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Unification of flavor, $\mathcal{CP}$, and modular symmetries

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ABSTRACT

Flavor symmetry plays a crucial role in the standard model of particle physics but its origin is still unknown. We develop a new method (based on outer automorphisms of the Narain space group) to determine flavor symmetries within compactified string theory. A picture emerges where traditional (discrete) flavor symmetries, $\mathcal{CP}$-like symmetries and modular symmetries (like T-duality) of string theory combine to unified flavor symmetries. The groups depend on the geometry of compact space and the geographical location of fields in the extra dimensions. We observe a phenomenon of "local flavor groups" with potentially different flavor symmetries for the various sectors of quarks and leptons. This should allow interesting connections to existing bottom-up attempts in flavor model building.

1. Introduction

Traditionally, $\mathcal{CP}$ transformations and flavor symmetries were assumed to be of different origin. More recently, however, it was suggested that $\mathcal{CP}$ should be considered as an outer automorphism of the flavor group [1–5]. It was shown that such a link between flavor and $\mathcal{CP}$ has a natural embedding in a string theory framework [6]. In the present paper we show that in string theory, where symmetries arise from the geometry of compactified extra dimensions and string selection rules [7–9], an even stronger link can be established: the $\mathcal{CP}$ and flavor transformations of the low-energy effective theory are unified in a common symmetry group. This observation has been made in the generalization of the discussion in [6] by including duality symmetries of string theory. In our examples, the full unified symmetry (including flavor and $\mathcal{CP}$) is a combination of the original flavor symmetry and the T-duality transformations [10–16] of the stringy extension. $\mathcal{CP}$, originally an outer automorphism of the flavor group, now becomes an element (inner automorphism) of the unified flavor group. As it contains T-duality transformations, this unified flavor group depends on the location in moduli space: it is enhanced at special points of moduli space (where $\mathcal{CP}$ may be unbroken). At a generic point in moduli space only the original flavor symmetry (possibly with $\mathcal{CP}$ as an outer automorphism) is present. This allows a connection to the concept of "local grand unification" [17,18], where the various fields of the standard model of particle physics (quarks, leptons, and Higgs bosons) live at different locations in compactified higher dimensions and thus feel different subgroups of the unified flavor group. It leads to the flexibility to have different flavor symmetries in the various sectors (e.g. quark and lepton sectors) of the theory. It also allows a connection to model constructions that use T-duality transformations, or more general subgroups of the modular transformations, as flavor symmetries [19–33], especially for the mixings in the lepton sector.

This unified picture of flavor and $\mathcal{CP}$ is rather common and can be derived through a general mechanism that allows a full classification of all flavor symmetries in the given string model. As we shall explain in this paper, the mechanism is based on the consideration of outer automorphisms of the Narain-lattice construction [34,35] of string theory with compactified extra dimensions. It is a powerful tool that generalizes previous attempts in the search for flavor symmetries. We shall present the mechanism in its general form and explain the results in the specific example of the two-dimensional $\mathbb{Z}_3$ orbifold already discussed in ref. [6]. There the flavor group was $\Delta(54)$, and a physical $\mathcal{CP}$ transformation had to be a non-trivial outer automorphism of this group. However, due to the specific group theoretical structure of $\Delta(54)$ as a "type I" group [4], the physical $\mathcal{CP}$ symmetry of the light spectrum was naturally broken by the presence of heavy $\Delta(54)$ doublet states. In the generalized picture, $\Delta(54)$ is still the symmetry at a generic

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point in moduli space, but it will be enhanced at specific lines and points of the moduli space. Enhancements here include the groups SG(108, 17), SG(216, 87) and even SG(324, 39).[1] For these enhancements, the $\mathcal{CP}$ transformation of the low-energy spectrum is no longer an outer automorphism, but it becomes an element (inner automorphism) of the unified flavor group. Thus, the low-energy $\mathcal{CP}$ transformation is conserved at special points, but it will be spontaneously broken at a generic point in moduli space.

The main results of our paper originate from the discussion of the outer automorphisms of the "Narain space group" [37–39], where we find that non-Abelian flavor symmetries, modular symmetries of T-duality and $\mathcal{CP}$ have a common origin in string theory. Hence, a complete classification of the unified flavor symmetry is possible. First we present a warm-up example by the consideration of the outer automorphisms of the geometrical space group and then generalize to the discussion of the outer automorphisms of the "Narain space group". The main results of the discussion include:

- The unification of flavor- and $\mathcal{CP}$-symmetries,
- The extension of the traditional flavor group with modular symmetries,
- A diversification of flavor symmetries to "local flavor groups" that depend on the location of fields in compactified extra dimensions (and thus could lead e.g. to different flavor groups for quarks and leptons).

We shall apply this to our example of the two-dimensional $\mathbb{Z}_3$ orbifold and discuss the interplay of the original $\Delta(54)$ flavor symmetry and the relevant part of the modular transformation of T-duality, here given by $\Gamma(3)$ (which is isomorphic to A_4). We explore different regions in moduli space and construct the enhanced unified flavor groups.

2. Outer automorphisms of the space group

As a warm-up example we consider the two-dimensional $\mathbb{Z}_3$ orbifold $\mathbb{R}^2/S$. For the generators of the space group S we can choose $(\theta, 0)$, $(\mathbb{1}, e_1)$, and $(\mathbb{1}, e_2)$. The vectors e_1 and e_2 enclose an angle of 120° and have the same length, $|e_1| = |e_2|$. They span a two-dimensional lattice which defines the two-torus $\mathbb{T}^2$. Furthermore, the so-called twist θ is a counter-clockwise 120° rotation matrix with $\theta^3 = \mathbb{1}$ that maps the torus lattice to itself, i.e. $\theta\, e_1 = e_2$. In this case, a general space group element $g \in S$ can be expressed as

$$g = (\theta^k, e\,n) \qquad \text{with} \quad k \in \{0, 1, 2\} \text{ and } n \in \mathbb{Z}^2\,. \tag{1}$$

Here, the vielbein e contains the two basis vectors e_1 and e_2 as columns. The element g acts on a coordinate $y \in \mathbb{R}^2$ of the two spatial extra dimensions as $y \overset{g}{\mapsto} g\,y = \theta^k\,y + e\,n$. Consequently, two space group elements $(\theta^k, e\,n)$ and $(\theta^\ell, e\,m)$ multiply as $(\theta^k, e\,n)\,(\theta^\ell, e\,m) = (\theta^{k+\ell}, \theta^k e\,m + e\,n)$. Finally, the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold is defined as a quotient space, i.e. $y_1 \sim y_2$ if there is a $g \in S$ such that $y_1 = g\,y_2$. This orbifold has three fixed points, i.e. points that are invariant (up to lattice translations) under the 120° rotation, see Fig. 1.

