Flavon alignments from orbifolding: $\mathrm{SU}(5) \times \mathrm{SU}(3)$ model with $\mathbb{T}^6/\Delta(54)$

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ABSTRACT: We systematically develop the formalism necessary for ensuring that boundary conditions of flavon fields in extra dimensions are consistent with heterotic string theory. Having developed a set of consistency conditions on the boundary conditions, we explore a series of examples of orbifolds in various dimensions to see which ones can satisfy them. In addition we impose the further phenomenological requirements of having non-trivial flavon vacuum alignments and also of having quarks and leptons located appropriately in extra dimensions. The minimal successful case seems to be a 10d theory with a $\mathrm{SU}(3)_{\mathrm{fl}}$ gauged flavour symmetry, where the six-dimensional torus is compactified on a $\mathbb{T}^6/\Delta(54)$ orbifold. We construct a realistic $\mathrm{SU}(5)$ grand unified theory along these lines, leading to tribimaximal-reactor lepton mixing, which we show to be consistent with current neutrino data.

Contents

1 Introduction

The Standard Model (SM), though remarkably successful, gives no understanding of either the origin of the three generations of quarks and leptons or their curious pattern of masses and mixings. In particular the observed very light neutrino masses and approximate tribimaximal lepton mixing requires new physics beyond the SM. To address some of these questions, it has been suggested that the three generations of quarks and leptons may be unified into a triplet of an $SU(3)_{fl}$ gauged flavour symmetry. The three generations are then analogous to the three colours of quarks in QCD. However, unlike QCD, the $SU(3)_{fl}$ gauged flavour symmetry must be broken spontaneously in such a way as to result in the observed quark and lepton masses and mixings [1–5].

In order to spontaneously break the $SU(3)_{fl}$ gauged flavour symmetry, one introduces additional Higgs-like scalars, usually referred to as flavons. Such flavons must have certain vacuum alignments in $SU(3)_{fl}$ flavour space in order to account for the observed quark and lepton masses and mixings. Unfortunately, the sectors introduced in order to achieve such vacuum alignments are typically rather ad hoc. However, there is a top-down method for achieving flavon vacuum alignments coming from string theory formulated in extra dimensions. For example, $E_8 \times E_8$ heterotic string theory in 10d can accommodate a larger gauge symmetry than the SM which in principle could also include a flavon sector whose vacuum alignments may be understood from a more robust theoretical point of view.

The approach to flavons suggested above is somewhat analogous to extra-dimensional grand unified theories (GUTs) compactified on orbifolds, often called "orbifold GUTs" [6–15]. They are typically based on $S^1/\mathbb{Z}_2$ or $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifolds, where Higgs doublet-triplet splitting may be achieved within the framework of extra dimensions [8]. Exactly the same approach can be applied to understanding flavon vacuum alignments. Indeed, a discrete subgroup of the $SU(3)_{fl}$ gauged flavour symmetry may result from the compactification of a 6d theory down to 4d [13, 16–21]. The connection of such field theory orbifold GUTs to string orbifolds [22, 23], especially the stringy origin of non-Abelian discrete flavour symmetries, has been discussed in ref. [24] (see also ref. [25]) and extended to the full string picture in refs. [26, 27].

A recent example of the above approach to flavons in extra dimensions was discussed in ref. [28]. There it was suggested that a $SU(3)_{fl}$ gauged flavour symmetry in 6d, when compactified on a torus with the orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and supplemented by a $\mathbb{Z}_6 \times \mathbb{Z}_3$ discrete symmetry, together with orbifold boundary conditions, may generate all the desired $SU(3)_{fl}$ breaking vacuum expectation values (VEVs). The analysis considered the $(0, 1, -1)$ and $(1, 3, -1)$ vacuum alignments (CSD3) in $SU(3)_{fl}$ flavour space of the Littlest Seesaw model [29, 30] suitable for atmospheric and solar neutrino mixing. Although this idea of using orbifold boundary conditions to dial the desired vacuum alignments of flavons is very attractive, it is non-trivial to ensure that such boundary conditions are consistent with the constraints arising from string theory. These constraints were ignored in [28], loosening the connection of that model with string theory.

In this paper we systematically develop the formalism necessary for ensuring that boundary conditions of flavon fields in extra dimensions are consistent with heterotic string

Orbifold	Flavon alignment	GUT breaking	SM matter localization	Section
$\mathbb{T}^2/\mathbb{Z}_2$	✗	✔	✔	3
$\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$	✗	✔	✗	4
$\mathbb{T}^6/S_4$	✔	✔	✗	5
$\mathbb{T}^6/\Delta(54)$	✔	✔	✔	6

Table 1. Orbifolds studied in this paper. We demand three necessary conditions to build a realistic and predictive model: a non-trivial flavon alignment, the possibility of orbifold GUT breaking and appropriate localizations for SM matter. Only $\mathbb{T}^6/\Delta(54)$ fulfils all of them.

theory. Having developed a set of consistency conditions on the boundary conditions, we explore a series of examples which satisfy them plus the further phenomenological requirements of yielding non-trivial flavon vacuum alignments and of having the SM fermions located on orbifold fixed points, so that their massless modes may include complete multiplets under the gauged flavour symmetry. It turns out to be non-trivial to satisfy all of these conditions (theoretical and phenomenological) together. For instance, the simple $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, while allowing SM fermion matter localisation on fixed points, does not permit non-trivial flavon vacuum alignments, consistently with the formal requirements of the boundary conditions. This motivates us to go to 10d models. However, the simple orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ fares no better than the previous case. We find that the boundary conditions must exhibit some non-Abelian structure so that we can have non-trivial VEV alignments. Hence, we are led to consider 10d non-Abelian orbifolds [31–33], where the torus is modded out, for example, by S_4 or $\Delta(54)$. The latter case is an example where we can locate the SM fermions on fixed points in extra dimensions. Since the $\Delta(54)$ orbifold is not so well studied in the literature, we develop this case in some detail, and eventually show that we can choose the extra dimensions in a way that we can build a complete, consistent, predictive and realistic model which is in principle compatible with string theory. The resulting model is capable of achieving the flavon vacuum alignments consistent with tri-bimaximal lepton mixing [34], which may be corrected to yield tribimaximal-reactor lepton mixing [35], which we show to be consistent with the latest neutrino data.

The layout of the paper is summarised in table 1 which not only summarises the foregoing situation but also gives the organisation of the main body of this paper in terms of the section numbers 3, 4, 5, and 6 shown. These sections are bracketed by the extra dimensional heterotic string friendly formalism in section 2 and a realistic model based on the $\Delta(54)$ orbifold in section 7. Section 8 concludes the paper.

2 Field theory orbifolds

2.1 Constraints on the gauge embedding

Let us consider a field theory with D extra dimensions z compactified on an orbifold $\mathbb{O} = \mathbb{R}^D/S$ with space group S, see appendix A for details. The geometric orbifold action $z \mapsto \theta z + \lambda$ of a space group element $h = (\theta, \lambda) \in S$ (where $\theta \in P$ and the so-called point

group P is a finite subgroup of $\mathrm{SO}(D)$) is embedded into an action on a general field $\Phi(x,z)$ of the theory as

$$\Phi(x,z) \;\overset{h}{\longmapsto}\; R_h\,\Phi(x,h^{-1}z)\,, \tag{2.1}$$

where for each $h \in S$ there is an element R_h of the symmetry group $\mathcal{G}$ of the theory. In general, the symmetry group $\mathcal{G}$ contains the internal part of the higher-dimensional Lorentz symmetry. For example, if $\Phi(x,z)$ is a four-dimensional scalar equipped with an internal vector-index we get $\Phi(x,z) \mapsto \theta\,\Phi(x,\theta^{-1}z)$ for $h = (\theta,0)$. In addition, $\mathcal{G}$ can contain a higher-dimensional gauged flavour symmetry such that R_h is given by a constant gauge transformation, see for example ref. [12]. We call R_h the (gauge) embedding of h and denote the associated discrete group by $R_S := \{R_h \,|\, h \in S\}$. One can apply two transformations g and h on a field $\Phi(x,z)$, either combined or one transformation after the other, i.e.

$$\Phi(x,z) \;\overset{g\,h}{\longmapsto}\; R_{g\,h}\,\Phi(x,(g\,h)^{-1}z)\,, \tag{2.2a}$$

$$\Phi(x,z) \;\overset{h}{\longmapsto}\; R_h\,\Phi(x,h^{-1}z) \;\overset{g}{\longmapsto}\; R_g\,R_h\,\Phi(x,h^{-1}g^{-1}z)\,. \tag{2.2b}$$

In both cases one has to obtain the same result. Hence, the consistency condition

$$R_{g\,h} = R_g\,R_h \tag{2.3}$$

follows. Mathematically, this condition says that R has to be a group homomorphism from the space group S into the (gauge) group $\mathcal{G}$. It is easy to obtain some immediate consequences of eq. (2.3), e.g.

$$R_{(\mathbb{1},0)} = \mathbb{1}\,, \quad R_{g^{-1}} = (R_g)^{-1}\,, \quad R_{g\,h\,g^{-1}} = R_g\,R_h\,(R_g)^{-1}\,, \tag{2.4}$$

for $g,h \in S$, and

$$R_{e_i}\,R_{e_j} = R_{e_j}\,R_{e_i}\,, \quad R_{\theta e_i} = R_{(\theta,\lambda)}\,R_{e_i}\,\big(R_{(\theta,\lambda)}\big)^{-1}\,, \quad \big(R_{(\theta,\lambda)}\big)^N = \mathbb{1}\,, \tag{2.5}$$

for a space group element $(\theta,\lambda) \in S$ of order N, i.e. $(\theta,\lambda)^N = (\mathbb{1},0)$, $i,j \in \{1,\dots,D\}$, and we have defined the embedding of a pure translation as $R_{e_i} = R_{(\mathbb{1},e_i)}$, see also section 2 of ref. [36]. Hence, the choices for the embedding R are strongly constrained as we will also see in more detail in the examples of the next sections.

2.2 Standard embedding

There exists a simple solution to the group homomorphism condition (2.3): the so-called standard embedding. In this case, (ignoring for a moment the higher-dimensional Lorentz symmetry for simplicity) one chooses a gauge group $\mathcal{G}$ such that the point group P is a discrete subgroup of it. Furthermore, for supersymmetric orbifolds in $D = 6$ dimensions we have $P \subset \mathrm{SU}(3)$ such that there exists a globally defined constant spinor on the orbifold $\mathbb{O}$ [22, 23]. Hence, in complex coordinates each point group element $\theta \in P$ is given by a 3×3 unitary matrix (with $\det(\theta) = 1$) and the choice

$$R_h = \theta \qquad \text{for} \qquad h = (\theta,\lambda) \in S\,, \tag{2.6}$$

trivially satisfies the group homomorphism condition (2.3) for $\mathcal{G} = \mathrm{SU}(3)_{\text{fl}}$. This $\mathrm{SU}(3)$ gauge symmetry can naturally be identified with a flavour symmetry, hence the subscript "fl". In other words, the geometrical element $h = (\theta, \lambda)$ that acts on the (complex) extra-dimensional coordinates $z \in \mathbb{C}^3$ as $z \mapsto \theta z + \lambda$ is accompanied by an identical gauge transformation $R_h = \theta$ in $\mathrm{SU}(3)_{\text{fl}}$ flavour space, in detail, $\Phi \mapsto \theta \Phi$ if Φ is a triplet of $\mathrm{SU}(3)_{\text{fl}}$. Note that eq. (2.6) implies for instance that $R_{e_i} = \mathbb{1}$ for $i \in \{1, \ldots, 6\}$, i.e. there are no Wilson lines along the torus-directions [12] in the case of standard embedding, and the discrete gauge embedding group is isomorphic to the point group, $R_S \cong P$.

As a final remark, in addition to the condition (2.3) on the (gauge) embedding, string theory orbifolds are constrained by world-sheet modular invariance of the one-loop string partition function [23]. However, modular invariance is automatically satisfied in the case of standard embedding. Thus, the standard embedding $R_S \cong P$ in our field theory discussion fulfils all necessary conditions for a full string completion.

2.3 Orbifold boundary conditions

Next, we discuss the various origins of fields $\Phi(x, z)$ on an orbifold $\mathbb{O}$ and the conditions to make these fields well-defined within the orbifold construction. This discussion is crucial in order to understand the orbifold-alignment of flavon VEVs in flavour space.

For each space group element $g \in S$ that has a non-trivial fixed point set

$$\mathrm{F}_g = \{z \in \mathbb{R}^D \mid g z = z\} \ne \emptyset \tag{2.7}$$

one can define a field $\Phi_g(x, z)$ that is localized on F_g, i.e.

$$\Phi_g(x, z) = 0 \quad \text{if} \quad z \notin \mathrm{F}_g . \tag{2.8}$$

This localized field $\Phi_g(x, z)$ transforms in some representation of $\mathcal{G}$. For example, for a flavour symmetry $\mathcal{G} = \mathrm{SU}(3)_{\text{fl}}$ we will mostly assume that $\Phi_g(x, z)$ is either a singlet or a triplet of $\mathrm{SU}(3)_{\text{fl}}$.

In the language of string theory, a field $\Phi_g(x, z)$ with $g \ne \mathbb{1}$ corresponds to a so-called twisted string localized at the fixed point set F_g of the so-called constructing element $g \in S$. In contrast, a field $\Phi_g(x, z)$ with $g = \mathbb{1}$ has a trivial fixed point set $\mathrm{F}_g = \mathbb{R}^D$ and, hence, lives in the full bulk $\mathbb{O}$ of the extra dimensions.

Importantly, a field $\Phi_g(x, z)$ from eq. (2.8) is in general not yet invariant under the orbifold action eq. (2.1), i.e. it transforms as

$$\Phi_g(x, z) \overset{h}{\longmapsto} R_h \Phi_g(x, h^{-1} z) , \tag{2.9}$$

for $h \in S$. Let us analyze eq. (2.9) in more detail. While the field $\Phi_g(x, z)$ on the left-hand side is localized at $z \in \mathrm{F}_g$, the field $\Phi_g(x, h^{-1}z)$ on the right-hand side is localized at $h^{-1}z \in \mathrm{F}_g$ which is equivalent to $z \in \mathrm{F}_{h g h^{-1}}$. However, the only field localized on the fixed point set $\mathrm{F}_{h g h^{-1}}$ is $\Phi_{h g h^{-1}}(x, z)$. Consequently, the fields $\Phi_g(x, h^{-1}z)$ and $\Phi_{h g h^{-1}}(x, z)$ have to be related,

$$\Phi_g(x, h^{-1} z) \sim \Phi_{h g h^{-1}}(x, z) . \tag{2.10}$$

By definition, the space group element $h\,g\,h^{-1}$ belongs to the conjugacy class $[g]$. Thus, fields from the same conjugacy class have to be identical up to some proportionality factors, i.e. up to a symmetry transformation from $\mathcal{G}$.

