a *compositional* approach for constructing abstractions. Recently, there have been several results on the compositional construction of (in)finite abstractions of deterministic control systems including [1], [2], [3], and of a class of stochastic hybrid systems [4]. These results use a small-gain type condition to enable the compositional construction of abstractions. However, as shown in [5], this type of condition is a function of the size of the network and can be violated as the number of subsystems grows.

Recently in [6], a compositional framework for the construction of infinite abstractions of networks of control systems has been proposed using dissipativity theory. In this result a notion of storage function is proposed which describes joint dissipativity-type properties of control systems and their abstractions. This notion is used to derive compositional conditions under which a network of abstractions approximate a network of concrete subsystems. Those conditions can be independent of the number of the subsystems under some properties on the interconnection topologies and joint dissipativity properties of subsystems and their abstractions. This approach was extended to a class of stochastic hybrid systems in [7].

All the aformentioned results in the context of (in)finite abstractions consider systems evolving over the Euclidean spaces. The state-space of many systems constitute Riemannian manifolds [8], and consequently their analysis requires techniques from differential geometry [9]. In this work, for the first time, we propose techniques for compositional construction of infinite abstractions for interconnected control systems evolving over smooth Riemannian manifolds. We introduce a notion of so-called differential storage functions, adopted from the notion of differential storage functions introduced in the context of differential dissipativity [10], describing joint differential dissipativity properties of control subsystems and their abstractions. Given a network of control subsystems and the differential storage functions between them and their abstractions, we derive sufficient conditions based on the interconnection topology, guaranteeing that a network of abstractions quantitatively approximates the original network of concrete subsystems.

II. Control Systems

A. Notation

The sets of non-negative integer and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Those symbols are subscripted to restrict them in the usual way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. The symbol $\mathbb{R}^{n\times m}$ denotes the

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vector space of real matrices with n rows and m columns. The symbols $\vec{1}_n, \vec{0}_n, I_n, 0_{n\times m}$ denote the vector with all its elements to be one, the zero vector, identity, and zero matrices in $\mathbb{R}^n$, $\mathbb{R}^n, \mathbb{R}^{n\times n}$, and $\mathbb{R}^{n\times m}$, respectively. For $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval in $\mathbb{R}$ is denoted by $[a, b]$. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b]$ to denote the corresponding interval in $\mathbb{N}$. Given $N \in \mathbb{N}_{\geq 1}$, vectors $x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}_{\geq 1}$ and $i \in [1; N]$, we use $x = [x_1; \ldots; x_N]$ to denote the concatenated vector in $\mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the Euclidean norm of x. Given matrices $M_1, \ldots, M_n$, the notation $\mathsf{diag}(M_1, \ldots, M_n)$ represents a block diagonal matrix with diagonal matrix entries $M_1, \ldots, M_n$. Given a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty :=$ (ess)sup$\{\|f(t)\|,\ t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the map $\beta(r, t)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed non zero r, the map $\beta(r, t)$ is decreasing with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$. An (n-dimensional) manifold $\mathcal{M}_n$ is a pair $(\mathcal{M}_n, \mathcal{A}^+)$ where $\mathcal{M}_n$ is a set and $\mathcal{A}^+$ is a maximal atlas into $\mathbb{R}^n$, such that the topology induced by $\mathcal{A}^+$ is Hausdorff and second countable. We denote the tangent space of $\mathcal{M}_n$ at $x \in \mathcal{M}_n$ by $\mathcal{T}_x\mathcal{M}_n$, and the tangent bundle of $\mathcal{M}_n$ by $\mathcal{T}\mathcal{M}_n = \bigcup_{x\in\mathcal{M}_n}\{x\} \times \mathcal{T}_x\mathcal{M}_n$. A curve on the manifold is a mapping $\gamma : I \subset \mathbb{R} \to \mathcal{M}_n$. A distance (or metric) $d : \mathcal{M}_n \times \mathcal{M}_n \to \mathbb{R}_{\geq 0}$ on a manifold $\mathcal{M}_n$ is a continuous positive function that satisfies $d(x, y) = 0$ if and only if $x = y$ for each $x, y \in \mathcal{M}_n$, and $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in \mathcal{M}_n$. A (pseudo) *Riemannian* metric [11] on a smooth manifold $\mathcal{M}_n$ is a smoothly varying inner product on the tangent bundle $\mathcal{T}\mathcal{M}_n$ of manifold $\mathcal{M}_n$. Given $\mathcal{M}_n$, and a matrix valued map $G : \mathcal{M}_n \to \mathbb{R}^{n\times n}$ such that $G(x)$ is a positive (semi) definite matrix for each $x \in \mathcal{M}_n$, the (pseudo) Riemannian metric corresponding to the (pseudo) Riemannian structure G is given by $\delta x^T G(x)\delta y$ for each $x \in \mathcal{M}_n$, $\delta x \in \mathcal{T}_x\mathcal{M}_n$ and $\delta y \in \mathcal{T}_x\mathcal{M}_n$. Given two points $x, y \in \mathcal{M}_n$, a smooth curve $\gamma : [0, 1] \to \mathcal{M}_n$ such that $\gamma(0) = x$, and $\gamma(1) = y$, and a (pseudo) Riemannian structure G defined on $\mathcal{M}_n$, we define the (pseudo) Riemannian energy functional as $E_G(\gamma) = \int_0^1 \frac{\partial\gamma}{\partial s}^T(s)G(\gamma(s))\frac{\partial\gamma}{\partial s}(s)ds$. For two points $z_1, z_2 \in \mathcal{M}_n$, $\Gamma(z_1, z_2)$ denotes the set of piecewise continuous curves connecting z_1 and z_2: $\Gamma(z_1, z_2) = \{\gamma : [0, 1] \to \mathcal{M}_n | \gamma \text{ is piecewise continuous }, \gamma(0) = z_1, \gamma(1) = z_2\}$. Given two points $x, y \in \mathcal{M}_n$, a Riemannian structure G defined on $\mathcal{M}_n$, $\underset{\gamma\in\Gamma(x,y)}{\arg\min} \int_0^1 \sqrt{\frac{\partial\gamma}{\partial s}^T(s)G(\gamma(s))\frac{\partial\gamma}{\partial s}(s)}ds$ is called a geodesic curve between x and y with respect to G. The n-dimensional manifold $\mathbb{S}^n$ is defined by $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