Each element $g = (\theta^k, e\,n) \in S$ of the space group describes a boundary condition for a closed string on the orbifold [40,41]. For example, for a worldsheet boson $y(\tau, \sigma)$ of a closed string we impose the boundary condition

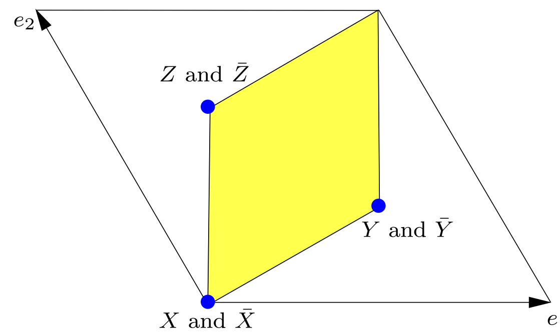

Fig. 1. The fundamental domain of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold is depicted in yellow and the three inequivalent fixed-points are blue. (X, Y, Z) denote three (left-chiral) twisted strings from the first twisted sector, while $(\bar{X}, \bar{Y}, \bar{Z})$ are three (right-chiral) twisted strings from the second twisted sector.

$$y(\tau, \sigma + 1) = g\,y(\tau, \sigma) \quad \Leftrightarrow \quad y(\tau, \sigma + 1) = \theta^k\,y(\tau, \sigma) + e\,n\,. \tag{2}$$

g is called the constructing element of the closed string. If g has a fixed point $y_\text{f} \in \mathbb{R}^2$, i.e. if $g\,y_\text{f} = y_\text{f}$, the corresponding string eq. (2) is localized at y_f in compactified higher dimensions. However, since $h\,y \sim y$ on the orbifold for any $h \in S$, the boundary condition eq. (2) and

$$y(\tau, \sigma + 1) = h\,g\,h^{-1}\,y(\tau, \sigma) \tag{3}$$

describe the same closed string on the orbifold. Hence, closed strings on orbifolds are associated to conjugacy classes $[g] = \{h\,g\,h^{-1} \mid h \in S\}$ of the space group. Then, each conjugacy class $[g]$ of the space group S corresponds to a class of boundary conditions and, thereby, to a distinct string of the theory. The $\mathbb{T}^2/\mathbb{Z}_3$ orbifold has seven conjugacy classes that yield massless strings at a generic point in moduli space, i.e. for a generic size of the orbifold and for generic value of the background B-field: The conjugacy class $[(\mathbb{1}, 0)]$ gives the trivial boundary condition of massless untwisted strings and twisted strings correspond to the conjugacy classes

$$X : [(\theta, 0)]\,, \qquad Y : [(\theta, e_1)]\,, \qquad Z : [(\theta, e_1 + e_2)]\,, \tag{4a}$$

$$\bar{X} : [(\theta^2, 0)]\,, \qquad \bar{Y} : [(\theta^2, e_1 + e_2)]\,, \qquad \bar{Z} : [(\theta^2, e_2)]\,, \tag{4b}$$

from the first (θ) and second (θ^2) twisted sector, respectively. We choose the convention that the string states (X, Y, Z) from the first twisted sector give rise to left-chiral degrees of freedom, while $(\bar{X}, \bar{Y}, \bar{Z})$ from the second twisted sector yield their right-chiral CPT-conjugates needed to form complete left-chiral superfields. Twisted strings are localized at the respective fixed points of their constructing elements. For example, the string X with constructing element from the conjugacy class $[(\theta, 0)]$ is localized at $y_\text{f} = 0$, see Fig. 1.

It is advantageous to change the basis from the so-called coordinate basis to the so-called lattice basis. This can be performed for a general space group element, eq. (1), as

$$\hat{g} = (e^{-1}, 0)\,(\theta^k, e\,n)\,(e, 0) = (\hat{\theta}^k, n) \;\in\; \hat{S}\,,$$

$$\text{where} \quad \hat{\theta} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \tag{5}$$

is defined via $\hat{\theta} := e^{-1}\theta\,e$, and $\hat{S}$ denotes the space group in the lattice basis. We can now look at the automorphisms of the space group. An inner automorphism,

$$\hat{g} \overset{\hat{h}}{\mapsto} \hat{h}\,\hat{g}\,\hat{h}^{-1} \in \hat{S} \quad \text{with} \quad \hat{h} \in \hat{S}\,, \tag{6}$$

[1] Here, we use the SmallGroup library of GAP [36] to denote groups.

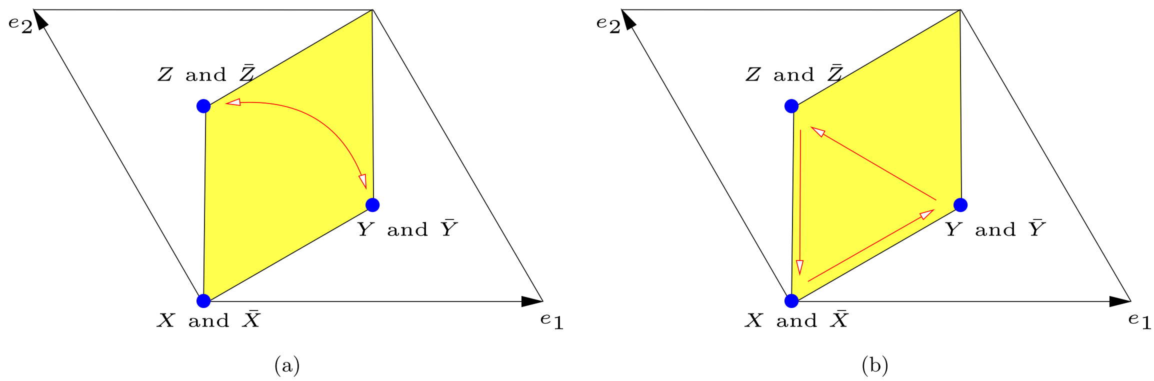

Fig. 2. The $\mathbb{T}^2/\mathbb{Z}_3$ orbifold and the actions of the outer automorphisms: (a) the 180° rotation $\hat{h}_1$ maps X to itself and interchanges Y and Z (and analogously for $(\bar{X}, \bar{Y}, \bar{Z})$) and (b) the $\mathbb{Z}_3$ translation $\hat{h}_2$ rotates the three twisted states.

