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Compositional Synthesis of Symbolic Models for (In)Finite Networks of Cyber-Physical Systems

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This dissertation is dedicated to my wife ***Fatma*** and my sons ***Mohammed***, ***Mahmud***, ***Anas***, and ***Hamza***.

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Abstract

This dissertation is motivated by the challenges arising in the analysis and synthesis of large-scale cyber-physical systems (CPSs). In the past decades, CPSs have received considerable attention as an important modeling framework describing many engineering systems and play major roles in many real-life applications including transportation systems, traffic networks, power systems, and so on. Automated verification and control synthesis for this type of complex systems with the aim to achieve high-level specifications is quite challenging. It is known that providing automated synthesis of correct-by-design controllers for CPSs is definitely crucial in several safety-critical applications such as autonomous driving. As a promising technique, symbolic models are introduced to cope with the computational complexity arising in the analysis of large-scale CPSs. More specifically, one method to deal with encountered complications is to first construct symbolic models for the concrete systems, then design controllers for the symbolic models. Finally one can refine the controllers back to the original systems based on some behavioral relation between original systems and their symbolic models such as approximate alternating simulation relations [PT09] or feedback refinement relations [RWR17]. Since the mismatch between the output of the concrete system and that of its symbolic model is formally quantified, one can guarantee that the original systems also satisfy the same specifications as the symbolic ones with guaranteed error bounds on their output trajectories.

Unfortunately, construction of symbolic models for large-scale CPSs in a monolithic fashion suffers acutely from the so-called curse of dimensionality. Specifically, the computational complexity of the construction of symbolic models grows exponentially with respect to the dimension of the state and input sets. Consequently, such a construction will become computationally intractable when dealing with large-scale systems. To resolve this issue, one promising technique is to consider the large-scale CPS as a network composed of many systems, and provide a compositional scheme for synthesizing a symbolic model for the given network using symbolic models of its local systems. This dissertation provides novel compositional methodologies to design symbolic models for large-scale CPSs in a constructive and formal manner.

The compositional methodologies in the dissertation are based on two approaches.

- The first approach utilizes some dissipativity type conditions which may enjoy specific interconnection topologies and provide scale-free compositional construction for symbolic models of the concrete networks. We show that if some dissipativity type conditions hold, one can construct symbolic models of a network composed of finitely many systems using symbolic models of those systems.

- The second approach leverages general small-gain type conditions to provide a compositional framework for constructing symbolic models for either infinite or finite concrete networks. We show that the proposed max type small-gain approach is more general than the classic one in the literature since it does not require linear growth on the gains of systems which is the case in the classic small-gain approach. We also show that the overall approximation error is proportional to the maximum of the approximation errors of symbolic models of systems. In our small-gain framework, the overall approximation error is determined independently of the number of systems that constitute the concrete network. Therefore, the results here can potentially provide symbolic models for a large network contains an in(finite) number of systems with much smaller approximation error in comparison with those proposed based on the classic small-gain and dissipativity approaches.

In addition, we provide a procedure for constructing symbolic models for a class of discrete-time control systems that are incrementally passive or incrementally input-to-state stable. Moreover, we extend the results from discrete-time control systems to switched ones by imposing those stability properties on each mode of the switched systems.

Zusammenfassung

Diese Dissertation ist motiviert durch die Herausforderungen, die sich bei der Analyse und Synthese hochdimensionaler cyberphysischer Systeme (CPS) stellen. In den vergangenen Jahrzehnten haben CPS erhebliche Beachtung erfahren als wichtiger Modellierungsrahmen zur Beschreibung vieler Engineeringsysteme. Zudem spielen CPS eine große Rolle in vielen realen Anwendungen wie zum Beispiel Transportsystemen, Verkehrsnetzwerken usw. Automatische Verifikation und Reglersynthese für diesen Typ von komplexen Systemen mit dem Ziel gewisse High-LevelSpezifikationen zu erfüllen ist sehr anspruchsvoll. Insbesondere ist es in verschiedenen sicherheitskritischen Anwendungen wie z.B. beim autonomen Fahren entscheidend, eine automatische Synthese von a priori korrekten Reglern bereit zu stellen. Für diese Aufgabe werden symbolische Modelle eingeführt als eine vielversprechende Methode um mit der rechnerischen Komplexität fertig zu werden, die sich in der Analyse von hochdimensionalen CPS ergibt. Spezifischer ist es eine vielversprechende Methode um mit den auftretenden Komplikationen umzugehen zuerst symbolische Modelle für die konkreten Systeme zu konstruieren, dann Regler für die symbolischen Modelle zu entwerfen und schließlich die Regler zur Anwendung auf die ursprünglichen Modelle zu verfeinern basierend auf behavioral relations zwischen den Originalsystemen und ihren symbolischen Modellen wie etwa approximativen alternierenden Simulationsrelationen [PT09] oder FeedbackRefinement-Relationen [RWR17]. Da der Unterschied zwischen dem Ausgang des konkreten Systems und der seines symbolischen Modells formal quantifiziert ist, kann man garantieren, dass die Originalsysteme dieselben Spezifikationen wie die symbolischen Systeme erfüllen mit garantierten Fehlerschranken für ihre Ausgangstrajektorien.

Unglücklicherweise leidet die Konstruktion von symbolischen Modellen für hochdimensionale CPS auf monolithische Art stark am sogenannten Fluch der Dimensionalität: die rechnerische Komplexität der Konstruktion von symbolischen Modellen wächst exponentiell mit der Dimension der Zustands- und Eingangsmengen; daher wird eine solche Konstruktion für hochdimensionale Systeme rechnerisch unlösbar. Ein vielversprechender Ansatz um dieses Problem zu lösen ist es, das hochdimensionale CPS als ein Netzwerk von vielen Systemen zu betrachten und eine kompositionelle Strategie zur Synthese von symbolischen Modellen für das gegebene Netzwerk bereit zu stellen, die symbolische Modelle von dessen lokalen Systemen verwendet. Diese Dissertation liefert neue kompositionelle Methoden um symbolische Modelle für hochdimensionale CPS auf konstruktive und formale Art zu konstruieren.

Die kompositionellen Methoden in dieser Dissertation basieren auf zwei Ansätzen.

- Der erste Ansatz verwendet gewisse Dissipativitätsbedingungen, die unter Umständen spezifische Netzwerktopologien genießen und eine skalierungsfreie kompositionelle Konstruktion für symbolische Modelle der konkreten Netzwerke liefern.

Wir zeigen, dass falls gewisse Dissipativitätsbedingungen erfüllt sind, symbolische Modelle eines Netzwerks konstruiert werden können, das aus abzählbar vielen Systemen zusammengesetzt ist, wobei wir symbolische Modelle dieser Systeme verwenden.

- Der zweite Ansatz verwendet hinreichend allgemeine SmallGain-artige Bedingungen, um ein kompositionelles Framework zur Konstruktion symbolischer Modelle für unendliche oder endliche konkrete Netzwerke bereit zu stellen. Wir zeigen, dass der vorgestellte Max-Small-Gain Ansatz allgemeiner ist als der klassische in der Literatur, da er kein lineares Wachstum der Gains des Systems fordert, was bei klassischem Small-Gain der Fall ist. Wir zeigen auch, dass der gesamte Approximationsfehler proportional zum Maximum der Approximationsfehler symbolischer Modelle von Systemen ist. Daher können diese Resultate potentiell symbolische Modelle für ein großes Netzwerk mit (un)endlicher Zahl an Systemen liefern mit viel kleinerem Approximationsfehler im Vergleich zu denen, die auf klassischer Small-Gain-Analyse und Dissipativitätsansätzen basieren

Zusätzlich liefern wir ein Verfahren zur Konstruktion symbolischer Modelle für eine Klasse zeitdiskreter Kontrollsysteme, die inkrementell passiv oder inkrementell Input-to-State stabil sind. Darüber hinaus erweitern wir die Resultate von zeitdiskreten zu geschalteten Systemen, indem wir diese Stabilitätseigenschaften für jeden Modus des geschalteten Systems fordern.

Publications by the Author during Ph.D.

Journal Papers

1. C. Kawan, A. Mironchenko, **A. Swikir**, N. Noroozi, and M. Zamani, "A Lyapunov-based Small-gain Theorem for Infinite Networks," *IEEE Transactions on Automatic Control*.

2. **A. Swikir**, A. Girard, and M. Zamani, "Symbolic Models for A Calss of Impulsive Systems," *IEEE Control Systems Letters*, 5(1), pp. 247-252, June 2021. (*The 59th Conference on Decision and Control 2020.*).

3. **A. Swikir** and M. Zamani, "Compositional Synthesis of Symbolic Models for Networks of Switched Systems," *IEEE Control Systems Letters*, 3(4), pp. 1056 - 1061, June 2019. (*The 58th Conference on Decision and Control 2019.*)

4. **A. Swikir** and M. Zamani, "Compositional Synthesis of Finite Abstractions for Networks of Systems: A small-gain approach," *Automatica*, 107, pp. 551 - 561, September 2019.

Conference Papers

5. M. Sharifi, **A. Swikir**, N. Noroozi and M. Zamani, "Compositional Construction of abstractions for infinite networks of switched systems," *The 59th IEEE Conference on Decision and Control (CDC), 2020.*

6. S. Liu, **A. Swikir**, and M. Zamani, "Verification of Initial-State Opacity for Switched Systems," *The 59th IEEE Conference on Decision and Control (CDC), 2020.*

7. M. Shahamat, J. Askari, **A. Swikir**, N. Noroozi and M. Zamani, "Construction of continuous abstractions for discrete-time time-delay systems," *The 59th IEEE Conference on Decision and Control (CDC), 2020.*

8. C. Kawan, A. Mironchenko, **A. Swikir**, N. Noroozi, and M. Zamani, "A Spectral Small-gain Condition for Input-to-state Stability of Infinite Networks," *The 21st IFAC World Congress 2020.*

9. **A. Swikir**, N. Noroozi, and M. Zamani, "Compositional Synthesis of Symbolic Models for Infinite Networks," *The 21st IFAC World Congress 2020.*

10. P. Jagtap, **A. Swikir**[*], and M. Zamani, "Compositional Construction of Control Barrier Functions for Interconnected Control Systems," *The 23rd ACM International Conference on Hybrid Systems: Computation and Control (HSCC) 2020.*

11. **A. Swikir** and M. Zamani, "Compositional Abstractions of Interconnected Discrete-Time Switched Systems," *The 18th European Control Conference*, pp. 1251-1256, 2019.

12. N. Noroozi, **A.Swikir**, F. R. Wirth, and M. Zamani, "Compositional Construction of Abstractions Via Relaxed Small-gain Conditions Part II: Discrete Case," *The 17th European Control Conference*, pp. 1-4, 2018.

13. **A. Swikir**, A. Girard, and M. Zamani, "From Dissipativity Theory to Compositional Synthesis of Symbolic models," *The 4th Indian Control Conference*, pp. 30-35, 2018. (**Best student paper award finalist**)

Under Submission and Review

1. S. Liu, **A. Swikir**, and M. Zamani, "Verification of Initial-State Opacity for Switched Systems: A Compositional Approach," *Nonlinear Analysis: Hybrid systems* submitted.

[*]Both authors have contributed equally.

Contents

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List of Abbreviations

δ-P	Incremental passivity
δ-ISS	Incremental input-to-state stability
LMI	Linear matrix inequality
LTL	Linear temporal logic
CPSs	Cyber-physical systems
DA	Dissipativity approach
SGA	Small-gain approach

List of Symbols

Sets and Spaces

$\mathbb{R}$	Set of real numbers
$\mathbb{R}_{>0}$	Set of positive real numbers
$\mathbb{R}_{\geq 0}$	Set of nonnegative real numbers
$\mathbb{N}:=\{1,2,3,\ldots\}$	Set of positive integers
$\mathbb{N}_0:=\{0,1,2,\ldots\}$	Set of nonnegative integers
$\mathbb{R}^{n\times m}$	Set of real matrices with n rows and m columns
I_n	Identity matrix in $\mathbb{R}^{n\times n}$
$0_{n\times m}$	Zero matrix in $\mathbb{R}^{n\times m}$
0_n	Zero vector in $\mathbb{R}^n$
ℓ^∞	Banach space of all infinite uniformly bounded sequences
ℓ^∞_+	Positive cone in ℓ^∞ consisting of all vectors $s\in\ell^\infty$ with $s_i\geq 0, i\in\mathbb{N}$
e_i	Sequence of zeros in ℓ^∞ with exception of position i (unit vectors)

For $a,b\in\mathbb{R}$ with $a\leq b$ and $x,y\in\mathbb{N}_0$ with $x\leq y$, we use following notations:

$\lvert a\rvert$	Absolute value of a
$[a,b]$	Closed interval in $\mathbb{R}$
(a,b)	Open interval in $\mathbb{R}$
$[a,b)$, $(a,b]$	Half-open intervals in $\mathbb{R}$
$[x;y]$	Closed interval in $\mathbb{N}$
$(x;y)$	Open interval in $\mathbb{N}$
$[x;y)$, $(x;y]$	Half-open intervals in $\mathbb{N}$

Matrices and Vectors

Given a matrix $A\in\mathbb{R}^{n\times m}$, a symmetric matrix $Z\in\mathbb{R}^{n\times n}$, and vectors $x\in\mathbb{R}^n$, $y\in\ell^\infty$ we have following notations:

$\{A\}_{ij}$	Individual element in A at ith row and jth column
$A^\top$	Transpose of matrix A
$\lvert A\rvert_2$	Euclidean norm of A
$\lambda_{\min}(Z)$	Minimum eigenvalue of Z
$\lambda_{\max}(Z)$	Maximum eigenvalue of Z
$\lvert x\rvert_2$	Euclidean norm of x, $x\in\mathbb{R}^n$
$\lvert x\rvert$	Infinity norm of x, $x\in\mathbb{R}^n$

$\|y\|$	Infinity norm of y, $y \in \ell^\infty$
$(x_i)_{i\in[1;N]}$	Column vector with N components, $N \leq \infty$
$[x_1; \cdots ; x_n]$	Column vector in $\mathbb{R}^n$

Functions

$f : X \to Y$	Ordinary map from set X into set Y
$f : X \rightrightarrows Y$	Set-valued map from set X into set Y
$g \circ f : X \to Y$	Function composition defined by $g \circ f(x) = g(f(x))$ for all x in X
$\mathcal{K}$	$\mathcal{K}:=\{\alpha:\mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \alpha$ is continuous, strictly increasing, and $\alpha(0)=0\}$
$\mathcal{K}_\infty$	$\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \mid \lim_{r\to\infty} \alpha(r) = \infty\}$
id	Identity function over $\mathbb{R}_{\geq 0}$ defined as $\mathsf{id}(r) = r, \forall r \in \mathbb{R}_{\geq 0}$
Γ^0	Identity operator on ℓ^∞

Sets

Given sets X and Y, we have following notations:

$X \cap Y$	Intersection of sets X and Y
$X \cup Y$	Union of sets X and Y
$\mathrm{card}(X)$	Cardinality of the set X
$\varnothing$	Empty set
$\prod_{i=1}^{N} X_i$	N-ary Cartesian product

Other

T	Transition system
Σ	Concrete system
$T(\Sigma)$	Transition system associated to Σ
$\mathcal{I}(\Sigma_i)_{i\in[1;N]}$	A network composed of N concrete systems, $N \leq \infty$
$\mathcal{I}(T_i(\Sigma_i))_{i\in[1;N]}$	A network composed of N transition systems, $N \leq \infty$

1 Introduction

1.1 Motivation and Contributions

Cyber-physical systems (CPSs) are complex interconnected models combining both cyber (computation and communication) and physical components, which tightly interact with each other in a feedback loop [LS16]. In the past decades, CPSs have received considerable attention as an important modeling framework describing many engineering systems and play major roles in many real-life applications including transportation systems, traffic networks, and so on Figure 1.1. Most CPSs are of hybrid nature: discrete dynamics model computation units including hardware and software, and continuous dynamics model physical components. The complexity raised by the interaction between computation units and physical components often makes it difficult to obtain analytical results for this type of complex systems. For instance, automated verification and control synthesis for CPSs to achieve some high-level specifications, e.g., those expressed as linear temporal logic (LTL) formulae [Pnu77], is quite challenging [DLS12, KK12]. In addition, many CPSs are safety critical or mission critical; hence, the satisfaction of safety or some desired specifications must be guaranteed.

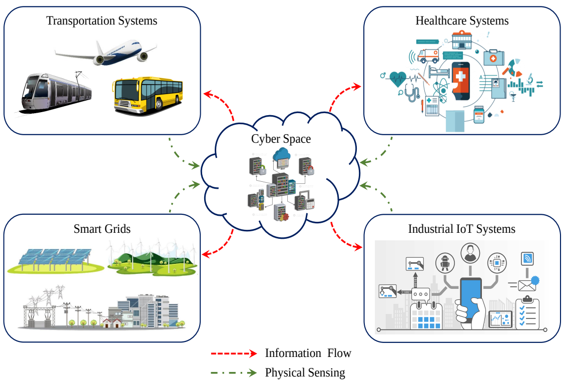

Figure 1.1: Application scenarios of CPSs.

Formal methods are known to provide fundamental tools for the synthesis of CPSs, as they give theoretical or rigorous mathematical frameworks which ensure that the system meets the desired specification [CW96, Win90, Ses15]. Although formal methods have been originally developed in software engineering as a framework to find bugs or security vulnerabilities in software, they have been recently identified to be useful in many other applications, including control design of CPSs. In particular, one of the most successful approaches that interface formal methods and control synthesis of CPSs is the so-called symbolic control [Tab09]. In this approach, symbolic models (a.k.a. finite abstractions) are commonly used to replace the concrete systems in the analysis and controller synthesis process. Symbolic models are abstract descriptions of the continuous-space control systems in which each discrete state and input correspond to an aggregate of continuous states and inputs of the original system, respectively. In general, there exist two types of symbolic models: *sound* ones whose behaviors (approximately) contain those of the concrete systems and *complete* ones whose behaviors are (approximately) equivalent to those of the concrete systems [Tab09]. Remark that existence of a complete symbolic model results in a sufficient and necessary guarantee in the sense that there exists a controller enforcing the desired specifications on the symbolic model *if and only if* there exists a controller enforcing the same specifications on the original control system. On the other hand, a sound symbolic model provides only a sufficient guarantee in the sense that failing to find a controller for the desired specifications on the symbolic model does not prevent the existence of a controller for the original control system. Since symbolic models are finite, controller synthesis problems can be algorithmically solved over them by resorting to automata-theoretic approaches [MPS95, Tho95, BJP$^+$12]. Then one can refine the synthesized controllers back to the original systems based on some behavioral relation between original systems and their symbolic models such as approximate alternating simulation relations [PT09] or feedback refinement relations [RWR17]. Figure 1.2 schematically describes the symbolic control scheme.

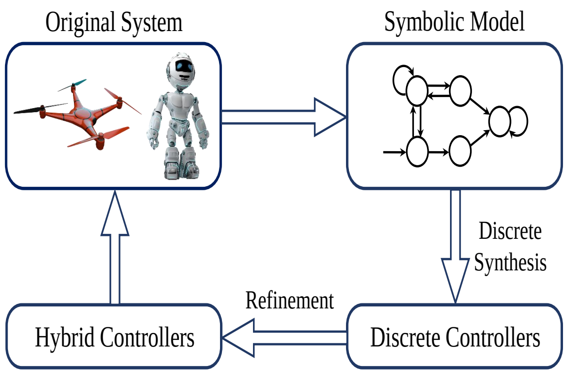

Figure 1.2: Symbolic control flowchart.

Large numbers of studies have been conducted on the computation of symbolic models for various classes of systems. In [TP06, TP03b, TP03a], complete symbolic models were constructed for controllable linear systems. In [BH06], a feedback control over facet was utilized to design complete symbolic models for nonlinear control affine systems, in [ADLB14] complete abstractions were constructed for switched linear systems using polyhedral sublevel sets of Lyapunov function, and in [AHLP00] equivalent discrete abstractions of different classes of hybrid systems were introduced. Approximately complete symbolic models were synthesized for different classes of systems, among many others, in the following papers: nonlinear systems [PGT08, PT09], switched systems [GPT10], singularly perturbed hybrid affine systems [KG19], time-delay systems [PPB10, PPB15], infinite dimensional systems [Gir14, JZ20], networked control systems [BPB19, ZMKA18], and stochastic systems [MA14, ZEM$^+$14]. There are other results in which symbolic models were constructed for concrete systems. A finite abstractions for continuous systems was proposed in [RO98] as a nondeterministic automaton. The notion of a strongest l-complete approximation was introduced in [MR99, MRO02] as a discrete abstraction for time invariant behavioral systems, and the applicability of this notion was enlarged in [SR14] using the notion of so called asynchronous l-complete approximations. Symbolic models for piecewise-affine hybrid system were introduced in [HCv06]. Results for stabilizable and incrementally forward complete nonlinear systems were established in [Tab08, ZPMT12, LLO15]. In [MGW19, CA17], symbolic models were designed for monotone and mixed monotone systems, and in [LTOM12] for differentially flat systems. Markov chain abstractions have been introduced and leveraged to safety verification and reachable sets computation in the framework of traffic networks and autonomous vehicles in [ASB07b, ASB07a, ASB08, ASB09b, ASB09a]. We refer interested reader to [Tab09, BYG17, GP11, PB19] for more details on some of the results mentioned above.

However, the computational complexity of constructing symbolic models often scales exponentially with the dimension of the state and input spaces. Several approaches have been proposed in the literature to overcome this scalability problem. Adaptive multi-resolution and multi-scale state-space discretization approaches have been proposed in [TI09, GGM16, HMMS18b] to compute symbolic models. A state-space discretization free approach was introduced in [CGG13, ZAG15, Gir14] where symbolic states are given by input sequences. In [WRR17], the size of symbolic models were minimized using optimal discretization parameters. In [GKA17], symbolic models were constructed by exploiting sparse interconnection structure of the dynamical systems. In [HMMS18a, HMMS18a], a lazy versions of multi-layered abstractions for nonlinear systems against safety and reachability specifications have been proposed. The authors in [CGG11, GGM16] introduced lazy safety synthesis for incrementally stable switched systems using multiscale symbolic models.

Unfortunately, all the aforementioned monolithic approaches for synthesizing symbolic models will become computationally intractable while dealing with large-scale systems. A convenient method to cope with this challenge is to provide a compositional framework for constructing symbolic models for networks of concrete systems. To do so, one should first *i*) partition the overall concrete network into a number of concrete systems and construct symbolic models of them individually; *ii*) then establish a compositional scheme that allows us to construct a symbolic model of the overall network using those individual ones. This divide-and-conquer scheme is illustrated in Figure 1.3.

The compositional framework for designing symbolic models based on a divide-and-conquer scheme [Kea11] is not new. Several results have already introduced compositional techniques for constructing symbolic models of networks of control systems. The results in [TI08, PPB16, MSSM19] provide techniques to approximate networks of control systems by networks of complete symbolic models by assuming some stability property of the concrete systems. Other compositional approaches provide techniques

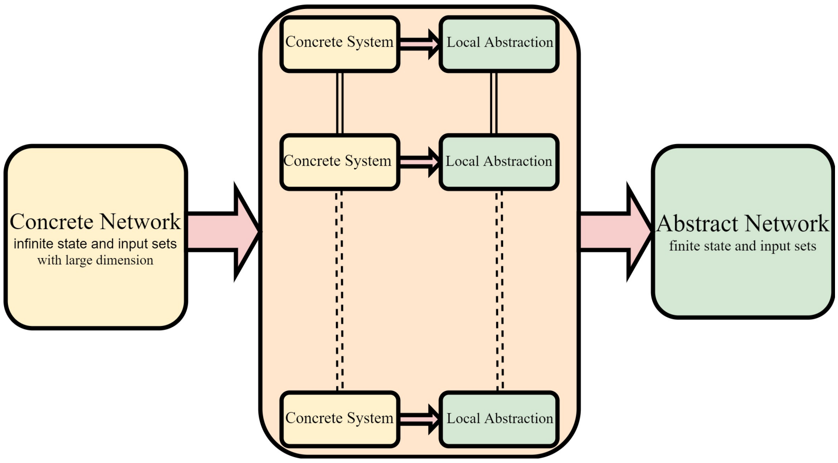

Figure 1.3: A divide and conquer strategy scheme.

to design sound symbolic models of concrete networks without requiring any stability property or condition on the gains of systems [MGW17, HAT17, KAZ18]. In addition, compositional approaches for constructing finite abstractions (a.k.a. finite Markov decision processes) for stochastic systems have been widely investigated in the recent years; see [LSZ20b, LSZ20a] and references therein.

Unfortunately, as we are concerned here with complete symbolic models for deterministic systems, all the results in [TI08, PPB16, MSSM19] have three main drawbacks: *i*) they deal only with networks composed of finite number of systems and can *not* be applied to networks consisting of infinite numbers of components. *ii*) those compositional results are not concerned with switched systems, and they do not provide any compositional framework for the construction of symbolic models for networks of switched systems. *iii*) they use *conservative* small-gain type conditions which implicitly require concrete systems to have a (nearly) linear behavior. Motivated by the above limitation, this dissertation aims at proposing a compositional framework for constructing symbolic models for (in)finite networks of concrete systems by considering more relaxed compositional conditions and also providing compositionality results for networks consisting of infinitely many finite-dimensional systems.

In this dissertation, we first propose compositional techniques based on dissipativity theory. The utilized dissipativity conditions may enjoy specific interconnection topologies and provide scale-free compositional construction for symbolic models of the networks of discrete-time control systems. Under the satisfaction of those conditions, we construct a symbolic model of a network composed of finitely many discrete-time control systems using their symbolic models. In particular, we use a notion of so-called sum-type simulation function between systems and their symbolic models to compositionally construct a so-called alternating simulation function as a relation between the network of symbolic models and that of control systems. The existence of such an alternating simulation function ensures that the output behavior of the network of discrete-time control systems is quantitatively approximated by the that one of their symbolic models.

We also leverage general small-gain type conditions to provide a compositional scheme for designing symbolic models for either infinite or finite networks of discrete-time control systems. We show that the proposed small-gain approach is more general than the classic one in the literature since it does not require linear growth on the gains of systems which is the case in the classic one. We also show that the overall approximation error is proportional to the maximum of the approximation errors of symbolic models of systems. In our small-gain framework, the overall approximation error is determined independently of the number of discrete-time control systems in the concrete network. Therefore, the proposed results can potentially provide symbolic models for a network composed of a large number of discrete-time control systems with much smaller approximation error in comparison with those proposed based on the classic small-gain and dissipativity approaches. Additionally, we introduce a compositional scheme based on robust small-gain conditions to construct symbolic models for networks consisting of infinitely many finite-dimensional discrete-time control systems.

Furthermore, using the same dissipativity and max small-gain conditions, we extend our compositionality results from finite networks of discrete-time control systems to the ones of switched systems whose switching signals satisfy a dwell-time condition. In addition, we provide a procedure for constructing symbolic models of local systems (discrete-time control and switched systems) that are incrementally passive or incrementally input-to-state stable. We also provide some linear matrix inequalities replacing those stability properties for some classes of concrete systems. We provide case studies illustrating efficiency of all proposed techniques.

1.2 Outline of the Thesis

This dissertation is divided into 5 chapters, the first of which is the current introduction. The rest is structured as follows:

Chapter 2 presents some mathematical notations and preliminaries, and also some systems definitions, propositions, lemmas that will be frequently used throughout the dissertation.

Chapter 3 studies compositional construction of symbolic models for infinite and finite networks of discrete-time control systems based on two different compositionality approaches, i.e., dissipativity and small-gain approaches. The results of this chapter are respectively presented based on [SGZ18, SZ19b, SNZ20].

Chapter 4 discusses compositional construction of symbolic models for finite networks of discrete-time switched systems with the same compositional techniques as the previous chapter. The results of this chapter are respectively presented based on [SZ19c, SZ19a].

Chapter 5 summarizes the results of this dissertation and outlines potential directions for the future research.

For more clarity of exposition, **Chapters 3** and **4** follow a common structure. They start with an introduction including a description of the problem addressed, a brief literature review, and a statement of the contributions made. The developed techniques are detailed in subsequent sections, followed by a section illustrating their efficiency on different case studies. The chapters are concluded with a summary section.

2 Notations and Preliminaries Results

2.1 Notations

We denote by $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ the sets of real numbers, non-negative integers, and positive integers, respectively. We denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a,b]$, (a,b), $[a,b)$, and $(a,b]$, respectively. For $a,b \in \mathbb{N}_0$ and $a \leq b$, we use $[a;b]$, $(a;b)$, $[a;b)$, and $(a;b]$ to denote the corresponding intervals in $\mathbb{N}_0$. Given any $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a. Given any $\nu = [\nu_1; \cdots; \nu_n] \in \mathbb{R}^n$, the infinity norm of ν is defined by $|\nu| = \max_{1 \leq i \leq n} |\nu_i|$. Elements of $\mathbb{R}^n$ are by default regarded as column vectors and we write $\nu^\top$ for the transpose of a vector $\nu \in \mathbb{R}^n$. Given a symmetric matrix A, $\lambda_{\max}(A)$, and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of A, respectively. By ℓ^∞ we denote the Banach space of all infinite uniformly bounded sequences $s := (s_i)_{i\in\mathbb{N}} \in \ell^\infty$, where s_i denotes the ith position of a sequence $s \in \ell^\infty$. Moreover, ℓ^∞_+ denotes the positive cone in ℓ^∞ consisting of all vectors $s \in \ell^\infty$ with $s_i \geq 0, i \in \mathbb{N}$. For all $s, s' \in \ell^\infty$ we say that $s \leq s'$ if $s_i \leq s'_i$ for all $i \in \mathbb{N}$, and that $s \not\geq s$ if there is $i \in \mathbb{N}$ such that $s_i < s'_i$. The standard unit vectors in ℓ^∞ are denoted by e_i, $i \in \mathbb{N}$; i.e., e_i is the sequence of zeros with exception of position i, where the entry is 1. Given an operator $\Gamma : \ell^\infty_+ \to \ell^\infty_+$, $k \geq 1 \in \mathbb{N}$, we define $\Gamma^k(\cdot) := \Gamma^{k-1} \circ \Gamma(\cdot)$, where Γ^0 is the identity operator on ℓ^∞. We denote by $\mathrm{card}(A)$ the cardinality of a set A and by $\varnothing$ the empty set. For any set $S \subseteq \mathbb{R}^n$ which is a finite union of boxes, e.g., $S = \bigcup_{j=1}^M S_j$ for some finite number $M \in \mathbb{N}$, where $S_j = \prod_{i=1}^n [c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$, and a positive constant $\eta \leq \mathit{span}(S)$, where $\mathit{span}(S) = \min_{j=1,\ldots,M} \eta_{S_j}$ and $\eta_{S_j} = \min\{|d_1^j - c_1^j|, \ldots, |d_n^j - c_n^j|\}$, we define $[S]_\eta = \{a \in S \mid a_i = k_i\eta, k_i \in \mathbb{N}, i = 1, \ldots, n\}$. The set $[S]_\eta$ will be used as a finite approximation of S with precision η. Note that $[S]_\eta \neq \varnothing$ for any $\eta \leq \mathit{span}(S)$. We use the notations $\mathcal{K}$ and $\mathcal{K}_\infty$ to denote different classes of comparison functions, as follows: $\mathcal{K} = \{\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | \ \alpha \text{ is continuous, strictly increasing, and } \alpha(0) = 0\}$; $\mathcal{K}_\infty = \{\alpha \in \mathcal{K} | \lim_{r\to\infty} \alpha(r) = \infty\}$. For $\alpha, \gamma \in \mathcal{K}_\infty$ we write $\alpha \leq \gamma$ if $\alpha(r) \leq \gamma(r)$, and, with abuse of the notation, $\alpha = c$ if $\alpha(r) = cr$ for all $r \geq 0$ and a given $c \geq 0$. Finally, we denote by id the identity function over $\mathbb{R}_{\geq 0}$, that is $\mathsf{id}(r) = r, \forall r \in \mathbb{R}_{\geq 0}$.

2.2 Transition Systems

In this section, we consider a general form of transition systems which allows us to model concrete systems and their symbolic models in a common framework.

Definition 2.2.1. *A transition system is a tuple* $T = (X, X_0, W, U, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$*, consisting of:*

- *a set of states* X*;*
- *a set of initial states* $X_0 \subseteq X$*;*
- *a set of external inputs* U*;*
- *a set of internal inputs* W*;*
- *transition function* $\mathcal{F} : X \times W \times U \rightrightarrows X$*;*
- *an external output set* Y^1*;*
- *an internal output set* Y^2*;*
- *an external output map* $\mathcal{H}^1 : X \to Y^1$*;*
- *an internal output map* $\mathcal{H}^2 : X \to Y^2$*.*

The transition $x^+ \in \mathcal{F}(x, w, u)$ means that the system can evolve from state x to state x^+ under the inputs w and u. Thus, the transition function defines the dynamics of the transition system. Sets X, W, U, and Y are assumed to be subsets of normed vector spaces with appropriate finite dimensions.

If for all $x \in X, w \in W, u \in U$, $\text{card}(\mathcal{F}(x, w, u)) \leq 1$ we say that T is deterministic, and non-deterministic otherwise. Additionally, T is called finite if X, W, U are finite sets and infinite otherwise. Furthermore, if for all $x \in X$ there exists $w \in W$ and $u \in U$ such that $\text{card}(\mathcal{F}(x, w, u)) \neq 0$ we say that T is non-blocking. In this work, we only deal with non-blocking transition systems.

2.2.1 Networks of Transition Systems

Let $\mathscr{N}$ be either infinite or finite set, i.e.,$\mathscr{N} := \mathbb{N}$, or $\mathscr{N} := [1; N]$ for a finite number $N \in \mathbb{N}$. As we consider T_i to be a part of a network, we define the following sets defining the neighbors of T_i. For each $i \in \mathscr{N}$ let $\mathcal{N}_i$ and $\mathcal{M}_i$ be finite subsets of $\mathscr{N}$. Here, the index sets $\mathcal{N}_i$ and $\mathcal{M}_i$ enumerate the neighbors of T_i, i.e., those systems $T_j, j \in \mathcal{N}_i$, $T_{j'}, j' \in \mathcal{M}_i$ that affect or are affected by T_i, respectively. By definition, we require that $i \notin \mathcal{N}_i \cup \mathcal{M}_i$, $\forall i \in \mathscr{N}$. Since $\mathcal{N}_i$ and $\mathcal{M}_i$ are finite subsets of $\mathscr{N}$, each T_i can have only a finite number of neighbors.

Now, we provide a formal definition of the network of transition systems based on two different compositional approaches, i.e., the dissipativity and small-gain approaches.

2.2.1.1 Dissipativity Approach Formulation

Definition 2.2.2. *Consider transition systems* $T_i = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$*,* $i \in \mathscr{N} := [1; N]$*,* $N \in \mathbb{N}$*. Let* M *be a static matrix of an appropriate dimension defining the coupling of these systems such that* $M \prod_{i\in\mathscr{N}} Y_i^2 \subseteq \prod_{i\in\mathscr{N}} W_i$*. The network of transition systems is a tuple* $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$*, denoted by* $T = \mathcal{I}_M(T_i)_{i\in\mathscr{N}}$*,* *where* $X = \prod_{i\in\mathcal{N}} X_i$, $X_0 = \prod_{i\in\mathcal{N}} X_{0_i}$, $U = \prod_{i\in\mathcal{N}} U_i$, $Y = \prod_{i\in\mathcal{N}} Y_i^1$. *Moreover,* $\mathcal{F}$ *and* $\mathcal{H}$ *are defined as*

$$\mathcal{F}(x,u) = \{(x_i^+)_{i\in\mathcal{N}} | x_i^+ \in \mathcal{F}_i(x_i, w_i, u_i)\},\ \mathcal{H}(x) = (\mathcal{H}_i^1(x_i))_{i\in\mathcal{N}},$$

where $x = (x_i)_{i\in\mathcal{N}}$, $u = (u_i)_{i\in\mathcal{N}}$, *and with the internal variables constrained by* $(w_i)_{i\in\mathcal{N}} = M(h_i^2(x_i))_{i\in\mathcal{N}}$.

Note that condition $M \prod_{i\in\mathcal{N}} Y_i^2 \subseteq \prod_{i\in\mathcal{N}} W_i$ is required to have a well-defined interconnection. The interconnection scheme of network T based on dissipativity approach formulation is illustrated in Figure 2.1.

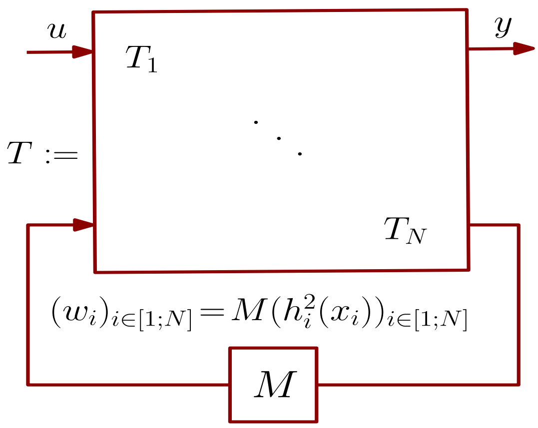

Figure 2.1: The interconnection scheme for network T composed of N systems based on the dissipativity approach formulation.

