TECHNISCHE UNIVERSITÄT MÜNCHEN

Fakultät für Mathematik
Lehr- und Forschungseinheit Algebra – Algebraische Geometrie

# Moduli of Bielliptic Surfaces

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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Prüfer der Dissertation: 1. Prof. Dr. Christian Liedtke
2. Prof. Dr. Andreas Rosenschon
Ludwig-Maximilians Universität München

Die Dissertation wurde am 02.12.2020 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 17.03.2021 angenommen.

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*To Mercedes, my wife, who always believes in me, even when I don't. Thank you for all your support. This was only possible because of you.*

*To Isabella, my daughter. You came in the right moment to encourage me every day with your beautiful smile.*

# Contents

# Introduction

Many problems in mathematics are concern with the classification of objects of a fixed kind. In order to carry out such a classification, one first has to agree on a notion of *equivalence* among this objects, that is, one has to decide when two of these objects will be considered as being essentially the same. The solution of the classification problem will then of course depend on the chosen notion of equivalence. After doing that, one ends up with a set of equivalence classes of objects, but just that: a *set*. This is just a partial solution to the classification problem: while it gives us some information about the objects we are trying to classify (e.g., how many different equivalent classes there are), it does not tell us anything about how different objects are related to each other. In other words, the set of equivalence classes does not give us any information about how the objects are allowed to vary.

To remedy this, one usually wants to endow this set with some extra *geometric* structure in order to use techniques form geometry to study the way in which the objects vary. This way of proceeding has lead mathematicians to the notion of *moduli space*. At first sight, a moduli space is just a parameter space, i.e., the points of the space parameterize (equivalence classes of) objects of a certain kind, and the geometry of the space should reflect the way in which the objects vary. However, depending on the additional properties one requires the moduli space to have, one comes to the different notions of *coarse moduli space*, *fine moduli space* and the more modern notion of *moduli stack*.

The objects we are interested in are a special type of surfaces called *(quasi-)bielliptic* surfaces. According to the *Kodaira-Enriques* classification of surfaces, which was extended by Bombieri and Mumford to positive characteristic, bielliptic and quasi-bielliptic surfaces form one of the four classes of minimal smooth projective algebraic surfaces of Kodaira dimension zero. More precisely, these four classes consist of K3 surfaces, Enriques surfaces, Abelian surfaces and (quasi-)bielliptic surfaces.

Moduli spaces of Abelian surfaces has been extensively studied as part of the more general study of moduli spaces of Abelian varieties and are well understood (see, for example, [**HKW93, MFK94, FC90a, Ols08**]). For K3 and Enriques surfaces moduli spaces have also been well studied and modern accounts can be found in [**Riz06**] and [**Lie15a**], respectively. On the other hand, moduli spaces of (quasi-)bielliptic surfaces have not yet been studied from a modern point of view. It is therefore natural to consider the problem of constructing and describing the moduli space of these surfaces from the point of view of stacks.

After defining and giving some background information on (quasi-)bielliptic surfaces, we will first give a short summary of preliminary work concerning the moduli spaces of bielliptic surfaces and then present our main results.

## 1. State of the Art and Previous Work

**1.1. Bielliptic and Quasi-bielliptic Surfaces.** To have a picture in mind, let us first introduce bielliptic surfaces in an informal way.

While studying surfaces, a very useful approach is to try to describe a given surface as a family of curves parameterized by another base curve. This way, the surface can be seen as a collection of fibers, each one corresponding to a given point of the base curve. If a surface admits such a description, it is said to be *fibered.* With this picture in mind, bielliptic surfaces can be roughly described as surfaces which are fibered by elliptic curves in two different (transversal) ways. Thus, the name *bi-elliptic.* Quasi-bielliptic surfaces can be described similarly.

1.1.1. *Definition and Properties of (Quasi-)Bielliptic Surfaces.*

According to the *Kodaira-Enriques* classification, smooth projective algebraic surfaces over the complex numbers can be divided into four classes, depending on their *Kodaira dimension* $\kappa \in \{-\infty, 0, 1, 2\}$ (see, for example, [**Bea96**]). *Bielliptic surfaces* (also known as *hyperelliptic surfaces* in a more classical terminology) belong to the class of surfaces of Kodaira dimension zero. They were classified over the complex numbers into seven different types by G. Bagnera and M. de Franchis in [**Bd08**] (see also [**ES09, ES10**]).

The extension of the Kodaira-Enriques classification to arbitrary characteristic $p \geq 0$ was carried out by E. Bombieri and D. Mumford in a series of articles ([**Mum69, BM77, BM76**]). Over fields of characteristic $p \geq 5$, the classification of bielliptic surfaces remains the same as over the complex numbers. However, in positive small characteristic, that is, over fields of characteristic $p \in \{2, 3\}$, a new class of surfaces appears, the so-called *quasi-bielliptic surfaces* (also known as *quasi-hyperelliptic surfaces*). Moreover, in characteristic $p = 2$ there is a new type of bielliptic surfaces and some types of bielliptic surfaces do not exist in small characteristic (see Table 1.1 below).

Let us now give a precise definition of (quasi-)bielliptic surfaces.

Definition 1.1. A *(quasi-)bielliptic* surface is a smooth, projective, minimal surface $X$ of Kodaira dimension 0 satisfying $b_2 = 2$, where $b_i$ denotes the i-th étale or crystalline Betti number.

(Quasi-)bielliptic surfaces have nice properties and can be described very explicitly. We refer to Badescu's book [**Bǎd01**] on algebraic surfaces and to the original articles of Bombieri and Mumford [**BM77, BM76**] for this properties. We recall some of them now.

First of all, it follows from the definition above that the Albanese variety $\mathrm{Alb}(X)$ of $X$ is an elliptic curve and that the Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$ gives $X$ the structure of a *genus one fibration.* This roughly means that there is a surjective map from $X$ to a smooth curve, in this case to $\mathrm{Alb}(X)$, such that the general fiber is an integral curve of arithmetic genus one. A genus one fibration is called *elliptic* or *quasi-elliptic*, depending on whether the general fiber is smooth or not. This allows us to distinguish between *bielliptic* surfaces, for which the Albanese fibration is elliptic, and *quasi-bielliptic* surfaces, for which the Albanese fibration is quasi-elliptic. As said before, quasi-bielliptic surfaces only exist in characteristic 2 and 3. Moreover, (quasi-)bielliptic surfaces always admit a second elliptic fibration to a projective line $\mathbb{P}^1$.

Using this two fibrations, it is possible to describe the structure of (quasi-)bielliptic surfaces very explicitly. We consider here only the case of bielliptic surfaces. For the case of quasi-bielliptic surfaces see [**BM76**, Theorem 1].

Theorem 1.2 ([**BM77**], Theorem 4, § 3). *Every bielliptic surface $X$ is of the form*

$$X = (E \times F)/G,$$

*where $E$ and $F$ are elliptic curves, $G$ is a finite subgroup scheme of $E$, and $G$ acts on the product $E \times F$ by*

$$g.(x, y) = (x + g, \alpha(g)(y))$$

*for some suitable injective homomorphism $\alpha\colon G \to \mathrm{Aut}(F)$. Moreover, the two elliptic fibration of $X$ are given by:*

$$f\colon X \to E/G \text{ (elliptic curve) } \text{ and } g\colon X \to F/\alpha(G) \cong \mathbb{P}^1.$$

This theorem can be used to classify all possible bielliptic surfaces and leads to the (extended) Bagnera-DeFranchis list in Table 1.1. The last column of the table indicates the characteristic of the fields over which a given type of bielliptic surface exists. For instance, bielliptic surfaces of type (a3), where $\mu_2$ denotes the group scheme of square roots of unity, only exist in characteristic two.

| Type | $G$ | $\mathrm{char}(k)$ |
|---|---|---|
| $(a1)$ | $\mathbb{Z}/2\mathbb{Z}$ | |
| $(a2)$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\neq 2$ |
| $(a3)$ | $\mu_2 \times (\mathbb{Z}/2\mathbb{Z})$ | 2 |
| $(b1)$ | $\mathbb{Z}/3\mathbb{Z}$ | |
| $(b2)$ | $(\mathbb{Z}/3\mathbb{Z})^2$ | $\neq 3$ |
| $(c1)$ | $\mathbb{Z}/4\mathbb{Z}$ | |
| $(c2)$ | $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ | $\neq 2$ |
| $(d)$ | $\mathbb{Z}/6\mathbb{Z}$ | |

TABLE 1.1. Possible types of bielliptic surfaces.

1.1.2. *(Quasi-)Bielliptic Surfaces in Small Characteristic.*

From the above discussion, it is clear that the study of (quasi-)bielliptic surfaces can be divided into two parts according to the characteristic of the ground field $k$. The first part consist of the study of (just) bielliptic surfaces over fields of characteristic $p \notin \{2, 3\}$. The second part deals with the study of bielliptic and quasi-bielliptic surfaces over fields of characteristic $p \in \{2, 3\}$.

It turns out that the second part is considerably harder than the first one. First of all, the quasi-bielliptic surfaces come into play increasing the number of different types of surfaces to be considered, as the following table shows [1] :

| | $p = 2$ | $p = 3$ | $p \notin \{2, 3\}$ |
|---|---|---|---|
| Bielliptic | 6 | 6 | 7 |
| Quasi-bielliptic | 8 | 5 | 0 |

TABLE 1.2. Number of types of (quasi-)bielliptic surfaces according to the characteristic of the ground field.

Secondly, quasi-bielliptic surfaces are less familiar than the bielliptic ones. Indeed, the Albanese fibration of quasi-bielliptic surfaces has cuspidal curves as fibers, instead of elliptic curves, for which there is a well established theory. Finally, some (quasi-)bielliptic surfaces in positive small characteristic may have a non-reduced Picard scheme, what makes the deformation theory of these surfaces more complicated.

[1] According to Bombieri and Mumford [**BM76**], in characteristic $p = 3$ there are 6 different types of quasi-bielliptic surfaces. However, in [**Lan79**, p. 489] Lang argues that one of these types does not exist.

For these reasons, we will restrict ourselves to the study of moduli spaces of bielliptic surfaces in characteristic different from 2 and 3.

**1.2. Moduli Spaces of Bielliptic Surfaces.** As a motivation and to show where our work on moduli spaces of bielliptic surfaces shall begin, we give here a short description of previous work on the subject. A more detailed description will be given in Section 1.1 of Chapter 2.

1.2.1. *Moduli Spaces of Complex Bielliptic Surfaces.*

Moduli spaces of complex bielliptic surfaces were studied by H. Tsuchihashi in [**Tsu79**] building on previous work of T. Suwa [**Suw69**].

Classically, it was already known that over the complex numbers $\mathbb{C}$ there are seven different types of bielliptic surfaces. In [**Suw69**] Suwa describes the seven types of bielliptic surfaces as quotient spaces of Abelian surfaces. He recovers this way the classification of complex bielliptic surfaces established earlier by Bagnera-DeFrancis [**Bd08**] and Enriques-Severi [**ES09, ES10**].

Moreover, he shows that the surfaces of each type form a complex analytic family, which can be parameterized by the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$. For two types of bielliptic surfaces, the ones corresponding to the types $(a1)$ and $(a2)$ in the notation of Table 1.1, each family is parameterized by two parameters $\tau, \omega \in \mathbb{H}$. For the remaining types only one parameter $\tau \in \mathbb{H}$ is needed.

An important result obtained by Suwa is the fact that the seven types of bielliptic surfaces are completely classified topologically: different types of bielliptic surfaces are topologically different. Thus, a complex analytic family of bielliptic surfaces (over a connected base) can only contain bielliptic surfaces of the same type as fibers, since all the fibers of such a family are diffeomorphic. Therefore, a moduli space for complex bielliptic surfaces has to split into a disjoint union of seven moduli spaces, each one parameterizing bielliptic surfaces of a given type.

With Suwa's explicit description of the seven types of complex bielliptic surfaces at hand, Tsuchihashi [**Tsu79**] construct coarse moduli spaces for each type of bielliptic surfaces as product of modular curves. Moreover, to construct fine moduli spaces, Tsuchihashi rigidify the moduli problem by considering bielliptic surfaces together with suitable base points satisfying certain conditions.

1.2.2. *Moduli Spaces of Bielliptic Surfaces in Positive Characteristic.*

Moduli spaces of bielliptic surfaces in positive characteristic were studied by W. Seiler in [**Sei87a, Sei87b**] as part of his study of global moduli spaces for elliptic surfaces. There, however, surfaces in characteristic 2 and 3 are excluded and the moduli spaces he obtains are all coarse moduli spaces.

In [**Sei87a**] Seiler shows the existence of coarse moduli spaces for elliptic surfaces with a section. In particular, he proves the existence of coarse moduli spaces for bielliptic surfaces with a section of the Albanese fibration (which we shall call *Jacobian* bielliptic surfaces). Furthermore, the author shows that the corresponding moduli functor splits in a natural way into a disjoint union of subfunctors parameterizing Jacobian bielliptic surfaces $f\colon X \to \mathrm{Alb}(X)$ for which the canonical bundle $\omega_X$ has order $n \in \{2, 3, 4, 6\}$ in the Picard group $\mathrm{Pic}(X)$, respectively. Thus, the coarse moduli space of Jacobian bielliptic surfaces splits into four disjoint components. This generalizes and agrees with Suwa's work over the complex numbers.

In [**Sei87b**] the existence of coarse moduli spaces for (numerically) polarized elliptic surfaces not necessarily admitting a section is shown. In particular, Seiler proves the existence of a Hilbert scheme for polarized bielliptic surfaces and shows that it splits into connected components parameterizing bielliptic surfaces a each type, respectively. Furthermore, he obtains the existence of coarse moduli spaces of polarized and numerically polarized bielliptic surfaces.

## 2. Objective and Results

**2.1. Objective and Guiding Questions.** As we have seen, much work has been done to construct moduli spaces of bielliptic surfaces, both over the complex numbers and in positive characteristic, if not in small characteristic. However, all the resulting moduli spaces are either coarse moduli spaces or fine moduli spaces obtained by rigidifying the original moduli problem. We want to give a solution of the classification (moduli) problem of bielliptic surfaces in terms of stacks.

The main goal of this Ph.D. thesis is to construct and study the moduli stack $\mathcal{M}_{biell}$ of bielliptic surfaces over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]$, whose geometric points correspond to bielliptic surfaces over algebraically closed fields of characteristic different from 2 and 3.

As a byproduct of the study needed to construct such a moduli stack, the thesis aims to bring together the otherwise scattered literature on bielliptic surfaces. It should present the theory of bielliptic surfaces in a modern language and in a new generality without the sometimes limiting assumptions placed on it in past. This way, it should serve as a general and modern reference for the topic.

The following questions will serve as a guide to our study of the moduli stack $\mathcal{M}_{biell}$ of bielliptic surfaces over Spec $\mathbb{Z}[\frac{1}{6}]$:

(a) Is it possible to separate the different types of bielliptic surfaces over fields of positive characteristic different from 2 and 3 in a similar way as they are separated over the complex numbers? This is a question in deformation theory which can be formulated more precisely as follows:

(a') Given a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over a connected Noetherian scheme $S$ over $\mathbb{Z}[\frac{1}{6}]$, can this family have bielliptic surfaces of different types as geometric fibers?

We will show that the answer to question (a') is negative: the geometric fibers of such a family must all be bielliptic surfaces of the same type. Consequently, the moduli stack $\mathcal{M}_{biell}$ splits into seven disjoint components

$$\mathcal{M}_{biell} = \bigsqcup \mathcal{M}_{biell}^{(i)},$$

each of which is a substack $\mathcal{M}_{biell}^{(i)}$ of $\mathcal{M}_{biell}$ parameterizing bielliptic surfaces of type $(i)$, where $(i)$ is one of the types in the list of Bagnera-DeFranchis (cf. Table 1.1). In particular, the moduli stacks $\mathcal{M}_{biell}^{(i)}$ can be studied individually.

(b) Which nice (geometric) properties do the stacks $\mathcal{M}_{biell}^{(i)}$ have? For instance,
- are they algebraic or even Deligne-Mumford stacks?
- are they (quasi-)separated, finite?
- are they irreducible, smooth, of which dimension?

We will center our attention on the study of the moduli stack $\mathcal{M}_{biell}^{(a1)}$ parameterizing families of bielliptic surfaces of type $(a1)$ as a guiding example for the further study of the remaining types. To investigate the moduli stack $\mathcal{M}_{biell}^{(a1)}$ we follow Tsuchihashi's work [**Tsu79**] mentioned before and relate the moduli stack in question to a product of *modular stacks*, that is, stacks parameterizing elliptic curves with some additional torsion data.

**2.2. Results.** Let us shortly summarize our main results.

2.2.1. *Separation of Families.* Our first main result is the negative answer to question (a'). We proceed by generalizing ideas used by Suwa [**Suw69**] in his topological classification of complex bielliptic surfaces. His classification can be seen, from a modern perspective, as relying on two facts: the invariance of the plurigenera under deformation and the knowledge of the first homology groups of complex bielliptic surfaces, which are as well invariant under deformation. The first fact is equivalent to the invariance of the order of the canonical bundle. Moreover, the first homology group of a complex bielliptic surface is isomorphic to its Néron-Severi group. Thus, we proceed by showing that for a bielliptic surface $X$ over an algebraically closed fields $k$ of characteristic different from 2 and 3, the following two facts hold (see Lemma 1.9 in Chapter 2):

(a) The order of the canonical bundle $\omega_X$ is invariant under deformation.
(b) The Néron-Severi group $\mathrm{NS}(X)$ is invariant under deformation.

Since the values of these invariants taken together are different for different types of bielliptic surfaces, the desired result follows. More precisely, we obtain the following result (see Theorem 1.8 in Chapter 2).

THEOREM 2.1. *Let* $\pi\colon \mathcal{X} \to S$ *be a family of bielliptic surfaces over a connected Noetherian base scheme* $S$ *over* $\mathbb{Z}[\frac{1}{6}]$*. If a geometric fiber of* $\pi$ *is a bielliptic surface of type* $(i)$*, then every geometric fiber of* $\pi$ *is a bielliptic surface of type* $(i)$*.*

Form this theorem, it follows that the moduli stack $\mathcal{M}_{biell}$ splits as described by the next result (see Proposition 2.2 in Chapter 2).

PROPOSITION 2.2. *The stack of bielliptic surfaces* $\mathcal{M}_{biell}$ *splits into a disjoint union*

$$\mathcal{M}_{biell} = \bigsqcup \mathcal{M}^{(i)}_{biell},$$

*where* $\mathcal{M}^{(i)}_{biell}$ *denotes the substack whose objects are families of bielliptic surfaces of type* $(i)$*.*

This proposition, in turn, allow us to study the moduli stacks $\mathcal{M}^{(i)}_{biell}$ individually. Accordingly, we turn our attention to the study of the moduli stack $\mathcal{M}^{(a1)}_{biell}$.

2.2.2. *Morphism from Modular Stacks.* Following Tsuchihashi [**Tsu79**], our strategy for studying the moduli stack $\mathcal{M}^{(a1)}_{biell}$ consist in relating this moduli stack to products of modular stacks parameterizing elliptic curves with extra torsion data.

In Section 2.2 of Chapter 2, we show that, by choosing a pair of relative elliptic curves with some extra torsion data on one of them, it is possible to construct a family of bielliptic surfaces whose fibers are bielliptic surfaces of type $(a1)$. In this manner, we obtain a morphism of stacks

$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)},$$

where $\mathcal{Y}_1(2)[1/6]$ denotes the algebraic stack over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ parameterizing elliptic curves $\mathcal{E}/S$ together with a $\Gamma_1(2)$-*structure* and $\mathcal{M}_{1,1}[1/6]$ the Deligne-Mumford stack over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ of elliptic curves (see Corollary 2.6 in Chapter 2).

The moduli stacks on the left-hand side have been well studied (see for example [**KM85**], [**DR73**]) and are known to have nice properties (e.g., both are algebraic stacks). If the morphism $\varphi$ turns out to be an isomorphism of stacks, then the moduli stack $\mathcal{M}_{biell}^{(a1)}$ will share some of this nice properties.

We take a first step in the study of the morphism $\varphi$ and show that it is in fact an epimorphism of stacks. This is Theorem 2.15 in Chapter 2. Unfortunately, we were not able to answer the question whether the morphism $\varphi$ is fully faithful, from which it would follow that $\varphi$ is an isomorphism of stacks.

2.2.3. *Marked and Numerically Polarized Bielliptic Surfaces.* In section 3 of Chapter 2, we present some results on marked and numerically polarized bielliptic surfaces, respectively. We considered marked and numerically polarized bielliptic surfaces for two reasons.

Firstly, we wanted to obtain a better behaved moduli stack than $\mathcal{M}_{biell}$. Since automorphism groups of bielliptic surfaces are not finite (see, for example, [**Mar20**]) the moduli stack $\mathcal{M}_{biell}$ of bielliptic surfaces can not be a Deligne-Mumford stack. Our original aim in studying families of bielliptic surfaces with extra structure was to obtained a moduli stack which could, in principle, be a Deligne-Mumford stack. However, as it turns out, every automorphism of a bielliptic surface fixes the Néron-Severi lattice (see Proposition 3.3 in Chapter 2). Consequently, a marking or a numerical polarization on a bielliptic surface does not ridigify the moduli problem.

Secondly, every bielliptic surfaces $X$ comes naturally equipped with a *marking*, that is, an isometry $\phi\colon U \to \mathrm{Num}(X)$, where $\mathrm{Num}(X)$ denotes the Néron-Severi lattice of $X$ and $U$ the hyperbolic lattice. Furthermore, from such a marking one can construct a unique numerical polarization of degree 2 on $X$. Thus, the question whether this extra structure should be taken into account when defining moduli spaces of bielliptic surfaces arises naturally.

Accordingly, we define the moduli stacks $\mathcal{M}_U^{(a1)}$ and $\mathcal{M}_{d,\mathrm{Num}}^{(a1)}$ of marked and of numerically $d$-polarized bielliptic surfaces of type $(a1)$, respectively. Moreover, our results show that certain morphisms between the different moduli spaces can be defined, as given by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] & \xrightarrow{\varphi_U} & \mathcal{M}_U^{(a1)} \\
 & \searrow^{\varphi_{2,\mathrm{Num}}} & \downarrow \psi_{2,\mathrm{Num}} \\
 & & \mathcal{M}_{2,\mathrm{Num}}^{(a1)} \\
 & \searrow^{\varphi} & \downarrow \mathit{forget} \\
 & & \mathcal{M}^{(a1)}.
\end{array}
$$

Both $\mathcal{M}_U^{(a1)}$ and $\mathcal{M}_{d,\mathrm{Num}}^{(a1)}$ are good candidates for being the "right" moduli space of bielliptic surfaces of type $(a1)$, since the extra structure required on them is naturally found in bielliptic surfaces. Moreover, we believe that the extra structure on these moduli stacks could be used to obtain more information about the morphisms $\varphi_U$ and $\varphi_{2,\mathrm{Num}}$, respectively. We, however, do not investigate this question further.

Although our study of marked bielliptic surfaces did not lead to the desired goal, we decided to include our results, as we think they still are of some interest for the study of moduli spaces of bielliptic surfaces.

2.2.4. *Further Results.* Finally, let us mention some further results.

- In Section 2.5 of Chapter 1 we compute the Hodge-Witt cohomology groups for bielliptic surfaces in characteristic different from 2 and 3. We use these groups to calculate the Brauer group of these surfaces in Section 1.5 of Chapter 2. As it turns out, the Brauer group coincides with the torsion subgroup of the Néron-Severi group. This was already known for complex bielliptic surfaces.
- In Section 3 of Chapter 1 we prove that the Neron-Severi lattice of a bielliptic surface in arbitrary characteristic is a unimodular lattice, which is isomorphic to the hyperbolic lattice. This allows us to generalize some result of Serrano [**Ser90**] on divisors of complex bielliptic surfaces.
- Every bielliptic surfaces admit a canonical cover, which is an Abelian surfaces. In Section 1.4.3 of Chapter 2 we show an analogous result for families of bielliptic surfaces in characteristic different from 2 and 3: under some mild assumptions, every such family admits, étale locally, a cover which is an Abelian scheme.

## Acknowledgments

First of all, I want to thank my doctoral advisor Christian Liedtke for his support, encouragement and patience, and for proposing the topic of this dissertation.

I also want to thank Gebhard Martin for taking the time to answer all my questions. Furthermore I would like to thank my past and current colleagues at the Department of Algebra at the Technical University Munich. In particular, thanks to Kai Behrens, Frank Gounelas, Oliver Gregory, Roberto Laface and Claudia Stadlmayr. Thank you all for all the helpful discussions.

Finally, I want to thank my family. Every day and every minute I spent working on this thesis was a day and a minute I took from you. I apologize for that.

CHAPTER 1

# Bielliptic Surfaces

## 1. (Quasi-)Bielliptic Surfaces

In this section we introduce bielliptic and quasi-bielliptic surfaces and recall some general facts about them. The main focus, however, is on bielliptic surfaces.

After recalling the definition of (quasi-)bielliptic surfaces and the canonical bundle formula, we present the structure theorem of bielliptic surfaces. Finally, we introduce the canonical cover of a bielliptic surface and its associated Jacobian fibration.

**1.1. Preliminaries.** Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p \geq 0$. According to the *Kodaira-Enriques classification* of surfaces, $X$ belongs to one of four classes depending on the value of its Kodaira dimension

$$\kappa(X) := \operatorname{tr.deg}_k \bigoplus_{n=0}^{\infty} H^0(X, \omega_X^{\otimes n}) - 1 \in \{-1, 0, 1, 2\}.$$

The surfaces of Kodaira dimension zero can be further classified into four classes:

- K3 surfaces.
- Enriques surfaces.
- Abelian surfaces.
- (Quasi-)Bielliptic surfaces.

We are interested in the last class of surfaces of Kodaira dimension zero.

Definition 1.1. A *(quasi-)bielliptic surface* is a smooth and projective surface $X$ of finite type over an algebraically closed field $k$ of characteristic $p \geq 0$ such that

$$\omega_X \equiv \mathcal{O}_X \quad \text{and} \quad b_2(X) = 2,$$

where $\equiv$ denotes numerical equivalence, and $b_i$ denotes the i-th étale or crystalline Betti number.

Recall that for a surface $X$ one has *Noether's formula*

$$\chi(\mathcal{O}_X) = \frac{1}{12}((K^2) + c_2,$$

where $K$ denotes the canonical divisor of $X$ and $c_2$ the second Chern class of the tangent sheaf of $X$. Moreover, if

$$q(X) \coloneqq \dim H^1(X, \mathcal{O}_X) \quad \text{and}$$
$$p_g(X) \coloneqq \dim H^2(X, \mathcal{O}_X) = \dim H^0(X, \omega_X)$$

denote the *irregularity* and the *geometric genus* of $X$, respectively, Noether's formula can be rewritten as follows (see, for example, [**Băd01**, Chapter 5]):

$$10 - 8q + 12p_g = (K^2) + b_2 + 2\Delta, \tag{1.1}$$

where $\Delta \coloneqq 2q - b_1$ is the "non-classical" term which measures the non-reducedness of the identity component $\mathbf{Pic}^0_{X/k}$ of the Picard scheme $\mathbf{Pic}_{X/k}$ of $X$, as explained in the remark below.

If $X$ is a (quasi-)bielliptic surface, the solutions of equation (1.1) show that $X$ has the following invariants:

$$b_1(X) = 2, \quad \chi(\mathcal{O}_X) = 0 \quad \text{and} \quad \begin{cases} q(X) = 1, & p_g(X) = 0, \quad \Delta = 0 \quad \text{or} \\ q(X) = 2, & p_g(X) = 1, \quad \Delta = 2. \end{cases}$$

Remark 1.2. Over a field of characteristic zero the identity component of the Picard scheme $\mathbf{Pic}^0_{X/k}$ is always reduced. Over a field of positive characteristic this does not hold in general. The dimension of the (classical) Picard variety $(\mathbf{Pic}^0_{X/k})_{red}$ is given by $s(X) \coloneqq \dim(\mathbf{Pic}^0_{X/k})_{red} = b_1(X)/2$ and the dimension of the tangent space at the identity of the Picard scheme $\mathbf{Pic}^0_{X/k}$ is given by $\dim T_{\mathbf{Pic}^0_{X/k},0} = \dim H^1(X, \mathcal{O}_X) = q(X)$. Thus, we have

$$q(X) - s(X) = \dim T_{\mathbf{Pic}^0_{X/k},0} - \dim T_{(\mathbf{Pic}^0_{X/k})_{red},0} \geq 0.$$

Moreover, by [**Mum66**, Lecture 27] the inequality $0 \leq q(X) - s(X) \leq p_g(X)$ always holds. Thus, setting

$$\Delta = 2q - b_1 = 2(q - s),$$

we see that $0 \leq \Delta \leq 2p_g(X)$ and that $\Delta = 0$ if and only if $\mathbf{Pic}^0_{X/k}$ is reduced.

In particular, if $X$ is a (quasi-)bielliptic surface, then $\mathbf{Pic}^0_{X/k}$ is reduced if and only if $p_g(X) = 0$. The latter is always the case, if the characteristic of the ground field $k$ is different from 2 and 3. However, there are some (quasi-)bielliptic surfaces in characteristic 2 and 3 for which $p_g(X) = 1$ holds.

As mentioned in the introduction, one important property of (quasi-)bielliptic surfaces is the fact that they are *fibered surfaces*, that is, they carry the structure of a *genus one fibration*, as defined below. In fact, these surfaces can be fibered in two different ways, which are transversal to each other.

Definition 1.3. Let $X$ be a smooth projective surface.

(1) A surjective morphism $f\colon X \to B$ from $X$ to a smooth projective curve $B$ is called a *genus one fibration* if $f_*\mathcal{O}_X \cong \mathcal{O}_B$ and all but finitely many fibers of $f$ are integral curves of arithmetic genus one. If, moreover, the fibers of $f$ do not contain any $(-1)$-curves, then $f$ is called a *relatively minimal* genus one fibration.
(2) A relatively minimal genus one fibration is called *elliptic* if its general fiber is smooth and *quasi-elliptic* otherwise.
(3) A genus one fibration $f\colon X \to B$ is said to be *Jacobian* if it admits a *section*, that is, a morphism $\sigma\colon B \to S$ such that $f \circ \sigma = \mathrm{id}_B$.

Before stating the next theorem let us recall the definition of the Albanese variety.

Definition 1.4. Let $X$ be a smooth projective variety and let $x \in X$ be a fixed closed point. The *Albanese variety* $\mathrm{Alb}(X)$ of $X$ is an Abelian variety together with a morphism $\alpha\colon X \to \mathrm{Alb}(X)$ with $\alpha(x) = 0_{\mathrm{Alb}(X)}$, where $0_{\mathrm{Alb}(X)}$ denotes the identity element of $\mathrm{Alb}(X)$, such that the following universal property holds: for every morphism $f\colon X \to B$ from $X$ to an Abelian variety $B$ with $f(x) = 0_B$, there exist a unique homomorphism $g\colon \mathrm{Alb}(X) \to B$ of Abelian varieties such that $g \circ \alpha = f$. The morphism $\alpha\colon \mathrm{Alb}(X) \to X$ will be called the *Albanese map*.

Being characterized by a universal property, it is clear that the Albanese variety $(\mathrm{Alb}(X), \alpha)$ is unique up to unique isomorphism. Moreover, it can be shown that it is isomorphic to the dual of the Picard variety $(\mathbf{Pic}^0_{X/k})_{red}$, that is, $\mathrm{Alb}(X) \cong (\mathbf{Pic}^0_{X/k})_{red}{}^\vee$ (see, for example, [**Băd01**, Theorem 5.3]). In particular, we have

$$\dim \mathrm{Alb}(X) = \dim(\mathbf{Pic}^0_{X/k})_{red} = b_1(X)/2.$$

The first structure of a genus one fibration on a (quasi-)bielliptic surface $X$ is given by the Albanese map. More precisely, we have the following result.

Theorem 1.5 ([**BM77**], Proposition p. 26, Proposition 5). *Let $X$ be a (quasi-)bielliptic surface. Then $b_1(X) = 2$, hence $\mathrm{Alb}(X)$ is an elliptic curve and the Albanese map*

$$f\colon X \to \mathrm{Alb}(X)$$

*is a genus one fibration. Moreover, the fibers of $f$ are either all smooth or all singular rational curves, each having exactly one singular point that is an ordinary cusp.*

This allows us to distinguish between *bielliptic* surfaces, for which the Albanese fibration is elliptic, and *quasi-bielliptic* surfaces, for which the Albanese fibration is quasi-elliptic.

Definition 1.6. Let $X$ be a (quasi-)bielliptic surface. We call $X$ *bielliptic* (resp. *quasi-bielliptic*), if the Albanese fibration $f\colon X \to \mathrm{Alb}(X)$ is elliptic (resp. quasi-elliptic).

Remark 1.7. By [**Băd01**, Theorem 7.18] quasi-elliptic fibrations only exist in characteristic 2 or 3 and the general fiber of a quasi-elliptic fibration is a singular rational curve with exactly one ordinary cusp.

Theorem 1.8 ([**BM77**], Theorem 3, § 2). *Let* $f\colon X \to \mathrm{Alb}(X)$ *be a bielliptic or quasi-bielliptic surface. Then there is an elliptic fibration* $g\colon X \to \mathbb{P}^1$*, which is transversal to* $f$ *(i.e., every fiber of* $g$ *intersect every fiber of* $f$ *positively).*

Since the Albanese fibration $f\colon X \to \mathrm{Alb}(X)$ of a bielliptic or quasi-bielliptic surface $X$ is given by the Albanese map, it is canonically defined, that is, it is unique up to unique isomorphism. It is then natural to ask how unique is the second elliptic fibration $g\colon X \to \mathbb{P}^1$ of $X$. This is answered by the following result.

Proposition 1.9. *The fibration* $g\colon X \to \mathbb{P}^1$ *transversal to* $f$ *is unique in the sense that any other such fibration differs from* $g$ *by an automorphism of* $\mathbb{P}^1$.

To prove Proposition 1.9 we will used a result about the *Picard number* $\rho(X) \coloneqq \mathrm{rankNS}(X)$ of $X$. We state this result as a separated lemma due to its importance and for later use.

Lemma 1.10. *Let* $X$ *be a bielliptic or quasi-bielliptic surface and let* $\rho(X) \coloneqq \mathrm{rankNS}(X)$ *be the* Picard number *of* $X$*. Then,* $\rho(X) = b_2(X) = 2$.

Proof. Let $f\colon X \to \mathrm{Alb}(X)$ be the Albanese fibration of $X$. Then, a closed fiber $F$ of $f$ and a hyperplane section $H$ on $X$ are linearly independent in $\mathrm{NS}(X)_{\mathbb{Q}} \coloneqq \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, since $(F^2) = 0$ and $(H^2) > 0$. This together with the Igusa-Severi inequality (cf. [**Igu60**]) $\rho(X) \leq b_2(X) = 2$ implies the claim. □

Proof of Proposition 1.9. Let $f\colon X \to B \coloneqq \mathrm{Alb}(X)$ be the Albanese fibration of $X$. Denote by $F_0$ a fiber of $f$ over some closed point $t_0 \in B$ and let $G$ be a non-multiple fiber of $g$. Then, the classes of $F_0$ and $G$ in $\mathrm{NS}(X)_{\mathbb{Q}} \coloneqq \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ are linearly independent, since $(F_0^2) = 0$ and $(G \cdot F_0) > 0$. Since $\rho(X) = 2$ by Lemma 1.10, the classes of $F_0$ and $G$ form a basis of $\mathrm{NS}(X)_{\mathbb{Q}}$.

Now let $g'\colon X \to \mathbb{P}^1$ be a second elliptic fibration transversal to $f$. Take a non-multiple fiber $G'$ of $g'$. It follows that

$$(G' \cdot F_t) > 0 \ \text{ for all } t \in B \ \text{ and } \ (G'^2) = 0. \tag{1.2}$$

Now write $[G'] = \alpha[F_0] + \beta[G]$ with $\alpha, \beta \in \mathbb{Q}$. From (1.2) it follows that $\alpha = 0$ and $\beta > 0$. Thus, $[G'] = \beta[G]$. It follows that $(G \cdot G') = 0$ and thus $G'$ must be contained in one of the fibers of $g$, since $G'$ is connected. Moreover, since $g$ has only finitely many multiple fibers, we may assume that $G' \subseteq G_0$, where $G_0$ is a irreducible and reduced fiber of $g$. Since both $G'$ and $G_0$ are irreducible and reduced, we get $G' = G_0$. Now, since $g$ is a fibration over $\mathbb{P}^1$, all fibers of $g$ are linearly equivalent and the same is true for $g'$. From $G' = G_0$ it follows that all fibers of $g$ are linearly equivalent to all fibers of $g'$. Thus, $g$ and $g'$ are the same fibration. □

Finally, let us state the following result for later use.

