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Diffraction at Thick Curved Layers with a Nonuniform Dielectric Permittivity

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Abstract—In this paper, we obtain an asymptotic solution for the problem of electromagnetic diffraction at a thick curved dielectric layer with a nonuniform dielectric permittivity. We show that, in the case of thick layers, the main asymptotic approximation already comprises the curvature correction, verify the results by comparison with a solution obtained with the integral equation method, and offer to approximate the piecewise constant dielectric permittivity of a stratified layer with a continuous function.

1. INTRODUCTION

The accurate computation of electromagnetic diffraction at thick dielectric layers is important for simulation of thick-wall antenna radomes [1–4], dielectric lenses [5], atmospheric propagation [6], and components of antennas [7, 8]. With the emergence of new manufacturing techniques, the attention of researchers has recently been drawn to layers with a variable dielectric permittivity [9–12].

It appears that the first theoretical investigation of the problem was undertaken by Bremmer [13, 14]. Bremmer studied plane-wave diffraction at thick dielectric layers with plane-parallel boundaries. He found the leading asymptotic term with the WKB-method and obtained the correcting terms via iterations by assuming that the dielectric constant's gradient is small.

Subsequently, Primakoff and Keller [15] applied the asymptotic analysis of the fields perturbed by a curved layer to a scalar problem, and then Keller [16] expanded it to the case of Maxwell's equations. Keller considered a homogeneous layer with equidistant boundaries and described the fields with integral equations derived using Green's formula. Fixing the free-space wavenumber k_0 and expanding all the functions in the integral equation into power series of the layer thickness, he got the integral representation of the leading asymptotic term and evaluated it with the stationary-phase method for a large k_0. However, the results obtained in this way describe only the layers whose thickness is much smaller than the wavelength.

Another approach assuming *a priori* relations between small parameters associated with the curvature and the layer thickness was offered for diffraction at thin layers [17, 18] and for layers backed by a perfectly conducting screen [19]. This approach is based on combining the ray-tracing method and the small-parameter expansions inside the layer. The same considerations were applied [20, 21] to scattering by good conductors and Sommerfeld's model of absorbers [22] and were verified in [23]. The method proposed here can be considered as the extension of our works [18] and [23].

The treatment of thick layers is mathematically very challenging due to the complex phase structure of the field inside a curved layer caused by multiple reflections between its boundaries (see [24, 25]).

The main goal of this paper is to derive the leading asymptotic term of the problem with a dielectric permittivity represented by a continuous function and to discuss the limitations of this solution. We

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show that for the inclined incidence of the wave, the leading term of the asymptotic solution already includes the curvature of boundaries, in contrast with the canonical geometric optics (GO) solution. We also offer to approximate the piecewise constant dielectric permittivity with a continuous function. The numerical results are validated in 2D by comparison with the results obtained with the Muller boundary integral equations (for homogeneous layers) and the volume integral equations (for layers with a piece-wise constant dielectric permittivity).

2. PROBLEM FORMULATION

A layer G bounded by smooth surfaces S_1 and S_2 (closed or extending to infinity) and characterized by complex constants $\varepsilon = \varepsilon(\mathbf{x})$ and $\mu = \mu(\mathbf{x})$ is placed in infinite space characterized by constants $\varepsilon_0 = 1$, $\mu_0 = 1$. Here and later on we denote by bold script ($\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}$, etc.) vectors in three-dimensional (3-D) space. In continuation of [17–25], we use the Gaussian units throughout the paper.

Surface S_1 separates G from the region G_1 which contains sources that excite time-periodic ($e^{-i\omega t}$) electromagnetic fields ($\mathbf{E}^i(\mathbf{x})$, $\mathbf{H}^i(\mathbf{x})$). Surface S_2 separates G from region G_2.

It is necessary to find the layer-perturbed field ($\mathbf{E}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$) in each region (G_1, G, G_2).

Let us introduce the curvilinear coordinates (σ_1, σ_2, n) in the layer and its vicinity, where (σ_1, σ_2) are any coordinates on the surface S_1, and $|n|$ is the distance from S_1, measured along a normal to it such that $n_1 > 0$ inside the layer G (Fig. 1).

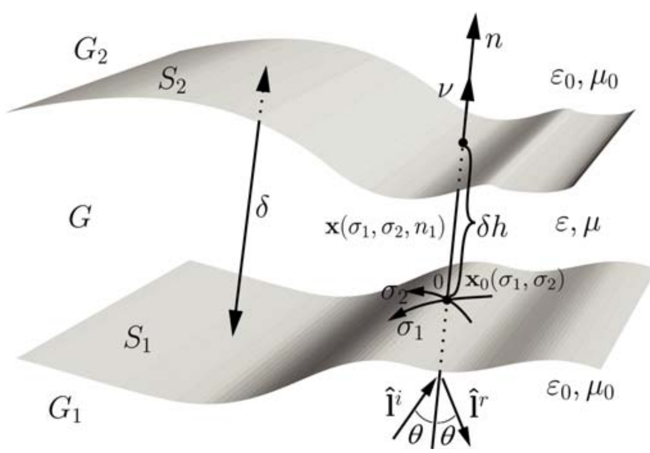


Figure 1. Problem geometry: regions G, G_1, G_2 and the coordinate system $(\sigma_1, \sigma_2, \nu)$.

The equation of the surface S_2 in these coordinates has the form:

$$n = \delta h(\sigma_1, \sigma_2) = \delta h(\mathbf{x_0}), \tag{1}$$

where $x = x_0(\sigma_1, \sigma_2)$ is the vector equation of the surface S_1; h is a continuous function ($0 < h_0 \leq h(\sigma_1, \sigma_2) \leq 1$); and δ is the maximal distance from S_1 to S_2 measured along the normal to S_1.

In addition to coordinates (σ_1, σ_2, n), we shall introduce in the layer G the coordinates $(\sigma_1, \sigma_2, \nu)$, where ν is dimensionless:

$$\nu = \frac{n}{\delta h(\sigma_1, \sigma_2)}. \tag{2}$$

Taking into account Eq. (1), we conclude that within the layer $0 \leq \nu \leq 1$ for any fixed (σ_1, σ_2), and the surfaces S_1 and S_2 are determined by equations $\nu = 0$ and $\nu = 1$.

Let $k_0 = \frac{\omega}{c}$ be the wave number in G_1 and G_2, $k = k_0\sqrt{\varepsilon\mu}$ be the wave number in G †. We also introduce the parameter $\varkappa_0 = \frac{1}{R_0}$, where R_0 is the characteristic linear dimension, related to the geometry of the surface S_1, the wavefront of the primary wave, and the scale of variation of the properties of the medium G in directions tangent to S_1.

† $\mathrm{Im}\varepsilon \geq 0$, $\mathrm{Im}\mu \geq 0$, $\mathrm{Im}k \geq 0$.

For example, in the important case of plane-wave diffraction at a homogeneous layer or at a layer, whose dielectric properties vary only in the normal direction, it is natural to choose $\varkappa_0$ as the largest value of the main curvatures of S_1; in the case of incidence of a cylindrical or a spherical wave onto a plane-parallel layer, R_0 is a distance from the source to S_1; and so on.

We assume that throughout the paper that R_0 is a maximal curvature radius of S_1.

Commonly, a choice of $\varkappa_0$ must be made in such a way that the value $\varkappa_0 = 0$ would correspond to the passage of a plane wave through a homogeneous or a stratified layer.

Let us assume that

$$\eta \equiv \varkappa_0/k_0 \ll 1, \quad \xi \equiv \varkappa_0 \delta \ll 1, \tag{3}$$

which means that the curvature radius of the layer's boundary is big compared both to the wavelength and the layer thickness.

We also assume that the properties of the layer and also of the incident wave vary slowly along S_1 ($\sqrt{|\varepsilon\mu|} \sim 1, \quad 1/(\varkappa_0 h_j^0)\,|\partial h/\partial\sigma_j| \sim 1, \quad j = 1, 2$, with $h_j^0 = \partial \mathbf{x_0}/\partial\sigma_j$).

In this case the pertubations imposed on the field by the layer G can be estimated by asymptotic methods. In this problem, which contains simultaneously two small parameters in Eq. (3), it is possible to speak about a correct asymptotic approach only when the *a priori* relationships between the orders of magnitude of small parameters are given. Therefore, an important role must be played by parameter

$$\zeta \equiv \xi/\eta = k_0\delta, \tag{4}$$

which measures the layer thickness in wavelengths.

In our study, we assume the layer to be "thick", that is

$$\xi = \alpha/\zeta, \quad \eta = \alpha/\zeta^2, \quad \alpha \sim 1. \tag{5}$$

The thicker layer in terms of wavelengths is, the smaller the parameter ξ is, and thus, the more accurate the small-parameter expansions are.

Our study is based on certain assumptions about the structure of incident and diffracted fields. Let us use $(\mathbf{E}^i, \mathbf{H}^i)$ to denote the incident field of sources and let us assume that it can be represented in the entire space or, at least, in a certain region containing the layer G, by means of the asymptotic expansion ($k_0 \to +\infty$)

$$\mathbf{E}^i(\mathbf{x}, k_0) \sim e^{ik_0\Phi(\mathbf{x})}\boldsymbol{\mathscr{E}}^i(\mathbf{x}, k_0), \quad \text{with} \quad \boldsymbol{\mathscr{E}}^i \sim \sum_{m=0}^{\infty} \boldsymbol{\mathscr{E}}^i_m(\mathbf{x})k_0^{-m}, \tag{6}$$

where $\Phi(\mathbf{x})$ and $\boldsymbol{\mathscr{E}}^i_m(\mathbf{x})$ are at least twice continuously differentiable. $\Phi(\mathbf{x})$ satisfies the eikonal equation $|\nabla\Phi(\mathbf{x})| = 1$, and the normal unit vector of the incident wave front $\hat{\mathbf{l}}^i = \nabla\Phi(\mathbf{x})$ forms at any point $\mathbf{x} \in S_1$ an acute angle θ with $\hat{\mathbf{n}}$, the unit normal to S_1 (Fig. 1). In each of the regions G_1, G and G_2 the vector $\mathbf{H}$ is represented by expressions analogous to those given for $\mathbf{E}$.

3. DIFFERENTIAL OPERATORS

Let (σ_1, σ_2, n) be orthogonal coordinates on surface S_1, read along the lines of the principal curvatures.

A position vector of any point (σ_1, σ_2, n) in the neighborhood of S_1 is

$$\mathbf{x} = \mathbf{x}_0(\sigma_1, \sigma_2) + n\hat{\mathbf{n}}(\sigma_1, \sigma_2), \tag{7}$$

where $\mathbf{x}_0(\sigma_1, \sigma_2)$ is a point on S_1, and $\hat{\mathbf{n}}(\sigma_1, \sigma_2)$ is a normal unit to S_1 directed to S_2.

Since

$$\frac{\partial\hat{\mathbf{n}}}{\partial\sigma_j} = \varkappa_j \frac{\partial\mathbf{x}_0}{\partial\sigma_j}, \quad (j = 1, 2), \tag{8}$$

where $\varkappa_1$ and $\varkappa_2$ are the principal curvatures, we see that

$$\frac{\partial\mathbf{x}}{\partial\sigma_j} = (1 + \varkappa_j n)\frac{\partial\mathbf{x}_0}{\partial\sigma_j}, \quad (j = 1, 2), \tag{9}$$

and the Lamé coefficients are

$$h_j(\sigma_1, \sigma_2, n) = (1 + \varkappa_j n) h_j^0(\sigma_1, \sigma_2), \quad h_j^0 = \left| \frac{\partial \mathbf{x}_0}{\partial \sigma_j} \right|, \quad (j = 1, 2). \tag{10}$$

Now let us introduce a differential operator

$$\mathbf{D} = \frac{1}{h_1} \hat{\mathbf{e}}_1 \frac{\partial}{\partial \sigma_1} + \frac{1}{h_2} \hat{\mathbf{e}}_2 \frac{\partial}{\partial \sigma_2}, \tag{11}$$

where e_j are tangential unit vectors to coordinate lines σ_1 and σ_2. Coordinates $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{n}})$ have right-handed orientation. Evidently,

$$\nabla = \mathbf{D} + \hat{\mathbf{n}} \frac{\partial}{\partial n}. \tag{12}$$

It can easily be checked that the operator $\mathbf{D}$ possesses the following properties:

$$(\hat{\mathbf{n}} \cdot \mathbf{D}) = 0, \tag{13}$$

$$(\mathbf{D} \cdot \hat{\mathbf{n}}) = \frac{\varkappa_1}{1 + \varkappa_1 n} + \frac{\varkappa_2}{1 + \varkappa_2 n}, \tag{14}$$

$$\mathbf{D} \times \hat{\mathbf{n}} = 0. \tag{15}$$

Suppose that $\mathbf{A}_\top(\sigma_1, \sigma_2, n)$ is some tangential smooth vector field

$$\mathbf{A}_\top = A_1(\sigma_1, \sigma_2, n) \hat{\mathbf{e}}_1 + A_2(\sigma_1, \sigma_2, n) \hat{\mathbf{e}}_2.$$

Then, by straightforward calculations, we get

$$\hat{\mathbf{n}} \times (\mathbf{D} \times \mathbf{A}_\top) = -\sum_{j=1,2} \frac{\varkappa_j}{1 + \varkappa_j n} A_j \hat{\mathbf{e}}_j = -(\varkappa) \mathbf{A}_\top, \tag{16}$$

where $(\varkappa)$ is the operator defined in the basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ by the matrix

$$(\varkappa) = \begin{pmatrix} \dfrac{\varkappa_1}{1 + \varkappa_1 n} & 0 \\ 0 & \dfrac{\varkappa_2}{1 + \varkappa_2 n} \end{pmatrix}. \tag{17}$$

In the sequel from now on, we use the following notation

$$\mathbf{A}_\perp = \hat{\mathbf{n}} \times \mathbf{A}, \quad \mathbf{A}_\top = \mathbf{A}_\perp \times \hat{\mathbf{n}}, \quad A_N = (\hat{\mathbf{n}} \cdot \mathbf{A}) \tag{18}$$

for any, not necessarily tangential vector field $\mathbf{A}$. Obviously, $\mathbf{A}_\top$ is a tangential component of the vector $\mathbf{A}$, and A_N is its normal projection. Likewise,

$$\mathbf{D}_\perp = \hat{\mathbf{n}} \times \mathbf{D}.$$

By definition, we set

$$(\varkappa_\perp) = \begin{pmatrix} \dfrac{\varkappa_2}{1 + \varkappa_2 n} & 0 \\ 0 & \dfrac{\varkappa_1}{1 + \varkappa_1 n} \end{pmatrix}. \tag{19}$$

Then the following relations hold:

$$\hat{\mathbf{n}} \times (\varkappa) \mathbf{A}_\top = (\varkappa_\perp) \mathbf{A}_\perp, \tag{20}$$

$$\mathbf{D}_\perp \cdot \mathbf{A}_\top = -\mathbf{D} \cdot \mathbf{A}_\perp.^{\ddagger} \tag{21}$$

Formulas (11) and (12) represent the operator ∇ in the coordinates (σ_1, σ_2, n).