2.2.1.2 Small-Gain Approach Formulation

Definition 2.2.3. *Consider transition systems* $T_i = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$, $i \in \mathcal{N}$. *The network of transition systems is a tuple* $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$, *denoted by* $T = \mathcal{I}(T_i)_{i\in\mathcal{N}}$, *where* $X = \{x = (x_i)_{i\in\mathcal{N}} : x_i \in X_i,\ \|x\| := \sup_{i\in\mathcal{N}}\{|x_i|\} < \infty\}$, $X_0 = \{x = (x_i)_{i\in\mathcal{N}} : x_i \in X_{0_i}\}$, $U = \{u = (u_i)_{i\in\mathcal{N}} : u_i \in U_i,\ \|u\| := \sup_{i\in\mathcal{N}}\{|u_i|\} < \infty\}$, $Y = \prod_{i\in\mathcal{N}} Y_i^1$, $\mathcal{F}(x,u) = \{(x_i^+)_{i\in\mathcal{N}} | x_i^+ \in \mathcal{F}_i(x_i, w_i, u_i)\}$, $\mathcal{H}(x) = (\mathcal{H}_i^1(x_i))_{i\in\mathcal{N}}$, *and with the internal variables constrained by* $w_i = (y_j^2)_{j\in\mathcal{N}_i} = (h_j^2(x_j))_{j\in\mathcal{N}_i}$, $\prod_{j\in\mathcal{N}_i} Y_j^2 \subseteq W_i$, $\forall j \in \mathcal{N}_i, \forall i \in \mathcal{N}$.

The interconnection scheme of network T based on small-gain formulation is illustrated in Figure 2.2.

Remark 2.2.4. *If* $\mathcal{N} := [1; N]$, $N \in \mathbb{N}$, *sets* X, X_0, *and* U *in Definition 2.2.3 can be also written as those in Definition 2.2.2.* ◇

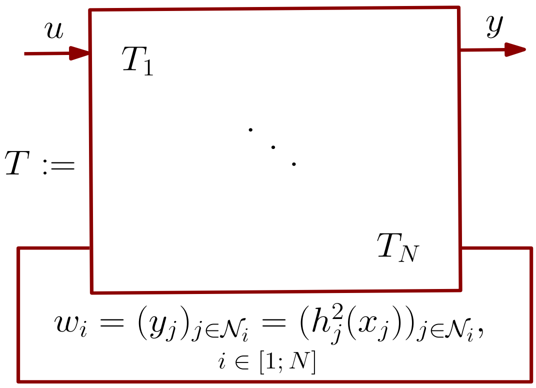

Figure 2.2: The interconnection scheme for network T composed of N systems based on the small-gain approach formulation.

2.2.2 Alternating Simulation Functions

Next we introduce a notion of so-called alternating simulation functions, inspired by [GP09, Definition 1], which quantitatively relates two network of transition systems as in Subsection 2.2.1.

Definition 2.2.5. *Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be networks of transition systems with $\hat{Y} \subseteq Y$. A function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called an alternating simulation function from $\hat{T}$ to T if there exist $\tilde{\alpha}, \tilde{\sigma} \in \mathcal{K}_\infty$, with $\tilde{\sigma} \leq \mathsf{id}$, $\tilde{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\tilde{\varepsilon} \in \mathbb{R}_{\geq 0}$ so that the following hold:*

- *For every $x \in X, \hat{x} \in \hat{X}$, one has*

$$\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|) \leq \tilde{\mathcal{S}}(x, \hat{x}). \tag{2.2.1}$$

- *For every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$, there exists $u \in U$ such that for every $x^+ \in \mathcal{F}(x, u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u})$ so that*

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \max\{\tilde{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})), \tilde{\rho}_u(\|\hat{u}\|), \tilde{\varepsilon}\}. \tag{2.2.2}$$

We say that $\hat{T}$ is an abstraction of T and write $\hat{T} \preceq_{\tilde{\mathcal{S}}} T$ if there exists an alternating simulation function from $\hat{T}$ to T. In addition, if $\hat{T}$ is finite ($\hat{X}$ and $\hat{U}$ are finite sets), we say that $\hat{T}$ is a symbolic model of T. Moreover, note that when T is a finite network, we have $\|\cdot\| = |\cdot|$. Hence, from now on, we use $|\cdot|$ instead of $\|\cdot\|$ in the conditions 2.2.1 and 2.2.2 if the network is finite.

Let us point out some differences between our notion of alternating simulation function and the one in Definition 1 in [GP09]. The notion of simulation function in [GP09, Definition 1] is defined between two continuous-time control systems, whereas in Definition 2.2.5, we define the alternating simulation function between two transition systems which can be used to represent several classes of systems including continuous-time control systems. Additionally, on the right-hand-side of (2.2.2), we introduce constant $\tilde{\varepsilon} \in \mathbb{R}_{\geq 0}$ to allow the relation to be defined between two systems defined over either infinite or finite state and input sets. The role of this constant will become clear later when we introduce symbolic models. Such a constant does not appear in [GP09, Definition 1] which makes it only suitable for systems defined over infinite sets. Furthermore, we formulate the decay condition (2.2.2) in a *max-form*, while in [GP09] the decay condition is formulated in an *implication-form*.

The following technical lemmas are used to prove some of the results in this chapter and the following ones.

Lemma 2.2.6. *For any $a, b \in \mathbb{R}_{>0}$, the following holds*

$$a + b \leq \max\{(\mathsf{id} + \lambda)(a), (\mathsf{id} + \lambda^{-1})(b)\}, \tag{2.2.3}$$

for any $\lambda \in \mathcal{K}_\infty$.

Proof. Define $c = \lambda^{-1}(b)$. Now, one has

$$a + b = \begin{cases} a + \lambda(c) \leq c + \lambda(c) = (\mathsf{id} + \lambda^{-1})(b) & \text{if } a \leq c, \\ a + \lambda(c) < a + \lambda(a) = (\mathsf{id} + \lambda)(a) & \text{if } a > c, \end{cases}$$

which implies (2.2.3). □

The next lemmas are borrowed from [Kel14].

Lemma 2.2.7. *Consider $\alpha \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$, where $(\chi - \mathsf{id}) \in \mathcal{K}_\infty$. Then for any $a, b \in \mathbb{R}_{\geq 0}$*

$$\alpha(a + b) \leq \alpha \circ \chi(a) + \alpha \circ \chi \circ (\chi - \mathsf{id})^{-1}(b).$$

Lemma 2.2.8. *For any function $\bar{\sigma} \in \mathcal{K}_\infty$, there exists a function $\hat{\sigma} < \mathsf{id} \in \mathcal{K}_\infty$ satisfying $\hat{\sigma} \leq \tilde{\sigma}$.*

The next theorem shows that the decay condition (2.2.2) of the alternating simulation function in Definition 2.2.5 can be also formulated in a sum-form.

Theorem 2.2.9. *Consider systems T and $\hat{T}$ and function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ as in Definition 2.2.5. Assume that there exist functions $\bar{\alpha}, \bar{\sigma} \in \mathcal{K}_\infty$, with $\bar{\sigma} < \mathsf{id}$, $\bar{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\bar{\varepsilon} \in \mathbb{R}_{\geq 0}$ such that for every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$, there exists $u \in U$ such that for every $x^+ \in \mathcal{F}(x, u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u})$ so that*

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \bar{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) + \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon}. \tag{2.2.4}$$

Then there exist $\tilde{\alpha}, \tilde{\sigma} \in \mathcal{K}_\infty$, with $\tilde{\sigma} < \mathsf{id}$, $\tilde{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\tilde{\varepsilon} \in \mathbb{R}_{\geq 0}$ such that $\tilde{\mathcal{S}}$ satisfies (2.2.2).

Proof. Since $\bar{\sigma} < \mathsf{id}$, define $\hat{\sigma} = \mathsf{id} - \bar{\sigma} \in \mathcal{K}_\infty$. Let ψ be a $\mathcal{K}_\infty$ function with $\psi < \mathsf{id}$, and define $c = \hat{\sigma}^{-1} \circ \psi^{-1}(\bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon})$. Let $D = \{(x, \hat{x}) \in X \times \hat{X} | \tilde{\mathcal{S}}(x, \hat{x}) \leq c\}$. First, assume $(x, \hat{x}) \in D$. Then $\tilde{\mathcal{S}}(x, \hat{x}) \leq c$, that is, $\psi \circ \hat{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) \leq \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon}$. Since $\hat{\sigma} < \mathsf{id}$, and $\psi \circ \hat{\sigma}(c) = \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon}$, and by using (2.2.4), one obtains

$$\begin{aligned}\tilde{\mathcal{S}}(x^+, \hat{x}^+) &\leq \bar{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) + \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon} \\ &\leq (\mathsf{id} - \hat{\sigma})(\tilde{\mathcal{S}}(x, \hat{x})) + \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon} \leq (\mathsf{id} - \hat{\sigma})(c) + \psi \circ \hat{\sigma}(c) \\ &\leq c - \hat{\sigma}(c) + \psi \circ \hat{\sigma}(c) \leq -(\mathsf{id} - \psi)(\hat{\sigma}(c)) + c \leq c,\end{aligned}$$

Using the definition of c and by utilizing Lemmas 2.2.6 and 2.2.7, we have the following inequality

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \hat{\sigma}^{-1} \circ \psi^{-1}(\bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon}) \leq \max\{\tilde{\rho}_u(\|\hat{u}\|), \tilde{\varepsilon}\}, \tag{2.2.5}$$

where $\tilde{\rho}_u = (\mathsf{id} + \lambda) \circ \hat{\sigma}^{-1} \circ \psi^{-1} \circ \chi \circ \bar{\rho}_u$, and $\tilde{\varepsilon} = (\mathsf{id} + \lambda^{-1}) \circ \hat{\sigma}^{-1} \circ \psi^{-1} \circ \chi \circ (\chi - \mathsf{id})^{-1}(\bar{\varepsilon})$, for some arbitrarily chosen $\lambda, \chi \in \mathcal{K}_\infty$ with $\chi > \mathsf{id}$.

Now assume $(x, \hat{x}) \notin D$. Then $\psi \circ \hat{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) > \bar{\rho}_u(\|\hat{u}\|) + \bar{\varepsilon}$, and one has

$$\begin{aligned}\tilde{\mathcal{S}}(x^+, \hat{x}^+) &\leq \tilde{\mathcal{S}}(x, \hat{x}) - \hat{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) + \psi \circ \hat{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})) \leq \tilde{\mathcal{S}}(x, \hat{x}) - (\mathsf{id} - \psi)(\hat{\sigma}(\tilde{\mathcal{S}}(x, \hat{x}))) \\ &\leq -\tilde{\psi}(\tilde{\mathcal{S}}(x, \hat{x})) + \tilde{\mathcal{S}}(x, \hat{x}) \leq (\mathsf{id} - \tilde{\psi})(\tilde{\mathcal{S}}(x, \hat{x})),\end{aligned} \tag{2.2.6}$$

for all $x^+ \in f(x, u)$ and some $\hat{x}^+ \in \hat{f}(\hat{x}, \hat{u})$, where $\tilde{\psi}(s) := (\mathsf{id} - \psi) \circ \hat{\sigma}$. Observe that $(\mathsf{id} - \tilde{\psi})$ is a $\mathcal{K}_\infty$ function since $\mathsf{id} - \psi$ and $\hat{\sigma}$ are $\mathcal{K}_\infty$ functions and $(\mathsf{id} - \tilde{\psi}) < \mathsf{id}$. From (2.2.6) and by defining $\tilde{\sigma} := (\mathsf{id} - \tilde{\psi})$, one gets

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \tilde{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})). \tag{2.2.7}$$

Combining (2.2.5) and (2.2.7), one gets

$$\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \max\{\tilde{\sigma}(\tilde{\mathcal{S}}(x, \hat{x})), \tilde{\rho}_u(\|\hat{u}\|), \tilde{\varepsilon}\},$$

which completes the proof. □

Before showing the next result, let us recall the definition of an alternating simulation relation introduced in [PT09].

Definition 2.2.10. *Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be networks of transition systems with $\hat{Y} \subseteq Y$. A relation $R \subseteq X \times \hat{X}$ is called an $\hat{\varepsilon}$-approximate alternating simulation relation from $\hat{T}$ to T if for any $(x, \hat{x}) \in R$*

(i) $\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\| \leq \hat{\varepsilon}$*;*

(ii) For any $\hat{u} \in \hat{U}$, there exists $u \in U$ such that for all $x^+ \in \mathcal{F}(x, u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u})$ satisfying $(x^+, \hat{x}^+) \in R$.

In addition, if *(ii)* still holds when reversing the role of T and $\hat{T}$, the relation R is in fact an $\hat{\varepsilon}$-approximate alternating bisimulation relation between T and $\hat{T}$ [PT09].

The next result shows that the existence of an alternating simulation function for networks of transition systems implies the existence of an approximate alternating simulation relation between them as defined above.

Proposition 2.2.11. *Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be networks of transition systems with $\hat{Y} \subseteq Y$. Assume $\tilde{\mathcal{S}}$ is an alternating simulation function from $\hat{T}$ to T as in Definition 2.2.5 and that there exists $r \in \mathbb{R}_{>0}$ such that $\|\hat{u}\| \leq r$ for all $\hat{u} \in \hat{U}$. Then, relation $R \subseteq X \times \hat{X}$ defined by*

$$R = \left\{(x, \hat{x}) \in X \times \hat{X} | \tilde{\mathcal{S}}(x, \hat{x}) \leq \max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\}\right\},$$

is an $\hat{\varepsilon}$-approximate alternating simulation relation from $\hat{T}$ to T with

$$\hat{\varepsilon} = \tilde{\alpha}^{-1}(\max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\}). \tag{2.2.8}$$

Proof. Item (i) in Definition 2.2.10 is a simple consequence of the definition of R and condition (2.2.1) (i.e. $\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|) \leq \tilde{\mathcal{S}}(x, \hat{x}) \leq \max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\}$), which results in $\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\| \leq \tilde{\alpha}^{-1}(\max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\}) = \hat{\varepsilon}$. Item (ii) in Definition 2.2.10 follows immediately from the definition of R, condition (2.2.2), and the fact that $\tilde{\sigma} \leq \mathsf{id}$. In particular, we have $\tilde{\mathcal{S}}(x^+, \hat{x}^+) \leq \max\{\tilde{\rho}_u(r), \tilde{\varepsilon}\}$ which implies $(x^+, \hat{x}^+) \in R$. □

The approximate alternating simulation relation guarantees that for each output behavior of T there exists one of $\hat{T}$ such that the distance between these output behaviors is uniformly bounded by $\hat{\varepsilon}$.

Remark 2.2.12. *Since the input set in all practical applications is bounded, requiring the control inputs to be bounded is not restrictive at all. Moreover, under certain stability properties of concrete systems, one can choose function $\tilde{\rho}_u$ in (2.2.8) to be identically zero which cancels the dependency on the size of control inputs in Proposition 2.2.11.* ⋄

2.3 Discrete-Time Control Systems

The discrete-time control systems considered here are defined below.

Definition 2.3.1. *A discrete-time control system Σ is defined by the tuple*

$$\Sigma = (\mathbb{X}, \mathbb{W}, \mathbb{U}, f, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2), \tag{2.3.1}$$

where

- $\mathbb{X} \subseteq \mathbb{R}^n, \mathbb{U} \subseteq \mathbb{R}^m, \mathbb{W} \subseteq \mathbb{R}^r, \mathbb{Y}^1 \subseteq \mathbb{R}^{q^1}$, *and* $\mathbb{Y}^2 \subseteq \mathbb{R}^{q^2}$ *are the state set, external input set, internal input set, external output set, and internal output set, respectively;*
- $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X}$ *is the transition function;*
- $h^1 : \mathbb{X} \to \mathbb{Y}^1$ *is the external output map;*
- $h^2 : \mathbb{X} \to \mathbb{Y}^2$ *is the internal output map.*

The discrete-time control system Σ *is described by difference equations of the form*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f(\mathbf{x}(k), \omega(k), \nu(k)), \\ \mathbf{y}^1(k) &= h^1(\mathbf{x}(k)), \\ \mathbf{y}^2(k) &= h^2(\mathbf{x}(k)), \end{cases} \tag{2.3.2}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$, $\omega : \mathbb{N} \to \mathbb{W}$, $\nu : \mathbb{N} \to \mathbb{U}$, $\mathbf{y}^1 : \mathbb{N} \to \mathbb{Y}^1$, *and* $\mathbf{y}^2 : \mathbb{N} \to \mathbb{Y}^2$ *are the state signal, internal input signal, external input signal, external output signal, and internal output signal, respectively.*

If Σ *is linear,* (2.3.2) *reduces to*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A\mathbf{x}(k) + D\omega(k) + B\nu(k), \\ \mathbf{y}^1(k) &= C^1\mathbf{x}(k), \\ \mathbf{y}^2(k) &= C^2\mathbf{x}(k), \end{cases} \tag{2.3.3}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C^1 \in \mathbb{R}^{q^1 \times n}$, $C^2 \in \mathbb{R}^{q^2 \times n}$, *and* $D \in \mathbb{R}^{n \times r}$. *We use the tuple* $\Sigma = (A, B, C^1, C^2, D)$ *to refer to the class of control systems of the form* (2.3.3).

2.3.1 Discrete-Time Control Systems as Transition Systems

Here, we represent discrete-time control systems as transition systems. Such a representation allows us to write discrete-time control systems and their symbolic models in a unified way.

Definition 2.3.2. *Given a discrete-time control system* $\Sigma = (\mathbb{X}, \mathbb{W}, \mathbb{U}, f, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$ *we define the associated transition system* $T(\Sigma) = (X, X_0, W, U, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$, *where:* $X = \mathbb{X}$, $X_0 = \mathbb{X}$, $W = \mathbb{W}$, $U = \mathbb{U}$, $Y^1 = \mathbb{Y}^1$, $Y^2 = \mathbb{Y}^2$, $\mathcal{H}^1 = h^1$, $\mathcal{H}^2 = h^2$, *and* $x^+ \in \mathcal{F}(x, w, u)$ *if and only if* $x^+ = f(x, w, u)$.

2.4 Discrete-Time Switched Systems

In this section, we consider discrete-time switched systems as defined below.

Definition 2.4.1. *A discrete-time switched system* Σ *is defined by the tuple*

$$\Sigma \quad = (\mathbb{X}, \mathbb{W}, P, F, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2), \tag{2.4.1}$$

where

- $\mathbb{X} \subseteq \mathbb{R}^n, \mathbb{W} \subseteq \mathbb{R}^r, \mathbb{Y}^1 \subseteq \mathbb{R}^{q^1}$, *and* $\mathbb{Y}^2 \subseteq \mathbb{R}^{q^2}$ *are the state set, internal input set, external output set, and internal output set, respectively;*
- $P = \{1, \ldots, m\}$ *is a finite set of modes;*
- $F = \{f_1, \ldots, f_m\}$ *is a collection of transition maps* $f_p : \mathbb{X} \times \mathbb{W} \to \mathbb{X}$ *for all* $p \in P$;
- $h^1 : \mathbb{X} \to \mathbb{Y}^1$ *is the external output map;*

- $h^2 : \mathbb{X} \to \mathbb{Y}^2$ *is the internal output map.*

The discrete-time switched system Σ *is described by difference equations of the form*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f_{\mathsf{p}(k)}(\mathbf{x}(k), \omega(k)), \\ \mathbf{y}^1(k) &= h^1(\mathbf{x}(k)), \\ \mathbf{y}^2(k) &= h^2(\mathbf{x}(k)), \end{cases} \tag{2.4.2}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$, $\omega : \mathbb{N} \to \mathbb{W}$, $\mathsf{p} : \mathbb{N} \to \mathbb{P}$, $\mathbf{y}^1 : \mathbb{N} \to \mathbb{Y}^1$, *and* $\mathbf{y}^2 : \mathbb{N} \to \mathbb{Y}^2$ *are the state signal, internal input signal, switching signal, external output signal, and internal output signal, respectively. We denote by* Σ_p *the system* (2.4.2) *with constant switching signal* $\mathsf{p}(k) = p \in P \ \forall k \in \mathbb{N}_{\geq 1}$. *Let* $\phi_k, k \in \mathbb{N}_{\geq 1}$, *denote the time when the k-th switching instant occurs and define* $\Phi := \{\phi_k : k \in \mathbb{N}_{\geq 1}\}$ *as the set of switching instants. We assume that signal* p *satisfies a dwell-time condition [Mor96] (i.e. there exists* $k_d \in \mathbb{N}_{\geq 1}$, *called the dwell-time, such that for all consecutive switching time instants* $\phi_k, \phi_{k+1} \in \Phi$, $\phi_{k+1} - \phi_k \geq k_d$*).*

If Σ *is an affine switched system,* (2.4.2) *reduces to*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A_{\mathsf{p}(k)}\mathbf{x}(k) + D_{\mathsf{p}(k)}\omega(k) + B_{\mathsf{p}(k)}, \\ \mathbf{y}^1(k) &= C^1\mathbf{x}(k), \\ \mathbf{y}^2(k) &= C^2\mathbf{x}(k), \end{cases} \tag{2.4.3}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^n$, $C^1 \in \mathbb{R}^{q^1\times n}$, $C^2 \in \mathbb{R}^{q^2\times n}$, *and* $D_p \in \mathbb{R}^{n\times r}$, *for all* $p \in P$. *We use the tuple* $\Sigma = (A, B, C^1, C^2, D)$ *to refer to the class of control systems of the form* (2.4.3), *where* $A = \{A_1, \ldots, A_m\}$, $B = \{B_1, \cdots, B_m\}$, $D = \{D_1, \ldots, D_m\}$.

2.4.1 Discrete-Time Switched Systems as Transition Systems

Similar to Subsection 2.3.1, we also define discrete-time switched systems as transition systems.

Definition 2.4.2. *Given a discrete-time switched system* $\Sigma = (\mathbb{X}, \mathbb{W}, P, F, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$ *we define the associated transition system* $T(\Sigma) = (X, X_0, W, U, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$, *where:*

- $X = \mathbb{X} \times P \times \{0, \cdots, k_d - 1\}$ *is the state set;*
- $X_0 = \mathbb{X} \times P \times \{0\}$ *is the initial state set;*
- $U = P$ *is the external input set;*
- $W = \mathbb{W}$ *is the internal input set;*
- $(x^+, p^+, l^+) \in \mathcal{F}((x, p, l), u, w)$ *if and only if* $x^+ = f_p(x, w), u = p$ *and the following scenarios hold:*
  - $l < k_d - 1$, $p^+ = p$ *and* $l^+ = l + 1$*: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;*

- $l = k_d - 1$, $p^+ = p$ *and* $l^+ = k_d - 1$*: switching is allowed but no switch occurs;*
- $l = k_d - 1$, $p^+ \neq p$ *and* $l^+ = 0$*: switching is allowed and a switch occurs;*

- $Y^1 = \mathbb{Y}^1$*;*
- $Y^2 = \mathbb{Y}^2$*;*
- $\mathcal{H}^1 : X \to Y^1$ *is the external output map defined as* $\mathcal{H}^1(x,p,l) = h^1(x)$*;*
- $\mathcal{H}^2 : X \to Y^2$ *is the external output map defined as* $\mathcal{H}^2(x,p,l) = h^2(x)$*.*

2.5 General Remark

1. We assume that there exists a unique solution to (2.3.2) and (2.4.2) describing the evolution of the discrete-time control and switched systems in (2.3.1) and (2.4.1), respectively.
2. The external and internal output map h^j, $j \in [1;2]$, for systems in (2.3.1) and (2.4.1), satisfy the following general Lipschitz assumption: there exist $\ell^j \in \mathcal{K}_\infty$ such that $|h^j(x) - h^j(x')| \leq \ell^j(|x-x'|)$ for all $x, x' \in \mathbb{X}$ and for all $j \in [1;2]$. Note that this assumption on h^j, $j \in [1;2]$ is not restrictive at all provided that one is interested to work on a compact subset of $\mathbb{X}$.
3. The network of the discrete-time control and switched systems in (2.3.1) and (2.4.1) based on the Dissipativity and small-gain approaches can be similarly defined to the network of transition systems in Definitions 2.2.1.1 and 2.2.1.2. Those definition will be formally given in the following chapters.
4. The use of transition systems as an alternative description for discrete-time control and switched systems allows one to define infinite and finite systems in a common framework. Moreover, it allows us to directly apply the compositional frameworks, which will be introduced in the following chapters.

3 Symbolic Models for (In)finite Networks of Discrete-Time Control Systems

3.1 Introduction

Large-scale networks appear in a wide variety of modern applications, including traffic networks [Ker19], transportation systems [Cas01], and power systems [Qua77]. Moreover, in many applications, a system is considered as a finite but very large network with possibly unknown number of systems; see [JB05, BPD02, Li11] and references therein. Hence, it is reasonable to over-approximate such a system by an *infinite network* which is seen as an interconnection of infinitely many finite-dimensional systems. In general, designing controllers enforcing sophisticated properties on those large networks is a challenging problem. Construction of symbolic models was introduced in recent years as a promising method to address these issues. Unfortunately, the construction of symbolic models for large-scale systems is itself computationally a complex and challenging task. An appropriate technique to overcome this challenge is to first construct symbolic models of the concrete systems individually and then establish a compositional framework using which one can construct a symbolic model of the overall network using those individual symbolic models. Motivated by the above limitation, this chapter is concerned with providing compositional approaches for designing symbolic models for (in)finite network of discrete-time systems. We propose a compositional technique for the construction of a notion of so-called alternating simulation function as a relation between a network of symbolic models and that of discrete-time systems. The alternating simulation function provides a formal upper bound for the mismatch between the output behaviors of the concrete and the abstract network.

3.1.1 Related Work

3.1.1.1 Finite Network

In the past few years, there have been several results on the compositional construction of symbolic models of networks of control systems. The framework introduced by [TI08] based on the notion of interconnection-compatible approximate bisimulation relation provides networks of symbolic models approximating networks of stabilizable linear control systems. This work was extended by [PPB16] to networks of incrementally input-to-state stable nonlinear control systems using the notion of approximate bisimulation relation. The recent result by [MSSM19] introduces a new system relation, called (approximate) disturbance bisimulation relation, as the basis for the compositional construction of symbolic models. The proposed results by [TI08], [PPB16], and [MSSM19] use the small-gain type conditions and provide *complete* symbolic models of interconnected systems compositionally. Unfortunately, those small-gain type conditions are *conservative*, in the sense that they are all formulated in terms of "almost" linear gains, which means the considered systems should have a (nearly) linear behavior. Those conditions may not hold in general for systems with nonlinear gain functions. Additionally, those small-gain type conditions depend essentially on the size of the network graph and can be violated as the number of systems in the network increases [DK04]. Moreover, all compositional techniques for the construction of symbolic models introduced in [TI08, PPB16, MSSM19] are tailored to networks composed of finitely many systems and can *not* be directly applied to networks consisting of an infinitely many components.

3.1.1.2 Infinite Network

Construction of symbolic models for infinite dimensional systems is already proposed in [PPB10, Gir14, PPB15, JZ20]. In [PPB10], symbolic models are constructed for nonlinear continuous time-delay systems with known and constant delays. This work was extended in [PPB15] to the same class of systems with unknown and time-varying delays. The results in [Gir14] provide a generic state-space discretization-free approach for computing symbolic models of finite or infinite dimensional systems which are incrementally stable. A state-space discretization-free approach was also introduced in [JZ20] for designing symbolic models for infinite dimensional stochastic systems, particularly, retarded jump-diffusion systems. While the results in [PPB10, PPB15, JZ20] deal with time-delay systems evolving over finite-dimensional state spaces, here we deal with an interconnection of infinitely many finite-dimensional systems evolving over infinite-dimensional state spaces. The result in [Gir14] deals with a single incrementally stable infinite-dimensional system with finite-dimensional input space and the finite abstraction is based on input sequences which is not the case in this work. Here both state and input spaces of the concrete network is infinite-dimensional and the construction of symbolic models is based on the discretization of both state and input spaces. Moreover, all the proposed results in [PPB10, Gir14, PPB15, JZ20] take a monolithic view of the systems while constructing symbolic models. However, our result provides a compositional approach on the construction of a symbolic model of the network using those of local systems.

3.1.2 Contributions

In the first part of this chapter, we introduce a compositional approach for the construction of symbolic models of finite networks of discrete-time control systems by leveraging techniques from dissipativity theory [AMP16]. First, we introduce a notion of so-called sum-type simulation function inspired by the one introduced in [ZA17] and use it to quantify the joint dissipativity-type properties of local discrete-time control systems and their symbolic models. Given a sum-type simulation function between systems and their symbolic models, we drive compositional conditions under which one can construct a so-called alternating simulation function as a relation between the network of symbolic models and that of control systems. The existence of such an alternating simulation function ensures that the output behavior of the concrete network is quantitatively approximated by that of its symbolic model. In addition, we provide a procedure for the construction of symbolic models together with their corresponding sum-type simulation functions for a class of discrete-time control systems satisfying some incremental passivity property. We also show that for a network of linear discrete-time control systems, the aforementioned incremental passivity property can be readily verified by checking a matrix inequality. Finally, we apply the proposed results to the temperature regulation in a circular building and construct compositionally a symbolic model of a network containing 1200 rooms. Moreover, we show the effectiveness of the proposed results in the case of fully connected network as well.

In the second part of the chapter, we introduce a compositional approach for the construction of symbolic models for network of discrete-time control systems using more general small-gain type conditions. First, we introduce a notion of so-called max-type simulation functions inspired by [GP09, Definition 1] as a system relation. Given max-type simulation functions between local systems and their symbolic models, we derive some small-gain type conditions to construct an overall alternating simulation function as a relation between the interconnected abstractions and the concrete network. Those general small-gain type compositional conditions are formulated in a general nonlinear form which can be applied to both linear and nonlinear gain functions without making any pre-assumptions on them. We show that our proposed small-gain type condition is much more general than the ones proposed in [PPB16, MSSM19]. Moreover, we introduce a compositional methodology for constructing symbolic models for a network composed of infinitely many finite-dimensional systems. Based on the recently developed small-gain theorem [DMSW19], we show that an alternating simulation function can be constructed by composing max-type simulation functions relating each finite-dimensional system and its symbolic model. In addition, we provide a framework for the construction of symbolic models together with their corresponding max-type simulation functions for discrete-time control systems satisfying an incremental input-to-state stability property [TRK16]. Finally, we illustrate results based on the small-gain approach in three case studies by compositionally constructing symbolic models of three networks of (linear and nonlinear) discrete-time control systems and their corresponding alternating simulation functions. The first and second case studies particularly elucidate the effectiveness of the proposed results in comparison with the compositionality result using dissipativity-type conditions in the first part of this chapter. The third case study shows the effectiveness of our proposed technique by applying it to a model of a road traffic network containing infinitely many cells (systems). We construct symbolic models for the original systems and compositionally construct an alternating simulation function from the interconnection of infinitely many symbolic models to the interconnection of the concrete systems. We also design controllers compositionally maintaining the density of traffic between 10 and 25 vehicles per cell.

3.2 Dissipativity Approach (DA)

3.2.1 Finite Networks of Discrete-Time Control Systems: DA Formulation

Definition 3.2.1. *Consider discrete-time control systems* $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, f_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$*,* $i \in \mathcal{N} := [1; N]$*,* $N \in \mathbb{N}$*. Let* M *be a static matrix of an appropriate dimension defining the coupling of these system such that* $M \prod_{i\in\mathcal{N}} \mathbb{Y}_i^2 \subseteq \prod_{i\in\mathcal{N}} \mathbb{W}_i$*. The network of discrete-time control system is a tuple* $\Sigma = (\mathbb{X}, \mathbb{U}, f, \mathbb{Y}, h)$*, denoted by* $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$*, where* $\mathbb{X} = \prod_{i\in\mathcal{N}} \mathbb{X}_i$*,* $\mathbb{U} = \prod_{i\in\mathcal{N}} \mathbb{U}_i$*,* $\mathbb{Y} = \prod_{i\in\mathcal{N}} \mathbb{Y}_i^1$*. Moreover,* f *and* h *are defined as*

$$f(x, u) = (f_i(x_i, w_i, u_i))_{i\in\mathcal{N}}, h(x) = (h_i^1(x_i))_{i\in\mathcal{N}},$$

where $x = (x_i)_{i\in\mathcal{N}}$*,* $u = (u_i)_{i\in\mathcal{N}}$*, and with the internal variables constrained by* $(w_i)_{i\in\mathcal{N}} = M(h_i^2(x_i))_{i\in\mathcal{N}}$*. The network is described by the difference equations*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f(\mathbf{x}(k), \nu(k)), \\ \mathbf{y}(k) &= h(\mathbf{x}(k)), \end{cases} \tag{3.2.1}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$*,* $\nu : \mathbb{N} \to \mathbb{U}$*, and* $\mathbf{y} : \mathbb{N} \to \mathbb{Y}$*.*

Note that condition $M \prod_{i\in\mathcal{N}} \mathbb{Y}_i^2 \subseteq \prod_{i\in\mathcal{N}} \mathbb{W}_i$ is required to have a well-defined interconnection.

3.2.2 Sum-Type Simulation Functions

Consider a network of discrete-time control systems $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$, or their equivalent network of transition systems $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$, where each $T_i(\Sigma_i)$ given as in Definition 2.3.2. Assume that each system $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ and $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ admit a sum-type simulation function as defined next.

Definition 3.2.2. *Consider systems* $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ *and* $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ *where* $\hat{Y}_i^1 \subseteq Y_i^1$*. A function* $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ *is called a sum-type simulation function from* $\hat{T}_i(\Sigma_i)$ *to* $T_i(\Sigma_i)$ *if there exist* $\alpha_i, \sigma_i \in \mathcal{K}_\infty$*,* $\rho_{u_i} \in \mathcal{K}_\infty \cup \{0\}$*, a symmetric matrix* R_i *of appropriate dimension with conformal block partitions* $R_i^{i'j'}$*,* $i', j' \in [1; 2]$*, and some* $\varepsilon_i \in \mathbb{R}_{\geq 0}$ *so that the following hold:*

- *For every* $x_i \in X_i, \hat{x}_i \in \hat{X}_i$*, one has*

$$\alpha_i(|\mathcal{H}_i^1(x_i) - \hat{\mathcal{H}}_i^1(\hat{x}_i)|) \leq \mathcal{S}_i(x_i, \hat{x}_i). \tag{3.2.2}$$

- *For every* $x_i \in X_i, \hat{x}_i \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$, *there exists* $u_i \in U_i$ *such that for every* $w_i \in W_i, \hat{w}_i \in \hat{W}_i, x_i^+ \in \mathcal{F}_i(x_i, w_i, u_i)$ *there exists* $\hat{x}_i^+ \in \hat{\mathcal{F}}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ *so that*

$$\mathcal{S}_i(x_i^+, \hat{x}_i^+) - \mathcal{S}_i(x_i, \hat{x}_i) \leq - \sigma_i(\mathcal{S}_i(x_i, \hat{x}_i)) + \rho_{u_i}(|\hat{u}_i|) + \varepsilon_i \quad (3.2.3)$$
$$+ \begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(x_i) - \hat{\mathcal{H}}_i^2(\hat{x}_i) \end{bmatrix}^\top \overbrace{\begin{bmatrix} R_i^{11} & R_i^{12} \\ R_i^{21} & R_i^{22} \end{bmatrix}}^{R_i :=} \begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(x_i) - \hat{\mathcal{H}}_i^2(\hat{x}_i) \end{bmatrix}.$$

System $\hat{T}_i(\Sigma_i)$ is called an abstraction of $T_i(\Sigma_i)$, denoted by $\hat{T}_i(\Sigma_i) \preceq_{\mathcal{S}_i^s} T_i(\Sigma_i)$, if there exists a sum-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$. Moreover, if $\hat{T}_i(\Sigma_i)$ is finite, it is called a symbolic model of $T_i(\Sigma_i)$.

3.2.3 Compositional Abstractions for Finite Networks of Discrete-Time Control Systems: DA

We assume that we are given $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, f_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$, or equivalently $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ as in Definition 2.3.2, together with their abstractions $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$, $i \in \mathscr{N}$, and sum-type simulation functions $\mathcal{S}_i$ from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$ as in Definition 3.2.2.

The next theorem provides a compositional approach on the construction of abstractions of the networks of transition systems $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i \in \mathscr{N}}$ associated to network of discrete-time control systems $\Sigma = \mathcal{I}_M(\Sigma_i)_{i \in \mathscr{N}}$ and that of the corresponding alternating simulation function.

Theorem 3.2.3. *Consider the network* $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i \in \mathscr{N}}$ *associated to the network of discrete-time control systems* $\Sigma = \mathcal{I}_M(\Sigma_i)_{i \in \mathscr{N}}$. *Suppose each transition system* $T_i(\Sigma_i)$ *admits an abstraction* $\hat{T}_i(\Sigma_i)$ *with the corresponding sum-type simulation function* $\mathcal{S}_i$. *If there exist* $\mu_i > 0$, $i \in \mathscr{N}$ *such that the matrix inequality and inclusion*

$$\begin{bmatrix} M \\ I_q \end{bmatrix}^\top \tilde{R} \begin{bmatrix} M \\ I_q \end{bmatrix} \preceq 0, \quad (3.2.4)$$

$$M \textstyle\prod_{i=1}^N \hat{Y}_i^2 \subseteq \prod_{i=1}^N \hat{W}_i, \quad (3.2.5)$$

are satisfied, where

$$\tilde{R} := \begin{bmatrix} \mu_1 R_1^{11} & & & \mu_1 R_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N R_N^{11} & & & \mu_N R_N^{12} \\ \mu_1 R_1^{21} & & & \mu_1 R_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N R_N^{21} & & & \mu_N R_N^{22} \end{bmatrix}, \quad (3.2.6)$$

and q is the number of rows in M, then

$$\tilde{\mathcal{S}}(x,\hat{x}) := \sum_{i\in\mathcal{N}} \mu_i \mathcal{S}_i(x_i,\hat{x}_i), \tag{3.2.7}$$

is an alternating simulation function, as in Definition 2.2.5, from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$.