Proposition 1.11. *Let $X$ be a bielliptic or quasi-bielliptic surface. Then on $X$ there are no $(-2)$-curves, i.e., irreducible rational curves $C$ with $(C^2) = -2$.*

Proof. Let $f\colon X \to \mathrm{Alb}(X)$ be the Albanese fibration and $C$ an irreducible curve on $X$ with $(C^2) = -2$. Then $C$ can not be contained in any fiber of $f$, since all the fibers of $f$ are irreducible by Theorem 1.5. It follows that the rational curve $C$ would have to dominate the elliptic curve $\mathrm{Alb}(X)$. But this can not be the case, by Hurwitz's formula (cf. [**Har77**], Corollary 2.4 and Example 2.5.4). □

**1.2. The Canonical Bundle Formula.** The canonical sheaf of a bielliptic or quasi-bielliptic surface plays a central role in their classification. As it turns out, to determine the canonical sheaf of a bielliptic or quasi-bielliptic surface one can use the fact that these surfaces have the structure of a genus one fibration. In general, the canonical sheaf of a relatively minimal genus one fibrations is determined by the so called *canonical bundle formula.* To state this formula we will now introduce some notation and refer to the original work of Bombieri and Mumford [**BM77**, §1] or to Badescu's Book [**Băd01**, Chapter 7] for the details.

Definition 1.12. Let $X$ be a minimal surface and $K = \mathcal{O}_X(\omega_X)$ be a canonical divisor. An effective divisor $D = \sum_{i=1}^{r} n_i E_i > 0$ on $X$ is called a *curve of canonical type* if $(K \cdot E_i) = (D \cdot E_i) = 0$ holds for all $i = 1, \dots r$. If $D$ is also connected and the greatest common divisor of the integers $n_1, \dots n_r$ is equal to 1, then $D$ is called an *indecomposable curve of canonical type.*

It is not difficult to show that every fiber of an elliptic or quasi-elliptic fibration $f\colon X \to B$ is a curve of canonical type (cf. [**Băd01**, p. 91]). Furthermore, among the fibers of $f$, one can distinguish between multiple and non-multiple fibers. Notice first that all but finitely many fibers of $f$ are integral curves. These are the non-multiple fibers of $f$. Indeed, since $B$ is a smooth curve, the condition $f_*\mathcal{O}_X \cong \mathcal{O}_B$ is equivalent to the rational function field $k(B)$ of $B$ being algebraically closed in the rational function field $k(X)$ of $X$. Then, the assertion follows from [**Băd01**, Corollary 7.3]. Thus, at finitely many closed points $b_1, \dots, b_r \in B$ the fibers $F_{b_i} = f^{-1}(b_i)$ are *multiple* fibers, that is, they are of the form

$$F_{b_i} = f^{-1}(b_i) = m_i P_i,$$

with $m_i \geq 2$ and $P_i$ is an idecomposable curve of canonical type.

Moreover, we have a decomposition of the first direct image $R^1 f_* \mathcal{O}_X$ of $\mathcal{O}_X$ as follows:

$$R^1 f_* \mathcal{O}_X \cong L \oplus T,$$

where $L$ is a locally free sheaf of finite rank and $T = Tors(R^1 f_* \mathcal{O}_X)$ is an $\mathcal{O}_B$-module supported at the points $b \in B$ at which $\dim H^0(F_b, \mathcal{O}_{F_b}) \leq 2$. Moreover, $\mathrm{Supp}(T) \subseteq \{b_1, \dots, b_r\}$. In particular, $T$ is an $\mathcal{O}_B$-module of finite length.

Definition 1.13. The fibers of $f$ over $\mathrm{Supp}(T)$ are called *wild fibers.*

We are now ready to state the *canonical bundle formula.*

THEOREM 1.14 ([**BM77**], Theorem 2, § 1). *Let $f\colon X \to B$ be an elliptic or quasi-elliptic fibration and let $R^1 f_* \mathcal{O}_X = L \oplus T$ be the decomposition introduced above. Then,*

$$\omega_X \cong f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X \left( \sum_{i=1}^{r} a_i P_i \right),$$

*where*

(a) $m_i P_i = F_{b_i}$ $(i = 1, \dots r)$ *are all the multiple fibers of* $f$,
(b) $0 \leq a_i < m_i$,
(c) $a_i = m_i - 1$ *if* $F_{b_i}$ *is not a wild fiber, and*
(d) $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + l(T)$, *where* $l(T)$ *is the length of* $T$.

If we now apply the canonical bundle formula to a bielliptic or quasi-bielliptic surface, we obtain the following Corollary.

COROLLARY 1.15. *Let $f\colon X \to B = \mathrm{Alb}(X)$ be a bielliptic or a quasi-bielliptic surface. Then*

(i) $\omega_X \cong f^*(L^{-1} \otimes \omega_B) \cong f^*(L^{-1})$.
(ii) $L = R^1 f_* \mathcal{O}_X$ *and* $\deg(L) = 0$.
(iii) *The order of* $L$ *in* $\mathrm{Pic}(B)$ *is the same as the order of* $\omega_X$ *in* $\mathrm{Pic}(X)$.
(iv) $\omega_X \in \mathrm{Pic}^0(X)$, *that is,* $\omega_X$ *is algebraically trivial.*

PROOF. Statements (i) and (ii) follow from the canonical bundle formula and the proof of Theorem 1.5, which can be found in [**BM77**, Proposition 5]. We recall the argument here. We have $\chi(\mathcal{O}_X) = 0$ and $\omega_X \equiv 0$, since $X$ is a bielliptic or a quasi-bielliptic surface. Moreover, since $B = \mathrm{Alb}(X)$ is an elliptic curve, we have $p_a(B) = 1$ and $\omega_B \cong \mathcal{O}_B$. By applying Theorem 1.14, we see that

$$\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + l(T) = l(T) \geq 0$$

and

$$\omega_X \cong f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X \left( \sum_{i=1}^{r} a_i P_i \right) \equiv 0.$$

Then, for any closed point $b \in B$ one has

$$l(T) \cdot f^{-1}(b) + \sum_{i=1}^{r} a_i P_i \equiv 0.$$

Since the left-hand side is an effective divisor which is numerically equivalent to zero, it has to be the zero divisor. Hence, $l(T) = a_i = 0$. In particular, the fibration $f$ does not have any multiple fibers. This shows (i) and (ii).

Since $f_*\mathcal{O}_X \cong \mathcal{O}_B$, the projection formula (cf. [**Har77**, Exercise 5.1(d)]) implies that the induced map

$$f^* \colon \mathrm{Pic}(B) \to \mathrm{Pic}(X)$$

is injective. Therefore, the order of $\omega_X \cong f^*(L^{-1})$ in $\mathrm{Pic}(X)$ equals the order of $L^{-1}$ in $\mathrm{Pic}(B)$, which is equal to the order of $L$ in $\mathrm{Pic}(B)$. This proves (iii).

From (ii) we have $L \in \mathrm{Pic}^0(B)$. Hence, $L^{-1} \in \mathrm{Pic}^0(B)$. Since $f$ is flat and by [**Ful84**, Proposition 10.3(b)] flat pullback preserves algebraic equivalence, we get that $\omega_X \cong f^*(L^{-1})$ is algebraically equivalent to zero, i.e., $\omega_X \in \mathrm{Pic}^0(X)$. □

**1.3. Structure Theorem for Bielliptic Surfaces.** As we have seen, bielliptic surfaces admit two different transversal elliptic fibrations: the Albanese fibration, given by the Albanese map, and a second fibration over $\mathbb{P}^1$. These two fibrations can be used to described bielliptic surfaces in a very explicit manner. As it turns out, every bielliptic surface can be written as a quotient of the product of two elliptic curves by a finite Abelian group scheme. More precisely, we have the following *structure theorem*.

Theorem 1.16 ([**BM77**], Theorem 4, § 3). *Every bielliptic surface $X$ is of the form*

$$X = (E \times F)/G,$$

*where $E$ and $F$ are elliptic curves, $G$ is a finite subgroup scheme of $E$, and $G$ acts on the product $E \times F$ by*

$$g.(x, y) = (x + g, \alpha(g)(y))$$

*for some suitable injective homomorphism $\alpha \colon G \to \mathrm{Aut}(F)$. Moreover, the two elliptic fibration of $X$ are given by*

$$f \colon X \to E/G \text{ (elliptic curve) } \text{ and } \; g \colon X \to F/\alpha(G) \cong \mathbb{P}^1.$$

This theorem can be used to classify all possible bielliptic surfaces and leads to the Bagnera-DeFranchis list.

In order to fix notation, we give here a rough sketch of the idea that leads to the list of Bagnera-DeFrancis and refer to [**BM77**, p. 36] for details:

- By choosing a base point $0_F \in F$ one gets an isomorphism $\mathrm{Aut}(F) \cong F \rtimes \mathrm{Aut}(F, 0_F)$, where $F$ is seen as the subgroup of translations of $F$ and $\mathrm{Aut}(F, 0_F)$ denotes the subgroup of automorphism fixing the identity element $0_F$ on $F$.
- Since $E_0/\alpha(G) \cong \mathbb{P}^1$, the image $\alpha(G)$ of $G$ in $\mathrm{Aut}(F)$ is not entirely contained in the group of translations $F$. Moreover, $\alpha(G)$ is commutative, as the group $G$ is commutative.

- Recall that the maximal Abelian subgroups of $\mathrm{Aut}(F, 0_F)$ are cyclic of order $2, 4$ or 6 (see, for example, [**Sil09**, Theorem III.10.1 and Theorem A.1.2]). Then, since $\alpha(G) \not\subset F$, there is a element $g \in G$ such that $\alpha(g)$ generates the image of the subgroup $\alpha(G)$ in the quotient group $\mathrm{Aut}(F)/F \cong \mathrm{Aut}(F, 0_F)$.
- Thus, after replacing $0_F$ by a fixed point of $\alpha(g)$, it follows that $\alpha(G)$ is a direct product
$$\alpha(G) \cong G_0 \times \mathbb{Z}/n\mathbb{Z}, \quad n \in \{2, 3, 4, 6\},$$
where $G_0$ is a finite subgroup scheme of translations of $F$ and $\mathbb{Z}/n\mathbb{Z} \cong \langle \phi \rangle$ is generated by an automorphism $\phi$ of the elliptic curve $F$ of order $n$.
- Finally, since $G_0$ must commute with $\phi$, we have that $G_0$ has to be contained in the fixed subgroup $F^\phi \subset F$ of $F$, that is,
$$G_0 \subset F^\phi := \{x \in F \mid \phi(x) = x\}.$$
This fixed subgroup can be calculated explicitly (cf. [**Lan87**, Appendix 1 by J. Tate]) and leads to the Bagnera-DeFranchis list classifying all possible bielliptic surfaces.

In the Table 1.1 below we present the Bagnera-DeFranchis list. It present the possible types of bielliptic surfaces according to the isomorphism class of the group $G$. For every type, we include the group of translation $G_0$, the characteristic of the ground field $k$ for which the given type of bielliptic surface exist as well as the $j$-invariant of the elliptic curve $F$. Moreover, after the table, we give an explicit description of the action of $G$ on the product $E \times F$ for every type of bielliptic surfaces (cf. [**BM77**, p.37] and [**Bǎd01**, List 10.27]).

LIST 1.17 (Bagnera-DeFranchis).

| Type | $G$ | $G_0$ | $n$ | $\mathrm{char}(k)$ | $j(F)$ |
|---|---|---|---|---|---|
| $(a1)$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{0\}$ | 2 | | |
| $(a2)$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | $\neq 2$ | |
| $(a3)$ | $\mu_2 \times (\mathbb{Z}/2\mathbb{Z})$ | $\mu_2$ | 2 | 2 | |
| $(b1)$ | $\mathbb{Z}/3\mathbb{Z}$ | $\{0\}$ | 3 | | 0 |
| $(b2)$ | $(\mathbb{Z}/3\mathbb{Z})^2$ | $\mathbb{Z}/3\mathbb{Z}$ | 3 | $\neq 3$ | 0 |
| $(c1)$ | $\mathbb{Z}/4\mathbb{Z}$ | $\{0\}$ | 4 | | 1728 |
| $(c2)$ | $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ | $\mathbb{Z}/2\mathbb{Z}$ | 4 | $\neq 2$ | 1728 |
| $(d)$ | $\mathbb{Z}/6\mathbb{Z}$ | $\{0\}$ | 6 | | 0 |

TABLE 1.1. Possible types of bielliptic surfaces.

REMARK 1.18. There is a similar structure theorem and a similar list for quasi-bielliptic surfaces (cf. [**BM76**, Theorem 1 and the list on p. 214]).

The action of the group $G$ on the product $E \times F$ in every case is given as follows:

$(a1)$ $(E \times F)/(\mathbb{Z}/2\mathbb{Z})$: where $\mathbb{Z}/2\mathbb{Z} \cong \langle a \rangle$ for a nontrivial 2-torsion point $a \in E[2]$ of $E$ and $\alpha(a) = -\operatorname{id}_F \in \operatorname{Aut}(F, 0_F)$. The action on the product is then given by

$$a.(x, y) = (x + a, -y).$$

$(a2)$ $(E \times F)/(\mathbb{Z}/2\mathbb{Z})^2$: where $(\mathbb{Z}/2\mathbb{Z})^2 \cong \langle b \rangle \times \langle a \rangle$ for two different nontrivial 2-torsion point $a, b \in E[2]$, $\alpha(a) = -\operatorname{id}_F \in \operatorname{Aut}(F, 0_F)$ and $\alpha(b)$ corresponds to a translation $t_c$ on $F$ by a nontrivial 2-torsion point $c \in F[2]$. In this case the action is given by

$$a.(x, y) = (x + a, -y) \text{ and}$$
$$b.(x, y) = (x + b, y + c).$$

$(a3)$ $(E \times F)/\mu_2 \times (\mathbb{Z}/2\mathbb{Z})$: where $\mu_2 \times \mathbb{Z}/2\mathbb{Z} \cong \mu_2 \times \langle a \rangle$ for a nontrivial 2-torsion point $a \in E[2]$ and $\alpha(a) = -\operatorname{id}_F \in \operatorname{Aut}(F, 0_F)$. The action in this case is given by

$$a.(x, y) = (x + a, -y)$$

and $\mu_2$ acts by translation on both factors. This case occurs only when $\operatorname{char}(k) = 2$.

$(b1)$ $(E \times F)/(\mathbb{Z}/3\mathbb{Z})$: where $\mathbb{Z}/3\mathbb{Z} \cong \langle a \rangle$ for a nontrivial 3-torsion point $a \in E$ and $\alpha(a) = \omega \in \operatorname{Aut}(F, 0_F)$ is an automorphism of order 3. This case is only possible when $j(F) = 0$. The action is given by

$$a.(x, y) = (x + a, \omega(y)).$$

$(b2)$ $(E \times F)/(\mathbb{Z}/3\mathbb{Z})^2$: where $(\mathbb{Z}/3\mathbb{Z})^2 \cong \langle b \rangle \times \langle a \rangle$ for two different nontrivial 3-torsion point $a, b \in E[3]$, $\alpha(a) = \omega \in \operatorname{Aut}(F, 0_F)$ is an automorphism of order 3 and $\alpha(b)$ corresponds to a translation $t_c$ on $F$ by a nontrivial 3-torsion point $c \in F[3]$ with $\omega(c) = c$. This case is only possible when $j(F) = 0$. The action is given by

$$a.(x, y) = (x + a, \omega(y)) \text{ and}$$
$$b.(x, y) = (x + b, y + c).$$

$(c1)$ $(E \times F)/(\mathbb{Z}/4\mathbb{Z})$: where $\mathbb{Z}/4\mathbb{Z} \cong \langle a \rangle$ for a nontrivial 4-torsion point $a \in E[4]$ and $\alpha(a) = i \in \operatorname{Aut}(F, 0_F)$ is an automorphism of order 4. This case is only possible when $j(F) = 1728$. The action is given by

$$a.(x, y) = (x + a, i(y)).$$

($c2$) $(E \times F)/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$: where $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \cong \langle b \rangle \times \langle a \rangle$ for a nontrivial 4-torsion point $a \in E[4]$ and a nontrivial 2-torsion point $b \in E[2]$, $\alpha(a) = i \in \mathrm{Aut}(E_0, 0)$ an automorphism of order 4 and $\alpha(b)$ corresponds to a translation $t_c$ on $F$ by a nontrivial 2-torsion point $c \in F[2]$ with $i(c) = c$. This case is only possible when $j(F) = 1728$. The action is given by

$$a.(x,y) = (x + a, i(y)) \text{ and}$$
$$b.(x,y) = (x + b, y + c).$$

(d) $(E \times F)/(\mathbb{Z}/6\mathbb{Z})$: where $\mathbb{Z}/6\mathbb{Z} \cong \langle a \rangle$ for a nontrivial 6-torsion point $a \in E[6]$ and $\alpha(a) = -\omega \in \mathrm{Aut}(F, 0_F)$ with $\omega$ an automorphism of order 3. This case is only possible when $j(F) = 0$. The action is given by

$$a.(x,y) = (x + a, -\omega(y)).$$

Theorem 1.16 can also be used to determine the order of the canonical sheaf $\omega_X$ in the Picard group $\mathrm{Pic}(X)$ of $X$, as explained in [**BM77**, §3, p. 37].

Proposition 1.19. *Let $X$ be a bielliptic surface. Then the order of the canonical sheaf $\omega_X$ of $X$ is given by the following table.*

| Type | $\mathrm{char}(k) \neq 2, 3$ | $\mathrm{char}(k) = 2$ | $\mathrm{char}(k) = 3$ |
|---|---|---|---|
| $(a)$ | 2 | 1 | 2 |
| $(b)$ | 3 | 3 | 1 |
| $(c)$ | 4 | 1 | 4 |
| $(d)$ | 6 | 3 | 2 |

Table 1.2. Order of $\omega_X$ in $\mathrm{Pic}(X)$.

In the following sections we will see that bielliptic surfaces with trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$ are the most *pathological* ones. As a first evidence for that we have the following easy result.

Corollary 1.20. *Let $X$ be a bielliptic surface. Then the following are equivalent:*

*(1) The identity component of the Picard scheme $\mathbf{Pic}^0_{X/k}$ is non-reduced.*
*(2) $p_g(X) = 1$.*
*(3) $\omega_X \cong \mathcal{O}_X$, that is, $\mathrm{ord}(\omega_X) = 1$.*

Proof. The equivalence of (1) and (2) was already discuss in Remark 1.2. If $p_g(X) = 1$, then $H^0(X, \omega_X) \neq 0$. This together with $\omega_X \equiv 0$ imply that $\omega_X \cong \mathcal{O}_X$. This proves that (2) implies (3). Finally, if $\omega_X \cong \mathcal{O}_X$, then $p_g(X) = \dim H^0(X, \omega_X) = \dim H^0(X, \mathcal{O}_X) = 1$. Hence, (3) implies (2). □

**1.4. The Canonical Cover of a Bielliptic Surface in Characteristic $p \notin \{2, 3\}$.** Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $\mathrm{char}(k) = p$ with $p \notin \{2, 3\}$. If $n$ denotes the order of the canonical sheaf $\omega_X$ of $X$ in $\mathrm{Pic}(X)$, then $p \nmid n$. We can then construct the *canonical cyclic cover* $\widetilde{X}$ of $X$ defined by $\omega_X$.

Indeed, if $\mu_n$ denotes the group scheme of $n$-roots of the unity over $X$, then by Kummer theory (cf. [**Mil80**, Chapter III,§4, p. 125]) there is a canonical identification between the étale cohomology group $H^1_{\text{ét}}(X, \mu_n)$ and the group of isomorphism classes of pairs $(L, \phi)$, where $L$ is an invertible sheaf on $X$ and $\phi\colon L^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ is a trivialization of the $n$-th power of $L$. The group $H^1_{\text{ét}}(X, \mu_n)$ can in turn be identified with the group of isomorphism classes of $\mu_n$-torsors over $X$ by [**Mil80**, Chapter III, Corollary 4.7]. Thus, from the isomorphism $\omega_X^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ by means of these identifications, we obtain up to isomorphism a $\mu_n$-torsor

$$\pi\colon \widetilde{X} \to X$$

over $X$. Since $k$ is algebraically closed with $\mathrm{char}(k) = p \nmid n$ and $X$ is a scheme over $\mathrm{Spec}\,(k)$, the group scheme $\mu_n$ is isomorphic (non-canonically) to the constant group scheme $\mathbb{Z}/n\mathbb{Z}$ over $X$. In particular, $\mu_n$ is a finite and étale group scheme of length $n$ over $X$. Hence, $\pi\colon \widetilde{X} \to X$ is a finite étale cover of $X$ of degree $n$. In fact, $\pi$ is a Galois cover of $X$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

More explicitly, the cover $\pi\colon \widetilde{X} \to X$ can be described as follows: let ${\omega_X}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ be a trivialization. Then $\mathcal{A} := \oplus_{i=0}^{n-1} {\omega_X}^{\otimes i}$ has a natural structure of an $\mathcal{O}_X$-algebra and defines a finite étale cover of degree $n$

$$\pi\colon \widetilde{X} := \underline{\mathrm{Spec}}_X \mathcal{A} \to X,$$

such that $\pi_*(\mathcal{O}_{\widetilde{X}}) \cong \mathcal{A} = \oplus_{i=0}^{n-1} {\omega_X}^{\otimes i}$. Note that $\widetilde{X}$ is uniquely defined up to isomorphism. Moreover, $\omega_{\widetilde{X}} \cong \pi^*\omega_X \cong \mathcal{O}_{\widetilde{X}}$ is trivial and there is a free action of the group $\mathbb{Z}/n\mathbb{Z}$ on $\widetilde{X}$, defined by twisting by $\omega_X$ on $\mathcal{A}$, such that the quotient $\widetilde{X}/(\mathbb{Z}/n\mathbb{Z})$ is naturally isomorphic to $X$.

By the Kodaira-Enriques classification of surfaces, $\widetilde{X}$ is an Abelian surface. Indeed, since $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$, we have $\kappa(\widetilde{X}) = 0$ and $p_g(\widetilde{X}) = 1$. Moreover, since $\pi$ is étale of degree $n$, we have $\chi(\mathcal{O}_{\widetilde{X}}) = n \cdot \chi(\mathcal{O}_X) = 0$. Thus $q(\widetilde{X}) = 2$. Therefore, $\widetilde{X}$ is an Abelian surface, since the characteristic of the ground field is different from 2 and 3.

Definition 1.21. Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p \notin \{2, 3\}$. By the *canonical Abelian cover* of $X$ we shall mean the unique Abelian surface $\widetilde{X}$ defined above together with the quotient morphism $\pi\colon \widetilde{X} \to X$.

Remark 1.22. In small characteristic, i.e., when $\mathrm{char}(k) \in \{2, 3\}$, even if the characteristic of the ground field does not divide the order of the canonical sheaf, it is not clear if the canonical cover of a bielliptic surface is an Abelian surface, since a surface with invariants $\kappa = 0$, $p_g = 1$ and $q = 2$ need not be an Abelian surface. Indeed, in characteristic 2 and 3 there exist bielliptic and quasi-bielliptic surfaces having those invariants.

**1.5. The Associated Jacobian Fibration of a Bielliptic Surface.** In general, to every genus one fibration, one can associate a Jacobian genus one fibration, see, for example [**CD89**, Proposition 5.2.5]. For bielliptic surfaces, this can be done using the theory of the relative Picard scheme. For the results on the relative Picard scheme we refer the reader to [**Kle05**] and [**BLR90**, Chapter 8].

First, note that a bielliptic surface $f\colon X \to B = \mathrm{Alb}(X)$ over $k$ is a projective and smooth elliptic fibration with (geometric) connected fibers, since $f$ is flat and its fibers are all smooth genus one curves by Theorem 1.5. It follows from the theory of the relative Picard functor (see, for example, [**BLR90**, §9.4, Proposition 4]) that the identity component $\mathbf{Pic}^0_{X/B}$ of the relative Picard scheme $\mathbf{Pic}_{X/B}$ of a bielliptic surface $f\colon X \to B$ is a projective Abelian $B$-scheme of relative dimension 1. In particular, $\mathbf{Pic}^0_{X/B} \to B$ is a Jacobian genus one fibration.

Definition 1.23. Let $f\colon X \to B = \mathrm{Alb}(X)$ be a bielliptic surface over $k$. Then the Jacobian genus one fibration $\mathbf{Pic}^0_{X/B} \to B$ is called the *Jacobian fibration associated to* $f\colon X \to B$ and we will denote it by

$$j\colon J_{X/B} \coloneqq \mathbf{Pic}^0_{X/B} \to B.$$

Remark 1.24. Our definition coincide in the case of smooth elliptic fibrations with the more general definition of the Jacobian fibration of a genus one fibration $f\colon X \to C$ as given in [**CD89**, Proposition 5.2.5]. Indeed, if $f\colon X \to C$ is a smooth elliptic fibration, then its generic fiber $X_\eta$ is a smooth genus 1 curve and its Jacobian $J(X_\eta) \coloneqq \mathbf{Pic}_{X_\eta/k(C)}$ is a smooth elliptic curve over the function field $k(C)$ of $C$. Thus, the Jacobian fibration $j\colon J \to C$ of $f$ as defined in [**CD89**, Proposition 5.2.5] is smooth and coincide therefore with the Néron model of the Jacobian $J(X_\eta)$ of $X_\eta$. By [**BLR90**, Theorem 9.5.1] $\mathbf{Pic}^0_{X/C}$ is a Néron model of its generic fiber, i.e., of the Jacobian $J(X_\eta)$ of $X_\eta$. Hence, both definitions coincide in the case of smooth elliptic fibrations.

Proposition 1.25. *Let $f\colon X \to B = \mathrm{Alb}(X)$ be a bielliptic surface over $k$. Then the associated Jacobian fibration $j\colon J_{X/B} \to B$ of $f$ is a bielliptic surface. Moreover, the order of the canonical sheaves $\omega_X$ and $\omega_{J_{X/B}}$ coincide.*

Proof. To simplify notation we set $J = J_{X/B}$. We first show that $R^1f_*\mathcal{O}_X \cong R^1j_*\mathcal{O}_J$. This follows from Definition 1.23 and the isomorphism $R^1f_*\mathcal{O}_X \cong \mathrm{Lie}(\mathbf{Pic}^0_{X/B})$, where $\mathrm{Lie}(\mathbf{Pic}^0_{X/B})$ denotes the Lie algebra of $\mathbf{Pic}^0_{X/B}$ (cf. [**LLR04**, Proposition 1.3] and [**LLR04**, Proposition 1.1(d)]). The same is true for $j$, i.e., $R^1j_*\mathcal{O}_J \cong \mathrm{Lie}(\mathbf{Pic}^0_{J/B})$. Moreover, there is a canonical isomorphism $J \cong \mathbf{Pic}^0_{J/B}$ by [**MFK94**, Proposition 6.9]. Thus, we get

$$R^1f_*\mathcal{O}_X \cong \mathrm{Lie}(\mathbf{Pic}^0_{X/B}) \cong \mathrm{Lie}(J) \cong \mathrm{Lie}(\mathbf{Pic}^0_{J/B}) \cong R^1j_*\mathcal{O}_J.$$

From the Leray spectral sequence for $f$ and $j$ we get

$$\chi(\mathcal{O}_X) = \chi(f_*\mathcal{O}_X) - \chi(R^1 f_*\mathcal{O}_X) = \chi(\mathcal{O}_B) - \chi(R^1 f_*\mathcal{O}_X)$$
$$= \chi(j_*\mathcal{O}_X) - \chi(R^1 j_*\mathcal{O}_X) = \chi(\mathcal{O}_J).$$

Thus, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J)$.

Now note that $f$ and $j$ are both smooth elliptic fibrations over the elliptic curve $B$. In particular, they don't have multiple fibers. Then, from the canonical bundle formula (Theorem 1.14) we get

$$\omega_J \cong j^*(L_j^{-1} \otimes \omega_B) \cong j^*(L_j^{-1}),$$

where $L_j = R^1 j_*\mathcal{O}_J$ is an invertible sheaf of degree $\chi(\mathcal{O}_J) = \chi(\mathcal{O}_X) = 0$. Hence, $\omega_J \in \mathrm{Pic}^0(J)$. In particular, we have $\omega_J \equiv 0$, from which it follows that $J$ is a minimal surface of Kodaria dimension 0. Moreover, by [**CD89**, Corollary 5.3.5], the Betti numbers of $X$ and $J$ coincide, i.e., $b_i(X) = b_i(J)$, $i \geq 0$. So $J$ is a bielliptic surface, since $\omega_J \equiv 0$, $b_2(J) = b_2(X) = 2$ and the fibers of $j\colon J \to B$ are smooth.

The claim about the order of the canonical sheaves follows from $R^1 f_*\mathcal{O}_X \cong R^1 j_*\mathcal{O}_J$ and the fact that the order of the canonical sheaf $\omega_X$ of a bielliptic surface $X$ coincides with the order of the invertible sheaf $R^1 f_*\mathcal{O}_X$, as stated in Corollary 1.15.

□

## 2. Cohomology of Bielliptic Surfaces

Given a smooth and projective variety $X$, one can define several different cohomology theories on it, depending on the nature of the ground field over which the variety is defined and the view point from which one decides to consider the variety. In this section we study different cohomology theories for bielliptic surfaces and present the corresponding cohomology groups. Most of the result we present are due to others and we will refer to the original works accordingly.

We refer to [**Lie16**, Section 1] for a short introduction to the different cohomology theories we will consider and to [**CDL20**, Section 0.10] for a more detailed discussion.

Note first that from the definition of a bielliptic surface and Poincaré duality in $l$-adic cohomology, where $l$ is a prime number, we have the following result.

PROPOSITION 2.1. *The $l$-adic Betti numbers of a bielliptic surface $X$ are as follows*

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $b_i(X)$ | 1 | 2 | 2 | 2 | 1 |

**2.1. Singular Cohomology and the Néron-Severi Group.** We first consider complex bielliptic surfaces.

LEMMA 2.2. *Let $X$ be a bielliptic surface over $\mathbb{C}$. Then the first singular homology group $H_1(X,\mathbb{Z})$ and the Néron-Severi group $\mathrm{NS}(X)$ are isomorphic.*

PROOF. According to the universal coefficient theorem for (singular) cohomology (see, for example, [**Hat02**, Theorem 3.2, Corollary 3.3]), the torsion of the second cohomology group and the torsion of the first homology group coincide, $H^2(X,\mathbb{Z})_{tors} \cong H_1(X,\mathbb{Z})_{tors}$. Since $b_1(X) = b_2(X)$, we get $H^2(X,\mathbb{Z}) \cong H_1(X,\mathbb{Z})$.

From the exact exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

one obtains the exact sequence

$$\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X).$$

Since $p_g(X) = \dim H^2(X,\mathcal{O}_X) = 0$, we have $H^2(X,\mathcal{O}_X) = 0$. It follows that the map $\delta\colon \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ is surjective. Therefore,

$$\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \cong \mathrm{Pic}(X)/\ker(\delta) \cong H^2(X,\mathbb{Z}).$$

Thus, $H_1(X,\mathbb{Z}) \cong \mathrm{NS}(X)$. □

The torsion of the first homology groups of the different types of complex bielliptic surfaces has been computed by Serrano in [**Ser91**, Theorem 4.3] and are given in Table 2.1.

| Case | Torsion of $H_1(X,\mathbb{Z})$ |
|---|---|
| $(a1)$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| $(a2)$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $(b1)$ | $\mathbb{Z}/3\mathbb{Z}$ |
| $(b2)$ | 0 |
| $(c1)$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $(c2)$ | 0 |
| $(d)$ | 0 |

TABLE 2.1. Torsion of $H_1(X,\mathbb{Z})$.

From Lemma 2.2 together with the results from Serrano, we can determine the Néron-Severi groups of complex bielliptic surfaces. These are given in Table 2.2.

| Type | $NS(X)$ |
|---|---|
| $(a1)$ | $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$ |
| $(a2)$ | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ |
| $(b1)$ | $\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $(b2)$ | $\mathbb{Z}^2$ |
| $(c1)$ | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ |
| $(c2)$ | $\mathbb{Z}^2$ |
| $(d)$ | $\mathbb{Z}^2$ |

TABLE 2.2. Néron-Severi group of complex bielliptic surfaces.

It is now easy to calculate the singular (co-)homology groups of $X$ in terms of the Néron-Severi group.

THEOREM 2.3. *Let $X$ be a bielliptic surface over $\mathbb{C}$ with Néron-Severi group* $\mathrm{NS}(X)$*. Then, $X$ has the following singular homology and cohomology groups.*

$$\begin{array}{ll} H_0(X,\mathbb{Z}) = \mathbb{Z} & H^0(X,\mathbb{Z}) = \mathbb{Z} \\ H_1(X,\mathbb{Z}) = \mathrm{NS}(X) & H^1(X,\mathbb{Z}) = \mathbb{Z}^2 \\ H_2(X,\mathbb{Z}) = \mathrm{NS}(X) & H^2(X,\mathbb{Z}) = \mathrm{NS}(X) \\ H_3(X,\mathbb{Z}) = \mathbb{Z}^2 & H^3(X,\mathbb{Z}) = \mathrm{NS}(X) \\ H_4(X,\mathbb{Z}) = \mathbb{Z} & H^4(X,\mathbb{Z}) = \mathbb{Z}. \end{array}$$

PROOF. Since $X$ is an orientable topological 4-manifold, we get the assertion about the (co-)homology groups for $i = 0, 4$. From Lemma 2.2 we have

$$H^2(X, Z) \cong H_1(X, Z) \cong \mathrm{NS}(X).$$

Using Poincaré duality and the universal coefficient theorems for cohomology one can compute the remaining (co-)homology groups. □

**2.2. $l$-adic Cohomology.** Let $l$ be a prime number. In what follows we describe the $l$-adic cohomology groups of a bielliptic surface over an algebraically closed field $k$.

2.2.1. *Characteristic zero.* Assume first that the characteristic of the ground field $k$ is zero. By the Lefschetz principle, we may assume $k = \mathbb{C}$. From the comparison theorem between singular and étale cohomology [**Mil80**, Ch. III, §3, Thm. 3.12] and the universal coefficient theorem for cohomology [**Hat02**, Theorem 3.2], we have

$$H^i_{\text{ét}}(X, \mathbb{Z}_l) \cong H^i(X, \mathbb{Z}_l) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(X, \mathbb{Z}), \mathbb{Z}_l) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}_l).$$

From this together with our knowledge of the singular cohomology groups we can compute the $l$-adic cohomology groups.

THEOREM 2.4. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic zero with Néron-Severi group* NS$(X)$*. Then, $X$ has the following $l$-adic cohomology groups.*

$$\begin{aligned} H^0_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l \\ H^1_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l^2 \\ H^2_{\text{ét}}(X, \mathbb{Z}_l) &= \text{NS}(X) \otimes \mathbb{Z}_l \\ H^3_{\text{ét}}(X, \mathbb{Z}_l) &= \text{NS}(X) \otimes \mathbb{Z}_l \\ H^4_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l. \end{aligned}$$

2.2.2. *Characteristic $p \geq 5$.* To calculate the $l$-adic cohomology groups of a bielliptic surface $X$ over a field $k$ of characteristic $p \geq 5$ one may argue as follows: as we will see in Section 5, bielliptic surfaces lift to characteristic zero, by Theorem 5.12. Moreover, in Section 1.3 of Chapter 2, we will show that the type of a bielliptic surface over $k$ is preserved under lifting (see Theorem 1.8). Then, the smooth base change theorem for étale cohomology ([**Mil80**, Ch. VI, 4, Cor. 4.2]) allows us to deduce the $l$-adic cohomology groups from the characteristic zero case.

2.2.3. *The second $l$-adic cohomology group in arbitrary characteristic.* For bielliptic surfaces over fields of characteristic 2 or 3, we can not use the argument above, since in that case we don't know if the surfaces lift to surfaces of the same type.

Nevertheless, the above result for the second $l$-adic cohomology group holds for every bielliptic surface in any characteristic, as the following theorem shows.