A special case appears if $h\,g\,h^{-1} = g$ for $h \neq g$, i.e. in the case when g and h commute. Hence, we define the set of commuting elements (the so-called centralizer C_g) of the constructing element g as

$$C_g := \{h \in S \mid g\,h = h\,g\} \subset S\,. \tag{2.11}$$

Now, we have to distinguish between two cases, depending on whether g and h commute or not: the case $h \notin C_g$ is not of great importance to us and is therefore relegated to appendix A.1. On the other hand, if $h \in C_g$ we get $\Phi_{h\,g\,h^{-1}}(x,z) = \Phi_g(x,z)$. Then, eqs. (2.9) and (2.10) yield a non-trivial boundary condition

$$\Phi_g(x,z) \overset{h}{\longmapsto} R_h\,\Phi_g(x,h^{-1}z) = \Phi_g(x,z) \qquad \text{for} \qquad h \in C_g \quad \text{and} \quad h \neq g\,. \tag{2.12}$$

Since z and $h^{-1}z$ are identified on the orbifold $\mathbb{O}$, this boundary condition ensures that the field Φ_g evaluated at identified points has a unique value up to a symmetry transformation with $R_h \in \mathcal{G}$. The orbifold boundary condition using the constructing element $h = g \in C_g$ itself is special, since

$$\Phi_g(x,g^{-1}z) = \Phi_g(x,z) = 0 \ \text{ if } z \notin \text{F}_g\,, \tag{2.13a}$$
$$\Phi_g(x,g^{-1}z) = \Phi_g(x,z) \qquad \text{if } z \in \text{F}_g\,. \tag{2.13b}$$

Hence, $\Phi_g(x,g^{-1}z) = \Phi_g(x,z)$ for all $z \in \mathbb{R}^D$ and the transformation (2.9) reads

$$\Phi_g(x,z) \overset{g}{\longmapsto} R_g\,\Phi_g(x,g^{-1}z) = R_g\,\Phi_g(x,z)\,, \tag{2.14}$$

while the relation (2.10) becomes trivial for $h = g$. In more detail, for $h = g$ we get $\Phi_g(x,h^{-1}z) = \Phi_g(x,z)$ on the left-hand side and $\Phi_{h\,g\,h^{-1}}(x,z) = \Phi_g(x,z)$ on the right-hand side of eq. (2.10). From the string construction, we know that a string with constructing element $g \in S$ survives the orbifold projection at least under the action of $g \in C_g$, see for example appendix A.5 in ref. [37]. Hence, eq. (2.14) imposes the boundary condition $R_g\,\Phi_g(x,z) = \Phi_g(x,z)$ on the localized field $\Phi_g(x,z)$.

In summary, in order to build an orbifold-invariant field $\Phi_g(x,z)$ that is localized on the fixed point set F_g we have to impose a non-trivial boundary condition (2.12) for each space group element $h \in C_g$ that commutes with the constructing element $g \in S$ of the localized field $\Phi_g(x,z)$.

Let us briefly discuss the trivial example with constructing element $g = \mathbb{1} \in S$. In this case, the fixed point set is given by $\text{F}_\mathbb{1} = \mathbb{R}^D$ and the field $\Phi_\mathbb{1}(x,z)$ lives in the full bulk $\mathbb{O}$ of the extra dimensions. Furthermore, the centralizer $C_\mathbb{1}$ equals the full space group, i.e. $C_\mathbb{1} = S$, and we have to impose boundary conditions (2.12) for all elements $h \in S$, i.e. for all generators of S.

2.4 VEV alignment from orbifold boundary conditions

Our main focus is to interpret $\mathcal{G}$ as a gauged flavour group in extra dimensions and the field $\Phi_g(x, z)$ as a flavon. Therefore, we separate the higher-dimensional Lorentz symmetry from $\mathcal{G}$ and consider $\mathcal{G}$ as a pure gauge symmetry. Then, orbifold boundary conditions (2.12) break the flavour symmetry $\mathcal{G}$ and simultaneously align the vacuum expectation values of the flavons, as we discuss next.

Consider a field $\Phi_g(x, z)$ localized at $z \in \mathrm{F}_g$ with constructing element $g \in S$. We take an element $h = (\theta_h, \lambda_h) \in C_g$, where the order of θ_h is denoted by N_h, i.e. $(\theta_h)^{N_h} = \mathbb{1}$. After choosing $\ell_h \in \{0, \ldots, N_h-1\}$, the field $\Phi_g(x, z)$ has to satisfy the boundary condition (2.12),

$$\Phi_g(x, z) \;\overset{h}{\longmapsto}\; \exp\left(\frac{2\pi\mathrm{i}\,\ell_h}{N_h}\right) R_h\, \Phi_g(x, h^{-1}z) = \Phi_g(x, z) \,. \tag{2.15}$$

The ℓ_h-dependent phase originates from diagonalizing the higher-dimensional Lorentz transformation, which is possible for all $h \in C_g$ simultaneously if the centralizer C_g is Abelian.

We denote the four-dimensional zero mode of the field $\Phi_g(x, z)$ by $\Phi_g(x)$. Then, the orbifold boundary condition (2.15) evaluated at the vacuum expectation value of the zero mode $\langle\Phi_g\rangle$ reads

$$\langle\Phi_g\rangle \;\overset{h}{\longmapsto}\; \exp\left(\frac{2\pi\mathrm{i}\,\ell_h}{N_h}\right) R_h\, \langle\Phi_g\rangle = \langle\Phi_g\rangle \,, \tag{2.16}$$

for all elements $h \in C_g$ from the centralizer of the constructing element g. This condition can align the VEV of a localized field to a specific direction in flavour space. In other words, the VEV of a flavon located in F_g must be an eigenvector of the matrices R_h with an ℓ_h-dependent phase as eigenvalue. However, the magnitude of the VEV cannot be constrained by orbifold boundary conditions.

For example, take an $\mathrm{SU}(3)_{\mathrm{fl}}$ flavour group and a triplet flavon Φ_g with constructing element $g \in S$ and assume that the centralizer contains an element $h \in C_g$ with $h^3 = \mathbb{1}$ such that $(R_h)^3 = \mathbb{1}$. By choosing a special embedding $R_h \in \mathrm{SU}(3)_{\mathrm{fl}}$ and $\ell_h = 0$, the flavon VEV $\langle\Phi_g\rangle$ is aligned according to

$$R_h = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \langle\Phi_g\rangle \;\propto\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,. \tag{2.17}$$

However, one can always choose a basis in flavour space such that a given embedding matrix R_h becomes diagonal and the VEV aligns, for example, into the first component of $\langle\Phi_g\rangle$. To avoid this rather trivial situation, one has to ensure that the discrete embedding group $R_S \subset \mathcal{G}$ is non-Abelian such that one cannot diagonalize all elements simultaneously. This is the key observation towards successful flavon alignment.
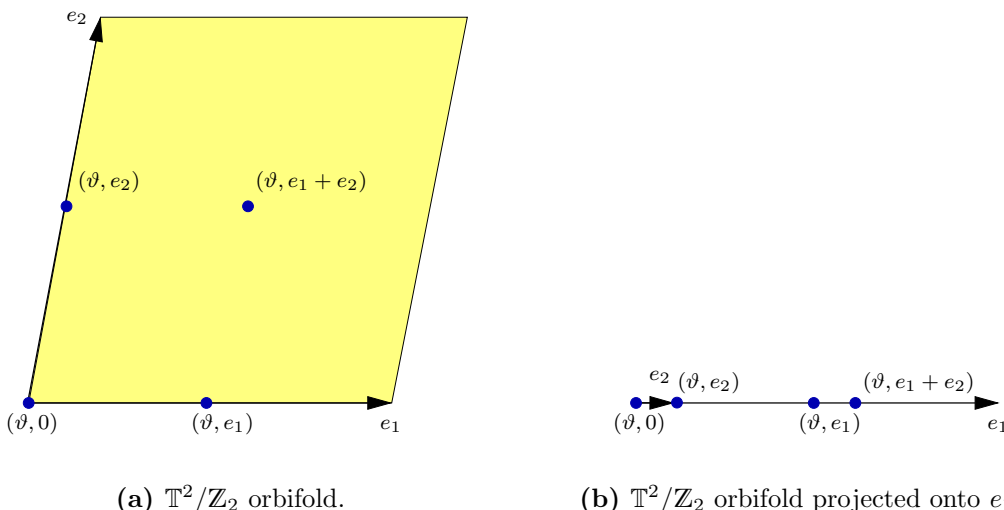
(a) $\mathbb{T}^2/\mathbb{Z}_2$ orbifold.

(b) $\mathbb{T}^2/\mathbb{Z}_2$ orbifold projected onto e_1.

Figure 1. (a) The two-torus $\mathbb{T}^2$ is spanned by the vectors e_1 and e_2 and its fundamental domain is highlighted in yellow. The $\mathbb{Z}_2$ orbifold action ϑ maps $z \mapsto -z$ resulting in four fixed points labeled by their constructing elements. (b) The same orbifold projected onto one dimension given by the e_1 direction (as a preparation for six-dimensional orbifolds projected onto three dimensions, see e.g. figure 2).

3   Flavour from a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold

Let us begin with an easy example with $D = 2$ extra dimensions, parametrized in complex coordinates by $z \in \mathbb{C}$. We choose a general two-torus $\mathbb{T}^2$ spanned by two vectors e_1 and e_2. Then, z is compactified on a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, i.e. with point group $P \cong \mathbb{Z}_2$, where the $\mathbb{Z}_2$ orbifold action ϑ is generated by $z \mapsto -z$. This orbifold has four inequivalent fixed points F_g,

$$\mathrm{F}_g \;\in\; \left\{ \{0\}, \;\; \left\{\frac{e_1}{2}\right\}, \;\; \left\{\frac{e_2}{2}\right\}, \;\; \left\{\frac{e_1 + e_2}{2}\right\} \right\} , \tag{3.1}$$

corresponding to the constructing elements g,

$$g \;\in\; \left\{ (\vartheta, 0) , \;\; (\vartheta, e_1) , \;\; (\vartheta, e_2) , \;\; (\vartheta, e_1 + e_2) \right\} , \tag{3.2}$$

respectively, see figure 1.

To define a field theory on this $\mathbb{Z}_2$ orbifold, we have to choose a (gauge) embedding R_ϑ, R_{e_1}, and R_{e_2}, for each generator of the $\mathbb{Z}_2$ space group S, i.e. for each $(\vartheta, 0)$, $(\mathbb{1}, e_1)$, and $(\mathbb{1}, e_2) \in S$. We have to ensure that this embedding satisfies the following conditions, obtained from eq. (2.3),

$$(R_\vartheta)^2 = \mathbb{1} , \quad R_{e_1} R_{e_2} = R_{e_2} R_{e_1} \quad \text{and} \quad R_\vartheta R_{e_i} R_\vartheta = (R_{e_i})^{-1} , \tag{3.3}$$

where $i \in \{1, 2\}$, see also eqs. (2.4) and (2.5).

3.1 Example of a trivial VEV alignment

A trivial solution to eq. (3.3) is given by the choice of a gauged flavour symmetry $\mathcal{G} = \mathrm{U}(2)$ and

$$R_\vartheta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R_{e_i} = \mathbb{1} \, . \tag{3.4}$$

In this case, the embedding group $R_S \cong \mathbb{Z}_2$ is Abelian and, consequently, VEVs can only be aligned trivially in flavour space. In detail, take a doublet flavon $\Phi_g(x, z)$ from the bulk (i.e. with trivial constructing element $g = (\mathbb{1}, 0) \in S$ and centralizer $C_g = S$). Then, its VEV can only be aligned according to

$$\langle \Phi_{g,+} \rangle \;\propto\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \langle \Phi_{g,-} \rangle \;\propto\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, , \tag{3.5}$$

originating from the boundary conditions

$$R_\vartheta \, \langle \Phi_{g,\pm} \rangle = \pm \, \langle \Phi_{g,\pm} \rangle \quad \text{and} \quad R_{e_i} \, \langle \Phi_{g,\pm} \rangle = \langle \Phi_{g,\pm} \rangle \, , \tag{3.6}$$

where the extra $\pm$ sign in the ϑ-boundary condition is motivated from the transformation properties of $\Phi_g(x, z)$ under higher-dimensional Lorentz symmetry, see section 2.4.

In order to avoid this trivial VEV alignment eq. (3.5) in the case of this $\mathbb{Z}_2$ orbifold, we have to choose non-trivial Wilson lines $R_{e_i} \neq \mathbb{1}$ for some $i \in \{1, 2\}$. However, choosing $R_{e_i} \neq \mathbb{1}$ with $(R_{e_i})^2 = \mathbb{1}$ in eq. (3.3) would also result in an Abelian embedding group[1] and, consequently, would yield a trivial VEV alignment.

3.2 Example of a non-trivial VEV alignment

Let us give a non-trivial example of matrices R_ϑ, R_{e_1} and R_{e_2} for a gauged flavour symmetry $\mathcal{G} = \mathrm{U}(3)$, where $(R_{e_i})^2 \neq \mathbb{1}$ for some translations e_i.

Since $(\vartheta)^2 = \mathbb{1}$ the gauge embedding R has to satisfy $(R_\vartheta)^2 = \mathbb{1}$. We might choose

$$R_\vartheta = T_{13} \quad \text{where} \quad T_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (T_{13})^2 = \mathbb{1} \, . \tag{3.7}$$

Furthermore, we have to satisfy condition (3.3), i.e. $(R_\vartheta R_{e_i})^2 = \mathbb{1}$ for $i \in \{1, 2\}$. To do so, we are left with the freedom to choose R_{e_i} for $i \in \{1, 2\}$ subject to the previous condition. One can check that a solution is given by

$$R_{e_1} = SU \, T_{13} \quad \text{and} \quad R_{e_2} = T_{13} \, SU = (R_{e_1})^{-1} \, , \tag{3.8}$$

using the matrices

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \, , \tag{3.9}$$

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[1]This choice corresponds to the Abelianization $R_S \cong S/[S, S]$ of the space group S, see e.g. [38–40].

where $(SU)^2 = \mathbb{1}$. These matrices are chosen since they are part of a specific representation of the discrete group S_4, which is known for generating predictive flavour structures [28, 41].

Importantly, for this choice the discrete gauge embedding group R_S is non-Abelian and, furthermore, the matrices R_{e_i} have infinite order, i.e. for all $N_i \in \mathbb{N}$ we have

$$(R_{e_i})^{N_i} \neq \mathbb{1} . \tag{3.10}$$

Hence, R_S is not a finite group, see ref. [42] and, furthermore, ref. [14] for a discussion on rank reduction in the case when the discrete gauge embedding group R_S is non-Abelian.

Now, consider a triplet flavon $\Phi_g(x, z)$ with constructing element $g = (\mathbb{1}, 0) \in S$. It is localized in the bulk $\mathbb{O}$ of the extra-dimensional space. Then, the corresponding centralizer C_g can be generated by

$$C_g = \langle (\vartheta, 0) , (\mathbb{1}, e_1) , (\mathbb{1}, e_2) \rangle \qquad \text{for} \qquad g = (\mathbb{1}, 0) . \tag{3.11}$$

The flavon is subject to boundary conditions (2.16) which result in the following non-trivial conditions on the VEV (choosing $\ell_h = 0$ for $h = (\vartheta, 0)$)

$$R_\vartheta \langle \Phi_g \rangle = \langle \Phi_g \rangle , \quad R_{e_1} \langle \Phi_g \rangle = \langle \Phi_g \rangle , \quad R_{e_2} \langle \Phi_g \rangle = \langle \Phi_g \rangle . \tag{3.12}$$

The solution is given by a fixed VEV alignment

$$\langle \Phi_g \rangle \propto \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} , \tag{3.13}$$

where the magnitude of the VEV cannot be determined by orbifold boundary conditions.

We can try to locate another flavon $\tilde{\Phi}_g$ in the bulk with a different $\mathbb{Z}_2$ phase, explicitly $R_\vartheta \langle \tilde{\Phi}_g \rangle = - \langle \tilde{\Phi}_g \rangle$, to obtain a different flavon alignment. However, it turns out that $\langle \tilde{\Phi}_g \rangle = 0$. In other words, $\tilde{\Phi}_g$ is projected out by the orbifold action in this case.

We have achieved the flavon alignment (3.13), which is necessary for the CSD3 setup [29]. This is highly predictive for the lepton sector and usually complicated to obtain through a vacuum alignment superpotential [21, 29, 43–46]. However, it is not enough by itself. We have shown that there are no other alignments we can obtain through boundary conditions in this setting. Consequently, after a brief discussion on GUT breaking and the localization of SM matter in the following, we will go to higher-dimensional orbifolds with larger point groups to align various flavons simultaneously.