Proof. First we show that inequality (2.2.1) holds. Consider any $x \in X$ and $\hat{x} \in \hat{X}$, one gets:

$$\begin{aligned}|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})| &= |(\mathcal{H}_i^1(x_i) - \hat{\mathcal{H}}_i^1(\hat{x}_i))_{i\in\mathcal{N}}| \\ &\leq \sum_{i=1}^{N} |\mathcal{H}_i^1(x_i) - \hat{\mathcal{H}}_i^1(\hat{x}_i)| \leq \sum_{i=1}^{N} \alpha_i^{-1}(\mathcal{S}_i(x_i,\hat{x}_i)) \leq \hat{\alpha}\big(\tilde{\mathcal{S}}(x,\hat{x})\big),\end{aligned}$$

where $\hat{\alpha}$ is a $\mathcal{K}_\infty$ function defined as

$$\hat{\alpha}(s) \quad = \max_{\hat{s}\geq 0} \left\{ \textstyle\sum_{i=1}^{N} \alpha_i^{-1}(s_i) | \mu^\top \hat{s} = s \right\},$$

where $\hat{s} = (s_i)_{i\in\mathcal{N}} \in \mathbb{R}_{\geq 0}^N$ and $\mu = (\mu_i)_{i\in\mathcal{N}}$. By defining the $\mathcal{K}_\infty$ function $\tilde{\alpha} = \hat{\alpha}^{-1}$, one obtains

$$\tilde{\alpha}(|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})|) \leq \tilde{\mathcal{S}}(x,\hat{x}),$$

satisfying inequality (2.2.1). Now we show that inequality (2.2.2) holds as well. Define

$$\bar{\sigma}(s) := \min_{\hat{s}\geq 0} \left\{ \sum_{i=1}^{N} \mu_i \sigma_i(s_i) | \mu^\top \hat{s} = s \right\}, \tag{3.2.8}$$

$$\bar{\rho}_u(s) := \max_{\hat{s}\geq 0} \left\{ \sum_{i=1}^{N} \mu_i \rho_{u_i}(|s_i|) \mid |\hat{s}| = s \right\}, \bar{\varepsilon} := \sum_{i=1}^{N} \mu_i \varepsilon_i,$$

where $\bar{\sigma} \in \mathcal{K}_\infty$ and $\bar{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$. Moreover, consider condition (3.2.4), and the definition of matrix $\tilde{R}$ in (3.2.6). Then, one gets the chain of inequalities in (3.2.9). Now, without loss of generality, we can consider $\bar{\sigma} < \mathsf{id}$ and write

$$\tilde{\mathcal{S}}(x^+,\hat{x}^+) \leq \sigma(\tilde{\mathcal{S}}\,(x,\hat{x})) + \bar{\rho}_u(|\hat{u}|) + \bar{\varepsilon},$$

where $\sigma = \mathsf{id} - \bar{\sigma}$. Otherwise, we can define $\sigma = \mathsf{id} - \hat{\sigma}$, where $\hat{\sigma}$ is given as in Lemma 2.2.8 for function $\bar{\sigma}$ appearing in (3.2.8). By using the result of Theorem 2.2.9, one obtains

$$\tilde{\mathcal{S}}(x^+,\hat{x}^+) \leq \max\{\tilde{\sigma}(\tilde{\mathcal{S}}\,(x,\hat{x})), \tilde{\rho}_u(|\hat{u}|), \tilde{\varepsilon}\},$$

satisfying (2.2.2) with $\tilde{\sigma} := (\mathsf{id} - (\mathsf{id} - \psi) \circ (\mathsf{id} - \sigma))$, $\tilde{\rho}_u = (\mathsf{id} + \lambda) \circ (\mathsf{id} - \sigma)^{-1} \circ \psi^{-1} \circ \chi \circ \bar{\rho}_u$, and $\tilde{\varepsilon} = (\mathsf{id} + \lambda^{-1}) \circ (\mathsf{id} - \sigma)^{-1} \circ \psi^{-1} \circ \chi \circ (\chi - \mathsf{id})^{-1}(\bar{\varepsilon})$, for some arbitrarily chosen $\lambda, \psi, \chi \in \mathcal{K}_\infty$ with $\psi < \mathsf{id}$ and $\chi > \mathsf{id}$. Hence, $\tilde{\mathcal{S}}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$. □

$$
\begin{aligned}
&\tilde{\mathcal{S}}(x^+,\hat{x}^+)-\tilde{\mathcal{S}}(x,\hat{x})=\sum_{i=1}^{N}\mu_i\big(\mathcal{S}_i(x_i^+,\hat{x}_i^+)-\mathcal{S}_i(x_i,\hat{x}_i)\big)\\
&\leq\sum_{i=1}^{N}\mu_i\bigg(-\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i))+\rho_{u_i}(|\hat{u}_i|)+\varepsilon_i+\begin{bmatrix}w_i-\hat{w}_i\\\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i)\end{bmatrix}^\top\overbrace{\begin{bmatrix}R_i^{11}&R_i^{12}\\R_i^{21}&R_i^{22}\end{bmatrix}}^{R_i:=}\begin{bmatrix}w_i-\hat{w}_i\\\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i)\end{bmatrix}\bigg)\\
&=\sum_{i=1}^{N}-\mu_i\big(\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i))+\rho_{u_i}(|\hat{u}_i|)+\varepsilon_i\big)+\begin{bmatrix}(w_i-\hat{w}_i)_{i\in\mathcal{N}}\\(\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i))_{i\in\mathcal{N}}\end{bmatrix}^\top\tilde{R}\begin{bmatrix}(w_i-\hat{w}_i)_{i\in\mathcal{N}}\\(\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i))_{i\in\mathcal{N}}\end{bmatrix}\\
&\leq\sum_{i=1}^{N}-\mu_i\big(\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i))\big)+\sum_{i=1}^{N}\mu_i\rho_{u_i}(|\hat{u}_i|)+\sum_{i=1}^{N}\mu_i\varepsilon_i\\
&\quad+\big[(\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i))_{i\in\mathcal{N}}\big]^\top\begin{bmatrix}M\\I_q\end{bmatrix}^\top\tilde{R}\begin{bmatrix}M\\I_q\end{bmatrix}\big[(\mathcal{H}_i^2(x_i)-\hat{\mathcal{H}}_i^2(\hat{x}_i))_{i\in\mathcal{N}}\big]\\
&\leq\sum_{i=1}^{N}-\mu_i\big(\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i))\big)+\sum_{i=1}^{N}\mu_i\rho_{u_i}(|\hat{u}_i|)+\sum_{i=1}^{N}\mu_i\varepsilon_i\\
&=-\bar{\sigma}(\tilde{\mathcal{S}}\,(x,\hat{x}))+\bar{\rho}_u(|\hat{u}|)+\bar{\varepsilon}.
\end{aligned}
\tag{3.2.9}
$$

Figure 3.1 schematically shows the results of Theorem 3.2.3.

Remark 3.2.4. *Condition* (3.2.4) *is a linear matrix inequality which can be verified by some semi-definite programming tools (e.g. YALMIP [Lof04] or SeDuMi [Stu98]). Note that condition* (3.2.5) *is required to have a well-defined interconnection of abstractions and is automatically fulfilled if one constructs the internal input sets of each abstractions* $\hat{T}_i(\hat{\Sigma}_i)$ *such that the equality* $M\prod_{i=1}^{N}\hat{Y}_{2i}=\prod_{i=1}^{N}\hat{W}_i$ *holds.* ◇

3.2.4 Construction of Symbolic Models

In the following, we introduce some stability properties for $\Sigma=(\mathbb{X},\mathbb{W},\mathbb{U},f,\mathbb{Y}^1,\mathbb{Y}^2,h^1,h^2)$ based on which one can construct its symbolic model along with the corresponding sum-type simulation functions between Σ and its symbolic model.

3.2.4.1 Incremental Passivity

Definition 3.2.5. *System* Σ *is called incrementally passive (δ-P) if there exist functions* $S:\mathbb{X}\times\mathbb{X}\rightarrow\mathbb{R}_{\geq0}$, $\varphi_x,\varphi_s,\varphi_u\in\mathcal{K}_\infty$, *with* $\varphi_s<\mathsf{id}$, *and a symmetric matrix* Q *of appropriate dimension, such that for all* $x,x'\in\mathbb{X}$, $u,u'\in\mathbb{U}$, *and for all* $w,w'\in\mathbb{W}$

$$\varphi_x(|x-x'|)\leq S(x,x'), \tag{3.2.10}$$

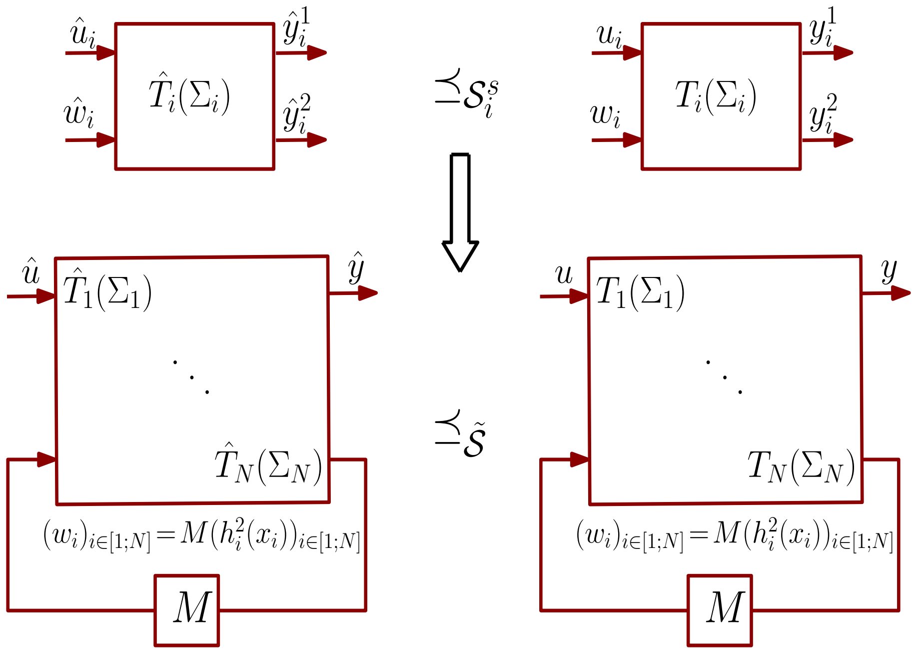

Figure 3.1: Compositionality results for constructing network of abstractions provided that condition (3.2.4) and (3.2.5) are satisfied.

$$S(f(x,w,u),f(x',w',u')) \leq \varphi_s(S(x,x')) + \varphi_u(|u-u'|) \qquad (3.2.11)$$
$$+ \begin{bmatrix} w-w' \\ h^2(x)-h^2(x') \end{bmatrix}^\top \overbrace{\begin{bmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{bmatrix}}^{Q:=} \begin{bmatrix} w-w' \\ h^2(x)-h^2(x') \end{bmatrix}.$$

We say that S and Q are δ-P storage function and supply rate, respectively, for system Σ if they satisfy (3.2.10) and (3.2.11).

3.2.4.2 Symbolic Models

In the following lines, we show how to construct a symbolic model $\hat{T}(\Sigma)$ of transition system $T(\Sigma)$ associated to a δ-P discrete-time control system Σ.

Definition 3.2.6. *Consider a transition system* $T(\Sigma) = (X, X_0, W, U, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$, *associated to a δ-P discrete-time control system* $\Sigma = (\mathbb{X}, \mathbb{W}, \mathbb{U}, f, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$. *Suppose* $\mathbb{X}, \mathbb{W}, \mathbb{U}$ *are finite unions of boxes (see Chapter 2, Section 2.1). Then one can construct a symbolic model* $\hat{T}(\Sigma) = (\hat{X}, \hat{X}_0, \hat{W}, \hat{U}, \hat{\mathcal{F}}, \hat{Y}^1, \hat{Y}^2, \hat{\mathcal{H}}^1, \hat{\mathcal{H}}^2)$ *where:*

- $\hat{X} = [X]_{\eta^x}$, *where* $0 < \eta^x \leq \mathrm{span}(X)$ *is the state set quantization parameter;*
- $\hat{X}_0 = [X_0]_{\eta^x}$;
- $\hat{W} = [W]_{\eta^w}$, *where* $0 < \eta^w \leq \mathrm{span}(W)$ *is the internal input set quantization parameter;*

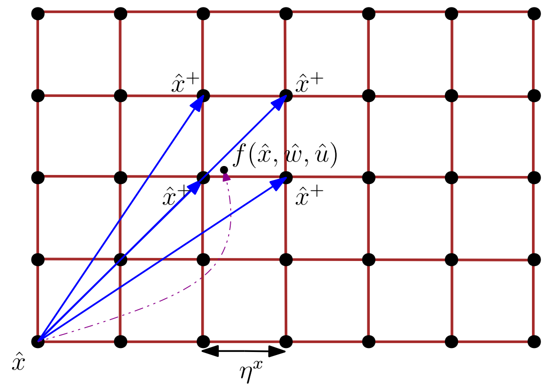

Figure 3.2: An illustration of the computation of the transitions of $\hat{T}(\Sigma)$ for particular $\hat{x}$, $\hat{u}$, $\hat{w}$.

- $\hat{U} = [U]_{\eta^u}$, *where* $0 < \eta^u \leq \text{span}(U)$ *is the external input set set quantization parameter;*
- $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{w}, \hat{u})$ *if and only if* $|\hat{x}^+ - f(\hat{x}, \hat{w}, \hat{u})| \leq \eta^x$*;*
- $\hat{Y}^1 = \{\mathcal{H}^1(\hat{x}) | \hat{x} \in \hat{X}\}$*;*
- $\hat{Y}^2 = \{\mathcal{H}^2(\hat{x}) | \hat{x} \in \hat{X}\}$*;*
- $\hat{\mathcal{H}}^1 = \mathcal{H}^1$*;*
- $\hat{\mathcal{H}}^2 = \mathcal{H}^2$.

An illustration of the computation of the transitions of $\hat{T}(\Sigma)$ is shown in Figure 3.2.

Remark 3.2.7. *In the context of networks of systems in which we consider* $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ *as a component of* $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i \in [1;N]}$*, the quantization parameter* η_i^w *of the set* W_i *should be chosen in such a way that the network* $\hat{T}(\Sigma)$ *is well-defined. Alternatively, if we directly choose* $\hat{W}_i$ *such that* $M \prod_{i=1}^N \hat{Y}_i^2 = \prod_{i=1}^N \hat{W}_i$*, condition* (3.2.5) *holds and* $\hat{T}(\Sigma)$ *is well-defined (cf. Remark 3.2.4).*

In the next subsection, we show the existence of sum-type simulation functions between $T(\Sigma)$ associated to δ-P discrete-time control systems Σ and their symbolic models $\hat{T}(\Sigma)$ constructed as in Definition 3.2.6.

3.2.4.3 Construction of Sum-Type Simulation Functions

Given a δ-P discrete-time control system Σ, we show that the δ-P storage function S in Definition 3.2.5 is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Theorem 3.2.8. *Consider a transition system $T(\Sigma)$, associated to a δ-P discrete-time control system Σ. Let $\hat{T}(\Sigma)$ be a symbolic model constructed as in Definition 3.2.6. Assume that there exists a $\mathcal{K}_\infty$ function γ such that for any $x, y, z \in X$*

$$S(x,y) \leq S(x,z) + \gamma(|y-z|), \tag{3.2.12}$$

for S in Definition 3.2.5. Then S is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Proof. Given the Lipschitz assumption on h^1 and since system Σ is incrementally passive, from (3.2.10), for any $x \in X$ and any $\hat{x} \in \hat{X}$, we have

$$|\mathcal{H}^1(x) - \hat{\mathcal{H}}^1(\hat{x})| = |h^1(x) - h^1(\hat{x})| \leq \ell^1(|x-\hat{x}|) \leq \ell^1 \circ \varphi_x^{-1}(S(x,\hat{x})).$$

By defining $\alpha = (\ell^1 \circ \varphi_x^{-1})^{-1}$, one obtains

$$\alpha(|\mathcal{H}^1(x) - \hat{\mathcal{H}}^1(\hat{x})|) \leq S(x,\hat{x}),$$

satisfying (3.2.2).

Now consider any $\hat{u} \in \hat{U}$ and choose $u = \hat{u}$. Then, using (3.2.12), for any $x \in X, \hat{x} \in \hat{X}$, any $\hat{u} \in \hat{U}$, and any $w \in W, \hat{w} \in \hat{W}$, we have

$$S(f(x,\hat{u},w),\hat{x}^+) \leq S(f(x,\hat{u},w), f(\hat{x},\hat{u},\hat{w})) + \gamma(|\hat{x}^+ - f(\hat{x},\hat{u},\hat{w})|),$$

for any $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x},\hat{u},\hat{w})$. Now, from Definition 3.2.6, the above inequality reduces to

$$S(f(x,\hat{u},w),\hat{x}^+) \leq S(f(x,\hat{u},w), f(\hat{x},\hat{u},\hat{w})) + \gamma(\eta^x).$$

Note that by (3.2.11), we get

$$S(f(x,\hat{u},w), f(\hat{x},\hat{u},\hat{w})) \leq \varphi_s(S(x,\hat{x})) + \begin{bmatrix} w-\hat{w} \\ \mathcal{H}^2(x) - \hat{\mathcal{H}}^2(\hat{x}) \end{bmatrix}^\top \overbrace{\begin{bmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{bmatrix}}^{Q:=} \begin{bmatrix} w-\hat{w} \\ \mathcal{H}^2 - \hat{\mathcal{H}}^2(\hat{x}) \end{bmatrix}.$$

It follows that for any $x \in X, \hat{x} \in \hat{X}$, any $\hat{u} \in \hat{U}$, and any $w \in W, \hat{w} \in \hat{W}$, one obtains

$$S(f(x,\hat{u},w),\hat{x}^+) \leq \varphi_s(S(x,\hat{x})) + \begin{bmatrix} w-\hat{w} \\ \mathcal{H}^2(x) - \hat{\mathcal{H}}^2(\hat{x}) \end{bmatrix}^\top \overbrace{\begin{bmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{bmatrix}}^{Q:=} \begin{bmatrix} w-\hat{w} \\ \mathcal{H}^2(x) - \hat{\mathcal{H}}^2(\hat{x}) \end{bmatrix} + \gamma(\eta^x),$$

for any $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x},\hat{u},\hat{w})$, satisfying (3.2.3) with $\varepsilon = \gamma(\eta^x)$, $\sigma = \mathsf{id} - \varphi_s$, $\rho_u = 0$, and $R = Q$. Hence, S is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. The rest of the proof follows similar argument. In particular, by the definition of $\hat{U}$, for any $u \in U$ there always exists $\hat{u} \in \hat{U}$ such that $\varphi_u(|u-\hat{u}|) \leq \varphi_u(\eta^u)$ which results in $\varepsilon = \varphi_u(\eta^u) + \gamma(\eta^x)$. Other terms in the sum-type simulation function S are the same as the first part of the proof. □

Remark 3.2.9. *Note that if system Σ is not δ-P, one can assume that Σ is incrementally passivable. That is there exists feedback controller $\mathcal{G}:\mathbb{X}\rightarrow\mathbb{U}$ such that* (3.2.11) *is satisfied with the left-hand side of* (3.2.11) *given as $S(f(x,w,\mathcal{G}(x)+u), f(x',w',\mathcal{G}(x')+u'))$.* $\diamond$

Remark 3.2.10. *Remark that condition* (3.2.12) *is not restrictive at all provided that one is interested to work on a compact subset of $\mathbb{X}$. We refer the interested readers to the explanation provided after equation (V.2) in [ZEM+ 14] on how to compute such function γ.* $\diamond$

Now we provide similar results as in Theorem 3.2.8 but tailored to linear control systems as in (2.3.3) which are computationally more efficient. In particular, the incremental passivity assumption in Definition 3.2.5 boils down in the linear case to the following assumption.

Assumption 3.2.11. *Consider linear control systems $\Sigma = (A, B, C^1, C^2, D)$. Assume that there exists matrix $Z \succ 0$, a symmetric matrix G of appropriate dimension with conformal block partitions G^{ij}, $i, j \in [1; 2]$ of appropriate dimensions such that the matrix inequality*

$$\begin{bmatrix}(1+\theta)A^\top ZA & A^\top ZD \\ D^\top AZ & (1+\theta)D^\top ZD\end{bmatrix} \preceq \begin{bmatrix}\varphi_c Z + {C^2}^\top G^{22}C^2 & {C^2}^\top G^{21} \\ G^{12}C^2 & G^{11}\end{bmatrix}, \tag{3.2.13}$$

holds for some constants $0 < \varphi_c < 1$, and $\theta \in \mathbb{R}_{>0}$.

Theorem 3.2.12. *Consider a transition system $T(\Sigma)$, associated to linear control systems Σ for which Assumption 3.2.11 holds. Let $\hat{T}(\Sigma)$ be a symbolic model constructed as in Definition 3.2.6. Then function S defined as*

$$S(x, \hat{x}) = (x - \hat{x})^\top Z(x - \hat{x}), \tag{3.2.14}$$

is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Proof. First, we show that condition (3.2.2) holds. Since $C^1 = \hat{C}^1$, we have

$$|C^1x - \hat{C}^1\hat{x}|^2 \leq n\lambda_{\max}({C^1}^\top C^1)|x - \hat{x}|^2,$$

and similarly

$$\lambda_{\min}(Z)|x - \hat{x}|^2 \leq (x - \hat{x})^\top Z(x - \hat{x}).$$

It can be verified that (3.2.2) holds for S defined in (3.2.14) with $\alpha(s) = \frac{\lambda_{\min}(Z)}{n\lambda_{\max}({C^1}^\top C^1)}s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We continue to show that (3.2.3) holds as well. Let $\hat{u}$ be given and choose $u = \hat{u}$. Consider any $x^+ = Ax + Dw + B\hat{u}$ and let $\hat{x}^+$ be defined as in Definition 3.2.6. Define $\Delta := A\hat{x} + D\hat{w} + B\hat{u} - \hat{x}^+$, and observe that $|\Delta| \leq \eta^x$ by Definition 3.2.6. Now, one obtains the chain of inequalities in (3.2.15) satisfying (3.2.3) with $\varepsilon = \frac{(2+n\theta)\lambda_{\max}(Z)(\eta^x)^2}{\theta}$, $\sigma = 1 - \varphi_c$, $\rho_u = 0$, and $R = G$. Hence, S is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. The rest of the proof follows similar argument. In particular, by the definition of $\hat{U}$, for any $u \in U$ there always exists $\hat{u} \in \hat{U}$ such that $\varphi_u(|u - \hat{u}|) \leq \varphi_u(\eta^u)$ which results in $\varepsilon = \frac{(2+n\theta)\lambda_{\max}(Z)(|B|\eta^u+\eta^x)^2}{\theta}$. Other terms are the same as before. $\square$

$$
\begin{aligned}
&S(x^+,\hat{x}^+)\\
=&(Ax+Dw+B\hat{u}-(A\hat{x}+D\hat{w}+B\hat{u})+(A\hat{x}+D\hat{w}+B\hat{u})-\hat{x}^+)^\top Z\\
&(Ax+Dw+B\hat{u}-(A\hat{x}+D\hat{w}+B\hat{u})+(A\hat{x}+D\hat{w}+B\hat{u})-\hat{x}^+)\\
=&(Ax+Dw+B\hat{u}-(A\hat{x}+D\hat{w}+B\hat{u})+\Delta)^\top Z\\
&(Ax+Dw+B\hat{u}-(A\hat{x}+D\hat{w}+B\hat{u})+\Delta)\\
\leq&(A(x-\hat{x})+D(w-\hat{w})+\Delta)^\top Z(A(x-\hat{x})+D(w-\hat{w})+\Delta)\\
\leq&(x-\hat{x})^\top A^\top ZA(x-\hat{x})+2(x-\hat{x})^\top A^\top ZD(w-\hat{w})+2(x-\hat{x})^\top A^\top Z\Delta\\
&(w-\hat{w})^\top D^\top ZD(w-\hat{w})+2(w-\hat{w})^\top D^\top Z\eta^x+\Delta^\top Z\Delta\\
\leq&(x-\hat{x})^\top A^\top ZA(x-\hat{x})+2(x-\hat{x})^\top A^\top ZD(w-\hat{w})+2|(x-\hat{x})^\top A^\top\sqrt{Z}|_2|\sqrt{Z}\Delta|_2\\
&(w-\hat{w})^\top D^\top ZD(w-\hat{w})+2|(w-\hat{w})^\top D^\top\sqrt{Z}|_2|\sqrt{Z}\Delta|_2+n\lambda_{\max}(Z)(\eta^x)^2\\
\leq&(x-\hat{x})^\top A^\top ZA(x-\hat{x})+2(x-\hat{x})^\top A^\top ZD(w-\hat{w})+\theta|(x-\hat{x})^\top A^\top\sqrt{Z}|_2^2+\frac{|\sqrt{Z}\Delta|_2^2}{\theta}\\
&(w-\hat{w})^\top D^\top ZD(w-\hat{w})+\theta|(w-\hat{w})^\top D^\top\sqrt{Z}|_2^2+\frac{|\sqrt{Z}\Delta|_2^2}{\theta}+n\lambda_{\max}(Z)(\eta^x)^2\\
\leq&(1+\theta)(x-\hat{x})^\top A^\top ZA(x-\hat{x})+2(x-\hat{x})^\top A^\top ZD(w-\hat{w})\\
&+(1+\theta)(w-\hat{w})^\top D^\top ZD(w-\hat{w})+\frac{(2+n\theta)\lambda_{\max}(Z)(\eta^x)^2}{\theta}\\
=&\begin{bmatrix}x-\hat{x}\\w-\hat{w}\end{bmatrix}^\top\begin{bmatrix}(1+\theta)A^\top ZA & A^\top ZD\\ D^\top AZ & (1+\theta)D^\top ZD\end{bmatrix}\begin{bmatrix}x-\hat{x}\\w-\hat{w}\end{bmatrix}+\frac{(2+n\theta)\lambda_{\max}(Z)(\eta^x)^2}{\theta}\\
\leq&\begin{bmatrix}x-\hat{x}\\w-\hat{w}\end{bmatrix}^\top\begin{bmatrix}\varphi_c Z+C^{2^\top}G^{22}C^2 & C^{2^\top}G^{21}\\ G^{12}C^2 & G^{11}\end{bmatrix}\begin{bmatrix}x-\hat{x}\\w-\hat{w}\end{bmatrix}+\frac{(2+n\theta)\lambda_{\max}(Z)(\eta^x)^2}{\theta}\\
\leq&\varphi_c S(x,\hat{x})+\begin{bmatrix}w-\hat{w}\\C^2x-\hat{C}^2\hat{x}\end{bmatrix}^\top\overbrace{\begin{bmatrix}G^{11} & G^{12}\\G^{21} & G^{22}\end{bmatrix}}^{G:=}\begin{bmatrix}w-\hat{w}\\C^2x-\hat{C}^2\hat{x}\end{bmatrix}+\frac{(2+n\theta)\lambda_{\max}(Z)(\eta^x)^2}{\theta}
\end{aligned}
\tag{3.2.15}
$$

Remark 3.2.13. *Note that if condition* (3.2.13) *can not be satisfied, one can still have the result in Theorem 3.2.12 by assuming that there exists matrices $Z \succ 0$, G, and K of appropriate dimensions such that the matrix inequality*

$$
\begin{bmatrix}(1+\theta)(A+BK)^\top ZA & (A+BK)^\top ZD\\ D^\top(A+BK)Z & (1+\theta)D^\top ZD\end{bmatrix}\preceq\begin{bmatrix}\varphi_c Z+C^{2^\top}G^{22}C^2 & C^{2^\top}G^{21}\\ G^{12}C^2 & G^{11}\end{bmatrix},
\tag{3.2.16}
$$

holds for some constants $0<\varphi_c<1$, and $\theta\in\mathbb{R}_{>0}$. Here, K is a state feedback controller gain. ◇

3.2.5 Case Studies

In this section we provide two case studies to illustrate the results of this section. First, we apply our results to the temperature regulation in a circular building of $n \geq 3$ rooms by constructing compositionally a symbolic model of the network. Then, we apply the proposed techniques to a fully connected network to show its applicability to strongly connected networks as well. The construction of symbolic models and controllers are performed using SCOTS [RZ16] on a PC with Intel i7@3.4GHz CPU and 16 GB of RAM.

3.2.5.1 Room Temperature Control

Here, we apply our results to the temperature regulation in a circular building of $n \geq 3$ rooms, each equipped with a heater. The dynamics of the temperature $\mathbf{x}$ for all rooms are described by the interconnected discrete-time model:

$$\Sigma : \begin{cases} \mathbf{x}(k+1) = A\mathbf{x}(k) + \hat{\beta}T_E + \bar{\beta}T_h\nu(k), \\ \quad \mathbf{y}(k) = \mathbf{x}(k), \end{cases} \tag{3.2.17}$$

adapted from [MGW17], where $A \in \mathbb{R}^{n\times n}$ is a matrix with elements $\{A\}_{i,i} = (1 - 2\beta - \hat{\beta} - \bar{\beta}\nu_i(k))$, $\{A\}_{i,i+1} = \{A\}_{i+1,i} = \{A\}_{1,n} = \{A\}_{n,1} = \beta$, $\forall i \in [1;n-1]$, and all other elements are identically zero, $\mathbf{x}(k) = [\mathbf{x}_1(k);\ldots;\mathbf{x}_n(k)]$, $\nu(k) = [\nu_1(k);\ldots;\nu_n(k)]$, $T_E = [T_{e1};\ldots;T_{en}]$, where $\nu_i(k)$, $\forall i \in [1;n]$, are taking values in $[0, 0.5]$. The other parameters are as follow: $\forall i \in [1;n]$, $T_{ei} = -1\,°C$ is the outside temperature, $T_h = 50\,°C$ is the heater temperature, and the conduction factors are given by $\beta = 0.45$, $\hat{\beta} = 0.045$, and $\bar{\beta} = 0.09$.
Now, by introducing Σ_i described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = (1 - 2\beta - \hat{\beta} - \bar{\beta}\nu_i(k))\mathbf{x}_i(k) + \omega_i(k) + \hat{\beta}T_{ei} + \bar{\beta}T_h\nu_i(k), \\ \quad \mathbf{y}_i^1(k) = \mathbf{x}_i(k), \\ \quad \mathbf{y}_i^2(k) = \mathbf{x}_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in[1,n]}$ where the elements of coupling matrix M are $\{M\}_{i,i+1} = \{M\}_{i+1,i} = \{M\}_{1,n} = \{M\}_{n,1} = \beta$, $i \in [1;n-1]$, and all other elements are identically zero.

Note that for any $i \in [1;n]$, condition (3.2.13) is satisfied with $Z_i = 1$, $\varphi_{c_i} = 0.95$, $\theta_i = 0.02$, and

$$G_i = \begin{bmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{bmatrix} = \begin{bmatrix} (1+\theta_i) & \lambda_i \\ \lambda_i & \pi_i \end{bmatrix}, \tag{3.2.18}$$

where $\lambda_i = (1 - 2\beta - \hat{\beta} - \bar{\beta}\nu_i(k))$, and $\pi = (1+\theta_i)\lambda_i^2 + \theta_i - \varphi_{c_i}$. Hence, $S_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^\top Z_i(x_i - \hat{x}_i)$, is a sum-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$, satisfying (3.2.2) and (3.2.3) with $\alpha_i(s) = s^2$ $\forall s \in \mathbb{R}_{\geq 0}$, $\sigma_i = 1 - \varphi_{c_i}$, $\rho_{u_i} = 0$, $R_i = G_i$, and $\epsilon_i = \frac{(2+\theta_i)\lambda_{\max}(Z_i)((\eta_i^x)^2/4)}{\theta_i}$.

By choosing $\mu_i = 1$ for all $i \in [1, n]$, matrix $\tilde{R}$ in (3.2.6) reduces to

$$\tilde{R} = \begin{bmatrix}(1+\theta)I_n & \lambda I_n \\ \lambda I_n & \pi I_n\end{bmatrix},$$

where $\lambda = \lambda_i$ and $\theta = \theta_i$ for all $i \in [1, n]$. Consequently, condition (3.2.4) reduces to

$$\begin{bmatrix}M \\ I_n\end{bmatrix}^\top \tilde{R} \begin{bmatrix}M \\ I_n\end{bmatrix} = (1+\theta)M^\top M + 2\lambda M - \pi I_n \preceq 0,$$

which, by Gershgorin circle theorem[Bel65], always holds without any restrictions on the number of the systems. Moreover, by choosing finite internal input sets $\hat{W}_i$ of $\hat{T}(\Sigma_i)$ in such a way that $\prod_{i=1}^{N} \hat{W}_i = M \prod_{i=1}^{N} \hat{X}_i$, condition (3.2.5) is satisfied. Now, one can verify that $\tilde{\mathcal{S}}(x, \hat{x}) = \sum_{i=1}^{n}(x_i - \hat{x}_i)^2$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in[1,n]}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in[1,n]}$ satisfying conditions (2.2.1) and (2.2.2) by $\tilde{\sigma} = (\mathsf{id} - (\mathsf{id} - \psi) \circ \sigma_i$, $\tilde{\rho}_u = 0$, and $\tilde{\varepsilon} = \sigma_i^{-1} \circ \psi^{-1}(\sum_{i=1}^{n} \frac{(2+\theta_i)((\eta_i^x)^2/4)}{\theta_i})$, for some arbitrarily chosen $\psi \in \mathcal{K}_\infty$ with $\psi < \mathsf{id}$.

Now, we synthesize a controller for Σ via abstractions $\hat{T}_i(\Sigma_i)$ such that the temperature of each room is maintained in the comfort zone $\mathscr{S} = [19,\ 20]$. The procedure is as the following: First, local controllers for abstractions $\hat{T}_i(\Sigma_i)$ are synthesized while assuming that the other systems meet their specifications. Then, those local controllers are refined to concrete systems $T_i(\Sigma_i)$. This approach is called assume-guarantee reasoning [HSR98], and it allows for the compositional synthesis of controllers. The computation times for constructing symbolic models and synthesizing controllers for $T_i(\Sigma_i)$ are 0.6s and 0.005s $\forall i \in [1, n]$. Figure 3.3 shows the maximum and minimum of the state trajectories of the closed-loop network Σ, consisting of 1200 rooms with the state quantization parameters $\eta_i^x = 0.01$ $\forall i \in [1; 1200]$.

3.2.5.2 Fully Connected Network

In order to show the applicability of our approach to strongly connected networks, we consider a nonlinear control network Σ described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) & = A\mathbf{x}(k) + \varphi(\mathbf{x}(k)) + \nu(k), \\ \mathbf{y}(k) & = \mathbf{x}(k), \end{cases}$$

where $A = I_n - \tau L$ for some Laplacian matrix $L \in \mathbb{R}^{n\times n}$ of an undirected graph [GR01], and constant $0 < \tau < 1/\Delta$, where Δ is the maximum degree of the graph [GR01]. Moreover $\mathbf{x}(k) = [\mathbf{x}_1(k); \ldots; \mathbf{x}_n(k)]$, $\nu(k) = [\nu_1(k); \ldots; \nu_n(k)]$, and $\varphi(\mathbf{x}(k)) = [\varphi_1(\mathbf{x}_1(k)); \ldots; \varphi_n(\mathbf{x}_n(k))]$, where $\varphi_i(x_i) = 0.1sin(x_i), \forall i \in [1; n]$. Assume L is the Laplacian matrix of a complete graph:

$$L = \begin{bmatrix} n-1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}.$$

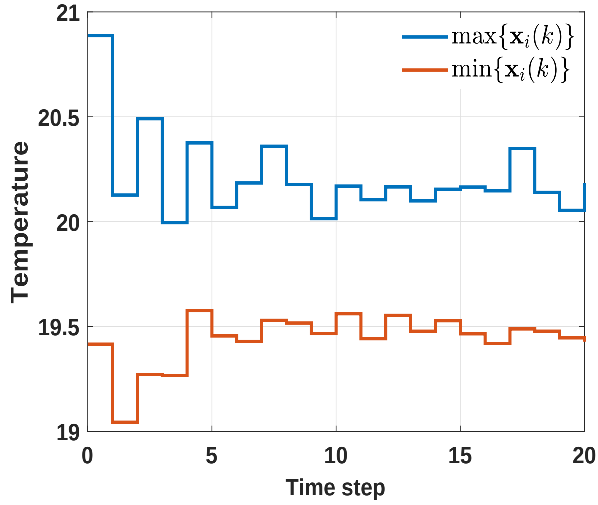

Figure 3.3: Bounds inside which state trajectories of the closed-loop network Σ consisting of 1200 rooms are evolving.