THEOREM 2.5. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p \geq 0$. Then, for every prime number $l \neq p$ we have*

$$H^2_{\text{ét}}(X, \mathbb{Z}_l) \cong H^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \cong \text{NS}(X) \otimes \mathbb{Z}_l.$$

*If $l = p$, the statement remains true after replacing the étale topology with the flat topology.*

PROOF. First, assume that $k$ is a field of characteristic zero. By the Lefschetz principle, we may assume $k = \mathbb{C}$. From the proof of Theorem 2.2 we know that NS$(X) \cong H^2(X, \mathbb{Z})$. By the comparison theorem between $l$-adic cohomology and singular cohomology we have

$$H^2_{\text{ét}}(X, \mathbb{Z}_l) \cong H^2(X, \mathbb{Z}_l).$$

Moreover, from the universal coefficient theorem we get

$$H^2(X, \mathbb{Z}_l) \cong H^2(X, \mathbb{Z}) \otimes \mathbb{Z}_l.$$

Thus, $H^2_{\text{ét}}(X, \mathbb{Z}_l) \cong \text{NS}(X) \otimes \mathbb{Z}_l$.

Now let us assume that $k$ is a field of positive characteristic $\mathrm{char}(k) = p > 0$. Here the result will follow from the fact that $\rho(X) = b_2(X)$. We have two cases:

(1) $l \neq p$: The Kummer sequence gives an exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \to T_l\mathrm{Br}(X) \to 0,$$

where $T_l\mathrm{Br}(X)$ denote the $l$-adic Tate module of the Brauer group of $X$. Moreover, $T_l\mathrm{Br}(X)$ is a finitely generated $\mathbb{Z}_l$-module of rank $t_l(X) := b_2(X) - \rho(X)$ (see, for example, [**Mil80**, Chapter V, Remark 3.29(d)]). Since $\rho(X) = b_2(X) = 2$, we have $T_l\mathrm{Br}(X) = 0$ and thus, $\mathrm{NS}(X) \otimes \mathbb{Z}_l \cong H^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \cong H^2_{\text{ét}}(X, \mathbb{Z}_l)$, since $k$ is algebraically closed.

(2) $l = p$: Here we have to work instead with the flat topology. From the Kummer exact sequence, we obtain the short exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_p \to H^2_{\text{fl}}(X, \mathbb{Z}_p(1)) \to T_pH^2_{\text{fl}}(X, \mathbb{G}_m) \to 0.$$

By [**Ill79**, Proposition II.5.12] $T_pH^2_{\text{fl}}(X, \mathbb{G}_m)$ is a free $\mathbb{Z}_p$-module. Let $t_p(X)$ denote its rank and set $h(x) := \dim_K(H^2(X, W\mathcal{O}_X) \otimes_W K)$. Then we have the Igusa-Artin-Mazur formula (see [**Ill79**, Proposition II.5.12]):

$$\rho(X) = b_2(X) - 2h(X) - t_p(X).$$

Since $\rho(X) = b_2(X) = 2$, we have $h(X) = t_p(X) = 0$. Thus, $T_pH^2_{\text{fl}}(X, \mathbb{G}_m) = 0$ and $\mathrm{NS}(X) \otimes \mathbb{Z}_p \cong H^2_{\text{fl}}(X, \mathbb{Z}_p(1)) \cong H^2_{\text{fl}}(X, \mathbb{Z}_p)$.

□

It turns out that Theorem 2.5 is enough to determine the $l$-adic cohomology groups of bielliptic surfaces in positive characteristic without lifting them to characteristic zero.

THEOREM 2.6. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p \geq 0$ with Néron-Severi group $\mathrm{NS}(X)$. Let $l \neq p$ be a prime number. Then, $X$ has the following $l$-adic cohomology groups.*

$$\begin{aligned}
H^0_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l \\
H^1_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l^2 \\
H^2_{\text{ét}}(X, \mathbb{Z}_l) &= \mathrm{NS}(X) \otimes \mathbb{Z}_l \\
H^3_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l^2 \oplus \mathrm{Ext}^1_{\mathbb{Z}_l}(H^2_{\text{ét}}(X, \mathbb{Z}_l), \mathbb{Z}_l) \\
H^4_{\text{ét}}(X, \mathbb{Z}_l) &= \mathbb{Z}_l.
\end{aligned}$$

PROOF. By [**Mil80**, Theorem V.3.23] we have $H^0_{\text{ét}}(X, \mathbb{Z}_l) = \mathbb{Z}_l = H^4_{\text{ét}}(X, \mathbb{Z}_l)$. Moreover, the Albanese map $f \colon X \to \mathrm{Alb}(X)$ induces an isomorphism $H^1_{\text{ét}}(\mathrm{Alb}(X), \mathbb{Z}_l) \cong H^1_{\text{ét}}(X, \mathbb{Z}_l)$ (see, for example, [**CDL20**, page 160]). Since the Albanese variety $\mathrm{Alb}(X)$ of $X$ is an elliptic curve, we obtain $H^1_{\text{ét}}(X,\mathbb{Z}_l)=\mathbb{Z}_l^2$. Finally, by Poincaré duality for $l$-adic cohomology [**Mil80**, Theorem VI.11.1], we have $H^3_{\text{ét}}(X,\mathbb{Z}_l)_{tors}\cong \operatorname{Ext}^1_{\mathbb{Z}_l}(H^2_{\text{ét}}(X,\mathbb{Z}_l),\mathbb{Z}_l)$. Since $b_3=2$, we obtain the result for the third cohomology group. □

REMARK 2.7. Unfortunately, we were not able to simplify the expression for the third cohomology group further, but we expect it to simplifies to $H^3_{\text{ét}}(X,\mathbb{Z}_l)=\operatorname{NS}(X)\otimes\mathbb{Z}_l$.

**2.3. Hodge and de Rham Cohomology.** Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p\geq 0$ and let $T_X=T_{X/k}:=\Omega^\vee_{X/k}$ denote the tangent sheaf of $X$. Moreover, set $h^i(T_X):=\dim_k H^i(X,T_X)$ and let $h^{i,j}:=\dim_k H^j(X,\Omega^i_{X/k})$ denote the Hodge numbers of $X$.

The Hodge and the de Rham cohomology groups of bielliptic surfaces were computed by Lang [**Lan79**] and later by Suwa [**Suw83**]. In what follows we present their results.

THEOREM 2.8 ([**Lan79**], Theorem 4.9). *The cohomology of the tangent and cotangent bundles of a bielliptic surface $X$ over a field of characteristic $\neq 2$ is given by the following table.*

| ord $\omega_X$ | $h^0(T_X)$ | $h^1(T_X)$ | $h^2(T_X)$ | $h^{0,1}$ | $h^{0,2}$ | $h^{1,0}$ | $h^{1,1}$ | $h^{1,2}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 2 | 2 | 1 | 2 | 4 | 2 |
| 2 | 1 | 2 | 1 | 1 | 0 | 1 | 2 | 1 |
| 3 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |
| 4 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |
| 6 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |

TABLE 2.3. Cohomology of the tangent and cotangent bundles.

Let $h^i_{\text{dR}}:=H^i_{\text{dR}}(X/k)$ denote the dimension of the i-th de Rham cohomology group.

THEOREM 2.9 ([**Lan79**], Theorem 4.10). *The de Rham cohomology of a bielliptic surface $X$ over a field of characteristic $\neq 2$ is given by the following table.*

| ord $\omega_X$ | $h^0_{dR}(X)$ | $h^1_{dR}(X)$ | $h^2_{dR}(X)$ | $h^3_{dR}(X)$ | $h^4_{dR}(X)$ |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 3 | 1 |
| 2 | 1 | 2 | 2 | 2 | 1 |
| 3 | 1 | 2 | 2 | 2 | 1 |
| 4 | 1 | 2 | 2 | 2 | 1 |
| 6 | 1 | 2 | 2 | 2 | 1 |

TABLE 2.4. Dimensions of the de Rham cohomology groups.

In characteristic 2 we have the following result.

THEOREM 2.10 ([**Lan79**], Theorem 4.11, p. 497). *The cohomology of the tangent and cotangent bundles and the de Rham cohomology of a bielliptic surface $X$ over a field of characteristic 2 is given by the following tables.*

| *Type* | $h^0(T_X)$ | $h^1(T_X)$ | $h^2(T_X)$ | $h^{0,1}$ | $h^{0,2}$ | $h^{1,0}$ | $h^{1,1}$ | $h^{1,2}$ |
|---|---|---|---|---|---|---|---|---|
| $(a1)$ | 2 | 4 | 2 | 2 | 1 | 2 | 4 | 2 |
| $(a3)$ | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 1 |
| $(b1)$ | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |
| $(b2)$ | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |
| $(c1)$ | 2 | 4 | 2 | 2 | 1 | 2 | 4 | 2 |
| $(d)$ | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 |

TABLE 2.5. Cohomology of the tangent and cotangent bundles.

| *Type* | $h^0_{dR}(X)$ | $h^1_{dR}(X)$ | $h^2_{dR}(X)$ | $h^3_{dR}(X)$ | $h^4_{dR}(X)$ |
|---|---|---|---|---|---|
| $(a1)$ | 1 | 4 | 6 | 4 | 1 |
| $(a3)$ | 1 | 3 | 4 | 3 | 1 |
| $(b1)$ | 1 | 2 | 2 | 2 | 1 |
| $(b2)$ | 1 | 2 | 2 | 2 | 1 |
| $(c1)$ | 1 | 3 | 4 | 3 | 1 |
| $(d)$ | 1 | 2 | 2 | 2 | 1 |

TABLE 2.6. Dimensions of the de Rham cohomology groups.

REMARK 2.11. Note that if $\omega_X \cong \mathcal{O}_X$, then

$$T_X = \Omega_{X/k}^\vee \cong \Omega_{X/k} \otimes \det(\Omega_{X/k}) \cong \Omega_{X/k} \otimes \omega_X \cong \Omega_{X/k},$$

since $\Omega_{X/k}$ is a locally free sheaf of rank 2 (cf. [**Har77**, Exercise II.5.16]). Thus, one get the equality $h^i(T_X) = h^i(\Omega_{X/k}) = h^{1,i}$.

**2.4. Crystalline Cohomology.** We now turn to the crystalline cohomology groups of bielliptic surfaces. For a detailed introduction to crystalline cohomology we refer to [**CL98**] and to [**Lie16**, Section 1] for a shorter basic introduction.

Let $k$ be an algebraically closed field of positive characteristic $p > 0$. And let $W = W(k)$ denote the ring of Witt vectors with coefficients in $k$. The crystalline cohomology groups $H^i_{\text{cris}}(X/W)$, $0 \leq i \leq 4$, of a bielliptic surface $X$ over $k$ were computed by Lang [**Lan79**]. Here we present his results.

Theorem 2.12 ([**Lan79**], Theorem 4.11, p. 499). *Let $X$ be a bielliptic surface over a field $k$ of characteristic $p \neq 2$. Then $H^2_{\mathrm{cris}}(X/W)_{tors}$ is killed by $p$, and its rank as a vector space over $k$ is $h^1_{dR}(X) - 2$.*

Corollary 2.13. *Let $X$ be a bielliptic surface over a field $k$ of characteristic $p \notin \{2,3\}$. Then the crystalline cohomology groups of $X$ are as follows*

$$\begin{aligned} H^0_{\mathrm{cris}}(X/W) &= W, \\ H^1_{\mathrm{cris}}(X/W) &= W^2, \\ H^2_{\mathrm{cris}}(X/W) &= W^2, \\ H^3_{\mathrm{cris}}(X/W) &= W^2, \\ H^4_{\mathrm{cris}}(X/W) &= W. \end{aligned}$$

**2.5. Hodge-Witt Cohomology in Characteristic $p \notin \{2,3\}$.** In this section we compute the Hodge-Witt cohomology groups

$$H^{i,j}_W(X) := H^j(X, W\Omega^i_{X/k})$$

of a bielliptic surface $X$ over an algebraically closed field $k$ of characteristic different from 2 and 3. In order to do so, we adapt the arguments of the proof of [**Ill79**, Section II, Proposition 7.3.6] to the case of bielliptic surfaces.

First let us recall a result by Lang inspired as well by Illusie's analysis in [**Ill79**, Section II, Proposition 7.3.2].

Proposition 2.14 ([**Lan79**], Proposition 4.3). *Let $X$ be an algebraic surface, and suppose the first Bockstein operation of Witt vector cohomology $\beta_1 \colon H^1(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is surjective. Then $H^2(X, W\mathcal{O}_X) \cong k^{p_g}$.*

Corollary 2.15. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p \notin \{2,3\}$. Then, $H^2(X, W\mathcal{O}_X) \cong 0$.*

Remark 2.16. In particular, $H^2(X, W\mathcal{O}_X)$ is torsion free and finitely generated. This is even true for any bielliptic surface $X$ in any characteristic. Thus, by a result of Nygaard (cf. [**Ill79**, Section II, Corollaire 3.14], the slope spectral sequence degenerate and $X$ is *Hodge-Witt*, i.e., all the cohomology groups $H^j(X, W\Omega^i_{X/k})$ are finitely generated $W$-modules (cf. [**Ill79**, Section II 2]).

Now, for Hodge-Witt varieties, one has a "Hodge decomposition" of the crystalline cohomology groups in terms of the Hodge-Witt cohomology groups. Indeed, according to [**Ill83**, Theorem 3.4.1], for any Hodge-Witt variety there is a canonical decomposition

$$H^n_{\mathrm{cris}}(X/W) \cong \bigoplus_{i+j=n} H^j(X, W\Omega^i_{X/k}). \tag{2.1}$$

Finally, we need one lemma about the action of the absolute Frobenius on $H^1(X, \mathcal{O}_X)$ for a bielliptic surfaces $X$.

Lemma 2.17. *Let $X$ be a bielliptic surfaces over an algebraically closed field $k$ of characteristic $p \notin \{2, 3\}$ and $f : X \to \mathrm{Alb}(X)$ the Albanese fibration. Consider the map*

$$F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$$

*induced by the absolute Frobenius $F$ on $X$. Then $F^*$ is bijective (resp. zero) on $H^1(X, \mathcal{O}_X)$ if and only if $\mathrm{Alb}(X)$ is an ordinary (resp. a supersingular) elliptic curve.*

Proof. Since $H^1(X, \mathcal{O}_X) \cong k$ and $k$ is algebraically closed and therefore perfect, the induce map $F^*$ is either zero or bijective on $H^1(X, \mathcal{O}_X)$. Consider now the Albanese fibration $f : X \to B = \mathrm{Alb}(X)$. Since $f_*\mathcal{O}_X \cong \mathcal{O}_B$, the Leray spectral sequence associated with the morphism $f$ induces an exact sequence

$$0 \to H^1(B, \mathcal{O}_B) \xrightarrow{f^*} H^1(X, \mathcal{O}_X) \to H^0(B, R^1 f_*\mathcal{O}_X) \to H^2(B, \mathcal{O}_B) = 0.$$

Moreover, since $B = \mathrm{Alb}(X)$ is an elliptic curve, we have $p_g(B) = \dim_k H^1(B, \mathcal{O}_B) = 1$. Thus, $f^* : H^1(B, \mathcal{O}_B) \to H^1(X, \mathcal{O}_X)$ is an isomorphism. Therefore, by considering the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{F} & B \end{array}$$

and taking cohomology, we see that $F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is bijective (resp. zero) if and only if the same holds for $F^* : H^1(B, \mathcal{O}_B) \to H^1(B, \mathcal{O}_B)$, that is, if and only if the elliptic curve $B$ is ordinary (resp. supersingular). □

Remark 2.18. The previous lemma states that the bielliptic surface $X$ is *ordinary* in the sense of Bloch-Kato [**BK86**, Definition 7.2] if and only if its Albanese variety $B = \mathrm{Alb}(X)$ is an ordinary elliptic curve. Here one uses $H^2(X, \mathcal{O}_X) = 0$.

Theorem 2.19. *Let $X$ be a bielliptic surfaces over an algebraically closed field $k$ of characteristic $p \notin \{2, 3\}$. Then the Hodge-Witt cohomology groups $H^{i,j}_W(X)$ of $X$ are given by the following table.*

| Type of $X$ | $H^{0,0}_W$ | $H^{1,0}_W$ | $H^{0,1}_W$ | $H^{2,0}_W$ | $H^{1,1}_W$ | $H^{0,2}_W$ | $H^{2,1}_W$ | $H^{1,2}_W$ | $H^{2,2}_W$ |
|---|---|---|---|---|---|---|---|---|---|
| ordinary | $W$ | $W$ | $W$ | $0$ | $W^2$ | $0$ | $W$ | $W$ | $W$ |
| not ordinary | $W$ | $W$ | $W$ | $0$ | $W^2$ | $0$ | $0$ | $W^2$ | $W$ |

Table 2.7. Hodge-Witt cohomology groups of $X$.

Proof. We have $H_W^{0,0} \cong H_W^{2,2} \cong W$, since $X$ is a smooth and proper surface.

According to [**Ill79**, II 3.11.2], the Albanese map $f\colon X \to B = \mathrm{Alb}(X)$ induces an isomorphism

$$f^*\colon H^1(\mathrm{Alb}(X)/W) \xrightarrow{\sim} H^1(X/W).$$

Moreover, there are isomorphisms

$$H^0(X, W\Omega^1_{X/k}) \cong H^0(\mathrm{Alb}(X), W\Omega^1_{\mathrm{Alb}(X)/k}) \cong W \quad \text{and}$$
$$H^1(X, W\mathcal{O}_X) \cong H^1(\mathrm{Alb}(X), W\mathcal{O}_{\mathrm{Alb}(X)}) \cong W,$$

since $\mathrm{Alb}(X)$ is an elliptic curve (cf. [**CDL20**, page 160]).

Since $\rho(X) = b_2(X)$, the $F$-isocrystal $H^2_{\mathrm{cris}}(X/W) \otimes K$ is of slope 1. This implies that $H^0(X, W\Omega^2_{X/k}) \otimes K$ is zero, since the latter is isomorphic to the sub-$F$-isocrystal of $H^2_{\mathrm{cris}}(X/W) \otimes K$ of slope 2. Since all $H_W^{i,0}$ are free $W$-modules of finite rank by [**Ill79**, Section II, Corollaire 2.17], it follows that $H_W^{2,0} = 0$. From Proposition 2.14, we know that $H_W^{0,2} = 0$. Since $X$ is Hodge-Witt, we have a canonical decomposition (for $n = 2$)

$$H^2_{\mathrm{cris}}(X/W) \cong \bigoplus_{i+j=2} H^j(X, W\Omega^i_{X/k}).$$

Thus, $H^1(X, W\Omega^1_{X/k}) \cong H^2_{\mathrm{cris}}(X/W) \cong W^2$, by Theorem 2.12.

Now for $n = 3$ we have the canonical decomposition

$$H^3_{\mathrm{cris}}(X/W) \cong H^1(X, W\Omega^2_{X/k}) \oplus H^2(X, W\Omega^1_{X/k}).$$

We now compute $H^1(X, W\Omega^2_{X/k})$ by following the proof of [**Ill79**, Section II, Proposition 7.3.6]: the Verschiebung $V$ induces a short exact sequence

$$0 \to W\Omega^2_{X/k} \xrightarrow{V} W\Omega^2_{X/k} \to W\Omega^2_{X/k}/VW\Omega^2_{X/k} \to 0.$$

Taking cohomology and using $H^2(X, W\Omega^2_{X/k}) \cong W$ and the Cartier operator $C$ in Hodge-Witt cohomology, we get an isomorphism

$$H^1(X, W\Omega^2_{X/k})/VH^1(X, W\Omega^2_{X/k}) \cong \varprojlim_C H^1(X, \omega_X),$$

where $\omega_X = \Omega^2_X$ is the canonical sheaf of $X$. Since $X$ is a surface, by [**Ill79**, Section II, Corollaire 2.18] the Frobenius $F$ is an automorphism on $H^1(X, W\Omega^2_{X/k})$ and we have

$$H^1(X, W\Omega^2_{X/k})/VH^1(X, W\Omega^2_{X/k}) \cong H^1(X, W\Omega^2_{X/k})/pH^1(X, W\Omega^2_{X/k}),$$

since $V = pF^{-1}$.

Now $\varprojlim_C H^1(X, \omega_X)$ is dual two $\varinjlim_F H^1(X, \mathcal{O}_X)$. Thus, if $F$ is bijective (resp. zero) on $H^1(X, \mathcal{O}_X) \cong k$, then $\varinjlim_F H^1(X, \mathcal{O}_X)$ equals $k$ (resp. 0). Therefore,

$$H^1(X, W\Omega^2_{X/k})/pH^1(X, W\Omega^2_{X/k}) \cong k \text{ (resp. =0)},$$

if $F$ is bijective (resp. zero) on $H^1(X, \mathcal{O}_X)$. Since $H^3_{\text{cris}}(X/W)$ is torsion free, we get

$$H^1(X, W\Omega^2_{X/k}) \cong W \quad \text{(resp. } = 0),$$

if $F$ is bijective (resp. zero) on $H^1(X, \mathcal{O}_X)$. □

## 3. Divisors of Bielliptic Surfaces and the Néron-Severi Lattice

Divisors of complex bielliptic surfaces were studied by Serrano in [**Ser90**]. One of his main results is an explicit description of the Néron-Sever lattice $\text{Num}(X)$ of a complex bielliptic surface $X$ (cf. [**Ser90**, Theorem 1.4]). In this section we will show that this result still holds true for bielliptic surfaces over fields of characteristic different from 2 and 3.

More precisely, for a bielliptic surface $X$ over an algebraically closed field $k$ of characteristic $\text{char}(k) = p \notin \{2, 3\}$ we will prove the following:

(1) The Néron-Severi lattice $\text{Num}(X)$ of $X$ is an even and *unimodular* lattice of rank 2 and signature $(1, 1)$. In particular, $\text{Num}(X)$ is isomorphic to the hyperbolic lattice $U$ (see Proposition 3.1).
(2) A basis of $\text{Num}(X)$ is given by (rational) multiples of the numerical classes of fibers of the two elliptic fibrations on $X$ (see Theorem 3.3).

Proposition 3.1. *Let $X$ be a bielliptic surface over an algebraically closed field $k$. Then*

$$\text{Num}(X) \cong U,$$

*which is an even and unimodular lattice of rank* 2 *and signature* $(1, 1)$. *Here $U$ denotes the hyperbolic lattice with intersection matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Assume first that $\text{Num}(X)$ is a unimodular lattice. Then we can argue as follows.

First, recall that $\rho(X) = \text{rankNS}(X) = b_2(X) = 2$, by Lemma 1.10. Hence, $\text{Num}(X)$ has rank 2. From the Hodge Index Theorem, we get that $\text{Num}(X)$ has signature $(1, 1)$. Moreover, from the Riemann-Roch theorem it follows that $\text{Num}(X)$ is an even lattice. Indeed, given an invertible $\mathcal{O}_X$-module $L$ on $X$, since $\chi(\mathcal{O}_X) = 0$ and $\omega_X \equiv \mathcal{O}_X$, the Riemann-Roch theorem yields

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2} \cdot ((L, L) - (L, \omega_X)) = \frac{1}{2} \cdot (L, L).$$

Thus, $L^2 = (L, L) = 2 \cdot \chi(L)$. So $\mathrm{Num}(X)$ is an even lattice. Now from the structure theory of unimodular lattices (see for example [**Ser73**, Chapter V, Theorem 5]) it follows that $\mathrm{Num}(X) \cong U$.

We proceed now to prove that $\mathrm{Num}(X)$ is indeed unimodular. We will give two different proofs of this fact. The first one is based on the proof of the corresponding statement for Enriques Surfaces in [**CDL20**, Proposition 1.5.1]. The second proof uses the fact that bielliptic surfaces lift to characteristic zero (cf. Theorem 5.12 in Chapter 2) together with the existence of the (injective) specialization map of the Néron-Severi lattice.

*First proof of unimodularity*:

Consider first the case where $k$ is a field of characteristic zero. By the Lefschetz principle we may assume $k = \mathbb{C}$. Since $p_g(X) = 0$, the first Chern class map $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is surjective and induces an isomorphism $\mathrm{NS}(X) \cong H^2(X, \mathbb{Z})$ which is compatible with the intersection pairing on $\mathrm{NS}(X)$ and the cup-product on $H^2(X, \mathbb{Z})$, thus

$$\mathrm{Num}(X) \cong H^2(X, \mathbb{Z})/Tors \cong \mathbb{Z}^2.$$

From Poincaré duality for $H^2(X, \mathbb{Z})$ it follows that $\mathrm{Num}(X)$ is a unimodular lattice.

Now consider the case where $k$ has positive characteristic $p > 0$. For all prime $l \neq p$ the Kummer exact sequence gives an exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_l \to H^2(X, \mathbb{Z}_l(1)) \to \mathrm{T}_l\mathrm{Br}(X) \to 0,$$

where $\mathrm{T}_l\mathrm{Br}(X)$ denotes the Tate module of the Brauer group, which is a finitely generated $\mathbb{Z}_l$-module of rank $t(x) = b_2(X) - \rho(X)$ (see [**Mil80**, Chapter V, Remark 3.29]). Since $\rho(X) = b_2(X)$, we have that $\mathrm{T}_l\mathrm{Br}(X) = 0$ and thus there is an isomorphism

$$\mathrm{Num}(X) \otimes \mathbb{Z}_l \cong H^2(X, \mathbb{Z}_l(1))/Tors,$$

that is compatible with intersection pairings on both sides. By Poincaré duality for $l$-adic cohomology it follows that $\mathrm{Num}(X) \otimes \mathbb{Z}_l$ is a unimodular lattice over $\mathbb{Z}_l$ for all $l \neq p$.

Now, if $l = p$, we saw in the proof of Theorem 2.5 that the Igusa-Artin-Mazur formula implies $t_p(X) = 0$ and thus there is an isomorphism $\mathrm{NS}(X) \otimes \mathbb{Z}_p \cong H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$. Now consider the crystalline first Chern class map

$$c_1 \colon \mathrm{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{\mathrm{cris}}(X/W),$$

which is the composition of the following injective maps (see [**Ill79**, Remarque II.5.21.4]):

$$c_1 \colon \mathrm{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \hookrightarrow H^2(X, W\Omega^{\geq 1}_X) = P^1 H^2_{\mathrm{cris}}(X/W) \hookrightarrow H^2_{\mathrm{cris}}(X/W).$$

Recall that $c_1$ is compatible with the intersection pairings. By Theorem 2.12, we know that the second crystalline cohomology groups of bielliptic surfaces are of the form

$$H^2_{\mathrm{cris}}(X/W) \cong W^2 \oplus k^{h^1_{\mathrm{dR}}(x)-2}.$$

From this it follows that the crystalline first Chern class map induces isomorphisms

$$\mathrm{Num}(X) \otimes_{\mathbb{Z}} W \cong (H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W)/Tors \cong H^2_{\mathrm{cris}}(X/W)/Tors,$$

which are compatible with intersection pairings. By Poincar Duality in crystalline cohomology, the pairing on the right-hand side is perfect. This implies that the pairing on $\mathrm{Num}(X)\otimes\mathbb{Z}_p$ is perfect.

We have shown that $\mathrm{Num}(X)\otimes\mathbb{Z}_l$ is a unimodular lattice over $\mathbb{Z}_l$ for all primes $l$, including $l = p$. Thus, $\mathrm{Num}(X)$ is a unimodular lattice over $\mathbb{Z}$.

*Second proof of unimodularity*:

Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p > 0$. Since $X$ lift to characteristic zero by Theorem 5.12, there exist a integral ring $(R, m)$ of characteristic zero and residue field $R/m \cong k$ and a scheme $\mathcal{X}$ flat over $R$, such that $X \cong X_0 := \mathcal{X} \times_{Spec(R)} \mathrm{Spec}\,(k)$. Let $K$ be the fraction field of $R$ and denote $X_\eta := \mathcal{X} \times_{Spec(R)} \mathrm{Spec}\,(K)$ the generic fiber. After choosing a discrete valuation ring dominating $(R, m)$ and after passing to the $m$-adic completion, we may assume that $(R, m)$ is a local and $m$-adically complete discrete valuation ring.

Consider the specialization morphism

$$sp_{\mathrm{Pic}} \colon \mathrm{Pic}(X_{\overline{\eta}}) \to \mathrm{Pic}(X_0),$$

where $X_{\overline{\eta}} := X_\eta \times_{\mathrm{Spec}\,(K)} \mathrm{Spec}\,(\overline{K})$ denotes the geometric generic fiber (cf. [**MP12**, Proof of Proposition 3.3] or the original reference [**SGA71**, Expose X, Appendice 7.13.3]). Note that $X_{\overline{\eta}}$ is a bielliptic surface over $\overline{K}$.

Now, the specialization map on the Picard groups induces a specialization map at the level of the Néron-Severi lattices

$$sp_{\mathrm{Num}} \colon \mathrm{Num}(X_{\overline{\eta}}) \to \mathrm{Num}(X_0),$$

which is compatible with the intersection product. Indeed, by [**SGA71**, Expose X, Appendice 7.9] this is the case if $sp_{\mathrm{Pic}}$ is an epimorphism modulo torsion. By the proof of Lemma 1.9 of Chapter 2, the relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/R}$ is smooth. Then, by Hensel's lemma (cf. [**Gro67**, Théorème 18.5.17 ]), the specialization map $sp_{\mathrm{Pic}}$ is surjective.

Since the intersection product on Num is non-degenerate, the last map is injective. Thus, $\mathrm{Num}(X_{\overline{\eta}}) \subseteq \mathrm{Num}(X_0)$ is a sublattice of finite index, say $N \in \mathbb{N}$, since both lattices have rank 2. For the discriminants we get

$$\mathrm{disc}(\mathrm{Num}(X_{\overline{\eta}})) = N^2 \cdot \mathrm{disc}(\mathrm{Num}(X_0)).$$

By Poincaré duality we have that $\mathrm{Num}(X_{\overline{\eta}})$ is a unimodular lattice, since $\overline{K}$ is a field of characteristic zero. Hence, $\mathrm{disc}(\mathrm{Num}(X_{\overline{\eta}}) = 1$. But then it follows $\mathrm{disc}(\mathrm{Num}(X_0)) = 1$. Thus, $\mathrm{Num}(X_0)$ is an unimodular lattice. □

### 3.1. Bases for the Neŕon-Severi Lattices.

Let $X = E \times F/G$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $\mathrm{char}(k) = p \notin \{2, 3\}$ and let

$$f\colon X \to E/G \text{ (elliptic curve) and } g\colon X \to F/\alpha(G) \cong \mathbb{P}^1,$$

be the two elliptic fibrations given by the natural projections of the product $E \times F$ (cf. Theorem 1.16). The following facts are easy to check:

(1) The fibration $f$ is smooth and isotrivial with all fibers being isomorphic to the elliptic curve $F$.
(2) The fibers of $g$ are all irreducible. The general fiber is a smooth elliptic curve isomorphic to $E$ and over a point $p \in F/\alpha(G)$ that is the image of a point $x \in F$ with stabilizer subgroup $G_x$, the fiber of $g$ over $p$ is isomorphic to $E/G_x$ with multiplicity $|G_x|$. The multiplicity is the one of $p$ under the map $F \to F/\alpha(G)$.
(3) The multiple fibers of $g$ occur as multiple-sections of $f$.
(4) In particular, all the fiber of $f$ are isomorphic to $F$ and all smooth fibers of $g$ are isomorphic to $E$.

The last point suggest the following notation (following Serrano): since the fibers of a fibration are all numerically equivalent, we will denote by $[F]$ and $[E] \in \mathrm{Num}(X)$ the numerical class of a fiber of $f$ and of $g$, respectively.

The classes $[F]$ and $[E]$ intersect as follows:

$$[F]^2 = [E]^2 = 0 \text{ and } [F] \cdot [E] = |G|.$$

From this, it is easy to see that the classes $\{[F], [E]\}$ form a basis of the 2-dimensional vector spaces

$$\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } \mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

For a complex bielliptic surface $X$ a basis of $\mathrm{Num}(X)$ was described by Serrano in [**Ser90**]. As we will see, his result also holds for bielliptic surfaces in characteristic different from 2 and 3.

LEMMA 3.2 ([**Ser90**], Lemma 1.3). *Let $X \cong E \times F/G$ be a bielliptic surface over an algebraically closed field $k$ of characteristic* $\mathrm{char}(k) = p \geq 0$ *and let $D$ be a divisor of numerical class $\alpha[E] + \beta[F] \in \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\alpha, \beta \in \mathbb{Q}$. Then:*

*(i)* $\chi(\mathcal{O}_X(D)) = \alpha\beta \cdot |G|$.
*(ii) $D$ is ample if and only if $\alpha > 0, \beta > 0$.*
*(iii) If $H^0(\mathcal{O}_X(D)) \neq 0$, then $\alpha \geq 0, \beta \geq 0$*
*(iv) Assume $p \notin \{2, 3\}$. If $D$ is ample, then $h^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D))$.*

PROOF. The only result used in Serrano's proof which is not true in (arbitrary) positive characteristic is Kodaira vanishing, which is used to prove statement (iv). As we will see in Section 5, bielliptic surfaces over fields of characteristic different from 2 and 3 lift to characteristic 0 over the Witt ring (see Proposition 5.6). In particular, they lift to the second Witt vectors $W_2(k)$. Hence, they satisfy Kodaira vanishing (see, for example, [**Ray78**] and [**EV92**, Lecture 11]). □

THEOREM 3.3 ([**Ser90**], Theorem 1.4). *Let $X \cong E \times F/G$ be a bielliptic surface over an algebraically closed field $k$ of characteristic different from 2 and 3. The following table shows the multiplicities of the multiple fibers of the second fibration $g\colon X \to E_0/\alpha(G) \cong \mathbb{P}^1$, their degree over the Albanese fibration $f$ and a basis of* $\mathrm{Num}(X)$ *for the seven types of bielliptic surfaces.*

| Type | Multiplicities of fibers | Degree over $E/G$ via $f$ | Basis of $\mathrm{Num}(X)$ |
|---|---|---|---|
| $(a1)$ | $\{2,2,2,2\}$ | $\{1,1,1,1\}$ | $\{\frac{1}{2}[E],[F]\}$ |
| $(a2)$ | $\{2,2,2,2\}$ | $\{2,2,2,2\}$ | $\{\frac{1}{2}[E],\frac{1}{2}[F]\}$ |
| $(b1)$ | $\{3,3,3\}$ | $\{1,1,1\}$ | $\{\frac{1}{3}[E],[F]\}$ |
| $(b2)$ | $\{3,3,3\}$ | $\{3,3,3\}$ | $\{\frac{1}{3}[E],\frac{1}{3}[F]\}$ |
| $(c1)$ | $\{2,4,4\}$ | $\{2,1,1\}$ | $\{\frac{1}{4}[E],[F]\}$ |
| $(c2)$ | $\{2,4,4\}$ | $\{4,2,2\}$ | $\{\frac{1}{4}[E],\frac{1}{2}[F]\}$ |
| $(d)$ | $\{2,3,6\}$ | $\{3,2,1\}$ | $\{\frac{1}{6}[E],[F]\}$ |

TABLE 3.1. Multiple fibers and basis of $\mathrm{Num}(X)$.

PROOF. For the possible multiplicities of the fibers, see [**BM77**]. For the basis of $\mathrm{Num}(X)$, the proof in Serrano's papers works in this general setting. The only change that has to be made is that, instead of using Poincaré duality, one uses the fact that $\mathrm{Num}(X)$ is a unimodular lattice, which is given by Proposition 3.1. □

REMARK 3.4. It is important to notice that while the divisibility of $[E]$ in $\mathrm{Num}(X)$ is justified by the existence of multiple fibers of $g$, this is not the case for $[F]$, since the fibration $f$ is smooth. For complex bielliptic surfaces Serrano relates the divisibility of a fiber with the torsion of the first homology group $H_1(X,\mathbb{Z})$ (see [**Ser90**] and [**Ser91**]). As we have seem, for complex bielliptic surfaces we have $\mathrm{NS}(X) \cong H_1(X,\mathbb{Z})$. Thus, it would be interesting to determine if there is any relation between the torsion of the Néron-Severi group and the divisibility of smooth fibers for bielliptic surfaces over fields of arbitrary characteristic.

From the information about the multiple fiber of the second fibration $g\colon X \to \mathbb{P}^1$ and its degree over $\mathrm{Alb}(X)$, one obtains the following corollary.