3.3 GUT breaking

We assume that the extra-dimensional theory before orbifolding contains a GUT gauge symmetry in addition to the gauged flavour symmetry $\mathcal{G}$, where we will choose $\mathcal{G} = \text{SU}(3)_\text{fl}$ or $\text{U}(3)_\text{fl}$ as our prime examples. Then, the full gauge symmetry in extra dimensions is given by

$$\text{SU}(5)_\text{GUT} \times \mathcal{G} \qquad \text{or} \qquad \text{SO}(10)_\text{GUT} \times \mathcal{G} . \tag{3.14}$$

Both, the GUT group and the flavour group, have to be broken by orbifold boundary conditions, determined by the gauge embedding R. In this specific $\mathbb{T}^2/\mathbb{Z}_2$ orbifold the

GUT-breaking boundary conditions can be chosen to correspond to any of the translations R_{e_i}, while the flavor symmetry can be broken by the gauge embedding R_ϑ of the orbifold twist ϑ.

If the symmetry is $SU(5)_{GUT}$ the GUT-breaking boundary condition can be chosen as

$$P_{SU(5)} = \text{diag}(1, 1, 1, -1, -1) \otimes \mathbb{1}_{\text{flavour}} , \qquad (3.15)$$

which breaks $SU(5)_{GUT} \to SU(3)_C \times SU(2)_L \times U(1)_Y$. Since $P_{SU(5)}$ does not act on the flavour symmetry $\mathcal{G}$, it commutes with all flavour breaking conditions and, hence, is consistent with eqs. (3.3) (using for example $R_{e_1} = P_{SU(5)}$ and $(P_{SU(5)})^2 = \mathbb{1}$).

In the case of an $SO(10)_{GUT}$ symmetry, $SO(10)$ can be broken by two independent $\mathbb{Z}_2$ boundary conditions [13]

$$P_{GG} = \text{diag}(1, 1, 1, 1, 1) \otimes \sigma_2 \otimes \mathbb{1}_{\text{flavour}} , \quad P_{PS} = \text{diag}(1, 1, 1, -1, -1) \otimes \sigma_0 \otimes \mathbb{1}_{\text{flavour}} , \ (3.16)$$

where σ_i are the Pauli matrices. Each condition separately breaks $SO(10)_{GUT}$ according to

$$P_{GG} : SO(10)_{GUT} \to SU(5) \times U(1), \quad P_{PS} : SO(10)_{GUT} \to SU(4) \times SU(2) \times SU(2), \ (3.17)$$

while together they break $SO(10)_{GUT} \to SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)$, see e.g. [18]. The two boundary conditions P_{GG} and P_{PS} commute with each other, as well as with any flavour-breaking condition. Thus, we can choose each one to be one of the R_{e_i} consistently.

3.4 SM fermion localization

The SM matter fields, as any other field in this $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, must be located somewhere in extra dimensions: i) either on a fixed point set F_g for $g \neq \mathbb{1}$, being points in compactified dimensions, see eq. (3.1), or ii) in the two-dimensional bulk $\mathbb{O}$. In principle, localized fields feel boundary conditions (2.16) with respect to their centralizers C_g. Consequently, some zero modes of SM matter fields can be projected out by the orbifold — depending on the respective centralizers C_g.

First, consider a localized field Φ_g with constructing element g from eq. (3.2). In this case, the centralizer $C_g = \{\mathbb{1}, g\}$ is trivial as discussed in section 2.3. Consequently, a localized field Φ_g in the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold is not subject to orbifold boundary conditions and the four-dimensional zero mode $\Phi_g(x)$ is not projected out. Therefore, SM matter fermions from localized fields appear in complete GUT multiplets. This is the field-theoretical analogue to string-theoretical local GUTs with complete SM generations [47–49].

In contrast, a bulk field is subject to non-trivial orbifold boundary conditions, especially to those that induce GUT breaking. Let us discuss the consequences of SM matter in the bulk for $SU(5)_{GUT}$ and $SO(10)_{GUT}$ in the following:

In the $SU(5)_{GUT}$, a complete SM generation is given by the representations $\bar{5} + 10 + 1$ of $SU(5)_{GUT}$. What happens if we assume that these matter fields live in the bulk of the orbifold? In this case, the singlet (of the right-handed neutrino) is not affected by the GUT-breaking boundary condition $P_{SU(5)}$. For the other $SU(5)_{GUT}$ representations each matter field can be a positive or negative eigenstate of $P_{SU(5)}$, which determines the

respective zero modes. For example, let us denote the $\bar{5}$ of SU(5) as $F = (d^c, \ell)$. Then, eq. (2.16) yields

$$F \overset{P_{\mathrm{SU(5)}}}{\longmapsto} P_{\mathrm{SU(5)}} F = +F \quad \Rightarrow \quad F \equiv \bar{5}^+ \to d^c \quad \text{or} \tag{3.18a}$$

$$F \overset{P_{\mathrm{SU(5)}}}{\longmapsto} P_{\mathrm{SU(5)}} F = -F \quad \Rightarrow \quad F \equiv \bar{5}^- \to \ell \,, \tag{3.18b}$$

while the 10 of $\mathrm{SU(5)_{GUT}}$ (being an anti-symmetric 5×5 matrix T) is subject to the boundary conditions $T \mapsto P_{\mathrm{SU(5)}} T P_{\mathrm{SU(5)}} = \pm T$. In total, we get

$$\bar{5}^+ \to d^c, \quad \bar{5}^- \to \ell, \quad 10^+ \to u^c + e^c, \quad 10^- \to q, \quad 1^+ \to n \,. \tag{3.19}$$

To have the full SM matter content after compactification, we have to have each eigenstate. Consequently, the number of SM matter bulk fields before compactification must be duplicated.

In the $\mathrm{SO(10)_{GUT}}$, a complete SM generation fits into the 16 of $\mathrm{SO(10)_{GUT}}$. Depending on the eigenstate of each boundary condition P_{GG} and P_{PS} we obtain the zero modes

$$16^{++} \to d^c + n, \quad 16^{+-} \to \ell, \quad 16^{-+} \to q, \quad 16^{--} \to u^c + e^c \,. \tag{3.20}$$

Hence, if the SM matter is supposed to originate from the extra-dimensional bulk we need four copies of 16-plets in the orbifold bulk for a single generation of SM quarks and leptons. Note that split GUT multiplets can be beneficial to explain the different masses for charged leptons and down quarks.

Beside GUT-breaking boundary conditions, SM matter fields will in general be subject to the boundary conditions of the flavour group $\mathrm{SU(3)_{fl}}$. Obviously, $\mathrm{SU(3)_{fl}}$ singlets are not affected by the flavour-breaking boundary conditions. However, in order to get non-trivial predictions from the $\mathrm{SU(3)_{fl}}$ flavour group, we assume that some SM matter fields transform as $\mathrm{SU(3)_{fl}}$ triplets corresponding to the number of three generations. Then, in order to keep the structure in the fermion mass matrices dictated by the flavour symmetry, these flavour-triplets must be kept as triplets after compactification. Therefore, the $\mathrm{SU(3)_{fl}}$ matter triplets must be localized at zero-dimensional fixed points in the extra dimensions of the orbifold with trivial centralizers such that they are not subject to flavour-breaking boundary conditions.

4 Flavour from a $\mathbb{T}^6 / \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Next, we consider a ten-dimensional theory with $\mathcal{N} = 1$ supersymmetry compactified on a $\mathbb{T}^6 / \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold [47, 50] to four-dimensional space-time. In this case, the six-dimensional torus $\mathbb{T}^6$ can be chosen to be factorized $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, where each two-torus $\mathbb{T}^2$ is specified by two vectors

$$e_{2a-1} \quad \text{and} \quad e_{2a} \quad \text{for} \quad a \in \{1, 2, 3\} \,. \tag{4.1}$$

Then, the point group $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by

$$z \mapsto \vartheta z \quad \text{and} \quad z \mapsto \omega z \,, \tag{4.2}$$

where $z = (z^1, z^2, z^3) \in \mathbb{C}^3$ denotes the complex coordinates on the three two-tori and ϑ and ω are given by

$$\vartheta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{4.3}$$

Since $\vartheta, \omega \in \mathrm{SU}(3)$, four-dimensional $\mathcal{N} = 1$ supersymmetry can be preserved in this orbifold.

This orbifold has $16 + 16 + 16 = 48$ inequivalent fixed point sets F_g corresponding to the constructing elements g being

$$\left(\vartheta, \sum_{a=2,3} (n_{2a-1}e_{2a-1} + n_{2a}e_{2a}) \right) \quad \text{where} \quad n_3, n_4, n_5, n_6 \in \{0, 1\} , \tag{4.4a}$$

$$\left(\omega, \sum_{a=1,3} (n_{2a-1}e_{2a-1} + n_{2a}e_{2a}) \right) \quad \text{where} \quad n_1, n_2, n_5, n_6 \in \{0, 1\} , \tag{4.4b}$$

$$\left(\vartheta\omega, \sum_{a=1,2} (n_{2a-1}e_{2a-1} + n_{2a}e_{2a}) \right) \quad \text{where} \quad n_1, n_2, n_3, n_4 \in \{0, 1\} . \tag{4.4c}$$

Note that each fixed point set F_g is two-dimensional, e.g.

$$\mathrm{F}_g = \{(z^1, 0, 0) \mid z^1 \in \mathbb{C}\} \qquad \text{for} \qquad g = (\vartheta, 0) , \tag{4.5a}$$

$$\mathrm{F}_g = \{(0, z^2, 0) \mid z^2 \in \mathbb{C}\} \qquad \text{for} \qquad g = (\omega, 0) , \tag{4.5b}$$

$$\mathrm{F}_g = \{(0, 0, z^3) \mid z^3 \in \mathbb{C}\} \qquad \text{for} \qquad g = (\vartheta\omega, 0) , \tag{4.5c}$$

see figure 2 for a three-dimensional illustration.

To define a field theory on this $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, we have to choose a (gauge) embedding R_ϑ, R_ω, and R_{e_i} for each generator of the space group S, i.e. for each $(\vartheta, 0)$, $(\omega, 0)$, and $(\mathbb{1}, e_i)$ for $i \in \{1, \ldots, 6\}$. We have to ensure that this embedding satisfies the following conditions, obtained from eq. (2.3),

$$(R_\vartheta)^2 = \mathbb{1} , \quad (R_\omega)^2 = \mathbb{1} , \quad R_\vartheta R_\omega = R_\omega R_\vartheta , \tag{4.6}$$

and

$$R_{e_i} R_{e_j} = R_{e_j} R_{e_i} \qquad\qquad \text{for } i, j \in \{1, \ldots, 6\} , \tag{4.7a}$$

$$R_\vartheta R_{e_i} R_\vartheta = (R_{e_i})^{-1} \qquad\qquad \text{for } i \in \{3, 4, 5, 6\} , \tag{4.7b}$$

$$R_\vartheta R_{e_i} R_\vartheta = R_{e_i} , \qquad\qquad \text{for } i \in \{1, 2\} , \tag{4.7c}$$

$$R_\omega R_{e_i} R_\omega = (R_{e_i})^{-1} \qquad\qquad \text{for } i \in \{1, 2, 5, 6\} , \tag{4.7d}$$

$$R_\omega R_{e_i} R_\omega = R_{e_i} , \qquad\qquad \text{for } i \in \{3, 4\} , \tag{4.7e}$$

see also eqs. (2.4) and (2.5). Then, using the homomorphism property (2.3) a general element of the space group $g \in S$ has a (gauge) embedding R_g given by

$$g = \left(\vartheta^k \omega^\ell, \sum_{i=1}^{6} n_i e_i \right) \;\Leftrightarrow\; R_g = \prod_{i=6}^{3} (R_{e_i})^{n_i} (R_\vartheta)^k (R_\omega)^\ell , \tag{4.8}$$
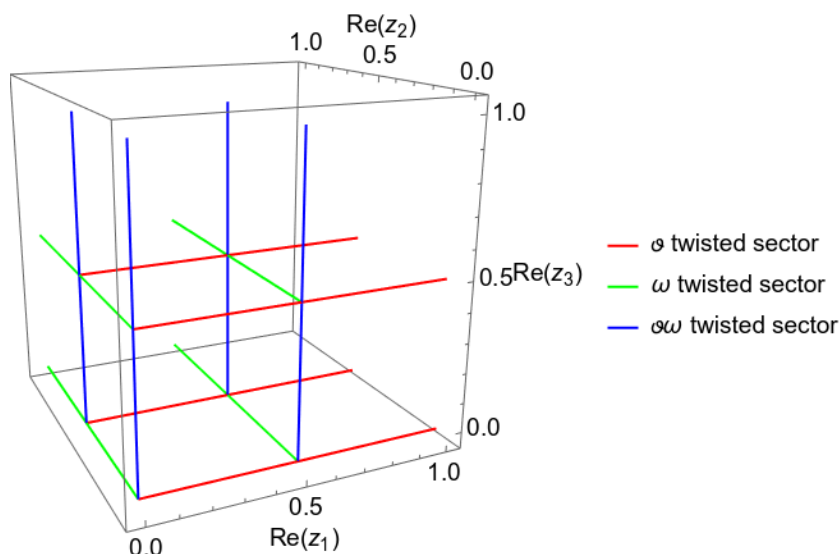
for $k, \ell \in \{0, 1\}$.

Figure 2. Illustration of the six-dimensional $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with an orthonormal six-torus, projected onto $(\mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Re}(z_3))$. The orbifold has $16 + 16 + 16$ two-dimensional fixed tori corresponding to the ϑ, ω and $\vartheta\omega$ twisted sectors, respectively. They become $4 + 4 + 4$ fixed lines in this projection, cf. ref. [51].

Compared to the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold in section 3 we have more R-matrices that could potentially allow us to fix more flavon alignments. However, the conditions (4.6) and (4.7) are quite restrictive. It turns out that if we were to build a flavon-setup as in section 3.2 the stringent conditions would not allow new useful alignments. We end up with the same alignment capabilities as the smaller orbifold $\mathbb{T}^2/\mathbb{Z}_2$. Furthermore, the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold only has two-dimensional fixed tori and no zero-dimensional fixed points. Consequently, the centralizers C_g are non-trivial and induce in general projection conditions on localized SM matter fields, compare to section 3.4.

5 Flavour from a $\mathbb{T}^6/S_4$ orbifold

In the previous section, we studied boundary conditions in the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold which do not allow for predictive flavour alignments. Hence, we enlarge the orbifolding symmetry. We know that the alignment CSD3 can be obtained with the flavour group S_4 [43], and noticing that $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$, the orbifold $\mathbb{T}^6/S_4$ seems a fair choice.

Consider an orthonormal basis e_i in the six extra dimensions $i \in \{1, \ldots, 6\}$, i.e.

$$e_i \cdot e_j = \delta_{ij} \,, \tag{5.1}$$

and define a corresponding orthonormal six-torus $\mathbb{T}^6$ (ignoring the possibility to change the overall radius of $\mathbb{T}^6$). In complex coordinates $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ the basis vectors e_{2a-1} and e_{2a} lie in the complex plane z_a for $a \in \{1, 2, 3\}$.

Next, we choose two rotational space group generators $(\vartheta, 0)$ and $(\omega, 0)$ with the following actions on $z = (z_1, z_2, z_3)$

$$\vartheta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \tag{5.2}$$

see appendix C.2 in ref. [31]. One can check that ϑ and ω generate the permutation group S_4, i.e.

$$S_4 \cong \langle \vartheta, \omega \mid \omega^4 = \vartheta^3 = (\vartheta\,\omega)^2 = \mathbb{1} \rangle . \tag{5.3}$$

Then, the $\mathbb{T}^6/S_4$ orbifold is defined as the quotient space

$$z \sim \vartheta z \quad \text{and} \quad z \sim \omega z \tag{5.4}$$

of the six-torus (5.1). Since $\vartheta, \omega \in \mathrm{SU}(3)$, four-dimensional $\mathcal{N} = 1$ supersymmetry can be preserved in this orbifold.