Note now that

$$\mathbf{D} = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \nabla) \tag{22}$$

and define the unit normal to S_2 at the point $(\sigma_1, \sigma_2, \delta h(\sigma_1, \sigma_2))$ by $\hat{\mathbf{n}}' = \hat{\mathbf{n}}'(\sigma_1, \sigma_2)$. Suppose that the operator

$$\mathbf{D}' = -\hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \nabla), \tag{23}$$

‡ Let us prove, for instance, (21):
$\mathbf{D}_\perp \cdot \mathbf{M}_\top = (\hat{\mathbf{n}} \times \mathbf{D}) \cdot (\mathbf{M}_\perp \times \hat{\mathbf{n}}) = -\mathbf{M}_\perp [(\hat{\mathbf{n}} \times \mathbf{D}) \times \hat{\mathbf{n}}] + [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{D})] \cdot \mathbf{M}_\perp = (\mathbf{M}_\perp \cdot \hat{\mathbf{n}})(\mathbf{D} \cdot \hat{\mathbf{n}}) - \mathbf{D} \cdot \mathbf{M}_\perp = -\mathbf{D} \cdot \mathbf{M}_\perp$, since $\mathbf{M}_\perp \cdot \hat{\mathbf{n}} = 0$.

is associated with the surface S_2 in the same way as $\mathbf{D}$ with S_1. Then, as in Eq. (12),

$$\nabla = \mathbf{D}' + \hat{\mathbf{n}}' \frac{\partial}{\partial n'}, \tag{24}$$

where $\frac{\partial}{\partial n'}$ means differentiation in the direction $\hat{\mathbf{n}}'$.

Since on the surface S_2 $\mathbf{x} = \mathbf{x}_0(\sigma) + \delta h(\sigma)\hat{\mathbf{n}}$, we get

$$\hat{\mathbf{n}}' = \frac{\dfrac{\partial \mathbf{x}}{\partial \sigma_1} \times \dfrac{\partial \mathbf{x}}{\partial \sigma_2}}{\left| \dfrac{\partial \mathbf{x}}{\partial \sigma_1} \times \dfrac{\partial \mathbf{x}}{\partial \sigma_2} \right|} = \frac{\hat{\mathbf{n}} - \delta \mathbf{D} h}{\sqrt{1 + \delta^2 |\mathbf{D} h|^2}}.$$

Hence

$$\hat{\mathbf{n}}' = \hat{\mathbf{n}} - \delta \mathbf{D} h + o(\delta) \tag{25}$$

and, consequently,

$$\frac{\partial}{\partial n'} \equiv (\hat{\mathbf{n}}' \cdot \nabla) = \frac{\partial}{\partial n} - \delta(\mathbf{D} h \cdot \mathbf{D}) + o(\delta), \tag{26}$$

$$\mathbf{D}' \equiv \nabla - \hat{\mathbf{n}}' \frac{\partial}{\partial n'} = \mathbf{D} + \delta \left[\mathbf{D}(h) \frac{\partial}{\partial n} + \hat{\mathbf{n}}(\mathbf{D} h \cdot \mathbf{D}) \right] + o(\delta). \tag{27}$$

After the change of variable $\xi = \varkappa_0 \delta$, we obtain

$$\hat{\mathbf{n}}' = \hat{\mathbf{n}} - \xi \mathbf{D}^{\mathbf{0}} h + o(\xi), \tag{28}$$

$$\frac{\partial}{\partial n'} = \frac{\partial}{\partial n} - \xi(\mathbf{D}^{\mathbf{0}} h \cdot \mathbf{D}) + o(\xi), \tag{29}$$

$$\mathbf{D}' = \mathbf{D} + \xi \left[\mathbf{D}^0(h) \frac{\partial}{\partial n} + \hat{\mathbf{n}}(\mathbf{D}^0 h \cdot \mathbf{D}) \right] + o(\xi), \tag{30}$$

with $\mathbf{D}^0 = \frac{1}{\varkappa_0} \mathbf{D}|_{\nu=0}$.

For $h(\sigma) \equiv const$, trivially, $\hat{\mathbf{n}}' \equiv \hat{\mathbf{n}}$ and $\mathbf{D}' \equiv \mathbf{D}$.

4. MAXWELL'S EQUATIONS IN A LAYER AND ITS NEIGHBORHOOD

We denote by $(\mathbf{E}, \mathbf{H})$ the field induced in a layer by an incident wave. From Maxwell's equation

$$ik_0 \mu \mathbf{H} = \nabla \times \mathbf{E}, \tag{31}$$

$$-ik_0 \varepsilon \mathbf{E} = \nabla \times \mathbf{H}, \tag{32}$$

using the representation in Eq. (12) of ∇, Eq. (15), and the obvious relation $\partial \hat{\mathbf{n}}/\partial n = 0$, we derive

$$ik_0 \mu \mathbf{H} = (\mathbf{D} + \hat{\mathbf{n}} \partial/\partial n) \times (\mathbf{E}_\top + \hat{\mathbf{n}} E_N) = (\hat{\mathbf{n}} \times \partial \mathbf{E}_\top/\partial n) + (\mathbf{D} \times \mathbf{E}_\top) - (\hat{\mathbf{n}} \times \mathbf{D}) E_N, \tag{33}$$

where $E_N = (\hat{\mathbf{n}} \cdot \mathbf{E})$.

Taking the vector product of $\hat{\mathbf{n}}$ and both sides of Eq. (33), and using Eq. (16), we obtain

$$ik_0 \mu \mathbf{H}_\perp = -\left[\partial/\partial n + (\varkappa)\right] \mathbf{E}_\top + \mathbf{D} E_N.$$

In the same manner, Eq. (32) yields the equation

$$-ik_0 \varepsilon \mathbf{E}_\perp = -\left[\partial/\partial n + (\varkappa)\right] \mathbf{H}_\top + \mathbf{D} H_N.$$

Taking the vector product $\hat{\mathbf{n}} \times$ one more time and using Eq. (20), we get

$$ik_0 \varepsilon \mathbf{E}_\top = -\left[\partial/\partial n + (\varkappa_\perp)\right] \mathbf{H}_\perp + \mathbf{D}_\perp H_N.$$

Now we form the termwise scalar product of Eqs. (31) and (32) with $\hat{\mathbf{n}}$ and see that

$$ik_0 \mu H_N = (\hat{\mathbf{n}} \times \mathbf{D}) \cdot (\mathbf{E}_\top + \hat{\mathbf{n}} E_N) = \mathbf{D}_\perp \cdot \mathbf{E}_\top + \hat{\mathbf{n}} \cdot [\mathbf{D} \times (\hat{\mathbf{n}} E_N)] = \mathbf{D}_\perp \cdot \mathbf{E}_\top; \tag{34}$$

$$ik_0 \varepsilon E_N = -\mathbf{D}_\perp \cdot \mathbf{H}_\top = \mathbf{D} \cdot \mathbf{H}_\perp \tag{35}$$

Eq. (34) is true, since $\hat{\mathbf{n}}\cdot[\mathbf{D}\times(\hat{\mathbf{n}}E_N)] = [\hat{\mathbf{n}}\cdot(\mathbf{D}\times\hat{\mathbf{n}})]E_N + \hat{\mathbf{n}}\cdot[\mathbf{D}E_N\times\hat{\mathbf{n}}]$, and both summands here vanish (the first one by Eq. (15)). To derive Eq. (35), we take into account of Eq. (21).

Finally, we obtain the following system of equations:

$$ik_0\mu\mathbf{H}_\perp = -\left[\partial/\partial n + (\varkappa)\right]\mathbf{E}_\top + \mathbf{D}E_N, \tag{36}$$

$$ik_0\varepsilon\mathbf{E}_\top = -[\partial/\partial n + (\varkappa_\perp)]\mathbf{H}_\perp + \mathbf{D}_\perp H_N, \tag{37}$$

$$ik_0\mu H_N = \mathbf{D}_\perp\cdot\mathbf{E}_\top, \tag{38}$$

$$ik_0\varepsilon E_N = \mathbf{D}\cdot\mathbf{H}_\perp. \tag{39}$$

The system of equations (36)–(39) is equivalent to the set (31) by derivation. At the same time, it is well suited to the specific character of diffraction problems considered.

Let us point out two facts related to the system (36)–(39). First, substituting E_N and H_N expressed by Eqs. (38) and (39) into Eqs. (36) and (37), we arrive at the autonomous set of equations with respect to the vectors $\mathbf{E}_\top$ and $\mathbf{H}_\perp$. Secondly, the system (36)–(39) describes not only the field in the layer, but also, if to set $\varepsilon = \mu = 1$, the field in the regions G_1 and G_2, at least in the neighborhood of G. We will use the latter fact in Section 6.1 to derive the boundary conditions in a special form.

Let the field at a point (σ,ν) inside the layer have the form

$$\mathbf{E} = \boldsymbol{\mathscr{E}}e^{ik_0\Phi_0(\sigma)}, \quad \mathbf{H} = \boldsymbol{\mathscr{H}}e^{ik_0\Phi_0(\sigma)}, \tag{40}$$

with $\Phi_0(\sigma) = \Phi(\sigma,0)$.[§]

Let us introduce the following notation:

$$\boldsymbol{\mathscr{E}}_\top = \mathbf{u}, \quad \boldsymbol{\mathscr{H}}_\perp = \mathbf{v}; \quad \nabla\Phi(\sigma,n) = \hat{\mathbf{l}}^i = \hat{\mathbf{l}}^i(\sigma,n);$$

$$\hat{\mathbf{l}}^\mathbf{i}(\sigma,0)\cdot\hat{\mathbf{n}}(\sigma) = \cos\theta, \quad \theta = \theta(\sigma); \quad \hat{\mathbf{l}}^\mathbf{i}(\sigma,0)\cdot\hat{\mathbf{e}}_\mathbf{j}(\sigma) = \cos\alpha_j(\sigma);$$

$$\mathbf{D}\Phi_0 = \sum_{j=1,2}\frac{\hat{\mathbf{e}}_j}{1+\varkappa_j n}\cos\alpha_j \equiv \mathbf{l}_\top(\sigma,n),$$

$$\mathbf{D}_\perp\Phi_0 \equiv \mathbf{l}_\perp(\sigma,n); \quad \mathbf{l}_\perp(\sigma,0) = \mathbf{l}_\perp(\sigma,n).$$

Obviously, $\mathbf{l}_\top(\sigma,0) = \mathbf{l}^i_\top(\sigma,0)$ and $\mathbf{l}_\perp(\sigma,0) = \mathbf{l}^i_\perp(\sigma,0)$.

Substituting Eq. (40) into Eqs. (36)–(39), we get

$$ik_0\mu\mathbf{v} = -\left[\partial/\partial n + (\varkappa)\right]\mathbf{u} + (\mathbf{D} + ik_0\mathbf{l}_\top)\mathscr{E}_N, \tag{41}$$

$$ik_0\varepsilon\mathbf{u} = -\left[\partial/\partial n + (\varkappa_\perp)\right]\mathbf{v} + (\mathbf{D}_\perp + ik_0\mathbf{l}_\perp)\mathscr{H}_N, \tag{42}$$

$$ik_0\mu\mathscr{H}_N = (\mathbf{D}_\perp + ik_0\mathbf{l}_\perp)\cdot\mathbf{u}, \tag{43}$$

$$ik_0\varepsilon\mathscr{E}_N = (\mathbf{D} + ik_0\mathbf{l}_\top)\cdot\mathbf{v}. \tag{44}$$

Eliminating $\mathscr{E}_N$ and $\mathscr{H}_N$ from Eqs. (41) and (42), and changing a variable n to ν, we obtain a system of two equations for $\mathbf{u}(\sigma,\nu)$ and $\mathbf{v}(\sigma,\nu)$ in the layer $(0\le\nu\le1)$ with functions $\varepsilon = \varepsilon(\nu,\sigma)$ and $\mu = \mu(\nu,\sigma)$

$$\begin{cases} \dfrac{1}{i\zeta\mu h}\dfrac{\partial\mathbf{u}}{\partial\nu} = -\mathbf{v} + \dfrac{1}{\varepsilon\mu}\varepsilon\mathbf{Z}_\top(\dfrac{1}{\varepsilon}\mathbf{Z}_\top\cdot\mathbf{v}) + \dfrac{i\eta}{\mu}(\overline{\varkappa})\mathbf{u}, \\ \dfrac{1}{i\zeta\varepsilon h}\dfrac{\partial\mathbf{v}}{\partial\nu} = -\mathbf{u} + \dfrac{1}{\varepsilon\mu}\mu\mathbf{Z}_\perp(\dfrac{1}{\mu}\mathbf{Z}_\perp\cdot\mathbf{u}) + \dfrac{i\eta}{\varepsilon}(\overline{\varkappa}_\perp)\mathbf{v}, \end{cases} \tag{45}$$

where $(\overline{\varkappa}) = \frac{1}{\varkappa_0}(\varkappa)$, $\quad (\overline{\varkappa}_\perp) = \frac{1}{\varkappa_0}(\varkappa_\perp)$, and

$$\mathbf{Z}_\top = \mathbf{l}_\top + 1/(ik_0)\mathbf{D}, \quad \mathbf{Z}_\perp = \mathbf{l}_\perp + 1/(ik_0)\mathbf{D}_\perp. \tag{46}$$

[§] $\Phi(\sigma,n)$ is the phase function of the incident wave, see Eq. (6). Thus, the phase function inside the layer differs from the GO phase, as it does not take into account the wave refraction. This distinction does not lead to an error, since the amplitude functions $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{H}}$ will compensate for the refraction of initial wavefront after the exact boundary conditions are satisfied.

5. ASYMPTOTIC EXPANSIONS OF THE FIELD INSIDE A LAYER

In the case of thin layers, vector functions $\mathbf{u}$ and $\mathbf{v}$ are smooth and nonoscillatory. For this reason, the role of the phase function is played by the phase function of the incident wave $\Phi(\mathbf{x})$ at a given point on surface S_1, and the phase incursion, induced by the passage of a wave through a layer, is taken into account automatically in calculating the amplitude vectors $\mathbf{u}$ and $\mathbf{v}$.

If however we have thick layers ($\zeta \gg 1$), the vectors $\mathbf{u}$ and $\mathbf{v}$ become highly oscillating due to the multiple internal reflections between S_1 and S_2. Moreover, the bigger characteristic parameter $k\delta$ is, the higher is the frequency of oscillations.

For this reason, we seek the vector functions $\mathbf{u}$ and $\mathbf{v}$ in the form of the following asymptotic expansions:

$$\vec{\vec{\mathbf{W}}}= \sum_{p=0}^{\infty} \vec{\vec{\mathbf{W}}}_p \, e^{i\zeta\Lambda_p(\sigma,\nu)}, \quad \text{with} \quad \vec{\vec{\mathbf{W}}}_p \sim \sum_{m=0}^{\infty} \frac{\vec{\vec{\mathbf{W}}}_{pm}\,(\sigma,\nu)}{\zeta^m}, \tag{47}$$

where $\vec{\vec{\mathbf{W}}}= \binom{\mathbf{u}}{\mathbf{v}}$, $\vec{\vec{\mathbf{W}}}_p= \binom{\mathbf{u}_p}{\mathbf{v}_p}$, and Λ_p are some functions of variables σ and ν. Below we establish the form of these functions and show the way to efficiently find the vectors $\mathbf{u}_p$ and $\mathbf{v}_p$ ($p = 0, 1, 2, \dots$) up to an arbitrary asymptotic order. The form of expansions in Eq. (47) is justified by solutions of the canonical problems, in particular, [26].