COROLLARY 3.5. *Let $X$ be a bielliptic surface over a field of characteristic $p \neq 2,3$. Then, the Albanese fibration $f\colon X \to \mathrm{Alb}(X)$ has a section if and only if $X$ is of type $(a1), (b1), (c1)$ or $(d)$.*

**3.2. Ample and Nef Cones of Bielliptic Surfaces.** With the results we have obtained so far, we can give a description of the ample and nef cones in the vector space of numerical classes of $\mathbb{R}$-divisors $\mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. For definitions and notation we refer the reader to Lazarsfeld's book [**Laz04**].

First of all, the real vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ has dimension 2 and is generated by the two classes $[F]$ and $[E]$. Taking this classes as basis of the real vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ and representing this vector space as the $[F]-[E]$-plane, Lemma 3.2 (ii) implies that the ample cone $\mathrm{Amp}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ lies within the first quadrant, i.e.,

$$\mathrm{Amp}(X) = \{[D] \in \mathrm{N}^1(X)_{\mathbb{R}} | [D] = \alpha[E] + \beta[F], \alpha, \beta > 0\}.$$

Moreover, since $F$ and $E$ are irreducible curves with $F^2 = E^2 = 0$, $[F]$ and $[E]$ are nef classes. Thus, the nonnegative multiples of this classes form the two boundaries of the nef cone $\mathrm{Nef}(X)$, i.e.,

$$\mathrm{Nef}(X) = \{[D] \in \mathrm{N}^1(X)_{\mathbb{R}} | [D] = \alpha[E] + \beta[F], \alpha, \beta \geq 0\}.$$

Now let $\overline{\mathrm{NE}}(X)$ denote the closed cone of curves of $X$, that is, the closure in $\mathrm{N}^1(X)_{\mathbb{R}}$ of the cone of curves

$$\mathrm{NE}(X) = \{\sum a_i[C_i] | C_i \subset X \ \text{ an irreducible curve}, \ a_i \in \mathbb{R}, \ a_i \geq 0\}.$$

For a smooth projective surface $X$, one always have the inclusion $\mathrm{Nef}(X) \subset \overline{\mathrm{NE}}(X)$. If $X$ is a bielliptic surface, then one even have $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$, since on a bielliptic surface there is no irreducible curves $C \subset X$ with negative self-intersection, as shown by the following result.

LEMMA 3.6. *Let $X$ be a bielliptic surfaces over an algebraically closed field $k$. Then, for every irreducible curve $C \subset X$ we have $(C^2) \geq 0$.*

Proof. Suppose $C \subset X$ is an irreducible curve with $(C^2) < 0$. Then, since $K_X \equiv 0$, from the genus formula we get

$$2p_a(C) - 2 = (C^2) + (C \cdot K_X) = (C^2) < 0.$$

Thus, $p_a(C) = 0$ and $(C^2) = -2$. But this is impossible, since on a bielliptic surface $X$ there are no $(-2)$-curves by Proposition 1.11. Hence, $(C^2) \geq 0$. □

Corollary 3.7. *Let $X$ be a bielliptic surfaces over an algebraically closed field $k$. Then, the closed cone of curves and the nef cone coincide, i.e.,* $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$*.*

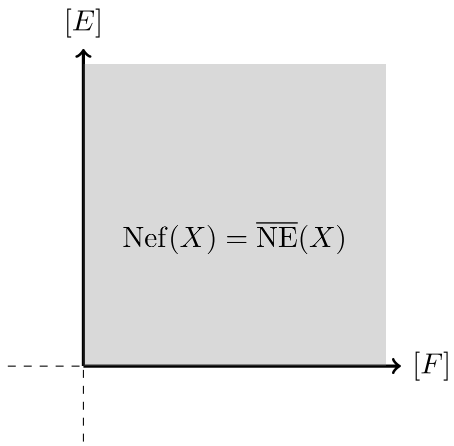



Figure 3.1. The effective cone of a bielliptic surface $X$.

Corollary 3.8. *If $D$ and $D'$ are two effective divisors on a bielliptic surface $X$, then $(D \cdot D') \geq 0$. In particular, for every effective divisor $D$ on $X$ we have $(D^2) \geq 0$.*

Corollary 3.9. *If $D$ is an $(\mathbb{R}-)$divisor on a bielliptic surface $X$ such that $(D^2) > 0$ (resp. $(D^2) \geq 0$), then either $D$ or $-D$ is ample (resp. nef). Equivalently, if a numerical class $\eta \in \mathrm{N}^1(X)_{\mathbb{R}}$ is such that $\eta^2 > 0$ (resp. $\eta^2 \geq 0$), then either $\eta$ or $-\eta$ is ample (resp. nef).*

Lemma 3.10. *Let $X$ be a bielliptic surface and $D$ a divisor on $X$. If $D^2 > 0$, then either $D$ or $-D$ is linear equivalent to an effective divisor.*

Proof. Take $H$ an ample divisor on $X$. Then, from the Hodge Index Theorem it follows that $D \cdot H \neq 0$.

If $D \cdot H > 0$, then $(K_X - D) \cdot H = -D \cdot H < 0$, since $K_X$ is numerically trivial. Thus, $K_X - D$ cannot be linear equivalent to an effective divisor. It follows that $h^0(K_X - D) = 0$ and by Serre Duality we get $h^2(D) = 0$.

Then by Riemann-Roch we have

$$h^0(D) - h^1(D) = \frac{1}{2}D^2 > 0.$$

Since $h^1(D) \geq 0$, we get $h^0(D) \geq \frac{1}{2}D^2 > 0$. Therefore, $D$ is linear equivalent to an effective divisor.

If $D \cdot H < 0$, then setting $D' \coloneqq -D$, we have $(D')^2 = (-D)^2 > 0$ and $D' \cdot H > 0$. Then, the same argument as above shows that $h^0(D') > 0$. Thus, $-D$ is linear equivalent to an effective divisor. □

Remark 3.11. The statement of the previous lemma holds in general for every surface $X$ with $K_X \equiv 0$, if one requires that $D^2 > -2\chi(\mathcal{O}_X)$.

## 4. Automorphism Schemes of Bielliptic Surfaces

The automorphism groups of complex bielliptic surfaces were determined by Bennett and Miranda in [**BM90a**]. For a complex bielliptic surface $X$ of the form $(E \times F)/G$ (cf. Theorem 1.16) the authors prove that the automorphism group $\mathrm{Aut}(X)$ of $X$ is given by

$$\mathrm{Aut}(X) = C_{\mathrm{Aut}(E)\times \mathrm{Aut}(F)}(G)/G, \tag{4.1}$$

where $C_{\mathrm{Aut}(E)\times \mathrm{Aut}(F)}(G)$ denotes the centralizer of $G$ in $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$.

They explicitly compute the group $C_{\mathrm{Aut}(E)\times \mathrm{Aut}(F)}(G)$ for every type of complex bielliptic surfaces and present its generators in [**BM90a**, Table 3.1]. In particular, their analysis shows that the elliptic curve $E$, seen as a subgroup of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$, centralizes $G$ and since the intersection $E \cap G$ is trivial, the elliptic curve $E$ embeds in the quotient $C_{\mathrm{Aut}(E)\times \mathrm{Aut}(F)}(G)/G$ as a normal subgroup. Thus,

$$E \subseteq \mathrm{Aut}(X).$$

Consequently, the group $\mathrm{Aut}(X)$ is not finite. However, it turns out that the quotient group $\mathrm{Aut}(X)/E$ is finite. Bennett and Miranda compute this quotient group explicitly for every complex bielliptic surface and give its generators (see [**BM90a**, Table 3.1]).

Recently, the work of Bennett and Miranda on the automorphism groups of complex bielliptic surfaces was generalized by G. Martin [**Mar20**]. Martin determines the automorphism schemes of bielliptic and quasi-bielliptic surfaces over algebraically closed fields of arbitrary characteristic. His work not only solves the problem of the classification of automorphisms of (quasi-)bielliptic surfaces in full generality, but it also shows that some subtle cases in characteristic 0 were missed in the work of Bennett and Miranda (see Remark 4.7 below).

We now present Martin's results on the automorphism schemes of bielliptic surfaces, as these are the surfaces we are interested in. Note however that his work, as already mentioned, also includes results on the automorphism schemes of quasi-bielliptic surfaces. For those results, as well as for the proofs and further details of the results we present, we refer to the article [**Mar20**].

Let us first recall some definitions.

DEFINITION 4.1. Let $X$ and $Y$ be two schemes over a base scheme $S$ and let $\pi\colon X \to Y$ be a morphism of $S$-schemes.

(i) The functor $\mathrm{Aut}_X\colon (Sch/S)^{op} \to (Groups)$ whose $T$-value points are given by

$$\mathrm{Aut}_X(T) \coloneqq \{T\text{-automorphisms of } X \times_S T \to T\},$$

is called the *automorphism functor* of $X$.

(ii) The functor $\mathrm{Aut}_\pi\colon (Sch/S)^{op} \to (Groups)$ whose $T$-value points are

$$\mathrm{Aut}_\pi(T) \coloneqq \{(h, h') \in \mathrm{Aut}_X(T) \times \mathrm{Aut}_Y(T) \mid \pi_T \circ h = h' \circ \pi_T\},$$

where $\pi_T$ denotes the pullback of $\pi$ along $T \to S$, is called the *automorphism functor of (the morphism)* $\pi$.

(iii) The subfunctor $\mathrm{Aut}_{X/Y}$ of $\mathrm{Aut}_\pi$ whose $T$-value points are given by

$$\mathrm{Aut}_{X/Y}(T) \coloneqq \{(h, \mathrm{id}) \in \mathrm{Aut}_X(T) \times \mathrm{Aut}_Y(T) \mid \pi_T \circ h = \mathrm{id} \circ \pi_T\}$$

is called the *automorphism functor of $X$ over $Y$*.

Recall that for a proper variety $X$ over an algebraically closed field $k$ the automorphism functor $\mathrm{Aut}_X$ is representable by a reduced group scheme locally of finite type over $k$, by [**MO67**, Theorem 3.6]. In particular, if $\pi\colon X \to Y$ is a morphism of proper varieties over an algebraically closed field $k$, then the functors $\mathrm{Aut}_Y$, $\mathrm{Aut}_\pi$ and $\mathrm{Aut}_{X/Y}$ are all group schemes as well.

Next, let us recall the notions of centralizer and the normalizer in the context of group schemes.

DEFINITION 4.2. Let $G$ be a group scheme and $H \subseteq G$ a subgroup scheme of $G$.

(i) The functor $N_G(H)\colon (Sch)^{op} \to (Set)$ which sends a scheme $S$ to the set

$$N_G(H)(S) \coloneqq \{g \in G(S) \mid g_T \circ h \circ g_T^{-1} \in H(T) \text{ for all } T \to S \text{ and } h \in H(T)\}$$

is called the *normalizer of $H$ in $G$*.

(ii) Similarly, the functor $C_G(H)\colon (Sch)^{op} \to (Set)$ sending a scheme $S$ to the set

$$C_G(H)(S) \coloneqq \{g \in G(S) \mid g_T \circ h = h \circ g_T \text{ for all } T \to S \text{ and } h \in H(T)\}$$

is called the *centralizer of $H$ in $G$*.

By [**ABD$^+$66**, Expos VIB, Proposition 6.2] both the normalizer $N_G(H)$ and the centralizer $C_G(H)$ of $H$ in $G$ are representable by closed subgroup schemes of $G$.

We are now ready to state Martin's results.

Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic $p \geq 0$. Recall that by the structure theorem of bielliptic surfaces (see Theorem 1.16) $X$ is of the form

$$X = E \times F/G$$

where, $E$ and $F$ are elliptic curves and $G \subseteq E$ is a finite subgroup scheme of $E$ which acts on $F$ via an injective homomorphism $\alpha\colon G \to \mathrm{Aut}_F$. Moreover, let

$$f\colon X \to E/G \text{ and } \; g\colon X \to F/\alpha(G) \cong \mathbb{P}^1.$$

denote the two elliptic fibrations of $X$.

THEOREM 4.3 ([**Mar20**], Theorem 1.1). *Let $X = (E \times F)/G$ be a bielliptic surface. Then, the following hold:*

*(1)* $\mathrm{Aut}_{X/(F/\alpha(G))} = C_{\mathrm{Aut}_E}(G)$.
*(2)* $\mathrm{Aut}_{X/(E/G)} = C_{\mathrm{Aut}_F}(\alpha(G))$.
*(3) There is a short exact sequence of group schemes*

$$0 \to (C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_F}(\alpha(G)))/G \to \mathrm{Aut}_X \to M \to 0,$$

*where $G$ is embedded via* $\mathrm{id} \times \alpha$ *and $M$ is a finite and étale subquotient of the groups* $\mathrm{Aut}_{E/G}/(f_*C_{\mathrm{Aut}_E}(G))$ *and* $N_{\mathrm{Aut}(F)}(\alpha(G)(k))/(C_{\mathrm{Aut}_F}(\alpha(G))(k))$.

The group scheme $M$ is interesting as it represents a small characteristic phenomenon, that is, it appears only if $p \in \{2, 3\}$. Indeed, Bennett and Miranda proved that $M$ is trivial in characteristic zero (cf. [**BM90a**, Section 2]) and the more general statement that $M$ is trivial if $p \notin \{2, 3\}$ was proved by Martin (cf. [**Mar20**, Proposition 4.4]). As we will see, Martin calculates the group $M$ in the cases where it is not trivial. His analysis shows that $M$ also comes from automorphism of $E \times F$ and obtains the following corollary:

COROLLARY 4.4 ([**Mar20**], Corollary 1.2). *Let $X = (E \times F)/G$ be a bielliptic surface. Then,*

$$\mathrm{Aut}_X = N_{\mathrm{Aut}_E \times \mathrm{Aut}_F}(G)/G.$$

REMARKS 4.5.
(1) Theorem 4.3 and Corollary 4.4 are also true for quasi-bielliptic surfaces and they are stated accordingly in [**Mar20**].
(2) In characteristic zero it was shown by Bennett and Miranda that every element of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ which normalizes $G$ in fact centralizes $G$ (see [**BM90b**, Lemma 2.6]). Hence, $C_{\mathrm{Aut}(E)\times\mathrm{Aut}(F)}(G) = N_{\mathrm{Aut}(E)\times\mathrm{Aut}(F)}(G)$. The equation (4.1) can be therefore be rewritten as

$$\mathrm{Aut}(X) = N_{\mathrm{Aut}(E)\times\mathrm{Aut}(F)}(G)/G.$$

Written this way, it is evident how Martin's result generalizes the one of Bennett and Miranda.

As in the characteristic zero case, the elliptic curve $E$ always centralizes $G \subseteq E$ and $E$ is normal in $\mathrm{Aut}_X$ (cf. [**Mar20**, Lemma 3.2]). In particular, $E \subset \mathrm{Aut}_X$ and $\mathrm{Aut}_X$ is not a finite group scheme. Moreover, the quotient $\mathrm{Aut}_X/E$ can be written as an extension of $M$ by $(C_{\mathrm{Aut}_E}(G)/E) \times (C_{\mathrm{Aut}_F}(\alpha(G))/\alpha(G))$, that is, there is a short exact sequence of group schemes

$$0 \to (C_{\mathrm{Aut}_E}(G)/E) \times (C_{\mathrm{Aut}_F}(\alpha(G))/\alpha(G)) \to \mathrm{Aut}_X/E \to M \to 0.$$

Thus, in order to determine $\mathrm{Aut}_X/E$, it is enough to calculate the group schemes $C_{\mathrm{Aut}_E}(G)/E$, $C_{\mathrm{Aut}_F}(\alpha(G))/\alpha(G)$ and $M$. This was carried out by Martin and its computations lead to Table 4.1 below, where the groups $S_n$, $A_n$, and $D_{2n}$ denote the symmetric, alternating, and dihedral group (of order $2n$), respectively, and $M_2$ is the $p$-torsion subscheme of a supersingular elliptic curve.

Some of the notation used in Table 4.1 is explained by Martin in a fundamental remark [**Mar20**, Remark 1.4], where he refers to some missing cases in the work of Bennett and Miranda. Due to its importance, we include that remark as Remark 4.7 below.

Corollary 4.6 ([**Mar20**], Corollary 1.3). *Let $X = (E \times F)/G$ be a bielliptic surface. Then, depending on the group scheme $G$ and the $j$-invariants $j(E)$ and $j(F)$, the group schemes $C_{\mathrm{Aut}_E}(G)/E$, $C_{\mathrm{Aut}_F}(\alpha(G))/\alpha(G)$ and $M$ are as in Table 4.1.*

Remark 4.7 ([**Mar20**], Remark 4.1). If $p \neq 2, 3$ and $j(E) = 1728$ (resp. $j(E) = 0$), then every automorphism $g$ of $(E, O)$ of order 4 (resp. 3) fixes a unique cyclic subgroup of $E$ of order 2 (resp. 3). The stars* after some $j$-invariants in Table 4.1 denote that the translation subgroup of $G$ or $\alpha(G)$ coincides with this cyclic subgroup. By [**Mar20**, Lemma 4.1], this implies that $g$ is in the corresponding centralizer. These cases seem to be missing from [**BM90a**], since they were not listed in [**BM90a**, Table 1.1], which is why [**BM90a**, Table 3.2] differs from Table 4.1. If $p \in \{2, 3\}$ and $j(E) = 0$, then every cyclic subgroup of order 2 (resp. 3) is fixed by some automorphism of order 4 (resp. 3), so there are no stars in these characteristics.

Remark 4.8. Note that, since $E \subseteq \mathrm{Aut}_X$, the automorphism scheme $\mathrm{Aut}_X$ is not a finite group scheme. From Corollary 4.6 and Table 4.1 it follows however, that the quotient $\mathrm{Aut}_X/E$ is finite.

Finally, recall that for a bielliptic surface $X$ the (identity component of the) Picard scheme $\mathbf{Pic}^0_{X/k}$ is reduced and thus smooth if and only if $p_g(X) = 0$ (cf. Remark 1.2). Since $\omega_X \equiv \mathcal{O}_X$, the last condition is equivalent to the condition $\omega_X \not\cong \mathcal{O}_X$. Thus, $\mathbf{Pic}^0_{X/k}$ is smooth if and only if $\omega_X \not\cong \mathcal{O}_X$. It turns out that the same holds for the automorphism scheme $\mathrm{Aut}_X$ of $X$: the smoothness of $\mathrm{Aut}_X$ for a bielliptic surface is related to the order of the canonical sheaf $\omega_X$, as the following result shows.

| $G$ | $j(E)$ | $C_{\mathrm{Aut}_E}(G)/E$ | $j(F)$ | $C_{\mathrm{Aut}_F}(\alpha(G))/\alpha(G)$ | $M$ | $p$ |
|---|---|---|---|---|---|---|
| $\mathbb{Z}/2\mathbb{Z}$ | *a)* *any*<br>*b)* $1728^*$ | *a)* $\mathbb{Z}/2\mathbb{Z}$<br>*b)* $\mathbb{Z}/4\mathbb{Z}$ | *i)* $\neq 0, 1728$<br>*ii)* $1728$<br>*iii)* $0$ | *i)* $(\mathbb{Z}/2\mathbb{Z})^2$<br>*ii)* $D_8$<br>*iii)* $A_4$ | $\{1\}$ | $\neq 2, 3$ |
| $(\mathbb{Z}/2\mathbb{Z})^2$ | *any* | $\mathbb{Z}/2\mathbb{Z}$ | *i)* *any*<br>*ii)* $1728^*$ | *i)* $\mathbb{Z}/2\mathbb{Z}$<br>*ii)* $(\mathbb{Z}/2\mathbb{Z})^2$ | $\{1\}$ | $\neq 2, 3$ |
| $\mathbb{Z}/3\mathbb{Z}$ | *a)* *any*<br>*b)* $0^*$ | *a)* $\{1\}$<br>*b)* $\mathbb{Z}/3\mathbb{Z}$ | $0$ | $S_3$ | $\{1\}$ | $\neq 2, 3$ |
| $(\mathbb{Z}/3\mathbb{Z})^2$ | *any* | $\{1\}$ | $0$ | $\{1\}$ | $\{1\}$ | $\neq 3$ |
| $\mathbb{Z}/4\mathbb{Z}$ | *any* | $\{1\}$ | $1728$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{1\}$ | $\neq 2$ |
| $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | *any* | $\{1\}$ | $1728$ | $\{1\}$ | $\{1\}$ | $\neq 2$ |
| $\mathbb{Z}/6\mathbb{Z}$ | *any* | $\{1\}$ | $0$ | $\{1\}$ | $\{1\}$ | $\neq 2, 3$ |
| $\mathbb{Z}/2\mathbb{Z}$ | *a)* $\neq 0$<br>*b)* $0$ | *a)* $\mathbb{Z}/2\mathbb{Z}$<br>*b)* $\mathbb{Z}/4\mathbb{Z}$ | *i)* $\neq 0$<br>*ii)* $0$ | *i)* $(\mathbb{Z}/2\mathbb{Z})^2$<br>*ii)* $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_3$ | $\{1\}$ | $3$ |
| $(\mathbb{Z}/2\mathbb{Z})^2$ | *any* | $\mathbb{Z}/2\mathbb{Z}$ | *i)* $\neq 0$<br>*ii)* $0$ | *i)* $\mathbb{Z}/2\mathbb{Z}$<br>*ii)* $(\mathbb{Z}/2\mathbb{Z})^2$ | $\{1\}$ | $3$ |
| $\mathbb{Z}/3\mathbb{Z}$ | $\neq 0$ | $\{1\}$ | $0$ | $\alpha_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $3$ |
| $\mathbb{Z}/6\mathbb{Z}$ | $\neq 0$ | $\{1\}$ | $0$ | $\{1\}$ | $\mathbb{Z}/2\mathbb{Z}$ | $3$ |
| $\mathbb{Z}/2\mathbb{Z}$ | $\neq 0$ | $\mathbb{Z}/2\mathbb{Z}$ | *i)* $\neq 0$<br>*ii)* $0$ | *i)* $\mu_2 \times \mathbb{Z}/2\mathbb{Z}$<br>*ii)* $M_2 \rtimes A_4$ | $\{1\}$ | $2$ |
| $\mu_2 \times \mathbb{Z}/2\mathbb{Z}$ | $\neq 0$ | $\mathbb{Z}/2\mathbb{Z}$ | $\neq 0$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{1\}$ | $2$ |
| $\mathbb{Z}/3\mathbb{Z}$ | *a)* $\neq 0$<br>*b)* $0$ | *a)* $\{1\}$<br>*b)* $\mathbb{Z}/3\mathbb{Z}$ | $0$ | $S_3$ | $\{1\}$ | $2$ |
| $\mathbb{Z}/4\mathbb{Z}$ | $\neq 0$ | $\{1\}$ | $0$ | $\alpha_2$ | $\mathbb{Z}/2\mathbb{Z}$ | $2$ |
| $\mathbb{Z}/6\mathbb{Z}$ | $\neq 0$ | $\{1\}$ | $0$ | $\{1\}$ | $\{1\}$ | $2$ |

TABLE 4.1. Automorphism group schemes of bielliptic surfaces

COROLLARY 4.9 ([**Mar20**], Corollary 1.6). *Let $X$ be a (quasi-)bielliptic surface. Then, the following hold:*

*(1)* $h^0(X, T_X) \leq 3$.
*(2) If $X$ is bielliptic or $p \neq 2$, then* $h^0(X, T_X) \leq 2$.
*(3)* $h^0(X, T_X) = 1$ *if and only if* $\omega_X \not\cong \mathcal{O}_X$ *if and only if* $\mathrm{Aut}_X$ *is smooth.*

## 5. Deformation Theory and Liftability of Bielliptic Surfaces

In [**Par13**] H. Partsch studied deformations of elliptic fiber bundles and classified deformations of bielliptic surfaces. It is not difficult to see that bielliptic surfaces are elliptic fiber bundles over elliptic curves, which are not Abelian surfaces. In what follows we present some of Partsch's main results concerning deformations of bielliptic surfaces and refer to the original article for proofs and details.

Definition 5.1. Let $S$ be a scheme over some ring $R$. An $R$-morphism $X \to S$ is called *elliptic fiber bundle* if there exist a surjective étale morphism $S' \to S$ and an elliptic curve $E$ over Spec $R$ such that

$$X \times_S S' \cong E \times_R S'.$$

An elliptic fiber bundle $X \to S$ is called *Jacobian*, if there exists a section $\sigma$ of $X \to S$.

Note that an elliptic fiber bundle is a proper and smooth morphism.

A deformation of an elliptic fiber bundle $X \to C$ over a field $k$ can be understood in two different senses. First, as a deformation of $X$ as a scheme, and secondly, as a deformation of the fibration $X \to C$. Moreover, if the elliptic fiber bundle $X/C$ is Jacobian, we can consider the deformations of $X/C$ which admit a lift of the section. To define this notions more precisely, let $\mathcal{A}lg_{\mathrm{W}}$ denote the category of local Artinian $W$-algebras with residue field $k$, where $W = W(k)$ denotes the ring of Witt vectors of $k$.

Definition 5.2. Let $k$ be a perfect field and let $A \in \mathcal{A}lg_{\mathrm{W}}$ be a local Artinian $W$-algebra with residue field $k$.

(1) Let $X$ be a scheme over $k$. A *deformation of $X$ over $A$* is a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $\mathcal{X}$ is a flat scheme over $A$. Note that $i$ is then necessarily a closed immersion and that by the definition of Cartesian diagram the induced morphism $X \to \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} k$ is an isomorphism. Two deformations $(\mathcal{X}, i)$ and $(\mathcal{X}', i')$ of $X$ over $A$ are *isomorphic*, if there exists a morphism $f \colon \mathcal{X} \to \mathcal{X}'$ over $A$ that induces the identity on the closed fiber $X$, i.e., such that $i' = f \circ i$. Note that by flatness, $f$ has to be an isomorphism.

Let

$$\mathcal{D}ef_X \colon \mathcal{A}lg_{\mathrm{W}} \longrightarrow (\text{Sets})$$

denote the functor of Artin rings given by

$$\mathcal{D}ef_X(A) = \{\text{isomorphism classes of deformations of } X \text{ over } A\},$$

for every $A \in \mathcal{A}lg_{\mathrm{W}}$.

(2) A *deformation of a fibration* $f\colon X \to C$ *over* $A$ is a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ C & \xrightarrow{\ j\ } & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $\pi$ and $\pi \circ F$ are flat. Thus, a deformation of a fibration $f\colon X \to C$ is a morphism $F\colon \mathcal{X} \to \mathcal{C}$ between deformations of $X$ and $Y$ together with an identification of the restriction of $F$ to the closed fiber with $f$, i.e., such that $F|_X = f$ (cf. [**Ser06**, Definition 3.4.1]). Two deformations $(\mathcal{X}/\mathcal{C}, F)$ and $(\mathcal{X}'/\mathcal{C}', F')$ of a fibration $X/C$ over $A$ are called *isomorphic* if there exist two isomorphisms of deformations $g\colon \mathcal{X} \to \mathcal{X}'$ and $h\colon \mathcal{C} \to \mathcal{C}'$ over $A$ of $X$ and $C$ respectively making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ g\ } & \mathcal{X}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{C} & \xrightarrow{\ h\ } & \mathcal{C}' \end{array}$$

Let

$$\mathcal{F}ib_{X/C}\colon \mathcal{A}lg_{\mathrm{W}} \to (\text{Sets})$$

denote the functor of Artin rings given by

$$\mathcal{F}ib_{X/C}(A) = \left\{ \begin{array}{l} \text{isomorphism classes of deformations} \\ \text{of the fibration } X/C \text{ over } A \end{array} \right\},$$

for every $A \in \mathcal{A}lg_{\mathrm{W}}$.

(3) Let $(X/C, \sigma)$ be a Jacobian elliptic fiber bundle with section $\sigma$. A *Jacobian deformation of* $(X/C, \sigma)$ *over* $A$ is a deformation $(\mathcal{X}/\mathcal{C}, F)$ of the fibration $X/C$ over $A$ together with a morphism $\Sigma\colon \mathcal{C} \to \mathcal{X}$, which is a lift of $\sigma$.

Let

$$\mathcal{J}ac_{X/C}\colon \mathcal{A}lg_{\mathrm{W}} \to (\mathrm{Sets})$$

denote the functor of Artin rings given by

$$\mathcal{J}ac_{X/C}(A) = \left\{ \begin{array}{l} \text{isomorphism classes of Jacobian deformations} \\ \text{of the Jacobian fibration } X/C \text{ over } A \end{array} \right\},$$

for every $A \in \mathcal{A}lg_{\mathrm{W}}$.

Note that, for a Jacobian elliptic fiber bundle $X/C$, the functor $\mathcal{J}ac_{X/C}$ can be considered as the subfunctor of $\mathcal{F}ib_{X/C}$ of deformations admitting a section, since different choices of a lift of $\sigma$ yield isomorphic Jacobian deformations.

**5.1. The Associated Jacobian Fibration of an Elliptic Fiber Bundle.** Given an elliptic fiber bundle $f\colon X \to C$ over $k$, we can associate to it a Jacobian elliptic fiber bundle in the same way as we did for bielliptic surfaces (cf. Definition 1.23). First, note that $f$ is a proper and smooth morphism and its geometric fibers are connected curves. By [**BLR90**, §9.4, Proposition 4] the identity component $\mathbf{Pic}^0_{X/C}$ of the Picard scheme $\mathbf{Pic}_{X/C}$ is an Abelian $C$-scheme. In particular, $\mathbf{Pic}^0_{X/C} \to C$ is a Jacobian elliptic fiber bundle over $k$. We called it the *associated Jacobian* of the fiber bundle $X/C$ and denote it by $J_{X/C} \coloneqq \mathbf{Pic}^0_{X/C}$.

Now let $\mathcal{X}/\mathcal{C}$ be a deformation of $X/C$ as a fibration. We can again consider the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/\mathcal{C}}$, which is again representable by an scheme $\mathbf{Pic}_{\mathcal{X}/\mathcal{C}}$ whose identity component $\mathbf{Pic}^0_{\mathcal{X}/\mathcal{C}}$ is an Abelian $\mathcal{C}$-scheme of relative dimension 1. Moreover, it is a Jacobian deformation of the associated Jacobian $J \coloneqq J_{X/C}$ of $X/C$. Thus, one get a natural morphism of functors

$$\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}, \quad \mathcal{X}/\mathcal{C} \mapsto \mathbf{Pic}^0_{\mathcal{X}/\mathcal{C}}.$$

Moreover, $\mathcal{X}/\mathcal{C}$ is in a natural way a torsor under $\mathbf{Pic}^0_{\mathcal{X}/\mathcal{C}}$.

THEOREM 5.3 ([**Par13**], Theorem 4.4). *The morphism of functors $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ is formally smooth.*

REMARK 5.4. Note that by the above theorem, an arbitrary bielliptic surfaces $X/C$ over $k$ has a formal deformation as a fibration if and only if its associated Jacobian $J/C$ has a formal deformation as a Jacobian fibration. In others words: $\mathcal{F}ib_{X/C}$ is unobstructed (or equivalently, smooth) if and only if the same is true for $\mathcal{J}ac_{J/C}$ (see, for example, [**Ser06**, Proposition 2.2.5.(iii)] and [**Ser06**, Proposition 2.2.10.(i)]).

Thus, we can restricted ourselves to the study of deformation of Jacobian bielliptic surfaces. Since a Jacobian bielliptic surface $J/C$ can be consider as an elliptic curve over $C$, to study the deformation functor $\mathcal{J}ac_{J/C}$ one first has to describe the deformations of elliptic curves over some scheme $S$.

For a local Artinian algebra $A \in \mathcal{A}lg_{\mathrm{W}}$, let $\mathcal{S}$ be a proper flat $A$-scheme such that the special fiber $S := \mathcal{S} \times_{\mathrm{Spec}\,(A)} \mathrm{Spec}\,(k)$ is regular. Let $\mathcal{J}/\mathcal{S}$ be an elliptic curve over $\mathcal{S}$. By [**Par13**, Proposition 3.2] there exist a finite étale cover $\mathcal{S}' \to \mathcal{S}$ with group $G$ such that

$$\mathcal{J} \cong (\mathcal{E} \times_A \mathcal{S}')/G,$$

where $\mathcal{E}$ is an elliptic curve over $\mathrm{Spec}\,(A)$ and the action is the diagonal action given by the Galois action on $\mathcal{S}'$ and by a homomorphism $G \to \mathrm{Aut}(\mathcal{E})$ on the left factor. Then, we have the following necessary and sufficient criterion for the existence of Jacobian deformations.

PROPOSITION 5.5 ([**Par13**], Corollary 3.3). *Let $J/S$ be an elliptic curve over $S$, given by $(E \times_{Spec(k)} S')/G$ for some finite étale Galois cover $S' \to S$ with group $G$. Denote the action of $G$ on $E$ by $\rho_0$. Then, a Jacobian deformation $\mathcal{J}/\mathcal{S}$ of $J/S$ exists if and only if there exist a deformation $\mathcal{E}$ of $E$ over $A$ together with an extension of the action of $\rho_0$.*

Now let $X = (E \times F)/G$ be a Jacobian elliptic surface over $k$, where the action of $G$ on the product $E \times F$ is given by $(x, y) \mapsto (\omega(x), y + c)$, where $\omega \in \mathrm{Aut}(E, 0)$ is an automorphism of $E$ of order $d \in \{2, 3, 4, 6\}$, which fixes the identity, and $c$ is a $d$-torsion point of $F$. By the previous proposition, the Jacobian deformations of $X$ are of the form

$$(\mathcal{E} \times \mathcal{F})/\Gamma,$$

where $\mathcal{E}$ is a deformation of $E$ extending the automorphism $\omega$ and $\mathcal{F}$ is a deformation of $F$ with a torsion point lifting the point $c$ on $F$. Denote by $(E, \omega)$ the functor of deformations of $E$ lifting the automorphism $\omega$ and by $(F, c)$ the functor of deformations of $F$ lifting the torsion point $c$ on $F$. Then we have

$$\mathcal{J}ac_{X/C} \cong (E, \omega) \times (F, c).$$

To write down a versal family for $\mathcal{J}ac_{X/C}$ Partsch treats the problem separately for each factor. He obtains the following result.

PROPOSITION 5.6 ([**Par13**], Proposition 6.2). *The versal hull $R$ of the functor $\mathcal{J}ac_{X/C}$ is given as follows.*

| | $p = 2$ | $p = 3$ | $p > 3$ |
|---|---|---|---|
| $d = 2$ | $W[[t_E]] \otimes W[[q-1]][\sqrt[2]{q}]$ | $W[[t_E]] \otimes W[[t_F]]$ | $W[[t_E]] \otimes W[[t_F]]$ |
| $d = 3$ | $W[[t_E]] \otimes W$ | $W[\pi] \otimes W[[q-1]][\sqrt[3]{q}]$ | $W \otimes W[[t_F]]$ |
| $d = 4$ | $W[i] \otimes W[[q-1]][\sqrt[4]{q}]$ | $W \otimes W[[t_F]]$ | $W \otimes W[[t_F]]$ |
| $d = 6$ | $W \otimes W[[q-1]][\sqrt[2]{q}]$ | $W[\pi] \otimes W[[q-1]][\sqrt[3]{q}]$ | $W \otimes W[[t_F]]$ |

TABLE 5.1. Versal hull of the functor $\mathcal{J}ac_{X/C}$.

*where $i$ is a primitive fourth root of unity, $\pi^2 = 3$ and $q$ is the* Serre-Tate parameter *of $F$ from the Serre-Tate theory of local moduli (cf.* [**KM85**, Section 8.9]*).*

This explicit description of the deformation ring gives us the following result.

COROLLARY 5.7. *Formal Jacobian deformations of Jacobian bielliptic surfaces are unobstructed, i.e., $\mathcal{J}ac_{X/C}$ is smooth for every Jacobian bielliptic surface $X/C$ over $k$.*

PROOF. Recall that a deformation functor is unobstructed if and only if the versal deformation $R$ is a regular local ring (see, for example, [**Har10**, Ex. 15.6]). □

COROLLARY 5.8. *Formal deformations of bielliptic surfaces as fibrations are unobstructed, i.e., the deformation functor $\mathcal{F}ib_{X/C}$ is smooth.*

PROOF. By the previous corollary, the functor $\mathcal{J}ac_{J/C}$, where $J/C$ is the associated Jacobian bielliptic surfaces of $X/C$ is smooth. By Theorem 5.3, the morphism of functors $\mathcal{F}ib_{X/C} \to \mathcal{J}ac_{J/C}$ is formally smooth. Thus, $\mathcal{F}ib_{X/C}$ is smooth by [**Ser06**, Proposition 2.2.5.(iii)]. □

For every bielliptic surface $X/C$ over $k$ there is a natural morphism of functors

$$\mathcal{F}ib_{X/C} \to \mathcal{D}ef_X, \ \mathcal{X}/\mathcal{C} \mapsto \mathcal{X},$$

given by forgetting the map $\mathcal{X} \to \mathcal{C}$. In his classification of deformations of bielliptic surfaces Partsch shows that this two deformation functors are actually isomorphic.