According to eq. (A.13) in appendix A the conjugacy classes (of the space group) give rise to the distinct sectors of the theory. Therefore, as a first step one needs to determine the conjugacy classes of the point group. As a result, the 24 elements of the S_4 point group decompose into five conjugacy classes, being

$$[\mathbb{1}] = \{\mathbb{1}\} , \tag{5.5a}$$
$$[\vartheta] = \{\vartheta, \vartheta^2, \vartheta\,\omega^2, \omega\,\vartheta^2\omega, \vartheta\,\omega^2\vartheta, \vartheta^2\omega^2, \omega^2\vartheta^2, \omega^2\vartheta\} , \tag{5.5b}$$
$$[\omega] = \{\omega, \omega\,\vartheta^2, \vartheta^2\omega, \vartheta^2\omega\,\vartheta, \vartheta\,\omega\,\vartheta, \vartheta\,\omega\,\vartheta^2\} , \tag{5.5c}$$
$$[\omega^2] = \{\omega^2, \vartheta\,\omega^2\vartheta^2, \vartheta^2\omega^2\vartheta\} , \tag{5.5d}$$
$$[\vartheta\,\omega] = \{\vartheta\,\omega, \omega\,\vartheta, \vartheta\,\omega^2\vartheta^2\omega, \vartheta^2\omega\,\vartheta^2, \omega^2\vartheta^2\omega, \omega\,\vartheta^2\omega^2\} . \tag{5.5e}$$

Subsequently, a full analysis of the conjugacy classes of the space group [33] reveals the distinct sectors as given in appendix B and indicated in table 2 by the so-called Hodge numbers $(h^{(1,1)}, h^{(2,1)})$. In addition, figure 3 illustrates the setup. We can interpret the Hodge numbers as follows: $h^{(1,1)}$ counts the number of distinct fixed point sets in the various twisted sectors, e.g. there are ten distinct fixed point sets in the ω^2 twisted sector. As a remark, $h^{(2,1)}$ counts how many of the $h^{(1,1)}$ fixed point sets are two-dimensional fixed tori where each two-torus is parametrised by a *non-frozen* complex structure modulus (in this case, one can modify the angle between the two basis vectors of the two-torus freely). In contrast, a twisted sector with $h^{(1,1)} > h^{(2,1)}$ contains $h^{(1,1)} - h^{(2,1)}$ fixed point sets that are either zero-dimensional points or two-dimensional tori but with a frozen complex structure.

After fixing the geometry, we have to choose the gauge embeddings R_ϑ, R_ω, and R_{e_i} corresponding to the twists $(\vartheta, 0)$ and $(\omega, 0)$ and the translations $(\mathbb{1}, e_i)$, respectively. The twists $(\vartheta, 0)$ and $(\omega, 0)$ generate the permutation group S_4. Since R must be a group homomorphism, see eq. (2.3), R_ϑ and R_ω can be chosen to generate also S_4 or a subgroup thereof (for example, ignoring world-sheet modular invariance from string theory one could also choose $R_\vartheta = R_\omega = \mathbb{1}$).

twisted sector of constr. element g	$(h^{(1,1)}, h^{(2,1)})$	eigenvalues of twist	extra dim.	centralizer C_g	generators of centralizer
$[\mathbb{1}]$	$(1,1)$	$(1,1,1)$	6	S_4	ϑ, ω
$[\vartheta]$	$(1,1)$	$(1, \alpha^2, \alpha^4)$	2	$\mathbb{Z}_3$	ϑ
$[\omega]$	$(4,4)$	$(1, \mathrm{i}, -\mathrm{i})$	2	$\mathbb{Z}_4$	ω
$[\omega^2]$	$(10,0)$	$(1,-1,-1)$	2	D_8	$\omega, \vartheta^2 \omega \vartheta^2$
$[\vartheta\omega]$	$(4,0)$	$(1,-1,-1)$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\vartheta\omega, \vartheta^2 \omega^2 \vartheta$

Table 2. Details of the S_4 orbifold: $\alpha = \exp\left(\frac{2\pi\mathrm{i}}{6}\right)$. D_8 is the dihedral group of order 8. The generators of the centralizer correspond to one specific g, for example, in the ω^2 twisted sector the centralizer is generated by ω and $\vartheta^2 \omega \vartheta^2$.
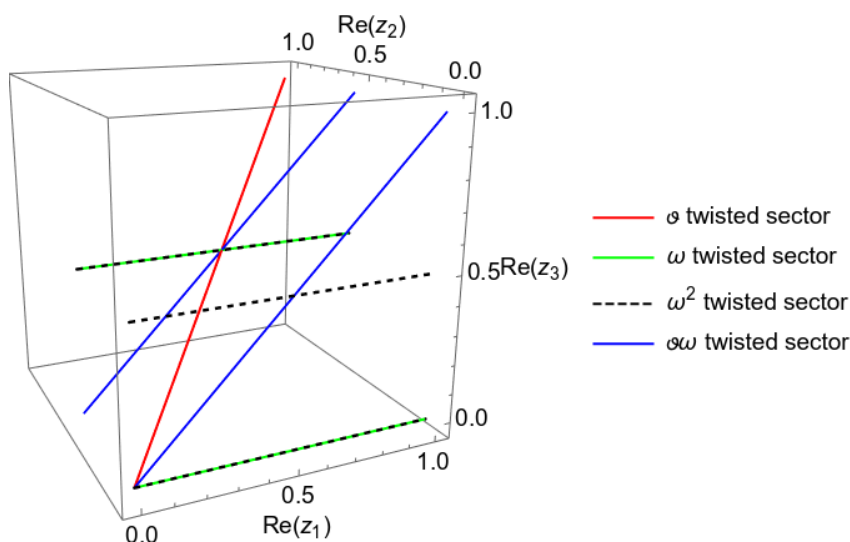

Figure 3. Illustration of the six-dimensional $\mathbb{T}^6/S_4$ orbifold projected onto $(\mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Re}(z_3))$. The orbifold has $1 + 4 + 10 + 4$ two-dimensional fixed tori corresponding to the ϑ, ω, ω^2 and $\vartheta\omega$ twisted sectors, respectively. They become $1 + 2 + 3 + 2$ fixed lines in this projection.

In order to render the gauge embeddings of twists and translations fully compatible, we have to find matrices R_ϑ, R_ω, R_{e_1} and R_{e_2} such that

$$(R_\omega)^4 = (R_\vartheta)^3 = (R_{\vartheta\omega})^2 = \mathbb{1}, \quad \text{i.e. } R_\vartheta, R_\omega \text{ generate } S_4, \tag{5.6a}$$

$$R_{e_1} R_{e_2} = R_{e_2} R_{e_1}, \tag{5.6b}$$

$$R_\omega R_{e_1} = R_{e_1} R_\omega, \tag{5.6c}$$

$$R_\omega R_{e_2} = R_{e_2} R_\omega, \tag{5.6d}$$

$$(R_{\vartheta^2 \omega \vartheta^2}) R_{e_1} = R_{e_1}^{-1} (R_{\vartheta^2 \omega \vartheta^2}) \quad \Leftrightarrow \quad ((R_{\vartheta^2 \omega \vartheta^2}) R_{e_1})^2 = \mathbb{1}, \tag{5.6e}$$

$$(R_{\vartheta^2 \omega \vartheta^2}) R_{e_2} = R_{e_2}^{-1} (R_{\vartheta^2 \omega \vartheta^2}) \quad \Leftrightarrow \quad ((R_{\vartheta^2 \omega \vartheta^2}) R_{e_2})^2 = \mathbb{1}, \tag{5.6f}$$

where the matrix $R_{\vartheta^2 \omega \vartheta^2} = (R_\vartheta)^2 R_\omega (R_\vartheta)^2$ is of order 2. If these conditions are satisfied

we can define R_{e_i} for $i = 3, 4, 5, 6$ using eqs. (B.4) in appendix B. This choice will satisfy all conditions, i.e. those from the presentation of S_4 in eq. (5.3) and, additionally, those from eqs. (B.2) and (B.3).

5.1 Alignment of flavon VEVs

We assume to have an $\mathrm{SU}(3)_{\mathrm{fl}}$ gauged flavour symmetry and choose the gauge embedding of the S_4 transformations using the known generators [41]

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} , \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^4 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} , \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{5.7}$$

as

$$R_\vartheta = T , \quad R_\omega = UTS = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\alpha^2 & 2\alpha^2 & -\alpha^2 \\ 2\alpha^4 & -\alpha^4 & 2\alpha^4 \end{pmatrix} \quad \text{and} \quad R_{e_i} = \mathbb{1} , \tag{5.8}$$

where $\alpha = \exp(\frac{2\pi \mathrm{i}}{6})$. This choice satisfies all gauge embedding conditions for an S_4 orbifold.

Next, we choose to have three flavons ϕ_1, ϕ_2, and ϕ_3 (each being a triplet of the $\mathrm{SU}(3)_{\mathrm{fl}}$ flavour group) and localize them in different sectors of the $\mathbb{T}^6/S_4$ orbifold as listed in table 3. Let us begin with specifying the flavon ϕ_1 in great detail so that our discussion for ϕ_2 and ϕ_3 can be shorter later on. We want the flavon ϕ_1 to be subject to the boundary condition $R_\vartheta R_\omega$. Hence, we localize it in the $\vartheta \omega$ twisted sector (e.g. on the fixed torus $z_{\mathrm{f}} \in \mathbb{C}^3$ given by the solutions of $\vartheta \omega z_{\mathrm{f}} = z_{\mathrm{f}}$). In order to identify all boundary conditions that act on ϕ_1 we have to compute the centralizer of $\vartheta \omega$, i.e. we have to identify all elements of S_4 that commute with $\vartheta \omega$. It turns out that the centralizer of $\vartheta \omega$ is generated by

$$\vartheta \omega \quad \text{and} \quad \vartheta^2 \omega^2 \vartheta , \tag{5.9}$$

and corresponds to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, the flavon ϕ_1 will feel the boundary conditions R_g of all elements g of the centralizer (up to some phases as introduced in section 2.4 that can be chosen freely). Hence, ϕ_1 is subject to

$$R_\vartheta R_\omega \langle \phi_1 \rangle = \pm \langle \phi_1 \rangle \quad \text{and} \quad (R_\vartheta)^2 (R_\omega)^2 R_\vartheta \langle \phi_1 \rangle = \pm \langle \phi_1 \rangle , \tag{5.10}$$

where the $\pm$ signs in both conditions can be chosen independently.

If this does not work out, there are ways to make the centralizer smaller. But then S_4 orbifold becomes more complicated. For example, one can use various different six-tori that can not be written as $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, cf. ref. [31].

After we have chosen the localization of each flavon, the flavon VEVs must comply with the respective boundary conditions. We assume that the flavons obtain a non-vanishing VEV through some other mechanism. However, the alignment of the flavon VEV in flavour space is fixed to a specific direction through the boundary conditions.

In more detail, the flavon ϕ_3 is chosen to be localized in the ϑ sector. Its VEV must comply with the boundary condition $R_\vartheta = T$. It has the freedom of having any of the three

flavon	localization	centralizer	generators of centralizer
ϕ_1	$\vartheta\,\omega \ \in \ [\vartheta\,\omega]$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\vartheta\,\omega$ and $\vartheta^2\,\omega^2\,\vartheta$
ϕ_2	$\omega\,\vartheta^2\,\omega^2 \ \in \ [\vartheta\,\omega]$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\omega\,\vartheta^2\,\omega^2$ and $\vartheta^2\,\omega^2\,\vartheta$
ϕ_3	$\vartheta \ \in \ [\vartheta]$	$\mathbb{Z}_3$	ϑ

Table 3. Localizations of the three flavons ϕ_1, ϕ_2 and ϕ_3 in the various sectors of the $\mathbb{T}^6/S_4$ orbifold and their centralizers, which indicates which boundary condition the respective flavon is feeling.

phases α^{2n} for $n = 0, 1, 2$, and we choose it to be α^4 so that

$$\langle\phi_3\rangle = \alpha^4 R_\vartheta\langle\phi_3\rangle = \alpha^4 T\langle\phi_3\rangle = \begin{pmatrix} \alpha^4 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\langle\phi_3\rangle \quad \rightarrow \quad \langle\phi_3\rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad (5.11)$$

which aligns the VEV completely. This VEV must be invariant (up to a phase) under the full centralizer of ϑ, which is generated by ϑ itself, so it is consistent.

The flavon ϕ_2 is chosen to be localized in the $\omega\vartheta^2\omega^2$ sector, so that its VEV must be invariant under $R_\omega R_\vartheta^2 R_\omega^2 = U$, up to a sign, which we choose to be negative. This enforces the VEV to be

$$\langle\phi_2\rangle = -R_\omega R_\vartheta^2 R_\omega^2\langle\phi_2\rangle = -U\langle\phi_2\rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\langle\phi_2\rangle \quad \rightarrow \quad \langle\phi_2\rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (5.12)$$

which aligns the VEV completely. The VEV must also be invariant (up to a sign) with the corresponding centralizer, which in this case is generated by $\omega\,\vartheta^2\,\omega^2$ and $\vartheta^2\,\omega^2\,\vartheta$. Hence, the VEV eq. (5.12) must also be invariant under the boundary condition using $R_\vartheta^2 R_\omega^2 R_\vartheta = S$, up to a sign. We choose the sign to be negative (the positive sign would force the VEV to vanish) so that

$$\langle\phi_2\rangle = -R_\vartheta^2 R_\omega^2 R_\vartheta\langle\phi_2\rangle = -S\langle\phi_2\rangle = \frac{-1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\langle\phi_2\rangle \quad \rightarrow \quad \langle\phi_2\rangle \propto \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}, \tag{5.13}$$

with arbitrary a, b. This alignment is compatible with the previous condition when $a = 0$. This fixes the VEV ϕ_2 completely and consistently through boundary conditions.

The flavon ϕ_1 obtains a VEV that, due to the choice of the localization in the sector $\vartheta\omega$, must be invariant under the boundary conditions $R_\vartheta R_\omega = SU$ up to a sign. We choose the positive sign so that

$$\langle\phi_1\rangle = R_\vartheta R_\omega\langle\phi_1\rangle = SU\langle\phi_1\rangle = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}\langle\phi_1\rangle \quad \rightarrow \quad \langle\phi_1\rangle \propto \begin{pmatrix} a \\ b \\ 2a-b \end{pmatrix}. \tag{5.14}$$

This VEV is aligned in the general CSDn direction which is defined with $a = 1$, $b = n$. It must also comply with the boundary conditions of the centralizer, up to a sign, which

is generated by $\vartheta\,\omega$ and $\vartheta^2\,\omega^2\,\vartheta$. This VEV must also be invariant under the boundary condition $R_\vartheta^2 R_\omega^2 R_\vartheta = S$, up to a sign. We choose the sign to be positive (the negative sign would force the VEV to vanish) so that

$$\langle\phi_1\rangle = R_\vartheta^2 R_\omega^2 R_\vartheta \langle\phi_1\rangle = S\langle\phi_1\rangle = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\langle\phi_1\rangle \rightarrow \langle\phi_1\rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad (5.15)$$

which is consistent with the previous condition fixing $a = b = 1$. This is the CSD1 alignment which is widely used in the tribimaximal (TBM) alignment [52].

We conclude that the flavon VEV alignments can be fixed completely and consistently to the TBM alignment and we can arrange for a situation with three flavons ϕ_i such that

$$\langle\phi_1\rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle\phi_2\rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \langle\phi_3\rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad (5.16)$$

These flavons are enough to fit all masses predictibly, specially in the lepton sector.