The expansion in Eq. (47) means that for any given $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that for $\zeta \to \infty$, uniformly with respect to ζ, holds

$$\left| \vec{\vec{\mathbf{W}}} - \sum_{p=0}^{N} e^{i\Lambda_p(\sigma,\nu)} \, \vec{\vec{\mathbf{W}}}_p \right| < \varepsilon. \tag{48}$$

Substituting Eq. (47) in Eq. (45), we get a system of equations with respect to $\mathbf{u}_p$ and $\mathbf{v}_p$

$$\frac{\partial \mathbf{u}_p}{\partial \nu} + i\zeta\lambda_p \mathbf{u}_p + \xi h(\overline{\varkappa})\mathbf{u}_p \;=\; -i\mu h\zeta \mathbf{v}_p + \frac{i\zeta h}{\varepsilon}\mathbf{Y}_{\top p}\mathbf{v}_p, \tag{49}$$

$$\frac{\partial \mathbf{v}_p}{\partial \nu} + i\zeta\lambda_p \mathbf{v}_p + \xi h(\overline{\varkappa}_\perp)\mathbf{v}_p \;=\; -i\varepsilon h\zeta \mathbf{u}_p + \frac{i\zeta h}{\mu}\mathbf{Y}_{\perp p}\mathbf{u}_p, \tag{50}$$

where

$$\lambda_p = \frac{\partial \Lambda_p}{\partial \nu}; \tag{51}$$

$$\begin{aligned} \mathbf{Y}_{\top p}\mathbf{v}_p &= \varepsilon e^{-i\zeta\Lambda_p}\mathbf{Z}_\top \left\{ \frac{1}{\varepsilon}\mathbf{Z}_\top (e^{i\zeta\Lambda_p}\mathbf{v}_p) \right\}, \\ \mathbf{Y}_{\perp p}\mathbf{u}_p &= \mu e^{-i\zeta\Lambda_p}\mathbf{Z}_\perp \left\{ \frac{1}{\mu}\mathbf{Z}_\perp (e^{i\zeta\Lambda_p}\mathbf{u}_p) \right\}. \end{aligned} \tag{52}$$

The expressions $\mathbf{Y}_{\top p}\mathbf{v}_p$ and $\mathbf{Y}_{\perp p}\mathbf{u}_p$ can be written as formal series with respect to the small parameter $\xi = \frac{\alpha}{\zeta}$, i.e.,

$$\mathbf{Y}_{\top p}\mathbf{v}_p = \sum_{m=0}^{\infty} \frac{\alpha^m \mathbf{Y}^{(m)}_{\top p}\mathbf{v}_p}{\zeta^m}, \quad \mathbf{Y}_{\perp p}\mathbf{u}_p = \sum_{m=0}^{\infty} \frac{\alpha^m \mathbf{Y}^{(m)}_{\perp p}\mathbf{u}_p}{\zeta^m}, \tag{53}$$

where $\mathbf{Y}^{(0)}_{\top p}$, $\mathbf{Y}^{(1)}_{\top p}$, $\mathbf{Y}^{(0)}_{\perp p}$, $\mathbf{Y}^{(1)}_{\perp p}$, $\dots$ are certain known matrix operators on 2D vectors. For instance,

$$\mathbf{Y}^{(0)}_{\top p}\mathbf{v}_p \;=\; (\mathbf{l}^i_\top \cdot \mathbf{v}_p)\mathbf{l}^i_\top, \tag{54}$$

$$\mathbf{Y}^{(1)}_{\top p}\mathbf{v}_p \;=\; (\mathbf{s}_p \cdot \mathbf{v}_p)\mathbf{l}^i_\top + (\mathbf{l}^i_\top \cdot \mathbf{v}_p)\mathbf{s}_p, \tag{55}$$

where $\mathbf{s}_p = \mathbf{D}^0\Lambda_p - h\nu(\overline{\varkappa})^0\mathbf{l}^i_\top$, $(\overline{\varkappa})^0 = (\overline{\varkappa})|_{n=0}$, and $\mathbf{D}^0 = \frac{1}{\varkappa_0}\mathbf{D}|_{n=0}$. A derivation of $\mathbf{Y}^{(m)}_{\top p}$ and $\mathbf{Y}^{(m)}_{\perp p}$ for $m = 0$ and $m = 1$ is given in Section A.2.

On the basis of these expansions and expansions of operators ($\overline{\varkappa}$) and ($\overline{\varkappa}_\perp$) (see Appendix), we can write the system of Equations (49) and (50) as follows

$$\frac{\partial \overset{\Rightarrow}{\mathbf{W}}_p}{\partial \nu} = \zeta \mathbf{L}_p \overset{\Rightarrow}{\mathbf{W}}_p + \mathbf{B}_{p0} \overset{\Rightarrow}{\mathbf{W}}_p + \frac{1}{\zeta}\mathbf{B}_{p1} \overset{\Rightarrow}{\mathbf{W}}_p + \ldots, \tag{56}$$

where $\mathbf{L}_p = (\mathbf{A}_p - i\lambda_p \mathbf{I})$, $\mathbf{A}_p$, $\mathbf{B}_{p0}$, $\mathbf{B}_{p1}, \ldots$ are four-dimensional matrix operators; $\mathbf{I}$ is a unit four-dimensional matrix; $\mathbf{I}^{(2\times2)}$ is a 2D unit matrix, and

$$\mathbf{A}_p = -ih \begin{pmatrix} 0 & \mu \mathbf{I}^{(2\times2)} - \frac{1}{\varepsilon}\mathbf{Y}^{(0)}_{\top p} \\ \varepsilon \mathbf{I}^{(2\times2)} - \frac{1}{\mu}\mathbf{Y}^{(0)}_{\perp p} & 0 \end{pmatrix}, \tag{57}$$

$$\mathbf{B}_{p0} = i\alpha h \begin{pmatrix} 0 & \frac{1}{\varepsilon}\mathbf{Y}^{(1)}_{\top p} \\ \frac{1}{\mu}\mathbf{Y}^{(1)}_{\perp p} & 0 \end{pmatrix}. \tag{58}$$

5.1. Operator $\mathbf{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and Matrix Operators in Terms of It

Let us consider a matrix operator acting on vectors tangential to S_1 according to the formula

$$\mathbf{T}(\alpha, \beta)\mathbf{a} = \alpha \mathbf{a} + \beta \mathbf{l}^i_\top (\mathbf{l}^i_\top \mathbf{a}), \quad \alpha, \beta = \text{const}. \tag{59}$$

Evidently, for any values of parameters α and β, the operator $\mathbf{T}(\alpha, \beta)$ applied to any vector tangential to S_1 is also a vector tangential to S_1.

It is important to note the following properties of the operator $\mathbf{T}(\alpha, \beta)$ easily deducible from its definition:

$\forall \mathbf{a} \in \mathbb{R}^3, \quad \forall \alpha, \beta \in \mathbb{R},$

$$1. \quad \mathbf{T}(k\alpha, k\beta)\mathbf{a} = k\mathbf{T}(\alpha, \beta)\mathbf{a}, \quad k = \text{const}; \tag{60}$$

$$2. \quad \{\mathbf{T}(\alpha_1, \beta_1) + \mathbf{T}(\alpha_2, \beta_2)\}\,\mathbf{a} = \mathbf{T}(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\mathbf{a}; \tag{61}$$

$$3. \quad \mathbf{T}(\alpha_1, \beta_1)\mathbf{T}(\alpha_2, \beta_2)\mathbf{a} = \tag{62}$$

$$\mathbf{T}(\alpha_1\alpha_2; \alpha_1\beta_2 + \beta_1\alpha_2 + \sin^2\theta\beta_1\beta_2)\mathbf{a}; \tag{63}$$

$$4. \quad \frac{\partial}{\partial x}\mathbf{T}(\alpha(x); \beta(x))\mathbf{a} = \mathbf{T}\left(\frac{\partial \alpha}{\partial x}; \frac{\partial \beta}{\partial x}\right)\mathbf{a} + \mathbf{T}(\alpha, \beta)\frac{\partial \mathbf{a}}{\partial x}; \tag{64}$$

$$5. \quad \mathbf{T}^{-1}(\alpha, \beta)\mathbf{a} = \mathbf{T}\left(\frac{1}{\alpha}; -\frac{\beta}{\alpha(\alpha + \beta\sin^2\theta)}\right)\mathbf{a}; \tag{65}$$

$$6. \quad \mathbf{T}(\alpha, \beta)\mathbf{a} = \alpha\mathbf{a}, \quad \text{if} \quad \theta = 0. \tag{66}$$

Matrix operators $\mathbf{A}_p$ and $\mathbf{L}_p$ take, on account of Eq. (59), the form

$$\mathbf{A}_p = -ih \begin{pmatrix} 0 & \mu\mathbf{T}\left(1; -\frac{1}{\varepsilon\mu}\right) \\ \varepsilon\mathbf{T}\left(\rho; \frac{1}{\varepsilon\mu}\right) & 0 \end{pmatrix}, \| \tag{67}$$

$$\mathbf{L}_p \overset{\Rightarrow}{\mathbf{W}}_p = (\mathbf{A}_p - i\lambda_p\mathbf{I}) \overset{\Rightarrow}{\mathbf{W}}_p = -i \begin{pmatrix} \lambda_p \mathbf{u_p} + \mu h \mathbf{T}(1; -\frac{1}{\varepsilon\mu})\mathbf{v_p} \\ \varepsilon h \mathbf{T}(\rho; \frac{1}{\varepsilon\mu})\mathbf{u_p} + \lambda_p \mathbf{v_p} \end{pmatrix}. \tag{68}$$

Using the operator $\mathbf{T}$ allows us to write all the derivations in the vector basis $(\mathbf{e}_1, \mathbf{e}_2)$ in a compact form. This has however one more important advantage over the method of [24, 25], where the leading

‖ It bears mentioning that $\mathbf{T}(\rho, \frac{1}{\varepsilon\mu}) = \mathbf{T}_\perp(1, -\frac{1}{\varepsilon\mu})$, where $\mathbf{T}_\perp$ is defined by analogy with the operator $\mathbf{T}$ as follows: $\forall \mathbf{a} \in \mathbb{R}^3, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \mathbf{T}_\perp(\alpha, \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{l}^i_\perp(\mathbf{l}^i_\perp \cdot \mathbf{a})$.

term of the asymptotic expansion was deduced on the basis $(\boldsymbol{l}_\top^i, \boldsymbol{l}_\perp^i)$. The formulas of [25] obviously lead in the case of normal incidence to an indeterminate form 0/0 to be analyzed apart. The property in Eq. (66) of the operator $\mathbf{T}(a,b)$ allows us to neglect the case when b has an indeterminate form at $\theta = 0$. As can be shown by straightforward calculations, the resulting solution in this case is a limit at $\theta \to 0$. We present thus a unified solution for all values of the incident angle.

5.2. Phases of Partial Waves

Using Eq. (47) and equating the coefficients of the same powers of ζ in Eq. (56), we obtain a system for the determination of the leading asymptotic term

$$\mathbf{L}_p \overset{\Rightarrow}{\mathbf{W}}_{p0} = 0, \tag{69}$$

$$\mathbf{L}_p \overset{\Rightarrow}{\mathbf{W}}_{p1} = \frac{\partial \overset{\Rightarrow}{\mathbf{W}}_{p0}}{\partial \nu} - \mathbf{B}_{p0} \overset{\Rightarrow}{\mathbf{W}}_{p0}, \tag{70}$$

and so on.

Obviously, the necessary and sufficient condition for a nontrivial solution of Eq. (69) is that quantities $i\lambda_p$ are the eigenvalues of operator $\mathbf{A}_p$. From the characteristic equation $\det|\mathbf{A}_p - i\lambda_p \mathbf{I}| = 0$, we obtain two solutions

$$\lambda_p^\pm = \pm h\sqrt{\varepsilon\mu\rho} \tag{71}$$

where $\rho = 1 - \frac{\sin^2\theta}{\varepsilon\mu}$. Hence, in virtue of Eq. (51), we obtain two families of phase functions, respectively,

$$\Lambda_p^\pm(\sigma,\nu) = \pm h(\sigma)\int_0^\nu \sqrt{\varepsilon\mu\rho}\,d\nu + l_p^\pm(\sigma), \tag{72}$$

where $l_p^\pm(\sigma)$ are some functions independent of ν.

In the sequel, we use the following notation interchangeably:

$$\Lambda_p^+ \equiv \Lambda_{2p}, \quad (p = 0, 1, \ldots), \quad \Lambda_p^- \equiv \Lambda_{2p-1} \quad (p = 1, 2, \ldots). \tag{73}$$

The constants $l_p^\pm(\sigma)$ in Eq. (72) can be determined, if Λ_p are subject to the physically justified conditions, such as

$$\Lambda_0^+|_{\nu=0} = 0, \quad \Lambda_1^-|_{\nu=1} = \Lambda_0^+|_{\nu=1}, \quad \Lambda_1^+|_{\nu=0} = \Lambda_1^-|_{\nu=0}, \quad \text{etc.},$$

that is a continuous joining of phases of partial waves. From these conditions, we get

$$\Lambda_p^\pm = \pm h(\sigma)\int_0^\nu \sqrt{\varepsilon\mu\rho}\,d\nu + 2ph(\sigma)\int_0^1 \sqrt{\varepsilon\mu\rho}\,d\nu \tag{74}$$

or, on account of Eq. (73),

$$\begin{aligned}\Lambda_{2p} &= h(\sigma)\int_0^\nu \sqrt{\varepsilon\mu\rho}\,d\nu + 2ph(\sigma)\int_0^1 \sqrt{\varepsilon\mu\rho}\,d\nu, \quad (p = 0, 1, \ldots),\\ \Lambda_{2p-1} &= h(\sigma)\int_\nu^1 \sqrt{\varepsilon\mu\rho}\,d\nu + (2p-1)h(\sigma)\int_0^1 \sqrt{\varepsilon\mu\rho}\,d\nu, \quad (p = 1, 2, \ldots).\end{aligned} \tag{75}$$

For the sake of preciseness, it bears mentioning that $\Lambda_p^\pm$ defined by Eq. (74) are not the true phase functions, rather their asymptotic approximations, which is quite natural considering that a layer is slightly curved.