Now, by introducing Σ_i described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) &= \mathbf{x}_i(k) + \varphi_i(\mathbf{x}_i(k)) + \omega_i(k) + \nu_i(k), \\ \mathbf{y}_i^1(k) &= \mathbf{x}_i(k), \\ \mathbf{y}_i^2(k) &= \mathbf{x}_i(k), \end{cases} \tag{3.2.19}$$

one can readily verify that $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in[1,n]}$ where the coupling matrix M is given by $M = -\tau L$. Note that, for any $i \in [1;n]$, conditions (3.2.10) and (3.2.11) are satisfied with $S_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^\top (x_i - \hat{x}_i)$, $\underline{\alpha}_i(s) = s^2$, $\varphi_{f_i} = 0.5$, $\rho_{u_i} = 0$, $\forall r \in \mathbb{R}_{\geq 0}$, where we have used function $\mathcal{G}$ in Remark 3.2.9 with $\mathcal{G}_i(x_i) = -0.5x_i, \forall i \in [1;n]$, and

$$G_i = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 0 \end{bmatrix}. \tag{3.2.20}$$

Hence, $S_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^\top (x_i - \hat{x}_i)$ is a sum-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$ associated to Σ_i.

By choosing $\mu_i = 1$ for all $i \in [1, n]$, matrix $\tilde{R}$ in (3.2.6) reduces to

$$\tilde{R} = \begin{bmatrix} 1.1I_n & 0.5I_n \\ 0.5I_n & 0_{n\times n} \end{bmatrix}.$$

Consequently, condition (3.2.4) reduces to

$$\begin{bmatrix} -\tau L \\ I_n \end{bmatrix}^\top X \begin{bmatrix} -\tau L \\ I_n \end{bmatrix} = \tau L\,(1.1\tau L - I_n) \preceq 0,$$

which always holds without any restrictions on the number of the systems with $\tau = \frac{0.5}{n-1}$. In order to show the above inequality, we used $L^\top = L \succeq 0$, and employing Gershgorin circle theorem [Bel65] to show that $1.1\tau L - I_n \preceq 0$. Moreover, by choosing finite internal input sets $\hat{W}_i$ of $\hat{T}(\Sigma_i)$ in such a way that $\prod_{i=1}^{N} \hat{W}_i = M \prod_{i=1}^{N} \hat{X}_i$, condition (3.2.5) is satisfied.

Now, one can verify that $\tilde{\mathcal{S}}(x, \hat{x}) = \sum_{i=1}^{n}(x_i - \hat{x}_i)^2$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in[1,n]}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in[1,n]}$ satisfying conditions (2.2.1) and (2.2.2) by $\tilde{\sigma} = 0.95$, $\tilde{\rho}_u = 0$, and $\tilde{\varepsilon} = 2\sum_{i=1}^{n}\gamma_i(\eta_i^x)$, where η_i^x is the state set quantization parameter of abstraction $\hat{T}_i(\Sigma_i)$ and γ_i is the $\mathcal{K}_\infty$ function satisfying (3.2.12) for S_i.

3.3 Small-Gain Approach (SGA)

3.3.1 (In)finite Networks of Discrete-Time Control Systems: SGA Formulation

Definition 3.3.1. *Let $\mathcal{N} := \mathbb{N}$, or $\mathcal{N} := [1; N]$, $N \in \mathbb{N}$. Consider discrete-time control systems $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, f_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$, $i \in \mathcal{N}$. The (in)finite network of discrete-time control system is a tuple $\Sigma = (\mathbb{X}, \mathbb{U}, f, \mathbb{Y}, h)$, denoted by $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathcal{N}}$, where $\mathbb{X} = \{x = (x_i)_{i\in\mathcal{N}} : x_i \in \mathbb{X}_i,\ \|x\| := \sup_{i\in\mathcal{N}}\{|x_i|\} < \infty\}$, $\mathbb{U} = \{u = (u_i)_{i\in\mathcal{N}} : u_i \in \mathbb{U}_i,\ \|u\| := \sup_{i\in\mathcal{N}}\{|u_i|\} < \infty\}$, and $\mathbb{Y} = \prod\limits_{i\in\mathcal{N}} \mathbb{Y}_i^1$. Moreover, f and h are defined as*

$$f(x, u) = (f_i(x_i, w_i, u_i))_{i\in\mathcal{N}}, h(x) = (h_i^1(x_i))_{i\in\mathcal{N}},$$

and with the internal variables constrained by $w_i = (y_j^2)_{j\in\mathcal{N}_i} = (h_j^2(x_j))_{j\in\mathcal{N}_i}$, $\prod\limits_{j\in\mathcal{N}_i} \mathbb{Y}_j^2 \subseteq \mathbb{W}_i$, $\forall j \in \mathcal{N}_i, \forall i \in \mathcal{N}$, where $\mathcal{N}_i$ is a finite subset of $\mathcal{N}$ enumerating the neighbors of Σ_i, i.e., those systems $\Sigma_j, j \in \mathcal{N}_i$ that affect Σ_i with $i \notin \mathcal{N}_i$. The network is described by the difference equations

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f(\mathbf{x}(k), \nu(k)), \\ \mathbf{y}(k) &= h(\mathbf{x}(k)), \end{cases} \tag{3.3.1}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$, $\nu : \mathbb{N} \to \mathbb{U}$, and $\mathbf{y} : \mathbb{N} \to \mathbb{Y}$.

We also assume that $f(x, u) \in \mathbb{X}$ for all pairs $(x, u) \in \mathbb{X} \times \mathbb{U}$ to ensure the network $\Sigma = (\mathbb{X}, \mathbb{U}, f, \mathbb{Y}, h)$ is well-defined which is automatically satisfied if $\mathcal{N} = [1; N]$. Note that if $\mathcal{N} := [1; N]$, $N \in \mathbb{N}$, sets $\mathbb{X}, \mathbb{U}$, and $\mathbb{Y}$ in Definition 3.3.1 can also be defined in the same way as those in Definition 3.2.1.

3.3.2 Max-Type Simulation Functions

Consider networks of discrete-time control systems $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathcal{N}}$, or their equivalent networks of transition systems $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$, where each $T_i(\Sigma_i)$ given as in Definition 2.3.2. Assume that each systems $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ and $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ admit a max-type simulation functions as defined next.

Definition 3.3.2. *Consider systems* $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ *and* $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ *where* $\hat{Y}_i^j \subseteq Y_i^j$, $j \in [1;2]$, $\hat{W}_i \subseteq W_i$. *A function* $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ *is called a max-type simulation function from* $\hat{T}_i(\Sigma_i)$ *to* $T_i(\Sigma_i)$ *if there exist* $\underline{\alpha}_i, \overline{\alpha}_i, \sigma_i, \rho_{w_i} \in \mathcal{K}_\infty$, $\rho_{u_i} \in \mathcal{K}_\infty \cup \{0\}$, *and some* $\varepsilon_i \in \mathbb{R}_{\geq 0}$ *so that the following hold:*

- *For every* $x_i \in X_i, \hat{x}_i \in \hat{X}_i$, $j \in [1;2]$, *one has*
$$\underline{\alpha}_i(|\mathcal{H}_i^j(x_i) - \hat{\mathcal{H}}_i^j(\hat{x}_i)|) \leq \mathcal{S}_i(x_i, \hat{x}_i) \leq \overline{\alpha}_i(|x_i - \hat{x}_i|) \tag{3.3.2}$$
- *For every* $x_i \in X_i, \hat{x}_i \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$, *there exists* $u_i \in U_i$ *such that for every* $w_i \in W_i, \hat{w}_i \in \hat{W}_i, x_i^+ \in \mathcal{F}_i(x_i, w_i, u_i)$ *there exists* $\hat{x}_i^+ \in \hat{\mathcal{F}}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ *so that*
$$\mathcal{S}_i(x_i^+, \hat{x}_i^+) \leq \max\{\sigma_i(\mathcal{S}_i(x_i, \hat{x}_i)), \rho_{w_i}(|w_i - \hat{w}_i|), \rho_{u_i}(|\hat{u}_i|), \varepsilon_i\}. \tag{3.3.3}$$

Here, $\hat{T}_i(\Sigma_i)$ is called an abstraction of $T_i(\Sigma_i)$, denoted by $\hat{T}_i(\Sigma_i) \preceq_{\mathcal{S}_i^m} T_i(\Sigma_i)$, if there exists a max-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$. Moreover, if $\hat{T}_i(\Sigma_i)$ is finite, it is called a symbolic model of $T_i(\Sigma_i)$.

Remark 3.3.3. *The upper bound* $\overline{\alpha}_i(|x_i - \hat{x}_i|)$ *in inequality* (3.3.2) *will be used later to ensure that the alternating simulation function composed of max-type simulation functions is well defined when* $\mathscr{N} = \mathbb{N}$. *However, if* $\mathscr{N} = [1; N]$, *such an upper bound can be omitted.* ◇

For functions σ_i, $\underline{\alpha}_i$, and ρ_{w_i} associated with $\mathcal{S}_i$, $\forall\, i \in \mathscr{N}$, given in Definition 3.3.2, we define $\forall i, j \in \mathscr{N}$
$$\gamma_{ij} := \begin{cases} \sigma_i & \text{if} \quad i = j, \\ \rho_{w_i} \circ \underline{\alpha}_j^{-1} & \text{if} \quad j \in \mathcal{N}_i, \\ 0 & \text{if} \quad i \neq j, j \notin \mathcal{N}_i. \end{cases} \tag{3.3.4}$$

Moreover, in the case we consider an infinite network, i.e., $\mathscr{N} = \mathbb{N}$, we additionally define an operator $\Gamma : \ell_+^\infty \to \ell_+^\infty$ by
$$\Gamma(s) = \big(\sup_{j \in \mathbb{N}}\{\gamma_{ij}(s_j)\}\big)_{i \in \mathbb{N}}, \quad s \in \ell_+^\infty. \tag{3.3.5}$$

Additionally, we assume that there exist $\hat{\sigma}, \hat{\rho}_w, \hat{\underline{\alpha}} \in \mathcal{K}_\infty$ such that $\sigma_i \leq \hat{\sigma}, \rho_{w_i} \leq \hat{\rho}_w, \underline{\alpha}_i \geq \hat{\underline{\alpha}}$ for all $i \in \mathscr{N}$, whenever $\mathscr{N} = \mathbb{N}$. This assumption guarantees that Γ is well-defined.

3.3.3 Compositional Abstractions for Finite Networks of Discrete-Time Control Systems: SGA

Let $\mathscr{N} := [1; N]$, $N \in \mathbb{N}$. Assume that we are given $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, f_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$, or equivalently $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ as in Definition 2.3.2, together with their abstractions $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$, $i \in \mathscr{N}$, and max-type simulation functions $\mathcal{S}_i$ from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$ as in Definition 3.3.2.

In order to establish the compositionality results for the finite network, we make the following small-gain type assumption.

Assumption 3.3.4. *Assume that functions γ_{ij} defined in* (3.3.4) *satisfy*

$$\gamma_{i_1 i^2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathsf{id}, \tag{3.3.6}$$

$\forall (i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$*, where* $r \in \{1, \ldots, N\}$.

Note that by Theorem 5.2 in [DRW10], the small-gain condition (3.3.6) implies the existence of $\psi_i \in \mathcal{K}_\infty$ $\forall i \in [1; N]$, satisfying

$$\max_{j \in \mathcal{N}} \{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\} < \mathsf{id}. \tag{3.3.7}$$

The next theorem provides a compositional approach to construct an alternating simulation function from the finite network of abstractions $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$, associated to the network of discrete-time control system $\Sigma = \mathcal{I}(\Sigma_i)_{i \in \mathcal{N}}$, via a max-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$.

Theorem 3.3.5. *Consider a finite network of transition systems $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$, associated to the network of discrete-time control system $\Sigma = \mathcal{I}(\Sigma_i)_{i \in \mathcal{N}}$. Suppose each transition system $T_i(\Sigma_i)$ admits an abstraction $\hat{T}_i(\Sigma_i)$ with the corresponding max-type simulation function $\mathcal{S}_i$. Suppose Assumption 3.3.4 holds. Then, function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ defined as*

$$\tilde{\mathcal{S}}(x, \hat{x}) := \max_{i \in \mathcal{N}} \{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\}, \tag{3.3.8}$$

is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$.

Proof. First, we show that (2.2.1) holds for some $\mathcal{K}_\infty$ function $\tilde{\alpha}$. Define $\tilde{\alpha} := (\max_i \{\underline{\alpha}_i^{-1} \circ \psi_i\})^{-1}$, and consider any $x \in X$, $\hat{x} \in \hat{X}$. Then, one gets

$$\begin{aligned} |\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})| &= \max_{i \in \mathcal{N}} \{|\mathcal{H}_i^1(x_i) - \hat{\mathcal{H}}_i^1(\hat{x}_i)|\} \leq \max_{i \in \mathcal{N}} \{\underline{\alpha}_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\} \\ &\leq \max_{i \in \mathcal{N}} \{\underline{\alpha}_i^{-1} \circ \psi_i\} \circ \max_{i \in \mathcal{N}} \{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\}. \end{aligned}$$

Hence, one obtains $\tilde{\alpha}(|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})|) \leq \tilde{\mathcal{S}}(x, \hat{x})$, satisfying (2.2.1).

Now, we show that (2.2.2) holds. Let $\tilde{\sigma} = \max_{i,j \in \mathcal{N}} \{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\}$. It follows from (3.3.7) that $\tilde{\sigma} < \mathsf{id}$. By defining $\tilde{\rho}_u$ and $\tilde{\varepsilon}$ as $\tilde{\rho}_u := \max_{i \in \mathcal{N}} \{\psi_i^{-1}\} \circ \max_{i \in \mathcal{N}} \{\rho_{u_i}\}$ and $\tilde{\varepsilon} := \max_{i \in \mathcal{N}} \{\psi_i^{-1}(\varepsilon_i)\}$, one gets the chain of inequalities in (3.3.9) which satisfies (2.2.2), and implies that $\tilde{\mathcal{S}}$ is indeed an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$. □

Figure 3.4 schematically shows the results of Theorem 3.3.5. Note that, similar technique was proposed in [RZ18] using nonlinear small-gain type condition to construct compositionally an approximate *infinite* abstraction of an interconnected *continuous-time* control system. Since in [RZ18, Definition 2] a simulation function between each

$$
\begin{aligned}
\tilde{\mathcal{S}}(x^+,\hat{x}^+) =& \max_{i\in\mathcal{N}}\{\psi_i^{-1}\circ\mathcal{S}_i(x_i^+,\hat{x}_i^+)\}\\
\leq& \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(|w_i-\hat{w}_i|),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
=& \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(|(y_j^2)_{j\in\mathcal{N}_i}-(\hat{y}_j^2)_{j\in\mathcal{N}_i}|),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
=& \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|y_j^2-\hat{y}_j^2|\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
\leq& \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|\mathcal{H}_j^2(x_j)-\hat{\mathcal{H}}_j^2(\hat{x}_j)|\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
\leq& \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{\underline{\alpha}_j^{-1}(\mathcal{S}_j(x_j,\hat{x}_j))\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
\leq& \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}(\mathcal{S}_j(x_j,\hat{x}_j)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
=& \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_j^{-1}(\mathcal{S}_j(x_j,\hat{x}_j)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
\leq& \max_{i,j,l\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_l^{-1}(\mathcal{S}_l(x_l,\hat{x}_l)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
=& \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j(\tilde{\mathcal{S}}(x,\hat{x})),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
=& \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(x,\hat{x})),\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}\circ\rho_{u_i}(|\hat{u}_i|),\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}(\varepsilon_i)\big\}\Big\}\\
\leq& \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(x,\hat{x})),\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}\circ\max_{i\in\mathcal{N}}\{\rho_{u_i}(|\hat{u}_i|)\},\max_{i}\big\{\psi_i^{-1}(\varepsilon_i)\big\}\Big\}\\
\leq& \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(x,\hat{x})),\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}\circ\max_{i\in\mathcal{N}}\{\rho_{u_i}(\max_{i\in\mathcal{N}}\{|\hat{u}_i|\})\},\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}(\varepsilon_i)\big\}\Big\}\\
=& \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(x,\hat{x})),\tilde{\rho}_u(|\hat{u}|),\tilde{\varepsilon}\Big\},
\end{aligned}
\tag{3.3.9}
$$

system and its abstraction is formulated in a *dissipative-form* [NGG+18], an extra operator (the operator D in [RZ18, equation (12)]) is required to formulate the small-gain condition and to construct what is called an Ω-path [DRW10, Definition 5.1], which is exactly the $\mathcal{K}_\infty$ functions $\psi_i, i\in\mathcal{N}$, that satisfy condition (3.3.7). However, the definition of the alternating simulation function in our work is formulated in a *max-form* [NGG+18] which results in not only simpler formulation of the small-gain condition but also the Ω-path construction can be achieved without the need of the extra operator; see [DRW10, Section 8.4].

Remark 3.3.6. *Here, we provide a general guideline on the computation of* $\mathcal{K}_\infty$ *functions* $\psi_i, i\in[1;N]$ *as the following:* (*i*) *In a general case of having* $N\geq 1$ *systems, functions* $\psi_i, i\in[1;N]$*, can be constructed numerically using the algorithm proposed in [Eav72] and the technique provided in [DRW10, Proposition 8.8], see [Ruf07, Chapter 4];* (*ii*) *Simple construction techniques are provided in [JMW96] and [DRW10, Section 9] for the*

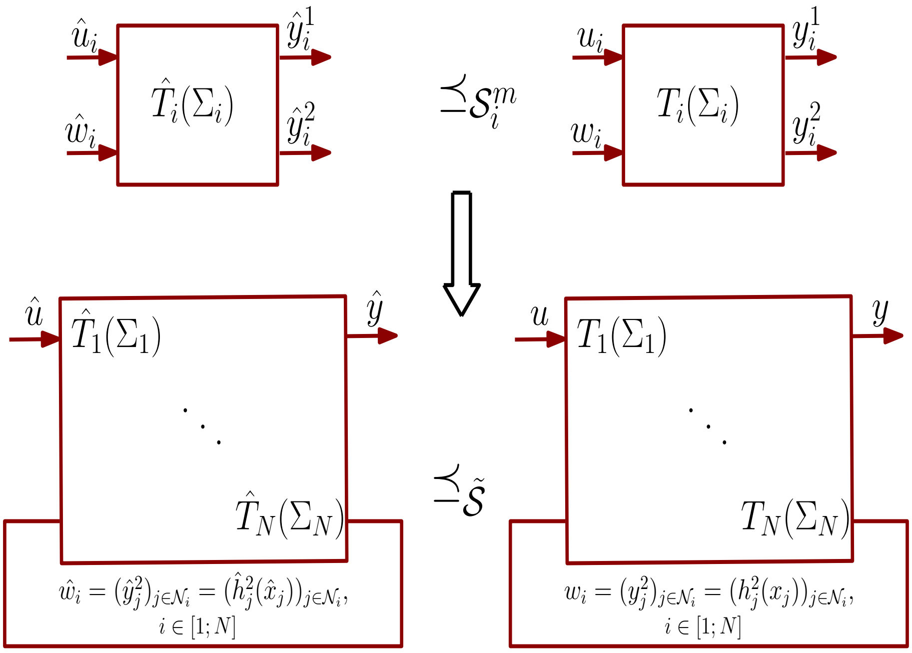

Figure 3.4: Compositionality results for constructing networks of abstractions provided that condition (3.3.6) is satisfied.

case of two and three systems, respectively; (iii) the $\mathcal{K}_\infty$ functions $\psi_i, i \in [1;N]$, can be always chosen as identity functions provided that $\gamma_{ij} < \mathsf{id}$, $\forall\ i,j \in [1;N]$, for functions γ_{ij} appeared in (3.3.4). ◇

Remark 3.3.7. *We emphasize that the proposed small-gain type condition in* (3.3.6) *is much more general than the one proposed in [PPB16]. To be more specific, consider deterministic transition system $T(\Sigma) = \mathcal{I}(T_1(\Sigma_1), T_2(\Sigma_2))$, in which the transition function for each system is given as the following:*

$$x_1^+ = \mathcal{F}_1(x_1, x_2, u_1) = a_1 x_1(k) + b_1\sqrt{|x_2(k)|} + u_1,$$
$$x_2^+ = \mathcal{F}_2(x_2, x_1, u_2) = a_2 x_2(k) + b_2 g(x_1(k)) + u_2,$$

where $0 < a_1 < 1$, $0 < a_2 < 1$, and function g satisfies the following quadratic Lipschitz assumption: there exists an $L \in \mathbb{R}_{>0}$ such that: $|g(x) - g(x')| \leq L|x - x'|^2$ for all $x, x' \in \mathbb{R}$. One can easily verify that functions $\mathcal{S}_1(x_1, \hat{x}_1) = |x_1 - \hat{x}_1|$ and $\mathcal{S}_2(x_2, \hat{x}_2) = |x_2 - \hat{x}_2|$ are max-type simulation functions from x_1-system to itself and x_2-system to itself, respectively. Here, one can not come up with gain functions satisfying Assumption (A2) in [PPB16] globally. In particular, those assumptions require existence of $\mathcal{K}_\infty$ functions being upper bounded by linear ones and lower bounded by quadratic ones which is impossible. On the other hand, the proposed small-gain condition (3.3.6) *is still applicable here showing that $\tilde{\mathcal{S}}(x, \hat{x}) := \max\{\psi_1^{-1} \circ \mathcal{S}_1(x_1, \hat{x}_1), \psi_2^{-1} \circ \mathcal{S}_2(x_2, \hat{x}_2)\}$ is an alternating simulation function from $T(\Sigma)$ to itself, for some appropriate $\psi_1, \psi_2 \in \mathcal{K}_\infty$ satisfying* (3.3.7) *which is guaranteed to exist if $|b_1|\sqrt{|b_2|L} < 1$ and $|b_2|(b_1 L)^2 < 1$.* ◇

3.3.4 Compositional Abstractions for Infinite Networks of Discrete-Time Control Systems

Let $\mathscr{N} := \mathbb{N}$. Assume that we are given $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, f_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$, or equivalently $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ as in Definition 2.3.2, together with their abstractions $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$, $i \in \mathscr{N}$, and max-type simulation functions $\mathcal{S}_i$ from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$ as in Definition 3.3.2. The compositionality result of the infinite network is based on the following robust small-gain type assumption, inspired by [DMSW19].

Assumption 3.3.8. *Consider operator Γ defined in* (3.3.5). *Assume that* $\sup_{j\in\mathbb{N}}\{\gamma_{ij}(s_j)\} > 0, \forall s_j > 0, \forall i,j \in \mathbb{N}$, *$\Gamma$ is continuous on ℓ_+^∞, $\lim_{k\to\infty} \Gamma^k(s) = 0, \forall s \in \ell_+^\infty$, and there exist positive constants c_1 and c_2 such that for all $i,j \in \mathbb{N}$ the operator $\Gamma_{i,j}(s) := \Gamma(s) + c_1 s_j e_i, s \in \ell_+^\infty$ satisfies*

$$\Gamma_{i,j}(s) \not\geq (1-c_2)s, \quad s \in \ell_+^\infty \backslash \{0\}. \tag{3.3.10}$$

Remark 3.3.9. *If for any $b \geq 0$ the set of all functions $\{\gamma_{ij}, i,j \in \mathbb{N}\}$ is uniformly equicontinuous in $[0,b]$, the operator Γ defined in* (3.3.5) *is continuous. That is, for any $\beta_1 > 0$ there exists $\beta_2 > 0$ such that for any $r_1, r_2 \in [0,b]$ with $|r_1 - r_2| < \beta_2$ it follows that $|\gamma_{ij}(r_1) - \gamma_{ij}(r_2)| < \beta_1, \forall i,j \in \mathbb{N}$. We refer the interested readers to [DMSW19] for more details on regularity properties of the operator Γ.* ⋄

Note that by using Lemma 4.5 in [DMSW19], the small-gain condition (3.3.10) implies that there exist a function $\psi := (\psi_i)_{i\in\mathbb{N}} : \mathbb{R}_{\geq 0} \to \ell_+^\infty$ with $\psi_i \in \mathcal{K}_\infty, i \in \mathbb{N}$, and $\epsilon \in (0,1)$ such that

$$\Gamma(\psi(r)) \leq (1-\epsilon)\psi(r), \quad r \in \mathbb{R}_{\geq 0}. \tag{3.3.11}$$

It follows from (3.3.11) that $\forall i \in \mathbb{N}$ and $\forall r \in \mathbb{R}_{\geq 0}$,

$$\sup_{j\in\mathbb{N}}\{\gamma_{ij} \circ \psi_j(r)\} \leq (1-\epsilon)\psi_i(r) \leq \psi_i(r).$$

Applying ψ_i^{-1} to both sides, one has

$$\psi_i^{-1}(\sup_{j\in\mathbb{N}}\{\gamma_{ij} \circ \psi_j(r)\}) = \sup_{j\in\mathbb{N}}\{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j(r)\} \leq r. \tag{3.3.12}$$

Since (3.3.12) holds for all $i \in \mathbb{N}$, one has

$$\sup_{i,j\in\mathbb{N}}\{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\} \leq \mathsf{id}. \tag{3.3.13}$$

Now we have all the ingredients to formulate the main result of this section. The next theorem provides a compositional approach to construct an alternating simulation function from the infinite network of abstractions $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathscr{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathscr{N}}$, associated to the network of discrete-time control system $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathscr{N}}$, via a max-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$.

Theorem 3.3.10. *Consider the infinite network* $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$ *associated to* $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathcal{N}}$. *Assume that each* $T_i(\Sigma_i)$ *and its abstraction* $\hat{T}_i(\Sigma_i)$ *admit a max-type simulation function* $\mathcal{S}_i$ *as in Definition 3.3.2. Suppose Assumption 3.3.8 holds and there exist* $\mathcal{K}_\infty$ *functions* $\underline{\psi}, \overline{\psi}, \hat{\alpha}, \overline{\rho}_u$, *and constant* $\overline{\varepsilon} \in \mathbb{R}_{\geq 0}$ *such that* $\underline{\psi} \leq \psi_i \leq \overline{\psi}$, $\overline{\alpha}_i \leq \hat{\alpha}$, $\rho_{u_i} \leq \overline{\rho}_u$, $\varepsilon_i \leq \overline{\varepsilon}$, $\forall i \in \mathbb{N}$. *Then, function* $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *defined as*

$$\tilde{\mathcal{S}}(x, \hat{x}) := \sup_{i\in\mathbb{N}}\{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\}, \tag{3.3.14}$$

is well-defined and it is also an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ *to* $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$.

Proof. First we show that function $\tilde{\mathcal{S}}$ given by (3.3.14) is well-defined. Note that $\forall x \in X$ and $\forall \hat{x} \in \hat{X}$ we have

$$\begin{aligned}
\tilde{\mathcal{S}}(x, \hat{x}) &:= \sup_{i\in\mathcal{N}}\{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\} \leq \sup_{i\in\mathcal{N}}\{\psi_i^{-1} \circ \overline{\alpha}_i(|x_i - \hat{x}_i|)\} \leq \sup_{i\in\mathcal{N}}\{\psi_i^{-1} \circ \overline{\alpha}_i(|x_i| + |\hat{x}_i|)\} \\
&\leq \sup_{i\in\mathcal{N}}\{\underline{\psi}^{-1} \circ \hat{\alpha}(|x_i|+|\hat{x}_i|)\} \leq \underline{\psi}^{-1} \circ \hat{\alpha}(\sup_{i\in\mathcal{N}}\{|x_i|+|\hat{x}_i|\}) \leq \underline{\psi}^{-1} \circ \hat{\alpha}(\sup_{i\in\mathcal{N}}\{|x_i|\} + \sup_{i\in\mathcal{N}}\{|\hat{x}_i|\}) \\
&\leq \underline{\psi}^{-1} \circ \hat{\alpha}(\|x\| + \|\hat{x}\|) < \infty.
\end{aligned}$$

Next, we show that (2.2.1) holds for some $\mathcal{K}_\infty$ function $\tilde{\alpha}$. Consider any $x \in X$, $\hat{x} \in \hat{X}$. One gets

$$\begin{aligned}
\|h(x) - \hat{h}(\hat{x})\| &\leq \sup_{i\in\mathcal{N}}\{|h_i^1(x_i) - \hat{h}_i^1(\hat{x}_i)|\} \leq \sup_{i\in\mathcal{N}}\{\underline{\alpha}_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\} = \sup_{i\in\mathcal{N}}\{\underline{\alpha}_i^{-1} \circ \psi_i \circ \psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\} \\
&\leq \underline{\hat{\alpha}}^{-1} \circ \overline{\psi}(\sup_{i\in\mathcal{N}}\{\psi_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i))\}) = \underline{\hat{\alpha}}^{-1} \circ \overline{\psi}(\tilde{\mathcal{S}}(x, \hat{x})).
\end{aligned}$$

Hence, inequality (2.2.1) holds with $\tilde{\alpha} := (\underline{\hat{\alpha}}^{-1} \circ \overline{\psi})^{-1}$.

Finally, we show that (2.2.2) holds. Let $\sigma := \sup_{i,j\in\mathcal{N}}\{\psi_i^{-1} \circ \gamma_{ij} \circ \psi_j\}$, $\rho_u := \underline{\psi}_i^{-1} \circ \overline{\rho}_u$, and $\varepsilon := \sup_{i\in\mathcal{N}}\{\psi_i^{-1}(\varepsilon_i)\}$. Observe that, by (3.3.13), $\sigma \leq \mathsf{id}$. Moreover, ε is well-defined since $\varepsilon \leq \underline{\psi}^{-1}(\sup_{i\in\mathcal{N}}\{\varepsilon_i\}) \leq \underline{\psi}^{-1}(\overline{\varepsilon}) < \infty$. Then, one gets the chain of inequalities in (3.3.15) which satisfies (2.2.2), and implies that $\tilde{\mathcal{S}}$ in (3.3.14) is indeed an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$. □

Remark 3.3.11. *If* $\gamma_{ij} \leq \mathsf{id}$ *for any* $i, j \in \mathcal{N}$, *inequality* (3.3.13) *holds with* $\psi_i = \mathsf{id}$ *for all* $i \in \mathcal{N}$, *and inequality* (3.3.14) *reduces to* $\tilde{\mathcal{S}}(x, \hat{x}) := \sup_{i\in\mathcal{N}}\{\mathcal{S}_i(x_i, \hat{x}_i)\}$, *and, consequently, the small-gain condition* (3.3.10) *is satisfied automatically.* ⋄

Remark 3.3.12. *Note that computing the symbolic models of infinite networks using those of their subsystems is not possible practically since it consumes infinite memory to store. However, our proposed compositional framework is still required even if controller synthesis problems can be solved compositionally using symbolic models of subsystems. In particular, if decentralized (or distributed) controllers exist for some types of specifications, one still needs to establish the compositional relation as in Theorem 3.3.10*

$$
\begin{aligned}
\tilde{\mathcal{S}}(x^+,\hat{x}^+) &= \sup_{i\in\mathcal{N}}\{\psi_i^{-1}\circ\mathcal{S}_i(x_i^+,\hat{x}_i^+)\}\\
&\le \sup_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(|w_i-\hat{w}_i|),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&= \sup_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(|(y_j^2)_{j\in\mathcal{N}_i}-(\hat{y}_j^2)_{j\in\mathcal{N}_i}|),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&= \sup_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(x_i,\hat{x}_i),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|y_j^2-\hat{y}_j^2|\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&= \sup_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|\mathcal{H}_j^2(x_j)-\hat{\mathcal{H}}_j^2(\hat{x}_j)|\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&\le \sup_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i(\mathcal{S}_i(x_i,\hat{x}_i)),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{\alpha_j^{-1}(\mathcal{S}_j(x_j,\hat{x}_j))\}),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&\le \sup_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}(V_j(x_j,\hat{x}_j)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&= \sup_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_j^{-1}(\mathcal{S}_j(x_j,\hat{x}_j)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&\le \sup_{i,j,l\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_l^{-1}(\mathcal{S}_l(x_l,\hat{x}_l)),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&= \sup_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j(V(x,\hat{x})),\rho_{u_i}(|\hat{u}_i|),\varepsilon_i\}\big)\Big\}\\
&\le \max\Big\{\sup_{i,j\in\mathcal{N}}\{\psi_i^{-1}\circ\gamma_{ij}\circ\psi_j(\tilde{\mathcal{S}}(x,\hat{x}))\},\sup_{i\in\mathcal{N}}\{\psi_i^{-1}\circ\rho_{u_i}(|\hat{u}_i|)\},\sup_{i\in\mathcal{N}}\{\psi_i^{-1}(\varepsilon_i)\}\Big\}\\
&\le \max\Big\{\sigma(\tilde{\mathcal{S}}(x,\hat{x})),\underline{\psi}^{-1}\circ\overline{\rho}_u(\sup_{i\in\mathcal{N}}\{|\hat{u}_i|\}),\sup_{i\in\mathcal{N}}\{\psi_i^{-1}(\varepsilon_i)\}\Big\}\\
&= \max\Big\{\sigma(\tilde{\mathcal{S}}(x,\hat{x})),\rho_u(\|\hat{u}\|),\varepsilon\Big\}, \qquad (3.3.15)
\end{aligned}
$$

to formally reason about the preservation and satisfaction of properties across related infinite networks. ◇

Remark 3.3.13. *In the context of stability analysis of infinite networks, condition* (3.3.10) *is used to show different stability properties (e.g., uniform global asymptotic stability or input-to-state stability) for the entire network by investigating stability criteria for subsystems. Moreover, condition* (3.3.10) *is also been shown to be tight and cannot be weakened in the context of stability verification of infinite networks. We refer interested readers to [DMSW19] for more details on the tightness analysis of small-gain condition* (3.3.10)*.* ◇

3.3.5 Construction of Symbolic Models

In the following, we introduce some stability properties for $\Sigma = (\mathbb{X}, \mathbb{W}, \mathbb{U}, f, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h^1, h^2)$ based on which one can construct a symbolic model for Σ along with the corresponding max-type simulation functions between Σ and its symbolic model.

3.3.5.1 Incremental Input-to-State Stability

Definition 3.3.14. *System Σ is incrementally input-to-state stable (δ-ISS) if there exist functions $V : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$, $\underline{\varphi}_x, \overline{\varphi}_x, \varphi_v, \varphi_w, \varphi_u \in \mathcal{K}_\infty$, with $\varphi_v < \mathsf{id}$ such that for all $x, x' \in \mathbb{X}$, $u, u' \in \mathbb{U}$, and for all $w, w' \in \mathbb{W}$*

$$\underline{\varphi}_x(|x - x'|) \leq V(x, x') \leq \overline{\varphi}_x(|x - x'|), \tag{3.3.16}$$

$$V(f(x, w, u), f(x', w', u')) \leq \varphi_v(V(x, \hat{x})) + \varphi_w(|w - w'|) + \varphi_u(|u - u'|). \tag{3.3.17}$$

We say that V is a δ-ISS Lyapunov function for system Σ if it satisfies (3.3.16) and (3.3.17). Observe that, any δ-ISS control system as in Definition 3.3.14 with $\varphi_w(r) = cr^2$, for some $c \in \mathbb{R}_{>0}$ and any $r \in \mathbb{R}_{\geq 0}$, is also δ-P as in Definition 3.2.5. We refer interested readers to [TRK16] for detailed information on incremental stability of discrete-time control systems.

3.3.5.2 Symbolic Models

The symbolic model of $T(\Sigma)$ associated to δ-ISS discrete-time control system Σ can be constructed similarly to the one in Definition 3.2.6. In particular when we consider a network of symbolic models, the symbolic model of $T(\Sigma)$ is the system $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ given by Definition 3.2.6 in which $\hat{W}_i$ should be constructed in such a way that the network $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathscr{N}}$ is well-defined. For example, choose $\hat{W}_i$ such that $\hat{W}_i = \prod\limits_{j \in \mathcal{N}_i} \hat{Y}_j^2$, $\forall j \in \mathcal{N}_i, \forall i \in \mathscr{N}$, where $\mathcal{N}_i$ is given as in Definition 3.3.1.

3.3.5.3 Construction of Max-Type Simulation Functions

In this subsection, we show how to construct a max-type simulation function between $T(\Sigma)$ associated to the δ-ISS discrete-time control systems Σ and its symbolic model $\hat{T}(\Sigma)$ constructed as in Definition 3.2.6.

Theorem 3.3.15. *Consider a transition system $T(\Sigma)$, associated to the δ-ISS discrete-time control system Σ. Let $\hat{T}(\Sigma)$ be a symbolic model constructed as in Definition 3.2.6. Suppose Assumption 3.2.12 holds for function V given in Definition 3.3.14. Then V is a max-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$.*

Proof. Recall that $\hat{\mathcal{H}}^j = \mathcal{H}^j = h^j, j \in [1; 2]$, by Definition 2.3.1 and 3.2.6. Hence, $\hat{\mathcal{H}}^j$ and $\mathcal{H}^j$ satisfy the Lipschitz assumption given on $h^j, j \in [1; 2]$ in the General Remark 2.5.

Consequently, and since system Σ is incrementally input-to-state stable, from (3.3.16), for any $x \in X$ and any $\hat{x} \in \hat{X}$, we have

$$|\mathcal{H}^j(x) - \hat{\mathcal{H}}^j(\hat{x})| \leq \ell^j(|x - \hat{x}|) \leq \ell^j \circ \underline{\varphi}_x^{-1}(V(x, \hat{x})).$$

By defining $\alpha = (\max_{j\in[1;2]}\{\ell^j\} \circ \underline{\varphi}_x^{-1})^{-1}$, one obtains $\underline{\alpha}(|\mathcal{H}^1(x) - \hat{\mathcal{H}}^1(\hat{x})|) \leq V(x, \hat{x})$. Furthermore, define $\overline{\alpha} := \overline{\varphi}_x$. Hence, (3.3.2) is satisfied.