B. Control Systems

Now, we define the class of control systems investigated in this paper.

Definition 2.1: The class of *control systems* studied in this paper is a tuple $\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$, where

- $\mathcal{M}_n$ is an n-dimensional state manifold containing the origin, while $\mathbb{R}^m, \mathbb{R}^p, \mathbb{R}^{q_1}$, and $\mathbb{R}^{q_2}$ are the external input, internal input, external output, and internal output (Euclidean) spaces of dimension m, p, q_1, and q_2 respectively;
- $\mathcal{U}$ and $\mathcal{W}$ are subsets of sets of all measurable functions of time taking values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively;
- $f : \mathcal{M}_n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{M}_n$ is the continuously differentiable state evolution map. We assume that $f(0, 0, 0) = 0$;
- $h_1 : \mathcal{M}_n \to \mathbb{R}^{q_1}$ is the continuously differentiable external output map;
- $h_2 : \mathcal{M}_n \to \mathbb{R}^{q_2}$ is the continuously differentiable internal output map.

A control system Σ satisfies

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)), \end{cases} \tag{II.1}$$

for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where a locally absolutely continuous curve $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}_n$ is called a *state trajectory* of Σ, $\zeta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_1}$ is called an external output trajectory of Σ, and $\zeta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_2}$ is called an internal output trajectory of Σ. We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the state trajectory at time $t \in \mathbb{R}_{\geq 0}$ under the input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$, where $a \in \mathcal{M}_n$. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to the state trajectory $\xi_{a\upsilon\omega}$.

Definition 2.2: Given any

$$\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

the *variational control system* of Σ is given by the tuple

$$\delta\Sigma = (\mathcal{T}\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, \delta f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, \delta h_1, \delta h_2),$$

where for every $[x; \delta x] \in \mathcal{T}\mathcal{M}_n, u \in \mathbb{R}^m, \delta u \in \mathbb{R}^m, w \in \mathbb{R}^p$, and $\delta w \in \mathbb{R}^p$:

$$\begin{aligned} \delta f(x, \delta x, u, \delta u, w, \delta w) &:= \frac{\partial f}{\partial x}(x, u, w)\delta x + \frac{\partial f}{\partial u}(x, u, w)\delta u \\ &\quad + \frac{\partial f}{\partial w}(x, u, w)\delta w \\ \delta h_1(x, \delta x) &:= \frac{\partial h_1}{\partial x}(x)\delta x \\ \delta h_2(x, \delta x) &:= \frac{\partial h_2}{\partial x}(x)\delta x. \end{aligned}$$

Remark 2.3: If the control system Σ does not have internal inputs and outputs, the definition of the control system in Definition 2.1 reduces to the tuple

$$\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h).$$

Correspondingly, the equation (II.1) describing the state and output trajectories reduces to:

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{II.2}$$

We use the notion of control system in (II.2) later to refer to an overall interconnected control system. The variational control system of Σ can be defined similar to Definition 2.2.

Definition 2.4: Let

$$\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\Sigma} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

be two control subsystems with the same external output space dimension. We define the *augmented system*

$$\tilde{\Sigma} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \tilde{h}_1, \tilde{h}_2),$$

where $\mathcal{M}_{\tilde{n}} = \mathcal{M}_n \times \mathcal{M}_{\hat{n}}$, $\tilde{\mathcal{U}} = \mathcal{U} \times \hat{\mathcal{U}}$, $\tilde{\mathcal{W}} = \mathcal{W} \times \hat{\mathcal{W}}$, $\tilde{m} = m + \hat{m}$, $\tilde{p} = p + \hat{p}$, $\tilde{q}_2 = q_2 + \hat{q}_2$, and for each $x \in \mathcal{M}_n, \hat{x} \in \mathcal{M}_{\hat{n}}$, $u \in \mathbb{R}^m, \hat{u} \in \mathbb{R}^{\hat{m}}$, $w \in \mathbb{R}^p$, and $\hat{w} \in \mathbb{R}^{\hat{p}}$:

$$\tilde{f}(\tilde{x}, \tilde{u}, \tilde{w}) := \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix},$$
$$\tilde{h}_1(\tilde{x}) := h_1(x) - \hat{h}_1(\hat{x}),$$
$$\tilde{h}_2(\tilde{x}) := \begin{bmatrix} h_2(x) \\ \hat{h}_2(\hat{x}) \end{bmatrix},$$

where $\tilde{x} = [x; \hat{x}]$, $\tilde{u} = [u; \hat{u}]$, and $\tilde{w} = [w; \hat{w}]$.