maps each conjugacy class $[\hat{g}]$ to itself and, consequently, each string to itself. Furthermore, for a (bosonic) string to be well-defined on an orbifold, it has to be invariant under the action of the space group that defines the orbifolded string theory. Hence, inner automorphisms act trivially on orbifold-invariant strings. In contrast, an outer automorphism of the space group $\hat{S}$ can be described by conjugation of the constructing element with an element that is not in $\hat{S}$, i.e. there is a transformation $\hat{h} := (\hat{\sigma}, t) \notin \hat{S}$ such that for all constructing elements $\hat{g} \in \hat{S}$ we have

$$\hat{g} \overset{\hat{h}}{\mapsto} \hat{h}\,\hat{g}\,\hat{h}^{-1} \overset{!}{\in} \hat{S}\,. \tag{7}$$

Spelled out explicitly, this is equivalent to the following conditions on $\hat{\sigma} \in \mathrm{GL}(2, \mathbb{Z})$ and t: For each $k \in \{0, 1, 2\}$ there must be a $k' \in \{0, 1, 2\}$ and $n' \in \mathbb{Z}^2$ such that[2]

$$\hat{\sigma}\,\hat{\theta}^k\,\hat{\sigma}^{-1} \overset{!}{=} \hat{\theta}^{k'}\,, \tag{8a}$$

$$\left(\mathbb{1} - \hat{\sigma}\,\hat{\theta}^k\,\hat{\sigma}^{-1}\right) t \overset{!}{=} n'\,. \tag{8b}$$

This is a special case of the general consistency conditions for outer automorphisms [5]. Solutions to these conditions can be written as shifts or rotations, the ones relevant for our illustration are generated by the two elements $\hat{h}_i$ given by

$$\hat{h}_1 := (-\mathbb{1}, 0)\,, \quad \hat{h}_2 := (\mathbb{1}, t) \qquad \text{with} \qquad t := \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}. \tag{9}$$

These elements of the outer automorphism group have the following geometrical interpretation: $\hat{h}_1$ yields a 180° rotation and $\hat{h}_2$ gives a $\mathbb{Z}_3$ translation, since $(\hat{h}_2)^3 = (\mathbb{1}, 3t)$ is an inner automorphism of $\hat{S}$, see Figs. 2a and 2b.

Together, the transformations $\hat{h}_1$ and $\hat{h}_2$ generate the permutation group

$$S_3 = \left\langle \hat{h}_1\,, \hat{h}_2 \;\middle|\; \left(\hat{h}_1\right)^2 = \left(\hat{h}_2\right)^3 = \left(\hat{h}_2\,\hat{h}_1\right)^2 = \mathbb{1} \right\rangle, \tag{10}$$

that acts geometrically as all permutations of the three fixed points of the $\mathbb{Z}_3$ orbifold. If one takes into account the $\mathbb{Z}_3 \times \mathbb{Z}_3$ space group selection rule of the two-dimensional $\mathbb{Z}_3$ orbifold [42,43], the combined flavor symmetry results as [7]

$$\Delta(54) = S_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)\,. \tag{11}$$

Note that in ref. [7] the S_3-symmetry had been postulated from geometrical considerations while here we have deduced it from the outer automorphisms of the space group. As a remark, our approach eq. (8) is similar to the identification of flavor symmetries in complete intersection Calabi-Yau manifolds [44,45]. Still, this is not the full picture. To obtain the complete flavor group of string theory we have to analyze the outer automorphisms of the Narain space group. This Narain approach will also reveal “non-geometric” symmetries that are not accessible in the geometrical approach: For example, also the $\mathbb{Z}_3 \times \mathbb{Z}_3$ space group selection rules (as discussed in our warm-up example here) will be part of the outer automorphisms of the Narain space group. In the more general picture there will be enhanced flavor symmetries, some of which originate from the modular symmetries of string theory and appear to be non-universal in moduli space.

3. Outer automorphisms of the Narain space group

In this section we extend the discussion of outer automorphisms of the geometrical space group to the Narain space group. It turns out that the outer automorphisms of the Narain space group give rise to the full (non-Abelian) unified flavor symmetry of the theory: it includes (i) possible permutation symmetries of the various fixed points and sectors of the orbifold, (ii) the space group selection rule of strings splitting and joining while moving on the surface of the orbifold, (iii) the target-space modular symmetries of T-duality and (iv) $\mathcal{CP}$-like transformations.[3] In fact, the total resulting flavor symmetry depends on the precise value of the Kähler and complex structure moduli, i.e. colloquially speaking, on the region in moduli space. In more detail, for certain shapes/sizes of the orbifold and for certain values of the background B-field the (non-Abelian) flavor symmetry gets enhanced.

We focus mainly on the bosonic string coordinates and consider a symmetric $\mathbb{T}^D/\mathbb{Z}_K$ orbifold with $\mathbb{Z}_K$ twist Θ in the Narain formulation, see appendix A. In this formulation the D-dimensional compactified string coordinates y are separated into D right- and D left-moving degrees of freedom, y_R and y_L, collectively denoted by $Y = (y_\mathrm{R}, y_\mathrm{L})^\mathrm{T}$. Now, in analogy to the geometrical construction discussed in section 2, Y is compactified on a $2D$-dimensional torus defined by a $2D$-dimensional Narain lattice with vielbein E, composed out of basis vectors E_i for $i = 1, \ldots, 2D$, and a metric with (D, D) signature $\eta = \mathrm{diag}(-\mathbb{1}, \mathbb{1})$. The $\mathbb{Z}_K$ Narain space group S_Narain is defined by its generators $(\Theta, 0)$, $(\mathbb{1}, E_i)$ for $i = 1, \ldots, 2D$. A general element $g \in S_\mathrm{Narain}$ reads

[2] Here, we have used $\hat{g} = (\hat{\theta}^k, n) \in \hat{S}$, $\hat{h} = (\hat{\sigma}, t) \notin \hat{S}$ and we have absorbed $\hat{\sigma}\,n \in \mathbb{Z}^2$ in the definition of n'.