Now consider any $\hat{u} \in \hat{U}$ and choose $u = \hat{u}$. Then, using (3.2.12), for any $x \in X, \hat{x} \in \hat{X}$, any $\hat{u} \in \hat{U}$, and any $w \in W, \hat{w} \in \hat{W}$, we have

$$V(f(x, \hat{u}, w), \hat{x}^+) \leq V(f(x, \hat{u}, w), f(\hat{x}, \hat{u}, \hat{w})) + \gamma(|\hat{x}^+ - f(\hat{x}, \hat{u}, \hat{w})|),$$

for any $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u}, \hat{w})$. Now, from Definition 3.2.6, the above inequality reduces to

$$V(f(x, \hat{u}, w), \hat{x}^+) \leq V(f(x, \hat{u}, w), f(\hat{x}, \hat{u}, \hat{w})) + \gamma(\eta^x).$$

Note that by (3.3.17), we get

$$V(f(x, \hat{u}, w), f(\hat{x}, \hat{u}, \hat{w})) \leq \varphi_v(V(x, \hat{x})) + \varphi_w(|w - \hat{w}|).$$

It follows that for any $x \in X, \hat{x} \in \hat{X}$, any $\hat{u} \in \hat{U}$, and any $w \in W, \hat{w} \in \hat{W}$, one obtains

$$V(f(x, \hat{u}, w), \hat{x}^+) \leq \varphi_v(V(x, \hat{x})) + \varphi_w(|w - \hat{w}|) + \gamma(\eta^x),$$

for any $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x}, \hat{u}, \hat{w})$. By using the result of Theorem 2.2.9, one obtains

$$V(f(x, \hat{u}, w), \hat{x}^+) \leq \max\{\tilde{\varphi}_v(V(x, \hat{x})), \tilde{\varphi}_w(|w - \hat{w}|), \tilde{\gamma}(\eta^x)\},$$

where $\tilde{\varphi}_v := (\mathsf{id} - (\mathsf{id} - \psi) \circ (\mathsf{id} - \varphi_v))$, $\tilde{\varphi}_w = (\mathsf{id} + \lambda) \circ (\mathsf{id} - \varphi_v)^{-1} \circ \psi^{-1} \circ \chi \circ \varphi_w$, and $\tilde{\gamma} = (\mathsf{id} + \lambda^{-1}) \circ (\mathsf{id} - \varphi_v)^{-1} \circ \psi^{-1} \circ \chi \circ (\chi - \mathsf{id})^{-1} \circ \gamma$, for some arbitrarily chosen $\lambda, \psi, \chi \in \mathcal{K}_\infty$ with $\psi < \mathsf{id}$ and $\chi > \mathsf{id}$. Thus, inequality (3.3.3) is satisfied with $\sigma = \tilde{\varphi}_v$, $\rho_w = \tilde{\varphi}_w$, $\rho_u = \tilde{\varphi}_u$, and $\varepsilon = \tilde{\gamma}(\eta^x)$. Hence, V is a max-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. The rest of the proof follows similar argument. In particular, by the definition of $\hat{U}$, for any $u \in U$ there always exists $\hat{u} \in \hat{U}$ such that $\varphi_u(|u - \hat{u}|) \leq \varphi_u(\eta^u)$ which results in $\varepsilon = (\mathsf{id} + \lambda^{-1}) \circ (\mathsf{id} - \varphi_v)^{-1} \circ \psi^{-1} \circ \chi \circ (\chi - \mathsf{id})^{-1} \circ \gamma(\varphi_u(\eta^u) + \gamma(\eta^x))$. □

Remark 3.3.16. *Observe that if φ_w and γ are linear functions in the previous theorem, $\tilde{\varphi}_w$ and $\tilde{\gamma}$ reduce to $\tilde{\varphi}_w = (\mathsf{id} + \lambda) \circ (\mathsf{id} - \varphi_v)^{-1} \circ \psi^{-1} \circ \varphi_w$ and $\tilde{\gamma} = (\mathsf{id} + \lambda^{-1}) \circ (\mathsf{id} - \varphi_v)^{-1} \circ \psi^{-1} \circ \gamma$, respectively.* ◇

Remark 3.3.17. *Although the choices of $\mathcal{K}_\infty$ functions λ, χ, and ψ in the previous theorem mainly depend on the dynamic of the given control systems, we provide a general guideline on choosing those functions as follows: (i) In order to reduce the undesirable effect of the inverse of* $\mathsf{id} - \varphi_v$ *and ψ in satisfying the small-gain condition in* (3.3.6)*, or in computing the value of the overall approximation error in* (2.2.8)*, one should choose those ψ to behave very close to the identity function, and φ_v as small as possible; (ii)*

Regarding λ and χ, one should choose those functions such that the small gain condition in (3.3.6) *is possibly satisfied, and then compute the overall approximation error in* (2.2.8)*. If the computed error is acceptable by the user, no further action is required; otherwise one should start slightly modifying those functions until a smaller error is achieved while ensuring that the small gain condition is not violated. For example, one can scale the $\mathcal{K}_\infty$ function λ by a linear function $\beta(s) = cs \in \mathcal{K}_\infty$, $\forall s \in \mathbb{R}_{\geq 0}, c > 1$, and then, using $\beta \circ \lambda$ instead of λ, start increasing the value of c until a smaller error is obtained. Same procedure can be simultaneously applied to the $\mathcal{K}_\infty$ function χ. It may be the case that the desired error is not achievable with the chosen λ and χ, then one should start over and choose different λ and χ and go through a similar procedure again.* ◇

Remark 3.3.18. *Note that if system Σ is not δ-ISS, one may assume that Σ is incrementally input-to-state stabilizable. That is there exists feedback controller $\mathcal{G} : \mathbb{X} \to \mathbb{U}$ such that* (3.3.17) *is satisfied with the left-hand side of* (3.3.17) *given as $V(f(x, w, \mathcal{G}(x) + u), f(x', w', \mathcal{G}(x') + u'))$.* ◇

If we consider linear control systems as in (2.3.3), similar results as in Theorem 3.3.15 can be provided in more computationally efficient way. In particular, the incremental input-to-state stability assumption in Definition 3.3.14 boils down in the linear case to the following assumption.

Assumption 3.3.19. *Consider linear control systems $\Sigma = (A, B, C^1, C^2, D)$. Assume that there exists matrix $Z \succ 0$ of appropriate dimensions such that the matrix inequality*

$$(1 + 2\theta)A^\top ZA \preceq \varphi_c Z, \tag{3.3.18}$$

holds for some constants $0 < \varphi_c < 1$, and $\theta \in \mathbb{R}_{>0}$.

Note that condition (3.3.18) is nothing more than asking matrix A being stable [AM07].

Theorem 3.3.20. *Consider a transition system $T(\Sigma)$, associated to the linear control systems Σ for which Assumption 3.3.19 holds. Let $\hat{T}(\Sigma)$ be a symbolic model constructed as in Definition 3.2.6. Then function V defined as*

$$V(x, \hat{x}) = \sqrt{(x - \hat{x})^\top Z(x - \hat{x})}, \tag{3.3.19}$$

is a max-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Proof. First, we show that condition (3.3.2) holds. Since $C = \hat{C}$, we have

$$|Cx - \hat{C}\hat{x}| \leq \sqrt{n\lambda_{\max}(C^\top C)}|x - \hat{x}|,$$

and similarity

$$\sqrt{\lambda_{\min}(Z)}|x - \hat{x}| \leq \sqrt{(x - \hat{x})^\top Z(x - \hat{x})}.$$

It can be readily verified that (3.3.2) holds for V defined in (3.3.19) with $\alpha(s) = \sqrt{\frac{\lambda_{\min}(Z)}{n\lambda_{\max}(C^\top C)}}s$ for any $s \in \mathbb{R}_{\geq 0}$. We continue to show that (3.3.3) holds as well. Let x, $\hat{x}$, $\hat{u}$, and $\hat{w}$ be given, and choose u as $u := \hat{u}$. Let $x^+ = Ax + Bu + Dw$, and $\hat{x}^+$ be defined as in Definition 3.2.6. Define $\Delta := A\hat{x} + B\hat{u} + D\hat{w} - \hat{x}^+$. Now, one obtains the chain of inequalities in (3.3.20). By following a similar argument as the one in the proof

$$
\begin{aligned}
&V(x^+, \hat{x}^+) \\
&= ((Ax + Bu + Dw - (A\hat{x} + B\hat{u} + D\hat{w}) + (A\hat{x} + B\hat{u} + D\hat{w}) - \hat{x}^+)^\top Z \\
&\quad (Ax + Bu + Dw - (A\hat{x} + B\hat{u} + D\hat{w}) + (A\hat{x} + B\hat{u} + D\hat{w}) - \hat{x}^+))^{\frac{1}{2}} \\
&= \big((x-\hat{x})^\top A^\top ZA(x-\hat{x}) + (w-\hat{w})^\top D^\top ZD(w-\hat{w}) + 2(w-\hat{w})^\top D^\top Z\Delta \\
&\quad + 2(x-\hat{x})^\top A^\top ZD(w-\hat{w}) + 2(x-\hat{x})^\top A^\top Z\Delta + \Delta^\top Z\Delta\big)^{\frac{1}{2}} \\
&\leq \big((x-\hat{x})^\top A^\top ZA(x-\hat{x}) + (w-\hat{w})^\top D^\top ZD(w-\hat{w}) + 2|(w-\hat{w})^\top D^\top \sqrt{Z}|_2|\sqrt{Z}\Delta|_2 \\
&\quad + 2|(x-\hat{x})^\top A^\top \sqrt{Z}|_2|\sqrt{Z}\Delta|_2 + 2|(x-\hat{x})^\top A^\top \sqrt{Z}|_2|\sqrt{Z}D(w-\hat{w})|_2 \\
&\quad + n\lambda_{\max}(Z)\eta^2\big)^{\frac{1}{2}} \\
&\leq \big((x-\hat{x})^\top A^\top ZA(x-\hat{x}) + 2\theta|(x-\hat{x})^\top A^\top \sqrt{Z}|_2^2 + (w-\hat{w})^\top D^\top ZD(w-\hat{w}) \\
&\quad + \frac{|(w-\hat{w})^\top D^\top \sqrt{Z}|_2^2}{\theta} + 2\frac{|\sqrt{Z}\Delta|_2^2}{\theta} + \theta|(w-\hat{w})^\top D^\top \sqrt{Z}|_2^2 + n\lambda_{\max}(Z)\eta^2\big)^{\frac{1}{2}} \\
&\leq \Bigg((1+2\theta)(x-\hat{x})^\top A^\top ZA(x-\hat{x}) + \frac{(1+\theta+\theta^2)(w-\hat{w})^\top D^\top ZD(w-\hat{w})}{\theta} \\
&\quad + \frac{n(2+\theta)\lambda_{\max}(Z)\eta^2}{\theta}\Bigg)^{\frac{1}{2}} \\
&\leq \sqrt{\varphi_c}V(x,\hat{x}) + \sqrt{\frac{1+\theta+\theta^2}{\theta}}|\sqrt{Z}D|_2|w-\hat{w}|_2 + \sqrt{\frac{n(2+\theta)\lambda_{\max}(Z)}{\theta}}\eta \\
&\leq \sqrt{\varphi_c}V(x,\hat{x}) + \sqrt{p\frac{1+\theta+\theta^2}{\theta}}|\sqrt{Z}D|_2|w-\hat{w}| + \sqrt{\frac{n(2+\theta)\lambda_{\max}(Z)}{\theta}}\eta.
\end{aligned}
\tag{3.3.20}
$$

of Theorem 2.2.9, one gets

$$
\begin{aligned}
V(x^+, \hat{x}^+) \leq \max\Bigg\{&\tilde{\varphi}_c\big((x-\hat{x})^\top Z(x-\hat{x})\big)^{\frac{1}{2}}, \frac{(1+\psi_c)}{(1-\sqrt{\varphi_c})\psi_c}\sqrt{p\frac{(1+\theta+\theta^2)}{\theta}}|\sqrt{Z}D|_2|w-\hat{w}|, \\
&\frac{(1+1/\psi_c)}{(1-\sqrt{\varphi_c})\psi_c}\sqrt{\frac{n(2+\theta)\lambda_{\max}(Z)}{\theta}}\eta\Bigg\},
\end{aligned}
$$

where $\tilde{\varphi}_c = (1-(1-\sqrt{\varphi_c})(1-\psi_c))$, satisfying (3.3.3) with $\sigma = \tilde{\varphi}_c$, $\rho_u = 0$, $\rho_w(s) = \frac{(1+\lambda_c)}{(1-\sqrt{\varphi_c})\psi_c}\sqrt{p\frac{(1+\theta+\theta^2)}{\theta}}|\sqrt{Z}D|_2$, $\varepsilon = \frac{(1+1/\lambda_c)}{(1-\sqrt{\varphi_c})\psi_c}\sqrt{\frac{n(2+\theta)\lambda_{\max}(Z)}{\theta}}\eta$, where ψ_c and λ_c can be chosen arbitrarily such that $0 < \psi_c < 1$ and $\lambda_c > 0$. Hence, the proposed V in (3.3.19) is a max-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. The rest of the proof follows similar argument. In particular, by the definition of $\hat{U}$, for any $u \in U$ there always exists $\hat{u} \in \hat{U}$ such that $|B||u-\hat{u}| \leq |B|\eta^u$ which results in $\varepsilon = \frac{(1+1/\lambda_c)}{(1-\sqrt{\varphi_c})\psi_c}\sqrt{\frac{n(2+\theta)\lambda_{\max}(Z)}{\theta}}(|B|\eta^u + \eta^x)$. Other terms are the same as before. □

Remark 3.3.21. *Note that if condition* (3.3.18) *can not be satisfied, one can assume that* $\Sigma = (A, B, C^1, C^2, D)$ *is stabilizable and still have the result in Theorem 3.3.20. That is there exists matrices* $Z \succ 0$ *and state feedback gain* K *of appropriate dimensions such that the matrix inequality*

$$(1+2\theta)(A+BK)^\top Z(A+BK) \preceq \varphi_c Z, \tag{3.3.21}$$

holds for some constants $0 < \varphi_c < 1$*, and* $\theta \in \mathbb{R}_{>0}$. ◇

Remark 3.3.22. *Given constants* φ_c *and* θ*, one can easily see that inequality* (3.3.21) *is not jointly convex on decision variables* Z *and* K *and, hence, not amenable to existing semidefinite tools for linear matrix inequalities (LMI). However, using Schur complement, one can easily transform inequality* (3.3.21) *to the following LMI over decision variables* E_1 *and* E_2*:*

$$\begin{bmatrix} -\varphi_c E_1 & E_1 A^\top + E_2^\top B^\top \\ AE_1 + BE_2 & -(1+2\theta)E_1 \end{bmatrix} \preceq 0, \;\; E_1 \succ 0,$$

where $E_1 = Z^{-1}$ *and* $E_2 = KE_1$. ◇

3.3.6 Case Studies

In this section we provide two case studies to illustrate the results of Section 3.3 and compare them with the results of Section 3.2. We first apply our results to the temperature regulation in a circular building by constructing compositionally a symbolic model of a finite network containing $n \geq 3$ rooms, each equipped with a heater. Then we apply the proposed techniques to a fully connected finite network to show its applicability to strongly connected networks as well. Moreover, we verify the effectiveness of proposed technique in Subsection 3.3.4 by applying it to a model of a road traffic network containing infinitely many cells (systems). We construct symbolic models for the original systems and compositionally construct an alternating simulation function from the infinite network containing infinitely many symbolic models to the infinite network of the concrete subsystems. We also design controllers compositionally maintaining the density of traffic between 10 and 25 vehicles per cell. The construction of symbolic models and controllers are performed using tool SCOTS [RZ16] on a PC with Intel i7@3.4GHz CPU and 16 GB of RAM.

3.3.6.1 Room Temperature Control

In this subsection, we apply our results to the temperature regulation in a circular building of $n \geq 3$ rooms, each equipped with a heater. The dynamic of the network Σ

is described by (3.2.17). By introducing

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = (1-2\beta-\hat{\beta}-\bar{\beta}\nu_i(k))\mathbf{x}_i(k) + [\beta;\beta]^\top \omega_i(k) + \hat{\beta}T_{ei} + \bar{\beta}T_h\nu_i(k), \\ \quad \mathbf{y}_i^1(k) = \mathbf{x}_i(k), \\ \quad \mathbf{y}_i^2(k) = \mathbf{x}_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}(\Sigma_i)_{i\in[1,n]}$, where $\omega_i(k) = [\mathbf{y}_{i-1}^2(k); \mathbf{y}_{i+1}^2(k)]$ with $\mathbf{y}_0^2 = \mathbf{y}_n^2$ and $\mathbf{y}_{n+1}^2 = \mathbf{y}_1^2$. Note that for any $i \in [1;n]$, conditions (3.3.16) and (3.3.16) are satisfied with $V_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$, $\underline{\varphi}_{x_i} = \overline{\varphi}_{x_i} = \mathsf{id}$, $\varphi_{f_i} = 1 - 2\beta - \hat{\beta}$, $\varphi_{w_i} = \beta$, and $\varphi_{u_i} = 0$, when $u = u'$. Furthermore, (3.2.12) is satisfied with $\gamma = \mathsf{id}$. Consequently, $V_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$ is a max-type simulation function from $\hat{T}_i(\Sigma_i)$, constructed as in Definition 3.2.6, to $T_i(\Sigma_i)$ associated to Σ_i.

Since we have $\gamma_{ij}(s) < \mathsf{id}$, $\forall i,j \in [1;n]$, $i \neq j$ and for any $n \geq 3$, the small-gain condition (3.3.6) is satisfied without any restriction on the number of rooms. Using the results in Theorem 3.3.5 with $\psi_i^{-1} = \mathsf{id}$, $\forall i \in [1;n]$, one can verify that $\tilde{\mathcal{S}}(x,\hat{x}) = \max_i\{|x_i - \hat{x}_i|\}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in[1,n]}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in[1,n]}$ associated to Σ, satisfying conditions (2.2.1) and (2.2.2) with $\tilde{\sigma} = \max\left\{(1-(1-2\beta+\hat{\beta})10^{-2}), \frac{2.02\beta}{1-(1-2\beta-\hat{\beta})}\right\}$, $\tilde{\alpha} = \mathsf{id}$, $\tilde{\rho}_u = 0$, $\tilde{\varepsilon} = \max_i\left\{\frac{2.02\eta_i^x}{1-(1-2\beta-\hat{\beta})}\right\}$, $\forall i \in [1;N]$, where η_i^x is the state set quantization parameter of abstraction $\hat{T}_i(\Sigma_i)$.

For the comparison, we compute error $\hat{\varepsilon}$ in the $\hat{\varepsilon}$-approximate alternating simulation relation as in (2.2.8) based on the dissipativity approach introduced in Section 3.2 and the small-gain approach introduced in Section 3.3. This error represents the mismatch between the output behavior of the concrete network Σ and that of its finite abstraction $\hat{T}(\Sigma)$. We evaluate $\hat{\varepsilon}$ for different numbers of systems n and different values of the state set quantization parameters η_i^x for abstractions $\hat{T}_i(\Sigma_i)$ $\forall i \in [1;n]$ as in Figure 3.5. As shown, the small-gain approach results in less mismatch errors than those obtained using the dissipativity based approach proposed in Section 3.2. The reason is that the error in (2.2.8) is computed based on the maximum of the errors between concrete systems and their symbolic models instead of being a linear combination of them which is the case in Section 3.2. Hence, by increasing the number of systems, the error computed based on the small-gain approach introduced in Section 3.3 does not change here whereas the error computed by the dissipativity based approach proposed in Section 3.2 will increase as shown in Figure 3.5.

Now, we synthesize a controller for Σ via abstractions $\hat{T}_i(\Sigma_i)$ such that the temperature of each room is maintained in the comfort zone $\mathscr{S} = [19,\ 21]$. The idea here is to design local controllers for abstractions $\hat{\Sigma}_i$, and then refine them to concrete systems Σ_i. To do so, the local controllers are synthesized while assuming that the other systems meet their own specifications. The computation times for constructing symbolic models and synthesizing controllers for Σ_i are 0.048s and 0.001s, respectively. Figure 3.6 shows the maximum and minimum of the state trajectories of the closed-loop network Σ, consisting of 1000 rooms, under control inputs u_i with the state and input quantization parameters $\eta_i^x = 0.01$ and $\eta_i^u = 0.01$, $\forall i \in [1;1000]$, respectively.

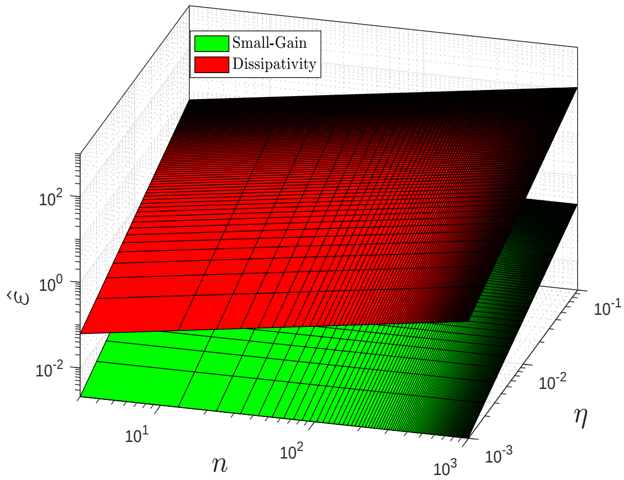

Figure 3.5: Temperature control: Comparison of errors in (2.2.8) resulted from the approach based on small-gain condition with those based on dissipativity-type condition for different values of $n \geq 3$ and $\eta = \eta_1^x = \cdots = \eta_n^x$.

3.3.6.2 Fully Connected Network

In order to show the applicability of the small-gain approach to strongly connected networks, we consider a nonlinear network Σ described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A\mathbf{x}(k) + \phi(x) + \nu(k), \\ \mathbf{y}(k) &= \mathbf{x}(k), \end{cases}$$

where $A = I_n - \tau L$ for some Laplacian matrix $L \in \mathbb{R}^{n \times n}$ of an undirected graph [GR01], and constant $0 < \tau < 1/\Delta$, where Δ is the maximum degree of the graph [GR01]. Moreover $\mathbf{x}(k) = [\mathbf{x}_1(k); \ldots; \mathbf{x}_n(k)]$, $\nu(k) = [\nu_1(k); \ldots; \nu_n(k)]$, and $\phi(x) = [\phi_1(x_1); \ldots; \phi_n(x_n)]$, where $\phi_i(x_i) = sin(x_i), \forall i \in [1; n]$. Assume L is the Laplacian matrix of a complete graph:

$$L = \begin{bmatrix} n-1 & -1 & \cdots & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & n-1 \end{bmatrix}.$$

Now, by introducing Σ_i described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) &= a_i\mathbf{x}_i(k) + \varphi_i(x_i) + d_i\omega_i(k) + \nu_i(k), \\ \mathbf{y}_i^1(k) &= \mathbf{x}_i(k), \\ \mathbf{y}_i^2(k) &= \mathbf{x}_i(k), \end{cases}$$

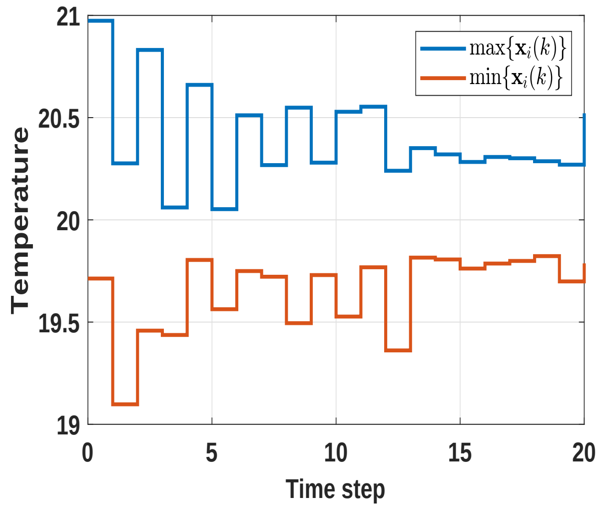

Figure 3.6: Bounds inside which state trajectories of the closed-loop network Σ consisting of 1000 rooms are evolving.

where $a_i = \{A\}_{i,i}$, $\omega_i(k) = (\mathbf{y}_j^2)_{j\in([1;n]\setminus i)}$, $d_i = [\{A\}_{i,1};\ldots;\{A\}_{i,i-1};\{A\}_{i,i+1};\ldots;\{A\}_{i,n}]^\top$, one can readily verify that $\Sigma = \mathcal{I}(\Sigma_i)_{i\in[1,n]}$. Clearly, for any $i \in [1;n]$, conditions (3.3.16) and (3.3.16) are satisfied with $V_i(x_i,\hat{x}_i) = |x_i - \hat{x}_i|$, $\mathcal{G}_i(x_i) = -c_i x_i$, where $\frac{a_i+1}{2} < c_i < a_i + 1$, $\underline{\varphi}_{x_i} = \overline{\varphi}_{x_i} = \mathsf{id}$, $\varphi_{f_i} = (1 + a_i - c_i)$, $\varphi_{w_i} = |d_i|$, and $\varphi_{u_i} = 0$. Note that we utilized feedback controller $\mathcal{G}$ as in Remark 3.3.18 to make systems Σ_i δ-ISS. Moreover, (3.2.12) is satisfied with $\gamma = \mathsf{id}$. Consequently, $V_i(x_i,\hat{x}_i) = |x_i - \hat{x}_i|$ is a max-type simulation function from $\hat{T}_i(\Sigma_i)$, constructed as in Definition 3.2.6, to $T_i(\Sigma_i)$ associated to Σ_i.

Fix $\tau = \frac{0.1}{\Delta} = \frac{0.1}{n-1}$. Since we have $\gamma_{ij}(s) < \mathsf{id}$, $\forall i,j \in [1;n]$, $i \neq j$, the small-gain condition (3.3.6) is satisfied without any restriction on the number of systems. Using the results in Theorem 3.3.5 with $\psi_i^{-1} = \mathsf{id}$, $\forall i \in [1;n]$, one can verify that $\tilde{\mathcal{S}}(x,\hat{x}) = \max_i\{|x_i - \hat{x}_i|\}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in[1,n]}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in[1,n]}$ associated to Σ satisfying conditions (2.2.1) and (2.2.2) with $\tilde{\alpha} = \mathsf{id}$, $\tilde{\rho}_u = 0$, $\tilde{\varepsilon} = \max_i\left\{\frac{2.02\eta_i^x}{1-(1+a_i-c_i)}\right\}$, $\tilde{\sigma}(s) = \max\left\{\max_i\left\{\left(1 - \frac{(1-(1+a_i-c_i))}{10^2}\right)s\right\}, \max_i\left\{\frac{2.02|d_i|}{1-(1+a_i-c_i)}s\right\}\right\}$, where η_i^x is the state set quantization parameter of abstraction $\hat{T}_i(\Sigma_i)$.

Similar to the previous case study, we compare the small-gain technique in Section 3.3 to the one proposed in Section 3.2. A comparison of the error $\hat{\varepsilon}$ in (2.2.8) resulted from the dissipativity approach proposed in Section 3.2 and the small-gain technique in Section 3.3 is shown in Figure 3.7. We compute $\hat{\varepsilon}$ for different n and different values of η_i^x for abstractions $\hat{T}_i(\Sigma_i)$ $\forall i \in [1;n]$. Clearly, the small-gain approach results in

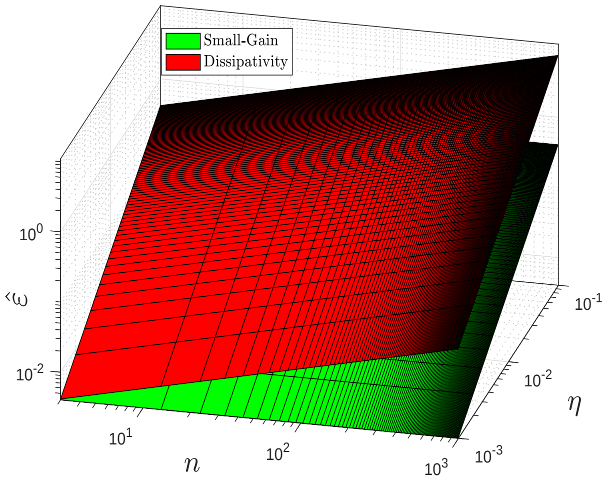

Figure 3.7: Fully connected network: Comparison of errors in (2.2.8) resulted from the approach based on small-gain condition with those based on dissipativity-type condition for different values of $n \geq 1$ and $\eta = \eta_1^x = \cdots = \eta_n^x$.

less mismatch errors than those obtained using the dissipativity based approach . The computation time for constructing abstractions for Σ_i is 0.9s after fixing $n = 1000$, $\eta_i^x = 0.01$, $\mu_i = 0.01$, $x_i \in [0, 10]$, $\nu_i \in [0, 1]$, $\forall i \in [1; n]$.

3.3.6.3 Infinite Road Traffic Model

In this case study, we apply the approach in Subsection 3.3.4 to a variant of the road traffic model from [dWOK12]. We consider a traffic network divided into infinitely many cells, indexed by $i \in \mathbb{N}$. Each cell i represents a one-dimensional subsystem $\Sigma_i = (X_i, W_i, U_i, f_i, X_i)$ described by a difference equation of the following form

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = & (1 - \frac{\tau v}{l} - e)\mathbf{x}_i(k) + d_i\omega_i(k) + b\nu_i(k), \\ \mathbf{y}_i^j(k) = & \mathbf{x}_i(k), \quad j \in [1; 2], \end{cases} \tag{3.3.22}$$

with the following structure

- $d_i = (\frac{1-e}{2})(\frac{\tau v}{l}, \frac{\tau v}{l})^\top, \omega_i = [\mathbf{y}_{i+1}^2, \mathbf{y}_{i+2}^2]$ if $i \in J_1 := \{1 + 2c : c \in \mathbb{N}_0\}$;
- $d_i = (1 - e)\frac{\tau v}{l}, \omega_i = \mathbf{y}_{i+1}^2$ if $i \in J_2 := \{2\}$;
- $d_i = (\frac{1-e}{2})(\frac{\tau v}{l}, \frac{\tau v}{l})^\top, \omega_i = [\mathbf{y}_{i-2}^2; \mathbf{y}_{i-1}^2]$ if $i \in J_3 := \{4 + 2c : c \in \mathbb{N}_0\}$.

In (3.3.22), τ is the sampling time interval in hours, l is the length of a cell in kilometers (km), and v is the flow speed of the vehicles in kilometers per hour (km/h). The state of each subsystem Σ_i, i.e. $\mathbf{x}_i$, is the density of traffic, given in vehicles per cell, for each cell i of the network. The scalar b represents the number of vehicles that can enter the

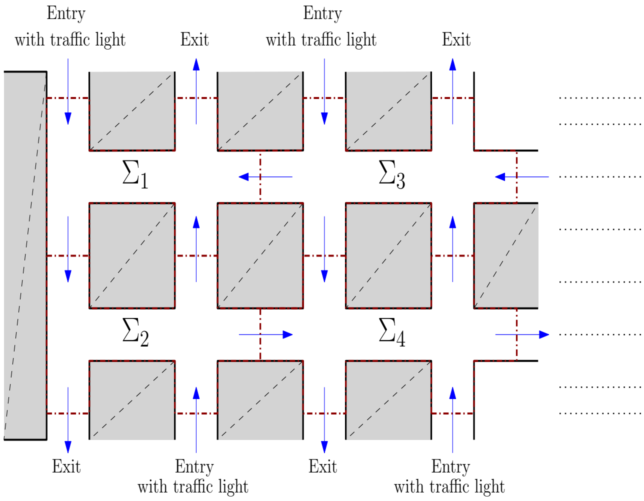

Figure 3.8: Model of a road traffic network composed of infinitely many systems.

cells through entries which are controlled by $\nu_i(\cdot)$. In particular, $\nu_i(\cdot) = 1$ means green light and $\nu_i(\cdot) = 0$ means red light. Moreover, the constant $e \in (0, 1)$ represents the percentage of vehicles that leave the cells using available exits. The infinite network and its cells are illustrated in Figure 3.8.

Let us first show that $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ is well-defined by showing that $\|f(x, u)\| < \infty$, where $f(x, u)$ is constructed as in Definition 3.3.1. Define $C_1 = |1 - \frac{\tau v}{l} - e|$, $C_2 = |(1-e)\frac{\tau v}{l}|$, $C_3 = |b|$, $C = \max_{1\leq i\leq 3}\{C_i\}$, then one has

$$
\begin{aligned}
\|f(x,u)\| &= \sup_{i\in\mathbb{N}}\{|f_i(x_i,w_i,u_i)|\} = \sup_{i\in\mathbb{N}}\{|(1-\frac{\tau v}{l}-e)x_i+d_iw_i+bu_i|\}\\
&\leq C_1\sup_{i\in\mathbb{N}}\{|x_i|\}+C_2\sup_{i\in\mathbb{N}}\{|x_i|\}+C_3\sup_{i\in\mathbb{N}}\{|u_i|\}\leq C(\sup_{i\in\mathbb{N}}\{|x_i|\}+\sup_{i\in\mathbb{N}}\{|x_i|\}+\sup_{i\in\mathbb{N}}\{|u_i|\})\\
&= C(\|x\| + \|x\| + \|u\|\}) < \infty
\end{aligned}
$$

Hence, $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ is well-defined.

Fix $\tau = \frac{10}{60\times 60}, v = 60, l = 0.5$, and $e = 0.1$, then for any $i \in \mathbb{N}$, system Σ_i is δ-ISS, where conditions (3.3.16) and (3.3.17) are satisfied with $V_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$, $\underline{\varphi}_{x_i} = \overline{\varphi}_{x_i} = \mathsf{id}$, $\varphi_{f_i} = (1 - (\frac{\tau v}{l} + e))$, $\varphi_{w_i} = |(1-e)\frac{\tau v}{l}|$, and $\varphi_{u_i} = 0$ with $u_i = \hat{u}_i$. Furthermore, (3.2.12) is satisfied with $\gamma_i = \mathsf{id}$. Consequently, $V_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$ is a max-type simulation function from $\hat{T}_i(\Sigma_i)$, constructed as in Definition 3.2.6, to $T_i(\Sigma_i)$ associated to Σ_i. Note that for the construction of symbolic models $\hat{T}_i(\Sigma_i)$, we have chosen the finite set $\hat{W}_i = \hat{X}_{i+1} \times \hat{X}_{i+2}$ for all $i \in J_1$, $\hat{W}_i = \hat{X}_{i+1}$ for all $i \in J_2$, and $\hat{W}_i = \hat{X}_{i-2} \times \hat{X}_{i-1}$ for all $i \in J_3$. Moreover, it can be readily verified that $\gamma_{ij} < \mathsf{id}$. Therefore, by remark 3.3.11, $\tilde{\mathcal{S}}(x, \hat{x}) := \sup_{i\in\mathbb{N}}\{|x_i - \hat{x}_i|\}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathbb{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathbb{N}}$ associated to Σ satisfying conditions (2.2.1) and (2.2.2) with $\tilde{\alpha} = \mathsf{id}$, $\tilde{\sigma} = 0.97$, $\tilde{\rho}_u = 0$, and $\tilde{\varepsilon} = 17\sup_{i\in\mathbb{N}}\{\eta_i^x\}$.

In order to guarantee that ε is well-defined, one should choose η_i^x such that there exists $\eta^x \in \mathbb{R}_{>0}$ so that $\eta_i^x \leq \eta^x, \forall i \in \mathbb{N}$.

Now we show how to use the constructed symbolic models $\hat{T}_i(\Sigma_i)$ to design a controller for Σ such that the density of traffic is maintained between 10 and 25 vehicles per cell (systems Σ_i). Based on assume-guarantee reasoning, we compositionally synthesize controllers for symbolic models, and then refine them to the ones for concrete systems. In particular, we design local controllers $\hat{u}_i$ for $\hat{T}_i(\Sigma_i)$ while assuming that the other systems $\hat{T}_i(\Sigma_j), j \neq i$, meet their specifications, and then refine $\hat{u}_i$ to u_i using $u_i = \hat{u}_i$. We leverage software tool SCOTS [RZ16] for constructing symbolic models and controllers for Σ_i compositionally with $b = 5$, state quantization parameter $\eta_i^x = 0.1$ and the computation times are amounted to $0.016s$ and $9 \times 10^{-4}s$, respectively. Figure 3.9 shows trajectories of sample system Σ_i starting from different initial conditions under input u_i. Finally, one can compute the mismatch between the output behavior of $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathbb{N}}$ and that of its symbolic model $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathbb{N}}$ by utilizing Proposition 2.2.11. In particular, using (2.2.8) and since $\tilde{\alpha} = \mathsf{id}, \rho_u = 0$, we have $\hat{\varepsilon} = \tilde{\alpha}^{-1}(\tilde{\varepsilon}) = \sup_{i\in\mathbb{N}}\{\tilde{\varepsilon}_i\} = 1.7$.