The functions Λ_p^+ and Λ_p^- and the expansion in Eq. (47) can be interpreted in terms of GO. Let a ray arrive at some point M_0 on the surface S_1 under an incidence angle θ and a refracted ray arrive at a point M_1 on the surface S_2 (Fig. 2). We denote by θ_1 a refractive angle and by N_0 where a normal

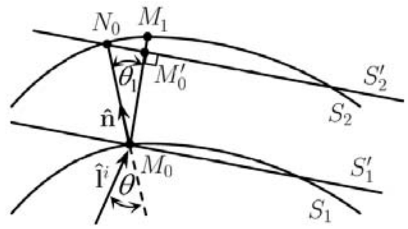


Figure 2. On the internal reflections inside a thick layer and a construction of the phase function.

to S_1 crosses the surface S_2. Now let us replace the layer G with a fictitious layer, whose plane-parallel boundaries S_1' and S_2' are constructed as follows. The plane S_2' goes through the point N_0 and is perpendicular to the reflected ray M_0M_1. The plane S_1' goes through a point M_0 and is parallel to S_2'.

Now let us calculate the distance M_0M_0'. Obviously, $M_0M_0' = \delta\cos\theta_1$, and, by Snell's law, $\sin\theta = \sqrt{\varepsilon\mu}\sin\theta_1$, hence $\cos\theta_1 = \sqrt{1 - \frac{\sin^2\theta}{\varepsilon\mu}} = \sqrt{\rho}$. Therefore, $M_0M_0' = \delta h\sqrt{\varepsilon\mu\rho}$.

By construction of the layer boundaries, a ray travels inside our fictitious layer from the point M_0 to the point M_0', then from M_0' to M_0, and so on; i.e., with each reflection from a boundary, the phase incursion is M_0M_0', which corresponds to the structure of formulas (75).

Therefore, $\zeta\Lambda_p^+$ can be interpreted as an increment in phase as a result of an even number of internal reflections from S_2, then from S_1, and so on, and $\zeta\Lambda_p^-$ is an increment produced by an odd number of such reflections.

The point to be emphasized is that the form of the phase functions in Eq. (75) is not initially imposed according to the geometrical consideration, but is deduced so that the geometric construction described above is just their illustrative interpretation.

Now we turn to the derivation of boundary conditions in a form appropriate for our purpose.

6. BOUNDARY CONDITIONS

6.1. Boundary Conditions in a Special Form

To get the boundary conditions on S_1, we write the equation for the field in the region G_1, then pass to the limit to the surface S_1 and, finally, equate the tangential components of the fields in the layer G and the region G_1. In Equations (36)–(39), we set $\varepsilon = \mu = 1$ and interpret $\mathbf{E}$, $\mathbf{H}$ as the field in the region G_1:

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^r, \quad \mathbf{H} = \mathbf{H}^i + \mathbf{H}^r,$$

where $\mathbf{E}^i = \boldsymbol{\mathscr{E}}^i e^{ik_0\Phi}$, $\mathbf{H}^i = \boldsymbol{\mathscr{H}}^i e^{ik_0\Phi}$, $\mathbf{E}^r = \boldsymbol{\mathscr{E}}^r e^{ik_0\Psi}$, $\mathbf{H}^r = \boldsymbol{\mathscr{H}}^r e^{ik_0\Psi}$.

Equation (36), after the limit passage from G_1 to S_1, generates the following boundary relation on S_1: ¶

$$\begin{aligned} ik_0(\boldsymbol{\mathscr{H}}^i_\perp + \boldsymbol{\mathscr{H}}^r_\perp) = &-\frac{\partial}{\partial n}(\boldsymbol{\mathscr{E}}^i_\top + \boldsymbol{\mathscr{E}}^r_\top) - ik_0\frac{\partial\Phi}{\partial n}(\boldsymbol{\mathscr{E}}^i_\top - \boldsymbol{\mathscr{E}}^r_\top) \\ &- (\varkappa)(\boldsymbol{\mathscr{E}}^i_\top + \boldsymbol{\mathscr{E}}^r_\top) + ik_0\mathbf{D}\Phi(\mathscr{E}^i_N + \mathscr{E}^r_N) + \mathbf{D}(\mathscr{E}^i_N + \mathscr{E}^r_N). \end{aligned} \tag{76}$$

Since on the surface S_1

$$\boldsymbol{\mathscr{H}}^i_\perp + \boldsymbol{\mathscr{H}}^r_\perp = \boldsymbol{\mathscr{H}}_\perp \equiv \mathbf{v}, \quad \boldsymbol{\mathscr{E}}^i_\top + \boldsymbol{\mathscr{E}}^r_\top = \boldsymbol{\mathscr{E}}_\top \equiv \mathbf{u}, \quad \mathscr{E}^i_N + \mathscr{E}^r_N = \varepsilon\mathscr{E}_N, \tag{77}$$

the relation in Eq. (76), with the aid of Eq. (44), takes the form

$$P_0\mathbf{u} - Q\mathbf{v}|_{S_1} = 2\mathbf{u}^i\cos\theta + \frac{1}{ik_0}\frac{\partial(\mathbf{u}^i + \mathbf{u}^r)}{\partial n}, \tag{78}$$

¶ We take into account that $\Phi = \Psi$, $\partial\Phi/\partial n = -\partial\Psi/\partial n$, and $\mathbf{D}\Phi = \mathbf{D}\Psi$ on S_1.

where P_0, Q are operators $P_0\mathbf{u} = \frac{\partial\Phi}{\partial n}\mathbf{u} - \frac{1}{ik_0}(\varkappa)\mathbf{u}$, $Q\mathbf{v} = \mathbf{v} - \mathbf{Z}_\top(\mathbf{Z}_\top \cdot \mathbf{v})$, with $\mathbf{u}^i = \boldsymbol{\mathscr{E}}^i_\top$, $\mathbf{u}^r = \boldsymbol{\mathscr{E}}^r_\top$, and $\mathbf{Z}_\top$ defined by Eq. (46).

In the same way, we find the condition on S_2:

$$P_1'\mathbf{u}' - Q'\mathbf{v}'\big|_{S_2} = \frac{1}{ik_0}\frac{\partial(\mathbf{u}^t)'}{\partial n'}e^{ik_0(\Phi-\Phi_0)}, \tag{79}$$

where $\mathbf{u}' = \boldsymbol{\mathscr{E}}_{\top'}$, $\mathbf{v}' = \boldsymbol{\mathscr{H}}_{\perp'}$, $(\mathbf{u}^t)' = \boldsymbol{\mathscr{E}}^t_{\top'}$, $\Phi = \Phi(\sigma, \delta h)$, $\Phi_0 = \Phi(\sigma, 0)$, $P_1'\mathbf{u}' = -\frac{\partial\Phi}{\partial n'}\mathbf{u}' - \frac{1}{ik_0}(\varkappa)'\mathbf{u}'$, $Q'\mathbf{v}' = \mathbf{v}' - \mathbf{Z}_{\top'}(\mathbf{Z}_{\top'} \cdot \mathbf{v}')$, $\mathbf{Z}_{\top'} = \mathbf{l}_{\top'} + \frac{1}{ik_0}\mathbf{D}'$.

The primes in $\top'$, $\perp'$, n', $(\varkappa)'$, etc., indicate that the corresponding operations or quantities are associated with the surface S_2. If the surfaces S_2 and S_1 are equidistant, the primes can be omitted.

Taking into account that $\eta = \varkappa_0/k_0$ and $\partial\Phi/\partial n = \cos\theta$, we write the boundary conditions in the following form:

$$P_0\mathbf{u} - Q\mathbf{v} \,=\, 2\mathbf{u}^i\cos\theta - \frac{i\eta}{\varkappa_0}\frac{\partial(\mathbf{u}^i + \mathbf{u}^r)}{\partial n}, \tag{80}$$

$$P_1'\mathbf{u}' - Q'\mathbf{v}' \,=\, -\frac{i\eta}{\varkappa_0}\frac{\partial(\mathbf{u}^t)'}{\partial n'}e^{ik_0(\Phi-\Phi_0)}, \tag{81}$$

where $P_0\mathbf{u} = \cos\theta\mathbf{u} + i\eta(\overline{\varkappa})\mathbf{u}$, $P_1'\mathbf{u}' = -\frac{\partial\Phi}{\partial n'}\mathbf{u}' + i\eta(\overline{\varkappa})'\mathbf{u}'$, and $\partial\Phi/\partial n'$ can be deduced from Eq. (29).

If functions $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$ are continuous in G, and the set of boundary conditions can be combined only from equalities in Eqs. (80), (81). If on the line $n =$ const at least one of these function has a discontinuity, then $\mathbf{E}_\top$ and $\mathbf{H}_\top$ must be continuous on this line. Therefore, we get an additional boundary condition $\mathbf{u}^+ = \mathbf{u}^-$, $\mathbf{v}^+ = \mathbf{v}^-$, where indices "+" and "−" mean the limiting values on the discontinuity line from both sides.

Let us note that we used Eq. (36) to derive equalities Eqs. (80) and (81). Alternatively, we could have derived the equivalent boundary conditions from Eq. (37).

The set of Equations (41)–(44) with the boundary conditions (80)–(81) is the starting point in our analysis of and solution to the diffraction problem.

As we show in Appendix B, the boundary conditions for the leading asymptotic term are given by the following relations on S_1:

$$\cos\theta\mathbf{u}^+_{00} - \mathbf{T}(1;-1)\mathbf{v}^+_{00} = 2\mathbf{u}^i\cos\theta, \tag{82}$$

$$\cos\theta(\mathbf{u}^+_{p0} + \mathbf{u}^-_{p0}) - \mathbf{T}(1;-1)(\mathbf{v}^+_{p0} + \mathbf{v}^-_{p0}) = 0, \quad p \ge 1, \tag{83}$$

and on S_2:

$$\cos\theta\mathbf{u}_{2p+\frac{1}{2},0} + \mathbf{T}(1;-1)\mathbf{v}_{2p+\frac{1}{2},0} = 0, \quad (p = 0, 1, 2, \dots). \tag{84}$$

where $\mathbf{v}_{2p+\frac{1}{2}} = \mathbf{v}^+_p + \mathbf{v}^-_{p+1}$, $\mathbf{u}_{2p+\frac{1}{2}} = \mathbf{u}^+_p + \mathbf{u}^-_{p+1}$.

7. ASYMPTOTIC SOLUTION OF THE PROBLEM

7.1. The Main Asymptotic Approximation

Because λ_p possesses two values $\lambda^\pm_p = \pm h\sqrt{\varepsilon\mu\rho}$, using Eqs. (69) and (70), we arrive at two systems of equations for the determination of the leading asymptotic term of the bivector $\overset{\Rightarrow}{\mathbf{W}}$

$$\mathbf{L}^\pm_p \overset{\Rightarrow}{\mathbf{W}}{}^\pm_{p0} \,=\, 0, \tag{85}$$

$$\mathbf{L}^\pm_p \overset{\Rightarrow}{\mathbf{W}}{}^\pm_{p1} \,=\, \frac{\partial \overset{\Rightarrow}{\mathbf{W}}{}^\pm_{p0}}{\partial\nu} - \mathbf{B}_{p0}\overset{\Rightarrow}{\mathbf{W}}{}^\pm_{p0}, \tag{86}$$

where the operator $\mathbf{L}_p$ is described by Eq (68).

Systems of Eqs. (85) and (86) determine two families of bivectors $\overset{\Rightarrow}{\mathbf{W}}{}^\pm_{p0} = \begin{pmatrix} \mathbf{u}^\pm_{p0} \\ \mathbf{v}^\pm_{p0} \end{pmatrix}$, where $\overset{\Rightarrow}{\mathbf{W}}{}^+_{p0}$ and $\overset{\Rightarrow}{\mathbf{W}}{}^-_{p0}$ are forward and backward waves, respectively.

As a preliminary, from the second equation of the system (85)

$$-i\varepsilon h\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)\mathbf{u}_{p0}^{\pm}-i\lambda_p^{\pm}\mathbf{v}_{p0}^{\pm}=0, \tag{87}$$

we express $\mathbf{v}_{p0}^{\pm}$ in terms of $\mathbf{u}_{p0}^{\pm}$.

$$\mathbf{v}_{p0}^{\pm}=-\frac{\varepsilon h}{\lambda_p^{\pm}}\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)\mathbf{u}_{p0}^{\pm}=\mp\frac{\Omega}{\rho}\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)\mathbf{u}_{p0}^{\pm}, \tag{88}$$

where $\Omega=\sqrt{\frac{\varepsilon\rho}{\mu}}$.

Now we proceed to find $\mathbf{u}_{p0}^{\pm}$. As we show in Section A.4, the equation $\mathbf{L}_p^{\pm}\,\overset{\Rightarrow}{\mathbf{W}}=\overset{\Rightarrow}{f}$, with $\overset{\Rightarrow}{f}=\left\{\begin{matrix}\mathbf{f}_1\\ \mathbf{f}_2\end{matrix}\right\}$, is solvable only if

$$\mathbf{f}_1=\frac{\mu h}{\lambda_p}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\mathbf{f}_2. \tag{89}$$

Thus, we apply the condition (89) to the right-hand side of Eq. (86) and get

$$\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}-\frac{i\alpha h}{\varepsilon}\mathbf{Y}_{\top p}^{(1)}\mathbf{v}_{p0}^{\pm}=\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\left[\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}-\frac{i\alpha h}{\mu}\mathbf{Y}_{\perp p}^{(1)}\mathbf{u}_{p0}^{\pm}\right]. \tag{90}$$

Using Eq. (88), we write the equality in Eq. (90) as follows

$$\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}-\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}=-\frac{i\alpha h^2}{\lambda_p^{\pm}}\left[\mathbf{Y}_{\top p}^{(1)}\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)+\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\mathbf{Y}_{\perp p}^{(1)}\right]\mathbf{u}_{p0}^{\pm}. \tag{91}$$

Taking into account the qualities of the operator $\mathbf{T}$ and the form of the operators $\mathbf{Y}_{\top p}^{(1)}$ and $\mathbf{Y}_{\perp p}^{(1)}$, it is not hard to show (see Section A5, Eq. (A13)) that for any tangential vector $\mathbf{a}$, holds

$$\left[\mathbf{Y}_{\top p}^{(1)}\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)+\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\mathbf{Y}_{\perp p}^{(1)}\right]\mathbf{a}=2\left(\mathbf{l}_{\top}^{i}\cdot\mathbf{s}_p\right)\mathbf{a}. \tag{92}$$

Therefore, Eq. (91) takes the form

$$\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}-\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}=-\frac{2i\alpha h^2}{\lambda_p^{\pm}}\left(\mathbf{l}_{\top}^{i}\cdot\mathbf{s}_p^{\pm}\right)\mathbf{u}_{p0}^{\pm}. \tag{93}$$

Equation (93) is obtained from the consistency condition of the system (85) and is obviously not sufficient to find $\mathbf{u}_{p0}^{\pm}$. To eliminate the term $\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}$ from Eq. (93), we differentiate the first equation of the system (85) by the convention in Eq. (64) and get

$$\mathbf{u}_{p0}^{\pm}=-\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\mathbf{v}_{p0}^{\pm}. \tag{94}$$