THEOREM 5.9 ([**Par13**], Theorem 6.7). *Every deformation $\mathcal{X}$ of a bielliptic surface $X$ induces a lifting of the elliptic fibration $X \to \mathrm{Alb}(X)$, i.e.,*

$$\mathcal{F}ib_{X/C} \cong \mathcal{D}ef_X.$$

Moreover, a versal deformation of a bielliptic surface is algebraizable.

PROPOSITION 5.10 ([**Par13**], Proposition 6.8). *Let $X$ be a bielliptic surface over $k$. Let $\mathcal{X} \to \mathrm{Spf}\,(R)$ be a formal versal deformation of $\mathcal{D}ef_X$. Then there exist a projective scheme $\mathcal{Y}$ over $R$ such that $\mathcal{X}$ is the formal completion of $\mathcal{Y}$ along the special fiber.*

As we have seen, the Albanese fibration of a bielliptic surface extends to any deformation. Since bielliptic surfaces has two elliptic fibration, it is natural to ask if the second fibration also extends to any deformation. This is indeed the case.

PROPOSITION 5.11 ([**Par13**], Proposition 6.9). *Let $X$ be a bielliptic surface. Then every deformation $\mathcal{X}$ of $X$ extends both fibrations.*

THEOREM 5.12. *Every bielliptic surface over $k$ lifts projectively to characteristic zero.*

PROOF. Corollary 5.8 and Theorem 5.9 imply that for every bielliptic surface $X$, the deformation functor $\mathcal{D}ef_X$ is smooth. By Proposition 5.10, every formal lifting is algebraizable. □

Remark 5.13. It is important to clarify here, that although every bielliptic surface lifts to characteristic zero, it is not true, that every bielliptic surface lift over the Witt ring. This was already observed by Lang in [**Lan95**], who give examples of bielliptic surfaces in characteristic 2 and 3 which does not lift over the Witt ring $W(k)$ but only over a ramified extension.

CHAPTER 2

# Moduli of Bielliptic Surfaces

## 1. Families of Bielliptic Surfaces in Characteristic $p \notin \{2,3\}$

### 1.1. Previous Work on Moduli of Bielliptic Surfaces.

1.1.1. *Moduli Spaces of Complex Bielliptic Surfaces.*

Over the complex numbers $\mathbb{C}$, moduli spaces of bielliptic surfaces were studied by H. Tsuchihashi in [**Tsu79**] building on previous work of T. Suwa [**Suw69**]. In what follows, we describe the main results of their work and refer to the original papers for the details.

**T. Suwa: On Hyperelliptic Surfaces**

Classically, it was already known that over the complex numbers $\mathbb{C}$ there are seven different types of bielliptic surfaces. Viewing bielliptic surfaces as elliptic bundles over an elliptic curve whose total spaces have first Betti number $b_1 = 2$, Suwa describes the seven types of bielliptic surfaces as quotient spaces of Abelian surfaces.

More precisely, for a bielliptic surface $X$ of a given type, he gives an Abelian surface $A$ (described by a period matrix) and an automorphism $g$ of $A$ such that $X \cong A/\langle g\rangle$, where $\langle g\rangle$ denotes the group generated by the automorphism $g$. He recovers this way the classification of bielliptic surfaces established earlier by Bagnera-DeFrancis [**Bd08**] and Enriques-Severi [**ES09, ES10**].

Theorem 1.1 ([**Suw69**], Theorem, p. 473). *Any complex bielliptic surface can be expressed as the quotient space of an Abelian variety $A$ by the group generated by an automorphism $g$ of $A$. The period matrix of $A$ and the automorphism $g$ are given in Table 1.1, with $\rho = \exp(2\pi i/3)$ and $\tau, \omega \in \mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$.*

Moreover, the surfaces of each type form an everywhere effectively parameterized and complete[1] complex analytic family (see [**Tsu79**, Remark 1] for a proof of this fact). For the first two types of bielliptic surfaces, i.e., the ones corresponding to the types $(a1)$ and $(a2)$,

[1]Here "effectively parameterize" means that the Kodaira-Spencer map is injective and "complete" means that the family contains all sufficiently small deformations of each of its fibers. The family is complete, for example, if the infinitesimal Kodaira-Spence map is surjective (Theorem of completeness).

| Type | Period matrix of $A$ | Automorphism $g$ |
|---|---|---|
| $(a1)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix}$ | $[x,y] \mapsto [x+\frac{1}{2}, -y]$ |
| $(a2)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{2} & \omega \end{pmatrix}$ | |
| $(b1)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix}$ | $[x,y] \mapsto [x+\frac{1}{3}, \rho y]$ |
| $(b2)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1-\rho}{3} & \rho \end{pmatrix}$ | |
| $(c1)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & i \end{pmatrix}$ | $[x,y] \mapsto [x+\frac{1}{4}, iy]$ |
| $(c2)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1+i}{2} & i \end{pmatrix}$ | |
| $(d)$ | $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix}$ | $[x,y] \mapsto [x+\frac{1}{6}, -\rho y]$ |

TABLE 1.1. Possible types of complex bielliptic surfaces.

each family is parameterized by two parameters $\tau, \omega \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For the remaining types only one parameter $\tau \in \mathbb{H}$ is needed.

Although not explicitly mentioned by Suwa, this already suggest that $\mathbb{H}^2$ and $\mathbb{H}$ provide good candidates of parameter spaces for the different types of bielliptic surfaces. They can not be moduli spaces though, since two different points can correspond to isomorphic surfaces. This will be worked out further by Tsuchihashi.

An important point showed by Suwa is that the different types of bielliptic surfaces are topologically different (cf. [**Suw69**, Remark 3]). Indeed, by considering the first homology groups and using results of S. Itaka [**Iit69**] on the fundamental group of these surfaces, it is shown that the seven types of complex bielliptic surfaces are completely classified topologically. From this, Suwa concludes that the plurigenera of bielliptic surfaces are topological invariants.

REMARK 1.2. Nowadays, the invariance of plurigenera under smooth projective deformations is known to be true by results of Siu [**Siu98, Siu02**] (see also [**Tsu02**]). Moreover, already in 1970 Itaka [**Iit70**] proved the invariance of plurigenera for deformations of compact complex surfaces.

From Suwa's topological classification of complex bielliptic surfaces it follows that a complex analytic family of bielliptic surfaces (over a connected base) can only contain bielliptic surfaces of the same type as fibers, since all the fibers of such a family are diffeomorphic. Therefore, a moduli space for complex bielliptic surfaces has to split into a disjoint union of seven moduli spaces, each one parameterizing one type of bielliptic surfaces.

**H. Tsuchihashi: Compactifications of the Moduli Spaces of Hyperelliptic Surfaces**

Building up on the work of Suwa and his explicit description of the seven types of complex bielliptic surfaces, Tsuchihashi constructs coarse moduli spaces for each type of bielliptic surfaces as product of modular curves.

Let us shortly recall the definition of modular curves. Consider the modular group

$$\Gamma \coloneqq \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$$

and the congruence subgroup

$$\Gamma_0(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod N \right\} \Big/ \{\pm I\}.$$

Then $\Gamma$ acts on the upper half plane $\mathbb{H}$ from the left by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}, \quad \tau \in \mathbb{H}.$$

The quotient space $\Gamma\backslash\mathbb{H}$ can then be identified with the coarse moduli space of complex elliptic curves $M_{1,1}$. Moreover, we define the *modular curve* $Y_0(N)$ to be the quotient space $\Gamma_0(N)\backslash\mathbb{H}$ corresponding to the congruence subgroup $\Gamma_0(N)$. Note that $M_{1,1}$ and $Y_0(N)$ are both Riemann surfaces (cf. [**DS05**, Chapter 2]).

Tsuchihashi's result on the coarse moduli spaces of complex bielliptic surfaces is the following.

THEOREM 1.3 ([**Tsu79**], Proposition 2). *Let $M_{1,1} \coloneqq \Gamma\backslash\mathbb{H}$ denote the coarse moduli space of complex elliptic curves and $Y_0(N) \coloneqq \Gamma_0(N)\backslash\mathbb{H}$ the modular curve corresponding to the congruence subgroup $\Gamma_0(N)$. Then, the coarse moduli spaces for complex bielliptic surfaces of each type are given in Table 1.2.*

In order to construct fine moduli spaces, Tsuchihashi rigidify the moduli problem by considering complex bielliptic surfaces together with suitable base points satisfying certain conditions (cf. [**Tsu79**, Proposition 3 and Theorem 2]). Moreover, a universal family is given for each one of these fine moduli spaces.

Finally, the author consider natural compactifications of these fine moduli spaces and using the theory of torus embeddings (today known as toric varieties), he shows that the points

| Type | Coarse moduli space |
|---|---|
| $(a1)$ | $M^{(a1)} := Y_0(2) \times M_{1,1}$ |
| $(a2)$ | $M^{(a2)} := Y_0(2)^2$ |
| $(b1)$ | $M^{(b1)} := Y_0(3)$ |
| $(b2)$ | $M^{(b2)} := Y_0(3)$ |
| $(c1)$ | $M^{(c1)} := Y_0(4)$ |
| $(c2)$ | $M^{(c2)} := Y_0(4)$ |
| $(d)$ | $M^{(d)} := Y_0(6)$ |

TABLE 1.2. Coarse moduli spaces of complex bielliptic surfaces.

of the compactified moduli spaces correspond naturally to possibly degenerate bielliptic surfaces with base points.

1.1.2. *Moduli Spaces of Bielliptic Surfaces in Positive Characteristic.*

Moduli spaces of bielliptic surfaces in positive characteristic were studied by W. Seiler in [**Sei87a, Sei87b**] as part of his study of global moduli spaces for elliptic surfaces. There, however, surfaces in characteristic 2 and 3 are excluded and the moduli spaces he obtains are all coarse moduli spaces.

**W. Seiler: Global Moduli for Elliptic Surfaces with a Section**

In [**Sei87a**] Seiler shows the existence of coarse moduli spaces for elliptic surfaces with a section (cf. [**Sei87a**, Theorem 7]). In particular, he proves the existence of coarse moduli spaces for bielliptic surfaces with a section of the Albanese fibration (which we have called *Jacobian* bielliptic surfaces). Furthermore, the author shows (cf. [**Sei87a**, Lemma 9]) that the corresponding moduli functor splits in a natural way into a disjoint union of subfunctors parameterizing Jacobian bielliptic surfaces $f\colon X \to \mathrm{Alb}(X)$ for which the canonical bundle $\omega_X$ has order $n \in \{2, 3, 4, 6\}$ in the Picard group $\mathrm{Pic}(X)$, respectively. Thus, the coarse moduli space of Jacobian bielliptic surfaces splits into four disjoint components. This generalizes and agrees with Suwa's work over the complex numbers.

**W. Seiler: Global Moduli for Polarized Elliptic Surfaces**

In [**Sei87b**] the existence of coarse moduli spaces for (numerically) polarized elliptic surfaces not necessarily admitting a section is shown (cf. [**Sei87b**, Theorem 2.15]). In particular, Seiler proves the existence of a Hilbert scheme for polarized bielliptic surfaces and shows that it splits into connected components parameterizing bielliptic surfaces a each type, respectively (cf. [**Sei87b**, Lemma 2.3]). Furthermore, he obtains the existence of coarse moduli spaces of polarized and numerically polarized bielliptic surfaces.

**1.2. Separation of Families.** As we have seen, for all moduli spaces of bielliptic surfaces consider by Suwa, Tsuchihashi and Seiler, there is a result about the splitting of the moduli space into disjoint components.

Suwa's work, for instance, shows that a family of bielliptic surfaces over a connected base can only have bielliptic surfaces of the same type as fibers. Hence, it is possible to separate families of bielliptic surfaces of different types. This result is what we refer to as *separation of families*. The splitting of the moduli spaces in disjoint components is an important consequence of the separation of families.

In the following sections we will prove that the separation of families also holds in characteristic different from 2 and 3. This generalizes Suwa's result for complex bielliptic surfaces. Moreover, it also generalizes Seiler's results, as we do not impose any extra condition in the families of bielliptic surfaces we consider, i.e., our families not necessarily come with a (global) section or a polarization.

The separation of families of bielliptic surfaces over the complex numbers relies on the following two results, where $X$ is a bielliptic surface over $\mathbb{C}$.

(1) *Invariance of the plurigenera.* For every positive integer $n$ the $n^{th}$-plurigenus of $X$, denoted by $p_n(X) \coloneqq \dim H^0(X, {\omega_X}^{\otimes n})$, is invariant under (holomorphic) deformation (see, for example, [**Siu98**], [**Siu02**] and [**Tsu02**]). Since bielliptic surfaces have torsion canonical sheaf, this is equivalent to the invariance of the order of $\omega_X$ in $\mathrm{Pic}(X)$ under deformation.

(2) *Invariance of the torsion of the first homology group:* The first homology group $H_1(X, \mathbb{Z})$ is invariant under deformation, as it is a topological invariant, and so is its torsion subgroup $H_1(X, \mathbb{Z})_{tors}$. Since $X$ have geometric genus $p_g(X) = 0$, the first homology group of $X$ is isomorphic to the Néron-Severi group $\mathrm{NS}(X)$ by Lemma 2.2 of Chapter 1. Thus, the invariance of the torsion of $H_1(X, \mathbb{Z})_{tors}$ is equivalent to the invariance of the torsion of the Néron-Severi group $\mathrm{NS}^{\tau}(X) \coloneqq \mathrm{NS}(X)_{tors}$.

The first result allow us to separate bielliptic surfaces into four classes according to the order of the canonical sheaf $\omega_X$ in $\mathrm{Pic}(X)$ (cf. Table 1.2 in Chapter 1). However, it could still be the case that two surfaces, whose canonical sheaves have the same order, deform into each other. For instance a surface of type $(a1)$ could, in principle, deform into a surface of type $(a2)$. The second result excludes such possibility, as surface of different types whose canonical sheaf have the same order have different first homology groups (see Table 2.1 in Chapter 1).

The previous discussion suggests the following strategy to generalize the result on the separation of families to bielliptic surfaces over field of characteristic $p \notin \{2, 3\}$. First, we will prove that for a bielliptic surface $X$ over an algebraically closed field $k$ of characteristic different from 2 and 3, the following two facts hold:

(a) The order of the canonical sheaf $\omega_X$ in $\mathrm{Pic}(X)$ is invariant under deformation.
(b) The torsion of the Néron-Severi group $\mathrm{NS}^\tau(X)$ is invariance under deformation.

This is Proposition 1.11. Second, we will calculate the Néron-Severi group for bielliptic surfaces in characteristic different from 2 and 3. This is done in Proposition 1.14.

**1.3. Families of Bielliptic Surfaces in Characteristic $p \notin \{2, 3\}$.** When dealing with moduli problems, it is usual to consider proper and flat families. The following proposition shows that in our case it is enough to consider families where all fibers are smooth.

PROPOSITION 1.4. *Let $X$ be a proper and normal surface over an algebraically closed field $k$ with at most rational double points as singularities. Consider the minimal resolution*

$$\pi \colon \widetilde{X} \to X$$

*of $X$. If $\widetilde{X}$ is a bielliptic surface over $k$, then $\pi$ is an isomorphism and $X$ is smooth.*

PROOF. This is a consequence of the fact that on a bielliptic surface there are no $(-2)$-curves, i.e., irreducible rational curves $C$ with $C^2 = -2$. (cf. Proposition 1.11 in Chapter 1).

□

REMARK 1.5. Note that Proposition 1.4 is also true for quasi-bielliptic surfaces.

DEFINITION 1.6. By a *family of bielliptic surfaces over a (Noetherian) base scheme $S$* we will mean an algebraic space $\mathcal{X}$ together with a proper and smooth morphism of algebraic spaces $\pi \colon \mathcal{X} \to S$, whose geometric fibers are bielliptic surfaces.

We will need some results on the relative Picard functor of a family of bielliptic surfaces. Since one of this results holds for families of bielliptic surfaces over an arbitrary base scheme $S$, we state them now, before we start considering families in characteristic different from 2 and 3.

PROPOSITION 1.7. *Let $S$ be a scheme and $\pi \colon \mathcal{X} \to S$ a family of bielliptic surfaces over $S$. Then,*

(a) *The relative Picard functor* $\mathrm{Pic}_{\mathcal{X}/S}$ *is representable by an algebraic space* $\mathbf{Pic}_{\mathcal{X}/S}$ *over $S$, which is separated and locally of finite type.*
(b) *Moreover, if $S$ is a scheme over* $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]$, *then* $\mathbf{Pic}_{\mathcal{X}/S}$ *is smooth over $S$.*

PROOF. The geometric fibers of the proper and smooth morphism $\pi \colon \mathcal{X} \to S$ are by definition bielliptic surfaces and hence reduced and irreducible. Thus, $\pi$ is cohomologically flat in dimension zero (cf. [**Gro63**, Proposition 7.8.6]). Then, the representability of the relative Picard functor by an algebraic space $\mathbf{Pic}_{\mathcal{X}/S}$ follows from [**BLR90**, §8.3, Theorem 1]. Moreover, $\mathbf{Pic}_{\mathcal{X}/S}$ is separable by [**BLR90**, §8.4, Theorem 3] and locally of finite presentation by [**Sta19**, Tag 0DNI]. In particular, $\mathbf{Pic}_{\mathcal{X}/S}$ is of finite type over $S$. This proves (a).

Now we prove (b). For every point $s \in S$, we have $H^2(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) = 0$, since the geometric fibers of $\pi \colon \mathcal{X} \to S$ are bielliptic surfaces over fields of characteristic different from 2 and 3. Then, $\mathbf{Pic}_{\mathcal{X}/S}$ is formally smooth over $S$ by [**BLR90**, §8.4, Proposition 2]. Moreover, since $\mathbf{Pic}_{\mathcal{X}/S}$ is of finite presentation over $S$, it is smooth over $S$ by the Infinitesimal Lifting Criterion [**Sta19**, Tag 04AM]. □

From now on, we will consider only bielliptic surfaces over fields of characteristic different from 2 and 3.

We are ready to state the main result of this section. By a bielliptic surface of type $(i)$ we will mean a bielliptic surface of one of the types in the Bagnera-DeFranchis list.

THEOREM 1.8. *Let $\pi \colon \mathcal{X} \to S$ be a family of bielliptic surfaces over a connected Noetherian base scheme $S$ over $\mathbb{Z}[\frac{1}{6}]$. If a geometric fiber of $\pi$ is a bielliptic surface of type $(i)$, then every geometric fiber of $\pi$ is a bielliptic surface of the same type $(i)$.*

In order to prove this theorem, we will first introduce some notation and prove some partial result, which will then be used in the proof of the theorem.

**Notation:** Consider a smooth and proper family of bielliptic surfaces $\pi \colon \mathcal{X} \to S = \mathrm{Spec} R$, where $R$ is a complete discrete valuation ring with residue field $k$ of characteristic $p \notin \{2, 3\}$ and fraction field $K$. Let $X_k = \mathcal{X} \times_R k$ and $X_K = \mathcal{X} \times_R K$ denote the special fiber and the generic fiber of $\pi$, respectively. Fix an algebraic closure $\overline{k}$ of $k$ and let $X_{\overline{k}} = X_0 \times_k \overline{k}$ denotes the geometric special fiber. Similarly, by fixing an algebraic closure $\overline{K}$ of $K$, we denote the geometric generic fiber by $X_{\overline{K}} = X_K \times_K \overline{K}$. We thus have the following diagram:

$$
\begin{array}{ccccccccc}
X_{\overline{k}} & \longrightarrow & X_k & \longrightarrow & \mathcal{X} & \longleftarrow & X_K & \longleftarrow & X_{\overline{K}} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\
\operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k & \longrightarrow & S = \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} K & \longleftarrow & \operatorname{Spec} \overline{K}
\end{array}
$$

LEMMA 1.9. *With the notation above we have:*

*(a) The identity component* $\mathbf{Pic}^0_{\mathcal{X}/S}$ *of the relative Picard scheme* $\mathbf{Pic}_{\mathcal{X}/S}$ *is an elliptic curve over* $S$.

*(b) The* torsion Néron-Severi scheme $\mathbf{NS}^{\tau}_{\mathcal{X}/S}$ *represents the quotient fpqc-sheaf*

$$\mathbf{Pic}^{\tau}_{\mathcal{X}/S}/\mathbf{Pic}^{0}_{\mathcal{X}/S}{}_{(fpqc)}$$

*and is a finite and étale* $S$*-group scheme.*

PROOF. We first show that the family $\pi\colon \mathcal{X} \to S$ is projective and that $\mathcal{X}$ is in fact a scheme. Indeed, the geometric special fiber $X_{\overline{k}}$ is projective. By [**Gro61**, Corollaire 6.6.5] projectiveness descends along field extensions. Therefore, the special fiber $X_k$ is projective as well. In particular, $X_k$ admits an ample line bundle $L$. Now, from Proposition 1.7, we know that $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth. Moreover, since $S = \operatorname{Spec} R$ with $R$ a complete discrete valuation ring, the reduction map

$$\mathbf{Pic}_{\mathcal{X}/S}(R) \to \mathbf{Pic}_{X_k/k}(k)$$

is surjective by Hensel's Lemma (cf. [**Gro67**, Théorème 18.5.17 ]). Thus, the line bundle $L$ on $X_k$ extends to a line bundle $\mathcal{L}$ on $\mathcal{X}$. Since ampleness is an open condition by [**Laz04**, Theorem 1.2.17], we see that the line bundle $\mathcal{L}$ on $\mathcal{X}$ is relatively ample. Hence, $\pi\colon \mathcal{X} \to S$ is projective. Finally, since $S$ is a scheme and $\pi\colon \mathcal{X} \to S$ is projective (actually, quasi-projective is enough), $\mathcal{X}$ is a scheme by [**Knu71**, II.7.6].

Note finally that, since $\pi\colon \mathcal{X} \to S$ is projective and flat with integral geometric fibers, by [**Kle05**, Theorem 9.4.8] $\mathbf{Pic}_{\mathcal{X}/S}$ is in fact a scheme over $S$.

We now prove the claims of the lemma.

(a) For every $s \in S$ let $\overline{k(s)}$ denote an algebraic closure of the residue field $k(s)$ of $s \in S$. The geometric fiber $X_{\overline{s}}$ of $\pi$ over $s$ is a bielliptic surface over a field of characteristic different from 2 and 3. Thus, the Picard scheme $\mathbf{Pic}^0_{X_{\overline{s}}/\overline{k(s)}}$ is an elliptic curve, since it is reduced and of dimension 1. By faithfully flat descent, it follows that for every $s \in S$ all the $\mathbf{Pic}^0_{X_s/k(s)}$ are complete, smooth and of dimension one, since

$$\mathbf{Pic}^0_{X_{\overline{s}}/\overline{k(s)}} \cong \mathbf{Pic}^0_{X_s/k(s)} \times_{k(s)} \overline{k(s)}.$$

Then, by [**Kle05**, Theorem 9.5.20], $\mathbf{Pic}^0_{\mathcal{X}/S}$ exist and is an open and closed subgroup scheme of $\mathbf{Pic}_{\mathcal{X}/S}$, which is smooth and proper over $S$. Since the geometric fibers of $\mathbf{Pic}^0_{\mathcal{X}/S}$ are given by the Picard schemes $\mathbf{Pic}^0_{X_{\overline{s}}/\overline{k(s)}}$ and these are all elliptic curves, $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an elliptic curve over $S$.

(b) The following proof is taken from [**Ant11**, Section 2]. We reproduce it here for the reader's convenience.

Since $\pi\colon \mathcal{X} \to S$ is projective and smooth with irreducible geometric fibers, by [**Kle05**, Theorem 9.6.16] and [**Kle05**, Exercise 9.6.18] $\mathbf{Pic}^\tau_{\mathcal{X}/S}$ is an open and closed group subscheme of $\mathbf{Pic}_{\mathcal{X}/S}$, projective (in particular proper) over $S$. Since $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth over $S$, so is $\mathbf{Pic}^\tau_{\mathcal{X}/S}$. Now, the existence of $\mathbf{NS}^\tau_{\mathcal{X}/S}$ as a scheme over $S$ follows from [**Gab63**, Théorème 7.1].

According to [**Ber63**, Proposition 9.2] $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is separated, flat and of finite type over $S$, and the canonical projection $p\colon \mathbf{Pic}^\tau_{\mathcal{X}/S} \to \mathbf{NS}^\tau_{\mathcal{X}/S}$ is faithfully flat.

Now, since the canonical projection $p\colon \mathbf{Pic}^\tau_{\mathcal{X}/S} \to \mathbf{NS}^\tau_{\mathcal{X}/S}$ is surjective, $\mathbf{Pic}^\tau_{\mathcal{X}/S}$ is proper over $S$ and $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is separated and of finite type over $S$. Then $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is proper over $S$ by [**GW10**, Proposition 12.59].

For every $s \in S$ the fiber of $\mathbf{NS}^\tau_{\mathcal{X}/S}$ over $s$ is given by $\mathbf{NS}^\tau_{X_s/k(s)}$, which is the étale finite $k(s)$-group scheme of connected components $\pi_0(\mathbf{Pic}^\tau_{X_s/k(s)})$ (see [**DG80**, Chapter II, 5, no 1]). Since $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is proper and quasi-finite (as the fibers $\mathbf{NS}^\tau_{X_s/k(s)}$ are finite $k(s)$-group schemes), it is finite. Since $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is flat over $S$ with smooth fibers, it is smooth over $S$. Moreover, $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is even étale over $S$ by [**Sta19**, Tag 02GM], since its fibers are étale.

□

Remark 1.10. The claim of Lemma 1.9 (a) still holds under the more general assumption that $\pi\colon \mathcal{X} \to S$ is a family of bielliptic surfaces over a *reduced* Noetherian scheme $S$. However, to proof requires one to work with algebraic spaces rather than schemes. We will prove the claim in this more general setting in the next section (see Proposition 1.17) in order to define the Albanese scheme of the family $\pi\colon \mathcal{X} \to S$.

Proposition 1.11. *With the notation above we have:*

*(a)* $\operatorname{ord}(\omega_{X_{\overline{K}}}) = \operatorname{ord}(\omega_{X_{\overline{k}}})$*, that is, the order of the canonical bundle is invariant under deformation.*

*(b)* $\mathrm{NS}^\tau(X_{\overline{K}}) \cong \mathrm{NS}^\tau(X_{\overline{k}})$*, that is, the torsion of the Néron-Severi group is invariant under deformation.*

Proof. (a) First, recall that by Corollary 1.15 the canonical sheaf $\omega_X$ of a bielliptic surface $X$ is algebraically trivial, that is, $\omega_X \in \mathrm{Pic}^0(X)$. Moreover, note that the order of $\omega_X$ always divide 12, hence $\omega_X$ is always a 12-torsion point of $\mathrm{Pic}^0(X)$, that is, $\omega_X \in \mathrm{Pic}^0(X)[12]$.

From Lemma 1.9, we know that $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an elliptic curve over $S$. Since 12 is invertible on $S$, it follows that the kernel $\mathbf{Pic}^0_{\mathcal{X}/S}[12]$ of the multiplication by 12 map on $\mathbf{Pic}^0_{\mathcal{X}/S}$ is a finite and étale $S$-group scheme (see, for example, [**KM85**, Theorem 2.3.1]). Then, by Hensel's lemma, the reduction map

$$\mathrm{Pic}^0(X_{\overline{K}})[12] \cong \mathbf{Pic}^0_{\mathcal{X}/S}[12](\overline{K}) \to \mathbf{Pic}^0_{\mathcal{X}/S}[12](\overline{k}) \cong \mathrm{Pic}^0(X_{\overline{k}})[12]$$

sending $\omega_{X_{\overline{K}}}$ to $\omega_{X_{\overline{k}}}$ is bijective.

(b) Similar as in (a), since $\mathbf{NS}^\tau_{\mathcal{X}/S}$ is finite and étale over $S$ by the previous lemma, the reduction map

$$\mathrm{NS}^\tau(X_{\overline{K}}) \cong \mathbf{NS}^\tau_{\mathcal{X}/S}(\overline{K}) \to \mathbf{NS}^\tau_{\mathcal{X}/S}(\overline{k}) \cong \mathrm{NS}^\tau(X_{\overline{k}})$$

is bijective.

□

REMARKS 1.12.
(1) Since the Betti numbers are constant in smooth families, it follows from part (b) of the previous proposition that the Néron-Severi group of bielliptic surfaces in characteristic different from 2 and 3 is constant under deformation.
(2) Part (b) of Proposition 1.11 can be related to the following result by D. Maulik and B. Poonen.

PROPOSITION 1.13 (Proposition 3.6, [**MP12**]). *Let $S$ be a Noetherian scheme. Let $s, t \in S$ be such that $s$ is a specialization of $t$. Let $p = \mathrm{char}(k(s))$. Let $\pi\colon \mathcal{X} \to S$ be a smooth proper morphism. Then it is possible to choose a homomorphism*

$$\mathrm{sp}_{\overline{t},\overline{s}}\colon \mathrm{NS}(\mathcal{X}_{\overline{t}}) \to \mathrm{NS}(\mathcal{X}_{\overline{s}})$$

*with the following properties:*
*(a) If $p = 0$, then $\mathrm{sp}_{\overline{t},\overline{s}}$ is injective and its cokernel is torsion-free.*
*(b) If $p > 0$, then (a) holds after tensoring with $\mathbb{Z}[1/p]$.*

If $\pi\colon \mathcal{X} \to S$ is a family of bielliptic surfaces, then the cokernel of the specialization map $\mathrm{sp}_{\overline{t},\overline{s}}$ vanishes. Indeed, in their proof, Maulik and Poonen show the torsion-freeness in (a) and (b) by using the following fact (cf [**SGA71**, Expose X, Appendice 7.13.10]): for any prime $l \neq p$ there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{NS}(\mathcal{X}_{\overline{t}}) \otimes \mathbb{Z}_l & \hookrightarrow & H^2_{\text{ét}}(\mathcal{X}_{\overline{t}}, \mathbb{Z}_l(1)) \\
\downarrow & & \| \\
\mathrm{NS}(\mathcal{X}_{\overline{s}}) \otimes \mathbb{Z}_l & \hookrightarrow & H^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \mathbb{Z}_l(1)).
\end{array}$$

From this diagram one obtains the following inclusion

$$\mathrm{coker}(\mathrm{sp}_{\bar{t},\bar{s}}) \otimes \mathbb{Z}_l \subseteq \mathrm{coker}(\mathrm{NS}(\mathcal{X}_{\bar{t}}) \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_l(1))) = T_l\mathrm{Br}(\mathcal{X}_{\bar{t}}),$$

where $T_l\mathrm{Br}(\mathcal{X}_{\bar{t}})$ denotes the Tate module of the Brauer group of $\mathcal{X}_{\bar{t}}$. The last equality can be shown using the Kummer sequence as in [**Mil80**, V.3.29(d)].

Now, if $\mathcal{X}_{\bar{t}}$ is a bielliptic surface, then $\rho(\mathcal{X}_{\bar{t}}) = b_2(\mathcal{X}_{\bar{t}})$. As explained in the proof of Theorem 2.5 in Chapter 1, this implies $T_l\mathrm{Br}(\mathcal{X}_{\bar{t}}) = 0$. Hence, $\mathrm{sp}_{\bar{t},\bar{s}}$ is bijective, i.e., $\mathrm{NS}(\mathcal{X}_{\bar{t}}) \cong \mathrm{NS}(\mathcal{X}_{\bar{s}})$.

So far we have prove the invariance under deformation of the order of the canonical sheaf and of the Néron-Severi group. We still have to calculate the Néron-Severi group of bielliptic surfaces in characteristic different from 2 and 3. In order to do so, we will use the fact that, according to Theorem 5.12 in Chapter 1, bielliptic surfaces lift to characteristic zero.

Proposition 1.14. *The torsion of the Néron-Severi group of a bielliptic surface $X$ in characteristic $p \geq 5$ coincide with the torsion of the Néron-Severi group of a bielliptic surface of the same type in characteristic zero.*

Proof. Let $X$ be a bielliptic surface over a field of characteristic $p \geq 5$ and denote by $B = \mathrm{Alb}(X)$ the Albanese of $X$. We consider two cases:

Case 1: $X$ is of type $(a1), (b1), (c1)$ or $(d)$. In this case $X$ is of the form $(E \times F)/G$ with $G$ finite and cyclic. Then, by [**Lan79**, Proposition 4.1], there is a split exact sequence

$$0 \to \mathrm{Pic}^0(B) \to \mathrm{Pic}^\tau(X) \to F^G \to 0.$$

After choosing a base point on $B$, we have

$$\mathrm{Pic}^0(B) \cong B = \mathrm{Alb}(X) \cong \mathrm{Pic}^0(X),$$

since $p_g(X) = 0$. Thus, $\mathrm{NS}^\tau(X) = \mathrm{Pic}^\tau/\mathrm{Pic}^0(X) \cong F^G$. This group can be calculate directly (cf. [**Băd01**, pp. 159-160]) and coincide with the results in characteristic zero.

Case 2: $X$ is of type $(a2), (b2)$ or $(c2)$. Then $f\colon X \to \mathrm{Alb}(X)$ has no section. By Theorem 5.12 in Chapter 1, we can lift $X$ to characteristic zero. Since $X$ has no section, the lift cannot have a section. This together with the invariance of the order of the canonical sheaf under deformation implies that $X$ lifts to a bielliptic surfaces of the same type. Since the torsion of the Néron-Severi is invariant under deformation and it is known in characteristic zero, we get the result. □

Corollary 1.15. *Let $X$ be a bielliptic surface over a field of characteristic $p \notin \{2,3\}$. Then the Néron-Severi group $\mathrm{NS}(X)$ of $X$ is given by the following table.*

| *Type* | NS($X$) |
|---|---|
| *(a1)* | $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$ |
| *(a2)* | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ |
| *(b1)* | $\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$ |
| *(b2)* | $\mathbb{Z}^2$ |
| *(c1)* | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ |
| *(c2)* | $\mathbb{Z}^2$ |
| *(d)* | $\mathbb{Z}^2$ |

TABLE 1.3. Néron-Severi group of $X$.

We are now ready to prove Theorem 1.8.

PROOF OF THEOREM 1.8. Since $S$ is a connected Noetherian scheme, it is sufficient to show that for two points $s_0, s_1 \in S$, such that $s_1$ specializes to $s_0$, the two geometric fibers $X_{\overline{s_0}}$ and $X_{\overline{s_1}}$ are bielliptic surfaces of the same type. Indeed, since $S$ is connected, we may assume that $S$ is irreducible and we can then compare the geometric fibers over any two points with the geometric fiber over the generic point of $S$.

Since $S$ is Noetherian, by [**GW10**, Proposition 15.7], we can always find a morphism $\operatorname{Spec} R \to S$ from a discrete valuation ring $R$ to $S$ such that the generic point $\eta$ of $\operatorname{Spec} R$ maps to $s_1 \in S$ and the special point maps to $s_0$. Thus, after making a base change $\operatorname{Spec} R \to S$, it is sufficient to consider the case $S = \operatorname{Spec} R$, with $R$ a discrete valuation ring.