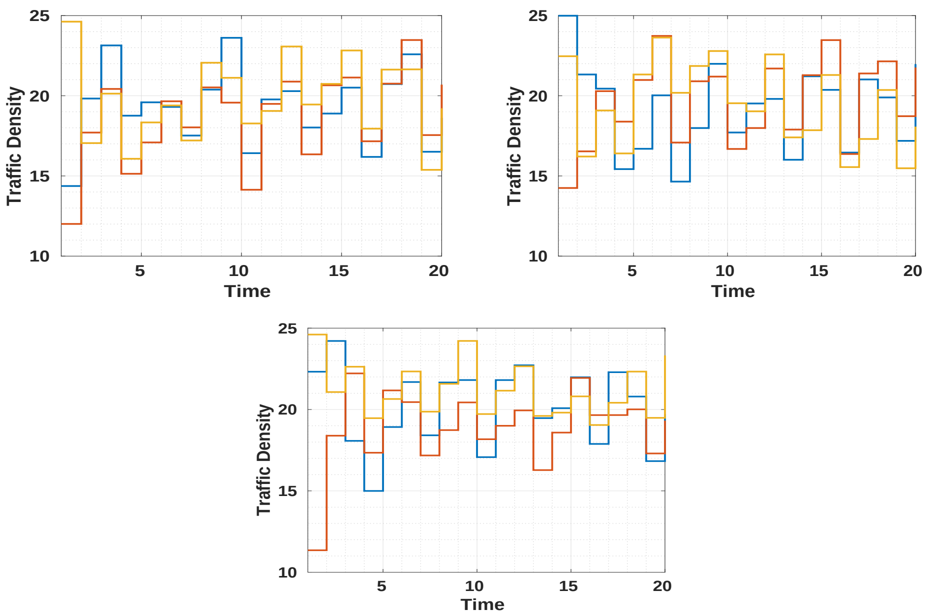

Figure 3.9: Trajectories of sample subsystem Σ_i starting from different initial conditions with (up-left) $i \in J_1$, (up-right) $i \in J_2$, and (down-middle) $i \in J_3$.

3.4 General Remark

Given that function S in Theorem 3.2.12, similarly in Theorem 3.2.8, is a sum-type simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$ and from $T(\Sigma)$ to $\hat{T}(\Sigma)$, it can be readily verified that function $\tilde{S}$ defined in (3.2.7) is also an alternating simulation function from

$T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$ to $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$. Note that the same argument is valid for function V in Theorem 3.3.15, similarly in Theorem 3.3.20. Thus, function $\tilde{\mathcal{S}}$ defined in (3.3.8) or in (3.3.14) is an alternating simulation function from $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$ to $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$. Hence, $\hat{T}(\Sigma)$ is a complete symbolic model [Tab09] for the concrete network $T(\Sigma)$. In other words, there exists a controller enforcing the desired specifications on the abstract network $\hat{T}(\Sigma)$ *if and only if* there exists a controller enforcing the same specifications on the original network $T(\Sigma)$.

3.5 Summary

In the first part of this chapter, we proposed a compositional framework based on dissipativity-type conditions for the construction of symbolic models for finite network of discrete-time control systems. First, we used a notion of so-called sum-type simulation functions in order to construct compositionally a notion of so-called alternating simulation functions that is used to quantify the error between the output behavior of the overall concrete network and that its symbolic model. Furthermore, we provided an approach to construct symbolic models together with their corresponding sum-type simulation functions for a class of discrete-time control systems under some incremental passivity property. We apply our results to the temperature regulation in a circular building by constructing compositionally a symbolic model of a finite network containing 1200 rooms. We use the constructed symbolic models as substitutes to synthesize controllers compositionally maintaining room temperatures in a comfort zone.

In the second part of the chapter, we proposed a compositional framework using two different nonlinear small-gain conditions for the construction of symbolic models for (in)finite network of discrete-time control systems. First, we used a notion of so-called max-type simulation functions in order to construct compositionally an alternating simulation function that is used to quantify the error between the output behavior of the overall concrete network and that of its symbolic model. Furthermore, we provided a technique to construct symbolic models together with their corresponding max-type simulation functions for discrete-time control systems under incremental input-to-state stability property. Finally, we illustrated the proposed results by constructing symbolic models for three networks of (linear and nonlinear) discrete-time control systems and their corresponding alternating simulation functions in a compositional fashion. The first two case studies elucidated the effectiveness of our compositionality results in comparison with the ones using dissipativity-type reasoning. The third case study shows the effectiveness of our compositionality technique when dealing with infinite networks.

4 Symbolic Models for Finite Networks of Discrete-Time Switched Systems

4.1 Introduction

Switched systems serve as an important modeling framework describing several engineering systems in which physical processes have various operation modes [Lib03]. Despite considerable number of studies that have been conducted regarding stability of switched systems (see for example [LHM11, YL15]), the fast grow in computational technology requires us to make same progresses with respect to more sophisticated objectives such as those expressed as linear temporal logic (LTL) formulae [BK08]. One particular technique to address complex objectives is based on the construction of symbolic models of switched systems. However, as the complexity of constructing symbolic models grows exponentially in the number of state variables in the switched system, the approaches proposed for constructing symbolic models for switched system so far in the literature are limiting the applications of symbolic models to only low-dimensional switched systems. This chapter proposes a compositional framework based on dissipativity and small-gain reasoning for synthesizing symbolic models for finite networks of switched systems.

4.1.1 Related Work

In recent years, there have been several results on the construction of symbolic models of switched systems. The work by [GPT10] provides a symbolic model that is related by an approximate bisimulation relation to the original incrementally stable switched system. In [CGG13, ZAG15], an approximate bisimulation relation was established between a symbolic model and incrementally stable switched system in which the symbolic states are sequences of modes of a given length. Recently, the result in [GPT10] has been extended to the case of multi-rate symbolic models in [SG17], multi-scale symbolic models computed using non-uniform adaptive space discretization in [GGM16], and to switched systems with aperiodic time sampling in [KGS18]. Note that all the proposed results in [GPT10, CGG13, ZAG15, GGM16, SG17, KGS18] take a monolithic view of switched systems when abstracting the entire system. Hence, the construction of symbolic models for large-scale networks of switched systems is very complex from a computational point of view. Although the result in [CGG13] provides a state-space discretization-free approach for computing symbolic models of incrementally stable switched systems, this approach is still monolithic and reduces the computational complexity only for switched systems with small number modes, see [CGG13, Section IV(D)].

4.1.2 Contributions

In the first part of this chapter, we provide a compositional methodology for the construction of symbolic models of finite networks of discrete-time switched systems based on dissipativity theory [AMP16]. We first define a notion of so-called sum-type augmented simulation functions to relate switched systems and their symbolic models. Then, by leveraging dissipativity-type compositional conditions, we construct a notion of so-called alternating simulation function as a relation between the finite network of switched systems and that of their symbolic models. This alternating simulation function allows one to determine quantitatively the mismatch between the output behavior of the network of switched systems and that of their symbolic models. Moreover, we provide an approach to construct symbolic models together with their corresponding sum-type augmented simulation functions for discrete-time switched systems under some assumptions ensuring incremental passivity of each mode of switched systems. Finally, we apply these results to a model of road traffic by constructing compositionally a symbolic model of a network containing 50 cells of 1000 meters each. We use the constructed symbolic models as substitutes to design controllers compositionally maintaining the density of traffic lower than 30 vehicles per cell. Additionally, we apply those results to a finite network of switched systems admitting multiple incremental passive storage functions.

In the second part of the chapter, we introduce a compositional methodology based on small-gain type reasoning for the construction of symbolic models of networks of switched systems. The proposed approach leverages sufficient small-gain type conditions to establish the compositionality results which rely on the existence of max-type augmented simulation functions as relations between switched systems and their symbolic models. In particular, based on some small-gain type conditions, we use those max-type augmented simulation functions to construct compositionally an alternating simulation function as a relation between a finite network of symbolic models and that of original switched systems. Furthermore, under standard assumptions ensuring incremental input-to-state stability of a switched system (i.e., existence of a common incremental input-to-state Lyapunov function, or multiple incremental input-to-state Lyapunov functions with dwell-time), we show that one can construct symbolic models of switched systems in general nonlinear settings. Moreover, we show that the incremental input-to-state stability assumption boils down to a linear matrix inequality for a specific class of nonlinear switched systems. We also use the result based on small-gain type reasoning to construct symbolic models and design controllers for the model of road traffic introduced in the first part of this chapter. Moreover, we apply those results to a network of switched systems admitting multiple incremental input-to-state Lyapunov funtions.

4.2 Dissipativity Approach (DA)

4.2.1 Networks of Discrete-Time Switched Systems: DA Formulation

Definition 4.2.1. *Consider discrete-time switched systems* $\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$*,* $i \in \mathscr{N} := [1; N]$*,* $N \in \mathbb{N}$*, and a static matrix* M *of an appropriate dimension* *defining the coupling of these systems, where* $M \prod_{i\in\mathcal{N}} \mathbb{Y}_i^2 \subseteq \prod_{i\in\mathcal{N}} \mathbb{W}_i$. *The network of discrete-time switched system* $\Sigma = (\mathbb{X}, P, F, \mathbb{Y}, h)$, *denoted by* $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$, *is defined by* $\mathbb{X} = \prod_{i\in\mathcal{N}} \mathbb{X}_i$, $P = \prod_{i\in\mathcal{N}} P_i$, $F = \prod_{i\in\mathcal{N}} F_i$, $\mathbb{Y} = \prod_{i\in\mathcal{N}} \mathbb{Y}_i^1$, $h(x) = (h_i^1(x_i))_{i\in\mathcal{N}}$, *where* $x = (x_i)_{i\in\mathcal{N}}$, *with the internal inputs constrained according to* $(w_i)_{i\in\mathcal{N}} = M(h_i^2(x_i))_{i\in\mathcal{N}}$. *The network of discrete-time switched systems is defined by the difference equations*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f_{\mathsf{p}(k)}(\mathbf{x}(k)), \\ \mathbf{y}(k) &= h(\mathbf{x}(k)), \end{cases} \tag{4.2.1}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$, $\mathsf{p} : \mathbb{N} \to \mathbb{P}$, $\mathbf{y} : \mathbb{N} \to \mathbb{Y}$, *and* $f_{\mathsf{p}(k)} = (f_{\mathsf{p}_i(k)}(\mathbf{x}_i(k), \omega_i(k)))_{i\in\mathcal{N}}$ *with* $\mathsf{p}(k) = (\mathsf{p}_i(k))_{i\in\mathcal{N}}$.

4.2.2 Sum-Type Augmented Simulation Functions

Consider a network of discrete-time switched systems $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$, or the equivalent network of transition systems $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$, where each $T_i(\Sigma_i)$ given as in Definition 2.4.2. Assume that each systems $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ and $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ admit a sum-type augmented simulation function as defined next.

Definition 4.2.2. *Consider systems* $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ *and* $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ *where* $\hat{Y}_i^1 \subseteq Y_i^1$. *A function* $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ *is called a sum-type augmented simulation function from* $\hat{T}_i(\Sigma_i)$ *to* $T_i(\Sigma_i)$ *if there exist* $\alpha_i \in \mathcal{K}_\infty$, $0 < \sigma_i < 1$, *a symmetric matrix* R_i *of appropriate dimension with conformal block partitions* $R_i^{jj'}$, $j, j' \in [1;2]$, *and some* $\varepsilon_i \in \mathbb{R}_{\geq 0}$ *so that the following hold:*

- *For every* $(x_i, p_i, l_i) \in X_i, (\hat{x}_i, p_i, l_i) \in \hat{X}_i$, *one has*

$$\alpha_i(|\mathcal{H}_i^1(x_i, p_i, l_i) - \hat{\mathcal{H}}_i^1(\hat{x}_i, p_i, l_i)|) \leq \mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)). \tag{4.2.2}$$

- *For every* $(x_i, p_i, l_i) \in X_i, (\hat{x}_i, p_i, l_i) \in \hat{X}_i, \hat{u}_i \in \hat{U}_i, w_i \in W_i, \hat{w}_i \in \hat{W}_i, (x_i^+, p_i^+, l_i^+) \in \mathcal{F}_i(x_i, w_i, u_i)$, *there exists* $(\hat{x}_i^+, p_i^+, l_i^+) \in \hat{\mathcal{F}}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ *so that*

$$\mathcal{S}_i((x_i^+, p_i^+, l_i^+), (\hat{x}_i^+, p_i^+, l_i^+)) \leq \sigma_i \mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) + \varepsilon_i \tag{4.2.3}$$
$$+ \begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(x_i, p_i, l_i) - \hat{\mathcal{H}}_i^2(\hat{x}_i, p_i, l_i) \end{bmatrix}^\top \overbrace{\begin{bmatrix} R_i^{11} & R_i^{12} \\ R_i^{21} & R_i^{22} \end{bmatrix}}^{R_i :=} \begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(x_i, p_i, l_i) - \hat{\mathcal{H}}_i^2(\hat{x}_i, p_i, l_i) \end{bmatrix}.$$

Here, $\hat{T}_i(\Sigma_i)$ is called an abstraction of $T_i(\Sigma_i)$ if there exists a sum-type augmented simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$. Moreover, if $\hat{T}_i(\Sigma_i)$ is finite, it is called a symbolic model of $T_i(\Sigma_i)$.

4.2.3 Compositional Abstractions for Networks of Discrete-Time Switched Systems: DA

We assume that we are given $\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_i^1, h_i^2)$, or equivalently $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ as in Definition 2.3.2, together with their abstractions $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$, $i \in \mathcal{N}$, and sum-type augmented simulation functions $\mathcal{S}_i$ from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$ as in Definition 4.2.2.

The next theorem provides a compositional approach on the construction of abstractions of the networks of transition systems $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$ associated to network of discrete-time switched system $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$ and that of the corresponding alternating simulation functions.

Theorem 4.2.3. *Consider the network $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$ associated to the network of discrete-time switched system $\Sigma = \mathcal{I}_M(\Sigma_i)_{i\in\mathcal{N}}$. Suppose each transition system $T_i(\Sigma_i)$ admits an abstraction $\hat{T}_i(\Sigma_i)$ with the corresponding sum-type augmented simulation function $\mathcal{S}_i$. If there exist $\mu_i > 0$, $i \in \mathcal{N}$, such that the matrix inequality and inclusion*

$$\begin{bmatrix} M \\ I_q \end{bmatrix}^\top \tilde{R} \begin{bmatrix} M \\ I_q \end{bmatrix} \preceq 0, \tag{4.2.4}$$

$$M \prod_{i=1}^N \hat{Y}_i^2 \subseteq \prod_{i=1}^N \hat{W}_i, \tag{4.2.5}$$

are satisfied, where

$$\tilde{R} := \begin{bmatrix} \mu_1 R_1^{11} & & & \mu_1 R_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N R_N^{11} & & & \mu_N R_N^{12} \\ \mu_1 R_1^{21} & & & \mu_1 R_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N R_N^{21} & & & \mu_N R_N^{22} \end{bmatrix}, \tag{4.2.6}$$

and q is the number of rows in M, then

$$\tilde{\mathcal{S}}((x,p,l),(\hat{x},p,l)) := \sum_{i\in\mathcal{N}} \mu_i \mathcal{S}_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i)), \tag{4.2.7}$$

is an alternating simulation function, as in Definition 2.2.5, from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$.

Proof. First, define $z = (z_i)_{i\in\mathcal{N}}$, $\hat{z} = (\hat{z}_i)_{i\in\mathcal{N}}$, $z^+ = (z_i^+)_{i\in\mathcal{N}}$, and $\hat{z}^+ = (\hat{z}_i^+)_{i\in\mathcal{N}}$, where $z_i = (x_i, p_i, l_i)$, $\hat{z}_i = (\hat{x}_i, p_i, l_i)$ $z_i^+ = (x_i^+, p_i^+, l_i^+)$, and $\hat{z}_i^+ = (\hat{x}_i^+, p_i^+, l_i^+)$, $\forall i \in \mathcal{N}$. Now we show that inequality (2.2.1) holds. Consider any $z \in X$ and $\hat{z} \in \hat{X}$, one gets:

$$|\mathcal{H}(z) - \hat{\mathcal{H}}(\hat{z})| = |(\mathcal{H}_i^1(z_i) - \hat{\mathcal{H}}_i^1(\hat{z}_i))_{i\in\mathcal{N}}| \leq \sum_{i\in\mathcal{N}} |\mathcal{H}_i^1(z_i) - \hat{\mathcal{H}}_i^1(\hat{z}_i)| \leq \sum_{i\in\mathcal{N}} \alpha_i^{-1}(\mathcal{S}_i(z_i,\hat{z}_i)) \leq \hat{\alpha}\big(\tilde{\mathcal{S}}(z,\hat{z})\big),$$

where $\hat{\alpha}$ is a $\mathcal{K}_\infty$ function defined as $\hat{\alpha}(s) = \max_{\hat{s}\geq 0}\left\{\sum_{i=1}^{N}\alpha_i^{-1}(s_i)|\mu^\top \hat{s} = s\right\}$, where $\hat{s} = (s_i)_{i\in\mathcal{N}}$ and $\mu = (\mu_i)_{i\in\mathcal{N}}$. By defining the $\mathcal{K}_\infty$ function $\tilde{\alpha} = \hat{\alpha}^{-1}$, one obtains

$$\tilde{\alpha}(|\mathcal{H}(z) - \hat{\mathcal{H}}(\hat{z})|) \leq \tilde{\mathcal{S}}(z, \hat{z}),$$

satisfying inequality (2.2.1). Now we show that inequality (2.2.2) holds as well. Define $\bar{\sigma} := \max_{i\in[1,N]}\{\sigma_i\}$, $\bar{\varepsilon} := \sum_{i\in\mathcal{N}}\mu_i\varepsilon_i$, and consider condition (4.2.4), and the definition of matrix $\tilde{R}$ in (4.2.6). Then, one gets the chain of inequalities in (4.2.8). Now by using

$$\begin{aligned}
&\tilde{\mathcal{S}}(z^+, \hat{z}^+) = \sum_{i\in\mathcal{N}}\mu_i\big(\mathcal{S}_i(z_i^+, \hat{z}_i^+)\big)\\
&\leq \sum_{i\in\mathcal{N}}\mu_i\left(\sigma_i\mathcal{S}_i(z_i, \hat{z}_i) + \varepsilon_i + \begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(z_i) - \hat{\mathcal{H}}_i^2(\hat{z}_i)\end{bmatrix}^\top \begin{bmatrix} R_i^{11} & R_i^{12} \\ R_i^{21} & R_i^{22}\end{bmatrix}\begin{bmatrix} w_i - \hat{w}_i \\ \mathcal{H}_i^2(z_i) - \hat{\mathcal{H}}_i^2(\hat{z}_i)\end{bmatrix}\right)\\
&= \sum_{i\in\mathcal{N}}\mu_i\big(\sigma_i\mathcal{S}_i(z_i, \hat{z}_i) + \varepsilon_i\big) + \begin{bmatrix} (w_i - \hat{w}_i)_{i\in\mathcal{N}} \\ (\mathcal{H}_i^2(z_i) - \hat{\mathcal{H}}_i^2(\hat{z}_i))_{i\in\mathcal{N}}\end{bmatrix}^\top \tilde{R}\begin{bmatrix} (w_i - \hat{w}_i)_{i\in\mathcal{N}} \\ (\mathcal{H}_i^2(z_i) - \hat{\mathcal{H}}_i^2(\hat{z}_i))_{i\in\mathcal{N}}\end{bmatrix}\\
&\leq \sum_{i\in\mathcal{N}}\mu_i\big(\sigma_i\mathcal{S}_i(z_i, \hat{z}_i)+\varepsilon_i\big)+\big[(\mathcal{H}_i^2(z_i)-\hat{\mathcal{H}}_i^2(\hat{z}_i))_{i\in\mathcal{N}}\big]^\top \begin{bmatrix} M \\ I_q\end{bmatrix}^\top \tilde{R}\begin{bmatrix} M \\ I_q\end{bmatrix}\big[(\mathcal{H}_i^2(z_i)-\hat{\mathcal{H}}_i^2(\hat{z}_i))_{i\in\mathcal{N}}\big]\\
&\leq \sum_{i\in\mathcal{N}}\mu_i\big(\sigma_i\mathcal{S}_i(z_i, \hat{z}_i)\big) + \sum_{i\in\mathcal{N}}\mu_i\varepsilon_i = \bar{\sigma}\tilde{\mathcal{S}}\,(z, \hat{z}) + \bar{\varepsilon}.
\end{aligned} \tag{4.2.8}$$

the result of Theorem 2.2.9, one obtains

$$\tilde{\mathcal{S}}(z^+, \hat{z}^+) \leq \max\{\tilde{\sigma}\tilde{\mathcal{S}}\,(z, \hat{z})\,, \tilde{\varepsilon}\}.$$

Thus, $\tilde{\mathcal{S}}$ satisfies (2.2.2) with $\tilde{\sigma} := (1-(1-\psi)(1-\bar{\sigma}))$, and $\tilde{\varepsilon} = (1-\bar{\sigma})^{-1}\psi^{-1}(\bar{\varepsilon})$, for some arbitrarily chosen positive constant ψ with $\psi < 1$. Hence, $\tilde{\mathcal{S}}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in\mathcal{N}}$. □

4.2.4 Construction of Symbolic Models

In the following, we introduce some stability properties for the subsystems (mode) $\Sigma_p, p \in P$, described in Definition 2.4.2, based on which one can construct a symbolic model for Σ along with the corresponding sum-type augmented simulation functions between Σ and its symbolic model.

4.2.4.1 Incremental Passivity

Definition 4.2.4. *Subsystem (mode) Σ_p is δ-P if there exist functions $S_p : \mathbb{X}\times\mathbb{X} \to \mathbb{R}_{\geq 0}$, $\underline{\varphi}_{x_p} \in \mathcal{K}_\infty$, a symmetric matrix Q_p of appropriate dimension, and constant $0 < \varphi_{c_p} < 1$, such that for all $x, x' \in \mathbb{X}$, and for all $w, w' \in \mathbb{W}$*

$$\underline{\varphi}_{x_p}(|x - x'|) \leq S_p(x, x'), \tag{4.2.9}$$

$$S_p(f_p(x,w),f_p(x',w'))\leq\varphi_{c_p}S_p(x,x')+\begin{bmatrix}w-w'\\h^2(x)-h^2(x')\end{bmatrix}^\top\overbrace{\begin{bmatrix}Q_p^{11}&Q_p^{12}\\Q_p^{21}&Q_p^{22}\end{bmatrix}}^{Q_p:=}\begin{bmatrix}w-w'\\h^2(x)-h^2(x')\end{bmatrix}. \tag{4.2.10}$$

We say that S_p and Q_p, $\forall p\in P$, are multiple δ-P storage functions and supply rates, respectively, for system Σ if they satisfy (4.2.9) and (4.2.10). Moreover, if $S_p = S_{p'}$ and $Q_p = Q_{p'}$, $\forall p,p'\in P$, we omit the index p in (4.2.9), (4.2.10), and say that S and Q are common δ-P storage function and supply rate for system Σ.

4.2.4.2 Symbolic Models

In the following, we show how to construct a symbolic model $\hat{T}(\Sigma)$ of transition system $T(\Sigma)$ associated to switched system Σ in which its modes Σ_p are δ-P.

Definition 4.2.5. *Consider a transition system $T(\Sigma)=(X,X_0,W,U,\mathcal{F},Y^1,Y^2,\mathcal{H}^1,\mathcal{H}^2)$, associated to the switched system $\Sigma=(\mathbb{X},P,\mathbb{W},F,\mathbb{Y}^1,\mathbb{Y}^2,h^1,h^2)$, where $\mathbb{X},\mathbb{W}$ are assumed to be finite unions of boxes. Let the modes Σ_p, for all $p\in P$ of Σ be δ-P as in Definition 4.2.4. Then one can construct a symbolic model $\hat{T}(\Sigma)=(\hat{X},\hat{X}_0,\hat{U},\hat{W},\hat{\mathcal{F}},\hat{Y}^1,\hat{Y}^2,\hat{\mathcal{H}}^1,\hat{\mathcal{H}}^2)$, where:*

- $\hat{X}=\hat{\mathbb{X}}\times P\times\{0,\cdots,k_d-1\}$, *where* $\hat{\mathbb{X}}=[\mathbb{X}]_{\eta^x}$ *and* $0<\eta^x\leq \mathrm{span}(\mathbb{X})$ *is the state set quantization parameter;*
- $\hat{X}_0=\hat{\mathbb{X}}\times P\times\{0\}$;
- $\hat{U}=U=P$ *is the external input set;*
- $\hat{W}=[\mathbb{W}]_{\eta^w}$, *where* $0\leq\eta^w\leq\mathrm{span}(\mathbb{W})$ *is the internal input set quantization parameter.*
- $(\hat{x}^+,p^+,l^+)\in\hat{\mathcal{F}}((\hat{x},p,l),\hat{u},\hat{w})$ *if and only if* $|f_p(\hat{x},\hat{w})-\hat{x}^+|\leq\eta^x$, $\hat{u}=p$ *and the following scenarios hold:*
  - $l<k_d-1$, $p^+=p$ *and* $l^+=l+1$;
  - $l=k_d-1$, $p^+=p$ *and* $l^+=k_d-1$;
  - $l=k_d-1$, $p^+\neq p$ *and* $l^+=0$;
- $\hat{Y}^1=\{\mathcal{H}^1(\hat{x},p,l)|(\hat{x},p,l)\in\hat{X}\}$;
- $\hat{Y}^2=\{\mathcal{H}^2(\hat{x},p,l)|(\hat{x},p,l)\in\hat{X}\}$;
- $\hat{\mathcal{H}}^1:\hat{X}\rightarrow\hat{Y}^1$ *is the external output map defined as* $\hat{\mathcal{H}}^1(\hat{x},p,l)=\mathcal{H}_1(\hat{x},p,l)=h^1(\hat{x})$;
- $\hat{\mathcal{H}}^2:\hat{X}\rightarrow\hat{Y}^2$ *is the internal output map defined as* $\hat{\mathcal{H}}^2(\hat{x},p,l)=\mathcal{H}_2(\hat{x},p,l)=h^2(\hat{x})$;

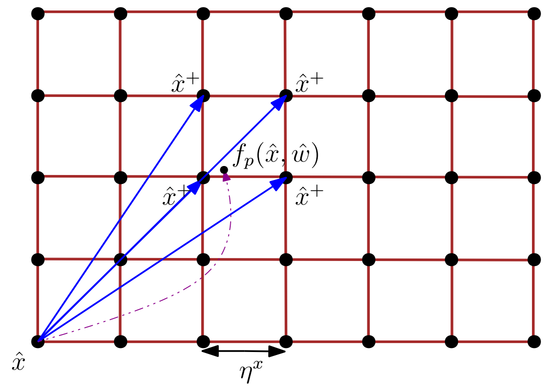

Figure 4.1: An illustration of the computation of the transitions of $\hat{T}(\Sigma)$ for particular $\hat{x}$, p, $\hat{w}$.

Note that the finite set $\hat{W}$ should be constructed in a similar way as discussed in Remark 3.2.7. An illustration of the computation of the transitions of $\hat{T}(\Sigma)$ is shown in Figure 4.1.

Let us point out some differences between the symbolic model in Definition 4.2.5 and the one proposed in [GPT10]. There is no distinction between internal and external inputs and outputs in the symbolic model defined in [GPT10], whereas their distinctions in our work play a major role in defining the networks of switched systems and providing the compositionality results of this chapter.

4.2.4.3 Construction of Sum-Type Augmented Simulation Functions

In this subsection, we show how to construct a sum-type augmented simulation function between a symbolic model $\hat{T}(\Sigma)$ of transition system $T(\Sigma)$ associated to the switched system Σ where Σ_p is δ-P. In the following, we impose assumptions on function S_p in Definition 4.2.4 which are used to prove some of the main results later.

Assumption 4.2.6. *There exists $\hat{\mu} \geq 1$ such that*

$$\forall x, y \in \mathbb{X}, \quad \forall p, p' \in P, \quad S_p(x, y) \leq \hat{\mu} S_{p'}(x, y). \tag{4.2.11}$$

Assumption 4.2.6 is an incremental version of a similar assumption that is used to prove input-to-state stability of switched systems under constrained switching assumptions [VCL07].

Assumption 4.2.7. *Assume that $\forall p \in P$, $\exists \gamma_p \in \mathcal{K}_\infty$ such that*

$$\forall x, y, z \in \mathbb{X}, \quad S_p(x, y) \leq S_p(x, z) + \gamma_p(|y - z|). \tag{4.2.12}$$

Now, we establish the relation between $T(\Sigma)$ and $\hat{T}(\Sigma)$, introduced above, via the notion of sum-type augmented simulation function as in Definition 4.2.2.

Theorem 4.2.8. *Consider a switched system* $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$ *with its equivalent transition system* $T(\Sigma) = (X, U, W, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$*. Let,* $\forall p \in P$, Σ_p *be* δ-*P as in Definition 4.2.4. Consider a symbolic model* $\hat{T}(\Sigma) = (\hat{X}, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}^1, \hat{Y}^2, \hat{\mathcal{H}}^1, \hat{\mathcal{H}}^2)$ *constructed as in Definition 4.2.5. Suppose that Assumptions 4.2.6 and 4.2.7 hold. Let* $\epsilon > 1$ *and define* $\varphi_c = \max_{p\in P}\left\{\varphi_{c_p}\right\}$*. If,* $k_d \geq \epsilon\frac{\ln(\hat{\mu})}{\ln(\frac{1}{\varphi_c})} + 1$*, and there exists a symmetric matrix* $\tilde{Q}$ *such that* $\forall q \in \{1, \dots, k_d - 1\}, \tilde{Q} - \varphi_c^{\frac{-q}{\epsilon}} \sum_{p=1}^{m} Q_p \succeq 0$*, then function* $\mathcal{V}$ *defined as*

$$\mathcal{V}((x, p, l), (\hat{x}, p, l)) := \varphi_c^{\frac{-l}{\epsilon}} \sum_{p=1}^{m} S_p(x, \hat{x}), \tag{4.2.13}$$

is a sum-type augmented simulation function from $\hat{T}(\Sigma)$ *to* $T(\Sigma)$*.*

Proof. Given the Lipschitz assumption on h^1 and since, $\forall p \in P$, Σ_p is δ-P , from (4.2.9), $\forall (x, p, l) \in X$ and $\forall (\hat{x}, p, l) \in \hat{X}$, we have

$$\begin{aligned}
&|\mathcal{H}^1(x, p, l) - \hat{\mathcal{H}}^1(\hat{x}, p, l)| = |h^1(x) - h^1(\hat{x})| \leq \ell(|x - \hat{x}|) \leq \ell \circ \varphi_{x_p}^{-1}(S_p(x, \hat{x}))\\
&\leq \ell \circ \varphi_{x_p}^{-1}\Big(\sum_{p=1}^{m} S_p(x, \hat{x})\Big) = \ell \circ \varphi_{x_p}^{-1}\left(\varphi_c^{\frac{l}{\epsilon}} \mathcal{V}((x, p, l), (\hat{x}, p, l))\right)\\
&\leq \ell \circ \varphi_{x_p}^{-1}\left(\mathcal{V}((x, p, l), (\hat{x}, p, l))\right) \leq \hat{\alpha}\left(\mathcal{V}((x, p, l), (\hat{x}, p, l))\right),
\end{aligned}$$

where $\hat{\alpha} = \max\limits_{p\in P}\{\ell \circ \varphi_{x_p}^{-1}\}$. Hence (4.2.2) is satisfied with $\alpha = \hat{\alpha}^{-1}$.

Now from (4.2.12) and Definition 4.2.5, $\forall x \in \mathbb{X}, \forall \hat{x} \in \hat{\mathbb{X}}, \forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$, we have

$$\begin{aligned}
S_p(f_p(x, w), \hat{x}^+) &\leq S_p(f_p(x, w), f_p(\hat{x}, \hat{w})) + \gamma_p(|\hat{x}^+ - f_p(\hat{x}, \hat{w})|)\\
&\leq S_p(f_p(x, w), f_p(\hat{x}, \hat{w})) + \gamma_p(\eta^x),
\end{aligned}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, p^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, p, l), \hat{u}, \hat{w})$. Let $\mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) := [w - \hat{w}; h^2(x) - h^2(\hat{x})]^\top Q_p [w - \hat{w}; h^2(x) - h^2(\hat{x})]$ and note that by (4.2.10), one gets

$$S_p(f_p(x, w), f_p(\hat{x}, \hat{w})) \leq \varphi_{c_p} S_p(x, \hat{x}) + \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p).$$

Hence, $\forall x \in \mathbb{X}, \forall \hat{x} \in \hat{\mathbb{X}}$, and $\forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$, one obtains

$$S_p(f_p(x, w), \hat{x}^+) \leq \varphi_{c_p} S_p(x, \hat{x}) + \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) + \gamma_p(\eta^x), \tag{4.2.14}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, p^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, p, l), \hat{u}, \hat{w})$. Now, in order to show function $\mathcal{V}$ defined in (4.2.13) satisfies (4.2.3), we consider the different scenarios in Definition 4.2.5 as follows.

- $l < k_d - 1$, $p^+ = p$ and $l^+ = l + 1$, using (4.2.14), we have

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) = \frac{\sum_{p^+=1}^m S_{p^+}(x^+, \hat{x}^+)}{\varphi_c^{\frac{l^+}{\epsilon}}} = \frac{\sum_{p=1}^m S_p(f_p(x, w), \hat{x}^+)}{\varphi_c^{\frac{l+1}{\epsilon}}}$$
$$\leq \frac{\sum_{p=1}^m \varphi_{c_p} S_p(x, \hat{x})}{\varphi_c^{\frac{1}{\epsilon}} \varphi_c^{\frac{l}{\epsilon}}} + \frac{\sum_{p=1}^m (\mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) + \gamma_p(\eta^x))}{\varphi_c^{\frac{l+1}{\epsilon}}}$$
$$\leq \varphi_c^{\frac{\epsilon-1}{\epsilon}} \mathcal{V}((x, p, l), (\hat{x}, p, l)) + \frac{\sum_{p=1}^m \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p)}{\varphi_c^{\frac{l+1}{\epsilon}}} + \frac{\sum_{p=1}^m \gamma_p(\eta^x)}{\varphi_c^{\frac{k_d}{\epsilon}}}.$$

- $l = k_d - 1$, $p^+ = p$ and $l^+ = k_d - 1$, using (4.2.14) and $\frac{\epsilon-1}{\epsilon} < 1$, one gets

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) = \frac{\sum_{p^+=1}^m S_{p^+}(x^+, \hat{x}^+)}{\varphi_c^{\frac{l^+}{\epsilon}}} = \frac{\sum_{p=1}^m S_p(f_p(x, w), \hat{x}^+)}{\varphi_c^{\frac{l}{\epsilon}}}$$
$$\leq \frac{\sum_{p=1}^m \varphi_{c_p} S_p(x, \hat{x})}{\varphi_c^{\frac{l}{\epsilon}}} + \frac{\sum_{p=1}^m (\mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) + \gamma_p(\eta^x))}{\varphi_c^{\frac{l}{\epsilon}}}$$
$$\leq \varphi_c^{\frac{\epsilon-1}{\epsilon}} \mathcal{V}((x, p, l), (\hat{x}, p, l)) + \frac{\sum_{p=1}^m \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p)}{\varphi_c^{\frac{l}{\epsilon}}} + \frac{\sum_{p=1}^m \gamma_p(\eta^x)}{\varphi_c^{\frac{k_d}{\epsilon}}}.$$

- $l = k_d - 1$, $p^+ \neq p$ and $l^+ = 0$, using (4.2.14), $k_d \geq \epsilon \frac{\ln(\hat{\mu})}{\ln(\frac{1}{\varphi_c})} + 1 \Leftrightarrow \hat{\mu} \varphi_{c_p}^{\frac{k_d-1}{\epsilon}} \leq 1$, and $\frac{\epsilon-1}{\epsilon} < 1$, one has

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) = \frac{\sum_{p^+=1}^m S_{p^+}(x^+, \hat{x}^+)}{\varphi_c^{\frac{l^+}{\epsilon}}} \leq \hat{\mu} \sum_{p=1}^m S_p(f_p(x, w), \hat{x}^+)$$
$$\leq \frac{\hat{\mu} \varphi_c^{\frac{k_d-1}{\epsilon}} \left(\sum_{p=1}^m (\varphi_{c_p} S_p(x, \hat{x}) + \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) + \gamma_p(\eta^x)) \right)}{\varphi_c^{\frac{k_d-1}{\epsilon}}}$$
$$\leq \frac{\sum_{p=1}^m \varphi_{c_p} S_p(x, \hat{x})}{\varphi_c^{\frac{k_d-1}{\epsilon}}} + \frac{\sum_{p=1}^m (\mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p) + \gamma_p(\eta^x))}{\varphi_c^{\frac{k_d-1}{\epsilon}}}$$
$$\leq \varphi_c^{\frac{\epsilon-1}{\epsilon}} \mathcal{V}((x, p, l), (\hat{x}, p, l)) + \frac{\sum_{p=1}^m \mathcal{T}(w, x, \hat{w}, \hat{x}, Q_p)}{\varphi_c^{\frac{k_d-1}{\epsilon}}} + \frac{\sum_{p=1}^m \gamma_p(\eta^x)}{\varphi_c^{\frac{k_d}{\epsilon}}}.$$

Let $\tilde{\gamma} = \varphi_c^{\frac{-k_d}{\epsilon}} \sum_{p=1}^m \gamma_p$. Since $\hat{\mathcal{H}}^2(\hat{x}, p, l) = h^2(\hat{x})$ and $\mathcal{H}_2(x, p, l) = h^2(x)$, $\forall (x, p, l) \in X$, $\forall (\hat{x}, p, l) \in \hat{X}$, $\forall w \in W$, and $\forall \hat{w} \in \hat{W}$, one obtains

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) \leq \varphi_c^{\frac{\epsilon-1}{\epsilon}} \mathcal{V}((x, p, l), (\hat{x}, p, l)) + \tilde{\gamma}(\eta^x)$$
$$+ \begin{bmatrix} w - \hat{w} \\ \mathcal{H}^2(x, p, l) - \mathcal{H}^2(\hat{x}, p, l) \end{bmatrix}^\top \tilde{Q} \begin{bmatrix} w - \hat{w} \\ \mathcal{H}^2(x, p, l) - \hat{\mathcal{H}}^2(\hat{x}, p, l) \end{bmatrix}.$$

Hence, inequality (4.2.3) is satisfied with $\sigma = \varphi_c^{\frac{\epsilon-1}{\epsilon}}$, $R = \tilde{Q}$, $\varepsilon = \tilde{\gamma}(\eta^x)$. Thus, $\mathcal{V}$ is a sum-type augmented simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. □

Observe that using exactly the same argument, we can show the $\mathcal{V}$ is a sum-type augmented simulation function from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Remark 4.2.9. *If equation* (4.2.10) *is satisfied with the same* $Q_p, \forall p \in P$*, then function* $\mathcal{V}$ *in Theorem 4.2.8 reduces to* $\mathcal{V}((x,p,l),(\hat{x},p,l)) := \varphi_c^{\frac{-l}{\epsilon}} S_p(x,\hat{x})$*. In addition, if* Σ *admits a common* δ*-P storage function, function* $\mathcal{V}$ *reduces to* $\mathcal{V}((x,p,l),(\hat{x},p,l)) := S(x,\hat{x})$*.*

Remark 4.2.10. *For affine switched systems* $\Sigma = (A, B, C^1, C^2, D)$ *as in Definition 2.4.3, we can restrict our attention to* δ*-P storage functions of the form* $S_p(x,\hat{x}) = (x-\hat{x})^\top Z_p\ (x-\hat{x}), Z_p \succ 0$*. It can be readily seen that such functions always satisfy* (4.2.9) *and* (4.2.11) *with* $\hat{\mu} = \max\left\{\frac{\lambda_{\max}(Z_p)}{\lambda_{\min}(Z_p)}, \frac{\lambda_{\max}(Z_{p'})}{\lambda_{\min}(Z_{p'})}\right\}$*, for all* $p, p' \in P$*. Moreover, inequality* (4.2.10) *reduces to the linear matrix inequality*

$$\begin{bmatrix}\theta_p A_p^\top Z_p A_p & A_p^\top Z_p D_p\\ D_p^\top Z_p A_p & \theta_p D_p^\top Z_p D_p\end{bmatrix} \preceq \begin{bmatrix}\varphi_{c_p} Z_p + C_2^\top Q_p^{22} C_2 & C_2^\top Q_p^{21}\\ Q_p^{12} C_2 & Q_p^{11}\end{bmatrix} \tag{4.2.15}$$

in which Z_p *and* Q_p *can be determined by semi-definite programming, where* $\theta_p > 1, 0 < \varphi_{c_p} < 1$*. Consequently, it can be readily verified that* ε *in* (2.2.2) *can be defined as* $\varepsilon = c_p \lambda_{\max}(Z_p)$*, for some* $c_p > 0$ *depending on* θ_p *and the dimensions of* Z_p*.*

4.2.5 Case Studies

Here we apply the proposed results of this section to a model of road traffic by constructing compositionally a symbolic model of a network containing 50 cells of 1000 meters each. We also design controllers compositionally maintaining the density of traffic lower than 30 vehicles per cell. Additionally, we apply those results to a network of switched systems admitting multiple incrementally passive storage functions.