Taking into account of Eq. (60), we obtain

$$-\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}=\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}+h\mathbf{T}\left(\frac{\partial}{\partial\nu}\frac{\mu}{\lambda_p^{\pm}};-\frac{\partial}{\partial\nu}\frac{1}{\varepsilon\lambda_p^{\pm}}\right)\mathbf{v}_{p0}^{\pm}, \tag{95}$$

or, by Eq. (88),

$$-\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1;-\frac{1}{\varepsilon\mu}\right)\frac{\partial\mathbf{v}_{p0}^{\pm}}{\partial\nu}=\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}-\frac{\varepsilon h^2}{\lambda_p^{\pm}}\mathbf{T}\left(\frac{\partial}{\partial\nu}\frac{\mu}{\lambda_p^{\pm}};-\frac{\partial}{\partial\nu}\frac{1}{\varepsilon\lambda_p^{\pm}}\right)\mathbf{T}\left(\rho;\frac{1}{\varepsilon\mu}\right)\mathbf{u}_{p0}^{\pm}. \tag{96}$$

Recasting the superposition of operators $\mathbf{T}$ by the rule in Eq. (63) and substituting the resulting expression to Eq. (93), we get finally the equation for finding $\mathbf{u}_{p0}^{\pm}$

$$2\frac{\partial\mathbf{u}_{p0}^{\pm}}{\partial\nu}=\mathbf{T}(a,b)\mathbf{u}_{p0}^{\pm}, \tag{97}$$

where

$$a=-\frac{\partial\ln\Omega}{\partial\nu}-2\frac{i\alpha h^2}{\lambda_p^\pm}(\mathbf{l}_\top^i\cdot\mathbf{s}_p^\pm),\quad b=-\frac{1}{\rho}\frac{\partial}{\partial\nu}\frac{1}{\varepsilon\mu}.^+$$

Seeking the solution of this equation in the form $\mathbf{u}_{p0}^\pm=A_p^\pm\mathbf{w}$, we arrive at the equation

$$2\frac{\partial A_p^\pm}{\partial\nu}\mathbf{w}+2A_p^\pm\frac{\partial\mathbf{w}}{\partial\nu}=\mathbf{T}(aA_p^\pm;bA_p^\pm)\mathbf{w}. \tag{99}$$

We set $2\frac{\partial A_p^\pm}{\partial\nu}=aA_p^\pm$, then $A_p^\pm$ can be written up to a constant in the form

$$\begin{aligned}A_p^+(\nu)&=\frac{1}{\sqrt{\Omega}}e^{-i\alpha h\int\limits_0^\nu(\mathbf{l}_\top^i\cdot\mathbf{s}_p^+)\frac{d\nu}{\sqrt{\varepsilon\mu\rho}}},\\A_p^-(\nu)&=\frac{1}{\sqrt{\Omega}}e^{-i\alpha h\int\limits_\nu^1(\mathbf{l}_\top^i\cdot\mathbf{s}_p^-)\frac{d\nu}{\sqrt{\varepsilon\mu\rho}}}.\end{aligned} \tag{100}$$

From Eq. (99), we obtain the equation for determination of $\mathbf{w}$

$$2\frac{\partial\mathbf{w}}{\partial\nu}=b\mathbf{l}_\top^i(\mathbf{l}_\top^i\cdot\mathbf{w}). \tag{101}$$

Decomposing $\mathbf{w}$ in the vector basis $(\hat{\boldsymbol{\tau}},\hat{\boldsymbol{\tau}}_\perp)$: $\mathbf{w}=w_\tau\hat{\boldsymbol{\tau}}+w_{\tau_\perp}\hat{\boldsymbol{\tau}}_\perp$, and taking into account that $|\mathbf{l}_\top^i|=|\mathbf{l}_\perp^i|=\sin\theta$ and $\hat{\boldsymbol{\tau}}=\frac{\mathbf{l}_\top^i}{|\mathbf{l}_\top^i|}$, we get from Eq. (101) two equations

$$\frac{\partial w_\tau}{\partial\nu}=\frac{b\sin^2\theta}{2}w_\tau,\quad\frac{\partial w_{\tau_\perp}}{\partial\nu}=0. \tag{102}$$

whose solutions are $w_\tau=c_\tau B_p^\pm(\nu)$ and $w_{\tau_\perp}=c_{\tau_\perp}$, with $c_\tau=\text{const}$, $c_{\tau_\perp}=\text{const}$, and

$$B_p^+(\nu)=e^{\frac{\sin^2\theta}{2}\int\limits_0^\nu\frac{\partial}{\partial\nu}\left(\frac{1}{\varepsilon\mu}\right)\frac{d\nu}{\rho}},\quad B_p^-(\nu)=e^{\frac{\sin^2\theta}{2}\int\limits_\nu^1\frac{\partial}{\partial\nu}\left(\frac{1}{\varepsilon\mu}\right)\frac{d\nu}{\rho}}. \tag{103}$$

Consequently,

$$\mathbf{w}=B_p^\pm(\mathbf{C}_{p0}^\pm\cdot\hat{\boldsymbol{\tau}})\hat{\boldsymbol{\tau}}+(\mathbf{C}_{p0}^\pm\cdot\hat{\boldsymbol{\tau}}_\perp)\hat{\boldsymbol{\tau}}_\perp=B_p^\pm(\mathbf{C}_{p0}^\pm\cdot\hat{\boldsymbol{\tau}})\hat{\boldsymbol{\tau}}+\mathbf{C}_{p0}^\pm-(\mathbf{C}_{p0}^\pm\cdot\hat{\boldsymbol{\tau}})\hat{\boldsymbol{\tau}}=\mathbf{T}\left(1;\frac{B_p^\pm-1}{\sin^2\theta}\right)\mathbf{C}_{p0}^\pm, \tag{104}$$

where $\mathbf{C}_{p0}^\pm$ is some constant vector.

Thus, we can write finally the solution of Eq. (97) in the form

$$\mathbf{u}_{p0}^\pm=A_p^\pm\mathbf{T}\left(1;\frac{B_p^\pm-1}{\sin^2\theta}\right)\mathbf{C}_{p0}^\pm, \tag{105}$$

where functions $A_p^\pm(\nu)$ and $B_p^\pm(\nu)$ are determined by Eqs. (100) and (103). The vector $\mathbf{v}_{p0}$ can be expressed by Eq. (88).

It is easy to see that $B^+(0)\equiv B^-(1)\equiv 1$, and $B^+(1)\equiv B^-(0)$, and $B_p^\pm$ can be easily derived in a closed form.

$^+$ Let us derive, for instance, b. Using (71) and the quality of the operator $\mathbf{T}$ (63), we get

$$b=\frac{\varepsilon h^2}{\lambda_p^\pm}\left[\frac{1}{\varepsilon\mu}\frac{\partial}{\partial\nu}\left(\frac{\mu}{\lambda_p^\pm}\right)-\frac{\partial}{\partial\nu}\left(\frac{1}{\varepsilon\lambda_p^\pm}\right)\right]=\frac{\varepsilon h^2}{\lambda_p^\pm}\left[\frac{1}{\varepsilon}\frac{\partial}{\partial\nu}\frac{1}{\lambda_p^\pm}+\frac{1}{\varepsilon\mu\lambda_p^\pm}\frac{\partial\mu}{\partial\nu}-\frac{1}{\varepsilon}\frac{\partial}{\partial\nu}\frac{1}{\lambda_p^\pm}-\frac{1}{\lambda_p^\pm}\frac{\partial}{\partial\nu}\frac{1}{\varepsilon}\right]=\frac{1}{\rho}\left(\frac{1}{\varepsilon\mu^2}\frac{\partial\mu}{\partial\nu}-\frac{1}{\mu}\frac{\partial}{\partial\nu}\frac{1}{\varepsilon}\right)=-\frac{1}{\rho}\frac{\partial}{\partial\nu}\frac{1}{\varepsilon\mu}$$

7.2. Determination of $\mathbf{C}_{p0}^{\pm}$

The vectors $\mathbf{C}_{p0}^{\pm}$ can be determined from the boundary conditions for the leading asymptotic term of Eqs. (B21), (B22), and (B35) on S_1 ($\nu = 0$)

$$\cos\theta \mathbf{u}_{00}^{+}(0) - \mathbf{T}(1;-1)\mathbf{v}_{00}^{+} = 2\mathbf{u}_0^i \cos\theta, \quad (p = 0), \tag{106}$$

$$\cos\theta[\mathbf{u}_{p0}^{+}(0) + \mathbf{u}_{p0}^{-}(0)] - \mathbf{T}(1;-1)[\mathbf{v}_{p0}^{+}(0) + \mathbf{v}_{p0}^{-}(0)] = 0, \quad (p = 1, 2, \ldots), \tag{107}$$

and on S_2 ($\nu = 1$), for $p = 0, 1, 2, \ldots$

$$\cos\theta[\mathbf{u}_{p0}^{+}(1) + \mathbf{u}_{p+1,0}^{-}(1)] - \mathbf{T}(1;-1)[\mathbf{v}_{p0}^{+}(1) + \mathbf{v}_{p+1,0}^{-}(1)] = 0. \tag{108}$$

From Eq. (105), follows $\mathbf{u}_{00}^{+}\big|_{\nu=0} = \frac{1}{\sqrt{\Omega(0)}}\mathbf{C}_{00}^{+}$, therefore, using Eq. (88), we obtain from Eq. (106) the equation for finding $\mathbf{C}_{00}^{+}$

$$\cos\theta \mathbf{C}_{00}^{+} + \frac{\varepsilon h}{\lambda_p^{+}}\mathbf{T}(1;-1)\mathbf{T}\left(\rho_0; \frac{1}{\varepsilon_0\mu_0}\right)\mathbf{C}_{00}^{+} = 2\mathbf{u}_0^i\sqrt{\Omega_0}\cos\theta, \tag{109}$$

where $\Omega_0 = \Omega(0)$, $\quad \rho_0 = \rho(0)$.

The superposition of operators $\mathbf{T}$ in the left-hand part of Eq. (109) can be recast by Eq. (63):

$$\cos\theta \mathbf{C}_{00}^{+} + \frac{\Omega_0}{\rho_0}\mathbf{T}\left\{\rho_0; \frac{1}{\varepsilon_0\mu_0} - 1\right\}\mathbf{C}_{00}^{+} = 2\mathbf{u}_0^i\sqrt{\Omega_0}\cos\theta, \tag{110}$$

that is,

$$\mathbf{T}\left\{\Omega_0 + \cos\theta; \frac{\Omega_0}{\rho_0}\left(\frac{1}{\varepsilon_0\mu_0} - 1\right)\right\}\mathbf{C}_{00}^{+} = 2\mathbf{u}_0^i\sqrt{\Omega_0}\cos\theta. \tag{111}$$

By applying the operator $\mathbf{T}^{-1}$ defined by Eq. (65) to the right and left sides of Eq. (111), we get

$$\mathbf{C}_{00}^{+} = 2\cos\theta \mathbf{T}\{\alpha_0(0), \beta_0(0)\}\mathbf{u}_0^i, \tag{112}$$

where

$$\alpha_0(\nu) = \frac{\sqrt{\Omega(\nu)}}{\Omega(\nu) + \cos\theta}, \quad \beta_0(\nu) = -\frac{\alpha_0(\nu)\left(\dfrac{1}{\varepsilon(\nu)\mu(\nu)} - 1\right)}{\left(\dfrac{\rho(\nu)}{\Omega(\nu)} + \cos\theta\right)\cos\theta}. \tag{113}$$

Likewise, from Eqs. (108) and (107), we deduce a recurrent formula to find $\mathbf{C}_{p0}^{\pm}$ $\quad (p = 1, 2, \ldots)$.

$$\mathbf{C}_{p+1,0}^{-} = \mathbf{T}\{a_p^{+}(1), b_p^{+}(1)\}\mathbf{C}_{p0}^{+}, \quad (p = 0, 1, 2, \ldots), \tag{114}$$

$$\mathbf{C}_{p0}^{+} = \mathbf{T}\{a_p^{-}(0), b_p^{-}(0)\}\mathbf{C}_{p0}^{-}, \quad (p = 1, 2, \ldots), \tag{115}$$

where

$$a_p^{\pm}(\nu) = \alpha_0(\nu)\alpha_p^{\pm}(\nu), \tag{116}$$

$$b_p^{\pm}(\nu) = -\alpha_0(\nu)\beta_p^{\pm}(\nu) + \beta_0(\nu)\alpha_p^{\pm}(\nu) - \beta_0(\nu)\beta_p^{\pm}(\nu)\sin^2\theta, \tag{117}$$

$$\alpha_p^{\pm}(\nu) = A_p^{\pm}(\nu)(\Omega - \cos\theta), \tag{118}$$

$$\beta_p^{\pm}(\nu) = A_p^{\pm}\left[\cos\theta\frac{B_p^{\pm} - 1}{\sin^2\theta} - \frac{\Omega}{\rho}\left(\frac{B_p^{\pm} - \rho}{\sin^2\theta} - B_p^{\pm}\right)\right], \tag{119}$$

$\alpha_0(\nu)$ and $\beta_0(\nu)$ are determined by Eq. (113).

From a physics perspective, $\mathbf{C}_{00}^{-} \equiv 0$, meanwhile $\mathbf{C}_{00}^{+}$ is given by Eq. (112). Then, the cyclic use of Eqs. (114) and (115) allows all the successive values of $\mathbf{C}_{p0}^{\pm}$ to be found for any p.

8. APPROXIMATION OF THE STRATIFIED MEDIUM WITH A CONTINUOUS FUNCTION

In our consideration, ε and μ are assumed to be continuous. It allows us not to introduce additional boundary conditions at the medium interfaces. For the stratified media, one can use certain continuous approximation of the dielectric permittivity. Let

$$\varepsilon(\nu) = \begin{cases} \varepsilon_1, & 0 \le \nu \le \nu_1 - \Delta, \\ \varepsilon_k, & \nu_{k-1} + \Delta \le \nu \le \nu_1 - \Delta, \quad k > 1, \\ P_k(\nu), & \nu_k - \Delta \le \nu \le \nu_k + \Delta, \quad k \ge 1 \end{cases} \tag{120}$$

where $\nu_k \equiv \text{const}$ are interfaces between mediums with ε_k and ε_{k+1}, and

$$P(\nu) = \varepsilon_k (1-t)^2 (1+2t) + \varepsilon_{k+1} t^2 (3-2t), \tag{121}$$

with $t = (\nu - \nu_k + \Delta)/2\Delta$.

Such parametrization is smooth; the derivative $\varepsilon'(\nu)$ is continuous; $\varepsilon'(\nu_k \pm \Delta) \equiv 0$ for any k. By decreasing the value of Δ, we may approximate the piece-wise constant dielectric permittivity as close, as we wish (see Fig. 3(b)).