Definition 2.5: Let $\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$ be two control systems without internal inputs and outputs, and with the same external output space dimension. We define the *augmented system* $\tilde{\Sigma} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \tilde{f}, \mathbb{R}^{q_1}, \tilde{h})$, where $\mathcal{M}_{\tilde{n}} = \mathcal{M}_n \times \mathcal{M}_{\hat{n}}$, $\tilde{\mathcal{U}} = \mathcal{U} \times \hat{\mathcal{U}}$, $\tilde{m} = m + \hat{m}$, and for each $x \in \mathcal{M}_n, \hat{x} \in \mathcal{M}_{\hat{n}}$, $u \in \mathbb{R}^m$, and $\hat{u} \in \mathbb{R}^{\hat{m}}$:

$$\tilde{f}(\tilde{x}, \tilde{u}) := \begin{bmatrix} f(x, u) \\ \hat{f}(\hat{x}, \hat{u}) \end{bmatrix},$$
$$\tilde{h}(\tilde{x}) := h(x) - \hat{h}(\hat{x}),$$

where $\tilde{x} = [x; \hat{x}]$, and $\tilde{u} = [u; \hat{u}]$.

III. Differential Storage and Simulation Functions

In this section, we introduce a notion of so-called differential storage functions, adapted from the notion of differential storage function introduced in [10] in the context of differential dissipativity.

Definition 3.1: Consider two control subsystems

$$\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$$

and

$$\hat{\Sigma} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$$

and the corresponding augmented system

$$\tilde{\Sigma} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \tilde{h}_1, \tilde{h}_2)$$

as in Definition 2.4. Let

$$\delta\tilde{\Sigma} = (\mathcal{T}\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \delta\tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \delta\tilde{h}_1, \delta\tilde{h}_2)$$

be the variational control system of $\tilde{\Sigma}$ as defined in Definition 2.2. Suppose there exists some positive constants α and λ, a matrix valued function $G : \mathcal{M}_{\tilde{n}} \to \mathbb{R}^{\tilde{n} \times \tilde{n}}$, such that $G(\tilde{x})$ is a positive (semi) definite matrix for all $\tilde{x} \in \mathcal{M}_{\tilde{n}}$, some matrices $W, \hat{W}, X^{ij}, i, j \in [1; 2]$, of appropriate dimensions, a function $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ and a continuously differentiable function[1] $k : \mathcal{M}_{\tilde{n}} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^m$ which satisfies $k(0, 0) = 0$, such that the following two conditions hold[2]:

- For any $\tilde{x} \in \mathcal{M}_{\tilde{n}}$:
$$G(\tilde{x}) \succeq \alpha \left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}} \right)^T \left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}} \right).$$
- For any $[\tilde{x}; \delta\tilde{x}] \in \mathcal{T}\mathcal{M}_{\tilde{n}}$, $\hat{u} \in \mathbb{R}^{\hat{m}}$, and $\delta\hat{u} \in \mathbb{R}^{\hat{m}}$, if we choose u using the map $u = k(\tilde{x}, \hat{u})$, then for any $\tilde{w} \in \mathbb{R}^{\tilde{p}}$, and any $\delta\tilde{w} \in \mathbb{R}^{\tilde{p}}$:
$$\begin{aligned}&\delta\tilde{x}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G(\tilde{x}) + G(\tilde{x}) \frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}} \tilde{f}(\tilde{x}, \tilde{u}, \tilde{w}) \right) \delta\tilde{x} \\ &+ 2\delta\tilde{w}^T \frac{\partial \tilde{f}}{\partial \tilde{w}}^T G(\tilde{x})\delta\tilde{x} + 2\delta\tilde{u}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G(\tilde{x})\delta\tilde{x} \\ &\leq -\lambda \delta\tilde{x}^T G(\tilde{x}) \delta\tilde{x} \\ &+ \begin{bmatrix} W\delta w - \hat{W}\delta\hat{w} \\ \delta y_2 - H\delta\hat{y}_2 \end{bmatrix}^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \begin{bmatrix} W\delta w - \hat{W}\delta\hat{w} \\ \delta y_2 - H\delta\hat{y}_2 \end{bmatrix} \\ &+ \psi_{\text{ext}}(\|\delta\hat{u}\|),\end{aligned}$$
where $\delta y_2 = \frac{\partial h_2(x)}{\partial x}\delta x$, $\delta\hat{y}_2 = \frac{\partial \hat{h}_2(\hat{x})}{\partial \hat{x}}\delta\hat{x}$, $\delta\tilde{u} = [\delta u; \delta\hat{u}]$, and $\delta u = \frac{\partial k}{\partial \tilde{x}}\delta\tilde{x} + \frac{\partial k}{\partial \hat{u}}\delta\hat{u}$.

Then $\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T G(\tilde{x})\delta\tilde{x}$ is a *differential storage function* from $\hat{\Sigma}$ to Σ. We call $\hat{\Sigma}$ (preferably with $\hat{n} < n$) an *abstraction* of Σ if there exists a differential storage function from $\hat{\Sigma}$ to Σ.