4.2.5.1 Road Traffic Model

Consider the network of switched systems Σ which is adapted from [dWOK12] and described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B_{\mathsf{p}(k)},\\ \mathbf{y}(k) &= \mathbf{x}(k),\end{cases} \tag{4.2.16}$$

where $A \in \mathbb{R}^{50\times 50}$ is a matrix with elements $\{A\}_{q,q} = 0.9 - \frac{\tau v}{d}$ if $q \in Q_1 = \{q \text{ is odd } | q \in [1;50]\}$ and $\{A\}_{q,q} = 0.65 - \frac{\tau v}{d}$ if $q \in Q_2 = \{q \text{ is even } | q \in [1;50]\}$, $\{A\}_{q+1,q} = \{A\}_{1,50} = \frac{\tau v}{d}$, $\forall q \in [1;50]$, and all other elements are identically zero, where $\tau = \frac{10}{60\times 60}$, $d = 1$, and $v = 120$ are sampling time interval in hours, length in kilometers, and the flow speed of the vehicles in kilometers per hour, respectively. The vector $B_p \in \mathbb{R}^{50}$ is defined

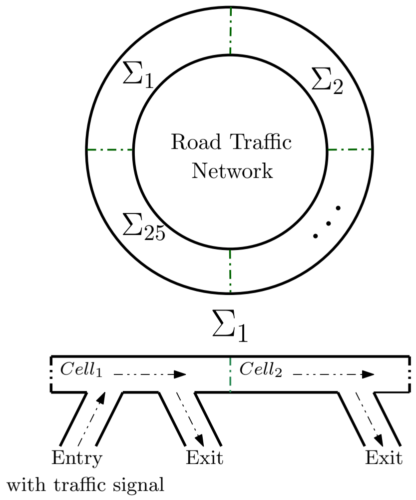

Figure 4.2: Model of a road traffic network in a circular highway composed of 25 identical links, each link has two cells.

as $B_p = [b_{1p_1}; \ldots; b_{25p_{25}}]$ such that $b_{ip_i} = [0; 0]$ if $p_i = 1$, and $b_{ip_i} = [0; 12]$ if $p_i = 2$, $\forall i \in [1, 25]$, $[p_1; \ldots; p_{25}] \in P = \{1, 2\}^{25}$, where P is the set of modes of Σ.

The chosen switched system Σ here is the model of a circular road around a city (Highway) divided in 50 cells of 1000 meters each. The road has 25 entries and 50 exits. A cell q has an entry and exit if $q \in Q_1$ and has an exit and no entry if $q \in Q_2$. All the entries are controlled by traffic signals, denoted by $s_r, r \in [1; 25]$. In Σ, the dynamic we want to observe is the density of traffic, given in vehicles per cell, for each cell q of the road. During the sampling time interval τ, we assume that 12 vehicles can pass the entry controlled by a traffic signal s_r when it is green. Moreover, 10% of vehicles that are in cells $q \in Q_1$, and 35% of vehicles that are in cells $q \in Q_2$ go out using available exits.

Now, in order to apply the compositionality result, we introduce systems Σ_i, $\forall i \in [1; 25]$. Each switched system Σ_i represents the dynamic of one link of the entire highway, where each link contains 2 cells, one entry, and two exits, as schematically illustrated in Figure 4.2. The switched system Σ_i, $\forall i \in [1; 25]$, is described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) &= A_i \mathbf{x}_i(k) + D_i w_i(k) + B_{i\mathsf{p}_i(k)}, \\ \mathbf{y}_i^1(k) &= \mathbf{x}_i(k), \\ \mathbf{y}_i^2(k) &= C_i^2 \mathbf{x}_i(k), \end{cases} \tag{4.2.17}$$

$$A_i = \begin{bmatrix} 0.9 - \frac{\tau v}{d} & 0 \\ \frac{\tau v}{d} & 0.65 - \frac{\tau v}{d} \end{bmatrix}, D_i = \begin{bmatrix} \frac{\tau v}{d} \\ 0 \end{bmatrix}, B_{i1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_{i2} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}, C_i^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^\top,$$

and the set of modes is $P_i = \{1, 2\}$, $\forall i \in [1; 25]$. Clearly, $\Sigma = \mathcal{I}_M(\Sigma_i)_{i \in [1,25]}$, where the elements of the coupling matrix M are $\{M\}_{i+1,i} = \{M\}_{1,25} = 1$, $\forall i \in [1; 25]$, and all other elements are identically zero. Note that, for any $i \in [1; 25]$, conditions (4.2.9) and (4.2.10) are satisfied with $S_{ip_i}(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^\top Z_{ip_i}(x_i - \hat{x}_i)$, $Z_{ip_i} = I_2$, $\varphi_{x_{ip_i}}(s) = s^2$,

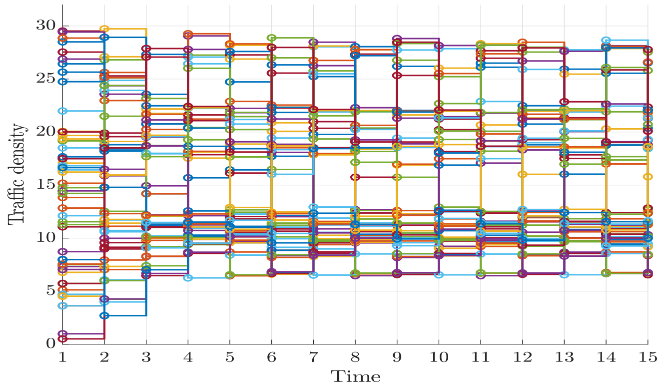

Figure 4.3: Closed-loop state trajectories of network Σ consisting of 50 subsystems.

$\varphi_{c_{ip_i}} = 0.98$, $Q^{11}_{ip} = 0.3527$, $Q^{12}_{ip} = Q^{21}_{ip} = 0.0937$, $Q^{22}_{ip} = -0.6785\ \forall p_i \in P_i$. Moreover, since $S_{ip_i} = S_{ip'_i}, \forall p, p' \in P$, and according to Remarks 4.2.9 and 4.2.10, function $\mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = S_i(x_i, \hat{x}_i)$ is a sum-type augmented simulation function from $\hat{T}_i(\Sigma_i)$, constructed as in Definition 4.2.5, to $T_i(\Sigma_i)$, defined in Definition 2.4.2. Now, by choosing $\mu_i = 1, \forall i \in [1; 25]$ and finite internal input sets $\hat{W}_i$ of $\hat{T}_i(\hat{\Sigma}_i)$ in such a way that $\prod_{i=1}^{25} \hat{W}_i = M \prod_{i=1}^{25} \hat{X}_i$, conditions (4.2.4) and (4.2.5) are satisfied. Therefore, applying Theorem 4.2.3, function $\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) = \sum_{i=1}^{25} \mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i))$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in[1,25]}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in[1,25]}$.

Let us now design a controller for Σ via symbolic models $\hat{T}_i(\Sigma_i)$ such that controllers maintain the density of traffic lower than 30 vehicles per cell (safety constraint), and to allow only 2 consecutive red lights for each traffic signal (fairness constraint). The former constraint implies that each vehicle can keep a 30-meter safe distance from the one directly in front. The latter constraint is a way to avoid the trivial solution (always red) of the safety constraint and ensures fairness between modes 1 and 2. The idea here is to design local controllers for symbolic models $\hat{T}_i(\Sigma_i)$, and then refine them to the ones for concrete switched systems Σ_i. To do so, the local controllers are designed while assuming that the other systems meet their specifications.

Note that the direct computation of the symbolic model for the original 50-dimensional system Σ is not possible monolithically. We leverage software tool `SCOTS` [RZ16] for constructing symbolic models and controllers for Σ_i compositionally with the state quantization parameter $\eta^x_i = 0.03$ and the computation times are amounted to $10.2s$ and $0.014s$, respectively. Figure 4.3 shows the closed-loop state trajectories of Σ, consisting of 50 cells.

4.2.5.2 Fully Connected Network

In this example, we apply our results to a network Σ composed of $N \geq 2$ linear switched systems $\Sigma_i, i \in [1; N]$, admitting multiple δ-P storage functions and supply rates. In this respect, we choose the dynamics' parameters such that neither condition (4.2.9) nor (4.2.10) holds with common δ-P storage functions and supply rates for all systems. In particular, as all systems are affine switched systems, we choose their dynamics' parameters such that the solution of the linear matrix inequality (4.2.15) with common Z_i and Q_i (i.e. $Z_{ip_i} = Z_{ip'_i}$ and $Q_{ip_i} = Q_{ip'_i}$, $\forall p, p' \in P, i \in [1;N]$) is infeasible. Hence, none of the switched systems admits a common δ-P storage function and supply rate. The dynamic of the network of switched systems Σ has the set of modes $P = \{1,2\}^N, N \in \mathbb{N}_{\geq 2}$, and it is given by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) & = A_{\mathsf{p}(k)}\mathbf{x}(k) + B_{\mathsf{p}(k)}, \\ \mathbf{y}(k) & = \mathbf{x}(k). \end{cases}$$

The vector $B_p \in \mathbb{R}^n$, where $n = 2N$, is defined as $\{B\}_{i,1} = B_{p_i}$ such that $B_{p_i} = [-0.9; 0.5]$ if $p_i = 1$, and $B_{p_i} = [0.9; -0.2]$ if $p_i = 2$, $\forall i, j \in [1;N], i \neq j$. The elements of the matrix $A_p \in \mathbb{R}^{n \times n}$ are as follows:

$$\{A\}_{i,j} = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \{A\}_{i,i} = A_{p_i} = \begin{cases} \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix} & \text{if } p_i = 1, \\ \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix} & \text{if } p_i = 2. \end{cases}$$

Now, by introducing Σ_i described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) & = A_{i\mathsf{p}_i(k)}\mathbf{x}_i(k) + \omega_i(k) + B_{i\mathsf{p}_i(k)}, \\ \mathbf{y}_i^1(k) & = \mathbf{x}_i(k), \\ \mathbf{y}_i^2(k) & = \mathbf{x}_i(k), \end{cases}$$

$$A_{i1} = \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix}, A_{i2} = \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix}, B_{i1} = \begin{bmatrix} -0.9 \\ 0.5 \end{bmatrix}, B_{i2} = \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix},$$

and the set of modes as $P_i = \{1,2\}$, one can readily verify that $\Sigma = \mathcal{I}_M(\Sigma_i)_{i \in [1,N]}$, where the elements of the coupling matrix M are $\{M\}_{i,i} = 0_2$ and $\{M\}_{i,j} = \{A\}_{i,j}$, $\forall i, j \in [1;N], i \neq j$. Note that, for any $i \in [1;N]$, conditions (4.2.9) and (4.2.10) are satisfied with $S_{ip_i}(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^\top Z_{ip_i}(x_i - \hat{x}_i)$,

$$Z_{i1} = \begin{bmatrix} 0.3030 & 0.0087 \\ 0.0087 & 0.4938 \end{bmatrix}, Z_{i2} = \begin{bmatrix} 0.4899 & -0.0033 \\ -0.0033 & 0.4291 \end{bmatrix},$$

$Q_{i1} = 10^{-3}L_{i1}, \varphi_{c_{i1}} = 0.7, \varphi_{x_{i1}}(s) = 0.3s^2, Q_{i2} = 10^{-3}L_{i2}, \varphi_{c_{i2}} = 0.7, \varphi_{xi2}(s) = 0.4s^2$, where

$$L_{i1} = \begin{bmatrix} 2.7 & 0 & -1 & -3 \\ 0 & 1 & -3 & 0 \\ -1 & -3 & -201.3 & -17 \\ -3 & 0 & -1.7 & 270.8 \end{bmatrix}, L_{i2} = \begin{bmatrix} 2.9 & 0 & -1.4 & 2.7 \\ 0 & 1.6 & 2.7 & 0 \\ -1.4 & 2.7 & 156 & 17.5 \\ 2.7 & 0 & 17.5 & -294 \end{bmatrix}.$$

Since Assumption 4.2.6 and $k_d \geq \epsilon \frac{\ln(\hat{\mu})}{\ln(1/\varphi_{cp})} + 1$ hold with $\hat{\mu} = 1.63$, $k_d = 3$, $\epsilon = 1.01$, one can easily find a matrix $\tilde{Q}$ such that $\forall q \in \{1,2\}, \tilde{Q} - 0.7^{\frac{-q}{\epsilon}} \sum_{p=1}^{2} Q_p \succeq 0$ by using semi-definite programming such that function $\mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = \sum_{i=1}^{N} S_{ip_i}(x_i, \hat{x}_i)\kappa_{p_i}^{-l/\epsilon}$ is a sum-type augmented simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$. Choose an arbitrary N, then by choosing $\mu_i = 1, \forall i \in [1; N]$, and finite internal input sets $\hat{W}_i$ of $\hat{T}_i(\hat{\Sigma}_i)$ in such a way that $\prod_{i=1}^{N} \hat{W}_i = M \prod_{i=1}^{N} \hat{X}_i$, conditions (4.2.4) and (4.2.5) are satisfied. Hence, utilizing the result of Theorem 4.2.3, one can see that function $\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) = \sum_{i=1}^{N} \mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i))$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in[1,N]}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in[1,N]}$.

Given $N \geq 5$, a set of state $X_i = [0, 1]$, and $\eta_i^x = 0.1$, we observe that constructing the symbolic model for the original system Σ is only possible compositionally even with this small range of state set and coarse quantization parameters. The computation time for constructing symbolic models of Σ_i is amounted to $0.53s$, using tool `SCOTS` [RZ16] with the state quantization parameter $\eta_i^x = 0.1$.

4.3 Small-Gain Approach (SGA)

4.3.1 Networks of Discrete-Time Switched Systems: SGA Formulation

Definition 4.3.1. *Consider discrete-time switched systems* $\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i^1, \mathbb{Y}_i^2, h_{1_i}, h_{2_i})$, $i \in \mathscr{N} := [1; N]$, $N \in \mathbb{N}$. *The finite network of discrete-time switched systems* $\Sigma = (\mathbb{X}, P, F, \mathbb{Y}, h)$, *denoted by* $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathscr{N}}$, *is defined by* $\mathbb{X} = \prod_{i\in\mathscr{N}} \mathbb{X}_i$, $P = \prod_{i\in\mathscr{N}} P_i$, $F = \prod_{i\in\mathscr{N}} F_i$, $\mathbb{Y} = \prod_{i\in\mathscr{N}} \mathbb{Y}_i^1$, $h(x) = (h_i^1(x_i))_{i\in\mathscr{N}}$, *where* $x = (x_i)_{i\in\mathscr{N}}$, *with the internal variables constrained by* $w_i = (y_j^2)_{j\in\mathcal{N}_i} = (h_j^2(x_j))_{j\in\mathcal{N}_i}$, $\prod_{j\in\mathcal{N}_i} \mathbb{Y}_j^2 \subseteq \mathbb{W}_i$, $\forall j \in \mathcal{N}_i, \forall i \in \mathscr{N}$, *where* $\mathcal{N}_i$ *is a finite subset of* $\mathscr{N}$ *that enumerates the neighbors of* Σ_i. *The network of discrete-time switched system is defined by the difference equations*

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= f_{\mathsf{p}(k)}(\mathbf{x}(k)), \\ \mathbf{y}(k) &= h(\mathbf{x}(k)), \end{cases} \tag{4.3.1}$$

where $\mathbf{x} : \mathbb{N} \to \mathbb{X}$, $\mathsf{p} : \mathbb{N} \to \mathbb{P}$, $\mathbf{y} : \mathbb{N} \to \mathbb{Y}$, *and* $f_p(x) = (f_{p_i}(x_i, w_i))_{i\in\mathscr{N}}$ *with* $p = (p_i)_{i\in\mathscr{N}}$.

4.3.2 Max-Type Augmented Simulation Functions

Consider network of discrete-time switched systems $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathscr{N}}$, or their equivalent network of transition systems $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathscr{N}}$, where each $T_i(\Sigma_i)$ given as in Definition 2.4.2. Assume that each system $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ and $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ admit a max-type augmented simulation function as defined next.

Definition 4.3.2. *Consider systems* $T_i(\Sigma_i) = (X_i, X_{0_i}, W_i, U_i, \mathcal{F}_i, Y_i^1, Y_i^2, \mathcal{H}_i^1, \mathcal{H}_i^2)$ *and* $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ *where* $\hat{Y}_i^j \subseteq Y_i^j, j \in [1; 2]$, $\hat{W}_i \subseteq W_i$. *A* *function* $\mathcal{S}_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ *is called a max-type augmented simulation function from* $\hat{T}_i(\Sigma_i)$ *to* $T_i(\Sigma_i)$ *if there exist* $\alpha_i, \rho_{w_i} \in \mathcal{K}_\infty$, $0 < \sigma_i < 1$, *and some* $\varepsilon_i \in \mathbb{R}_{\geq 0}$ *so that the following hold:*

- *For every* $(x_i, p_i, l_i) \in X_i, (\hat{x}_i, p_i, l_i) \in \hat{X}_i$, $j \in [1; 2]$, *one has*

$$\alpha_i(|\mathcal{H}_i^j(x_i, p_i, l_i) - \hat{\mathcal{H}}_i^j(\hat{x}_i, p_i, l_i)|) \leq \mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)). \tag{4.3.2}$$

- *For every* $(x_i, p_i, l_i) \in X_i, (\hat{x}_i, p_i, l_i) \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$, $w_i \in W_i, \hat{w}_i \in \hat{W}_i, (x_i^+, p_i^+, l_i^+) \in \mathcal{F}_i(x_i, w_i, u_i)$, *there exists* $(\hat{x}_i^+, p_i^+, l_i^+) \in \hat{\mathcal{F}}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ *so that*

$$\mathcal{S}_i((x_i^+, p_i^+, l_i^+), (\hat{x}_i^+, p_i^+, l_i^+)) \leq \max\{\sigma_i \mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)), \rho_{w_i}(|w_i - \hat{w}_i|), \varepsilon_i\}. \tag{4.3.3}$$

Here, $\hat{T}_i(\Sigma_i)$ is called an abstraction of $T_i(\Sigma_i)$ if there exists a max-type augmented simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$. Moreover, if $\hat{T}_i(\Sigma_i)$ is finite, it is called a symbolic model of $T_i(\Sigma_i)$.

The following small-gain assumption is needed to provide the compositionality results for this section.

Assumption 4.3.3. *Functions* γ_{ij} *defined in* (3.3.4) *for functions* α_i, *and* ρ_{w_i} *and constant* σ_i *associated with* $\mathcal{S}_i$, $\forall\, i \in \mathcal{N}$, *given in Definition 4.3.2 satisfy the small-gain condition* (3.3.6).

4.3.3 Compositional Abstractions for Finite Networks of Discrete-Time Switched Systems: SGA

In the following, we show how to construct an alternating simulation function from the finite network of abstractions $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$, associated to network of discrete-time switched systems $\Sigma = \mathcal{I}(\Sigma_i)_{i \in \mathcal{N}}$, via max-type augmented simulation functions from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$.

Theorem 4.3.4. *Consider the finite network of transition systems* $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$, *associated to the network of discrete-time switched systems* $\Sigma = \mathcal{I}(\Sigma_i)_{i \in \mathcal{N}}$. *Suppose each transition system* $T_i(\Sigma_i)$ *admits an abstraction* $\hat{T}_i(\Sigma_i)$ *with the corresponding max-type augmented simulation function* $\mathcal{S}_i$. *Let Assumption 4.3.3 holds. Then, for the* $\mathcal{K}_\infty$ *functions* ψ_i *given in* (3.3.7), *function* $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ *defined as*

$$\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) := \max_{i \in \mathcal{N}}\{\psi_i^{-1}(\mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)))\}, \tag{4.3.4}$$

is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i \in \mathcal{N}}$ *to* $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i \in \mathcal{N}}$.

Proof. Consider $z, \hat{z}, z^+, \hat{z}^+$, $\tilde{\sigma}$, and $\tilde{\varepsilon}$ defined in the proof of Theorem 4.2.3. Now, we show that (2.2.1) holds for some $\mathcal{K}_\infty$ function $\tilde{\alpha}$. Consider any $z_i \in X_i$, $\hat{z}_i \in \hat{X}_i$, $\forall i \in [1; N]$. Then, one gets

$$\begin{aligned}
|\mathcal{H}(z) - \hat{\mathcal{H}}(\hat{z})| &= \max_i\{|\mathcal{H}_i^1(z_i) - \hat{\mathcal{H}}_i^1(\hat{z}_i)|\} \leq \max_i\{\alpha_i^{-1} \circ \mathcal{S}_i(z_i, \hat{z}_i)\} \\
&\leq \max_i\{\alpha_i^{-1} \circ \psi_i\} \circ \max_i\{\psi_i^{-1} \circ \mathcal{S}_i(z_i, \hat{z}_i)\} = \max_i\{\alpha_i^{-1} \circ \psi_i\} \circ \tilde{\mathcal{S}}(z, \hat{z}).
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{S}}(z^+,\hat{z}^+) &= \max_{i\in\mathcal{N}}\{\psi_i^{-1}\circ\mathcal{S}_i(z_i^+,\hat{z}_i^+)\} \\
&\leq \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(z_i,\hat{z}_i),\rho_{w_i}(|w_i-\hat{w}_i|),\varepsilon_i\}\big)\Big\} \\
&= \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(z_i,\hat{z}_i),\rho_{w_i}(|(y_j^2)_{j\in\mathcal{N}_i}-(\hat{y}_j^2)_{j\in\mathcal{N}_i}|),,\varepsilon_i\}\big)\Big\} \\
&= \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(z_i,\hat{z}_i),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|y_j^2-\hat{y}_j^2|\}),,\varepsilon_i\}\big)\Big\} \\
&\leq \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(z_i,\hat{z}_i),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{|\mathcal{H}_j^2(z_j)-\hat{\mathcal{H}}_j^2(\hat{z}_j)|\}),\varepsilon_i\}\big)\Big\} \\
&\leq \max_{i\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\sigma_i\mathcal{S}_i(z_i,\hat{z}_i),\rho_{w_i}(\max_{j\in\mathcal{N}_i}\{\alpha_j^{-1}(\mathcal{S}_j(z_j,\hat{z}_j))\}),\varepsilon_i\}\big)\Big\} \\
&\leq \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}(\mathcal{S}_j(z_j,\hat{z}_j)),\varepsilon_i\}\big)\Big\} \\
&= \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_j^{-1}(\mathcal{S}_j(z_j,\hat{z}_j)),\varepsilon_i\}\big)\Big\} \\
&\leq \max_{i,j,l\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j\circ\psi_l^{-1}(\mathcal{S}_l(z_l,\hat{z}_l)),\varepsilon_i\}\big)\Big\} \\
&= \max_{i,j\in\mathcal{N}}\Big\{\psi_i^{-1}\big(\max\{\gamma_{ij}\circ\psi_j(\tilde{\mathcal{S}}(z,\hat{z})),\varepsilon_i\}\big)\Big\} \\
&\leq \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(z,\hat{z})),\max_{i\in\mathcal{N}}\big\{\psi_i^{-1}(\varepsilon_i)\big\}\Big\} \\
&= \max\Big\{\tilde{\sigma}(\tilde{\mathcal{S}}(z,\hat{z})),\tilde{\varepsilon}\Big\},
\end{aligned}
\tag{4.3.5}
$$

Hence, condition (2.2.1) is satisfied with $\tilde{\alpha} = (\max_i\{\alpha_i^{-1}\circ\psi_i\})^{-1}$. Now consider the chain of inequalities in (4.3.5), which satisfies (2.2.2), and implies that $\tilde{\mathcal{S}}$ is indeed an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in\mathcal{N}}$ to $T(\Sigma) = \mathcal{I}(T_i(\Sigma_i))_{i\in\mathcal{N}}$. □

4.3.4 Construction of Symbolic Models

Here, we show that if each subsystem (mode) $\Sigma_p, p\in P$, of $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$ is δ-ISS and some mild assumptions hold, one can construct a symbolic model for Σ along with the corresponding max-type augmented simulation functions between Σ and its symbolic model.

4.3.4.1 Incremental Input-to-State Stability

Definition 4.3.5. *System Σ_p is δ-ISS if there exist functions $V_p : \mathbb{X}\times\mathbb{X}\to\mathbb{R}_{\geq 0}$, $\underline{\varphi}_{x_p}, \overline{\varphi}_{x_p}, \varphi_{w_p} \in \mathcal{K}_\infty$, and constant $0 < \varphi_{c_p} < 1$, such that for all $x, x' \in \mathbb{X}$, and for all*

$w, w' \in \mathbb{W}$

$$\underline{\varphi}_{x_p}(|x - x'|) \leq V_p(x, x') \leq \overline{\varphi}_{x_p}(|x - x'|), \tag{4.3.6}$$

$$V_p(f_p(x, w), f_p(\hat{x}, w')) \leq \varphi_{c_p} V_p(x, x') + \varphi_{w_p}(|w - w'|). \tag{4.3.7}$$

We say that V_p, $\forall p \in P$, are multiple δ-ISS Lyapunov functions for system Σ if it satisfies (4.3.6) and (4.3.7). Moreover, if $V_p = V_{p'}, \forall p, p' \in P$, we omit the index p in (4.3.6), (4.3.7), and say that V is a common δ-ISS Lyapunov function for system Σ. We refer interested readers to [Lib03] for more details on common and multiple Lyapunov functions for switched systems.

Now, we show how to construct a symbolic model $\hat{T}(\hat{\Sigma})$ of transition system $T(\Sigma)$ associated to the switched system Σ in which Σ_p is δ-ISS.

4.3.4.2 Symbolic Models

The symbolic model of $T(\Sigma)$ associated with the switched system Σ in which Σ_p is δ-ISS can be constructed similarly to the one in Definition 4.2.5. Particularly, in the contest of networks of systems, the symbolic model of $T(\Sigma)$ is system $\hat{T}_i(\Sigma_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{W}_i, \hat{U}_i, \hat{\mathcal{F}}_i, \hat{Y}_i^1, \hat{Y}_i^2, \hat{\mathcal{H}}_i^1, \hat{\mathcal{H}}_i^2)$ given by Definition 4.2.5 in which $\hat{W}_i$ should be constructed in such a way that the finite network $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in[1;N]}$ is well-defined. For example, one may choose $\hat{W}_i$ such that $\hat{W}_i = \prod_{j\in\mathcal{N}_i} \hat{Y}_j^2, \forall j \in \mathcal{N}_i, \forall i \in [1; N]$, where $\mathcal{N}_i$ is given as in Definition 4.3.1.

4.3.4.3 Construction of Max-Type Simulation Functions

In this subsection, we show how to construct a max-type augmented simulation function between $T(\Sigma)$, associated to the switched system Σ in which Σ_p is δ-ISS, and its symbolic models $\hat{T}(\Sigma)$ constructed as in Definition 4.2.5.

Theorem 4.3.6. *Consider a switched system $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}^1, \mathbb{Y}^2, h^1, h^2)$ with its equivalent transition system $T(\Sigma) = (X, X_0, W, U, \mathcal{F}, Y^1, Y^2, \mathcal{H}^1, \mathcal{H}^2)$. Let Σ_p be δ-ISS as in Definition 4.3.5. Consider a symbolic system $\hat{T}(\Sigma) = (\hat{X}, \hat{X}_0, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}^1, \hat{Y}^2, \hat{\mathcal{H}}^1, \hat{\mathcal{H}}^2)$ constructed as in Definition 4.2.5. Assume that Assumptions 4.2.6 and 4.2.7 hold for function V_p in Definition 4.3.5. Let $\epsilon > 1$. If, $\forall p \in P$, $k_d \geq \epsilon \frac{\ln(\hat{\mu})}{\ln(\frac{1}{\varphi_{c_p}})} + 1$, then function $\mathcal{V}$ defined as*

$$\mathcal{V}((x, p, l), (\hat{x}, p, l)) := \frac{V_p(x, \hat{x})}{\varphi_{c_p}^{\frac{l}{\epsilon}}}, \tag{4.3.8}$$

is an alternating simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$.

Proof. Recall that $\hat{\mathcal{H}}^j = \mathcal{H}^j = h^j, j \in [1;2]$, by Definition 2.4.2 and 4.2.5. Hence, $\hat{\mathcal{H}}^j$ and $\mathcal{H}^j$ satisfy the Lipschitz assumption given on $h^j, j \in [1;2]$, in Remark 2.5. Since, $\forall p \in P$, Σ_p is δ-ISS , from (4.3.6), $\forall (x,p,l) \in X$ and $\forall (\hat{x},p,l) \in \hat{X}$, we have

$$\begin{aligned}
&|\mathcal{H}^j(x,p,l) - \hat{\mathcal{H}}^j(\hat{x},p,l)| = |h^j(x) - h^j(\hat{x})| \leq \ell^j(|x-\hat{x}|) \\
&\leq \ell^j \circ \underline{\varphi}_{x_p}^{-1}(V_p(x,\hat{x})) = \ell^j \circ \underline{\varphi}_{x_p}^{-1}\left(\varphi_{c_p}^{\frac{l}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l))\right) \\
&\leq \ell^j \circ \underline{\varphi}_{x_p}^{-1}\left(\mathcal{V}((x,p,l),(\hat{x},p,l))\right) \leq \hat{\alpha}\left(\mathcal{V}((x,p,l),(\hat{x},p,l))\right),
\end{aligned}$$

where $\hat{\alpha} = \max\limits_{p\in P}\{\max\limits_{j\in[1;2]}\{\ell^j\} \circ \underline{\varphi}_{x_p}^{-1}\}$. By defining $\alpha = \hat{\alpha}^{-1}$, one obtains

$$\alpha(|\mathcal{H}(x,p,l) - \hat{\mathcal{H}}(\hat{x},p,l)|) \leq \mathcal{V}((x,p,l),(\hat{x},p,l)),$$

satisfying (4.3.2).

Now from (4.2.12), $\forall x \in \mathbb{X}, \forall \hat{x} \in \hat{\mathbb{X}}, \forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$, we have

$$V_p(f_p(x,w),\hat{x}^+) \leq V_p(f_p(x,w), f_p(\hat{x},\hat{w})) + \gamma_p(|\hat{x}^+ - f_p(\hat{x},\hat{w})|),$$

for any $\hat{x}^+$ such that $(\hat{x}^+,p^+,l^+) \in \hat{\mathcal{F}}((\hat{x},p,l),\hat{u},\hat{w})$. Now, from Definition 4.2.5, the above inequality reduces to

$$V_p(f_p(x,w),\hat{x}^+) \leq V_p(f_p(x,w), f_p(\hat{x},\hat{w})) + \gamma_p(\eta).$$

Note that by (4.3.7), one gets

$$V_p(f_p(x,w), f_p(\hat{x},\hat{w})) \leq \varphi_{c_p} V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|).$$

Hence, $\forall x \in \mathbb{X}, \forall \hat{x} \in \hat{\mathbb{X}}$, and $\forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$, one obtains

$$V_p(f_p(x,w),\hat{x}^+) \leq \varphi_{c_p} V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta), \tag{4.3.9}$$

for any $\hat{x}^+$ such that $(\hat{x}^+,p^+,l^+) \in \hat{\mathcal{F}}((\hat{x},p,l),\hat{u},\hat{w})$. Now, in order to show function $\mathcal{V}$ defined in (4.3.8) satisfies (4.3.3), we consider different scenarios in Definition 4.2.5:

- $l < k_d - 1$, $p^+ = p$ and $l^+ = l+1$, using (4.3.9) and $k_d > l+1$, we have

$$\begin{aligned}
\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) &= \frac{V_{p^+}(x^+,\hat{x}^+)}{\varphi_{c_p}^{\frac{l^+}{\epsilon}}} = \frac{V_p(f_p(x,w),\hat{x}^+)}{\varphi_{c_p}^{\frac{l+1}{\epsilon}}} \\
&\leq \frac{\varphi_{c_p} V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l+1}{\epsilon}}} \\
&\leq \frac{\varphi_{c_p}}{\kappa_p^{\frac{1}{\epsilon}}}\frac{V_p(x,\hat{x})}{\varphi_{c_p}^{\frac{l}{\epsilon}}} + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l+1}{\epsilon}}} \\
&\leq \varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{k_d}{\epsilon}}}.
\end{aligned}$$

- $l = k_d - 1$, $p^+ = p$ and $l^+ = k_d - 1$, using (4.3.9) and $\frac{\epsilon-1}{\epsilon} < 1$, one gets

$$\begin{aligned}\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) &= \frac{V_{p^+}(x^+,\hat{x}^+)}{\varphi_{c_p}^{\frac{l^+}{\epsilon}}} = \frac{V_p(f_p(x,w),\hat{x}^+)}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\ &\leq \frac{\varphi_{c_p}V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\ &\leq \varphi_{c_p}\frac{V_p(x,\hat{x})}{\varphi_{c_p}^{\frac{l}{\epsilon}}} + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\ &\leq \varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{k_d}{\epsilon}}}.\end{aligned}$$

- $l = k_d - 1$, $p^+ \neq p$ and $l^+ = 0$, using (4.3.9), $\hat{\mu}\varphi_{c_p}^{\frac{k_d-1}{\epsilon}} \leq 1$, and $\frac{\epsilon-1}{\epsilon} < 1$, one has

$$\begin{aligned}\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) &= \frac{V_{p^+}(x^+,\hat{x}^+)}{\varphi_{v_{p^+}}^{\frac{l^+}{\epsilon}}} \leq \hat{\mu}V_p(f_p(x,w),\hat{x}^+)\\ &\leq \frac{\hat{\mu}\varphi_{c_p}^{\frac{k_d-1}{\epsilon}}\left(\varphi_{c_p}V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)\right)}{\varphi_{c_p}^{\frac{k_d-1}{\epsilon}}}\\ &\leq \frac{\varphi_{c_p}V_p(x,\hat{x}) + \varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\ &\leq \varphi_{c_p}\frac{V_p(x,\hat{x})}{\varphi_{c_p}^{\frac{l}{\epsilon}}} + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\ &\leq \varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\varphi_{w_p}(|w-\hat{w}|) + \gamma_p(\eta)}{\varphi_{c_p}^{\frac{k_d}{\epsilon}}}.\end{aligned}$$

Note that $\forall p \in P, \hat{\mu}\varphi_{c_p}^{\frac{k_d-1}{\epsilon}} \leq 1$ since $k_d \geq \epsilon\frac{\ln(\hat{\mu})}{\ln(\frac{1}{\varphi_{c_p}})} + 1$. By defining $\varphi_c = \max_{p\in P}\left\{\varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}}\right\}$, $\varphi_w = \max_{p\in P}\left\{\varphi_{c_p}^{-\frac{k_d}{\epsilon}}\varphi_{w_p}\right\}$, $\hat{\gamma} = \max_{p\in P}\left\{\varphi_{c_p}^{-\frac{k_d}{\epsilon}}\gamma_p\right\}$, $\forall(x,p,l)\in X$, $\forall(\hat{x},p,l)\in\hat{X}$, $\forall w\in W$, and $\forall\hat{w}\in\hat{W}$, one obtains

$$\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) \leq \varphi_c\mathcal{V}((x,p,l),(\hat{x},p,l)) + \varphi_w(|w-\hat{w}|) + \hat{\gamma}(\eta).$$

By using the result of Theorem 2.2.9, one obtains

$$\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) \leq \max\{\tilde{\varphi}_c\mathcal{V}((x,p,l),(\hat{x},p,l)), \tilde{\varphi}_w(|w-\hat{w}|), \tilde{\gamma}(\eta)\},$$

where $\tilde{\varphi}_c = 1-(1-\psi)(1-\varphi_c)$, $\tilde{\varphi}_w = (\mathcal{I}_d+\lambda)\circ\left(\frac{1}{(1-\varphi_c)\psi}\chi\circ\varphi_w\right)$, $\tilde{\gamma} = (\mathcal{I}_d+\lambda^{-1})\circ\left(\frac{1}{(1-\varphi_c)\psi}\chi\circ(\chi-\mathcal{I}_d)^{-1}\circ\hat{\gamma}\right)$, where λ,χ,ψ are some arbitrarily chosen $\mathcal{K}_\infty$ functions and positive constant with $\chi > \mathsf{id}$, $0 < \psi < 1$. Hence, inequality (4.3.3) is satisfied with $\sigma = \tilde{\varphi}_c$, $\rho_w = \tilde{\varphi}_w$, $\varepsilon = \tilde{\gamma}(\eta)$. Thus, $\mathcal{V}$ is a max-type augmented simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$. □

Note that by using exactly the same argument, we can show the that $\mathcal{V}$ is a max-type augmented simulation function from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Remark 4.3.7. *If Σ admits a common δ-ISS Lyapunov function satisfying Assumption 4.2.7, then function $\mathcal{V}$ in Theorem 4.3.6 reduces to $\mathcal{V}((x,p,l),(\hat{x},p,l)) := V(x,\hat{x})$.*

Now we provide similar results as in the first part of this subsection but tailored to a class of nonlinear switched systems which are computationally more efficient. Consider the class of discrete-time nonlinear switched systems described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A_{\mathsf{p}(k)}\mathbf{x}(k) + E_{\mathsf{p}(k)}\phi_{\mathsf{p}(k)}(G_{\mathsf{p}(k)}\mathbf{x}(k)) + D_{\mathsf{p}(k)}\omega(k) + B_{\mathsf{p}(k)}, \\ \mathbf{y}^1(k) &= C^1\mathbf{x}(k), \\ \mathbf{y}^2(k) &= C^2\mathbf{x}(k), \end{cases} \tag{4.3.10}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times 1}$, $D_p \in \mathbb{R}^{n\times b}$, $C^1 \in \mathbb{R}^{q^1\times n}$, $C^2 \in \mathbb{R}^{q^2\times n}, E_p \in \mathbb{R}^{n\times 1}$, $G_p \in \mathbb{R}^{1\times n}, \forall p \in P = \{1\cdots,m\}$, and $\phi_p : \mathbb{R} \to \mathbb{R}$ satisfying

$$0 \leq \frac{\phi_p(c) - \phi_p(d)}{c-d} \leq \overline{a}_p \quad \forall c,d \in \mathbb{R}, c \neq d, \tag{4.3.11}$$

for some $\overline{a}_p \in \mathbb{R}_{>0} \cup \{\infty\}$.