[3] A $\mathcal{CP}$-like transformation is a transformation that acts like a physical $\mathcal{CP}$ transformation on some but not all states of a theory [4].

$$g = (\Theta^k, E\,\hat{N})\,, \quad \text{with} \quad \Theta = \begin{pmatrix} \theta_\mathrm{R} & 0 \\ 0 & \theta_\mathrm{L} \end{pmatrix}, \quad k \in \{0, \ldots, K-1\}$$
$$\text{and} \quad \hat{N} \in \mathbb{Z}^{2D}\,, \tag{12}$$

where $\hat{N}$ contains the D winding and D Kaluza-Klein (KK) quantum numbers of the string. The $\mathbb{Z}_K$ Narain orbifold can be described as an extension of the geometrical construction discussed in section 2 from coordinates y to right- and left-movers Y. Therefore, one identifies

$$Y \sim g\,Y = \Theta^k\, Y + E\,\hat{N}\,. \tag{13}$$

A symmetric $\mathbb{Z}_K$ orbifold is obtained under the assumption that the $2D$-dimensional rotation matrix Θ acts left-right-symmetrically on the $2D$-dimensional Narain lattice, i.e. $\theta_\mathrm{R} = \theta_\mathrm{L} = \theta$ for symmetric $\mathbb{Z}_K$ orbifolds. Finally, we change the basis to the lattice basis (denoted by hatted quantities),

$$\hat{g} = (E^{-1}, 0)\,(\Theta^k, E\,\hat{N})\,(E, 0) = (\hat{\Theta}^k, \hat{N}) \;\in\; \hat{S}_\text{Narain}\,,$$
$$\text{where} \quad \hat{\Theta} := E^{-1}\Theta\, E\,. \tag{14}$$

Similar to the geometric space group, a conjugacy class $[\hat{g}]$ of the Narain space group defines a class of boundary conditions and, thereby, gives rise to a distinct string. Strings on orbifolds have to be invariant under inner automorphisms of the Narain space group $\hat{S}_\text{Narain}$. Hence, inner automorphisms of $\hat{S}_\text{Narain}$ act trivially on orbifold-invariant strings. In contrast, outer automorphisms of $\hat{S}_\text{Narain}$ correspond to the symmetries of the full (bosonic) string theory on orbifolds. Outer automorphisms of the Narain space group $\hat{S}_\text{Narain}$ are given by transformations $\hat{h} := (\hat{\Sigma}, \hat{T}) \;\notin\; \hat{S}_\text{Narain}$ which act on each element $\hat{g} \in \hat{S}_\text{Narain}$ such that

$$\hat{g} \;\overset{\hat{h}}{\mapsto}\; \hat{h}\,\hat{g}\,\hat{h}^{-1} \;\overset{!}{\in}\; \hat{S}_\text{Narain}\,. \tag{15}$$

Spelled out explicitly, this is equivalent to a set of consistency conditions requiring that for each k there must be an $k' \in \{0, \ldots, K-1\}$ and $\hat{N}' \in \mathbb{Z}^{2D}$ such that $\hat{\Sigma} \in \mathrm{GL}(2D, \mathbb{Z})$ and

$$\hat{\Sigma}\,\hat{\Theta}^k\,\hat{\Sigma}^{-1} \overset{!}{=} \hat{\Theta}^{k'}\,, \tag{16a}$$
$$\left(\mathbb{1} - \hat{\Sigma}\,\hat{\Theta}^k\,\hat{\Sigma}^{-1}\right)\hat{T} \overset{!}{=} \hat{N}'\,, \tag{16b}$$

in analogy to eq. (8). The translational part can be fractional, i.e. $\hat{T} \notin \mathbb{Z}^{2D}$, and $\hat{\Sigma}$ may not be a $\mathbb{Z}_K$ rotation, i.e. $\hat{\Sigma} \neq \hat{\Theta}^\ell$ for $\ell = 1, \ldots, K-1$. In addition to the consistency conditions, $\hat{\Sigma}$ must be a modular transformation (i.e. it must satisfy $\hat{\Sigma}^\mathrm{T}\hat{\eta}\,\hat{\Sigma} = \hat{\eta}$) in order to be a symmetry of the Narain lattice.

As a solution to these conditions one can find a set of generators of the outer automorphism group of the form

$$\left\{(\hat{\Sigma}_1, 0),\; (\hat{\Sigma}_2, 0),\; \ldots,\; (\mathbb{1}, \hat{T}_1),\; (\mathbb{1}, \hat{T}_2),\; \ldots\right\}, \tag{17}$$

i.e. the outer automorphism group can be generated by pure rotations $(\hat{\Sigma}_i, 0) \notin \hat{S}_\text{Narain}$ and pure translations $(\mathbb{1}, \hat{T}_j) \notin \hat{S}_\text{Narain}$ – roto-translations $(\hat{\Sigma}, \hat{T})$ are not needed as generators.[4]

Finally, it is instructive to transform the matrix $\hat{\Sigma}$ back to the coordinate basis such that

$$\hat{\Sigma} = E^{-1}\Sigma\, E\,, \quad \text{subject to the condition} \quad \Sigma^\mathrm{T}\eta\,\Sigma \overset{!}{=} \eta\,. \tag{18}$$

[4] Assuming we had a non-trivial roto-translation $(\hat{\Sigma}, \hat{T})$ as a solution to eq. (16). Then also $(\hat{\Sigma}, 0)$ and $(\mathbb{1}, \hat{T})$ are solutions.

A flavor symmetry transformation Σ should leave $(p_\mathrm{R})^2$ and $(p_\mathrm{L})^2$ invariant (and consequently the right- and left-moving string masses). Thus, we have to demand

$$\Sigma \overset{!}{=} \begin{pmatrix} \sigma_\mathrm{R} & 0 \\ 0 & \sigma_\mathrm{L} \end{pmatrix} \quad \text{where} \quad \sigma_\mathrm{R}, \sigma_\mathrm{L} \in \mathrm{O}(D) \quad \Leftrightarrow \quad \Sigma^\mathrm{T}\Sigma = \mathbb{1}\,. \tag{19}$$

Demanding this condition on the outer automorphism $(\Sigma, 0)$ leaves the compactification moduli invariant. Such a transformation belongs to the traditional flavor symmetry.