Moreover, we may assume that $R$ is complete. Indeed, since $R$ is a discrete valuation ring, the completion $\widehat{R}$ of $R$ is also a discrete valuation ring. Then, the base change $\mathcal{X}_{\widehat{R}}$ of $\mathcal{X}$ to the completion $\widehat{R}$ preserves the (geometric) special fiber and the geometric generic fiber of $\pi \colon \mathcal{X} \to \operatorname{Spec} R$ in the following sense: let $k$ denote the residue field of $R$, $K$ its field of fractions and $\overline{K}$ an algebraic closure of $K$. Since the residue fields of $R$ and $\widehat{R}$ are isomorphic, the special fiber of $\mathcal{X}_{\widehat{R}} \to \widehat{R}$ is isomorphic to the special fiber $\mathcal{X}_k$ of $\mathcal{X} \to R$. On the other hand, after choosing an algebraic closure $\overline{\widehat{K}}$ of the fraction field $\widehat{K}$ of $\widehat{R}$ containing $\overline{K}$, the geometric generic fiber of $\mathcal{X}_{\widehat{R}}$ coincide we the pullback of the geometric generic fiber $\mathcal{X}_{\overline{K}}$ along the field extension $\overline{K} \to \overline{\widehat{K}}$, that is, there is an isomorphism

$$\mathcal{X}_{\widehat{R}} \times_{\widehat{R}} \overline{\widehat{K}} \cong (\mathcal{X}_{\overline{K}}) \times_{\overline{K}} \overline{\widehat{K}}.$$

But both, the order of the canonical bundle and the Néron-Severi group of a smooth proper variety are preserved by algebraically closed field extensions, by [**Sta19**, Tag 0CC5] and [**MP12**, Proposition 3.1], respectively. Thus, after a base change to the completion of $R$, we may assume that $R$ is complete.

In this manner, we can reduce to the case where $\pi \colon \mathcal{X} \to S = \operatorname{Spec} R$ is a family of bielliptic surfaces over a complete discrete valuation ring $R$. Then, the result follows from Proposition

1.11 and Corollary 1.15. Indeed, according to Proposition 1.11, the order of the canonical bundle and the Néron-Severi group are invariant. Moreover, the order of the canonical bundle of bielliptic surfaces is given in Table 1.2 in Chapter 1 and Corollary 1.15 gives the values of the Néron-Severi groups. Since the values of these invariants taken together are different for different types of bielliptic surfaces, the desired result follows.

□

COROLLARY 1.16. *Let $S$ be a Noetherian scheme over $\mathbb{Z}[\frac{1}{6}]$ and $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces over $S$. Let $S^{(i)} \subseteq S$ denote the set of point of $S$ such that $\mathcal{X}_{\overline{s}}$ is a bielliptic surface of type $(i)$, that is,*

$$S^{(i)} := \{s \in S \mid \mathcal{X}_{\overline{s}} \textit{ is a bielliptic of type } (i)\} \subseteq S.$$

*Then, $S^{(i)}$ is a finite union of connected components of $S$ and is therefore a closed subset of $S$. Moreover, $S$ splits into a disjoint union of schemes $S^{(i)}$ for every type $(i)$, that is,*

$$S = \bigsqcup_{(i)} S^{(i)}.$$

PROOF. By Theorem 1.8, the fibers over a connected component of $S$ are all bielliptic surfaces of the same type. Thus, $S^{(i)}$ is a union of connected components of $S$. Since $S$ is Noetherian, it has finitely many connected components. Thus, $S^{(i)}$ is a finite union of connected components and thus closed. By endowing $S^{(i)}$ with the reduced induced scheme structure we obtain a reduced closed subscheme of $S$. From Theorem 1.8 it also follows that two different connected component having fibers of different type must be disjoint. Thus, $S$ is the disjoint union of the closed subschemes $S^{(i)}$. □

### 1.4. The Albanese Scheme and the Canonical Cover of a Family.

1.4.1. *The Albanese of a Family and the Albanese Morphism.*

For a bielliptic surface $X$ over an algebraically closed field $k$ we have seen that one of its elliptic fibrations is given by the Albanese map

$$f\colon X \to \mathrm{Alb}(X).$$

Recall that the Albanese variety $\mathrm{Alb}(X)$ of $X$ coincides with the dual of the Picard variety $(\mathbf{Pic}^0_{X/k})_{red}$. Moreover, the Albanese map in this case is projective (as a morphism between projective schemes over a field) and smooth by Theorem 1.5 of Chapter 1.

In this section we want to define in an analogous way for a family of bielliptic surfaces the *Albanese scheme* of the family together with the *Albanese morphism*.

Let $S$ be a Noetherian scheme over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ and consider a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over $S$. We will define the Albanese scheme $\mathbf{Alb}_{\mathcal{X}/S}$ as the dual Abelian scheme of the Abelian scheme $\mathbf{Pic}^0_{\mathcal{X}/S}$. Thus, the first step for defining the Albanese scheme is to show that $\mathbf{Pic}^0_{\mathcal{X}/S}$ is indeed an Abelian scheme over $S$.

If $R$ is a discrete valuation ring and $S = \operatorname{Spec} R$, then $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an elliptic curve over $S$ by Lemma 1.9). In the proof of that lemma we used the fact that the family $\pi\colon \mathcal{X} \to S = \operatorname{Spec} R$ is projective to prove the existence of the relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/S}$. For an arbitrary (Noetherian) scheme $S$ however the family $\pi\colon \mathcal{X} \to S$ is only proper. Nevertheless, one can still prove that the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space $\mathbf{Pic}_{\mathcal{X}/S}$. This, however, turns out to be sufficient to prove that the identity component $\mathbf{Pic}^0_{\mathcal{X}/S}$ of $\mathbf{Pic}_{\mathcal{X}/S}$ is an Abelian scheme. The only technical condition one has to impose on the base scheme $S$ is that it has to be reduced.

PROPOSITION 1.17. *Let $S$ be a Noetherian scheme over* $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ *and* $\pi\colon \mathcal{X} \to S$ *a family of bielliptic surfaces over $S$. If $S$ is reduced, then the identity component* $\mathbf{Pic}^0_{\mathcal{X}/S}$ *of* $\mathbf{Pic}_{\mathcal{X}/S}$ *is an Abelian $S$-scheme.*

PROOF. To prove the claim we used [**Kle05**, Proposition 9.5.20]. A careful study of Kleiman's proof of [**Kle05**, Proposition 9.5.20] reveals that the same result also holds in the category of algebraic spaces. Thus, we may argue as in the proof of part (a) of Lemma 1.9, to show that $\mathbf{Pic}^0_{\mathcal{X}/S}$ is a smooth and proper group algebraic space over $S$. Note that the reduceness hypothesis on $S$ is used to prove the smoothness. Furthermore, for every geometric point $\overline{s} \in S$ the geometric fiber $\mathbf{Pic}^0_{\mathcal{X}_{\overline{s}}/\overline{k(s)}}$ of $\mathbf{Pic}^0_{\mathcal{X}/S} \to S$ is an elliptic curves (remember: $\mathcal{X}_{\overline{s}}$ is a bielliptic surface over a field $\overline{k(s)}$ of characteristic different from 2 and 3). In particular, $\mathbf{Pic}^0_{\mathcal{X}_{\overline{s}}/\overline{k(s)}}$ is connected. Thus, $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an Abelian algebraic space $S$. Finally, by a result of Raynaud (cf. [**FC90b**, Theorem 1.9]), any Abelian algebraic space over a scheme is an Abelian scheme. Hence, $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an Abelian scheme over $S$. □

DEFINITION 1.18. Let $S$ be a reduced Noetherian scheme over $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ and $\pi\colon \mathcal{X} \to S$ a family of bielliptic surfaces over $S$. The *Albanese scheme* $\mathbf{Alb}_{\mathcal{X}/S}$ of the family $\pi\colon \mathcal{X} \to S$ is the dual Abelian scheme of $\mathbf{Pic}^0_{\mathcal{X}/S}$, i.e.,

$$\mathbf{Alb}_{\mathcal{X}/S} \coloneqq (\mathbf{Pic}^0_{\mathcal{X}/S})^\vee \coloneqq \mathbf{Pic}^0_{(\mathbf{Pic}^0_{\mathcal{X}/S})/S}.$$

Since $\mathbf{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space, Yoneda's lemma yields the existence of a universal sheaf $\mathcal{P}$ on $\mathcal{X} \times \mathbf{Pic}_{\mathcal{X}/S}$ called the *Poincaré sheaf* (see for example [**Kle05**, Exercise 9.4.3]).

Assume now that the family $\pi\colon \mathcal{X} \to S$ admits a section $\sigma\colon S \to \mathcal{X}$. Similarly as in [**Ant11**, Proposition 2.6], using the Poincaré sheaf and the section $\sigma\colon S \to \mathcal{X}$ we can construct an $S$-morphism

$$f\colon \mathcal{X} \to \mathbf{Alb}_{\mathcal{X}/S}$$

which send the section $\sigma$ to the identity section of $\mathbf{Alb}_{\mathcal{X}/S}$ and which induces the Albanese fibration on each surface in the family.

DEFINITION 1.19. Let $S$ be a reduced Noetherian scheme over Spec $\mathbb{Z}[\frac{1}{6}]$ and $\pi\colon \mathcal{X} \to S$ a family of bielliptic surfaces over $S$ together with a section $\sigma\colon S \to \mathcal{X}$. Then, we call

$$f\colon \mathcal{X} \to \mathbf{Alb}_{\mathcal{X}/S}$$

the *Albanese morphism* of the family $\pi\colon \mathcal{X} \to S$ with respect to $\sigma$.

As it turns out, the Albanese morphism have similar nice properties to the Albanese map of a bielliptic surface, as shown by the next result.

LEMMA 1.20. *The Albanese morphism $f\colon \mathcal{X} \to \mathbf{Alb}_{\mathcal{X}/S}$ is proper and smooth.*

PROOF. Since both $\mathcal{X}$ and $\mathbf{Alb}_{\mathcal{X}/S}$ are proper over $S$ the Albanese morphism $f$ is proper by [**Sta19**, Tag 04NX]. Now note that $f$ is a morphism of relative dimension 1. Thus, the fibers of $f$ have all dimension 1. Moreover, both $\mathcal{X}$ and $\mathbf{Alb}_{\mathcal{X}/S}$ are smooth over $S$ and therefore regular over $S$. Then $f$ is flat by miracle flatness [**Mat86**, Theorem 23.1]. Since $f$ is flat and have smooth geometric fibers, it is smooth. □

1.4.2. *Two Results of Seiler: The Second Fibration and Multiple Fibers of a Family.*

The Albanese scheme $\mathbf{Alb}_{\mathcal{X}/S}$ and the Albanese morphism $f\colon \mathcal{X} \to \mathbf{Alb}_{\mathcal{X}/S}$ of a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces are the generalizations to families of the Albanese variety and the Albanese fibration of a bielliptic surface. Recall that a bielliptic surface always has a second fibration transversal to the Albanese fibration. It turns out that, for a *polarized* family of bielliptic surfaces, i.e., a family $\pi\colon \mathcal{X} \to S$ together with a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$, one can also globalize the second fibration of the bielliptic surfaces in the family to obtain a second fibration of the family. Moreover, the multiple fibers of the second fibration of the bielliptic surfaces of the family can also be put together in a flat family. Hence, one obtains multiple fibers of the second fibration of the family.

The following results are due to W. Seiler [**Sei87b**].

Let us start with the globalization of the second fibration of a family. For a bielliptic surface $X$ over an algebraically closed field $k$, the construction of the second elliptic fibration boils down to the following fact (see the proof of [**BM77**, Theorem 3]): taking the fiber $F_0$ of the Albanese fibration $f\colon X \to \mathrm{Alb}(X)$ over the identity element of $\mathrm{Alb}(X)$ and an ample divisor $L$ on $X$ (recall that $X$ is projective), one can construct an idecomposable curve of canonical type $C$, which is transversal to the Albanese fibration, i.e., such that $C^2 = 0$ and $(C \cdot F_0) > 0$. Then, by [**Mum69**, §2, Step (II) and Step (III)] there is an integer $n > 0$ such that $nC$ yields the existence of the second elliptic fibration $g\colon X \to \mathbb{P}^1$.

Consider now a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over a Noetherian scheme $S$ together with a polarization $\mathcal{L}$ on $\mathcal{X}$. Then, Seiler's argues as follows to obtain the second elliptic fibration of the family (cf. see proof of [**Sei87b**, Lemma 1.6]): using the polarization $\mathcal{L}$ and the fiber $\mathcal{F}_0 := \mathcal{X} \times_{\mathbf{Alb}\mathcal{X}/S} S$ of the Albanese morphism over the zero section, one can construct a line bundle $\mathcal{D} := \mathcal{L}^{\otimes a} \otimes \mathcal{F}_0^{\otimes b}$ on $\mathcal{X}$ with self-intersection zero on every geometric fiber of $\pi\colon \mathcal{X} \to S$. Then, for every geometric fiber of $\pi$, some multiple of the restriction of $\mathcal{D}$ to that fibers yields the second fibration on that geometric fiber. Since $S$ is Noetherian, it is possible to choose a global $n$ such that the restrictions of $\mathcal{D}^{\otimes n}$ to the geometric fibers induce the second elliptic fibration on every bielliptic surface of the family.

PROPOSITION 1.21 ([**Sei87b**], Lemma 1.6). *Let $S$ be a Noetherian scheme over* $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ *and let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces over $S$ together with a polarization $\mathcal{L}$ on $\mathcal{X}$. Then there exist a smooth family $\gamma\colon \mathcal{C} \to S$ and an $S$-morphism $g\colon \mathcal{X} \to \mathcal{C}$ that induces the second elliptic fibration on each bielliptic surface in the family.*

Using the fact that the normal sheaf of a multiple fiber of the second elliptic fibration of a bielliptic surface in characteristic different from 2 and 3 is nontrivial, since the second fibration has no wild fibers, Seiler further shows the following result.

PROPOSITION 1.22 ([**Sei87b**], Lemma 1.8). *Let $S$ be an irreducible Noetherian scheme over* $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ *and let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces over $S$ together with a polarization $\mathcal{L}$ on $\mathcal{X}$. Let $\gamma\colon \mathcal{C} \to S$ and $g\colon \mathcal{X} \to \mathcal{C}$ be as above. Suppose that for a geometric point $\overline{s} \in S$, the bielliptic surface $\mathcal{X}_{\overline{s}}$ has a multiple fiber $mG_{\overline{s}}$ of the second elliptic fibration $g_{\overline{s}}\colon \mathcal{X}_{\overline{s}} \to \mathbb{P}^1_{\overline{k(s)}}$. Then, there exist a unique flat family $\mathcal{G} \to S$ of curves, a closed $S$-immersion $\mathcal{G} \to \mathcal{X}$, and a section $\sigma\colon S \to \mathcal{C}$, such that $G_{\overline{s}}$ is the fiber of $\mathcal{G} \to S$ over $\overline{s} \in S$, and for every $\overline{t} \in S$, $\mathcal{X}_{\overline{t}}$ has a multiple fiber $mG_{\overline{t}}$ with base point $\sigma(\overline{t})$.*

1.4.3. *The Canonical Cover of a Family.*

Let $S$ be a Noetherian connected scheme over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ and consider a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over $S$. Since $\pi$ is a smooth morphism, the *relative dualizing sheaf* $\omega_{\mathcal{X}/S}$ exist and it is a flat invertible sheaf equal to the determinant $\det \Omega^1_{\mathcal{X}/S}$ of the sheaf of differentials $\Omega^1_{\mathcal{X}/S}$ of $\mathcal{X}$ over $S$ (see, for example, [**Kle80**, Proposition (22)]). Moreover, since the geometric fibers are bielliptic surfaces, we have the following

PROPOSITION 1.23. *Let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces and $\omega_{\mathcal{X}/S}$ its relative canonical sheaf. Then,*

$$\omega^{\otimes n}_{\mathcal{X}/S} \cong \mathcal{O}_{\mathcal{X}} \otimes \pi^*\mathcal{N},$$

*for an invertible sheaf $\mathcal{N}$ on $S$, and an integer $n \in \{2, 3, 4, 6\}$, depending on the type of the bielliptic surfaces in the family. In particular, the element $\xi \in \operatorname{Pic}_{\mathcal{X}/S}(S)$ induced by $\omega^{\otimes n}_{\mathcal{X}/S}$ is trivial.*

PROOF. We follow the proof of [**Vak**, Proposition 28.1.11] and use the general version of Cohomology and Base Change Theorem for algebraic spaces [**Sta19**, Tag 08JR].

Recall that, according to Theorem 1.8, the geometric fibers of the family $\pi\colon \mathcal{X} \to S$ are all bielliptic surfaces of the same type and let $n$ denotes the order of the canonical sheaf $\omega_{\mathcal{X}_{\overline{s}}}$ of the bielliptic surface $\mathcal{X}_{\overline{s}}$ for every geometric point $\overline{s} \in S$. Thus, for every geometric point $\overline{s} \in S$, one has

$$\omega_{\mathcal{X}_{\overline{s}}}^{\otimes n} \cong \mathcal{O}_{\mathcal{X}_{\overline{s}}}. \tag{1.1}$$

By [**Kle05**, Exercise 9.3.11] it follows that $\pi_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$ holds universally, since $\pi$ is proper and flat with reduced and connected geometric fibers. Hence, for every invertible sheaf $\mathcal{M}$ on $S$, the natural morphism $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$ is an isomorphism. Set $\mathcal{F} := \omega_{\mathcal{X}/S}^{\otimes n}$. By the Cohomology and Base Change theorem [**Sta19**, Tag 08JR], $\pi_*\mathcal{F}$ is locally free of rank 1 and for every $s \in S$ the natural map

$$\pi_*\mathcal{F} \otimes_{\mathcal{O}_S} k(s) \to H^0(\mathcal{X}_s, \mathcal{F}_s)$$

is an isomorphism. Setting $\mathcal{N} := \pi_*\mathcal{F}$, there is a natural map of invertible sheaves

$$\pi^*\mathcal{N} = \pi^*\pi_*\mathcal{F} \to \mathcal{F}. \tag{1.2}$$

To prove that this map is an isomorphism, it is enough to show that it is surjective (cf. [**Har77**, Exercise II.7.1]). The last claim can then be proven by showing that the map in (1.2) is surjective on the geometric fibers of $\pi$. But this is clearly the case by (1.1). Indeed, if the cokernel of the map in (1.2) is not 0, then it is not 0 above some geometric point of $S$, but this contradicts (1.1).

Finally, recall that an element $\xi \in \mathrm{Pic}_{\mathcal{X}/S}(S)$ is trivial if it is induced by the pullback to $\mathcal{X}$ of a line bundle on $S$ (see, for example, [**BLR90**, §8.1]).

□

We want to construct a finite étale cover of $\mathcal{X}$ which will play the role of the canonical cover of the family $\pi\colon \mathcal{X} \to S$. In order to do so, we will use the proposition just proved and we need some further results. The first one is due to Raynaud [**Ray70**, Proposition (6.2.1)] and can be found in Milne's Étale Cohomology book stated as follows.

THEOREM 1.24 ([**Mil80**], Proposition III, 4.16). *Let $\pi\colon \mathcal{X} \to S$ be a proper, flat and such that $\pi_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$. For any finite flat commutative group scheme $G$ over $S$*

$$\mathrm{R}^1\, \pi_*(G_{\mathcal{X}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_S(G^{\mathrm{D}}, \mathrm{Pic}_{\mathcal{X}/S}),$$

*where $\mathrm{Pic}_{\mathcal{X}/S}$ is the relative Picard functor of $\mathcal{X}$ over $S$, $G_{\mathcal{X}} = G \times_S \mathcal{X}$ and $G^{\mathrm{D}}$ is the Cartier dual of $G$.*

Recall that in the present situation, the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space $\mathbf{Pic}_{\mathcal{X}/S}$ over $S$ (cf. Lemma 1.17 (a)). From the last theorem by taking global sections, we get

$$H^0(S, \mathrm{R}^1\,\pi_*(G_{\mathcal{X}})) = \mathrm{R}^1\,\pi_*(G_{\mathcal{X}})(S) = \underline{\mathrm{Hom}}_S(G^{\mathrm{D}}, \mathrm{Pic}_{\mathcal{X}/S})(S) = \mathrm{Hom}_S(G^{\mathrm{D}}, \mathbf{Pic}_{\mathcal{X}/S}).$$

Moreover, if $\pi\colon \mathcal{X} \to S$ has a section $\sigma\colon S \to \mathcal{X}$, we have the following proposition.

PROPOSITION 1.25. *Let $\pi\colon \mathcal{X} \to S$ be a proper, flat morphism together with a section $\sigma\colon S \to \mathcal{X}$ and such that $\pi_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$ holds universally. Let $G$ be a finite, flat commutative group scheme over $S$. Then,*

$$H^0(S, \mathrm{R}^1\,\pi_*(G_{\mathcal{X}})) \cong H^1(X, G_{\mathcal{X}})/H^1(S, G).$$

PROOF. Consider the Leray spectral sequence

$$E_2^{p,q} := H^p(S, R^q\pi_*(G_{\mathcal{X}})) \Rightarrow H^{p+q}(\mathcal{X}, G_{\mathcal{X}}).$$

We can form its exact sequence of low-degree terms

$$0 \to H^1(S, \pi_*(G_{\mathcal{X}})) \to H^1(X, G_{\mathcal{X}}) \to H^0(S, R^1\pi_*(G_{\mathcal{X}})) \to H^2(S, \pi_*(G_{\mathcal{X}})) \to H^2(\mathcal{X}, G_{\mathcal{X}}).$$

Now the section $\sigma\colon S \to \mathcal{X}$ of $\pi$ induces a left inverse of the map

$$H^2(S, \pi_*(G_{\mathcal{X}})) \to H^2(\mathcal{X}, G_{\mathcal{X}})$$

induced by $\pi$. Thus, that map is injective. From the exactness of the sequence of low-degree terms it follows that the map

$$H^1(\mathcal{X}, G_{\mathcal{X}}) \to H^0(S, R^1\pi_*(G_{\mathcal{X}}))$$

is surjective. Hence,

$$H^0(S, \mathrm{R}^1\,\pi_*(G_{\mathcal{X}})) \cong H^1(\mathcal{X}, G_{\mathcal{X}})/H^1(S, \pi_*G_{\mathcal{X}}).$$

Finally, note that $\pi_*G_{\mathcal{X}} \cong \pi_*\pi^*G \cong G$, since $\pi_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$ holds universally. □

Recall that the group $H^1(\mathcal{X}, G_{\mathcal{X}})$ classifies isomorphism classes of $G_{\mathcal{X}}$-torsors over $\mathcal{X}$. It turns out that the quotient group

$$H^1(\mathcal{X}, G_{\mathcal{X}})/H^1(S, G)$$

can be interpreted as the group of isomorphism classes of *pointed* $G_{\mathcal{X}}$-torsor over $\mathcal{X}$ in the following sense. Let

$$H^1{}_\bullet(\mathcal{X}, G_{\mathcal{X}}) \subseteq H^1(\mathcal{X}, G_{\mathcal{X}})$$

denotes the subgroup of $H^1(\mathcal{X}, G_{\mathcal{X}})$ of isomorphism classes of $G_{\mathcal{X}}$-torsors $\mathcal{Y} \to \mathcal{X}$ over $\mathcal{X}$ together with an $S$-valued point $y \in \mathcal{Y}_\sigma(S)$, where $\mathcal{Y}_\sigma = \mathcal{Y} \times_S \mathcal{X}$ is the pullback of $\mathcal{Y}$ along the section $\sigma\colon S \to \mathcal{X}$. Then, by [**Ant11**, Lemma 3.3], there is an isomorphism

$$H^1{}_\bullet(\mathcal{X}, G_{\mathcal{X}}) \cong H^1(\mathcal{X}, G_{\mathcal{X}})/H^1(S, G).$$

Hence, we have proved the following (cf. [**Ant11**, Proposition 3.2])):

PROPOSITION 1.26. *Let $\pi \colon \mathcal{X} \to S$ be a proper, flat morphism together with a section $\sigma \colon S \to \mathcal{X}$ and such that $\pi_* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$ holds universally. Let $G$ be a finite, flat commutative group scheme over $S$. Then, there is an isomorphism*

$$H^1{}_{\bullet}(\mathcal{X}, G_{\mathcal{X}}) \xrightarrow{\sim} \mathrm{Hom}_S(G^{\mathrm{D}}, \mathbf{Pic}_{\mathcal{X}/S}).$$

We are interested in the case when the $S$-group scheme $G$ is the group scheme of $n$-th roots of the unity over $S$, i.e., $G = \mu_{n,S}$ for an integer $n \in \{2, 3, 4, 6\}$. Recall that the Cartier dual of $G = \mu_{n,S}$ is the constant group scheme $(\mathbb{Z}/n\mathbb{Z})_S$. Then Proposition 1.26 yields an isomorphism

$$H^1{}_{\bullet}(\mathcal{X}, \mu_{n,\mathcal{X}}) \xrightarrow{\sim} \mathrm{Hom}_S((\mathbb{Z}/n\mathbb{Z})_S, \mathbf{Pic}_{\mathcal{X}/S}).$$

Using Proposition 1.23 and the relative canonical sheaf $\omega_{\mathcal{X}/S}$ we can now define a homomorphism

$$\phi \colon (\mathbb{Z}/n\mathbb{Z})_S \to \mathbf{Pic}_{\mathcal{X}/S}$$

as follows: for every $S$-scheme $T$, set

$$\phi(T) \colon (\mathbb{Z}/n\mathbb{Z})_S(T) \to \mathbf{Pic}_{\mathcal{X}/S}(T),\ 1 \mapsto \omega_{\mathcal{X}_T/T}.$$

By Proposition 1.26, the homomorphism $\phi \in \mathrm{Hom}_S((\mathbb{Z}/n\mathbb{Z})_S, \mathbf{Pic}_{\mathcal{X}/S})$ defined by $\omega_{\mathcal{X}/S}$ determines (uniquely, up to isomorphism) a $\mu_n$-torsor

$$\tau \colon \widetilde{\mathcal{X}} \to \mathcal{X}.$$

together with a section $S \to \widetilde{\mathcal{X}}_\sigma := \widetilde{\mathcal{X}} \times_{\mathcal{X}} S$. Now, if $n \in \{2, 3, 4, 6\}$, then $n$ is invertible on $S$ and $\mu_{n,S}$ is a finite étale $S$-group scheme. Therefore, $\mu_{n,\mathcal{X}}$ is also a finite étale $S$-group scheme and the map $\tau \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is finite and étale (cf. [**Mil80**, Proposition III 4.2]).

On the other hand, since the construction above is functorial, the restriction of $\tau$ to the geometric fibers coincide with the cyclic canonical cover of $\mathcal{X}_{\overline{s}}$ defined by $\omega_{\mathcal{X}_{\overline{s}}}$ (see Definition 1.21 in Chapter 1) for every geometric point $\overline{s} \in S$. In other words: for every geometric point $\overline{s} \in S$, the $\mu_n$-torsor

$$\tau_{\overline{s}} \colon \widetilde{\mathcal{X}}_{\overline{s}} \to \mathcal{X}_{\overline{s}}$$

is the canonical Abelian cover of the bielliptic surface $\mathcal{X}_{\overline{s}}$. In particular, $\widetilde{\mathcal{X}}_{\overline{s}}$ is an Abelian surface.

Finally, we are ready to defined the canonical cover of a family of bielliptic surfaces with a section:

DEFINITION 1.27. Let $S$ be a Noetherian connected scheme over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]$ and $\pi \colon \mathcal{X} \to S$ a family of bielliptic surfaces together with a section $\sigma \colon S \to \mathcal{X}$ of $\pi$, such that $\omega_{\mathcal{X}/S}$ has order $n \in \{2, 3, 4, 6\}$ in $\mathrm{Pic}_{\mathcal{X}/S}(S)$. By the *the canonical cover* of the family $\pi \colon \mathcal{X} \to S$ we mean the finite and étale $\mu_n$-torsor $\tau\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ obtained from the relative canonical sheaf $\omega_{\mathcal{X}/S}$, as constructed above.

Note that the composition $\pi \circ \tau\colon \widetilde{\mathcal{X}} \to \mathcal{X} \to S$ is smooth and proper (since $\tau$ is étale and finite) and that its geometric fibers are Abelian surfaces, since they are the canonical covers of the bielliptic surfaces in the family. However, $\widetilde{\mathcal{X}}/S$ need not be an Abelian scheme. For that a section is needed and in fact that condition is sufficient, as the following deformation result shows.

Theorem 1.28 ([**MFK94**], Theorem 6.14). *Let $S$ be a connected, locally Noetherian scheme. Let $\pi\colon X \to S$ be a smooth projective morphism, and let $\varepsilon\colon S \to X$ be a section of $\pi$. Assume that for one geometric point $s \in S$, the fiber $X_s$ of $\pi$ is an Abelian variety with identity $\varepsilon(s)$. Then $X$ is an Abelian scheme over $S$ with identity $\varepsilon$.*

Remark 1.29. The projectivity hypothesis in the above theorem is needed for the existence of the Hilbert and Hom schemes used in the proof of the theorem. Working with algebraic spaces one can relax the projectivity hypothesis to properness, since then those functors are representable without the need of the hypothesis of projectivity (cf. [**Sta19**, Tag 0D01] and [**Sta19**, Tag 0D1C]). Thus, the above theorem is true in the category of algebraic spaces if we consider smooth and proper morphisms instead of smooth projective ones. Moreover, Raynaud has proved that any Abelian algebraic space over a scheme is automatically an Abelian scheme (cf. [**FC90b**, Theorem 1.9]).

Corollary 1.30. *Let $S$ be a Noetherian connected scheme over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ and $\pi\colon \mathcal{X} \to S$ a family of bielliptic surfaces. Then, étale locally, the family $\pi\colon \mathcal{X} \to S$ admits a canonical cover which is an Abelian scheme. More precisely, there exist an affine scheme $S'$ and a surjective étale morphism $g\colon S' \to S$ of $S$ such that the base change $\tau'\colon \widetilde{\mathcal{X}}' \to \mathcal{X}'$ of the canonical cover $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ along $g$ is an Abelian scheme.*

Proof. Since $\pi\colon \mathcal{X} \to S$ is surjective and smooth, by [**Gro67**, Corollaire 17.16.3 (ii)] there exist an affine scheme $S'$ and a surjective étale morphism $g\colon S' \to S$ of $S$ such that the base change $\pi'\colon \mathcal{X}' \to S'$ of $\mathcal{X}$ along $g$ has a section. Thus, after an étale base change, we may assume that$S$ is affine and that $\pi\colon \mathcal{X} \to S$ has a section $\sigma\colon S \to \mathcal{X}$. Let $\tau\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ denote the canonical cover of $\pi\colon \mathcal{X} \to S$ with respect to $\sigma$. Then, the composition $\pi \circ \tau\colon \widetilde{\mathcal{X}} \to \mathcal{X} \to S$ is surjective and smooth. Hence, it admits a section étale locally, again by [**Gro67**, Corollaire 17.16.3 (ii)]. Then, the result follows from Theorem 1.28 and Remark 1.29.

□

### 1.5. The Brauer Group of Bielliptic Surfaces in Characteristic $p \notin \{2, 3\}$.

In what follows we calculate the Brauer group of bielliptic surfaces over fields of characteristic different from 2 and 3 as an application of our knowledge of the Néron-Severi group of these surfaces (cf. Corollary 1.15). As it turns out, the Brauer group of such surfaces coincide with the torsion part of the Néron-Severi group. For a discussion of the Brauer group of complex bielliptic surfaces and their relation to the Brauer group of their canonical covers we refer to the article [**BFTV19**].

For a scheme $X$ the *cohomological Brauer group* $\mathrm{Br}'(X)$ is defined as the torsion part of the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$, i.e.,

$$\mathrm{Br}'(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)_{tors}.$$

Some authors define the cohomological Brauer group of $X$ as the whole group $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Note, however, that for a regular integral Noetherian scheme $X$, the cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is torsion (cf. [**Poo17**, Proposition 6.6.7]). Thus, for regular integral Noetherian schemes both definitions of the cohomological Brauer group coincide.

On the other hand, the *Brauer group* $\mathrm{Br}(X)$ of a scheme $X$ is defined to be the group of similarity classes of Azumaya algebras on $X$ (cf. [**Mil80**, Chapter IV, §2]). By an unpublished result of Gabber (see [**dJ05**] for a proof) the two Brauer groups are isomorphic for any quasi-compact and separated scheme $X$ with an ample line bundle. In particular, if $X$ is a quasi-projective scheme over $\mathrm{Spec}\, R$ for some Noetherian ring $R$, then $\mathrm{Br}(X) \cong \mathrm{Br}'(X)$.

Thus, for a bielliptic surface $X$ we have

$$\mathrm{Br}(X) \cong \mathrm{Br}'(X) \cong H^2_{\text{ét}}(X, \mathbb{G}_m)_{tors}.$$

The main result of this section is the following.

THEOREM 1.31. *Let $X$ be a bielliptic surface over an algebraically closed field of characteristic $p \notin \{2,3\}$. Then the Brauer group $\mathrm{Br}(X)$ of $X$ is a finite Abelian group isomorphic to the torsion subgroup of the Néron-Severi group of $X$, i.e.,*

$$\mathrm{Br}(X) \cong \mathrm{NS}(X)_{tors}.$$

*Thus, the Brauer group $\mathrm{Br}(X)$ of $X$ is given by Table 1.4.*

| *Type* | $\mathrm{Br}(X)$ |
|---|---|
| *(a1)* | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ |
| *(a2)* | $\mathbb{Z}/2\mathbb{Z}$ |
| *(b1)* | $\mathbb{Z}/3\mathbb{Z}$ |
| *(b2)* | 0 |
| *(c1)* | $\mathbb{Z}/2\mathbb{Z}$ |
| *(c2)* | 0 |
| *(d)* | 0 |

TABLE 1.4. Brauer group of $X$.

Before proving the theorem let us introduce and recall some notation. Let $p$ be a prime number and $A$ an Abelian group. For an integer $n \in \mathbb{Z}$ let $[n]\colon A \to A$ denote the multiplication by $n$ map on $A$. Then, we set

$$\begin{aligned} {}_nA &:= \ker([n]\colon A \to A), \\ A^{(n)} &:= \operatorname{coker}([n]\colon A \to A), \\ {}_{p^\infty}A &:= \varinjlim \ker([p^n]\colon A \to A) \quad \text{and} \\ T_p(A) &:= \varprojlim \ker([p^n]\colon A \to A). \end{aligned}$$

We call ${}_{p^\infty}A$ the *p-torsion subgroup* of $A$ and $T_p(A)$ the *p-adic Tate module* of $A$.

Proof of Theorem 1.31. The following proof is an adaptation of the proof of the analogous result for Enriques surfaces in [**CDL20**, Theorem 1.2.17].

We first proof the result in characteristic zero. By the Lefschetz principle, we may assume $k = \mathbb{C}$. For complex varieties, the Brauer group is isomorphic to the torsion subgroup of $H^2(X, \mathcal{O}_X^*)$ in the analytic topology. After taking cohomology in the exponential sequence, we get $H^2(X, \mathcal{O}_X^*) \cong H^3(X, \mathbb{Z})$. Thus, $\mathrm{Br}(X) \cong H^3(X, \mathbb{Z})_{tors}$. By Theorem 2.3 of Chapter 1, we know that $H^3(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \cong \mathrm{NS}(X)$. Hence,

$$\mathrm{Br}(X) \cong H^3(X, \mathbb{Z})_{tors} \cong H^2(X, \mathbb{Z})_{tors} \cong \mathrm{NS}(X)_{tors}.$$

Suppose now that $p \geq 5$. To calculate the Brauer group in this situation we consider the $p$-torsion and the prime-to-$p$-torsion of the Brauer group separately. By Theorem 2.5 in Chapter 1, for every prime number $l \neq p$ there is an isomorphism

$$H^2_{\text{ét}}(X, \mathbb{Z}_l) \cong H^2_{\text{ét}}(X, \mathbb{Z}_l(1)) \cong \mathrm{NS}(X) \otimes \mathbb{Z}_l.$$

Moreover, that statement remains true for $l = p$ after replacing the étale topology with the flat topology. Let $t_l(X)$ denote the rank of the Tate module $T_l\mathrm{Br}(X)$ of the Brauer group $\mathrm{Br}(X)$ and $t_p(X)$ the rank of $T_pH^2_{\text{fl}}(X, \mathbb{G}_m)$. As in the proof of Theorem 2.5 of Chapter 1, we have $t_l(X) = 0$ for every prime number $l$, including $l = p$. Form this it follows that $\mathrm{Br}(X)$ is a finite Abelian group.