Remark 3.2: For linear subsystems, one can use the differential storage function given by

$$\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T \begin{bmatrix} \widehat{M} & -\widehat{M}P \\ -P^T\widehat{M} & P^T\widehat{M}P \end{bmatrix} \delta\tilde{x},$$

where $\widehat{M} \in \mathbb{R}^{n \times n}$ is a positive definite matrix, and $P \in \mathbb{R}^{n \times \hat{n}}$, satisfying the conditions given in [6] together with the associated linear interface map, for the construction of abstractions of subsystems.

Now we introduce the notion of simulation functions used in the paper.

Definition 3.3: Let $\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$ be two control systems without internal inputs and outputs and let $\tilde{\Sigma} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \tilde{f}, \mathbb{R}^q, \tilde{h})$ be the corresponding augmented control system as defined in Definition 2.5. Let $\delta\tilde{\Sigma} = (\mathcal{T}\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \delta\tilde{f}, \mathbb{R}^q, \delta\tilde{h})$ be the variational control system of $\tilde{\Sigma}$. Suppose there exist some positive constants α and λ, some function $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, some matrix valued function $G : \mathcal{M}_{\tilde{n}} \to \mathbb{R}^{\tilde{n} \times \tilde{n}}$, where $G(\tilde{x})$ is a positive (semi) definite matrix for each $\tilde{x} \in \mathcal{M}_{\tilde{n}}$, and a continuously differentiable function $k : \mathcal{M}_{\tilde{n}} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^m$ which satisfies $k(0, 0) = 0$, such that the following two conditions hold[2]:

[1] We refer to k as the *interface* map.

[2] Here, for brevity, we do not write the arguments of the partial derivatives explicitly.

- For all $\tilde{x} \in \mathcal{M}_{\tilde{n}}$:

$$G(\tilde{x}) \succeq \alpha \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^T \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right). \qquad \text{(III.1)}$$

- For any $[\tilde{x}; \delta\tilde{x}] \in \mathcal{TM}_{\tilde{n}}$, $\hat{u} \in \mathbb{R}^{\hat{m}}$, and $\delta\hat{u} \in \mathbb{R}^{\hat{m}}$, if we select u using the map $u = k(\tilde{x}, \hat{u})$:

$$\begin{aligned}&\delta\tilde{x}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G(\tilde{x}) + G(\tilde{x})\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}}\tilde{f}(\tilde{x}, \tilde{u})\right)\delta\tilde{x}\\ &+ 2\delta\tilde{u}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G(\tilde{x})\delta\tilde{x}\\ &\leq -\lambda\delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \psi_{\text{ext}}(\|\delta\hat{u}\|), \qquad \text{(III.2)}\end{aligned}$$

where $\delta\tilde{u} = [\delta u; \delta\hat{u}]$, and $\delta u = \frac{\partial k}{\partial \tilde{x}}\delta\tilde{x} + \frac{\partial k}{\partial \hat{u}}\delta\hat{u}$,

then

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma}\in\Gamma(\tilde{x},0)} \int_0^1 \frac{\partial}{\partial s}\tilde{\gamma}(s)^T G(\tilde{\gamma}(s))\frac{\partial}{\partial s}\tilde{\gamma}(s)ds,$$

is called a *simulation function* from Σ to $\hat{\Sigma}$ with respect to the (pseudo) Riemannian structure G.

The next theorem shows the usefulness of the existence of a simulation function in quantifying the closeness of two control systems.

Theorem 3.4: Consider two control systems $\Sigma = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$. Suppose V_G, associated with the (pseudo) Riemannian structure G, is a simulation function from $\hat{\Sigma}$ to Σ, and k is the associated interface map, then there exists $\beta \in \mathcal{KL}$, and $\bar{\psi}_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ such that for any $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, $\hat{\upsilon} \in \hat{\mathcal{U}}$, if we choose $\upsilon \in \mathcal{U}$ using the interface map k, then the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\|\zeta_{x\upsilon}(t) - \hat{\zeta}_{\hat{x}\hat{\upsilon}}(t)\| \leq \beta(V_G(x, \hat{x}), t) + \bar{\psi}_{\text{ext}}(\|\hat{\upsilon}\|_\infty). \quad \text{(III.3)}$$

Proof: Consider two points $\tilde{x} = [x; \hat{x}] \in \mathcal{M}_{\tilde{n}}$ and $0 \in \mathcal{M}_{\tilde{n}}$, and a geodesic $\chi : [0, 1] \to \mathbb{R}^{\tilde{n}}$, with respect to the (pseudo) Riemannian structure G, such that $\chi(0) = 0$, and $\chi(1) = \tilde{x}$. The energy functional corresponding to this geodesic is given by

$$V_G(\tilde{x}) = E_G(\tilde{x}, 0) = \int_0^1 \frac{\partial}{\partial s}\chi(s)^T G(\chi(s))\frac{\partial}{\partial s}\chi(s)ds.$$

Let $\tilde{\xi}_{\tilde{x}\tilde{\nu}} = [\xi_{x\nu}; \hat{\xi}_{\hat{x}\hat{\nu}}]$ be the solution trajectory of $\tilde{\Sigma}$ for any initial condition $\tilde{x} \in \mathcal{M}_{\tilde{n}}$, and under the input trajectory $\tilde{\nu} = [\nu; \hat{\nu}]$, where $\nu(t) = k(\tilde{\xi}_{\tilde{x}\tilde{\nu}}(t), \hat{\nu}(t))$, for all $t \in \mathbb{R}_{\geq 0}$, for any $\hat{\nu} \in \hat{\mathcal{U}}$.