5.2 Roto-translations

Another realization of the S_4 orbifold is based on a space group with roto-translations

$$\left(\vartheta, \frac{1}{4}(e_1 + e_3)\right) \qquad \text{and} \qquad \left(\omega, \frac{1}{4}(e_1 + 3e_2)\right). \qquad (5.17)$$

In this case, there are only three sectors corresponding to $[\mathbb{1}]$, $[\vartheta]$ and $[\vartheta\,\omega]$, where the later two, twisted sectors have trivial centralizers. Thus, one can localize the flavons ϕ_1 and ϕ_2 in the sector $[\vartheta\,\omega]$, while ϕ_3 is localized in $[\vartheta]$.

This way we can obtain the general CSDn complete alignment. However, we did not find a mechanism to fix $n = 2, 3$ so that we have a highly predictive fermion mass setup as in refs. [21, 44, 46].

5.3 GUT breaking

Up to now, we have only chosen specific embedding matrices R_ϑ and R_ω in eq. (5.8) to align flavons in the CSDn or TBM alignment. We have not fixed any R_{e_i} in the process. From eqs. (5.6), we see that there are only two free matrices (R_{e_1} and R_{e_2}) to choose and they must comply with their specific conditions. Hence, one option is to choose R_{e_1} and R_{e_2} to break the GUT but not flavour.

In this case, the matrices R_{e_1} and R_{e_2} would commute with R_ϑ and R_ω. The only remaining constraints from eqs. (5.6) state that R_{e_1} and R_{e_2} must commute with each other and be of order two. Hence, they generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, we are allowed to apply the GUT breaking mechanism as discussed in section 3.3.

5.4 SM fermion localization

We have shown that this S_4 orbifold is enough to align three different flavons in the TBM or the CSDn setups and break GUTs.

However, we want some of the SM fermions (e.g. the charged leptons ℓ) to form a triplet of $SU(3)_{\text{fl}}$ flavour. Thus, these fermions must be located where they are not affected by the flavour breaking conditions associated to R_ϑ or R_ω. In this S_4 orbifold the only places to locate fields are the bulk and invariant tori, see table 2, which all are affected by the conditions R_ϑ or R_ω. Hence, some components of $SU(3)_{\text{fl}}$ flavour triplets for SM matter are necessarily projected out in this S_4 orbifold. This would destroy any predictability coming from the flavon alignments (5.16). Consequently, we move on to another orbifold that allows for suitable fermion localizations.

6 Flavour from a $\mathbb{T}^6/\Delta(54)$ orbifold

We want to enlarge the orbifolding symmetry once again, to allow a place to locate the fermions consistently but keep the alignments we have achieved. We can note that $S_4 \simeq \Delta(24)$ so that we can continue in the $\Delta(6n^2)$ discrete group series by choosing the next one, $\Delta(54)$.

Consider a factorized six-torus $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ where the a-th torus is spanned by basis vectors e_{2a-1} and e_{2a} of length r, where

$$|e_{2a-1}| = |e_{2a}| = r \quad \text{and} \quad e_{2a-1} \cdot e_{2a} = -\frac{r^2}{2} \quad \text{for } a = 1, 2, 3 . \tag{6.1}$$

In complex coordinates $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ the basis vectors e_{2a-1} and e_{2a} lie in the complex plane z_a for $a = 1, 2, 3$. As a remark, this orbifold has only a single Kähler modulus T which parameterizes the overall size r and the overall B-field [33].

Next, we choose three space group generators $(\vartheta, 0)$, $(\omega, 0)$ and $(\rho, 0)$ with the following actions on $z = (z_1, z_2, z_3)$

$$\vartheta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \\ 1 & 0 & 0 \end{pmatrix} , \tag{6.2}$$

where $\alpha = \exp(\frac{2\pi i}{6})$. Since

$$\vartheta^2 = \omega^3 = (\omega\vartheta)^2 = \mathbb{1} , \tag{6.3}$$

we see that ϑ and ω generate an S_3 subgroup. Furthermore,

$$\omega^2 \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \end{pmatrix} \quad \text{and} \quad \rho\omega^2 = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{6.4}$$

generate a $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup such that, finally, we can write $\Delta(54) = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes S_3$, see e.g. [53]. Finally, since $\vartheta, \omega, \rho \in SU(3)$, four-dimensional $\mathcal{N} = 1$ supersymmetry can be preserved in this orbifold.

The 54 elements of $\Delta(54)$ decompose into ten conjugacy classes being

$$\left[\mathbb{1}\right] = \{\mathbb{1}\} \,, \tag{6.5a}$$

$$\left[\vartheta\right] = \{\vartheta, \vartheta\omega, \rho\vartheta\rho^2, \omega\vartheta, \omega\rho\vartheta\omega, \vartheta\rho\omega, \rho\omega\vartheta, \vartheta\rho^2\omega, \omega\vartheta\rho\} \,, \tag{6.5b}$$

$$\left[\omega\right] = \{\omega, \omega^2, \rho\omega\rho^2, \rho\vartheta\omega\rho\vartheta, \vartheta\omega\rho\vartheta\rho, \rho^2\omega\rho\} \,, \tag{6.5c}$$

$$\left[\rho\right] = \{\rho, \vartheta\rho\vartheta, \omega\rho\omega^2, \vartheta\omega\rho\vartheta\omega, \rho^2, \vartheta\rho^2\vartheta\} \,, \tag{6.5d}$$

$$\left[\vartheta\,\rho\right] = \{\vartheta\rho, \rho\vartheta, \rho^2\vartheta\omega, \vartheta\rho^2\omega^2, \vartheta\omega\rho^2, \omega\rho\vartheta\omega\rho, \vartheta\omega\rho\omega, \rho\vartheta\omega\rho, \rho\omega\rho^2\vartheta\} \,, \tag{6.5e}$$

$$\left[\omega\,\rho\right] = \{\omega\rho, \vartheta\omega\rho\vartheta, \vartheta\rho^2\vartheta\omega, \rho\omega, \rho^2\omega^2, \vartheta\rho\omega\vartheta\} \,, \tag{6.5f}$$

$$\left[\vartheta\,\omega\,\rho\right] = \{\vartheta\omega\rho, \omega\rho\vartheta, \vartheta\rho\omega^2, \rho\vartheta\omega, \vartheta\rho^2, \omega\rho\vartheta\rho, \rho^2\vartheta, \rho\vartheta\rho\omega, \rho\vartheta\rho\} \,, \tag{6.5g}$$

$$\left[\omega^2\,\rho\right] = \{\omega^2\rho, \rho^2\omega, \rho\omega^2, \rho\omega\rho, \omega\rho^2, \omega\rho\omega\} \,, \tag{6.5h}$$

$$\left[(\vartheta\,\rho)^2\right] = \{(\vartheta\,\rho)^2\} \,, \tag{6.5i}$$

$$\left[(\vartheta\,\omega\,\rho)^2\right] = \{(\vartheta\,\omega\,\rho)^2\} \,. \tag{6.5j}$$

To define a field theory on this $\Delta(54)$ orbifold, we have to choose a (gauge) embedding

$$R_\vartheta \,, R_\omega \,, R_\rho \,, \quad \text{and} \quad R_{e_i} \,, \quad \text{for} \quad i \in \{1,\ldots,6\} \,, \tag{6.6}$$

for each generator of the $\Delta(54)$ space group S, i.e. for each

$$(\vartheta, 0) \,, (\omega, 0) \,, (\rho, 0) \,, (\mathbb{1}, e_i) \quad \text{for} \quad i \in \{1,\ldots,6\} \,. \tag{6.7}$$

We have to ensure that this embedding satisfies the following conditions, obtained from eq. (2.3),

$$(R_\vartheta)^2 = (R_\omega)^3 = (R_\rho)^3 = \mathbb{1} \tag{6.8a}$$

$$R_\vartheta\, R_\omega\, (R_\vartheta)^{-1}\, (R_\omega)^{-1} = R_\omega \,, \tag{6.8b}$$

$$R_\vartheta\, R_\rho\, (R_\vartheta)^{-1}\, (R_\rho)^{-1} = R_\omega\, R_\rho\, (R_\omega)^2 \,, \tag{6.8c}$$

$$R_\omega\, R_\rho\, (R_\omega)^{-1}\, (R_\rho)^{-1} = (R_\vartheta\, R_\rho)^2 \,, \tag{6.8d}$$

such that R_ϑ, R_ω and R_ρ generate $\Delta(54)$ or a subgroup thereof. Furthermore,

$$R_{e_i}\, R_{e_j} = R_{e_j}\, R_{e_i} \,, \tag{6.9a}$$

$$R_\vartheta\, R_{e_i}\, R_\vartheta = R_{\vartheta\, e_i} \,, \tag{6.9b}$$

$$R_\omega\, R_{e_i}\, (R_\omega)^{-1} = R_{\omega\, e_i} \,, \tag{6.9c}$$

$$R_\rho\, R_{e_i}\, (R_\rho)^{-1} = R_{\rho\, e_i} \,, \tag{6.9d}$$

where $i,j \in \{1,\ldots,6\}$, see also eqs. (2.4) and (2.5). Explicitly, eqs. (6.9b), (6.9c) and (6.9d) read

$$R_\vartheta\, R_{e_i}\, R_\vartheta = (R_{e_i})^{-1} \qquad\qquad \text{for} \quad i \in \{1,2\} \,, \tag{6.10a}$$

$$R_\vartheta\, R_{e_i}\, R_\vartheta = \left(R_{e_{i+2}}\right)^{-1} \qquad\qquad \text{for} \quad i \in \{3,4\} \,, \tag{6.10b}$$

$$R_\vartheta\, R_{e_i}\, R_\vartheta = \left(R_{e_{i-2}}\right)^{-1} \qquad\qquad \text{for} \quad i \in \{5,6\} \,, \tag{6.10c}$$

and

$$R_\omega \, R_{e_i} \, (R_\omega)^{-1} = R_{e_{i+4}} \qquad \text{for} \quad i \in \{1, 2\} \, , \tag{6.11a}$$

$$R_\omega \, R_{e_i} \, (R_\omega)^{-1} = R_{e_{i-2}} \qquad \text{for} \quad i \in \{3, 4\} \, , \tag{6.11b}$$

$$R_\omega \, R_{e_i} \, (R_\omega)^{-1} = R_{e_{i-2}} \qquad \text{for} \quad i \in \{5, 6\} \, , \tag{6.11c}$$

and

$$R_{\omega^2 \rho} \, R_{e_i} \, \left(R_{\omega^2 \rho}\right)^{-1} = R_{e_i} \qquad \text{for} \quad i \in \{1, 2\} \, , \tag{6.12a}$$

$$R_{\omega^2 \rho} \, R_{e_3} \, \left(R_{\omega^2 \rho}\right)^{-1} = R_{e_4} \tag{6.12b}$$

$$R_{\omega^2 \rho} \, R_{e_4} \, \left(R_{\omega^2 \rho}\right)^{-1} = (R_{e_3})^{-1} \, (R_{e_4})^{-1} \tag{6.12c}$$

$$R_{\omega^2 \rho} \, R_{e_5} \, \left(R_{\omega^2 \rho}\right)^{-1} = (R_{e_5})^{-1} \, (R_{e_6})^{-1} \tag{6.12d}$$

$$R_{\omega^2 \rho} \, R_{e_6} \, \left(R_{\omega^2 \rho}\right)^{-1} = R_{e_5} \, , \tag{6.12e}$$

where we use $R_{\omega^2 \rho}$ instead of R_ρ in order to keep the conditions (6.12) simple.

One possibility to solve eqs. (6.10), (6.11) and (6.12) is given by assuming that

$$R_\theta \, R_{e_i} = R_{e_i} \, R_\theta \quad \text{for} \quad i \in \{1, \ldots, 6\} \, . \tag{6.13}$$

for all point group elements $\theta \in \Delta(54)$. Importantly, one can show that in this case the (gauge) embeddings of the translations have to be trivial, i.e.

$$R_{e_i} = \mathbb{1} \quad \text{for} \quad i \in \{1, \ldots, 6\} \, . \tag{6.14}$$

Then, we are left with R_ϑ, R_ω and R_ρ that have to satisfy eq. (6.8). In the following, we will choose standard embedding $R_\vartheta = \vartheta$, $R_\omega = \omega$ and $R_\rho = \rho$ with gauged flavour symmetry $\mathcal{G} = \mathrm{SU}(3)_{\mathrm{fl}}$, cf. section 2.2.

6.1 VEV alignment

Consider a (flavon) field $\Phi_g(x, z)$ localized at $z \in \mathrm{F}_g$ with constructing element $g \in S$. We denote the order of g by N_g, i.e. $g^{(N_g)} = \mathbb{1}$. Then, the field $\Phi_g(x, z)$ has to satisfy the boundary conditions

$$\Phi_g(x, g\, z) = \exp\left(\frac{2\pi \mathrm{i}\, k}{N_g}\right) R_g \, \Phi_g(x, z) \, , \tag{6.15a}$$

$$\Phi_g(x, h\, z) = \exp\left(\frac{2\pi \mathrm{i}\, \ell_h}{N_h}\right) R_h \, \Phi_g(x, z) \, , \tag{6.15b}$$

where $h \in C_g$ has to be taken from the centralizer of g. We choose standard embedding eq. (2.6), where the gauge embedding R_g is identical to the geometrical action, i.e. $R_g = \theta$ for $g = (\theta, \lambda) \in S$, and the flavon $\Phi_g(x, z)$ is a triplet of $\mathrm{SU}(3)_{\mathrm{fl}}$. Note that the additional phases in eq. (6.15) (which dependent on k and ℓ_h, respectively) can originate from additional U(1) charges or from higher-dimensional Lorentz symmetry. These boundary

twisted sector of constr. element g	$(h^{(1,1)}, h^{(2,1)})$	eigenvalues of twist	extra dim.	centralizer C_g	generators of centralizer
$[\mathbb{1}]$	$(1,0)$	$(1,1,1)$	6	$\Delta(54)$	$\vartheta,\,\omega,\,\rho$
$[\vartheta]$	$(2,1)$	$(-1,-1,1)$	2	$\mathbb{Z}_6$	$\rho\vartheta\rho$
$[\omega]$	$(9,0)$	$(\alpha^2,\alpha^4,1)$	2	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\omega,\,(\vartheta\rho)^2$
$[\rho]$	$(1,0)$	$(\alpha^2,\alpha^4,1)$	2	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\rho,\,\vartheta\rho\vartheta$
$[\omega\rho]$	$(1,0)$	$(\alpha^2,\alpha^4,1)$	2	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\omega\rho,\,(\vartheta\rho)^2$
$[\omega^2\rho]$	$(1,0)$	$(\alpha^2,\alpha^4,1)$	2	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\omega^2\rho,\,(\vartheta\rho)^2$
$[(\vartheta\omega\rho)^2]$	$(7,0)$	$(\alpha^4,\alpha^4,\alpha^4)$	0	$\Delta(54)$	$\vartheta,\,\omega,\,\rho$
$[(\vartheta\rho)^2]$	$-$	$(\alpha^2,\alpha^2,\alpha^2)$	0	$\Delta(54)$	$\vartheta,\,\omega,\,\rho$
$[\vartheta\rho]$	$(3,0)$	(α^4,α,α)	0	$\mathbb{Z}_6$	$\vartheta\rho$
$[\vartheta\omega\rho]$	$-$	$(\alpha^2,\alpha^5,\alpha^5)$	0	$\mathbb{Z}_6$	$\vartheta\omega\rho$

Table 4. The various (twisted) sectors of the $\Delta(54)$ orbifold are labelled by their constructing elements g. The Hodge numbers $(h^{(1,1)}, h^{(2,1)})$ count the number of fixed point sets F_g (and their deformations). Each eigenvalue of $+1$ indicates two extra dimensions of F_g such that, for example, $\mathrm{F}_{\vartheta\rho}$ yields zero-dimensional fixed points while F_ϑ gives two-dimensional fixed tori (where $\alpha = \exp\left(\frac{2\pi\mathrm{i}}{6}\right)$).