For $l \neq p$: Since $p \nmid l^n$, we can consider the Kummer exact sequence

$$0 \to \mu_{l^n} \to \mathbb{G}_m \xrightarrow{\cdot l^n} \mathbb{G}_m \to 0.$$

After taking cohomology it yields the short exact sequence

$$0 \to \mathrm{Pic}(X)^{(l^n)} \to H^2_{\text{ét}}(X, \mu_{l^n}) \to {}_{l^n}H^2_{\text{ét}}(X, \mathbb{G}_m) \to 0, \tag{1.3}$$

where $\mathrm{Pic}(X)^{(l^n)}$ denotes the cokernel of the multiplication by $l^n$ map on $\mathrm{Pic}(X)$. On the other hand, from the exact sequence

$$0 \to \mathbb{Z}/l^n\mathbb{Z} \to \mathbb{Z}/l^{n+m}\mathbb{Z} \to \mathbb{Z}/l^m\mathbb{Z} \to 0$$

after taking cohomology and passing to the projective limit in $n$ we get the short exact sequence

$$0 \to H^2_{\text{ét}}(X, \mathbb{Z}_l(1))^{(l^m)} \to H^2_{\text{ét}}(X, (\mathbb{Z}/l^m\mathbb{Z})(1)) \to {}_{l^m}H^3_{\text{ét}}(X, \mathbb{Z}_l(1)) \to 0. \tag{1.4}$$

Moreover, since $l \neq p$, there is a (non-canonical) isomorphism $\mu_{l^m} \cong (\mathbb{Z}/l^m\mathbb{Z})(1)$. Combining the sequences (1.3) and (1.4) and using the snake lemma, we get an exact sequence for all $m \geq 1$ including $m = \infty$

$$0 \to (\mathbb{Z}/l^m\mathbb{Z})^{t_l} \to {}_{l^m}\mathrm{Br}(X) \to {}_{l^m}H^3_{\text{ét}}(X, \mathbb{Z}_l(1)) \to 0.$$

Since $\rho(X) = b_2(X)$, the same argument as in the proof of Theorem 2.5 of Chapter 1 shows that $t_l = 0$. Thus, we obtain an isomorphism

$${}_{l^m}\mathrm{Br}(X) \cong {}_{l^m}H^3_{\text{ét}}(X, \mathbb{Z}_l(1)).$$

For $m = \infty$ and using Poincaré duality for étale cohomology we get

$${}_{l^\infty}\mathrm{Br}(X) \cong {}_{l^\infty}H^3_{\text{ét}}(X, \mathbb{Z}_l(1)) \cong {}_{l^\infty}H^2_{\text{ét}}(X, \mathbb{Z}_l(1)).$$

This together with Theorem 2.5 in Chapter 1 yields the isomorphism

$${}_{l^\infty}\mathrm{Br}(X) \cong {}_{l^\infty}(\mathrm{NS}(X) \otimes \mathbb{Z}_l).$$

For $l = p$: A similar argument as above shows that there exist a short exact sequence for all $m \geq 1$ including $m = \infty$

$$0 \to (\mathbb{Z}/p^m\mathbb{Z})^{t_p} \to {}_{p^m}\mathrm{Br}(X) \to {}_{p^m}H^3_{\text{fl}}(X, \mathbb{Z}_p(1)) \to 0.$$

Again, since $\rho(X) = b_2(X)$, we see that $t_p(X) = 0$ and we get an isomorphism

$${}_{p^m}\mathrm{Br}(X) \cong {}_{p^m}H^3_{\text{fl}}(X, \mathbb{Z}_p(1)).$$

By [**Ill79**, Section II(5.22.5)] there exist a short exact sequence

$$0 \to H^3_{\text{fl}}(X, \mathbb{Z}_p(1)) \to H^2(X, W\Omega^1_{X/k}) \xrightarrow{F-1} H^2(X, W\Omega^1_{X/k}).$$

Thus

$${}_{p^\infty}\mathrm{Br}(X) \cong {}_{p^\infty}H^3_{\text{fl}}(X, \mathbb{Z}_p(1)) \cong {}_{p^\infty}\ker(F - 1\colon H^2(X, W\Omega^1_{X/k}) \to H^2(X, W\Omega^1_{X/k})).$$

CLAIM 1.32. *The p-torsion of the Brauer group of X is zero, that is,* ${}_{p^\infty}\mathrm{Br}(X) = 0$.

PROOF OF THE CLAIM: By [**Lan79**, Proposition 4.3] we know that $H^2(X, W\mathcal{O}_X) = 0$, so in particular it is a finitely generated $W$-module. Moreover, by [**Ill79**, Section II.2.D.] all the other cohomology groups $H^j(X, W\Omega^i_{X/k})$ are finitely generated $W$-modules. Thus, $X$ is

Hodge-Witt. According to Illusie (cf. [**Ill83**, Theorem 3.4.1]), for any Hodge-Witt variety there is a canonical decomposition

$$H^n_{\text{cris}}(X/W) \cong \bigoplus_{i+j=n} H^j(X, W\Omega^i_{X/k}).$$

In particular, for $n = 3$ we get

$$H^3_{\text{cris}}(X/W) \cong H^1(X, W\Omega^2_{X/k}) \oplus H^2(X, W\Omega^1_{X/k}).$$

Since $H^3_{\text{cris}}(X/W) \cong W^2$ by Theorem 2.12 in Chapter 1, we see that $H^2(X, W\Omega^1_{X/k})$ is isomorphic to 0, $W$ or $W^2$. In all these cases $H^2(X, W\Omega^1_{X/k})$ has no torsion and so does the kernel of $F - 1$ on $H^2(X, W\Omega^1_{X/k})$. Thus,

$${}_{p^\infty}\mathrm{Br}(X) = {}_{p^\infty}\ker(F - 1\colon H^2(X, W\Omega^1_{X/k}) \to H^2(X, W\Omega^1_{X/k})) = 0.$$

□

So far we have shown that ${}_{l^\infty}\mathrm{Br}(X) \cong {}_{l^\infty}(\mathrm{NS}(X) \otimes \mathbb{Z}_l)$ for $l \neq p$ and ${}_{p^\infty}\mathrm{Br}(X) = 0$. Moreover, according to table 1.3, the torsion of the Néron-Severi group $\mathrm{NS}(X)$ of a bielliptic surface $X$ in characteristic different from 2 and 3 can be $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/3\mathbb{Z}$ or 0. Thus, for $l \neq 2, 3$ we get ${}_{l^\infty}\mathrm{Br}(X) = 0$. Putting all these results together we get

$$\mathrm{Br}(X) = \bigoplus_{l \in \{2,3\}} {}_{l^\infty}\mathrm{Br}(X) = \bigoplus_{l \in \{2,3\}} {}_{l^\infty}(\mathrm{NS}(X) \otimes \mathbb{Z}_l).$$

Now using table 1.3 it is easy to compute $\mathrm{Br}(X)$ and we obtain $\mathrm{Br}(X) \cong \mathrm{NS}(X)_{tors}$. □

## 2. Moduli Stacks of Bielliptic Surfaces

In this section we define and describe the moduli stack $\mathcal{M}_{biell}$ of bielliptic surfaces over $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$, whose geometric points correspond to bielliptic surfaces over algebraically closed fields of characteristic different from 2 and 3. To simplify notation we will denote $\mathcal{M}_{biell}$ by $\mathcal{M}$.

Consider the following moduli problem:

$$\mathcal{M}\colon (\mathrm{Sch}/\mathbb{Z}[1/6])^{op} \longrightarrow (\mathrm{Set}),$$

$$S \longmapsto \mathcal{M}(S) := \left\{ \begin{array}{l} \text{proper and smooth morphism } \mathcal{X} \to S \text{ of} \\ \text{algebraic spaces, whose geometric fibers} \\ \text{are bielliptic surfaces} \end{array} \right\}.$$

More precisely, let $\mathcal{M}$ denote the category fibered in groupoids over $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ whose objects are families of bielliptic surfaces $\pi\colon \mathcal{X} \to S$, where $S$ is a $\mathbb{Z}[\frac{1}{6}]$-scheme, and whose morphisms are Cartesian diagrams

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S' & \longrightarrow & S. \end{array}$$

The projection

$$\mathcal{M} \to (\mathrm{Sch}/\mathbb{Z}[\tfrac{1}{6}]),\ (\pi\colon \mathcal{X} \to S) \mapsto S$$

makes $\mathcal{M}$ into a category fibered in groupoids over the category $(\mathrm{Sch}/\mathbb{Z}[\frac{1}{6}])$ of schemes over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]$. Moreover, $\mathcal{M}$ is a stack.

Proposition 2.1. *The category fibered in groupoids $\mathcal{M}$ is a stack for the étale topology.*

Proof. This is just a special case of the general fact that relative algebraic spaces, the bases of which are schemes, form a stack. More precisely, the category $\mathcal{X}$ whose objects are pairs $(X, U)$, where $U$ is an affine $S$-scheme and $X$ is a $U$-algebraic space, i.e., an algebraic space $X$ together with a morphism of algebraic spaces $X \to U$, and whose morphisms are Cartesian diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ U' & \longrightarrow & U \end{array}$$

is a stack for the étale topology, and even for the fpqc topology on $(\mathrm{Aff}/S)$ (see, for example, [**LMB00**, (3.4.6) and (9.4)]).

□

**2.1. Splitting of the Moduli Stack $\mathcal{M}$.** An important consequence of the separation of families studied in the previous section, that is, of Theorem 1.8, is that the moduli stack $\mathcal{M}$ splits into seven disjoint components corresponding to the stacks parameterizing families of bielliptic surfaces of each type. More precisely, we have the following result.

Proposition 2.2. *The stack of bielliptic surfaces $\mathcal{M}$ splits into a disjoint union*

$$\mathcal{M} = \bigsqcup \mathcal{M}^{(i)},$$

*where $\mathcal{M}^{(i)}$ denotes the substack of $\mathcal{M}$ whose objects are families of bielliptic surfaces of type $(i)$.*

Proof. Let $\mathcal{M}^{(i)}$ denote the substack of $\mathcal{M}$ parameterizing families of bielliptic surfaces of type $(i)$ and denote by $j\colon \mathcal{M}^{(i)} \to \mathcal{M}$ the inclusion morphism. We will show that $j$ is representable by a closed immersion. Hence, $\mathcal{M}^{(i)}$ is a closed substack of $\mathcal{M}$. Let $S$ be a

Noetherian scheme and let $\pi\colon X \to S$ be a family of bielliptic surfaces view as a morphism $\pi\colon S \to \mathcal{M}$. Consider now the fiber product $\mathcal{M}^{(i)} \times_{j,\mathcal{M},\pi} S$, which we will denote by $S^{(i)}$ to simplify notation.

For a scheme $T$, an element of $S^{(i)}(T)$ is given by a triple $(a\colon Y \to T, t\colon T \to S, \sigma)$, where $(a\colon Y \to T) \in \mathcal{M}^{(i)}(T)$ is a family of bielliptic surfaces of type $(i)$, $t\colon T \to S$ is a morphism from $T$ to $S$ and $\sigma$ is an isomorphism $\sigma\colon Y \xrightarrow{\sim} X_T$ over $T$, where $X_T$ denotes the pullback of the family $\pi\colon X \to S$ along $t\colon T \to S$. That is, we have the following diagram

$$\begin{array}{ccccc} Y & \xrightarrow[\sigma]{\sim} & X_T & \longrightarrow & X \\ & {\scriptstyle\tau}\searrow & \downarrow{\scriptstyle\pi_T} & & \downarrow{\scriptstyle\pi} \\ & & T & \longrightarrow & S. \end{array}$$

Now let $S'$ be the scheme whose functor of points is given by

$$S'(T) = \{T \to S \in S(T) \mid X_T \to T \text{ is a family of bielliptic surfaces of type } (i)\}.$$

Clearly, the map $S^{(i)}(T) \to S'(T)$, $(a, t, \sigma) \mapsto t$ is surjective. Moreover, injectivity follows from the definition of the fiber product of categories fibered in groupoids (see, for example, [**LMB00**, (2.2.2)]). Hence, the fiber product $S^{(i)}$ is representable by the scheme $S'$. By Corollary 1.16, $S'$ is a finite union of connected components of $S$. Thus, $S'$ is a closed subscheme of $S$. Hence, the canonical projection $S^{(i)} = \mathcal{M}^{(i)} \times_{j,\mathcal{M},\pi} S \to S$ is representable by a closed immersion and $\mathcal{M}^{(i)}$ is therefore a closed substack of $\mathcal{M}$ (cf. [**LMB00**, Définition (3.14)]). □

From Proposition 2.2 it follows that one may study the moduli stacks $\mathcal{M}^{(i)}$ individually. Accordingly, we will focus on the study of the moduli stack $\mathcal{M}^{(a1)}$. Moreover, following the work of Tsuchihashi on the moduli spaces of complex bielliptic surfaces (cf. Theorem 1.3), we will relate the moduli stack $\mathcal{M}^{(a1)}$ to a product of stacks parameterizing elliptic curves equipped with a certain *level structure*. More precisely, we will define a morphism of stack

$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)},$$

where $\mathcal{Y}_1(2)[1/6]$ denotes the algebraic stack over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ parameterizing elliptic curves $\mathcal{E}/S$ together with a $\Gamma_1(2)$-*structure* and $\mathcal{M}_{1,1}[1/6]$ the Deligne-Mumford stack over $\operatorname{Spec}\mathbb{Z}[\frac{1}{6}]$ of elliptic curves.

**2.2. The Moduli Stack $\mathcal{M}^{(a1)}$.** In order to give a precise definition of the moduli stacks mentioned above, we first recall the notion of an elliptic curve over an scheme $S$ and the definitions of some *level structures* that can be defined on such elliptic curves. For a detailed exposition we refer to the work of Katz and Mazur [**KM85**] and Deligne-Rapoport [**DR73**].

DEFINITION 2.3. Let $S$ be an arbitrary scheme and $N$ be an integer.

(1) An *elliptic curve over* $S$ is a pair $(f\colon \mathcal{E} \to S, e)$, where $f$ is a smooth and proper morphism, $e\colon S \to \mathcal{E}$ is a section of $f$, i.e., $f \circ e = \mathrm{id}_S$, and for every geometric point $\overline{s}\colon \operatorname{Spec} k \to S$ the pullback $(\mathcal{E}_{\overline{s}}, e_{\overline{s}})$ is an elliptic curve over $k$.

(2) A $\Gamma_1(N)$-*structure* on an elliptic curve $\mathcal{E}/S$ is a homomorphism

$$\phi\colon \mathbb{Z}/N\mathbb{Z} \to \mathcal{E}[N](S),$$

such that the effective Cartier divisor in $\mathcal{E}$

$$\sum_{a \mod N} [\phi(a)]$$

is a subgroup scheme of $\mathcal{E}$, where $[\phi(a)]$ denotes the effective Cartier divisor in $\mathcal{E}/S$ defined by the section $\phi(a) \in \mathcal{E}(S)$. The point $P \coloneqq \phi(1)$ in $\mathcal{E}[N](S)$ is called a point of *exact order* $N$.

Equivalently, a $\Gamma_1(N)$-structure on $\mathcal{E}/S$ is an $N$-*isogeny* of elliptic curves over $S$

$$\pi\colon \mathcal{E} \to \mathcal{E}',$$

that is, a homomorphism which is finite locally free of degree $N$) together with a point $P \in (\ker \pi)(S) \subset \mathcal{E}[N](S)$ such that for the corresponding homomorphism

$$\phi\colon \mathbb{Z}/N\mathbb{Z} \to \ker \pi,\ a \mapsto aP$$

we have an equality of effective Cartier divisors in $\mathcal{E}$

$$\ker \pi = \sum_{a \mod N} [aP].$$

(3) A $\Gamma_0(N)$-*structure* on an elliptic curve $\mathcal{E}/S$ is an $N$-isogeny

$$\pi\colon \mathcal{E} \to \mathcal{E}'$$

which is *cyclic* in the sense that, locally fppf on $S$, the kernel $\ker \pi$ admits a generator.

Equivalently, a $\Gamma_0(N)$-structure on $\mathcal{E}/S$ is a finite flat subgroup scheme $K \subset \mathcal{E}[N]$, locally free of rank $N$, which is *cyclic* in the sense that, locally fppf on $S$, it admits a generator.

(4) The *moduli stack of elliptic curves* $\mathcal{M}_{1,1}$ is the stack parameterizing elliptic curves. It is a smooth and separated Deligne-Mumford stack of finite type over $\operatorname{Spec} \mathbb{Z}$ (cf. [**Ols16**, Theorem 13.1.2]). The stack $\mathcal{M}_{1,1} \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ will be denoted by $\mathcal{M}_{1,1}[1/N]$.

(5) The stack parameterizing elliptic curves $\mathcal{E}/S$ together with a $\Gamma_1(N)$-structure, i.e., together with a point $P \in \mathcal{E}[N](S)$ of exact order $N$, will be denoted by $\mathcal{Y}_1(N)$. According to [**DR73**, IV, (4.8)] the stack $\mathcal{Y}_1(N)[1/N] \coloneqq \mathcal{Y}_1(N) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ is an algebraic stack.

Consider the following data:

- $S \in (\mathrm{Sch}/\mathbb{Z}[\frac{1}{6}])$,
- $(f_1 \colon \mathcal{E}_1 \to S, e_1, P) \in \mathcal{Y}_1(2)[1/6](S)$, where $e_1 \colon S \to \mathcal{E}_1$ is a section of $f_1$, and $P$ a point of exact order 2 in $\mathcal{E}_1/S$, and
- $(f_0 \colon \mathcal{E}_0 \to S, e_0) \in \mathcal{M}_{1,1}[1/6](S)$, where $e_0 \colon S \to \mathcal{E}_0$ is a section of $f_0$.

Consider the fiber product

$$\begin{array}{ccc} \widetilde{\mathcal{X}} := \mathcal{E}_1 \times_S \mathcal{E}_0 & \xrightarrow{p_0} & \mathcal{E}_0 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_0} \\ \mathcal{E}_1 & \xrightarrow[f_1]{} & S. \end{array}$$

Note that $\widetilde{\mathcal{X}}$ is a smooth and proper scheme over $S$, since $f_0$ and $f_1$ are smooth and proper morphisms. We denote its structure morphism by $\widetilde{\pi} \colon \widetilde{\mathcal{X}} \to S$.

Remark 2.4. If $S = \operatorname{Spec} R$, with $R$ a Dedekind domain (e.g., a DVR), then the elliptic curves $\mathcal{E}_1, \mathcal{E}_0$ are projective over $S$ (see [**Lic68**, Theorem 2.8]), and thus $\widetilde{\mathcal{X}}$ is projective over $S$.

Let $G_P \subset \mathcal{E}_1$ be the cyclic subgroup of rank 2 generated by $P$, i.e., $G_P = e_1(S) + P(S)$. Since 2 is invertible on $S$, $G_P$ is a finite and étale group scheme over $S$, which is isomorphic to the constant $S$-group scheme $(\mathbb{Z}/2\mathbb{Z})_S$ by [**KM85**, Lemma 1.4.4]. Moreover, we can define a free action

$$\sigma_P \colon G_P \times_S \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$$

of $G_P$ on $\widetilde{\mathcal{X}}$ as follows: for every $S$-scheme $T$, let $(\gamma_1 \colon T \to \mathcal{E}_1, \gamma_0 \colon T \to \mathcal{E}_0)$ be a $T$-point of $\widetilde{\mathcal{X}}$ and set

$$P_T \cdot (\gamma_1, \gamma_0) := \sigma_P(P_T, (\gamma_1, \gamma_0)) = (\gamma_1 + P_T, -\gamma_0),$$

where $P_T$ is the point on $\mathcal{E}_1(T)$ induced by the base change $T \to S$.

We can now consider the quotient of $\widetilde{\mathcal{X}}$ by the free action of $G_P$ on $\widetilde{\mathcal{X}}$.

Proposition 2.5. *The quotient $\mathcal{X} := \widetilde{\mathcal{X}}/G_P$ is a smooth and proper algebraic space over $S$, whose geometric fibers are bielliptic surfaces of type $(a1)$, i.e., $\mathcal{X} \to S \in \mathcal{M}^{(a1)}(S)$.*

Proof. For the reader familiar with algebraic spaces this may be a straightforward result. Nevertheless, we give a proof, which amounts to checking the properties using the theory of algebraic spaces as explained, for example, in [**Sta19**, Tag 025R].

Since the action of $G_P$ on $\widetilde{\mathcal{X}}$ is free, the map

$$j\colon G_P \times_S \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}} \times_S \widetilde{\mathcal{X}}$$

is a monomorphism and is indeed an étale equivalence relation on $\mathcal{X}$. Then, by [**Sta19**, Tag 02WW] the quotient $\mathcal{X} := \widetilde{\mathcal{X}}/G_P$ is an algebraic space and the quotient map $q\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is étale and surjective. Thus, we have a commutative diagram of algebraic spaces

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{\quad q \quad} & \mathcal{X} \\ & {\scriptstyle \widetilde{\pi}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & S & \end{array}$$

where $q$ is étale and surjective, and $\widetilde{\pi}$ is smooth and proper. Then $\mathcal{X}$ is smooth over $S$ by [**Sta19**, Tag 0AHE].

According to [**Sta19**, Tag 08AJ], to show that $\mathcal{X}$ is proper over $S$ it is enough to show that it is separated and of finite type over $S$, since $\widetilde{\mathcal{X}}$ is proper over $S$. That $\pi$ is separated follows from the fact that $\widetilde{\pi}$ is separated and the group $G_P$ is finite, by [**Sta19**, Tag 02Z4]. Moreover, $\mathcal{X}$ is quasi-compact by definition, since $q\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is an étale cover with $\widetilde{\mathcal{X}}$ a quasi-compact scheme. Thus, $\pi$ is of finite type if it is locally of finite type, which is true by descent (see [**Sta19**, Tag 0AHC]). Hence, $\mathcal{X}$ is proper over $S$.

Finally, the quotient $\mathcal{X} = \widetilde{\mathcal{X}}/G_P$ is a universal categorical quotient, see, for example [**Ryd13**, Theorem 2.16]. In particular, forming the quotient $\mathcal{X}$ commutes with arbitrary base change. Thus, for a geometric point $\overline{s} = \operatorname{Spec} k$ in $S$ the geometric fiber $\mathcal{X}_{\overline{s}}$ of $\pi\colon \mathcal{X} \to S$ is isomorphic to the quotient $\mathcal{X}_{\overline{s}}/(G_P)_{\overline{s}} \cong (\mathcal{E}_{1\overline{s}} \times_k \mathcal{E}_{0\overline{s}})/(G_P)_{\overline{s}}$, which is a bielliptic surface of type (a1).

□

As a consequence of the last proposition we obtain the following corollary.

COROLLARY 2.6. *There is a well-defined morphism of stacks*

$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)}$$

*given (in the above notation) by* $\varphi((\mathcal{E}_1, P), \mathcal{E}_0) = \mathcal{X}$.

PROPOSITION 2.7. *The morphism* $\varphi$ *is surjective on geometric points.*

PROOF. This follows directly from Bombieri and Mumford's structure theorem of bielliptic surfaces, see Theorem 1.16 in Chapter 1. □

**2.3. The Morphism of Stacks $\varphi$.** We want to investigate the morphism $\varphi$ defined in the previous section. In order to do so, we will first study the situation over algebraically closed fields of characteristic different from 2 and 3. More precisely, we will show that, if $k$ is an algebraically closed field with $\mathrm{char}(k) \neq 2, 3$, then the induced morphism

$$\varphi_k \colon \mathcal{Y}_1(2)[1/6](k) \times \mathcal{M}_{1,1}[1/6](k) \to \mathcal{M}^{(a1)}(k)$$

is an isomorphism. On the other hand, we expect the morphism $\varphi$ to be an isomorphism of stacks. We take a first step towards that conjecture an show that $\varphi$ is an epimorphism of stacks.

2.3.1. *The Morphism $\varphi$ on Geometric Points.*

Recall the structure theorem of bielliptic surfaces over algebraically closed fields (cf. Theorem 1.16 in Chapter 1 and [**BM77**, Theorem 4, §3]). According to it, every bielliptic surface $X$ over an algebraically closed field $k$ is of the form

$$X = (E \times F)/G,$$

where $E$ and $F$ are elliptic curves and $G$ is a finite group subscheme of $E$ acting diagonally on the product $E \times F$.

If $X$ is of type $(a1)$ and the characteristic of $k$ is different from 2, then $G$ is isomorphic to the constant group scheme $\mathbb{Z}/2\mathbb{Z}$. Then, $G(k) = \mathbb{Z}/2\mathbb{Z}$ has a unique generator, which is a two torsion point $a \in E[2](k)$. Thus, from $X$ we obtain an elliptic curve $E$ together with a two torsion point $a \in E[2](k)$ and a second elliptic curve $F$, such that

$$X \cong (E \times F)/\langle a \rangle,$$

where the action of $\langle a \rangle$ on the product $E \times F$ is given by $a \cdot (x, y) = (x + a, -y)$.

Thus, the induced map

$$\varphi_k \colon \mathcal{Y}_1(2)[1/2](k) \times \mathcal{M}_{1,1}[1/2](k) \to \mathcal{M}^{(a1)}(k), \quad ((E, a), F) \mapsto E \times F/\langle a \rangle.$$

is surjective.

Moreover, the map $\varphi_k$ is even injective. To see this, it is enough to go through the proof by Bombieri and Mumford of theorem [**BM77**, Theorem 4, §3] and note that every choice of their construction is canonical, in the sense that it is unique up to isomorphism.

Proposition 2.8. *Let $k$ be an algebraically closed field of characteristic different from 2 and 3. Then the map*

$$\varphi_k \colon \mathcal{Y}_1(2)[1/6](k) \times \mathcal{M}_{1,1}[1/6](k) \to \mathcal{M}^{(a1)}(k)$$

*is an isomorphism.*

Proof. As mentioned before, the above map is surjective by Theorem 1.16 of Chapter 1. Now consider a bielliptic surface $f\colon X \to B \coloneqq \mathrm{Alb}(X)$ of type $(a1)$. After choosing a base point $b_0 \in B$, the proof of [**BM77**, Theorem 4, §3] shows that

$$X \cong (B \times F)/B[2],$$

where $F \coloneqq f^{-1}(b_0)$ is the fiber of $f$ over the base point $b_0 \in B$ and the action of $B[2]$ on $F$ is defined via the Albanese fibration $f$.

Recall that the Albanese variety $B \coloneqq \mathrm{Alb}(X)$ and the Albanese map $f\colon X \to B$ are unique up to isomorphism. Moreover, since the fibration $f\colon X \to B$ is isotrivial, i.e., all its fibers are isomorphic, the elliptic curve $F$ is also unique up to isomorphism and independent of the choice of the base point $b_0 \in B$.

Let $\alpha\colon B[2] \to \mathrm{Aut}(F)$ denote the action of $B[2]$ on $F$. Since it is defined via the Albanese map $f$, this action is also defined in a canonical way. However, this action is not faithful. So we have to quotient out by the kernel of $\alpha$ and set

$$E \coloneqq B/\ker(\alpha) \text{ and } G \coloneqq B[2]/\ker(\alpha).$$

Then, $E$ and $G$ are also uniquely defined. Since $G$ is isomorphic to the constant group scheme $\mathbb{Z}/2\mathbb{Z}$, there is a unique generator $a \in G(k)$ of $G(k)$.

In conclusion, all the choices involved in the construction of Bombieri and Mumford in the proof of [**BM77**, Theorem 4, §3] are canonical, i.e., unique up to isomorphism, since they are based on the Albanese variety $\mathrm{Alb}(X)$ of $X$. □

2.3.2. *Construction via $R^1 f_* \mathcal{O}_X$.*

There is a more explicit description of the elliptic curves $E$ and $F$, the group $G$ and the two torsion point $a \in G(k)$, which allow us to see that they are defined in a canonical way. Thus, the preimage of a bielliptic surface $X$ of type $(a1)$ under $\varphi_k$ is unique up to isomorphism.

Let $f\colon X \to B \coloneqq \mathrm{Alb}(X)$ be a bielliptic surface of type $(a1)$ together with its Albanese fibration. Consider the invertible sheaf $L \coloneqq R^1 f_* \mathcal{O}_X$ on $B$. It is a torsion invertible sheaf of order equal to the order of canonical sheaf $\omega_X$ on $X$ (cf. Corollary 1.15 (iii) in Chapter 1). Since $X$ is of type $(a1)$, we have $\mathrm{ord}(L) = \mathrm{ord}(\omega_X) = 2$. We can thus consider the étale cyclic cover

$$\pi_B\colon E \coloneqq \underline{\mathrm{Spec}}_B \oplus_{i=0}^{1} (L^{-1})^{\otimes i} \to B$$

of $B$ defined by $L$. It is a finite étale map of degree 2. Note that $E$ is unique up to isomorphism from the description of the relative spectrum as a functor (cf. [**Sta19**, Tag 01LQ]).

Moreover, by the Serre-Lang theorem (cf. [**Mum08**, Chapter IV,§18]) $E$ has a structure of Abelian variety such that $\pi_B$ is a separable isogeny. Thus, $E$ is an elliptic curve.

Now consider the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi_B} & B. \end{array}$$

PROPOSITION 2.9. *With the above notation we have: $\omega_{X'} \cong \mathcal{O}_{X'}$ and $X'$ is an Abelian surface. Moreover, $X'$ is in fact isomorphic to the Abelian canonical cover $\widetilde{X}$ of $X$.*

PROOF. Since $\pi_B$ is finite étale and $f$ is flat, then $\pi_X$ is also finite étale of the same degree. In particular, $\omega_{X'} \cong \pi_X^*\omega_X$ holds. From the canonical bundle formula, we know that $\omega_X \cong f^*(L^{-1})$. Finally, by the definition of the étale cyclic cover $\pi_B$, we have $\pi_B^*(L^{-1}) \cong \mathcal{O}_E$. Putting all these results together and using the commutativity of the Cartesian diagram above one gets

$$\omega_{X'} \cong \pi_X^*\omega_X \cong \pi_X^*(f^*(L^{-1})) \cong (f')^*(\pi_B^*(L^{-1})) \cong (f')^*(\mathcal{O}_E) \cong \mathcal{O}_{X'}.$$

From $\omega_{X'} \cong \mathcal{O}_{X'}$, it follows that $X'$ is minimal, it has Kodaira dimension zero and its geometric genus is given by $p_g(X') = h^0(\omega_{X'}) = h^0(\mathcal{O}_{X'}) = 1$. Moreover, since $\chi(\mathcal{O}_X) = 0$, we have $\chi(\mathcal{O}_{X'}) \cong \deg(\pi_X) \cdot \chi(\mathcal{O}_X) = 0$. Thus, $q(X) = 2$. From the classification of algebraic surfaces it follows that $X'$ is an Abelian surface (since we are excluding characteristic 2).

An alternative way to show that $X'$ is an Abelian surface is the following: consider the Abelian canonical cover $\pi\colon \widetilde{X} \to X$ of $X$ (cf. Definition 1.21 in Chapter 1). Since $\pi_X^*\omega_X \cong \mathcal{O}_{X'}$, it follows from the universal property of the canonical cover $\widetilde{X}$ that there is a unique map $g\colon X' \to \widetilde{X}$, such that the following diagram commutes

$$\begin{array}{ccc} & & \widetilde{X} \\ & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\pi_X} & X. \end{array}$$

Since both $\omega_X$ and $L^{-1}$ have the same order, then $\pi_X$ and $\pi$ have the same degree. Thus, $g$ must have degree 1 and is therefore an isomorphism. Hence, $X'$ is isomorphic to the Abelian canonical cover of $X$. □

Now choose a base point $\widetilde{x_0} \in \widetilde{X}$ such that $f'$ becomes a homomorphism of Abelian varieties. Let $e_0 \in E$ be the identity on $E$ and let $F := \ker(f')$ be the kernel of $f'$, i.e., $F$ is the fiber of $f'$ over $e_0$. Note that it is isomorphic to the fiber of $f$ over $\pi_B(e_0)$. We obtain an exact sequence of Abelian varieties

$$0 \to F \to X' \xrightarrow{f'} E \to 0.$$

Since $X$ is of type $(a1)$, the Albanese fibration $f$ admits a section $\lambda\colon B \to X$, which induces a section $\lambda'\colon E \to X'$ of $f'$. Thus, the above exact sequence splits and we obtain an isomorphism

$$X' \cong E \times F.$$

Let $G := \ker(\pi_B) \subset E$ be the kernel of $\pi_B$. Then $G$ acts on $E$ by translations and $B \cong E/G$. Moreover, $G$ acts on $X' = X \times_B E$ via the second factor such that

$$X \cong X'/G \cong (E \times F)/G.$$

Finally, since $\pi_B$ is a separable isogeny, $G$ is a finite étale group scheme of rank 2. Hence, $G$ is isomorphic to the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ and there is a unique generator $a \in G(k)$ of $G(k)$.

In this manner we get from $X$ uniquely, in a canonical way, the data $(E, a), F$ such that $X \cong E \times F/\langle a \rangle$.

Remark 2.10. It turns out that the elliptic curve $E$ from above can also be described via the Stein factorization of the composition of the Abelian canonical cover with the Albanese fibration. Indeed, by definition, the Stein factorization $B'$ of the composition

$$\widetilde{X} \xrightarrow{\pi} X \xrightarrow{f} B$$

is given by $B' = \underline{\operatorname{Spec}}_B (f \circ \pi)_* \mathcal{O}_{\widetilde{X}}$. But $\pi_* \mathcal{O}_{\widetilde{X}} \cong \oplus_{i=0}^{n-1} \omega_X^{\otimes i}$ and $\omega_X \cong f^*(L^{-1})$. Thus, we have the following chain of isomorphisms:

$$\begin{aligned}
(f \circ \pi)_* \mathcal{O}_{\widetilde{X}} &\cong f_*(\pi_* \mathcal{O}_{\widetilde{X}}) \\
&\cong f_*(\oplus_{i=0}^{n-1} \omega_X^{\otimes i}) \\
&\cong f_*(\oplus_{i=0}^{n-1} (f^*(L^{-1}))^{\otimes i}) \\
&\cong f_*(\oplus_{i=0}^{n-1} f^*((L^{-1})^{\otimes i})) \\
&\cong f_*(f^*(\oplus_{i=0}^{n-1} (L^{-1})^{\otimes i})) \\
&\cong f_* \mathcal{O}_X \otimes \left(\oplus_{i=0}^{n-1} (L^{-1})^{\otimes i}\right) \\
&\cong \mathcal{O}_B \otimes \left(\oplus_{i=0}^{n-1} (L^{-1})^{\otimes i}\right) \\
&\cong \oplus_{i=0}^{n-1} (L^{-1})^{\otimes i}.
\end{aligned}$$

Hence, $B' = \underline{\operatorname{Spec}}_B (f \circ \pi)_* \mathcal{O}_{\widetilde{X}} \cong \underline{\operatorname{Spec}}_B \oplus_{i=0}^{n-1} (L^{-1})^{\otimes i} = E.$

Moreover, on can directly show that the map $B' \to B$ is étale, so that $B'$ is an elliptic curve, by the Serre-Lang theorem [**Mum08**, Chapter IV,§18]. Indeed, since $L$ has the same order than $\omega_X$, the map $B' \to B$ has the same degree as the Abelian canonical cover $\pi\colon \widetilde{X} \to X$. From this, it follows that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \longrightarrow & B \end{array}$$

is Cartesian, i.e., $\widetilde{X} \cong X \times_B B'$. Since $\pi$ is étale and $f$ is faithfully flat, it follows that $B' \to B$ is étale.

2.3.3. *The Epimorphism Property.*

As mentioned before, we expect the morphism $\varphi$ to be an isomorphism of stacks according to the following definition.

Definition 2.11 ([**LMB00**], Définition (3.6)). Let $S$ be a scheme and $F\colon \mathcal{X} \to \mathcal{Y}$ a morphism of stacks over $(\mathrm{Sch}/S)$. Then $F$ is a *monomorphism* (resp. an *isomorphism*) if for every $S$-scheme $U \in (\mathrm{Sch}/S)$ the induced functor $F_U\colon \mathcal{X}(U) \to \mathcal{Y}(U)$ is fully faithful (resp. is an equivalence of categories, that is, if it is fully faithful and essentially surjective).

Explicitly, we state the following conjecture.

Conjecture 2.12. The morphism of stacks

$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[1/6]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)}$$

is an isomorphism of stacks, i.e., for every scheme $S \in (\mathrm{Sch}/\mathbb{Z}[1/2])$ the induced functor

$$\varphi_S\colon \mathcal{Y}_1(2)[1/6](S) \times \mathcal{M}_{1,1}[1/6](S) \to \mathcal{M}^{(a1)}(S)$$

is an equivalence of categories.

In order to prove that a morphism of stacks is an isomorphism, one may use the following definition and result about morphism of stacks.