Let us now specialize to traditional flavor symmetries in two compactified dimensions $D = 2$. Since in this case σ_R and σ_L are two-dimensional orthogonal matrices, $\sigma_\mathrm{R}, \sigma_\mathrm{L} \in \mathrm{O}(2)$, each of them can be uniquely parametrized by one angle, α_R and α_L, respectively. For example

$$\sigma_\mathrm{R}(\alpha_\mathrm{R}) = \begin{pmatrix} \cos(\alpha_\mathrm{R}) & \mp\sin(\alpha_\mathrm{R}) \\ \sin(\alpha_\mathrm{R}) & \pm\cos(\alpha_\mathrm{R}) \end{pmatrix}, \tag{20}$$

where the different signs correspond to a rotation or a reflection, respectively. As a consequence of (16a), for $\mathbb{Z}_K$ orbifolds with $K \neq 2$ in $D = 2$ the matrices σ_R and σ_L have to have the same determinant,

$$\det(\sigma_\mathrm{R}) = \det(\sigma_\mathrm{L})\,. \tag{21}$$

Hence, the matrices σ_R and σ_L describe either both rotations or both reflections, labeled by subscripts "rot." or "refl.", respectively. In both cases, Σ can be symmetric (i.e. $\sigma_\mathrm{R} = \sigma_\mathrm{L}$ thus $\alpha = \alpha_\mathrm{R} = \alpha_\mathrm{L}$) or asymmetric (i.e. $\sigma_\mathrm{R} \neq \sigma_\mathrm{L}$ thus $\alpha_\mathrm{R} \neq \alpha_\mathrm{L}$) and we denote the corresponding $\hat{\Sigma}$-matrix in the lattice basis by $\hat{S}(\alpha)$ or $\hat{A}(\alpha_\mathrm{R}, \alpha_\mathrm{L})$, respectively. Consequently, going back to the lattice basis, the outer automorphisms $(\hat{\Sigma}, 0)$ fall into four categories, where $\hat{\Sigma}$ can be either

$$\hat{S}_\text{rot.}(\alpha)\,,\; \hat{S}_\text{refl.}(\alpha)\,,\; \hat{A}_\text{rot.}(\alpha_\mathrm{R}, \alpha_\mathrm{L})\,, \quad \text{or} \quad \hat{A}_\text{refl.}(\alpha_\mathrm{R}, \alpha_\mathrm{L})\,. \tag{22}$$

Reflections are obviously of order 2, i.e. $\hat{S}_\text{refl.}(\alpha)^2 = \hat{A}_\text{refl.}(\alpha_\mathrm{R}, \alpha_\mathrm{L})^2 = \mathbb{1}$. On the other hand, rotations must map the four-dimensional Narain lattice to itself. Thus, the order of four-dimensional rotations is restricted to the values

$$\{1, 2, 3, 4, 5, 6, 8, 10, 12\}\,, \tag{23}$$

using the Euler-ϕ function [46].

In general, the 4×4 matrices $\hat{\Sigma} = E^{-1}\Sigma\, E$ obtained from eq. (22) are integer matrices only for special values of the moduli that parametrize the vielbein E of the Narain lattice. Thus, the traditional flavor symmetry obtained from the outer automorphism group of the Narain space group depends on the value of the moduli.

4. Flavor symmetries of the $\mathbb{Z}_3$ orbifold

In this section we shall use the method based on the automorphisms of the Narain space group to determine the traditional flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. We will see that the non-Abelian flavor symmetry $\Delta(54)$ is a subgroup of the unified flavor group: $\Delta(54)$ will be enhanced in certain regions of moduli space by the modular symmetries of the underlying string theory.

To obtain the symmetric $\mathbb{Z}_3$ orbifold of the $(2,2)$-dimensional Narain lattice Γ we choose a special Narain vielbein E (see appendix A) by setting

$$e = R\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \quad \text{and} \quad B = b\,\alpha'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{24}$$

The real parameters b and $r := {}^{R^2}/_{\alpha'}$ are combined into the Kähler modulus T and the complex structure modulus U of the two-torus, i.e.

$$T = b + \mathrm{i}\,\frac{\sqrt{3}}{2}\,r \quad \text{and} \quad U = \exp\left(\frac{2\pi\,\mathrm{i}}{3}\right) . \tag{25}$$

We see that the complex structure modulus U is frozen and the Kähler modulus T parametrizes the (b, r)-moduli space of the symmetric $\mathbb{Z}_3$ Narain orbifold in $D = 2$. This means that the Narain lattice Γ can be deformed by freely choosing the Kähler modulus T while the $\mathbb{Z}_3$ symmetry of Γ is kept intact. Now, the (left-right-symmetric) $\mathbb{Z}_3$ Narain orbifold can be specified by the Narain space group defined by the elements

$$\hat{g} = (\hat{\Theta}^k, \hat{N}) \in \hat{S}_{\text{Narain}} , \quad \text{with} \quad \hat{\Theta} = E^{-1}\Theta\, E = \begin{pmatrix} \hat{\theta} & 0 \\ 0 & \hat{\theta}^{-T} \end{pmatrix} , \tag{26}$$

where the two-dimensional twist matrix $\hat{\theta} = e^{-1}\theta\, e$ is given in eq. (5). Due to the choice of the Narain vielbein E specified in eq. (24) and appendix A, the Narain twist $\hat{\Theta}$ in the lattice basis is a symmetry of the Narain lattice $\hat{\Theta}^{\mathrm{T}}\hat{\eta}\,\hat{\Theta} = \hat{\eta}$, as necessary. The symmetries of the $(2, 2)$-dimensional Narain lattice are isomorphic to $\mathrm{SL}(2, \mathbb{Z})_T \times \mathrm{SL}(2, \mathbb{Z})_U$ and $B \mapsto -B$ (together with $G_{12} \mapsto -G_{12}$ [13]). After the orbifolding the complex structure modulus U is fixed, thus $\mathrm{SL}(2, \mathbb{Z})_U$ is broken. Therefore, we can focus on the remaining $\mathrm{SL}(2, \mathbb{Z})_T$ modular symmetry and on the $\mathcal{CP}$-like transformation $B \mapsto -B$.

Under $\mathrm{SL}(2, \mathbb{Z})_T$ modular transformations untwisted and twisted strings transform non-trivially [11]. Acting on massless strings only, the modular group turns out to be T′, i.e. the double covering group of $A_4 \simeq \Gamma(3)$ [13]. Combined with the transformation $B \mapsto -B$ the group gets further enhanced to $\mathrm{GL}(2, 3)$ (i.e. SG(48, 29)).