For a fixed $t \in \mathbb{R}_{\geq 0}$, consider the straight line $\hat{\eta}(s, t) = s\hat{\nu}(t)$ in s, where $s \in [0, 1]$. For any fixed $t \in \mathbb{R}_{\geq 0}$, the curve $\hat{\eta}(\cdot, t) : [0, 1] \to \mathbb{R}^{\hat{m}}$ is a geodesic, with respect to the Euclidean metric, on $\mathbb{R}^{\hat{m}}$ joining $\hat{\eta}(0, t) = 0$ and $\hat{\eta}(1, t) = \hat{\nu}(t)$.

For any $s \in [0, 1]$, let $\tilde{\phi}(s, \cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{n}}$ be the solution trajectory of $\tilde{\Sigma}$ from initial condition $\chi(s)$ under the input $\tilde{\eta}(s, \cdot)$, where $\tilde{\eta}(s, t) = \begin{bmatrix} k(\tilde{\phi}(s, t), \hat{\eta}(s, t)) \\ \hat{\eta}(s, t) \end{bmatrix}, \forall t \in \mathbb{R}_{\geq 0}$. Note that $\tilde{\phi}(0, t) = 0$, and $\tilde{\phi}(1, t) = \tilde{\xi}_{\tilde{x}\tilde{\nu}}(t)$.

For brevity, we denote $\frac{\partial}{\partial s}\tilde{\phi}(s, t) =: \tilde{w}(s, t)$. Note that

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{w}(s, t) &= \frac{\partial^2}{\partial t\partial s}\tilde{\phi}(s, t) = \frac{\partial^2}{\partial s\partial t}\tilde{\phi}(s, t)\\ &= \frac{\partial}{\partial s}\tilde{f}(\tilde{\phi}(s, t), \tilde{\eta}(s, t)) = \frac{\partial \tilde{f}}{\partial \tilde{x}}\frac{\partial}{\partial s}\tilde{\phi}(s, t) + \frac{\partial \tilde{f}}{\partial \tilde{u}}\frac{\partial}{\partial s}\tilde{\eta}(s, t)\\ &= \frac{\partial \tilde{f}}{\partial \tilde{x}}\tilde{w}(s, t) + \frac{\partial \tilde{f}}{\partial \tilde{u}}\begin{bmatrix} \frac{\partial k}{\partial \tilde{x}}\tilde{w}(s, t) + \frac{\partial k}{\partial \hat{u}}\hat{\nu}(t) \\ \hat{\nu}(t) \end{bmatrix}.\end{aligned}$$

Define

$$l(t) = \int_0^1 \tilde{w}(s, t)^T G(\phi(s, t))\tilde{w}(s, t)ds,$$

i.e. $l(t)$ is the energy functional of the curve $\tilde{\phi}(\cdot, t)$, with respect to G. We have

$$\begin{aligned}\frac{d}{dt}l(t) &= \int_0^1 \frac{\partial}{\partial t}\tilde{w}(s, t)^T G(\tilde{\phi}(s, t))\tilde{w}(s, t)ds\\ &= \int_0^1 \tilde{w}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G + G\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}}f\right)\tilde{w}ds\\ &+ 2\int_0^1 \begin{bmatrix} \frac{\partial k}{\partial \tilde{x}}\tilde{w} + \frac{\partial k}{\partial \hat{u}}\hat{\nu} \\ \hat{\nu} \end{bmatrix}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G\tilde{w}ds,\end{aligned}$$

where, again, we have dropped explicit arguments for clarity in the last expression. From (III.2), one has:

$$\begin{aligned}\frac{d}{dt}l(t) &\leq -\lambda\int_0^1 \tilde{w}(s, t)^T G(\tilde{\phi}(s, t))\tilde{w}(s, t)ds\\ &+ \int_0^1 \psi_{\text{ext}}\left(\left\|\frac{\partial\hat{\eta}(s, t)}{\partial s}\right\|\right)ds\\ &\leq -\lambda\int_0^1 \tilde{w}(s, t)^T G(\tilde{\phi}(s, t))\tilde{w}(s, t)ds\\ &+ \psi_{\text{ext}}(\|\hat{\nu}(t)\|)\int_0^1 ds\\ &\leq -\lambda l(t) + \psi_{\text{ext}}(\|\hat{\nu}\|_\infty).\end{aligned}$$

It follows from the comparison lemma [12] that

$$l(t) \leq \mathrm{e}^{-\lambda t}l(0) + \frac{1}{\lambda}\psi_{\text{ext}}(\|\hat{\nu}\|_\infty).$$