9. NUMERICAL VERIFICATION OF THE METHOD

For the illustration and verification of the method proposed, consider the following canonical 2D problem. Let a tapered plane E-polarized wave with $u^i = E_z^i = \cos\frac{\pi A}{2}$ be incident on a circular dielectric layer, as in Fig. 3(a). We compute only the transmitted field without taking into account secondary reflections from S_1 to the region G_1⋆ with the aid of formulas (105), (114)–(115) (red line in Figs. 3–8) and compare the field value at the surface S_2 with the numerical solutions of integral equations (green line). For a stratified layer, as in Fig. 9(c), we employ the volume integral equation, and same as in [23], the Müller boundary integral equations for homogeneous layers ($\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, Figs. 4–8). The oscillations in the exact numerical solution curves are hard to track with our method, since they are caused not solely by multiple reflections from S_1, but also by the waveguiding effect and the diffraction from wedges, which is not taken into account by our asymptotic theory. Still, in all the cases considered, the "asymptotic" curve represents an envelope of the highly-oscillating curves. As it can be seen from plots, the level of oscillations decreases for bigger sizes of the dielectric layer and for smaller values of dielectric permittivity. These oscillations almost vanish, if to introduce small losses to the dielectric permittivity, as in Fig. 8.

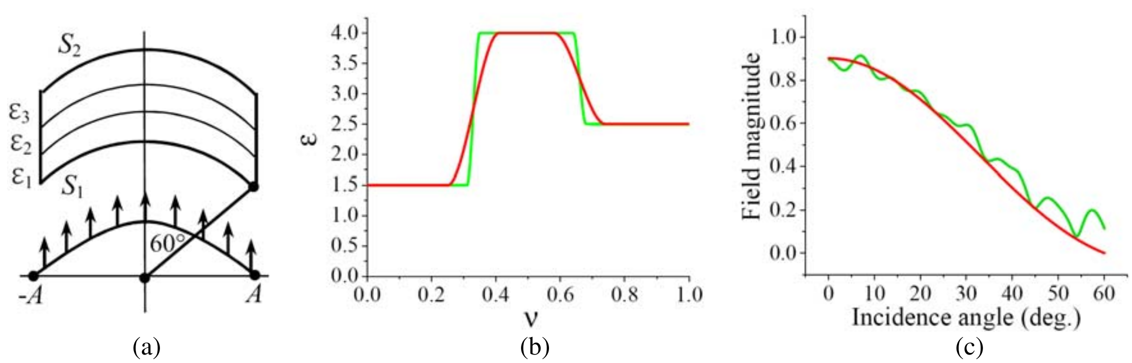
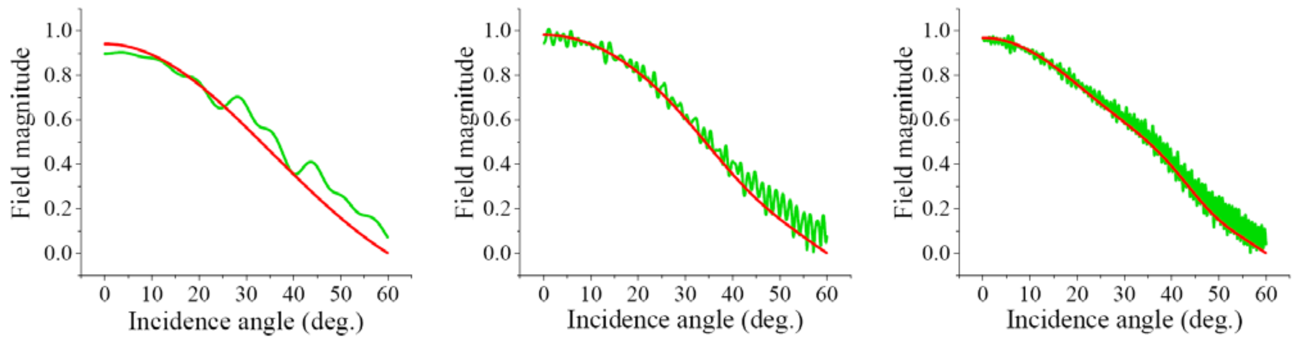


Figure 3. The geometry of a canonical problem (a), and approximation of the piecewise constant ε ($\varepsilon_1 = 1.5$, $\varepsilon_2 = 4$, and $\varepsilon_3 = 2.5$; $\nu_1 = 0.33$ and $\nu_2 = 0.66$) by a continuous function with $\Delta = 0.02$ (green line) and $\Delta = 0.08$ (red line) (b), and the field magnitude on S_2 for a circular layer with $R_0 = 5\lambda$, $\delta = 0.45\lambda$, and $\varepsilon_1 = 3$, $\varepsilon_2 = 4$ and $\varepsilon_3 = 5$: asymptotic (red line) and integral equation solutions (green line).

⋆ These reflections are negligible in this particular problem.

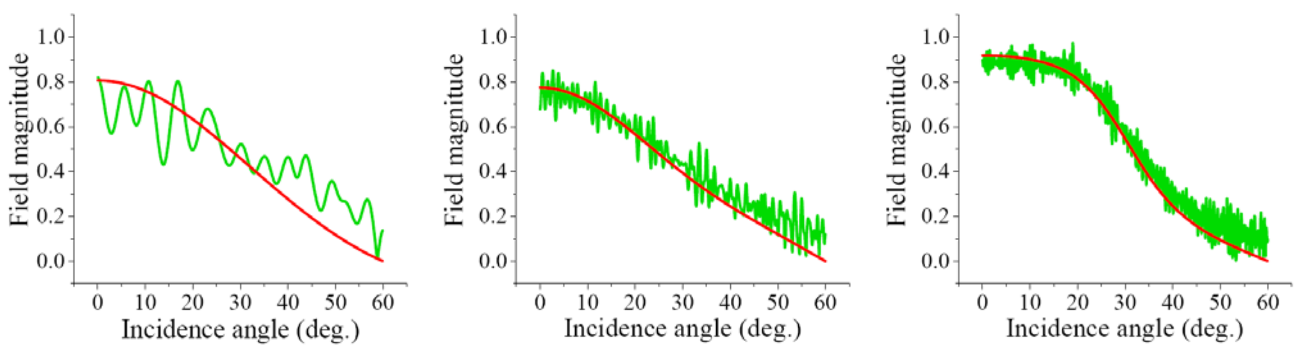
The relation between the curvature of the layer and its thickness can be efficiently controlled by the parameter α, as it is shown in Fig. 7.

Summarizing the calculations, we may say that the asymptotic technique offered in this paper provides valid results for dielectric permittivities up to $\varepsilon \sim 10$ and for the layer thickness from 0.5λ to 10λ and more.



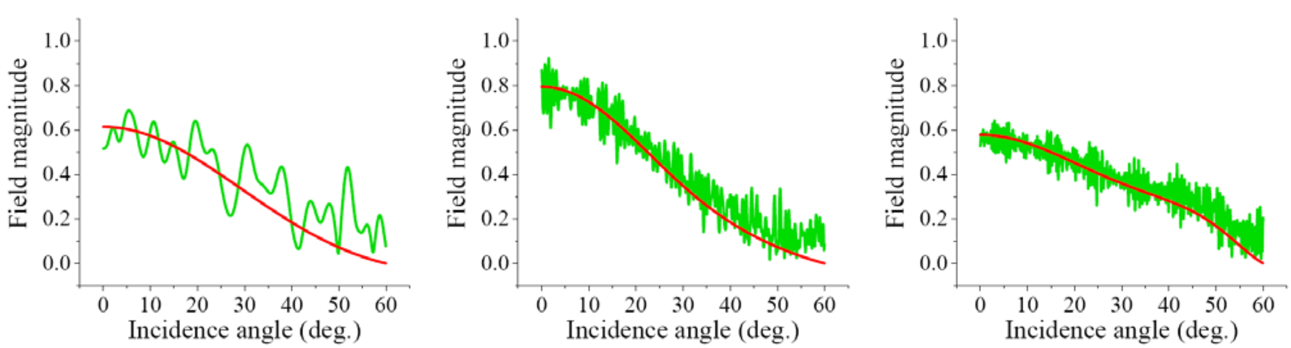
(a) (b) (c)

Figure 4. The field magnitude on S_2 for a circular layer with $\varepsilon = 2$, and (a) $R = 5\lambda$ and $\delta = 0.892\lambda$ ($\zeta = 5.6$, $\xi = 0.178$), (b) $R = 30\lambda$ and $\delta = 2.185\lambda$ ($\zeta = 13.729$, $\xi = 0.073$), and (c) $R = 60\lambda$ and $\delta = 3.09\lambda$ ($\zeta = 19.42$, $\xi = 0.052$): asymptotic solution (red line) vs. integral equations (green line). In all the cases considered $\alpha = 1$.



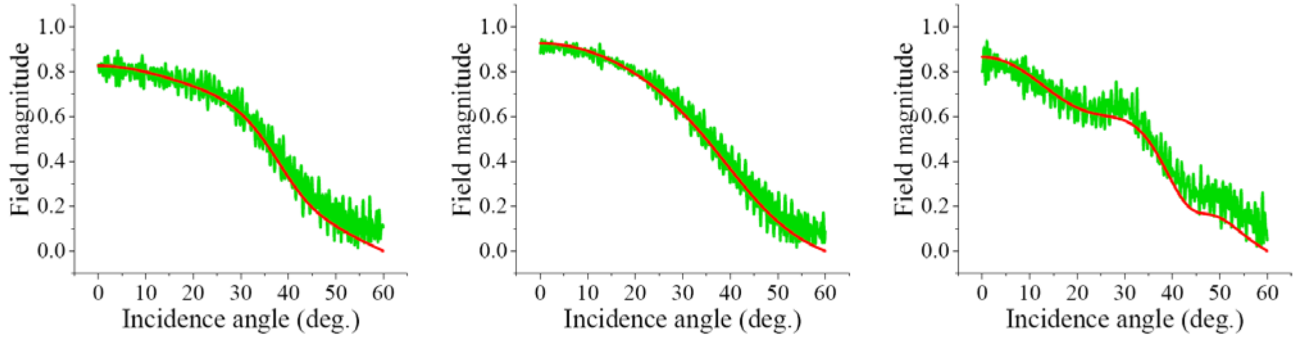
(a) (b) (c)

Figure 5. Same, as in Fig. 4, but for $\varepsilon = 6$.



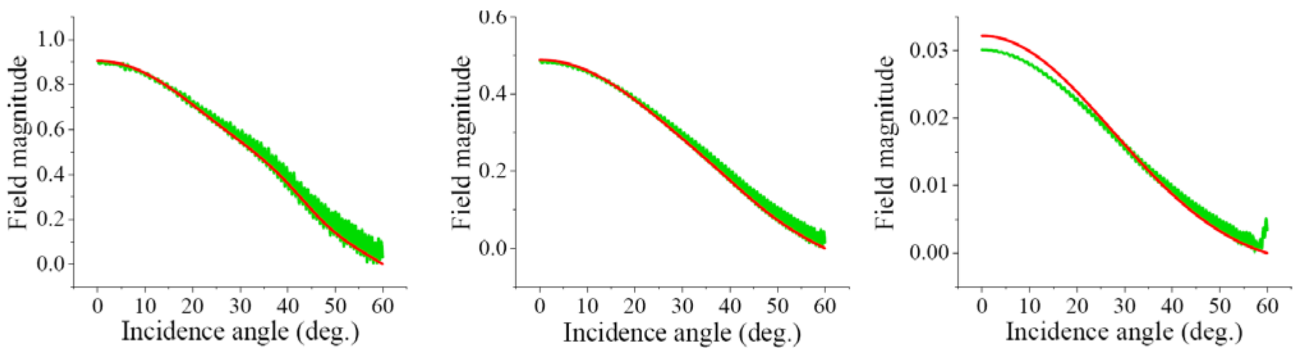
(a) (b) (c)

Figure 6. Same, as in Fig. 4, but for $\varepsilon = 10$.



(a) (b) (c)

Figure 7. The field magnitude on S_2 for a circular layer with $\varepsilon = 4$, $R = 60\lambda$, and (a) $\delta = 3.09\lambda$ ($\zeta = 19.42$, $\xi = 0.052$; $\alpha = 1$), (b) $\delta = 1.545\lambda$ ($\zeta = 9.708$, $\xi = 0.026$; $\alpha = 0.25$), and (c) $\delta = 6.18\lambda$ ($\zeta = 38.83$, $\xi = 0.103$; $\alpha = 4$): asymptotic solution (red line) vs. integral equations (green line).



(a) (b) (c)

Figure 8. Same, as in Fig. 7 (a), but for (a) $\varepsilon = 2 + 0.01i$, (b) $\varepsilon = 2 + 0.1i$, and (c) $\varepsilon = 2 + 0.5i$.

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APPENDIX A. DERIVATION OF FORMAL ASYMPTOTIC EXPANSIONS OF OPERATORS

A.1. Matrix Operator $(\varkappa)$ and Operator D

Operators $(\varkappa)$ and $\mathbf{D}$ given by Eqs. (17) and (11) take, by transition to the dimensionless coordinate $\nu = \frac{n}{\delta h(\sigma_1,\sigma_2)}$, the following view:

$$(\varkappa) = \begin{pmatrix} \dfrac{\varkappa_1}{1+\overline{\varkappa_1}\nu h\xi} & 0 \\ 0 & \dfrac{\varkappa_2}{1+\overline{\varkappa_2}\nu h\xi} \end{pmatrix}, \tag{A1}$$

$$\mathbf{D} = \begin{pmatrix} \dfrac{1}{1+\overline{\varkappa_1}\nu h\xi} & 0 \\ 0 & \dfrac{1}{1+\overline{\varkappa_2}\nu h\xi} \end{pmatrix} \mathbf{D}|_{\nu=0}, \tag{A2}$$

where $\overline{\varkappa}_j = \varkappa_j/\varkappa_0$ for $j = 1, 2$. Then, $(\varkappa)$ and $(\varkappa_\perp)$ can be expanded in series with respect to the small parameter ξ

$$(\overline{\varkappa}) = \frac{1}{\varkappa_0}(\varkappa) = (\overline{\varkappa})^0 - \nu h\xi[(\overline{\varkappa})^0]^2 + \dots \tag{A3}$$

$$\mathbf{D} = \left\{1 - (\overline{\varkappa})^0 \nu h\xi + +o(\xi)\right\} \mathbf{D}|_{\nu=0}. \tag{A4}$$

with $(\overline{\varkappa})^0 = (\overline{\varkappa})|_{n=0}$ and $(\overline{\varkappa}_\perp)^0 = (\overline{\varkappa}_\perp)|_{n=0}$.

Since $\mathbf{l}_\top = \mathbf{D}\Phi_0 = \mathbf{D}\Phi(\sigma, 0) = \mathbf{l}^i_\top(\sigma, 0)$, then, on account of Eq. (A4), we obtain

$$\mathbf{l}_\top = \mathbf{l}^i_\top - \nu h\xi\widetilde{\mathbf{l}}^i_\top + o(\xi), \tag{A5}$$

where $\widetilde{\mathbf{l}}^i_\top = (\overline{\varkappa})^0\mathbf{l}^i_\top$.