conditions (6.15) result in the following conditions on the VEV of the zero mode $\Phi_g(x)$,

$$\langle\Phi_g\rangle = \exp\left(\frac{2\pi\mathrm{i}\,k}{N_g}\right) R_g \,\langle\Phi_g\rangle \;, \tag{6.16a}$$

$$\langle\Phi_g\rangle = \exp\left(\frac{2\pi\mathrm{i}\,\ell_h}{N_h}\right) R_h \,\langle\Phi_g\rangle \;. \tag{6.16b}$$

For each sector g from table 4 we find some k and ℓ_h such that the VEV $\langle\Phi_g\rangle$ is non-trivial. For example, consider the sector $g = (\vartheta, 0)$ with $N_g = 2$. Then, eq. (6.16a) has two non-trivial solutions

$$\langle\Phi_g\rangle = v \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{for} \quad k = 0 \;, \tag{6.17a}$$

$$\langle\Phi_g\rangle = \begin{pmatrix} v \\ w \\ w \end{pmatrix} \qquad \text{for} \quad k = 1 \;, \tag{6.17b}$$

for $v, w \in \mathbb{C}$. Next, we have to ensure that these VEV alignments are invariant under transformations $h \in C_g$ from the centralizer. In this case, the centralizer C_g is generated by $h = (\rho\vartheta\rho, 0)$ which is of order $N_h = 6$. One can verify that eq. (6.16b) has the same

non-trivial solutions as before provided that ℓ_h takes some special values, i.e.

$$\langle \Phi_g \rangle = v \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{for} \quad k = 0 \quad \text{and} \quad \ell_h = 2 \,, \tag{6.18a}$$

$$\langle \Phi_g \rangle = \begin{pmatrix} v \\ w \\ w \end{pmatrix} \qquad \text{for} \quad k = 1 \quad \text{and} \quad \ell_h = 5 \,. \tag{6.18b}$$

We repeat this analysis for the other sectors of the $\Delta(54)$ orbifold, listed in table 4, and find the following invariant VEV directions. The ϑ-sector allows for two different boundary conditions that yield two flavon VEV alignments

$$\vartheta \langle \phi \rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \langle \phi \rangle = \pm \langle \phi \rangle \rightarrow_+ \langle \phi \rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \,, \tag{6.19a}$$

$$\rightarrow_- \langle \phi \rangle \propto \begin{pmatrix} a \\ b \\ b \end{pmatrix} \,. \tag{6.19b}$$

For the ω-sector we obtain three different VEV alignments

$$\omega \langle \phi \rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \langle \phi \rangle = \alpha^{2n} \langle \phi \rangle \rightarrow_{n=0} \langle \phi \rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,, \tag{6.20a}$$

$$\rightarrow_{n=1} \langle \phi \rangle \propto \begin{pmatrix} 1 \\ \alpha^2 \\ \alpha^4 \end{pmatrix} \,, \tag{6.20b}$$

$$\rightarrow_{n=2} \langle \phi \rangle \propto \begin{pmatrix} 1 \\ \alpha^4 \\ \alpha^2 \end{pmatrix} \,, \tag{6.20c}$$

while the ρ-sector yields

$$\rho \langle \phi \rangle = \begin{pmatrix} 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \\ 1 & 0 & 0 \end{pmatrix} \langle \phi \rangle = \alpha^{2n} \langle \phi \rangle \rightarrow_{n=0} \langle \phi \rangle \propto \begin{pmatrix} 1 \\ \alpha^4 \\ 1 \end{pmatrix} \,, \tag{6.21a}$$

$$\rightarrow_{n=1} \langle \phi \rangle \propto \begin{pmatrix} 1 \\ 1 \\ \alpha^4 \end{pmatrix} \,, \tag{6.21b}$$

$$\rightarrow_{n=2} \langle \phi \rangle \propto \begin{pmatrix} \alpha^4 \\ 1 \\ 1 \end{pmatrix} \,. \tag{6.21c}$$

The matrices ω and ρ are very similar only involving different phases, and we can only obtain one different matrix built from them

$$\omega^2 \rho \langle\phi\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \end{pmatrix} \langle\phi\rangle = \alpha^{2n} \langle\phi\rangle \rightarrow_{n=0} \langle\phi\rangle \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.22a}$$

$$\rightarrow_{n=1} \langle\phi\rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.22b}$$

$$\rightarrow_{n=2} \langle\phi\rangle \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{6.22c}$$

Multiplying ω to ρ or ϑ would only cyclicly rotate the entries of the VEVS. The only other possibility would be to study the matrix

$$\omega^2 \rho \vartheta \langle\phi\rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\alpha^2 \\ 0 & \alpha^2 & 0 \end{pmatrix} \langle\phi\rangle = \pm \langle\phi\rangle \rightarrow_+ \langle\phi\rangle \propto \begin{pmatrix} 0 \\ 1 \\ -\alpha^4 \end{pmatrix}, \tag{6.23a}$$

$$\rightarrow_- \langle\phi\rangle \propto \begin{pmatrix} a \\ b \\ b\alpha^4 \end{pmatrix}. \tag{6.23b}$$

We conclude that we can completely fix three flavons to have the TBM alignment choosing them to be eigenvectors of

$$\langle\phi_1\rangle = \omega \langle\phi_1\rangle, \quad \langle\phi_2\rangle = \vartheta \langle\phi_2\rangle, \quad \langle\phi_3\rangle = \alpha^2 \, \omega^2 \rho \langle\phi_3\rangle, \tag{6.24}$$

while adding other matrices can introduce powers of α^2 in any entry while keeping the same alignment.

6.2 GUT breaking

As we stated before, in principle we could choose the gauge embedding R_{e_i} of the translations to break the GUT, for example, to break SU(5). However, in this $\Delta(54)$ orbifold with standard embedding the consistency conditions force them to be unity, see (6.14). The simplest choice to avoid this situation in this orbifold would be to enlarge the $\mathbb{Z}_2$ generator

$$R_\vartheta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes P_{\text{SU}(5)}, \tag{6.25}$$

which is consistent with all conditions and breaks SU(5). Our model outlined in section 7 is based on this GUT breaking.

6.3 SM fermion localization

Quarks and leptons that transform as triplets under the $\mathrm{SU}(3)_{\mathrm{fl}}$ flavour symmetry should not feel any flavour breaking boundary conditions. Otherwise, some of them would be projected out by the orbifold and, hence, we would lose the predictivity from the flavon VEV alignments.

We can see from table 4 that this specific $\Delta(54)$ orbifold has specific locations with zero-dimensional fixed points. Any field localized at such a point in extra dimensions is already a 4d field and, hence, is not be subject to any boundary condition. Consequently, we localize our $\mathrm{SU}(3)_{\mathrm{fl}}$ lepton triplet at such a fixed point.

7 $\mathbf{SU(5) \times SU(3)_{fl}}$ model in $\mathbb{R}^4 \times \mathbb{T}^6/\Delta(54)$

We start with $\mathcal{N} = 1$ SUSY in 10 dimensions and with a $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{SU}(3)_{\mathrm{fl}}$ gauge symmetry. In addition, we impose a U(1) shaping symmetry that allows the required Yukawa sector. Then, we define the $\Delta(54)$ orbifold boundary conditions as

$$
R_\vartheta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes P_{\mathrm{SU}(5)}, \qquad R_\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{5\times5},
$$
$$
R_\rho = \begin{pmatrix} 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \\ 1 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{5\times5}, \qquad R_{e_i} = \mathbb{1}_{3\times3} \otimes \mathbb{1}_{5\times5},
$$

(7.1)

where $\alpha = \exp(\frac{2\pi i}{6})$ and $P_{\mathrm{SU}(5)} = \mathrm{diag}(1,1,-1,-1,-1)$. Since the embedding R acts as standard embedding on $\mathrm{SU}(3)_{\mathrm{fl}}$, one can check easily that these matrices fulfil all the necessary conditions of the $\mathbb{T}^6/\Delta(54)$ orbifold.

As also discussed in section 6.2 these boundary conditions break $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{SU}(3)_{\mathrm{fl}} \to \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$ with only the MSSM superfields and some pure flavons left after compactification.

The list of chiral superfield is given in table 5. There, the $\pm$ superscript indicates that there are two copies of each ten-plet T_i, $i = 1, 2, 3$, i.e. one for each parity under the boundary condition $P_{\mathrm{SU}(5)}$.

We assume a standard Kähler potential with canonical normalized fields (and without large corrections [54]). Then, the Yukawa sector after compactification reads

$$
\begin{aligned}
\mathcal{W}_Y = \; & Y_{ij}^u\, h_u\, q_i\, u_j^c \\
& + \frac{y_{33}^+}{\Lambda} h_d \left(\ell \cdot \phi_\tau^0\right) e_3^c + \frac{y_{22}^+}{\Lambda} h_d \left(\ell \cdot \phi_\mu^0\right) e_2^c + \frac{y_{11}^+}{\Lambda} h_d \left(\ell \cdot \phi_e^0\right) e_1^c \\
& + \frac{y_{23}^+ \tilde{\xi}}{\Lambda} h_d \left(\ell \cdot \phi_\tau^0\right) e_2^c + \frac{y_{13}^+ \tilde{\xi}^2 + y_{13}'^+ \tilde{\xi}'^2}{\Lambda} h_d \left(\ell \cdot \phi_\tau^0\right) e_1^c + \frac{y_{12}^+ \tilde{\xi}}{\Lambda} h_d \left(\ell \cdot \phi_\mu^0\right) e_1^c \\
& + \frac{y_{33}^-}{\Lambda} h_d \left(d^c \cdot \phi_\tau^0\right) q_3 + \frac{y_{22}^-}{\Lambda} h_d \left(d^c \cdot \phi_\mu^0\right) q_2 + \frac{y_{11}^-}{\Lambda} h_d \left(d^c \cdot \phi_e^0\right) q_1 \\
& + \frac{y_{23}^- \tilde{\xi}}{\Lambda} h_d \left(d^c \cdot \phi_\tau^0\right) q_2 + \frac{y_{13}^- \tilde{\xi}^2 + y_{13}'^- \tilde{\xi}'^2}{\Lambda} h_d \left(d^c \cdot \phi_\tau^0\right) q_1 + \frac{y_{12}^- \tilde{\xi}}{\Lambda} h_d \left(d^c \cdot \phi_\mu^0\right) q_1
\end{aligned}
$$

(7.2)

Field	Representation			Localization	extra dim.	Zero mode
	$SU(3)_{fl}$	$SU(5)_{GUT}$	$U(1)$			
F	3	$\bar{5}$	11	$[\vartheta\rho]$	0	d^c, ℓ
T_1^+	1	10	6	$[\vartheta]$	2	u_1^c, e_1^c
T_1^-	1	10	6	$-[\vartheta]$	2	q_1
T_2^+	1	10	4	$[\vartheta]$	2	u_2^c, e_2^c
T_2^-	1	10	4	$-[\vartheta]$	2	q_2
T_3^+	1	10	2	$[\vartheta]$	2	u_3^c, e_3^c
T_3^-	1	10	2	$-[\vartheta]$	2	q_3
N_a	1	1	1	$[\omega]$	2	n_a
N_s	1	1	2	$[\vartheta]$	2	n_s
H_5	1	5	-4	$[\mathbb{1}]$	6	h_u
$H_{\bar{5}}$	1	$\bar{5}$	11	$[\mathbb{1}]$	6	h_d
ξ	1	1	-2	$[\mathbb{1}]$	6	ξ^0
ξ'	1	1	-2	$[\vartheta\rho]$	0	ξ'^0
ϕ_s	$\bar{3}$	1	-7	$[\omega]$	2	$\phi_s^0 \propto (1,1,1)^T$
ϕ_a	$\bar{3}$	1	-8	$[\vartheta]$	2	$\phi_a^0 \propto (0,1,-1)^T$
ϕ_τ	$\bar{3}$	1	-2	$\alpha^2[\omega^2\rho]$	2	$\phi_\tau^0 \propto (0,0,1)^T$
ϕ_μ	$\bar{3}$	1	-4	$\alpha^4[\omega^2\rho]$	2	$\phi_\mu^0 \propto (0,1,0)^T$
ϕ_e	$\bar{3}$	1	-6	$[\omega^2\rho]$	2	$\phi_e^0 \propto (1,0,0)^T$

Table 5. Complete list of chiral superfields in the model. The U(1) is a shaping symmetry.

$$+ y_a^N \xi n_a n_a + \frac{y_s^N \xi^2 + y_s'^N \xi'^2}{\Lambda} n_s n_s + \frac{y_a^\nu}{\Lambda} h_u \left(\ell \cdot \phi_a^0\right) n_a + \frac{y_s^\nu \tilde{\xi}}{\Lambda} h_u \left(\ell \cdot \phi_s^0\right) n_s$$

$$+ \frac{y_e^\nu \tilde{\xi}'}{\Lambda} h_u \left(\ell \cdot \phi_e^0\right) n_a + \frac{y_\mu^\nu \tilde{\xi}\tilde{\xi}'}{\Lambda} h_u \left(\ell \cdot \phi_\mu^0\right) n_a + \frac{y_\tau^\nu \tilde{\xi}'\tilde{\xi}^2 + y_\tau'^\nu \tilde{\xi}'^3}{\Lambda} h_u \left(\ell \cdot \phi_\tau^0\right) n_a \,,$$

where

$$Y^u = \begin{pmatrix} y_{11}^u \tilde{\xi}^4 & y_{12}^u \tilde{\xi}^3 & y_{13}^u \tilde{\xi}^2 \\ y_{21}^u \tilde{\xi}^3 & y_{22}^u \tilde{\xi}^2 & y_{23}^u \tilde{\xi} \\ y_{31}^u \tilde{\xi}^2 & y_{32}^u \tilde{\xi} & y_{33}^u \end{pmatrix} + \tilde{\xi}'^2 \begin{pmatrix} y_{11}'^u \tilde{\xi}^2 & y_{12}'^u \tilde{\xi} & y_{13}'^u \\ y_{21}'^u \tilde{\xi} & y_{22}'^u & 0 \\ y_{31}'^u & 0 & 0 \end{pmatrix} + \tilde{\xi}'^4 \begin{pmatrix} y_{11}''^u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \qquad (7.3)$$

We have defined

$$\tilde{\xi} = \langle \xi \rangle / \Lambda \quad \text{and} \quad \tilde{v}_i = \langle v_i \rangle / \Lambda \,. \qquad (7.4)$$

The U(1) shaping symmetry allows only these terms and there are no higher order contributions.

Note that the terms coming from $H_5 T_i^- T_j^+$ and $H_{\bar{5}} F \phi_k T_i^\pm \xi^\ell$ satisfy the basic string selection rule for allowed interactions: the point group selection rule demands that the point group elements of the respective constructing elements multiply to the identity element. For example, $H_{\bar{5}}$ and ξ originate from the bulk with constructing element $[\mathbb{1}]$, F is localized

in the $[\vartheta\rho]$ sector, the fields ϕ_k for $k = \tau, \mu, e$ live in the $[\omega^2\rho]$ sector and, finally, the fields $T_i^\pm$ for $i = 1, 2, 3$ are localized in the $[\vartheta]$ sector. Then,

$$[\mathbb{1}] \; [\vartheta\rho] \; [\omega^2\rho] \; [\vartheta] \; [\mathbb{1}]^\ell \; \supset \; \mathbb{1} \, \vartheta\omega\rho^2 \, \rho\omega^2 \, \vartheta \, \mathbb{1}^\ell = \mathbb{1} \;, \tag{7.5}$$

using the conjugacy classes of $\Delta(54)$ given in eq. (6.5).

We assume all dimensionless couplings to be $\mathcal{O}(1)$ complex numbers, so that all the hierarchies are due to the flavon VEVs

$$1000 \, \tilde{v}_s \sim 1000 \, \tilde{v}_a \sim 100 \, \tilde{v}_e \sim 10 \, \tilde{v}_\mu \sim 10 \, \tilde{\xi}' \sim \tilde{v}_\tau \sim \tilde{\xi} \sim 0.1, \tag{7.6}$$

which is an assumption.