Note that $l(0) = V_G(\xi_{x\nu}(0), \hat{\xi}_{\hat{x}\hat{\nu}}(0)) = V_G(\tilde{x})$. Now using the fact that for any $t \in \mathbb{R}_{\geq 0}$, $l(t)$ is not necessarily the minimum energy functional corresponding to a geodesic because $\tilde{\phi}(s, t)$ is not necessarily a geodesic, i.e. $V_G(\xi_{x\nu}(t), \hat{\xi}_{\hat{x}\hat{\nu}}(t)) \leq l(t)$, one has:

$$\begin{aligned}V_G(\xi_{x\nu}(t), \hat{\xi}_{\hat{x}\hat{\nu}}(t)) &\leq \mathrm{e}^{-\lambda t}V_G(\xi_{x\nu}(0), \hat{\xi}_{\hat{x}\hat{\nu}}(0))\\ &+ \frac{1}{\lambda}\psi_{\text{ext}}(\|\hat{\nu}\|_\infty). \qquad \text{(III.4)}\end{aligned}$$

For every $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}$, we use (III.1) and the Schwarz inequality to obtain:

$$\alpha\|h_1(x)-\hat{h}_1(\hat{x})\|^2 = \alpha\|\tilde{h}(\tilde{x})\|^2$$
$$\leq \alpha\left(\int_0^1 \sqrt{\frac{\partial}{\partial s}\chi(s)^T\frac{\partial\tilde{h}_1}{\partial\tilde{x}}(\chi(s))^T\frac{\partial\tilde{h}_1}{\partial\tilde{x}}(\chi(s))\frac{\partial}{\partial s}\chi(s)ds}\right)^2$$
$$\leq \left(\int_0^1 \sqrt{\frac{\partial}{\partial s}\chi(s)^T G(\chi(s))\frac{\partial}{\partial s}\chi(s)ds}\right)^2$$
$$\leq \int_0^1 \frac{\partial}{\partial s}\chi(s)^T G(\chi(s))\frac{\partial}{\partial s}\chi(s)ds = V_G(\tilde{x}), \quad \text{(III.5)}$$

where $\tilde{x}=[x;\hat{x}]$. Combining (III.5) with (III.4), one can conclude that (III.3) is satisfied with $\beta(r,s)=\sqrt{\frac{\tau}{\alpha}}\mathrm{e}^{-\frac{\lambda}{2}s}$ and $\bar{\psi}_{\text{ext}}(r)=\sqrt{\frac{1}{\alpha\lambda}\psi_{\text{ext}}(r)}, \forall s,r\in\mathbb{R}_{\geq 0}$. ■

IV. Interconnected Systems

Here we define the interconnected system consisting of control subsystems interconnected via a constant interconnection topology.

Definition 4.1: Consider $N\in\mathbb{N}_{\geq 1}$ control subsystems

$$\Sigma_i = (\mathcal{M}_{n_i},\mathbb{R}^{m_i},\mathbb{R}^{p_i},\mathcal{U}_i,\mathcal{W}_i,f_i,\mathbb{R}^{q_{1i}},\mathbb{R}^{q_{2i}},h_{1i},h_{2i}),$$

where $i\in[1;N]$, and an interconnection matrix M of appropriate dimension defining the coupling of these subsystems. The interconnected control system $\Sigma=(\mathcal{M}_n,\mathbb{R}^m,\mathcal{U},f,\mathbb{R}^q,h)$ denoted by $\mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$, follows by $\mathcal{M}_n=\Pi_{i=1}^N\mathcal{M}_{n_i}$, $m=\sum_{i=1}^N m_i$, $q=\sum_{i=1}^N q_{1i}$, and the functions

$$f(x,u)=\left[f_1(x_1,u_1,w_1);\ldots;f_N(x_N,u_N,w_N)\right],$$
$$h(x)=[h_{11}(x);\ldots;h_{1N}(x_N)],$$

where $u=[u_1;\ldots;u_N]$, $x=[x_1;\ldots;x_N]$, and with the internal inputs constrained by

$$[w_1;\ldots;w_N]=M[h_{21}(x_1);\ldots;h_{2N}(x_N)].$$

In the next theorem, the proof of which is omitted due to lack of space, we derive sufficient conditions under which an interconnection of abstractions of control subsystems, interconnected via another (possibly simpler) interconnection topology, is an abstraction of the original interconnected system.

Theorem 4.2: Consider the interconnected control system $\Sigma=\mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$, induced by N control subsystems and a coupling matrix M. Suppose each subsystem Σ_i admits an abstraction $\hat{\Sigma}_i$ with a corresponding differential storage function $\mathcal{S}_i$. If there exists $\mu_i\geq 1$ and the matrix $\hat{M}$ such that the following matrix (in)equalities hold:

$$\begin{bmatrix} WM \\ I_{\bar{q}}\end{bmatrix}^T X(\mu_1X_1,\ldots,\mu_NX_N)\begin{bmatrix} WM \\ I_{\bar{q}}\end{bmatrix}\preceq 0, \quad \text{(IV.1)}$$

$$WMH=\hat{W}\hat{M}, \quad \text{(IV.2)}$$

where $\bar{q}=\sum_{i=1}^N q_{2i}$ and

$$W=\mathsf{diag}(W_1,\ldots,W_N), \hat{W}=\mathsf{diag}(\hat{W}_1,\ldots,\hat{W}_N),$$
$$H=\mathsf{diag}(H_1,\ldots,H_N),$$