As described in section 3, the relevant outer automorphisms of the Narain lattice are described by pure rotations $(\hat{\Sigma}_i, 0)$ from the modular symmetries $\hat{\Sigma}_i \in \mathrm{SL}(2, \mathbb{Z})_T$ and pure translations $(\mathbb{1}, \hat{T}_j)$ of the four-dimensional Narain lattice that fulfill the consistency condition (16). Generally, the transformations $\hat{\Sigma}_i$ act non-trivially on the T-modulus. However, at some specific points in moduli space some transformations $\hat{\Sigma}_i$ might leave the vacuum expectation value (VEV) $\langle T\rangle$ of the Kähler modulus invariant. Then, $\hat{\Sigma}_i$ is an element of the traditional flavor symmetry at the point $\langle T\rangle$ in T-moduli space. If $\hat{\Sigma}_i$ leaves $\langle T\rangle$ invariant we get $\Sigma_i \in \mathrm{O}(2) \times \mathrm{O}(2)$ and these rotations fall into the four categories specified in eq. (22): symmetric and asymmetric rotations and symmetric and asymmetric reflections. In the following we shall discuss these specific cases. We shall only present the results here and provide the more technical derivation in a future publication, which will include a derivation of the transformation behavior of twisted and untwisted strings under the action of the outer automorphisms of the Narain space group, and a full discussion of the origin of the modular symmetry $\mathrm{GL}(2, 3)$.

4.1. Generic point in $\langle T\rangle$

At a generic point $\langle T\rangle$ in the T-moduli space, the outer automorphisms that leave the Kähler modulus T invariant can be generated by two translations A and B and a symmetric rotation C given by $\hat{S}_{\text{rot.}}(\pi) = -\mathbb{1}$ in eq. (22), i.e.

$$\mathrm{A} := (\mathbb{1}, \hat{T}_1) , \ \mathrm{B} := (\mathbb{1}, \hat{T}_2) \quad \text{with}$$
$$\hat{T}_1 := \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix} , \ \hat{T}_2 := \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} , \quad \text{and} \quad \mathrm{C} := (-\mathbb{1}, 0) . \tag{27}$$

The automorphism A shifts the winding number and B shifts the KK number. Comparing eq. (27) to the generators of the outer automorphisms of the geometrical space group $\hat{h}_1 = (-\mathbb{1}, 0)$ and $\hat{h}_2 = (\mathbb{1}, t)$, listed in eq. (9), we can identify the correspondences

$$\mathrm{A}^2 \leftrightarrow \hat{h}_2 \quad \text{and} \quad \mathrm{C} \leftrightarrow \hat{h}_1 . \tag{28}$$

The $\mathbb{Z}_3$ Narain outer automorphism B is not accessible in the geometrical case. Acting with B on untwisted and twisted strings, we observe that $\mathrm{A}^2\mathrm{B}^2\mathrm{AB}$ and B give rise to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ point and space group selection rules [42,43]. Altogether, A, B, and C generate the flavor symmetry $\Delta(54)$ at a generic point in moduli space from the outer automorphisms of the Narain space group. This should be compared to the result in our warm-up example discussed earlier, where only the S_3 subgroup could be obtained from the outer automorphisms of the geometrical space group.

4.2. Special B-field with $b = \frac{1}{2} \times$ integer

Let us now show that for generic radii r but quantized values of the B-field, $b = {}^{n_B}/_2$ with $n_B \in \mathbb{Z}$, the flavor symmetry gets enhanced. Consider the left-right-symmetric reflective outer automorphism transformation

$$\mathrm{D}(n_B) := (\hat{S}_{\text{refl.}}({}^{2\pi}/_6), 0) , \quad \text{with}$$
$$\hat{S}_{\text{refl.}}(\alpha = {}^{2\pi}/_6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -n_B & n_B & 1 & 1 \\ n_B & 0 & 0 & -1 \end{pmatrix} , \tag{29}$$

where $\hat{S}_{\text{refl.}}$ has been introduced in equation (22). This transformation has a residual moduli dependence (here in the form of n_B), which is a possibility already noted at the end of section 3. Nonetheless, $\mathrm{D}(n_B)^2 = (\mathbb{1}, 0)$ as expected for a reflection. However, $\mathrm{D}(n_B)$ is a symmetry of the Narain lattice only if n_B takes integer values (cf. appendix A). The T-modulus transforms under $\mathrm{D}(n_B)$ as

$$T \ \mapsto \ n_B - \overline{T} . \tag{30}$$

Hence, the VEV $\langle T\rangle$ is invariant

$$\langle T\rangle \ \mapsto \ n_B - \overline{\langle T\rangle} = \langle T\rangle \quad \text{for} \quad \langle T\rangle = \frac{n_B}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\,\langle r\rangle . \tag{31}$$

Therefore, at regions in moduli space where $b = {}^{n_B}/_2$ with $n_B \in \mathbb{Z}$ there appears an unbroken $\mathbb{Z}_2$ transformation generated by $\mathrm{D}(n_B)$. This enhances the $\Delta(54)$ flavor symmetry to SG(108,17), see Fig. 3.

The six-dimensional representation $\mathbf{6}$ of SG(108,17) acts faithfully on the six twisted strings $(X, Y, Z, \bar{X}, \bar{Y}, \bar{Z})$. The $\mathbf{6}$ branches into $\mathbf{3} \oplus \bar{\mathbf{3}}$ of the $\Delta(54)$ subgroup, implying that A, B, and C only act separately in the barred and unbarred subspaces. On the contrary, $\mathrm{D}(n_B)$ acts as an interchange of (X, Y, Z) and their $\mathcal{CP}$-partners $(\bar{X}, \bar{Y}, \bar{Z})$ (possibly with n_B-dependent phases). This is backed-up by that fact that, geometrically, $\mathrm{D}(n_B)$ acts as a reflection on the axis perpendicular to e_2 and, consequently, it corresponds to complex conjugation in the extra dimensions [47]. Thus, from a $\Delta(54)$ point of view, the $\mathbb{Z}_2$ transformation $\mathrm{D}(n_B)$ acts as a $\mathcal{CP}$-like transformation. In the previous work [6] outer automorphisms of the flavor group $\Delta(54)$ were considered as candidates