A.2. Operators $\mathbf{Z}_\top$ and $\mathbf{Y}_\top$

From Eq. (A4), we get

$$\frac{1}{ik_0}\mathbf{D} = -i\left\{\eta\mathbf{D}^0 - (\overline{\varkappa})^0\nu h\xi\eta\mathbf{D}^0 + \dots\right\}, \tag{A6}$$

where $\mathbf{D}^0 = \frac{1}{\varkappa_0}\mathbf{D}|_{\nu=0}$. We assume the following relations between the orders of magnitude of small parameters of the problem: $\zeta = \alpha/\xi(\alpha \sim 1)$ $(\xi = \alpha/\zeta, \quad \eta = \alpha/\zeta^2)$. Taking into account Eqs. (A5) and (A6), we may write the operator $\mathbf{Z}_\top = \mathbf{l}_\top + \frac{1}{ik_0}\mathbf{D}$ in the form of power series in inverse orders of parameter ζ

$$\mathbf{Z}_\top \sim \mathbf{l}^i_\top - \frac{\nu h}{\zeta}\widetilde{\mathbf{l}}^i_\top. \tag{A7}$$

Applying $\frac{1}{\varepsilon}\mathbf{Z}_\top$ to a harmonics $e^{i\Lambda_p\zeta}\mathbf{v}_p$ and rearranging the terms, we get

$$\frac{1}{\varepsilon}\mathbf{Z}_\top(e^{i\Lambda_p\zeta}\mathbf{v}_p) \sim \frac{1}{\varepsilon}e^{i\Lambda_p\zeta}\left\{(\mathbf{l}^i_\top\cdot\mathbf{v}_p) + \frac{1}{\zeta}(\mathbf{s}_p\cdot\mathbf{v}_p)\right\}, \tag{A8}$$

where $\mathbf{s}_p = \mathbf{D}^0\Lambda_p - h\nu\widetilde{\mathbf{l}}^i_\top$. Let us now apply the operator $\varepsilon(e^{-i\Lambda_p\zeta})\mathbf{Z}_\top$ to the expression obtained and separate the terms of expansion with ζ^0:

$$(\mathbf{l}^i_\top\cdot\mathbf{v}_p)\mathbf{l}^i_\top,$$

and with ζ^{-1}:

$$(\mathbf{s}_p\cdot\mathbf{v}_p)\mathbf{l}^i_\top + (\mathbf{l}^i_\top\cdot\mathbf{v}_p)\mathbf{D}^0\Lambda_p - \frac{\nu h}{\zeta}\widetilde{\mathbf{l}}^i_\top(\mathbf{l}^i_\top\cdot\mathbf{v}_p) = (\mathbf{s}_p\cdot\mathbf{v}_p)\mathbf{l}^i_\top + (\mathbf{l}^i_\top\cdot\mathbf{v}_p)\mathbf{s}_p. \tag{A9}$$

Therefore, rearranging the associated terms, we may write the coefficients in Eqs. (54) and (55) of the asymptotic series for $\mathbf{Y}_{\top p}\mathbf{v}_p$.

A.3. Operators $(\varkappa_\perp)$, $\mathbf{D}_\perp$, $\mathbf{l}_\perp$, $\mathbf{Z}_\perp$, and $\mathbf{Y}_{\perp p}$

These operators can be obtained from formulas (A1)–(A7) by the following replacements:

$$(\varkappa) \to (\varkappa_\perp), \quad (\overline{\varkappa})^0 \to (\overline{\varkappa}_\perp)^0, \quad \mathbf{l}_\top \to \mathbf{l}_\perp, \quad \mathbf{l}^i_\top \to \mathbf{l}^i_\perp,$$

$$\mathbf{D} \to \mathbf{D}_\perp, \quad \mathbf{D}^0 \to \mathbf{D}^0_\perp = \frac{1}{\varkappa_0}\mathbf{D}_\perp|_{\nu=0},$$

$$\mathbf{s}_p \to \mathbf{s}_{p\perp} = \mathbf{D}^0_\perp\Lambda_p - h\nu(\overline{\varkappa}_\perp)^0\mathbf{l}^i_\perp.$$

A.4. Relation between Operators $\mathbf{T}$ and $\mathbf{L}_p^{\pm}$.

Let the operator $\mathbf{L}_p^{\pm}$ be defined by the formula

$$\mathbf{L}_p^{\pm}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}-i\lambda_p^{\pm}\mathbf{u} - ih\mu\mathbf{v} + ih/\varepsilon\mathbf{l}_{\top}^{i}(\mathbf{l}_{\top}^{i}\cdot\mathbf{v})\\ -i\lambda_p^{\pm}\mathbf{v} - ih\varepsilon\mathbf{u} + ih/\mu\mathbf{l}_{\perp}^{i}(\mathbf{l}_{\perp}^{i}\cdot\mathbf{u})\end{pmatrix}, \tag{A10}$$

with $\lambda_p^{\pm} = \pm h\sqrt{\varepsilon\mu\rho}$. Then, the equation $\mathbf{L}_p^{\pm}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{pmatrix}$ is solvable only if

$$\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1; -\frac{1}{\varepsilon\mu}\right)\mathbf{f}_2 = \mathbf{f}_1. \tag{A11}$$

Indeed,

$$\begin{aligned}\frac{\mu h}{\lambda_p^{\pm}}\mathbf{T}\left(1; -\frac{1}{\varepsilon\mu}\right)\mathbf{f}_2 &= -i\mu h\mathbf{v} - \frac{i\mu\varepsilon h^2}{\lambda_p^{\pm}}\mathbf{u} + \frac{ih^2}{\lambda_p^{\pm}}\mathbf{l}_{\perp}^{i}(\mathbf{l}_{\perp}^{i}\cdot\mathbf{u}) + \frac{ih}{\varepsilon}\mathbf{l}_{\top}^{i}(\mathbf{l}_{\top}^{i}\cdot\mathbf{v}) + \frac{ih^2}{\lambda_p^{\pm}}\mathbf{l}_{\top}^{i}(\mathbf{l}_{\top}^{i}\cdot\mathbf{u})\\ &= -i\mu h\mathbf{v} + \frac{ih}{\varepsilon}\mathbf{l}_{\top}^{i}(\mathbf{l}_{\top}^{i}\cdot\mathbf{v}) + \frac{ih^2}{\lambda_p^{\pm}}(|\mathbf{l}_{\top}^{i}|^2 - \mu\varepsilon)\mathbf{u}.\end{aligned} \tag{A12}$$

Since $\rho = 1 - \frac{\sin^2\theta}{\varepsilon\mu}$ and $|\mathbf{l}_{\top}^{i}| = \sin\theta$, we obtain, by Eq. (71), the equality in Eq. (A11).

A.5. Relation between Operators $\mathbf{Y}_{\top p}^{(1)}$, $\mathbf{Y}_{\perp p}^{(1)}$, and $\mathbf{T}$

Let us show that

$$\Upsilon\mathbf{a} \equiv \left[\mathbf{T}\left(1; -\frac{1}{\varepsilon\mu}\right)\mathbf{Y}_{\perp p}^{(1)} + \mathbf{Y}_{\top p}^{(1)}\mathbf{T}_{\perp}\left(1; -\frac{1}{\varepsilon\mu}\right)\right]\mathbf{a} = 2(\mathbf{l}_{\top}^{i}\cdot\mathbf{s}_p)\mathbf{a}, \tag{A13}$$

Here $\mathbf{Y}_{\top p}^{(1)}$ and $\mathbf{Y}_{\perp p}^{(1)}$ are defined by Eq. (55), $\mathbf{s}_p = \mathbf{D}^0\Lambda_p - h\nu(\overline{\varkappa})^0\mathbf{l}_{\top}^{i}$, and the operator $\mathbf{T}_{\perp}(\alpha,\beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{l}_{\perp}^{i}(\mathbf{l}_{\perp}^{i}\cdot\mathbf{a})$. It bears mentioning that $\mathbf{T}_{\perp}\left(1; -\frac{1}{\varepsilon\mu}\right) = \mathbf{T}\left(\rho, \frac{1}{\varepsilon\mu}\right)$.

Consider each term in Eq. (A13) separately:

$$\mathbf{T}\left(1; -\frac{1}{\varepsilon\mu}\right)\mathbf{Y}_{\perp p}^{(1)} = \mathbf{l}_{\perp}^{i}(\mathbf{s}_{p\perp}\cdot\mathbf{a}) + \mathbf{s}_{p\perp}(\mathbf{l}_{\perp}^{i}\cdot\mathbf{a}) - \frac{1}{\varepsilon\mu}\mathbf{l}_{\top}^{i}(\mathbf{l}_{\top}^{i}\cdot\mathbf{s}_{p\perp})(\mathbf{l}_{\perp}^{i}\cdot\mathbf{a}); \tag{A14}$$

$$\mathbf{Y}_{\top p}^{(1)}\mathbf{T}\left(\rho; \frac{1}{\varepsilon\mu}\right) = \mathbf{l}_{\top}^{i}\left[\rho\left(\mathbf{s}_p\cdot\mathbf{a}\right) + \frac{1}{\varepsilon\mu}\left(\mathbf{s}_p\cdot\mathbf{l}_{\top}^{i}\right)\left(\mathbf{l}_{\top}^{i}\cdot\mathbf{a}\right)\right] + \mathbf{s}_p\left[\rho\left(\mathbf{l}_{\top}^{i}\cdot\mathbf{a}\right) + \frac{\sin^2\theta}{\varepsilon\mu}(\mathbf{l}_{\top}^{i}\cot\mathbf{a})\right]. \tag{A15}$$

We represent then vectors $\mathbf{s}_p$, $\mathbf{s}_{p\perp}$, and $\mathbf{a}$ in Eqs. (A14)–(A15) in the basis $(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\tau}}_{\perp})$, where $\hat{\boldsymbol{\tau}} = \frac{\mathbf{l}_{\top}^{i}}{|\mathbf{l}_{\top}^{i}|}$:

$$\mathbf{s}_p = s_1\hat{\boldsymbol{\tau}} + s_2\hat{\boldsymbol{\tau}}_{\perp}, \quad \mathbf{s}_{p\perp} = -s_2\hat{\boldsymbol{\tau}} + s_1\hat{\boldsymbol{\tau}}_{\perp}, \quad \mathbf{a} = a_1\hat{\boldsymbol{\tau}} + a_2\hat{\boldsymbol{\tau}}_{\perp}. \tag{A16}$$

Now, by adding Eqs. (A14) and (A15), we get

$$\begin{aligned}\Upsilon\mathbf{a} &= \sin\theta\hat{\boldsymbol{\tau}}_{\perp}(-a_1s_2 + a_2s_1) + (-s_2\hat{\boldsymbol{\tau}} + s_1\hat{\boldsymbol{\tau}}_{\perp})\sin\theta a_2 + \frac{\sin^3\theta}{\varepsilon\mu}s_2a_2\hat{\boldsymbol{\tau}}\\ &\quad + \sin\theta\left[\rho(a_1s_1 + a_2s_2) + \frac{1}{\varepsilon\mu}\sin^2\theta s_1a_1\right]\hat{\boldsymbol{\tau}} + (s_1\hat{\boldsymbol{\tau}} + s_2\hat{\boldsymbol{\tau}}_{\perp})\left[\rho\sin\theta a_1 + \frac{\sin^3\theta}{\varepsilon\mu}a_1\right]\\ &= 2s_1\sin\theta\mathbf{a} = 2(\mathbf{l}_{\top}^{i}\cdot\mathbf{s}_p)\mathbf{a}.\end{aligned} \tag{A17}$$

APPENDIX B. BOUNDARY CONDITION

B.1. Boundary Condition on S_1

The boundary conditions on S_1 and S_2 in Eqs. (80), (81) contain $\frac{\partial \mathbf{u}^r}{\partial n}\big|_{S_1}$ and $\frac{\partial \mathbf{u}^t}{\partial n}\Big|_{S_2}$. Now we show that the limiting values of these normal derivatives are of order ζ. For this purpose, we derive them in the main asymptotic approximation.

We have assumed above that the field within the layer has the form Eq. (47)

$$\mathbf{u}(\sigma,\nu,\zeta)=\sum_{p=0}^{\infty}e^{i\zeta\Lambda_p}\mathbf{u}_p(\sigma,\nu,\zeta).$$

Here, the functions Λ_{2p} and Λ_{2p-1} are determined by Eq. (75); moreover, they possess the following values:

on S_1 ($\nu=0$) $\Lambda_{2p}(\sigma,0)=\Lambda_{2p-1}(\sigma,0)=2phM$,

on S_2 ($\nu=1$) $\Lambda_{2p}(\sigma,1)=\Lambda_{2p+1}(\sigma,1)=(2p+1)hM$, with $M=\int\limits_0^1\sqrt{\varepsilon\mu\rho}\,d\nu$.

Bearing this in mind, we can write the expressions for $\mathbf{u}(\sigma,\nu,\zeta)$ at $\nu=0$ and at $\nu=1$ as

$$\mathbf{u}(\sigma,\nu,\zeta)|_{\nu=0}\;=\;\mathbf{u}^i|_{\nu=0}+\sum_{p=1}^{\infty}e^{2ip\zeta hM}(\mathbf{u}_p^{+}+\mathbf{u}_p^{-})|_{\nu=0}, \tag{B1}$$

$$\mathbf{u}(\sigma,\nu,\zeta)|_{\nu=1}\;=\;\sum_{p=0}^{\infty}e^{i(2p+1)\zeta hM}(\mathbf{u}_p^{+}+\mathbf{u}_{p+1}^{-})|_{\nu=1}. \tag{B2}$$

The functions $\mathbf{v}(\sigma,\nu,\zeta)$ have the identical structure on boundaries.

The field in G_1 can be represented as a sum $\widetilde{\mathbf{u}}^i+\widetilde{\mathbf{u}}^r$, where $\widetilde{\mathbf{u}}^i$ is the incident wave and $\widetilde{\mathbf{u}}^r=e^{ik_0\Psi(\mathbf{x})}\mathbf{u}^r(\mathbf{x},\zeta)$ is the reflected wave which meet the boundary condition at $\nu=0$

$$\widetilde{\mathbf{u}}^i+\widetilde{\mathbf{u}}^r=\widetilde{\mathbf{u}}. \tag{B3}$$

Due to Eqs. (6), (47), and (B3), it is logical to search $\widetilde{\mathbf{u}}^r(\mathbf{x})$ in the form

$$\widetilde{\mathbf{u}}^r=e^{ik_0\Psi(\mathbf{x})}\sum_{p=0}^{\infty}e^{i\zeta\psi_p(\sigma,n)}\mathbf{u}_p^r(\mathbf{x},\zeta)\equiv e^{ik_0\Psi(\mathbf{x})}\mathbf{u}^r, \tag{B4}$$

where $k_0\Psi(\mathbf{x})$ is a GO phase of a wave reflected from S_1[♯] possessing the well-known properties

$$\Psi(\mathbf{x})|_{S_1}=\Phi(\sigma),\qquad \frac{\partial\Psi}{\partial n}\bigg|_{S_1}=-\frac{\partial\Phi}{\partial n}, \tag{B5}$$

and the functions ψ_p meet the conditions

$$\psi_p|_{S_1}=2phM. \tag{B6}$$

For the evaluation of $\mathbf{u}^r$ in the boundary conditions, we need to know the value of the derivative $\frac{\partial\psi_p}{\partial n}$ on S_1. Therefore, we must define the functions ψ_p at least in the vicinity of S_1.