$$X(\mu_1X_1,\ldots,\mu_NX_N)=$$
$$\begin{bmatrix} \mu_1X_1^{11} & & & \mu_1X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_NX_N^{11} & & & \mu_NX_N^{12} \\ \mu_1X_1^{21} & & & \mu_1X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_NX_N^{21} & & & \mu_NX_N^{22}\end{bmatrix},$$

then

$$V_G(\tilde{x})=\inf_{\tilde{\gamma}\in\Gamma(\tilde{x},0)}\int_0^1\frac{\partial}{\partial s}\tilde{\gamma}(s)^TG(\tilde{\gamma}(s))\frac{\partial}{\partial s}\tilde{\gamma}(s)ds,$$

is a simulation function from the interconnected control system $\hat{\Sigma}=\mathcal{I}(\hat{\Sigma}_1,\ldots,\hat{\Sigma}_N)$ with coupling matrix $\hat{M}$ to Σ, where

$$G(\tilde{x})=\begin{bmatrix}\mu_1G_1(\tilde{x}_1) & 0 & \ldots & 0\\ 0 & \mu_2G_2(\tilde{x}_2) & & 0\\ \vdots & & \ddots & \vdots\\ 0 & \ldots & 0 & \mu_NG_N(\tilde{x}_N)\end{bmatrix},$$

$\tilde{x}=[\tilde{x}_1;\ldots;\tilde{x}_N]$, and $\tilde{x}_i=[x_i;\hat{x}_i]\in\mathcal{M}_{n_i}\times\mathcal{M}_{\hat{n}_i}$ $\forall i\in[1;N]$.

V. Example

Consider an interconnection of $N\in\mathbb{N}$ subsystems Σ_i, $i\in[1;N]$, where each Σ_i is given by $\Sigma_i=(\mathbb{S}^{n_i},\mathbb{R}^{n_i},\mathbb{R}^{n_i},\mathcal{U}_i,\mathcal{W}_i,f_i,\mathbb{R},\mathbb{R}^{n_i},h_{1i},h_{2i})$, where for each $\theta_i=[\theta_{i1};\ldots;\theta_{in_i}]\in\mathbb{S}^{n_i}, u_i\in\mathbb{R}^{n_i}, w_i\in\mathbb{R}^{n_i}$:

$$f_i(\theta_i,u_i,w_i):=\frac{1}{n_i}\begin{bmatrix}\sum_{k=1}^{n_i}\sin(\theta_{ik}-\theta_{i1})\\ \vdots\\ \sum_{k=1}^{n_i}\sin(\theta_{ik}-\theta_{in_i})\end{bmatrix}+w_i+u_i,$$
$$h_{1i}(\theta_i):=\theta_{i1}$$
$$h_{2i}(\theta_i):=\theta_i.$$

The variational control system of Σ_i is given by the tuple $\delta\Sigma_i=(\mathbb{S}^{n_i}\times\mathbb{R}^{n_i},\mathbb{R}^{n_i},\mathbb{R}^{n_i},\mathcal{U}_i,\mathcal{W}_i,\delta f_i,\mathbb{R},\mathbb{R}^{n_i},\delta h_{1i},\delta h_{2i})$, where for each $[\theta_i;\delta\theta_i]\in\mathbb{S}^{n_i}\times\mathbb{R}^{n_i}$, $u_i\in\mathbb{R}^{n_i},\delta u_i\in\mathbb{R}^{n_i}$, $w_i\in\mathbb{R}^{n_i}$, and $\delta w_i\in\mathbb{R}^{n_i}$, $\delta f_i,\delta h_{1i}$, and δh_{2i} are defined in (V.1). We assume that the interconnection topology is given by

$$M=\begin{bmatrix}-2 & 1 & 0 & 0 & \ldots & 1\\ 1 & -2 & 1 & 0 & \ldots & 0\\ 0 & 1 & -2 & 1 & \ldots & 0\\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \ldots & 1 & -2\end{bmatrix}.$$

For each Σ_i, we consider that the abstract subsystems are given by the tuple $\hat{\Sigma}_i=(\mathbb{S},\mathbb{R},\mathbb{R},\hat{\mathcal{U}}_i,\hat{\mathcal{W}}_i,\hat{f}_i,\mathbb{R},\mathbb{R},1,1)$, where for each $\hat{\theta}_i\in\mathbb{S}$, $\hat{u}_i\in\mathbb{R}$, and $\hat{w}_i\in\mathbb{R}$:

$$\hat{f}_i(\hat{\theta}_i,\hat{u}_i,\hat{w}_i)=-\frac{1}{n_i}\sin(\hat{\theta}_i)+\hat{w}_i+\hat{u}_i.$$