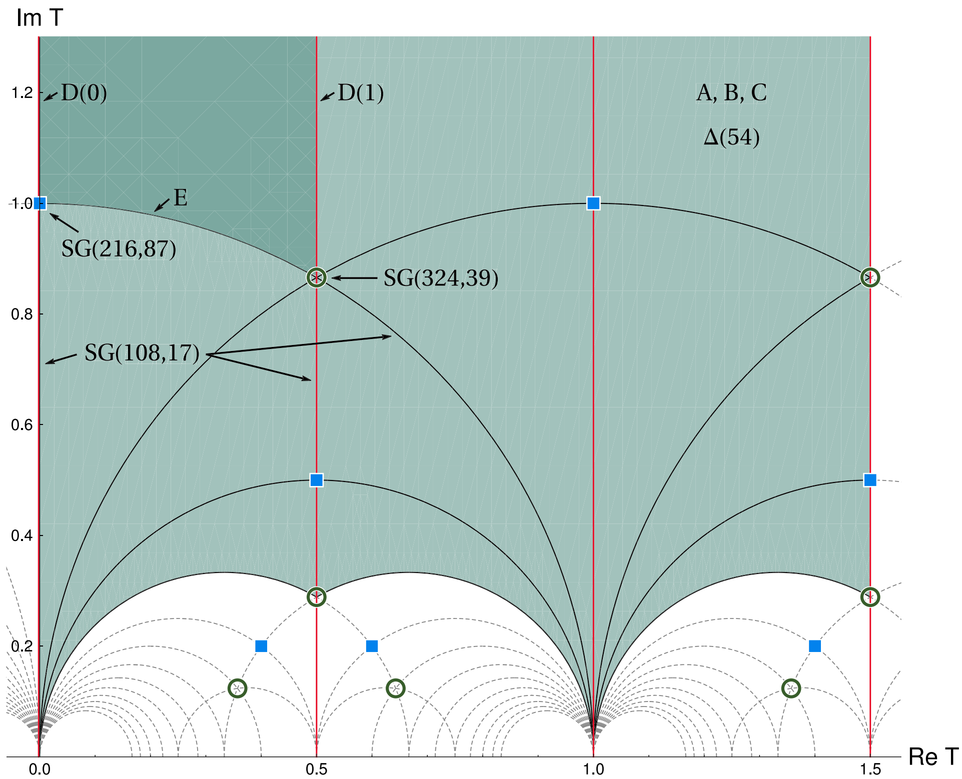

Fig. 3. Points and curves of flavor symmetry enhancement in the moduli space $T = b + \mathrm{i}\,\sqrt{3}/2\,r$. The dark teal area is the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})_T$, while the light teal area is the one of T′ (for both, modulo $B \mapsto -B$). On the vertical red lines and on the black semicircles the flavor group $\Delta(54)$ gets enhanced to SG(108, 17), see sections 4.2 and 4.3. When two lines intersect, i.e. points marked by blue squares, the symmetry is further enhanced to SG(216, 87), as discussed for $(b, r) = (0, 2/\sqrt{3})$ in section 4.4. When three lines intersect, i.e. points marked by small green circles, the flavor group is even further enhanced to SG(324, 39), see section 4.5 for the point $(b, r) = (1/2, 1)$.

for $\mathcal{CP}$-like symmetries. And indeed, the $\mathbb{Z}_2$ transformation $\mathrm{D}(n_B)$ is contained in the outer automorphism group S_4 of $\Delta(54)$. Altogether, we see that at the specific lines $b = n_B/2$ in moduli space the flavor- and $\mathcal{CP}$-symmetries are unified into a single symmetry group.

Moreover, deflecting the VEV of the T-modulus away from the symmetry-enhanced point

$$\langle T\rangle = \frac{n_B}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\,\langle r\rangle \quad \text{to} \quad \langle T'\rangle = \langle T\rangle + \delta T \quad \text{with} \quad \mathrm{Re}(\delta T) \neq 0\,, \tag{32}$$

induces a spontaneous symmetry breaking of SG(108,17) to $\Delta(54)$. This shows that the unified flavor symmetry, and more specifically the $\mathcal{CP}$-like transformation $\mathrm{D}(n_B)$, can be broken spontaneously.

4.3. Black circles

There are more regions in the T-moduli space with an enhanced flavor symmetry. For example, on the semicircle

$$|T|^2 = 1 \quad \text{with} \quad \mathrm{Im}(T) > 0\,, \tag{33}$$

a specific left-right-asymmetric reflection outer automorphism $\hat{A}_{\text{refl.}}(\alpha_\mathrm{R}, \alpha_\mathrm{L})$ becomes an element of the symmetry group. Here, α_R and α_L depend on T precisely in such a way as to ensure that the additional symmetry generated by

$$\mathrm{E} := (\hat{A}_{\text{refl.1}}, 0)\,, \quad \text{with} \quad \hat{A}_{\text{refl.1}} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,, \tag{34}$$

holds everywhere on the semi-circle.

Similar to the case in section 4.2, amending the generators A, B, C by the $\mathbb{Z}_2$ transformation E enhances the $\Delta(54)$ flavor symmetry to SG(108,17) everywhere on the semicircle. Despite being isomorphic, the two SG(108,17) groups here and in section 4.2 are not identical, i.e. they are different extensions of $\Delta(54)$. Analogous enhancements happen on the other black semicircles depicted in Fig. 3.

4.4. Special point at $b = 0$ and $r = 2/\sqrt{3}$

Let us now consider a case where two lines meet, for example the point $(b, r) = (0, 2/\sqrt{3})$ in the T-moduli space, marked by a blue square in Fig. 3. At this point, the unbroken generators, in addition to the usual ones of $\Delta(54)$, are D(0) and E. The total symmetry group at this point then can simply be computed as the closure of all generators, and the result is SG(216, 87).

We remark that other, apparently independent, transformations might be conserved at this point as well. For example, the left-right-asymmetric 4-fold rotation $\hat{A}_{\text{rot.}}(2\pi/4, -2\pi/4)$. However, none of these additional transformations is independent of the transformations above as all of them are already contained in SG(216, 87).

4.5. Special point at $b = 1/2$ and $r = 1$

Finally, we consider a point in the T-moduli space where three lines meet, for example $(b, r) = (1/2, 1)$. There, we identify the following flavor symmetries[5]: A, B, and C originate from the generic

[5] At the point $(b, r) = (1/2, 1)$ there is an additional $\mathrm{U}(1)^2$ gauge symmetry enhancement.

case, D(1) appears on the vertical line at $b = 1/2$, while E is the additional flavor symmetry on the semicircle $|T|^2 = 1$. In addition, for the semicircle with center $(b, r) = (1, 0)$ we identify the unbroken reflection

$$\mathrm{F} := (\hat{A}_{\mathrm{refl.2}}, 0)\,, \quad \text{with} \quad \hat{A}_{\mathrm{refl.2}} := \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (35)$$

This transformation is not independent of the others, as $\mathrm{F} = \mathrm{C\,E\,D(1)\,E}$. Also all other possible additional symmetries out of the set (22), which may be envisaged at this specific point, turn out to be dependent. Thus, the total flavor symmetry at $(b, r) = (1/2, 1)$ can be computed as the closure of $\{\mathrm{A, B, C, D(1), E}\}$ and the result is SG(324, 39), see Fig. 3.