With these assumptions we may approximate $\tilde{\xi} + \tilde{\xi}' \approx \tilde{\xi}$. We write the up quark mass matrix, coming from the first line of eq. (7.2)[2]

$$M^u = v_u \begin{pmatrix} y_{11}^u \tilde{\xi}^4 & y_{12}^u \tilde{\xi}^3 & y_{13}^u \tilde{\xi}^2 \\ y_{21}^u \tilde{\xi}^3 & y_{22}^u \tilde{\xi}^2 & y_{23}^u \tilde{\xi} \\ y_{31}^u \tilde{\xi}^2 & y_{32}^u \tilde{\xi} & y_{33}^u \end{pmatrix}. \tag{7.7}$$

The next lines of eq. (7.2) give masses to down quarks and charged leptons. The down quark matrix is

$$M^d = v_d \begin{pmatrix} y_{11}^- \tilde{v}_e & y_{12}^- \tilde{v}_\mu \tilde{\xi} & y_{13}^- \tilde{v}_\tau \tilde{\xi}^2 \\ 0 & y_{22}^- \tilde{v}_\mu & y_{23}^- \tilde{v}_\tau \tilde{\xi} \\ 0 & 0 & y_{33}^- \tilde{v}_\tau \end{pmatrix}, \tag{7.8}$$

while the charged lepton mass matrix is

$$(M^e)^* = v_d \begin{pmatrix} y_{11}^+ \tilde{v}_e & 0 & 0 \\ y_{12}^+ \tilde{v}_\mu \tilde{\xi} & y_{22}^+ \tilde{v}_\mu & 0 \\ y_{13}^+ \tilde{v}_\tau \tilde{\xi}^2 & y_{23}^+ \tilde{v}_\tau \tilde{\xi} & y_{33}^+ \tilde{v}_\tau \end{pmatrix}. \tag{7.9}$$

Since e^c comes from T^+ and q comes from T^- the Yukawa terms have different and independent couplings $y_{ij}^\pm$ for each one. This way the charged lepton mass matrix is completely independent of the down quark mass matrix.

The final two lines in eq. (7.2) give the Dirac neutrino mass matrix and the right handed neutrino Majorana mass matrix

$$M_D^\nu = v_u \begin{pmatrix} y_e^\nu \tilde{v}_e \tilde{\xi}' & y_s^\nu \tilde{v}_s \tilde{\xi} \\ y_a^\nu \tilde{v}_a + y_\mu^\nu \tilde{v}_\mu \tilde{\xi}\tilde{\xi}' & y_s^\nu \tilde{v}_s \tilde{\xi} \\ -y_a^\nu \tilde{v}_a + \tilde{v}_\tau (y_\tau^\nu \tilde{\xi}^2 \tilde{\xi}' + y_\tau'^\nu \tilde{\xi}\tilde{\xi}'^2) & y_s^\nu \tilde{v}_s \tilde{\xi} \end{pmatrix}, \quad M^N = \begin{pmatrix} y_a^N \tilde{\xi} & 0 \\ 0 & y_s^N \tilde{\xi}^2 \end{pmatrix} \langle\xi\rangle. \tag{7.10}$$

From the assumed VEV hierarchies from eq. (7.6), the $\tilde{\xi}$ terms in the first column are expected to be one order of magnitude smaller than the ones without $\tilde{\xi}$, so we may safely ignore them, leading to

$$M_D^\nu \simeq v_u \begin{pmatrix} y_e'^\nu \tilde{v}_e \tilde{\xi}' & y_s^\nu \tilde{v}_s \tilde{\xi} \\ y_a^\nu \tilde{v}_a & y_s^\nu \tilde{v}_s \tilde{\xi} \\ -y_a^\nu \tilde{v}_a & y_s^\nu \tilde{v}_s \tilde{\xi} \end{pmatrix} \sim \begin{pmatrix} \epsilon & b \\ a & b \\ -a & b \end{pmatrix}. \tag{7.11}$$

[2]All the mass matrices are given in the LR convention.

In the limit that the small entry denoted by ϵ is ignored, the Dirac mass matrix is of the CSD form and leads to tribimaximal neutrino mixing [52]. The presence of ϵ has the effect of switching on the reactor angle θ_{13}, without modifying very much the solar and atmospheric angles from their tribimaximal values [34]. This corresponds to so called tribimaximal-reactor lepton mixing [35].

The RHN are very heavy so that the left handed neutrinos become very light after the Seesaw mechanism has been implemented,

$$M^\nu = M_D^\nu (M^N)^{-1} (M^\nu)^{\mathrm{T}},$$

$$M^\nu \frac{\langle \xi \rangle}{v_u^2} \simeq \frac{(y_a^\nu)^2 \tilde{v}_a^2}{y_a^N} \begin{pmatrix} (y_{12}^\nu \tilde{\xi}')^2 & y_{12}^\nu \tilde{\xi}' & -y_{12}^\nu \tilde{\xi}' \\ y_{12}^\nu \tilde{\xi}' & 1 & -1 \\ -y_{12}^\nu \tilde{\xi}' & -1 & 1 \end{pmatrix} + \tilde{\xi} \frac{(y_s^\nu)^2 \tilde{v}_s^2}{y_s^N} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{7.12}$$

where $y_{12}^\nu = y_e^\nu \tilde{v}_e / y_a^\nu \tilde{v}_a$.

Looking at all the mass matrices, as noted above, we may see that the VEV $\tilde{\xi}'$ only appears in the neutrino mass matrix. Knowing that the charged lepton correction to the PMNS are negligible, if we sent this $\tilde{\xi}' \to 0$, we would have the tribimaximal (TBM) setup for the PMNS. Therefore the sole role of the $\tilde{\xi}'$ is to deviate from the TBM [35, 55, 56].

7.1 Numerical fit in the neutrino sector

In this section we perform a fit to the PMNS observables and the neutrino masses assuming a diagonal charged-lepton mass matrix (the off-diagonal elements in eq. (7.9) are negligible due to the appearance of $\tilde{\xi}$). The complex parameters in the up-type and down-type quark mass matrices in eqs. (7.7) and (7.8) have enough freedom to fit all the quark masses and the observed CKM mixing angles. Therefore, in this section we focus only on the neutrino sector.

The effective neutrino mass matrix in eq. (7.12) can be rewritten in terms of input parameters as

$$M^\nu \simeq \mu_a \begin{pmatrix} (\tilde{\xi}_{12} e^{i\eta_{\tilde{\xi}}})^2 & \tilde{\xi}_{12} e^{i\eta_{\tilde{\xi}}} & -\tilde{\xi}_{12} e^{i\eta_{\tilde{\xi}}} \\ \tilde{\xi}_{12} e^{i\eta_{\tilde{\xi}}} & 1 & -1 \\ -\tilde{\xi}_{12} e^{i\eta_{\tilde{\xi}}} & -1 & 1 \end{pmatrix} + \mu_s e^{i\eta} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{7.13}$$

where we have defined the input parameters

$$\mu_a = \left| \frac{v_u^2}{\langle \xi \rangle} \frac{(y_a^\nu)^2 \tilde{v}_a^2}{y_a^N}, \right| \quad \tilde{\xi}_{12} = |y_{12}^\nu \tilde{\xi}'|, \quad \eta_{\tilde{\xi}} = \arg\left(y_{12}^\nu \tilde{\xi}\right),$$

$$\mu_s = \left| \frac{v_u^2}{\langle \xi \rangle} \tilde{\xi} \frac{(y_s^\nu)^2 \tilde{v}_s^2}{y_s^N} \right| \quad \text{and} \quad \eta = \arg\left(\frac{(y_s^\nu)^2 \tilde{v}_s^2 \tilde{\xi}}{y_s^N} \frac{y_a^N}{(y_a^\nu)^2 \tilde{v}_a^2} \right) \tag{7.14}$$

We implement a numerical fit using a χ^2 test function

$$\chi^2 = \sum_n \left(\frac{P_n(x) - P_n^{\mathrm{obs}}}{\sigma_n} \right)^2, \tag{7.15}$$

Observable	Data		Model
	Central value	3σ range	Best fit
θ_{12}^l /°	33.82	$31.61 \to 36.27$	35.00
θ_{13}^l /°	8.610	$8.220 \to 8.990$	8.615
θ_{23}^l /°	48.30	$40.80 \to 51.30$	45.66
δ^l /°	222.0	$141.0 \to 370.0$	225.3
Δm_{21}^2 /$10^{-5}\,\mathrm{eV}^2$	7.390	$6.790 \to 8.010$	7.393
Δm_{31}^2 /$10^{-3}\,\mathrm{eV}^2$	2.523	$2.432 \to 2.618$	2.525
m_1 /meV			0
m_2 /meV			8.599
m_3 /meV			50.25
$\sum m_i$ /meV		$\lesssim 230$	58.85
α_{23} /°			221.3
$m_{\beta\beta}$ /meV		$\lesssim 60$–200	2.754

Table 6. Latest values of PMNS observables and neutrino masses given by NuFit4.1 [57] together with the model predictions with $\chi^2 \approx 4.8$. The neutrino masses m_i as well as the Majorana phase α_{23} are pure predictions of our model. The bound on $\sum m_i$ is taken from [60]. The bound on $m_{\beta\beta}$ is taken from [59]. There is only one physical Majorana phase since $m_1 = 0$.

where we sum over the 6 observables given by $P_n^{\mathrm{obs}} = \{\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell, \delta^l, \Delta m_{21}^2, \Delta m_{31}^2\}$ with statistical errors σ_n. The predictions of the model for these observables are given by $P_n(x)$, where $x = \{\mu_a, \tilde{\xi}_{12}, \eta_{\tilde{\xi}}, \mu_s, \eta\}$ refers to the different input parameters. We are doing the numerical fit in terms of the effective neutrino mass matrix in eq. (7.13) and we ignore any renormalisation group running corrections.

We use the recent global fit values of neutrino data from NuFit4.1 [57]. Most of the observables follow an almost Gaussian distribution and we take a conservative approach using the smaller of the given uncertainties in our computations except for θ_{23}^l and δ^l. The best fit from NuFit4.1 is for normal mass ordering with inverted ordering being disfavoured with a $\Delta\chi^2 = 6.2(10.4)$ without (with) the Super-Kamiokande atmospheric neutrino data analysis.

The model predictions are shown in table 6. The neutrino mass matrix in eq. (7.13) predicts near maximal atmospheric mixing angle $\theta_{23}^l = 45.66°$ and solar mixing angle $\theta_{12} = 35°$ as expected since we only have a small correction from tribimaximal mixing which allows a correct non-zero reactor angle $\theta_{13}^l = 8.615°$. The CP violation prediction is $\delta^l \approx 225°$. All the model predictions for the PMNS observables and the neutrino mass-squared differences are within the 3σ region from the latest neutrino oscillation data and reproduce a $\chi^2 = 4.8$ value. Furthermore, since we only have 2RH neutrinos, $m_1 = 0$ and there is only one physical Majorana phase α_{23} [58]. The bound on the effective Majorana mass $m_{\beta\beta}$ is taken from [59] while the prediction is also given in table 6.

Table 7 shows the input parameter values. There are 3 real parameters $\{\mu_a, \tilde{\xi}_{12}, \mu_s\}$ plus two additional phases $\{\eta_{\tilde{\xi}}, \eta\}$, a total of 4 input parameters to fit 6 data points. Naively, we can measure the goodness of the fit computing the reduced χ^2, i.e. the χ^2 per

Parameter	Value
$\mu_a/10^{-2}\,\mathrm{meV}$	2.454
$\tilde{\xi}_{12}$	0.221
$\eta_{\tilde{\xi}'}$	-2.392
$\mu_s/10^{-3}\,\mathrm{meV}$	2.887
η	-2.422

Table 7. Input parameter values reproducing the best fit point with $\chi^2 \sim 4.8$.

degree of freedom $\chi^2_\nu = \chi^2/\nu$. The number of degrees of freedom is given by $\nu = n - n_i$, where $n = 6$ is the number of measured observables, while $n_i = 4$ is the number of input parameters. A good fit is expected to have $\chi^2_\nu \sim 1$. We have 2 degrees of freedom and the best fit has a reduced $\chi^2_\nu \simeq 2.4$. We view this as a good fit and it also remarks the predictivity of the model, not only fitting to all available quark and lepton data but also fixing the neutrino masses and Majorana phases.

Using the definition of the input parameters in eq. (7.14) and their values in table 7 for the best fit point, we can give a naive estimation of the value of $\tilde{\xi}$. If we assume the dimensionless parameters to be $\mathcal{O}(1)$ and the VEVs $\tilde{v}_a \sim \tilde{v}_s$, then we find $\mu_s/\mu_a \approx \tilde{\xi} \approx 0.1$, which justifies the assumption of an approximate diagonal charged-lepton matrix in eq. (7.9) and the values of the VEVs in eq. (7.6). Also note that $|\tilde{\xi}_{12}| \approx 0.22$ which is exactly the value of Cabbibo angle [35, 55, 56].

8 Conclusions

The flavour puzzle, in particular the large mixing observed in the lepton sector, provides a strong motivation for going beyond the Standard Model. The literature is replete with flavour models involving some family symmetry spontaneously broken by flavon fields with certain vacuum alignments motivated by phenomenological considerations, but highly non-trivial to achieve without resort to extra symmetries and driving fields. In order to overcome this obstacle, one promising approach is to attempt to formulate such theories in extra dimensions, where the desired vacuum alignments may emerge from orbifold boundary conditions.

We have systematically developed the formalism necessary for ensuring that boundary conditions of flavon fields in extra dimensions are consistent with heterotic string theory. Having developed a set of consistency conditions on the boundary conditions, we have then explored a series of examples of orbifolds in various dimensions to see which ones can satisfy them. In addition we have imposed the further phenomenological requirement of having non-trivial flavon vacuum alignments. We have also demanded that quarks and leptons be located appropriately in extra dimensions so that their massless modes may include complete multiplets under the gauged flavour symmetry.

It turns out that it is highly non-trivial to satisfy all of these conditions (theoretical and phenomenological) together. For instance, the simple $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, while allowing SM fermion matter localisation on fixed points, does not permit non-trivial flavon vacuum

alignments, consistently with the formal requirements of the boundary conditions. This motivates us to go to 10d models. However, the simple orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ fares no better than the previous case, since it too can only provide one non-trivial alignment. We find that the boundary conditions must exhibit some non-Abelian structure so that we can have non-trivial VEV alignments.

Following the above logic, we were led to consider 10d non-Abelian orbifolds, where the torus is modded out by a non-Abelian group. We have studied the orbifold $\mathbb{T}^6/S_4$ which can fix flavons into the highly predictive CSDn structure. This orbifold however, does not have 4d branes where the SM matter could be localized. This motivated us to consider the orbifold $\mathbb{T}^6/\Delta(54)$, as an example where we can locate the SM fermions on fixed points in extra dimensions. Since the $\Delta(54)$ orbifold is not so well studied in the literature, we have developed this case in some detail, and eventually shown that we can choose the extra dimensions in such a way that we can build a realistic model.

The minimal successful flavour theory seems to be a 10d theory with a $\mathrm{SU}(3)_{\mathrm{fl}}$ gauged flavour symmetry, where the six extra dimensions are compactified on a $\mathbb{T}^6/\Delta(54)$ orbifold. The $\mathrm{SU}(3)_{\mathrm{fl}}$ flavour symmetry is broken by flavon VEVs which are completely aligned by the boundary conditions of the orbifold. The vacuum alignment of the flavons is of the tribimaximal form, but the theory can allow for some small corrections leading to tribimaximal-reactor lepton mixing, which we have shown to be consistent with current neutrino data. We have constructed a fully realistic $\mathrm{SU}(5) \times \mathrm{SU}(3)_{\mathrm{fl}}$ grand unified theory along these lines, which is complete, predictive and in principle consistent with heterotic string theory.