We use the tuple $\Sigma = (A,B,C^1,C^2,D,E,G,\Phi,P)$ to refer to the class of switched systems of the form (4.3.10), where $A = \{A_1,\cdots,A_m\}$, $B = \{B_1,\cdots,B_m\}$, $D = \{D_1,\cdots,D_m\}$, $E = \{E_1,\cdots,E_m\}$, $G = \{G_1,\cdots,G_m\}$, and $\Phi = \{\phi_1,\cdots,\phi_m\}$. Note that the nonlinear function ϕ_p in (4.3.10) has been widely used for modeling many physical systems including fuel cell [AGPV03], active magnetic bearing [AK01], underwater vehicles [AAFK01], and so on.

Remark that the incremental input-to-state stability assumption on system Σ_p in Subsection 4.3.4.1 boils down in this specific nonlinear case to the following assumption.

Assumption 4.3.8. *Let $\Sigma = (A,B,C^1,C^2,D,E,G,\Phi,P)$. Assume that $\forall p \in P$ there exist constants $0 < \varphi_{c_p} < 1$, $\theta_p \in \mathbb{R}_{>0}$, and matrices $Z_p \succ 0$ of appropriate dimensions such that the following matrix inequality hold*

$$\begin{bmatrix} (1+2\theta_p)A_p^\top Z_p A_p & A_p^\top Z_p E_p \\ E_p^\top Z_p A_p & (1+2\theta_p)E_p^\top Z_p E_p \end{bmatrix} \preceq \begin{bmatrix} \varphi_{c_p} Z_p & -G_p^\top \\ -G_p & 2/\overline{a}_p \end{bmatrix}. \tag{4.3.12}$$

Now, consider the quadratic function $V_p, p \in P$, defined as

$$V_p(x,\hat{x}) = (x-\hat{x})^\top Z_p(x-\hat{x}). \tag{4.3.13}$$

Note that for any function defined as in (4.3.13), one can always find $\hat{\mu}$ satisfying Assumption 4.2.6 (e.g., $\hat{\mu} := \frac{\max_{p,\in P}\{\lambda_{\max}(Z_p)\}}{\min_{p,\in P}\{\lambda_{\min}(Z_p)\}}$). Then, by employing V_p in (4.3.13), Theorem 4.2.8 reduces to the following one for this specific class of nonlinear switched systems.

Theorem 4.3.9. *Consider $T(\Sigma)$ associated to $\Sigma = (A, B, C^1, C^2, D, E, G, \Phi, P)$ and the symbolic model $\hat{T}(\Sigma)$ constructed as in Definition 4.2.5. Suppose Assumption 4.3.8 holds. Let $\epsilon > 1$ and consider V_p given in (4.3.13). If $\forall p \in P, k_d \geq \epsilon \frac{\ln \hat{\mu}}{\ln \frac{1}{\varphi c_p}} + 1$, then function $\mathcal{V}$ defined as*

$$\mathcal{V}((x,p,l),(\hat{x},p,l)) = \frac{V_p(x,\hat{x})}{\varphi c_p^{\frac{l}{\epsilon}}}, \tag{4.3.14}$$

is a max-type augmented simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$.

Proof. First, we show that condition (4.3.2) holds. Since $C^j = \hat{C}^j$, $j \in [1;2]$, we have $\mathcal{H}^j((x,p,l)) - \hat{\mathcal{H}}^j((\hat{x},p,l))|^2 = |C^j x - \hat{C}^j \hat{x}|^2 \leq n\lambda_{\max}({C^j}^\top C^j)|x-\hat{x}|^2$, and similarly $\lambda_{\min}(Z_p)|x-\hat{x}|^2 \leq (x-\hat{x})^\top Z_p(x-\hat{x})$. From the previous inequalities, one has

$$\begin{aligned}
&\frac{\lambda_{\min}(Z_p)}{n\lambda_{\max}(C^\top C)}|\mathcal{H}((x,p,l)) - \hat{\mathcal{H}}((\hat{x},p,l))|^2 \leq (x-\hat{x})^\top Z_p(x-\hat{x}) \\
&= V_p(x,\hat{x}) = \mathcal{V}((x,p,l),(\hat{x},p,l))\varphi c_p^{\frac{l}{\epsilon}} \leq \mathcal{V}((x,p,l),(\hat{x},p,l)),
\end{aligned}$$

which implies that (4.3.2) holds for $\mathcal{V}$ defined in (4.3.14) with $\alpha(s) = \min_p \left\{ \frac{\lambda_{\min}(Z_p)}{n\lambda_{\max}(C^\top C)} \right\} s^2$, for any $s \in \mathbb{R}_{\geq 0}$.

We continue to show that (4.3.3) holds as well. Define $c_1 = (1+\theta_p+\frac{2}{\theta_p}), c_2 = (1+\frac{3}{\theta_p})$, consider any $x^+ = A_p x + E_p\phi_p(G_p x) + D_p w + B_p$, and let $\hat{x}^+$ be defined as in Definition 4.2.5. Define $\Delta := A_p x + E_p\phi_p(G_p x) + D_p w + B_p - \hat{x}^+$, and observe that $|\Delta| \leq \eta^x$ by Definition 4.2.5.

Note that, from the slope restriction (4.3.11), $\phi_p(G_p x) - \phi_p(G_p\hat{x}) = \beta_p(G_p x - G_p\hat{x}) = \beta_p G_p(x-\hat{x})$, where β_p is a constant and depending on x and $\hat{x}$ takes values in the interval $[0, \overline{a}_p]$. Furthermore, consider the chain of inequalities in (4.3.15).

Now, in order to show function $\mathcal{V}$ defined in (4.3.14) satisfies (4.3.3), we consider the different scenarios in Definition 4.2.5. First, define $\rho_p = bc_1|\sqrt{Z_p}D_p|_2^2$, $\gamma_p = nc_2\lambda_{\max}(Z_p)$, $p \in P$, then consider

- $l < k_d - 1$, $p^+ = p$ and $l^+ = l+1$, using (4.3.15) and $k_d > l+1$ we have
$$\begin{aligned}
&\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) = \frac{V_p(x^+,\hat{x}^+)}{\varphi c_p^{\frac{l+1}{\epsilon}}} \leq \frac{\varphi c_p}{\varphi c_p^{\frac{1}{\epsilon}}}\frac{V_p(x,\hat{x})}{\varphi c_p^{\frac{l}{\epsilon}}} + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi c_p^{\frac{l+1}{\epsilon}}} \\
&\leq \varphi c_p^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi c_p^{\frac{k_d}{\epsilon}}}.
\end{aligned}$$

- $l = k_d - 1$, $p^+ = p$ and $l^+ = k_d - 1$, using (4.3.15) and $\frac{\epsilon-1}{\epsilon} < 1$ we have:
$$\begin{aligned}
&\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) = \frac{V_p(x^+,\hat{x}^+)}{\varphi c_p^{\frac{l}{\epsilon}}} \leq \varphi c_p\frac{V_p(x,\hat{x})}{\varphi c_p^{\frac{l}{\epsilon}}} + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi c_p^{\frac{l}{\epsilon}}} \\
&\leq \varphi c_p^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi c_p^{\frac{k_d}{\epsilon}}}.
\end{aligned}$$

$$
\begin{aligned}
&V_p(x^+,\hat{x}^+)\\
&= (A_px + E_p\phi_p(G_px) + D_pw + B_p - (A_p\hat{x} + E_p\phi_p(G_p\hat{x}) + D_p\hat{w} + B_p)\\
&\quad + (A_p\hat{x} + E_p\phi_p(G_p\hat{x}) + D_p\hat{w} + B_p) - x^+)^\top Z_p\\
&\quad (A_px + E_p\phi_p(G_px) + D_pw + B_p - (A_p\hat{x} + E_p\phi_p(G_p\hat{x}) + D_p\hat{w} + B_p)\\
&\quad + (A_p\hat{x} + E_p\phi_p(G_p\hat{x}) + D_p\hat{w} + B_p) - x^+)\\
&= (x-\hat{x})^\top(A_p+\beta_pE_pG_p)^\top Z(A_p+\beta_pE_pG_p)(x-\hat{x}) + (w-\hat{w})^\top D_p^\top Z_pD_p(w-\hat{w})\\
&\quad + 2(w-\hat{w})^\top D_p^\top Z_p\Delta_p + 2(x-\hat{x})^\top(A_p+\beta_pE_pG_p)^\top Z_pD_p(w-\hat{w}) + 2(x-\hat{x})^\top\\
&\quad (A_p+\beta_pE_pG_p)^\top Z_p\Delta_p + \Delta_p^\top Z_p\Delta_p\\
&\leq \begin{bmatrix} x-\hat{x}\\ \beta_pG_p(x-\hat{x})\end{bmatrix}^\top \begin{bmatrix}(1+2\theta_p)A_p^\top Z_pA_p & A_p^\top Z_pE_p\\ E_p^\top Z_pA_p & (1+2\theta_p)E_p^\top Z_pE_p\end{bmatrix}\begin{bmatrix} x-\hat{x}\\ \beta_pG_p(x-\hat{x})\end{bmatrix}\\
&\quad + bc_1|\sqrt{Z_p}D_p|_2^2|w-\hat{w}|^2 + nc_2\lambda_{\max}(Z_p)(\eta^x)^2\\
&\leq \begin{bmatrix} x-\hat{x}\\ \beta_pG_p(x-\hat{x})\end{bmatrix}^\top \begin{bmatrix}\varphi_{c_p}Z_p & -G_p^\top\\ -G_p & 2/\overline{a}_p\end{bmatrix}\begin{bmatrix} x-\hat{x}\\ \beta_pG_p(x-\hat{x})\end{bmatrix} + bc_1|\sqrt{Z_p}D_p|_2^2|w-\hat{w}|^2\\
&\quad + nc_2\lambda_{\max}(Z_p)(\eta^x)^2\\
&= \varphi_{c_p}V(x,\hat{x}) - 2\beta_p\left(1-\frac{\beta_p}{\overline{a}_p}\right)(x-\hat{x})^\top G^\top G(x-\hat{x}) + bc_1|\sqrt{Z_p}D_p|_2^2|w-\hat{w}|^2\\
&\quad + nc_2\lambda_{\max}(Z_p)(\eta^x)^2\\
&\leq \varphi_{c_p}V(x,\hat{x}) + bc_1|\sqrt{Z_p}D_p|_2^2|w-\hat{w}|^2 + nc_2\lambda_{\max}(Z_p)(\eta^x)^2.
\end{aligned}
\tag{4.3.15}
$$

- $l = k_d - 1$, $p^+ \neq p$ and $l^+ = 0$, using (4.3.15), $\varphi_{c_p}^{\frac{k_d-1}{\epsilon}}\hat{\mu} \leq 1$, and $\frac{\epsilon-1}{\epsilon} < 1$ we have:

$$
\begin{aligned}
&\mathcal{V}((x^+,p^+,l^+),(\hat{x}^+,p^+,l^+)) = \frac{V_{p^+}(x^+,\hat{x}^+)}{\varphi_{v_{p^+}}^{\frac{0}{\epsilon}}} \leq \hat{\mu}V_p(x^+,\hat{x}^+)\\
&\leq \frac{\varphi_{c_p}^{\frac{k_d-1}{\epsilon}}\hat{\mu}\Big(\varphi_{cp}V_p(x,\hat{x}) + \rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2\Big)}{\varphi_{c_p}^{\frac{k_d-1}{\epsilon}}} \leq \varphi_{c_p}\frac{V_p(x,\hat{x})}{\varphi_{c_p}^{\frac{l}{\epsilon}}} + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi_{c_p}^{\frac{l}{\epsilon}}}\\
&\leq \varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}}\mathcal{V}((x,p,l),(\hat{x},p,l)) + \frac{\rho_p|w-\hat{w}|^2 + \gamma_p(\eta^x)^2}{\varphi_{c_p}^{\frac{k_d}{\epsilon}}}.
\end{aligned}
$$

By defining $\varphi_c = \max_p \left\{ \varphi_{c_p}^{\frac{\epsilon-1}{\epsilon}} \right\}$, $\varphi_w = \max_p \left\{ \varphi_{c_p}^{-\frac{k_d}{\epsilon}} \rho_p \right\}$, $\gamma_c = \max_p \left\{ \varphi_{c_p}^{-\frac{k_d}{\epsilon}} \gamma_p \right\}$, $p \in P$, one has

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) \leq \varphi_c \mathcal{V}((x, p, l), (\hat{x}, p, l)) + \varphi_w |w - \hat{w}|^2 + \gamma_c (\eta^x)^2$$

for the all scenarios. Using the previous inequality and by following a similar argument as the one in the proof of Theorem 2.2.9, one obtains

$$\mathcal{V}((x^+, p^+, l^+), (\hat{x}^+, p^+, l^+)) \leq \max\{\tilde{\varphi}_c \mathcal{V}((x, p, l), (\hat{x}, p, l)), \tilde{\varphi}_w |w - \hat{w}|^2, \tilde{\gamma}_c (\eta^x)^2\},$$

where $\tilde{\varphi}_c = 1 - (1 - \psi)(1 - \varphi_c)$, $\tilde{\varphi}_w = (\mathsf{id} + \lambda) \circ \left(\frac{1}{(1-\varphi_c)\psi} \chi \circ \rho_c \right)$, $\tilde{\gamma}_c = (\mathsf{id} + \lambda^{-1}) \circ \left(\frac{1}{(1-\varphi_c)\psi} \chi \circ (\chi - \mathsf{id})^{-1} \circ \gamma_c \right)$, where λ, χ, ψ are some arbitrarily chosen $\mathcal{K}_\infty$ functions and positive constant with $\chi > \mathsf{id}$, $0 < \psi < 1$. Hence, inequality (4.3.3) is satisfied with $\sigma = \tilde{\varphi}_c$, $\rho(w) = \tilde{\varphi}_w s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\varepsilon = \tilde{\gamma}(\eta^x)^2$. Thus, $\mathcal{V}$ is a max-type augmented simulation function from $\hat{T}(\Sigma)$ to $T(\Sigma)$. □

Remark that by following the same argument in the previous proof, it can be readily verified that $\mathcal{V}$ is also a max-type augmented simulation function from $T(\Sigma)$ to $\hat{T}(\Sigma)$.

Remark 4.3.10. *For affine switched systems $\Sigma = (A, B, C^1, C^2, D)$ as in Definition 2.4.3, we can restrict our attention to δ-P storage functions of the form $S_p(x, \hat{x}) = (x - \hat{x})^\top Z_p \ (x - \hat{x}), Z_p \succ 0$. It can be readily seen that such functions always satisfy* (4.3.6) *and* (4.2.11) *with $\hat{\mu} = \max \left\{ \frac{\lambda_{\max}(Z_p)}{\lambda_{\min}(Z_p)}, \frac{\lambda_{\max}(Z_{p'})}{\lambda_{\min}(Z_{p'})} \right\}$, for all $p, p' \in P$. Moreover, inequality* (4.3.7) *reduces to the linear matrix inequality*

$$(1 + 2\theta_p) A_p^\top Z_p A_p \preceq \varphi_{c_p} Z_p, \tag{4.3.16}$$

where $\theta_p > 1$, and $0 < \varphi_{c_p} < 1$. Consequently, it can be readily verified that ε in (4.3.3) *would be defined as $\varepsilon = c_p \lambda_{\max}(Z_p)$, for some $c_p > 0$ depending on θ_p and the dimensions of Z_p.*

4.3.5 Case Studies

In this subsection, to demonstrate the effectiveness of our proposed results, we first apply our approaches to a road traffic network in a circular cascade ring composed of 50 identical cells, each of which has the length of 1000 meters with 1 entry and 2 exits, and construct compositionally a symbolic model of the network. We employ the constructed symbolic model as a substitute to compositionally synthesize controllers keeping the density of traffic lower than 30 vehicles per cell. Finally, to show the applicability of our results to switched systems accepting multiple Lyapunov functions with dwell-time, we apply our proposed techniques to a fully interconnected network.

4.3.5.1 Road Traffic Model

Consider the road traffic network $\Sigma = \mathcal{I}(\Sigma_i)_{i\in[1,25]}$ defined in (4.2.16), where each Σ_i is defined as in (4.2.17) with $\omega_i(k) = C^2_{i-1}\mathbf{x}_{i-1}(k)$ ($C^2_0 := C^2_N$, and $\mathbf{x}_0 := \mathbf{x}_N, N = 25$). Note that, for any $i \in [1; 25]$, conditions (4.3.6) and (4.3.7) are satisfied with $V_{ip_i}(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$, $\underline{\varphi}_{x_{ip_i}} = \overline{\varphi}_{x_{ip_i}} = \mathsf{id}$, $\varphi_{v_{ip_i}} = 0.65$, $\varphi_{w_{ip_i}} = 0.33$, $\forall p_i \in P_i$. Furthermore, condition (4.2.12) is satisfied with $\gamma_{ip_i} = \mathsf{id}$, $\forall p_i \in P_i$. Moreover, note that $V_{ip_i} = V_{ip'_i}, \forall p, p' \in P$. Consider systems $\hat{T}_i(\Sigma_i)$, constructed as in Definition 4.2.5, and $T_i(\Sigma_i)$, defined in Definition 2.4.2. According to Remark 4.3.7, function $\mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = |x_i - \hat{x}_i|$ is a sum-type augmented simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$, satisfying conditions (4.3.2) and (4.3.3) with $\alpha_i = \mathsf{id}$, $\sigma_i = 0.99$, $\rho_{w_i} = 0.98$, and $\varepsilon_i = 99\eta^x_i$, where η^x_i is the state set quantization parameter. Now, since we have $\gamma_{ij}(s) < \mathsf{id}$, $\forall i, j \in [1; 25]$, the small-gain condition (3.3.6) is satisfied. Using the results in Theorem 4.3.4 with $\psi_i^{-1} = \mathsf{id}$, $\forall i \in [1; 25]$, one can verify that $\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) = \max_i\{|x_i - \hat{x}_i|\}$ is an alternating simulation function from $\hat{T}(\Sigma) = \mathcal{I}_M(\hat{T}_i(\Sigma_i))_{i\in[1,25]}$ to $T(\Sigma) = \mathcal{I}_M(T_i(\Sigma_i))_{i\in[1,25]}$ satisfying conditions (2.2.1) and (2.2.2) with $\tilde{\sigma} = 0.98$, $\tilde{\alpha} = \mathsf{id}$, $\tilde{\varepsilon} = \max_i\{\varepsilon_i\}$.

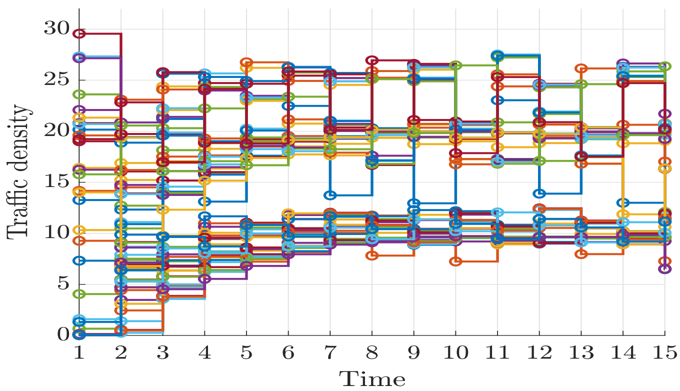

Figure 4.4: Closed-loop state trajectories of network Σ consisting of 50 systems.

Now we synthesize a controller for Σ via symbolic models $\hat{T}_i(\hat{\Sigma}_i)$ maintaining the density of traffic lower than 30 vehicles per cell, and allowing only 2 consecutive red light for each traffic signal. We design local controllers based on assume-guarantee reasoning for symbolic models $\hat{T}_i(\hat{\Sigma}_i)$, and then use them in concrete switched systems Σ_i. We leverage software tool `SCOTS` [RZ16] for constructing symbolic models and controllers for Σ_i. The computation times for constructing symbolic models and designing controllers for Σ_i with state quantization parameter $\eta_i = 0.03$ are $10.2s$ and $0.014s$, respectively. Figure 4.4 shows the closed-loop state trajectories of Σ, consisting of 50 cells.

Finally, one can compute the mismatch between the output behavior of the concrete network $\Sigma = \mathcal{I}(\Sigma_i)_{i\in[1,25]}$ and that of its symbolic model $\hat{T}(\Sigma) = \mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in[1,25]}$ by utilizing Proposition 2.2.11. In particular, using (2.2.8) and since $\tilde{\alpha} = \mathsf{id}$, we have $\hat{\varepsilon} = \tilde{\alpha}^{-1}(\tilde{\varepsilon}) = \max_{i\in[1;25]}\{\varepsilon_i\} = 2.9$.

4.3.5.2 Fully Connected Network

In order to show the applicability of our results to switched systems accepting multiple Lyapunov functions with dwell-time, we apply our proposed techniques to a fully interconnected network of N switched nonlinear systems (totally $2N$ dimensions) for any $N \geq 2$, and construct their symbolic models. The dynamic of the interconnected switched system Σ has a set of modes $P = \{1,2\}^N \ni p = [p_1; \cdots; p_N], N \in \mathbb{N}_{\geq 1}$, and it is described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) & = A_{\mathsf{p}(k)}\mathbf{x}(k) + E\varphi(G\mathbf{x}(k)) + B_{\mathsf{p}(k)}, \\ \mathbf{y}(k) & = \mathbf{x}(k). \end{cases}$$

The elements of the matrix $A_p \in \mathbb{R}^{n \times n}$, and the vector $B_p \in \mathbb{R}^n$, where $n = 2N$, are as follows:

$$\{A\}_{i,j} = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \ \{A\}_{i,i} = A_{p_i} = \begin{cases} \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix} & \text{if } p_i = 1, \\ \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix} & \text{if } p_i = 2, \end{cases}$$

$$\{B\}_{i,1} = B_{p_i} = \begin{cases} \begin{bmatrix} -0.9 \\ 0.5 \end{bmatrix} & \text{if } p_i = 1, \\ \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix} & \text{if } p_i = 2, \end{cases}$$

$\forall i,j \in [1;N], i \neq j$. Additionally, $E = [0.1; \cdots; 0.1] \in \mathbb{R}^n, G = [0.1, \cdots, 0.1] \in \mathbb{R}^n, \varphi(s) = sin(s), \forall s \in \mathbb{R}$. Now, by introducing Σ_i described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) & = A_{i\mathsf{p}_i(k)}\mathbf{x}_i(k) + E_i\varphi_i(G_i x_i(k)) + D_i\omega_i(k) + B_{i\mathsf{p}_i(k)}, \\ \mathbf{y}_i^1(k) & = \mathbf{x}_i(k) \\ \mathbf{y}_i^2(k) & = \mathbf{x}_i(k). \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}(\Sigma_i)_{i\in[1,N]}$, where, $\forall i \in [1;N]$, $\omega_i(k) = \sum_{j=1,j\neq i}^N \mathbf{y}_j^2(k)$, $\varphi_i(s) = sin(s), \forall s \in \mathbb{R}$,

$$A_{i1} = \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix}, A_{i2} = \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix}, D_i = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, B_{i1} = \begin{bmatrix} -0.9 \\ 0.5 \end{bmatrix},$$
$$B_{i2} = \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix}, E_i = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, G_i = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, P_i = \{1,2\}.$$

Following the same argument as in [Lib03] and [GPT10] for the case of continuous switched systems, one can observe that system Σ_i does not have a common δ-ISS Lyapunov function because it exhibits unstable behaviors for some switching signals (e.g., apply periodically mode 1 during 1 time unit, then mode 2 during 1 time unit and so on). However, system Σ_i has multiple δ-ISS Lyapunov function of the form $(x_i-\hat{x}_i)^\top Z_{ip_i}(x_i-\hat{x}_i)$. Accordingly, condition (4.3.12) is satisfied with

$$Z_{i1}=\begin{bmatrix}1.311 & 0.001\\0.001 & 0.492\end{bmatrix}, Z_{i2}=\begin{bmatrix}0.4 & 0.01\\0.012 & 1.49\end{bmatrix}, \varphi_{c_{i1}}=\varphi_{c_{i2}}=0.7, \theta_{i1}=\theta_{i2}=0.4.$$

Note that Assumption 4.2.6 and $k_d \geq \epsilon\frac{\ln(\hat{\mu})}{\ln(\frac{1}{\kappa_p})}+1$ hold with $\hat{\mu}=3.27$, $k_d=7$, and $\epsilon=1.7$. Now, consider function $\mathcal{V}$ defined as $\mathcal{V}_i((x_i,p_i,l_i),(\hat{x}_i,p_i,l_i))=(x_i-\hat{x}_i)^\top Z_{ip_i}(x_i-\hat{x}_i)\varphi_{c_{ip_i}}^{-l/\epsilon}$, systems $\hat{T}_i(\Sigma_i)$, constructed as in Definition 4.2.5, and $T_i(\Sigma_i)$, defined in Definition 2.4.2. Then according to Theorem 4.3.9, function $\mathcal{V}$ is a max-type simulation function from $\hat{T}_i(\Sigma_i)$ to $T_i(\Sigma_i)$, satisfying conditions (4.3.2) and (4.3.3) with $\alpha_i(s)=0.2s^2$, $\sigma_i=0.99$, $\rho_{w_i}(s)=0.19s^2$, and $\varepsilon_i=2.26\times10^3(\eta_i^x)^2$. Since we have $\gamma_{ij}(s)<\mathsf{id}$, $\forall i,j\in[1;N]$, $i\neq j$, the small-gain condition (3.3.6) is satisfied. Using the results in Theorem 4.3.4 with $\psi_i^{-1}=\mathsf{id}$, $\forall i\in[1;N]$, one can verify that $\tilde{\mathcal{S}}((x,p,l),(\hat{x},p,l))=\max_i\{(x_i-\hat{x}_i)^\top Z_{ip_i}(x_i-\hat{x}_i)\varphi_{c_{ip_i}}^{-l/\epsilon}\}$ is an alternating simulation function from $\hat{T}(\Sigma)=\mathcal{I}(\hat{T}_i(\Sigma_i))_{i\in[1,N]}$ to $T(\Sigma)=\mathcal{I}(T_i(\Sigma_i))_{i\in[1,N]}$ satisfying conditions (2.2.1) and (2.2.2) with $\tilde{\sigma}=0.99$, $\tilde{\alpha}(s)=0.2s^2$, $\forall s\in\mathbb{R}_{\geq0}$, $\tilde{\varepsilon}=\max_i\{\varepsilon_i\}$.

4.4 Summary

In this chapter, we proposed a compositional scheme based on dissipativity and small-gain type reasoning for the construction of symbolic models for networks of discrete-time switched systems. We used notions of a sum- and max-type augmented simulation functions in order to construct compositionally an alternating simulation function that is used to quantify the error between the output behavior of the network of switched system and that of its symbolic model. Furthermore, under some assumptions ensuring incremental passivity or incremental input-to-state stability of each mode of switched systems, we showed how to construct symbolic models together with their corresponding sum and max-type augmented simulations functions of the concrete systems. Finally, we applied our results to a model of road traffic by constructing compositionally a symbolic model of a network containing 50 cells of 1000 meters each. We used the constructed finite abstractions as substitutes to design controllers compositionally keeping the density of traffic lower than 30 vehicles per cell. We also applied our results to a network of switched systems admitting multiple incrementally passive storage functions, and also a network of switched systems accepting multiple incremental input-to-state Lyapunov functions with dwell-time.

5 Conclusions and Future Contributions

5.1 Conclusions

In this dissertation, we tackled the scalability issues that arise in the synthesis of symbolic models for large-scale cyber-physical systems. We proposed novel compositional techniques based on which one can construct symbolic models for networks composed of discrete-time control or switched systems. In the first approach, we leveraged techniques from dissipativity theory to provide the compositionality results. The proposed dissipativity conditions may enjoy specific interconnection topologies and provide scale-free compositional synthesis for symbolic models of networks of discrete-time control systems. We showed that if those dissipativity-type conditions hold, it is possible to design a symbolic model for a network composed of a finite number of discrete-time control systems using symbolic models of those systems. Particularly, we used a notion of so-called sum-type simulation function between concrete systems and their symbolic models to compositionally construct so-called alternating simulation functions that ensure that the output behaviors of the networks of discrete-time control systems are quantitatively approximated by the corresponding ones of their symbolic models.

In the second approach, we used two different small-gain type reasoning to provide the compositionality results. First, we utilized nonlinear max-type small-gain conditions to provide a compositional framework for constructing symbolic models for finite networks of discrete-time control systems. We showed that the proposed max small-gain approach is more general than the classic one in the literature since it does not require linear growth on the gains of systems which is the case in the classic small-gain. We also proved that the overall approximation error is independent of the number of discrete-time control systems in the concrete network. Specifically, this overall approximation error is proportional to the maximum of the approximation errors between local systems and their symbolic models. Hence, the proposed results can potentially provide symbolic models for a network composed of a large number of discrete-time control systems with a much smaller approximation error in comparison with those based on the classic small-gain and dissipativity approaches. Additionally, we introduced a compositional technique based on recently published robust small-gain conditions to synthesize symbolic models for concrete networks consisting of infinitely many finite-dimensional discrete-time control systems.

Moreover, we used the same dissipativity and max small-gain conditions mentioned above to provide a compositional construction of symbolic models for finite networks of discrete-time switched systems. In addition, we provided a technique for constructing symbolic models for discrete-time control systems that are incrementally passive or incrementally input-to-state stable. We extended our results from discrete-time control systems to switched ones whose switching signals accept a dwell-time condition with multiple δ-ISS Lyapunov functions (or multiple δ-P storage functions in the dissipativity setting). Moreover, we showed that those stability properties can be replaced by some linear matrix inequalities for particular classes of discrete-time control and switched systems.

Finally, we provided several case studies illustrating the efficiency of the proposed techniques. In all those case studies, we have assumed that the dynamics of the physical plants can be described by either discrete-time control or switched systems. If this is not the case and the states of physical plants evolve continuously in time (described by either continuous-time control or switched systems), a discrete-time models of those continuous-time systems still can be obtained through discretization. The obtained discrete-time models could be exact ones (e.g., available if continuous-time systems is linear) or approximate models (obtained from applying a numerical scheme, e.g., Euler method). In this case, a formal upper bound for the mismatch between the output behaviors of the concrete system and its symbolic model can be provided only at sampling instances. In addition, the closeness of those output behaviors during intersampling periods can be also analyzed using reachability analysis [GG10b, GG10a].

5.2 Recommendations for Future Research

In the following paragraphs, we propose some interesting subjects that could be considered as future research lines.

- **Network decomposition**. In this dissertation, we provided different compositional approaches for synthesizing symbolic models for networks of discrete-time control and switched systems. In order to provide such compositional approaches, we first *i*) decomposed the overall concrete network into a number of concrete systems and construct symbolic models of them individually; *ii*) then we used dissipativity and max small-gain conditions to provide compositionality results that allow us to construct a symbolic model of the overall network using those individual ones.

  Generally, a decomposition of a network can be derived directly from its physical description or its mathematical models. The physical decomposition suggests a “natural” grouping of the state variables which leads in numerical simplifications and provides information about the important structural properties of the system. However, a mathematical model of systems is usually what we have, and the properties of such a model provide little or no insight into how the a decomposition should be performed [Sil07]. A future direction is to develop a general framework for networks decomposition and investigate how the techniques used in this decomposition could be leveraged to increase the chance of the satisfaction of the dissipativity and max small-gain conditions used in this work.

- **Compositional controller synthesis**. From the standpoint of controllers design, designers more often aim to establish a decomposition framework which produces a number of interconnected subsystems that are controlled by local distinct control action. In other words, it is often desirable to formulate control laws that use only locally available states information. This strategy is computationally efficient, easy to implement, and can significantly reduce costly communication between subsystems [Sil10].

In this dissertation and based on the assume-guarantee reasoning approach, the compositionally constructed symbolic models might be used to synthesize controllers compositionally to enforce some decomposable specifications. However, when dealing with high-level properties that go beyond invariance, synthesizing controllers in a compositional fashion is far from being obvious. As one potential direction for future work, one may investigate possible compositional techniques for controller synthesis for networks of systems. In other words, given a general specification over a concrete network, one can explore how to compositionally design a controller enforcing those specification using local controllers for symbolic models.

- **New compositionality conditions**. Although the compositionality conditions used in this dissertation based on dissipativity and max small-type reasoning might be satisfied in many applications, there are still sufficient. Therefore, developing *necessary* compositionality conditions could be a possible direction of future research. The necessity of those compositionality conditions means that a compositionally constructed symbolic model for a concrete network exists if and only its monolithically constructed one does.

- **Enlarging classes of systems**. In this dissertation, we mainly focused on two types of networks of systems, namely, discrete-time control and switched systems. It would be interesting if one can design symbolic models for networks of other classes of hybrid systems, e.g., impulsive systems or state-dependent switched systems. In this direction, we recently provided a monolithic approach for constructing symbolic models for a class of time-dependent impulsive systems in [SGZ20]. We expect that it is possible to combine the proposed compositional framework in this dissertation with the results in [SGZ20] to provide compositional construction of symbolic models for networks of impulsive systems.

- **Compositional control barrier functions**. A discretization-free approach, based on control barrier functions, has shown the potential to solve the formal synthesis problems. Different compositionally results have been recently proposed on the construction of control barrier functions for finite networks of discrete-time control systems in [JSZ20], for large-scale interconnected stochastic systems in [ALZ20, NSZ20b], for finite networks of stochastic switched systems in [NSZ20a]. A future direction is to explore the compositional construction of control barrier functions for infinite networks while considering other classes of hybrid systems.

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