5. Conclusions and outlook

In the present paper we have given a unified description of $\mathcal{CP}$- and flavor-symmetries in string theory. This was possible through the development of a new tool to obtain the full classification of flavor symmetries. It is based on the investigation of outer automorphisms of the Narain space group of compactified string theory. Apart from the traditional flavor symmetries (as discussed in [7–9]), this approach includes string dualities as well. The unified flavor group has the peculiar property that it is non-universal in the moduli space of compactified extra dimensions. Different regions (points or lines) in moduli space might enjoy enhanced flavor symmetries. This allows the unification of $\mathcal{CP}$-transformations within the unified flavor symmetries. The spontaneous breakdown of $\mathcal{CP}$ is then controlled by the vacuum expectation values of the moduli fields. We have illustrated this in a specific example based on the $\mathbb{Z}_3$ orbifold. There we identify the traditional universal flavor group $\Delta(54)$ at generic points in moduli space, with enhancements to unified flavor symmetries SG(108,17), SG(216,87) up to SG(324,39). The enhanced groups are pretty large although our analysis only considered a two-dimensional compactified space, whereas in string theory we have altogether six additional compact space dimensions.

The picture discussed here makes contact to the previous work on $\mathcal{CP}$-violation described in ref. [6]. The phenomenological implications are still valid here, but we have gained a new perspective in the sense that the explicit breakdown of $\mathcal{CP}$ in [6] can now be understood as a spontaneous breakdown of $\mathcal{CP}$ within the unified picture. In addition, the new perspective presented here offers novel directions for flavor model building. As we find different flavor symmetries at different points in moduli space (in particular in six compact dimensions), fields that live at different locations in moduli space feel a different amount of flavor symmetry. Applied to the standard model of particle physics, this could explain, for example, why the observed flavor structure of quarks and leptons is so different. This is reminiscent of the concept of "local grand unification" [17,18] where we can identify different enhanced gauge groups at different "geographical" locations [48] in compact extra dimensions.

The enhanced unified flavor groups are pretty large (especially in the realistic case of six compact dimensions) and allow flexibility for a step-wise breakdown through Wilson lines [49–51] and the vacuum expectation values of the moduli of compact space. This could lead to a different flavor- and $\mathcal{CP}$-structure for the various sectors of the standard model like up- or down-quarks, charged leptons or neutrinos. Such a scheme would share similarities with flavor constructions discussed recently [52,53]. It would also connect to bottom-up constructions that use duality transformations for models of mixing in the quark and especially the lepton sector [21–33], although there are two major differences between these studies and our present point of view. The first one comes from the fact that in our picture the flavor symmetries are a hybrid combination of traditional and modular discrete symmetries, whereas in [21–33] the flavor symmetries are assumed to be completely contained within the modular group. The second difference is a consequence of the fact that in the string theory picture the Kähler potential (and thus the superpotential as well) transforms non-trivially under modular transformations [11–16] in contrast to the assumption of the papers mentioned above. We shall elaborate on the details of these differences in a future publication.

The present discussion shows that string theory naturally leads to a rich and flexible flavor structure that could explain many different aspects of flavor- and $\mathcal{CP}$-symmetry in the standard model. It is worthwhile to go ahead with future research in that direction, both from the bottom-up and top-down perspective.

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Appendix A. Narain lattice

We consider string compactifications on a (D, D)-dimensional Narain lattice Γ and its (symmetric) $\mathbb{T}^D/\mathbb{Z}_K$ orbifold in D extra dimensions, following the conventions of ref. [39]. In this case the Narain lattice is a $2D$-dimensional lattice. It is defined by $2D$ basis vectors $E_{i=1,\ldots,2D}$, which we combine into a $(2D \times 2D)$-dimensional vielbein matrix E. In more detail, the Narain lattice Γ can be defined by a torus compactification of right- and left-moving bosonic string coordinates y_{R} and y_{L}, respectively, i.e.

$$Y \sim Y + E\,\hat{N} \quad \text{with} \quad Y = \begin{pmatrix} y_{\mathrm{R}} \\ y_{\mathrm{L}} \end{pmatrix} \in \mathbb{R}^{2D} \quad \text{and}$$
$$\hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{Z}^{2D}\,, \qquad (36)$$

where $n \in \mathbb{Z}^D$ and $m \in \mathbb{Z}^D$ are the winding and Kaluza-Klein quantum numbers, respectively. The string coordinates y and their T-duals $\tilde{y}$ are related by

$$\begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} y_{\mathrm{R}} \\ y_{\mathrm{L}} \end{pmatrix}. \qquad (37)$$

For the string theory to have a modular invariant partition function, the vielbein E has to span an even self-dual lattice with signature (D, D) – called the Narain lattice Γ. Hence, in the absence of Wilson lines the vielbein E can be parametrized as

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}(G - B) & -\sqrt{\alpha'}\,e^{-\mathrm{T}} \\ \frac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}(G + B) & \sqrt{\alpha'}\,e^{-\mathrm{T}} \end{pmatrix}, \qquad (38)$$

where e is the geometrical vielbein of the D-dimensional torus $\mathbb{T}^D$ with metric $G := e^{\mathrm{T}} e$, $e^{-\mathrm{T}}$ denotes the transposed inverse of e, α' denotes the Regge slope, and $B = -B^{\mathrm{T}}$ is the anti-symmetric B-field. From eq. (38) it follows that

$$E^{\mathrm{T}} \eta\, E = \hat{\eta}\,, \quad \text{where} \quad \eta := \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \hat{\eta} := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \qquad (39)$$

A transformation of the vielbein $E \mapsto E' = E\,\hat{M}^{-1}$ is a symmetry of the Narain lattice Γ iff

$$\hat{M} \in \mathrm{GL}(2D,\mathbb{Z}) \qquad \text{and} \qquad \hat{M}^{\mathrm{T}}\hat{\eta}\,\hat{M} = \hat{\eta}\,. \tag{40}$$

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