Acknowledgments

A Definition of an orbifold by its space group

In order to define a D-dimensional toroidal orbifold $\mathbb{O}$ geometrically as a quotient space of D-dimensional space, i.e.

$$\mathbb{O} = \mathbb{R}^D/S \, , \tag{A.1}$$

one has to specify a space group S first. A general element g of a space group S consists of a rotation $\theta \in \mathrm{SO}(D)$ (also called twist) and a translation $\lambda \in \mathbb{R}^D$, i.e.

$$g = (\theta, \lambda) \ \in \ S \, . \tag{A.2}$$

By definition, g acts on the internal coordinates $z \in \mathbb{R}^D$ as

$$z \xmapsto{\ g\ } g \, z = (\theta, \lambda) \, z = \theta \, z + \lambda \, . \tag{A.3}$$

Consequently, two space group elements $g_1 = (\theta_1, \lambda_1) \in S$ and $g_2 = (\theta_2, \lambda_2) \in S$ multiply as

$$g_1 \, g_2 = (\theta_1, \lambda_1) \, (\theta_2, \lambda_2) = (\theta_1 \, \theta_2, \theta_1 \, \lambda_2 + \lambda_1) \,. \tag{A.4}$$

Furthermore, the inverse element $g^{-1} \in S$ of $g = (\theta, \lambda) \in S$ is given by

$$g^{-1} = (\theta^{-1}, -\theta^{-1} \, \lambda) \,, \tag{A.5}$$

and the neutral element is

$$g_{\mathbb{1}} = (\mathbb{1}, 0) \ \in \ S \,. \tag{A.6}$$

Hence, one can see that S is a discrete group, actually a discrete subgroup of the extra-dimensional Euclidean group.

Practically, one defines a space group by a (finite) list of generators, which are pure translations and rotations. In this work, we focus on the case of up to three rotational generators,[3] i.e.

$$(\mathbb{1}, e_i), \quad (\vartheta, 0), \quad (\omega, 0), \quad (\rho, 0) \,, \tag{A.7}$$

for $i \in \{1, \ldots, D\}$. The vectors $e_i \in \mathbb{R}^D$ span a D-dimensional lattice Γ that specifies a D-dimensional torus $\mathbb{T}^D = \mathbb{R}^D / \Gamma$ and the rotations ϑ, ω and ρ have to be symmetries of the lattice Γ, i.e.

$$\vartheta \, \Gamma = \Gamma \ , \ \ \omega \, \Gamma = \Gamma \ \ \text{and} \ \ \rho \, \Gamma = \Gamma \,. \tag{A.8}$$

The rotations ϑ, ω (and ρ) generate the so-called point group P, where we are dealing with the cases $P = \mathbb{Z}_2 \times \mathbb{Z}_2$ in section 4, $P = S_4$ in section 5 and $P = \Delta(54)$ in section 6.

Having defined a space group, the orbifold $\mathbb{O}$ given in eq. (A.1) is defined by identifying those points $z_{(1)}$ and $z_{(2)}$ in $\mathbb{R}^D$ that are mapped to each other by some element of the space group, i.e.

$$z_{(1)} \ \sim \ z_{(2)} \quad \Leftrightarrow \quad \text{there is } g \in S \text{ such that } z_{(1)} = g \, z_{(2)} \,. \tag{A.9}$$

This equivalence relation can be used to define a fundamental domain of the orbifold.

A given element $g \in S$ of the space group can have a set of fixed points F_g, defined by

$$\mathrm{F}_g := \left\{ z \in \mathbb{R}^D \mid g \, z = z \right\} \,. \tag{A.10}$$

For a given space group element $g = (\theta, \lambda) \in S$ (with appropriate translation λ) the dimension of F_g depends on the eigenvalues of the rotation matrix $\theta \in P$: each eigenvalue $+1$ corresponds to an invariant direction in F_g. Our main concern is the case of supersymmetric orbifolds in $D = 6$, where we find fixed point sets of dimensions six (i.e. the bulk $\mathbb{O}$ for $g = \mathbb{1}$), two (i.e. fixed tori) and zero (i.e. fixed points). By acting with $h \in S$ onto the fixed point equation $g \, z = z$ (i.e. $z \in \mathrm{F}_g$), one obtains

$$\left(h \, g \, h^{-1} \right) (h \, z) = (h \, z) \,. \tag{A.11}$$

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[3]Ignoring the possibility of roto-translations, i.e. rotations that are combined with fractional translations, for example (θ, λ) with $\lambda \notin \Gamma$.

Hence, $h\,z \in \mathrm{F}_{h\,g\,h^{-1}}$. However, due to eq. (A.9) points z and $h\,z$ are identified on the orbifold $\mathbb{O}$. Thus, the corresponding fixed point sets are identified on the orbifold $\mathbb{O}$ as well,

$$\mathrm{F}_g \;\sim\; \mathrm{F}_{h\,g\,h^{-1}}\,, \tag{A.12}$$

for all $h \in S$. Consequently, the inequivalent fixed point sets correspond to the conjugacy classes $[g]$ of S, where

$$[g] = \{h\,g\,h^{-1} \mid h \in S\}\,. \tag{A.13}$$

If the point group P is Abelian, each element $\tilde{g}$ of a conjugacy class $[g]$ has the same point group element $\theta \in P$, i.e.

$$g = (\theta, \lambda) \quad \Leftrightarrow \quad h\,g\,h^{-1} = (\theta, \lambda') \;\; \text{for all } h \in S \text{ and some } \lambda' \in \Gamma\,. \tag{A.14}$$

A.1 Orbifold-invariant fields

In this appendix we complete the discussion from section 2.3 in the case $g\,h \neq h\,g$. In this case, we can choose the proportionality in eq. (2.10) to be trivial, i.e.

$$\Phi_g(x, h^{-1}z) = \Phi_{h\,g\,h^{-1}}(x, z)\,, \tag{A.15}$$

where a possible phase has been absorbed in a redefinition of $\Phi_{h\,g\,h^{-1}}$. Consequently, all fields $\Phi_{h\,g\,h^{-1}}$ from the same conjugacy class $[g] = \{h\,g\,h^{-1} | h \in S\}$ are identified and eqs. (2.9) and (A.15) yield

$$\Phi_g(x, z) \;\overset{h}{\longmapsto}\; R_h\,\Phi_g(x, h^{-1}z) = R_h\,\Phi_{h\,g\,h^{-1}}(x, z)\,. \tag{A.16}$$

Then, we can construct an orbifold-invariant field, denoted by $\Phi_{[g]}(x, z)$, on the covering space $\mathbb{R}^D$ of $\mathbb{O}$. To do so, we have to build the following linear combination

$$\Phi_{[g]}(x, z) := \Phi_g(x, z) + \sum_{h \notin C_g} R_h\,\Phi_{h\,g\,h^{-1}}(x, z)\,, \tag{A.17}$$

ignoring the normalization of $\Phi_{[g]}(x, z)$. However, F_g and $\mathrm{F}_{h\,g\,h^{-1}}$ are identified on the orbifold $\mathbb{O}$. Hence, if we restrict $z \in \mathbb{O}$ (instead of $z \in \mathbb{R}^D$) we can ignore the contributions $\Phi_{h\,g\,h^{-1}}(x, z)$ in eq. (A.17) and use $\Phi_g(x, z)$ as a well-defined field on the orbifold $\mathbb{O}$. In this case, transformations (2.9) with $h \notin C_g$ are not considered as they would map a point z from the fundamental domain of the orbifold to a point outside of the fundamental domain.

B Details on the S_4 orbifold

Beside the untwisted sector with constructing element $\mathbb{1} \in S$, the $\mathbb{T}^6/S_4$ orbifold contains the following inequivalent constructing elements $g \in S$ from the various twisted sectors

$$g = (\vartheta, 0)\,, \tag{B.1a}$$

$$g = (\omega, (n_5 e_5 + n_6 e_6)) \qquad \text{for} \;\; n_5, n_6 \in \{0, 1\}\,, \tag{B.1b}$$

$$g = \left(\omega^2, (n_5 e_5 + n_6 e_6)\right) \qquad \text{for} \;\; n_5, n_6 \in \{0, 1\}\,, \tag{B.1c}$$

$$g = \left(\omega^2, (e_4 + n_5 e_5 + n_6 e_6)\right) \qquad \text{for} \;\; (n_5, n_6) \in \{(0,1), (1,0), (1,1)\}\,, \tag{B.1d}$$

$$g = \left(\omega^2, (e_3 + n_4 e_4 + e_5 + n_6 e_6)\right) \qquad \text{for} \;\; (n_4, n_6) \in \{(0,0), (0,1), (1,1)\}\,, \tag{B.1e}$$

$$g = (\vartheta\omega, (n_3 e_3 + n_4 e_4)) \qquad \text{for} \;\; n_3, n_4 \in \{0, 1\}\,. \tag{B.1f}$$

These $1+4+(4+3+3)+4$ constructing elements are the S_4 analogue of the four constructing elements of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold listed in eq. (3.2).

In order to identify the relations on the gauge embeddings between R_ϑ, R_ω and R_{e_i} we consider the action of the twists $(\vartheta, 0)$ and $(\omega, 0)$ on the basis vectors $(\mathbb{1}, e_i)$ explicitly and embed these relations into R_g. Thus, we obtain the conditions

$$(\vartheta, 0)\, (\mathbb{1}, e_1)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_5) \Rightarrow R_\vartheta\, R_{e_1} = R_{e_5}\, R_\vartheta \,, \tag{B.2a}$$

$$(\vartheta, 0)\, (\mathbb{1}, e_2)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_6) \Rightarrow R_\vartheta\, R_{e_2} = R_{e_6}\, R_\vartheta \,, \tag{B.2b}$$

$$(\vartheta, 0)\, (\mathbb{1}, e_3)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_1) \Rightarrow R_\vartheta\, R_{e_3} = R_{e_1}\, R_\vartheta \,, \tag{B.2c}$$

$$(\vartheta, 0)\, (\mathbb{1}, e_4)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_2) \Rightarrow R_\vartheta\, R_{e_4} = R_{e_2}\, R_\vartheta \,, \tag{B.2d}$$

$$(\vartheta, 0)\, (\mathbb{1}, e_5)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_3) \Rightarrow R_\vartheta\, R_{e_5} = R_{e_3}\, R_\vartheta \,, \tag{B.2e}$$

$$(\vartheta, 0)\, (\mathbb{1}, e_6)\, (\vartheta^{-1}, 0) = (\mathbb{1}, \quad e_4) \Rightarrow R_\vartheta\, R_{e_6} = R_{e_4}\, R_\vartheta \,, \tag{B.2f}$$

and

$$(\omega, 0)\, (\mathbb{1}, e_1)\, (\omega^{-1}, 0) = (\mathbb{1}, \quad e_1) \Rightarrow R_\omega\, R_{e_1} = R_{e_1}\, R_\omega \,, \tag{B.3a}$$

$$(\omega, 0)\, (\mathbb{1}, e_2)\, (\omega^{-1}, 0) = (\mathbb{1}, \quad e_2) \Rightarrow R_\omega\, R_{e_2} = R_{e_2}\, R_\omega \,, \tag{B.3b}$$

$$(\omega, 0)\, (\mathbb{1}, e_3)\, (\omega^{-1}, 0) = (\mathbb{1}, -e_5) \Rightarrow R_\omega\, R_{e_3} = R_{e_5}^{-1}\, R_\omega \,, \tag{B.3c}$$

$$(\omega, 0)\, (\mathbb{1}, e_4)\, (\omega^{-1}, 0) = (\mathbb{1}, -e_6) \Rightarrow R_\omega\, R_{e_4} = R_{e_6}^{-1}\, R_\omega \,, \tag{B.3d}$$

$$(\omega, 0)\, (\mathbb{1}, e_5)\, (\omega^{-1}, 0) = (\mathbb{1}, \quad e_3) \Rightarrow R_\omega\, R_{e_5} = R_{e_3}\, R_\omega \,, \tag{B.3e}$$

$$(\omega, 0)\, (\mathbb{1}, e_6)\, (\omega^{-1}, 0) = (\mathbb{1}, \quad e_4) \Rightarrow R_\omega\, R_{e_6} = R_{e_4}\, R_\omega \,. \tag{B.3f}$$

Let us assume that we have found two matrices R_{e_1} and R_{e_2} that commute with R_ω and, hence, eqs. (B.3a) and (B.3b) are satisfied. Then, we can solve eqs. (B.2a), (B.2b), (B.2c) and (B.2d) by defining

$$R_{e_3} = R_\vartheta^{-1}\, R_{e_1}\, R_\vartheta \,, \tag{B.4a}$$

$$R_{e_4} = R_\vartheta^{-1}\, R_{e_2}\, R_\vartheta \,, \tag{B.4b}$$

$$R_{e_5} = R_\vartheta\, R_{e_1}\, R_\vartheta^{-1} \,, \tag{B.4c}$$

$$R_{e_6} = R_\vartheta\, R_{e_2}\, R_\vartheta^{-1} \,. \tag{B.4d}$$

This choice automatically satisfies eqs. (B.2e) and (B.2f) using $(R_\vartheta)^3 = \mathbb{1}$. Consequently, we are left with the conditions (B.3c), (B.3d), (B.3e) and (B.3f). Let us start with eqs. (B.3c) and (B.3e), i.e. we have to demand

$$R_\omega\, R_{e_3} = R_{e_5}^{-1}\, R_\omega \,, \tag{B.5a}$$

$$R_\omega\, R_{e_5} = R_{e_3}\, R_\omega \,. \tag{B.5b}$$

Using the definitions of R_{e_3} and R_{e_5} from eq. (B.4) we see that this is equivalent to

$$\left(R_\vartheta^2\, R_\omega\, R_\vartheta^2\right)\, R_{e_1} = R_{e_1}^{-1}\, \left(R_\vartheta^2\, R_\omega\, R_\vartheta^2\right) \,, \tag{B.6a}$$

$$\left(R_\vartheta\, R_\omega\, R_\vartheta\right)\, R_{e_1} = R_{e_1}\, \left(R_\vartheta\, R_\omega\, R_\vartheta\right) \,, \tag{B.6b}$$

Since $R_\vartheta\, R_\omega\, R_\vartheta \;=\; R_{\vartheta\,\omega\,\vartheta} \;=\; R_{\omega^{-1}} \;=\; (R_\omega)^{-1}$ condition (B.6b) is trivially satisfied using our assumption $R_{e_1} R_\omega \;=\; R_\omega R_{e_1}$. As a remark, we see that $R_\vartheta^2\, R_\omega\, R_\vartheta^2 \;=\; R_{\vartheta^2\,\omega\,\vartheta^2}$ and $(\vartheta^2\,\omega\,\vartheta^2)^2 = \mathbb{1}$, thus

$$(R_{\vartheta^2\,\omega\,\vartheta^2})^2 = \mathbb{1} \; . \tag{B.7}$$

Now, we repeat these steps for eqs. (B.3d) and (B.3f) and obtain

$$\left(R_\vartheta^2\, R_\omega\, R_\vartheta^2\right)\, R_{e_2} = R_{e_2}^{-1}\, \left(R_\vartheta^2\, R_\omega\, R_\vartheta^2\right) \; , \tag{B.8a}$$

$$\left(R_\vartheta\, R_\omega\, R_\vartheta\right)\, R_{e_2} = R_{e_2}\, \left(R_\vartheta\, R_\omega\, R_\vartheta\right) \; . \tag{B.8b}$$

Again, using $R_\vartheta\, R_\omega\, R_\vartheta \;=\; (R_\omega)^{-1}$ and $R_{e_2} R_\omega \;=\; R_\omega R_{e_2}$ we see that condition (B.8b) is trivial.

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