The function $\Psi+\delta\psi_p$ plays a role of a phase function of $\widetilde{\mathbf{u}}^r$ in Eq. (B4), and thus, must satisfy the eikonal equation

$$|\nabla(\Psi+\delta\psi_p)|^2=1, \tag{B7}$$

whence

$$(\hat{\mathbf{l}}^r+\delta\nabla\psi_p)^2=1, \tag{B8}$$

where $\hat{\mathbf{l}}^r=\nabla\Psi$ is a normal unit of the reflected wave. From Eq. (B8), we obtain

$$2\delta(\hat{\mathbf{l}}^r\cdot\nabla\psi_p)+o(\delta)=0,$$

[♯] Here, we do not take into account multiple reflections of an incident wave from S_1 in the region G_1 assuming that they are negligible.

or, within small limits of the order $o(\delta)$,

$$(\hat{\mathbf{l}}^r \cdot \nabla\psi_p) = 0, \tag{B9}$$

in other notation

$$\frac{\partial \psi_p}{\partial \hat{\mathbf{l}}^r} = 0. \tag{B10}$$

Taking into account Eq. (12) and the formal expansion of the operator $\mathbf{D}$ (A4), we get from Eq. (B9)

$$\hat{\mathbf{l}}^r \left(\varkappa_0 \mathbf{D}^0 + \hat{\mathbf{n}} \frac{\partial}{\partial n} \right) \psi_p = 0, \tag{B11}$$

where $\mathbf{D}^0 = \frac{1}{\varkappa_0} \mathbf{D}|_{\nu=0}$. Since $(\hat{\mathbf{l}}^r \cdot \mathbf{n}) = -\cos\theta$, we can rewrite (B11) as

$$\varkappa_0 (\hat{\mathbf{l}}^r \cdot \mathbf{D}^0)\psi_p - \cos\theta \frac{\partial \psi_p}{\partial n} = 0, \tag{B12}$$

Thus, we conclude that in the main asymptotic approximation, i.e., with an accuracy to $o(\delta)$

$$\left.\frac{\partial \psi_p}{\partial n}\right|_{S_1} = \frac{\varkappa_0}{\cos\theta} (\hat{\mathbf{l}}^r \cdot \mathbf{D}^0)\psi_p \Big|_{S_1}, \tag{B13}$$

Because the operator $\mathbf{D}^0$ is tangential, and $\hat{\mathbf{l}}^r$ is distinct from $\hat{\mathbf{l}}^i$ only in its normal component ($\hat{\mathbf{l}}^r = \hat{\mathbf{l}}^i - 2\hat{\mathbf{n}}(\hat{\mathbf{l}}^i \cdot \hat{\mathbf{n}})$), we can replace ψ_p by its boundary value in Eq. (B6), and $\hat{\mathbf{l}}^r$ by $\hat{\mathbf{l}}^i$. Then Eq. (B13) takes the form

$$\left.\frac{\partial \psi_p}{\partial n}\right|_{S_1} = \frac{2\varkappa_0 p}{\cos\theta} \left(\hat{\mathbf{l}}^i \cdot \mathbf{t}^M \right), \tag{B14}$$

where $\mathbf{t}^M = \mathbf{D}^0(hM)$.

Specifically, when a layer has plane-parallel boundaries (and hence $h(\sigma) \equiv 1$, and $M(\sigma) \equiv \text{const}$), and an incident wave is plane, we obtain from Eqs. (B10) and (B6)

$$\psi_p(\mathbf{x}) = \text{const},$$

which conforms with the structure of a reflected wave, as the asymptotic terms $\mathbf{u}_p^r$ do not comprise oscillating components in this particular case.

From the boundary condition (B3), we obtain a sequence of boundary relations for $\mathbf{u}_p$ and $\mathbf{u}_p^r$ on S_1

$$\mathbf{u}^i + \mathbf{u}_0^r = \mathbf{u}_0, \tag{B15}$$

$$\mathbf{u}_p^r = \mathbf{u}_p^+ + \mathbf{u}_p^-, \quad p \geq 1. \tag{B16}$$

From Eqs. (B4) and (B14), we have at $\nu = 0$

$$\left.\frac{\partial \mathbf{u}^r}{\partial n}\right|_{S_1} = i\zeta \sum_{p=1}^{\infty} \frac{2\varkappa_0 p}{\cos\theta} \left(\mathbf{l}_\top^i \cdot \mathbf{t}^M \right) e^{i\zeta 2phM} \mathbf{u}_p^r(\sigma, 0) + o(1), \quad \zeta \to \infty, \tag{B17}$$

or taking into account Eqs. (B16) and (5),

$$\frac{1}{ik_0} \left.\frac{\partial \mathbf{u}^r}{\partial n}\right|_{S_1} = \frac{2\alpha}{\zeta\cos\theta} \left(\mathbf{l}_\top^i \cdot \mathbf{t}^M \right) \sum_{p=1}^{\infty} p e^{2ip\zeta Mh} \left(\mathbf{u}_p^+ + \mathbf{u}_p^- \right) + o\left(\frac{1}{\zeta}\right). \tag{B18}$$

We have, thus, proven that the leading asymptotic term of $\frac{1}{ik_0} \frac{\partial \mathbf{u}^r}{\partial n}\Big|_{S_1}$ is of order $\frac{1}{\zeta}$.

From the boundary condition on S_1 in Eq. (80), by using Eq. (B18), we derive the boundary conditions for functions $\mathbf{u}_p$ and $\mathbf{v}_p$ in the form

$$\cos\theta \mathbf{u}_0^+ + i\eta(\overline{\varkappa})\mathbf{u}_0^+ - \mathbf{v}_0^+ + \mathbf{Y}_{\top 0}\mathbf{v}_0^+ = 2\mathbf{u}^i \cos\theta + \frac{\eta}{i\varkappa_0} \frac{\partial \mathbf{u}^i}{\partial n}, \quad p = 0, \tag{B19}$$

$$\begin{aligned} &\cos\theta(\mathbf{u}_p^+ + \mathbf{u}_p^-) + i\eta(\overline{\varkappa})(\mathbf{u}_p^+ + \mathbf{u}_p^-) - (\mathbf{v}_p^+ + \mathbf{v}_p^-) + \\ &(\mathbf{Y}_{\top p}^+ \mathbf{v}_p^+ + \mathbf{Y}_{\top p}^- \mathbf{v}_p^-) = \frac{2\alpha p}{\zeta\cos\theta} \left[\left(\mathbf{l}_\top^i \cdot \mathbf{t}^M \right) \left(\mathbf{u}_p^+ + \mathbf{u}_p^- \right) + o\left(\frac{1}{\zeta}\right) \right], \quad p \geq 1. \end{aligned} \tag{B20}$$

The operator $\mathbf{Y}^{\pm}_{\top p}$ is defined by Eq. (53).

Substituting now the expansions of vectors $\mathbf{u}^{\pm}_{p0}$ and $\mathbf{v}^{\pm}_{p0}$, and the operator $\mathbf{Y}^{\pm}_{\top p}$ into Eqs. (B19) and (B20) and equating the corresponding coefficients of these series, we determine the boundary conditions on S_1 for the leading asymptotic term:

$$\cos\theta \mathbf{u}^{+}_{00} - \mathbf{T}(1;-1)\mathbf{v}^{+}_{00} = 2\mathbf{u}^i \cos\theta, \tag{B21}$$

$$\cos\theta(\mathbf{u}^{+}_{p0} + \mathbf{u}^{-}_{p0}) - \mathbf{T}(1;-1)(\mathbf{v}^{+}_{p0} + \mathbf{v}^{-}_{p0}) = 0, \quad p \geq 1. \tag{B22}$$

B.2. Boundary Condition on S_2

Now we determine the boundary conditions on S_2 with the aid of relations between vectors and operators associated with surfaces S_1 and S_2. Same as above, we use the prime notation to specify vectors and operators associated with the surface S_2.

With the aid of Eqs. (28)–(30), (43), (44), and (A5), it is easy to write the vectors $\mathbf{u}' = \boldsymbol{\mathscr{E}} - \hat{\mathbf{n}}'(\boldsymbol{\mathscr{E}} \cdot \hat{\mathbf{n}})$, $\mathbf{v}' = \hat{\mathbf{n}}' \times \boldsymbol{\mathscr{H}}$, $\mathbf{D}'\Phi_0|_{S_2}$ and $\mathbf{D}'\Phi_0|_{S_2}$, and the derivative $\frac{\partial\Phi}{\partial n'}\big|_{S_2}$ in the following form:

$$\mathbf{u}' = \mathbf{u} + \xi\left[\frac{1}{\varepsilon}\mathbf{t}(\mathbf{l}^i_\top \cdot \mathbf{v}) + \hat{\mathbf{n}}(\mathbf{t}\cdot\mathbf{u})\right] + o(\xi), \tag{B23}$$

$$\mathbf{v}' = \mathbf{v} + \xi\left[\frac{1}{\mu}\mathbf{t}_\perp(\mathbf{l}^i_\perp \cdot \mathbf{u}) + \hat{\mathbf{n}}(\mathbf{t}\cdot\mathbf{v})\right] + o(\xi), \tag{B24}$$

$$\mathbf{D}'\Phi_0|_{S_2} = \mathbf{l}^i_\top + \xi[\hat{\mathbf{n}}(\mathbf{t}\cdot\mathbf{l}^i_\top) - h\widetilde{\mathbf{l}}^i_\top] + o(\xi), \tag{B25}$$

$$\mathbf{D}'\Phi|_{S_2} = \mathbf{l}^i_\top + \xi[\mathbf{t}\cos\theta + \hat{\mathbf{n}}(\mathbf{t}\cdot\mathbf{l}^i_\top)] + o(\xi), \tag{B26}$$

$$\left.\frac{\partial\Phi}{\partial n'}\right|_{S_2} = \cos\theta + \xi\left[-(\mathbf{t}\cdot\hat{\mathbf{l}}^i) + \frac{h}{\varkappa_0}\left(\mathbf{n}\cdot\frac{\partial\hat{\mathbf{l}}^i}{\partial n}\right)\right] + o(\xi), \tag{B27}$$

where $\widetilde{\mathbf{l}}^i_\top = (\overline{\varkappa})^0\mathbf{l}^i_\top$, $\mathbf{t} = \mathbf{D}^0 h$, and $\mathbf{t}_\perp = \mathbf{D}^0_\perp h$.

Now we may pass to the derivation of the boundary condition on S_2. We continuously adjoin on S_2 the vector functions $\mathbf{u}'$ and $\mathbf{v}'$. Obviously,

$$\mathbf{u}' e^{ik_0\Phi_0} = (\mathbf{u}^t)' e^{ik_0\Phi}, \tag{B28}$$

$$\mathbf{u}' = (\mathbf{u}^t)' e^{ik_0(\Phi-\Phi_0)}. \tag{B29}$$

We recall that Φ is the phase of field on S_1, and Φ_0 is the phase of field inside the layer.

In the boundary condition (81), we take into account only the terms of order $\frac{1}{\zeta}$ and obtain

$$\mathbf{v}' + \frac{\partial\Phi}{\partial n'}\mathbf{u}' - \mathbf{D}'\Phi_0(\mathbf{D}'\Phi_0\mathbf{v}) - \frac{\eta}{i\varkappa_0}\mathbf{D}'\Phi_0(\mathbf{D}'\cdot\mathbf{v}) - \frac{\eta}{i\varkappa_0}\mathbf{D}'(\mathbf{D}'\Phi_0\mathbf{v}') = -\frac{\eta}{i\varkappa_0}\frac{\partial(\mathbf{u}^t)'}{\partial n'}e^{ik_0(\Phi-\Phi_0)} + o\left(\frac{1}{\zeta}\right). \tag{B30}$$

To draw a conclusion, we need to know the structure of $\partial(\mathbf{u}^t)'/\partial n'\big|_{S_2}$ and, hence, the structure of the fields $\mathbf{u}'$, $\mathbf{v}'$, and $(\mathbf{u}^t)'$.

In much the same way as in the derivation of the boundary condition on S_1, we conclude that the field $(\mathbf{u}^t)'$ in the region G_2 (at least in the neighborhood of S_2) has the form

$$(\mathbf{u}^t)' = \sum_{p=0}^{\infty} e^{i\zeta\varphi_p}(\mathbf{u}^t)'_p. \tag{B31}$$

With the aid of the condition (B29) and the expansion in Eq. (B2), it is easy to see that the functions φ_p must be connected to phase functions of the field inside the layer via a relation

$$\zeta\varphi_p + k_0(\Phi - \Phi_0) = (2p+1)\zeta h M,$$

that is,

$$\delta\varphi_p + (\Phi - \Phi_0) = (2p+1)\delta h M.$$

However, $\Phi - \Phi_0 = \delta h \cos\theta + o(\delta)$, hence

$$\varphi_p + h\cos\theta = (2p+1)hM + o(\delta). \tag{B32}$$

Like in the derivation of the boundary condition on S_1 for the functions ψ_p, it can be shown that

$$\left.\frac{\partial \varphi_p}{\partial n'}\right|_{S_2} = -\frac{\varkappa_0}{\cos\theta}(\mathbf{l}^i_\top \cdot \mathbf{D}^0\varphi_p) + O\left(\frac{1}{\zeta}\right), \tag{B33}$$

where φ_p is determined by Eq. (B32).

Substituting Eqs. (B2) and (B31) into Eq. (B30), we deduce now the boundary condition on S_2:

$$\begin{aligned}&\mathbf{v}'_{2p+\frac{1}{2}} + \frac{\partial\Phi}{\partial n'}\mathbf{u}'_{2p+\frac{1}{2}} - \mathbf{D}'\Phi_0(\mathbf{D}'\Phi_0\mathbf{v}_{2p+\frac{1}{2}}) - \xi(2p+1)(\mathbf{t}^M\mathbf{v}_{2p+\frac{1}{2}})\mathbf{D}'\Phi_0 - \\ &\frac{\xi}{\varkappa_0}(2p+1)\mathbf{D}'(hM)(\mathbf{D}'\Phi_0\mathbf{v}_{2p+\frac{1}{2}}) = -\frac{\xi}{\varkappa_0}\frac{\partial\varphi_p}{\partial n'}\mathbf{u}'_{2p+\frac{1}{2}} + o(\xi),\end{aligned} \tag{B34}$$

where $\mathbf{v}'_{2p+\frac{1}{2}} = \mathbf{v}'^{+}_{p} + \mathbf{v}'^{-}_{p+1}, \quad \mathbf{u}'_{2p+\frac{1}{2}} = \mathbf{u}'^{+}_{p} + \mathbf{u}'^{-}_{p+1}$.

Using Eqs. (B23), (B24), and (47), we derive from Eq. (B34) the boundary condition on S_2 for the leading asymptotic term

$$\cos\theta\,\mathbf{u}_{2p+\frac{1}{2},0} + \mathbf{T}(1;-1)\mathbf{v}_{2p+\frac{1}{2},0} = 0, \quad (p = 0, 1, 2, \dots). \tag{B35}$$

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