The variational control system of $\hat{\Sigma}_i$ is given by $\delta\hat{\Sigma}_i=(\mathbb{S}\times\mathbb{R},\mathbb{R},\mathbb{R},\hat{\mathcal{U}}_i,\hat{\mathcal{W}}_i,\delta\hat{f}_i,\mathbb{R},\mathbb{R},\delta\hat{h}_{1i},\delta\hat{h}_{2i})$, where for each

$$\delta f_i(\theta_i,\delta\theta_i,u_i,\delta u_i,w_i,\delta w_i) := \begin{bmatrix} -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{i1}) & \frac{1}{n_i}\cos(\theta_{i2}-\theta_{i1}) & \dots & \frac{1}{n_i}\cos(\theta_{in_i}-\theta_{i1}) \\ \frac{1}{n_i}\cos(\theta_{i1}-\theta_{i2}) & -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{i2}) & \dots & \\ \vdots & \ddots & & \vdots \\ \frac{1}{n_i}\cos(\theta_{i1}-\theta_{in_i}) & \dots & & -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{in_i}) \end{bmatrix}\begin{bmatrix}\delta\theta_{i1}\\ \vdots \\ \delta\theta_{in_i}\end{bmatrix} + \delta w_i + \delta u_i$$

$$\delta h_{1i}(\theta_i,\delta\theta_i) := \delta\theta_{i1}$$

$$\delta h_{2i}(\theta_i,\delta\theta_i) := \delta\theta_i \quad \text{(V.1)}$$

$[\hat{\theta}_i;\delta\hat{\theta}_i]\in\mathbb{S}\times\mathbb{R}$, $\hat{u}_i\in\mathbb{R},\delta\hat{u}_i\in\mathbb{R}$, $\hat{w}_i\in\mathbb{R}$, and $\delta\hat{w}_i\in\mathbb{R}$, $\delta\hat{f}_i,\delta\hat{h}_{1i}$, and $\delta\hat{h}_{2i}$ are given by:

$$\delta\hat{f}_i(\hat{\theta}_i,\delta\hat{\theta}_i,\hat{u}_i,\delta\hat{u}_i,\hat{w}_i,\delta\hat{w}_i) := -\frac{1}{n_i}\cos(\hat{\theta}_i)\delta\hat{\theta}_i+\delta\hat{w}_i+\delta\hat{u}_i$$

$$\delta\hat{h}_{1i}(\hat{\theta}_i,\delta\hat{\theta}_i) := \delta\hat{\theta}_i$$

$$\delta\hat{h}_{2i}(\hat{\theta}_i,\delta\hat{\theta}_i) := \delta\hat{\theta}_i.$$

Consider the following differential storage function with constant pseudo Riemannian structure:

$$\mathcal{S}_i(\delta\theta_i,\delta\hat{\theta}_i) = \begin{bmatrix}\delta\theta_{i1} & \dots & \delta\theta_{in_i} & \delta\hat{\theta}_i\end{bmatrix} G_i \begin{bmatrix}\delta\theta_{i1}\\ \vdots \\ \delta\theta_{in_i}\\ \delta\hat{\theta}_i\end{bmatrix},$$

where

$$G_i = \begin{bmatrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \ddots & & \\ -1 & -1 & \dots & n_i \end{bmatrix}.$$

For each $i\in[1;N]$, we choose $u_i=[u_{i1};\dots;u_{in_i}]\in\mathbb{R}^{n_i}$ according to the following interface map:

$$u_{ij} = -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\sin(\theta_{ik}-\theta_{ij}) - \frac{1}{2n_i}\theta_{ij} - \frac{1}{n_i}\sin(\hat{\theta}_i) + \frac{1}{2n_i}\hat{\theta}_i + \hat{u}_i,$$

where u_{ij} represents the j-th element of the vector u_i, and θ_{ij} represents the j-th element of the vector θ_i, $j=[1;n_i]$. It can be shown that $\mathcal{S}_i$ is a differential storage function from $\hat{\Sigma}_i$ to Σ_i with the following parameters

$$W_i = I_{n_i}, \hat{W}_i = \vec{1}_{n_i}, H_i = \vec{1}_{n_i}, X_i^{11} = X_i^{22} = 0_{n_i},$$
$$X_i^{12} = X_i^{21} = I_{n_i}, \alpha_i = 1, \lambda_i = \frac{1}{n_i}, \psi_{i\text{ext}} = 0,$$

where 0 represents the zero function. By selecting $\mu_1 = \dots = \mu_N = 1$, and $\hat{M}$ appropriately, it can be shown that (IV.1) and (IV.2) are satisfied and therefore one can conclude that

$$V_G(\tilde{\theta}) = \inf_{\tilde{\gamma}\in\Gamma(\tilde{\theta},0)}\int_0^1 \frac{\partial}{\partial s}\tilde{\gamma}(s)^T G\frac{\partial}{\partial s}\tilde{\gamma}(s)ds,$$

where $\tilde{\theta} = [\theta_1;\hat{\theta}_1;\dots;\theta_N,\hat{\theta}_N],\theta_i\in\mathbb{S}^{n_i},\hat{\theta}_i\in\mathbb{S},\forall i=[1;N]$, and

$$G = \mathsf{diag}(G_1,\dots,G_N),$$

is a simulation function, with respect to G, from $\mathcal{I}(\hat{\Sigma}_1,\dots\hat{\Sigma}_N)$ to $\mathcal{I}(\Sigma_1,\dots,\Sigma_N)$, with the interconnection matrix for $\hat{\Sigma}$ given by $\hat{M}$. For example, for $N=3, n_i=50,\forall i=[1;N]$, (i.e. $M\in\mathbb{R}^{150\times150}$), one can choose

$$\hat{M} = \begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}.$$

VI. Conclusion

In this work, we derived sufficient compositional conditions under which abstractions of interconnected systems evolving on smooth Riemannian manifolds can be constructed. Construction of abstractions for different classes of nonlinear subsystems evolving on Riemannian manifolds is a subject of future research.

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