Definition 2.13 ([**LMB00**], Définition (3.6)). Let $S$ be a scheme and $F\colon \mathcal{X} \to \mathcal{Y}$ a morphism of stacks over $(\mathrm{Sch}/S)$. Then $F$ is an *epimorphism* if for every $S$-scheme $U \in (\mathrm{Sch}/S)$ and every object $y \in \mathcal{Y}(U)$, there exist an étale covering $U' \to U$ and an object $x' \in \mathcal{X}(U')$, such that $F_{U'}(x')$ is isomorphic to the pullback $y_{U'} \in \mathcal{Y}(U')$ of $y \in \mathcal{Y}(U)$ by $U' \to U$.

Lemma 2.14 ([**LMB00**], Corollaire (3.7.1)). *Let $S$ be a scheme and $F\colon \mathcal{X} \to \mathcal{Y}$ a morphism of stacks over $(Sch/S)$. Then $F$ is an isomorphism if and only if $F$ is a monomorphism and an epimorphism.*

Thus, as a first step towards a proof of the above conjecture, we prove the following result.

Theorem 2.15. *The morphism of stacks*

$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[1/6]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)}$$

*is an epimorphism of stacks.*

First, we need a lemma.

Lemma 2.16. *Let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces over an affine scheme $S = \operatorname{Spec} R$. Then, there exist an étale cover $S'$ of $S$ such that the base change $\pi'\colon \mathcal{X}' \to S'$ along $S' \to S$ admits a polarization $\mathcal{L}'$.*

Proof. Let $s \in S$ be a closed point and consider the local ring at $s$ for the étale topology

$$\mathcal{O}^h_{S,s} \coloneqq \varinjlim_{(U_i,u_i)} \Gamma(U_i, \mathcal{O}_{U_i}),$$

where the limit is over the connected affine étale neighborhoods $(U_i, u_i)$ of $s$ (cf. [**Mil13**, p. 31]). Note that, since all the étale neighborhoods $U_i$ are affine, the transitions map between them are all affine.

Let $S^h \coloneqq \operatorname{Spec} \mathcal{O}^h_{S,s}$ denote the henselization of $S$ at $s$. For every $U_i$ the canonical homomorphism

$$\Gamma(U_i, \mathcal{O}_{U_i}) \to \varinjlim_{(U_i,u_i)} = \mathcal{O}^h_{S,s}$$

induces a morphisms

$$S^h \to \operatorname{Spec} \Gamma(U_i, \mathcal{O}_{U_i}) = U_i \to S,$$

where the last arrow is étale. We don't know however, if the first arrow is étale.

We will prove the claim of the lemma in three step. First, we prove that $\mathcal{X}^h \coloneqq \mathcal{X} \times_S S^h$ admits a polarization. Second, by writing $S^h$ as a direct limit of the étale connected affine neighborhoods $U_i$ of $S$ and using the fact that the Picard functor $\operatorname{Pic}_{\mathcal{X}/S}$ is limit preserving, we show that the polarization on $\mathcal{X}^h$ descends to a polarization on $\mathcal{X}_j = \mathcal{X} \times_S U_j$ for some étale neighborhood $U_j$ of $S$. Finally, the étale cover $S'$ will be constructed as a disjoint union of étale neighborhoods of the kind found in Step 2 for every $s \in S$.

*Step 1*: Let $\pi^h\colon \mathcal{X}^h \to S^h$ denote the pullback of $\pi\colon \mathcal{X} \to S$ along $S^h \to S$ and consider the special fiber $X_s = \mathcal{X} \times_{S^h} \mathcal{X}^h$. Since the special fiber is projective and the ring $\mathcal{O}^h_{S,s}$ is henselian, we can argue as in the beginning of the proof of Lemma 1.9: from the fact that $\mathbf{Pic}_{\mathcal{X}^h/S^h}$ is smooth (cf. Proposition 1.7 (b)) together with Hensel's lemma, it follows that an ample line bundle on $X_s$ extends to a relatively ample line bundle $\mathcal{L}^h$ on $\mathcal{X}^h$.

*Step 2*: Since all the $U_i$ are affine, the limit $\varprojlim U_i$ exist in the category of schemes and one has

$$S^h = \mathrm{Spec}\, \mathcal{O}^h_{S,s} = \mathrm{Spec}\, (\varinjlim \Gamma(U_i, \mathcal{O}_{U_i})) = \varprojlim U_i.$$

Then, by [**Sta19**, Tag 01YZ], $\mathcal{X}^h$ can be written as follows:

$$\mathcal{X}^h = \varprojlim \mathcal{X}_i = \varprojlim (\mathcal{X} \times_S U_i).$$

Consider now the relative Picard algebraic space $\mathbf{Pic}_{\mathcal{X}/S}$. It is locally of finite presentation by Proposition 1.7 (a). In other words, the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is limit preserving (see [**Sta19**, Tag 05N0]). Thus,

$$\mathbf{Pic}_{\mathcal{X}/S}(S^h) = \varinjlim \mathbf{Pic}_{\mathcal{X}/S}(U_i).$$

Now the line bundle $\mathcal{L}^h$ on $\mathcal{X}^h$ found in Step 1 defines an element $\xi \in \mathbf{Pic}_{\mathcal{X}/S}(S^h)$ which in turn determines an element $\xi_i \in \mathbf{Pic}_{\mathcal{X}/S}(U_i) = \mathbf{Pic}_{\mathcal{X}_i/U_i}(U_i)$ for some $i$. Let $\xi_i$ be represented by a line bundle $\mathcal{L}_i$ on $\mathcal{X}_i$. Since the pullback $\mathcal{L}^h$ of $\mathcal{L}_i$ to $\mathcal{X}^h$ is ample, then for some $U_j$ the pullback $\mathcal{L}_j$ of $\mathcal{L}_i$ to $\mathcal{X}_j$ is ample, by [**Sta19**, Tag 09MT]. Hence, $\mathcal{L}_j$ is a polarization of $X_j$. Note that $\mathcal{L}_j$ is relatively ample over $U_j$ because it is ample and $U_j$ is affine (cf. [**Sta19**, Tag 01VK]).

*Step 3*: For every $s \in S$ we have shown that we can find an étale neighborhood $U_s$ of $s$, such that $\mathcal{X}_s = \mathcal{X} \times_S U_s$ admits a polarization. We can now take the disjoint union

$$S' := \bigsqcup_{s\in S} U_s \to S,$$

which is an étale cover of $S$. Note that $S'$ is a finite union, since $S$ is Noetherian. Then, the base change $\pi' \colon \mathcal{X}' \to S'$ along $S' \to S$ admits a polarization $\mathcal{L}'$.

□

Proof of Theorem 2.15. Let $S$ be a connected and reduced Noetherian scheme over $\mathbb{Z}[\frac{1}{6}]$. Let $\pi \colon \mathcal{X} \to S$ be an object in $\mathcal{M}^{(a1)}(S)$.

*Step 1*: By Corollary 1.30, after an étale base change, we may assume that $S$ is an affine scheme and that the family $\pi \colon \mathcal{X} \to S$ admits a canonical cover $\tau \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ which is an Abelian scheme over $S$. Let $\widetilde{\sigma} \colon S \to \widetilde{\mathcal{X}}$ denote the identity section of $\widetilde{\mathcal{X}}$. Then, $\widetilde{\sigma}$ induces a section $\sigma \colon S \to \mathcal{X}$ of the family $\pi \colon \mathcal{X} \to S$ and we may define the Albanese scheme $\mathbf{Alb}_{\mathcal{X}/S}$, which we will denote by $\mathcal{B}$ to simplify notation, and the Albanese morphism

$$f\colon \mathcal{X} \to \mathcal{B} := \mathbf{Alb}_{\mathcal{X}/S}.$$

Furthermore, according to Lemma 2.16, after an étale base change, we may also assume that $\pi\colon \mathcal{X} \to S$ admits a polarization $\mathcal{L}$.

Claim 2.17. *the Albanese morphism*

$$f\colon \mathcal{X} \to \mathcal{B} := \mathbf{Alb}_{\mathcal{X}/S}.$$

*admits a section* $\lambda\colon \mathcal{B} \to \mathcal{X}$.

Proof of Claim 2.17. To prove this claim, we will use Seiler's results corresponding to Proposition 1.21 and Proposition 1.22.

Let $s \in S$ denote the closed point of $S$. By Proposition 1.21, there exist a smooth family $\gamma\colon \mathcal{C} \to S$ and an $S$-morphism $g\colon \mathcal{X} \to \mathcal{C}$ that induces the second elliptic fibration $g_{\overline{s}}\colon \mathcal{X}_{\overline{s}} \to \mathbb{P}^1$ on the geometric special fiber $\mathcal{X}_{\overline{s}}$ of the family $\pi\colon \mathcal{X} \to S$. Moreover, since $\mathcal{X}_{\overline{s}}$ is a bielliptic surface of type $(a1)$, the second elliptic fibration $g_{\overline{s}}$ has a fiber $2F'_{\overline{s}}$ of multiplicity 2, such that $F'_{\overline{s}}$ is a section of the Albanese fibration of $\mathcal{X}_{\overline{s}}$. Then, according to Proposition 1.22, there exist a unique flat family $\mathcal{F}' \to S$ of curves, a closed $S$-immersion $\mathcal{F}' \to \mathcal{X}$, and a section $\sigma\colon S \to \mathcal{C}$, such that $F'_{\overline{s}}$ is the fiber of $\mathcal{F}' \to S$ over $\overline{s} \in S$. Then, $\mathcal{F}' \to \mathcal{X}$ is a section of the Albanese morphism $f\colon \mathcal{X} \to \mathcal{B}$. Indeed, both $\mathcal{F}'$ and $\mathcal{B}$ are proper and flat over $S$ and the composition $\mathcal{F}' \to \mathcal{X} \xrightarrow{f} \mathcal{B}$ is fiberwise an isomorphism. Then, by the fibral isomorphism criterion (cf. [**Gro67**, Corollaire 17.9.5]), the composition $\mathcal{F}' \to \mathcal{X} \xrightarrow{f} \mathcal{B}$ is an isomorphism. We will denote this section by $\lambda$.

□

*Step 2*: Now consider the composition $\widetilde{f} := \tau \circ f\colon \widetilde{\mathcal{X}} \to \mathcal{B}$. It is proper, since both $f$ and $\tau$ are proper. Thus, we may consider the Stein factorization

$$\pi_{\mathcal{B}}\colon \mathcal{E} := \underline{\mathrm{Spec}}_{\mathcal{B}}(\widetilde{f}_*\mathcal{O}_{\widetilde{\mathcal{X}}}) \to \mathcal{B}.$$

Claim 2.18. *The commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{\tau} & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{B} \end{array}$$

*is Cartesian, that is,* $\widetilde{\mathcal{X}} \cong \mathcal{X} \times_{\mathcal{B}} \mathcal{E}$.

Proof of the claim 2.18. Let $\mathcal{X}' := \mathcal{X} \times_{\mathcal{B}} \mathcal{E}$ denote the fiber product. From the universal property of the fiber product, there exist a unique $S$-morphism $u\colon \widetilde{\mathcal{X}} \to \mathcal{X}'$. Note that both $\widetilde{\mathcal{X}}$ and $\mathcal{X}'$ are flat and proper over $S$. Indeed, this is clear for $\widetilde{\mathcal{X}}$, and for $\mathcal{X}'$ one can argue as follows: the composition $\widetilde{f} \coloneqq \tau \circ f \colon \widetilde{\mathcal{X}} \to \mathcal{B}$ is flat and has geometrically reduced fibers, since its geometric fibers are disjoint unions of elliptic curves. Moreover, since $\widetilde{f}$ is proper, it is of finite type. Now because $\mathcal{B}$ is Noetherian (as it is a finite type algebraic space over the Noetherian scheme $S$), we have that $\widetilde{f}$ is of finite presentation. By [**Sta19**, Tag 0E0D], we get then that $\pi_{\mathcal{B}} \colon \mathcal{E} \to \mathcal{B}$ is finite étale. From this one easily sees that $\mathcal{X}' \to \mathcal{E} \to \mathcal{B} \to S$ is smooth and proper and therefore flat and proper. Moreover, $u$ is fiberwise (on geometric fibers) an isomorphism. Then, by the fibral isomorphism criterion (cf. [**Gro67**, Corollaire 17.9.5]), the $S$-morphism $u \colon \widetilde{\mathcal{X}} \to \mathcal{X}'$ is an isomorphism.

□

*Step 3*: Since $\pi_{\mathcal{B}}$ is finite and étale, it is proper and smooth. Therefore, $\mathcal{E}$ is proper and smooth over $S$. Furthermore, its geometric fibers are elliptic curves (cf. Section 2.3.2). Denote by $f_{\mathcal{E}}$ the projection $f_{\mathcal{E}} \colon \widetilde{\mathcal{X}} \cong \mathcal{X} \times_{\mathcal{B}} \mathcal{E} \to \mathcal{E}$. From the section $\widetilde{\sigma}$ we get a section $\sigma_{\mathcal{E}} \coloneqq f_{\mathcal{E}} \circ \widetilde{\sigma}$ of $\mathcal{E}/S$, so that by Theorem 1.28 the curve $\mathcal{E}/S$ is an elliptic curve over $S$. Note that by definition $\sigma_{\mathcal{E}}$ is sent to the zero section of $\mathcal{B}$ by $\pi_{\mathcal{B}}$. Furthermore, the map $f_{\mathcal{E}} \colon \widetilde{\mathcal{X}} \to \mathcal{E}$ is then by construction a homomorphism of Abelian schemes, since $f_{\mathcal{E}}$ sends the section $\widetilde{\sigma}$ to the section $\sigma_{\mathcal{E}}$.

Now let $\mathcal{F}$ be the fiber of $f_{\mathcal{E}}$ over $\sigma_{\mathcal{E}}$, that is, $\mathcal{F} = \widetilde{\mathcal{X}} \times_{\mathcal{E}} S$. Note that $\mathcal{F}$ is isomorphic to the fiber of $f$ over the zero section of the Albanese scheme $\mathcal{B}$. Since $f$ is a smooth and proper morphism by Lemma 1.20, $\mathcal{F}$ is smooth and proper over $S$. Moreover, we get an induced section of $\mathcal{F}/S$ from the section $\lambda$ of the Albanese morphism $f$. Thus, by Theorem 1.28, $\mathcal{F}$ is an elliptic curve over $S$. (One can also get an induced section from the section $\widetilde{\sigma}$).

In this way we obtain an exact sequence of Abelian schemes over $S$

$$0 \to \mathcal{F} \to \widetilde{\mathcal{X}} \xrightarrow{f_{\mathcal{E}}} \mathcal{E} \to 0. \tag{2.1}$$

Moreover, since the section of the Albanese morphism $f$ induces a section $\lambda_{\mathcal{E}}$ of $f_{\mathcal{E}}$, the sequence (2.1) splits. Recall that the category of fppf sheaves of Abelian $S$-groups is an Abelian category and that $S$ is a final object. Thus, applying the splitting lemma we have

$$\widetilde{\mathcal{X}} \cong \mathcal{E} \times_S \mathcal{F}.$$

Now let $\mathcal{G} \coloneqq \ker \pi_{\mathcal{B}} \subset \mathcal{E}$ be the kernel of $\pi_{\mathcal{B}}$. Since $\pi_{\mathcal{B}}$ is finite étale of degree equal to the order of $\omega_{\mathcal{X}/S}$, $\mathcal{G}$ is a finite étale $S$-subgroup scheme of $\mathcal{E}$ of rank 2. Note that $\mathcal{G}$ acts on $\mathcal{E}$ by translations such that $\mathcal{B} \cong \mathcal{E}/\mathcal{G}$. Moreover, $\mathcal{G}$ acts on $\widetilde{\mathcal{X}} \cong \mathcal{X} \times_{\mathcal{B}} \mathcal{E}$ via the second factor such that $\mathcal{X} \cong \widetilde{\mathcal{X}}/\mathcal{G}$. Thus,

$$\mathcal{X} \cong (\mathcal{E} \times_S \mathcal{F})/\mathcal{G}.$$

Since $\mathcal{G}$ is a finite étale $S$-group scheme, it is étale locally on $S$ constant. Thus, after a (finite) étale base change, $\mathcal{G}$ is constant and isomorphic to the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_S$. Thus, there is a (unique) generator $P \in \mathcal{G}(S) \subseteq \mathcal{E}[2](S)$.

In conclusion, after replacing $S$ by an étale covering of $S' \to S$, there exist elliptic curves $\mathcal{E}$ and $\mathcal{F}$ over $S$ and a point $P \in \mathcal{G}(S) \subseteq \mathcal{E}[2](S)$ of exact order 2, such that

$$\mathcal{X} \cong \mathcal{E} \times_S \mathcal{F}/\mathcal{G},$$

where $\mathcal{G}$ is isomorphic to the cyclic group of rank 2 generated by $P$. Therefore, the morphism $\varphi$ is an epimorphism. □

REMARK 2.19. A crucial point in the proof of theorem above is the existence of the section of the Albanese morphism. We obtained that section by proving that a family of bielliptic surfaces admits a polarization after an étale base change. However, one could alternatively modify the moduli problem $\mathcal{M}^{(a1)}$ and consider polarized families of bielliptic surfaces of type $(a1)$. In this manner, every family would come with a polarization, which can then be used to obtain a section of the Albanese morphism.

## 3. Marked and Numerically Polarized Bielliptic Surfaces

In this section we present some results on marked and numerically polarized bielliptic surfaces, respectively.

There are two main reasons for consider marked bielliptic surfaces. First, as we will see, a bielliptic surface $X$ over an algebraically closed field $k$ of arbitrary characteristic comes naturally equipped with a *marking*, that is, an isometry $\phi\colon U \to \mathrm{Num}(X)$. Furthermore, from this marking one can construct a unique numerical polarization of degree 2 on $X$. Thus, it is worth considering the question, whether this extra structure should be taken into account when defining moduli spaces of bielliptic surfaces. Second, a family of bielliptic surfaces coming from a product of relative elliptic curves (as described at the beginning of Section 2.2) also comes with extra structure, such as a marking of the relative Néron-Severi lattice and a (numerical) polarization.

Our original aim in studying families of bielliptic surfaces with extra structure was to obtain a better behaved moduli stack. Recall that the automorphism groups of bielliptic surfaces are not finite (cf. Section 4). Consequently, the moduli stack $\mathcal{M}$ of bielliptic surfaces can not be a Deligne-Mumford stack. By considering marked bielliptic surfaces instead, we wanted to obtain better (e.g., finite) automorphism groups, so that the moduli stack could be in principle a Deligne-Mumford stack. However, we will see that every automorphism of a bielliptic surface fixes the Néron-Severi lattice (see Proposition 3.3). Thus, a marking or a numerical polarization on a bielliptic surface does not ridigify the moduli problem.

Although our study of marked bielliptic surfaces did not lead to the desired goal, we decided to include our results, as we think they still are of some interest for the study of moduli spaces of bielliptic surfaces.

### 3.1. Marked Bielliptic Surfaces.

DEFINITION 3.1. A *marked* bielliptic surface is a pair $(X, \phi)$ consisting of a bielliptic surface $X$ together with an isometry $\phi\colon U \to \mathrm{Num}(X)$.

PROPOSITION 3.2. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of arbitrary characteristic. Then, $X$ is a marked bielliptic surface. Moreover, if the characteristic of $k$ is different from 2 and 3 and $X = E \times F/G$, then we have the following:*

*(1) There is a standard marking given by rational multiples of the numerical classes $[F]$ and $[E]$ of a fiber of the Albanese fibrations and the second fibration of $X$, respectively.*
*(2) There is a unique numerical polarization of degree 2 on $X$.*

PROOF. The first claim is just a restatement of Proposition 3.1 of Chapter 1 according to which $\mathrm{Num}(X)$ is isomorphic to the hyperbolic lattice $U$. Thus, $X$ is a marked bielliptic surface. If $k$ is a field of characteristic different from 2 and 3 and $X = E \times F/G$, then Theorem 3.3 of Chapter 1 gives a basis $\{\alpha[E], \beta[F]\}$ with $\alpha, \beta \in \mathbb{Q}$. Denote this basis elements by $e = \alpha[E]$ and $f = \beta[F]$. By sending these basis elements to the standard (hyperbolic) basis of $U$ one obtains an isometry $\phi^{-1}\colon \mathrm{Num}(X) \to U$. Moreover, taking the sum of the basis elements of $\mathrm{Num}(X)$, we get an ample numerical class $\eta = e + f \in \mathrm{Num}(X)$ of degree 2. Indeed, $\eta$ is of degree two since

$$\eta^2 = e^2 + 2(e \cdot f) + f^2 = 1 + 0 + 1 = 2$$

and is an ample class according to Lemma 3.2 of Chapter 1.

Finally, a numerical ample class $\lambda \in \mathrm{Num}(X)$ of degree 2 is unique. Indeed, let $\lambda = ae + bf$ with $a, b \in \mathbb{Z}$. Then, again by Lemma 3.2 of Chapter 1, we have

$$\lambda \in \mathrm{Amp}(X) \iff a, b > 0.$$

From $\lambda^2 = 2ab = 2$ we get that $\alpha = \beta = 1$. Therefore, $\lambda = e + f = \eta$.

□

PROPOSITION 3.3. *Let $X$ be a bielliptic surface over an algebraically closed field $k$ of characteristic different from 2 and 3. Then, every automorphism of $X$ fixes the Néron-Severi lattice.*

PROOF. Let $\phi \in \mathrm{Aut}(X)$ be an automorphism of $X = E \times F/G$. By Theorem 3.3 of Chapter 1, there is a basis $\{\alpha[F], \beta[E]\}$ of $\mathrm{Num}(X)$ with $\alpha, \beta \in \mathbb{Q}$ and where $[F]$ and $[E]$ denote the numerical class of a fiber of the Albanese fibration and of the second elliptic fibration of $X$, respectively. We will work with this basis for $\mathrm{Num}(X)$.

The induced automorphism $\phi^*$ preserves the intersection product on $\mathrm{Num}(X)$, which is given by the intersection matrix

$$A \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\phi^*$ as to be given by one of the following matrices $\{\pm I, \pm A\}$, where $I$ denotes the identity matrix. Moreover, since the pullback of an ample class by $\phi$ has to be ample again, $\phi^*$ has to preserve the ample cone of $\mathrm{Num}(X)$, which lies within the first quadrant of the $[F]-[E]$-plane. Thus, $\phi^*$ is given either by the identity matrix $I$ or by $A$. Finally, recall from Chapter 1, Section 4, that according to Corollary 4.4, $\phi$ is induced from an element of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ which normalizes $G$. In particular, $\phi^*$ is diagonal and thus must be given by the identity matrix. □

We want to show that, for families of bielliptic surfaces coming from a product of relative elliptic curves as described in Section 2.2, an analogous result to Proposition 3.2 holds. Let us start with the definition of the relative Néron-Severi lattice.

Definition 3.4. Let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces. Let $\mathrm{Num}_{X/S}$ denotes the quotient fpqc-sheaf

$$(\mathbf{Pic}_{\mathcal{X}/S}/\mathbf{Pic}^{\tau}_{\mathcal{X}/S})_{(fpqc)}.$$

If $\mathrm{Num}_{X/S}$ is representable by a scheme (resp. by an algebraic space), we will denote by $\mathbf{Num}_{X/S}$ the scheme (resp. the algebraic space) that represents it. Moreover, we write

$$\mathrm{Num}(\mathcal{X}/S) \coloneqq \mathrm{Num}_{\mathcal{X}/S}(S)$$

for the $S$-valued points of $\mathrm{Num}_{\mathcal{X}/S}$ and call it the *relative Néron-Severi lattice.*

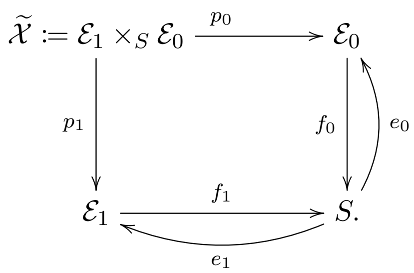
Let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces and $\overline{s}$ a geometric point of $S$. Then we have

$$h^2(\mathcal{O}_{\mathcal{X}_{\overline{s}}}) = h^1(\mathcal{O}_{\mathcal{X}_{\overline{s}}}) - b_1(\mathcal{X}_{\overline{s}})/2.$$

The following proposition appears in [**Lie15b**, Proposition 4.4] and [**EHS12**, Proposition 4.2] and applies to bielliptic surfaces.

Proposition 3.5 (Ekedahl−Hyland−Shepherd-Barron). *Let $\pi\colon \mathcal{X} \to S$ be a family of bielliptic surfaces. Then* $\mathrm{Num}_{\mathcal{X}/S}$ *is a locally constant sheaf of torsion-free finitely generated Abelian groups.*

We keep considering the case of bielliptic surfaces of type $(a1)$ and keep the notation of section 2.2, where $\pi\colon \mathcal{X} \to S$ is the family of bielliptic surfaces of type $(a1)$ given by $\mathcal{X} = \widetilde{\mathcal{X}}/G_P$ with



Denote by $[e_0] \coloneqq e_0(S)$ the relative effective Cartier divisor on $\mathcal{E}_0/S$ corresponding to the section $e_0 \in \mathcal{E}_0(S)$, and similarly for the relative effective Cartier divisor $[e_1]$ on $\mathcal{E}_1/S$ . Consider now the closed subscheme $\mathcal{E}_1 \times_S [e_0] \subset \widetilde{\mathcal{X}}$ obtained from $[e_0]$ after pulling it back along the map $f_1 \colon \mathcal{E}_1 \to S$. This is a relative effective Cartier divisor on $\widetilde{\mathcal{X}}/S$ which we denote by $\widetilde{\mathcal{F}_1} \coloneqq \mathcal{E}_1 \times_S [e_0]$. Similarly, we define the relative effective Cartier divisors $\widetilde{\mathcal{F}_0} \coloneqq [e_1] \times_S \mathcal{E}_0$ and $\widetilde{\mathcal{F}_P} \coloneqq [P] \times_S \mathcal{E}_0$ on $\widetilde{\mathcal{X}}/S$, where $[P] \coloneqq P(S)$ denotes the relative effective Cartier divisor corresponding to the point $P \in \mathcal{E}_1[2](S)$.

From the definition of the action $\sigma_P$ on $\widetilde{\mathcal{X}}$, we see that the divisors $\widetilde{\mathcal{F}_0} + \widetilde{\mathcal{F}_P}$ and $\widetilde{\mathcal{F}_1}$ are invariant under that action. Let us denote by $\mathcal{F}_0$ and $\mathcal{F}_1$ the scheme theoretic image under the morphism $q \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ of $\widetilde{\mathcal{F}_0} + \widetilde{\mathcal{F}_P}$ and $\widetilde{\mathcal{F}_1}$, respectively.

PROPOSITION 3.6. *The closed subspaces $\mathcal{F}_0$ and $\mathcal{F}_1$ of $\mathcal{X}$ are relative effective Cartier divisors on $\mathcal{X}/S$.*

PROOF. We prove the claim for $\mathcal{F}_1$. The same argument can be used to prove that $\mathcal{F}_0$ is a relatively effective Cartier divisor on $\mathcal{X}/S$.

Since $\widetilde{\mathcal{F}_1}$ is invariant under the action $\sigma_P$, the scheme theoretic inverse image of $\mathcal{F}_1$ coincide with $\widetilde{\mathcal{F}_1}$, that is, $\widetilde{\mathcal{F}_1} \cong \mathcal{F}_1 \times_{\mathcal{X}} \widetilde{\mathcal{X}}$. Moreover, for a closed subspace $\mathcal{Z} \subset \mathcal{X}$, the properties of being an effective Cartier divisor and the one of being flat over $\mathcal{X}$ can both be checked étale locally, by [**Sta19**, Tag 083C (3)] and [**Sta19**, Tag 03MK (5)], respectively. Recall that the quotient map $q \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is étale and surjective (cf. proof of Proposition 2.5). Since $\widetilde{\mathcal{F}_1} \cong \mathcal{F}_1 \times_{\mathcal{X}} \widetilde{\mathcal{X}}$ is a relative effective Cartier divisor on $\widetilde{\mathcal{X}}/S$, it follows that $\mathcal{F}_1$ is an effective Cartier divisor on $\mathcal{X}$ and that it is flat over $\mathcal{X}$. As $\mathcal{X}$ is flat over $S$, we see that $\mathcal{F}_1$ is flat over $S$. Thus, $\mathcal{F}_1$ is a relative effective Cartier divisor on $\mathcal{X}/S$. □

PROPOSITION 3.7. *Assume that $S$ is reduced and let $\overline{s} \in S$ be a geometric point. Consider the geometric fiber $\mathcal{X}_{\overline{s}}$. Then, we have the following:*

- *$\mathcal{F}_0|_{\mathcal{X}_{\overline{s}}}$ is a fiber of the Albanese fibration of $\mathcal{X}_{\overline{s}}$, and*
- *$\mathcal{F}_1|_{\mathcal{X}_{\overline{s}}}$ is contained in a fiber of the second elliptic fibration of $\mathcal{X}_{\overline{s}}$ over $\mathbb{P}^1$.*

*In particular, the sum $\mathcal{F}_0 + \mathcal{F}_1$ is a relatively ample divisor on $\mathcal{X}/S$, that is, a polarization.*

Proof. Note that $\widetilde{\mathcal{F}_1}$ is reduced, since it is flat over the reduced scheme $S$. Then, by [**Sta19**, Tag 0830], the scheme theoretic image of $\widetilde{\mathcal{F}_1}$ under the quotient morphism $q\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is the reduced induced algebraic space structure on the closure of the set theoretic image of $\widetilde{\mathcal{F}_1}$ under $q$. Moreover, since the morphism $q$ is finite and in particular universally closed, the scheme theoretic image $\mathcal{F}_1$ of $\widetilde{\mathcal{F}_1}$ coincides with the set theoretic image (with the reduced induced algebraic space structure). But taking the set theoretic image commute with pullbacks. Thus, we get

$$\mathcal{F}_1|_{\mathcal{X}_{\overline{s}}} = q(\widetilde{\mathcal{F}_1})|_{\mathcal{X}_{\overline{s}}} = q_{\overline{s}}(\widetilde{\mathcal{F}_1}|_{\mathcal{X}_{\overline{s}}}) = q_{\overline{s}}(\mathcal{E}_{1,\overline{s}} \times \{0_{\mathcal{E}_{0,\overline{s}}}\}),$$

where $q_{\overline{s}}$ denotes the pullback of the morphism $q$ along $\overline{s}\colon \operatorname{Spec} k \to S$ for an algebraically closed field $k$, and $0_{\mathcal{E}_{0,\overline{s}}}$ denotes the identity element of the elliptic curve $\mathcal{E}_{0,\overline{s}}$ over $k$. Finally, it is clear that $q_{\overline{s}}(\mathcal{E}_{1,\overline{s}} \times \{0_{\mathcal{E}_{0,\overline{s}}}\})$ is mapped to a point by the second fibration $g\colon \mathcal{X}_{\overline{s}} \to \mathbb{P}^1$ of the bielliptic surface $\mathcal{X}_{\overline{s}}$ and thus it is contained in a fiber of the fibration $g$ over $\mathbb{P}^1$.

Similarly, it can be shown that $\mathcal{F}_0|_{\mathcal{X}_{\overline{s}}} = q_{\overline{s}}(0_{\mathcal{E}_{1,\overline{s}}} \times \mathcal{E}_{0,\overline{s}})$ is mapped to a point by the Albanese fibration $f\colon \mathcal{X}_{\overline{s}} \to \operatorname{Alb}(\mathcal{X}_{\overline{s}})$. Since the fibers of $f$ are smooth, $\mathcal{F}_0|_{\mathcal{X}_{\overline{s}}}$ is a fiber of $f$.

Consider now the sum $\mathcal{L} \coloneqq \mathcal{F}_0 + \mathcal{F}_1$. By the discussion above, the pullback $\mathcal{L}_{\overline{s}}$ is the sum of a fiber of the Albanese fibration of $\mathcal{X}_{\overline{s}}$ and a divisor contained in a fiber of the second fibration over $\mathbb{P}^1$. Thus, the numerical class $[\mathcal{L}_{\overline{s}}]$ is of the form $[\mathcal{L}_{\overline{s}}] = [E] + [F]$, where $[F]$ and $[E]$ denote the numerical class of a fiber of the Albanese fibration and of the second fibration of $\mathcal{X}_{\overline{s}}$, respectively (this corresponds to the notation of Section 3 in Chapter 1). Thus, it follows from Lemma 3.2 of Chapter 1 that $\mathcal{L}_{\overline{s}}$ is an ample divisor. Since these is true for every geometric point $\overline{s} \in S$, we conclude that $\mathcal{L} \coloneqq \mathcal{F}_0 + \mathcal{F}_1$ is relatively ample, that is, a polarization on $\mathcal{X}$. □

By checking the intersection theory on geometric points, we get

$$\mathcal{F}_0^2 = \mathcal{F}_1^2 = 0, \text{ and } \mathcal{F}_0 \cdot \mathcal{F}_1 = 2 \quad \text{(the rank of } G_P\text{)}.$$

Proposition 3.8. *Let $[\mathcal{F}_i] \in \operatorname{Num}(\mathcal{X}/S)$ denote the numerical class of $\mathcal{F}_i$, for $i = 1, 2$. Then there exist integers $m_0, m_1 \in \mathbb{N}$ such that*

$$\frac{1}{m_i}[\mathcal{F}_i] \in \operatorname{Num}(\mathcal{X}/S) \quad \textit{and} \quad [\mathcal{F}_0] \cdot [\mathcal{F}_1] = m_0 m_1.$$

*In particular, $\frac{1}{m_1}[\mathcal{F}_1], \frac{1}{m_0}[\mathcal{F}_0]$ is a basis of $\operatorname{Num}(\mathcal{X}/S)$.*

Proof. We may assume that $S = \operatorname{Spec} R$, where $R$ is a complete Noetherian local ring with algebraically closed residue field. Let $s \in S$ be the closed point of $R$, and consider the specialization map

$$sp\colon \operatorname{Num}(\mathcal{X}/S) \hookrightarrow \operatorname{Num}(\mathcal{X}_s), \ [\mathcal{L}] \mapsto [\mathcal{L}]|_s. \tag{3.1}$$

This map is injective, since the intersection pairing is non-degenerate, and it is well-defined. Indeed, this is the case if the specialization map at the level of the relative Picard scheme is surjective up to torsion by [**SGA71**, Expose X, App. 7.9]. Since the relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth, Hensel's lemma implies that the specialization map $sp\colon \mathrm{Pic}(X/S) \to \mathrm{Pic}(\mathcal{X}_s)$ is surjective.

Moreover, since $\mathbf{Pic}^{\tau}_{\mathcal{X}/S}$ is an open and closed subscheme of the smooth scheme $\mathbf{Pic}_{\mathcal{X}/S}$, it is smooth. Thus $\mathbf{Num}_{\mathcal{X}/S}$ is smooth. Again, with Hensel's lemma we get that the specialization map (3.1) is surjective.

From Theorem 3.3 of Chapter 1 we have that there exist integers $m_0, m_1$, such that $\{\frac{1}{m_0}[\mathcal{F}_0]|_s, \frac{1}{m_1}[\mathcal{F}_1]|_s\}$ is a basis of $\mathrm{Num}(\mathcal{X}_s)$. In particular, $[\mathcal{F}_0]|_s \cdot [\mathcal{F}_1]|_s = m_0 m_1$.

The assertion now follows from the bijectivity of the specialization map (3.1). □

DEFINITION 3.9. A *marked family of bielliptic surfaces* over a Noetherian base scheme $S$ is a pair $(\mathcal{X}, \phi)$ consisting of a family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over $S$ together with a lattice isomorphism (isometry)

$$\phi\colon \underline{U}_S \to \mathbf{Num}_{\mathcal{X}/S},$$

where $\underline{U}_S$ denotes the constant sheaf associated to the hyperbolic plane $U$.

REMARK 3.10. Note that $\mathbf{Num}_{\mathcal{X}/S}$ is constant for a marked family of bielliptic surfaces $(\mathcal{X}, \phi)$. Moreover, from Proposition 3.5, after an étale base change this is always the case for every (not necessarily marked) family of bielliptic surfaces.

From Proposition 3.8 we see that $\{\frac{1}{m_1}[\mathcal{F}_1], \frac{1}{m_0}[\mathcal{F}_0]\}$ is a basis of $\mathrm{Num}(\mathcal{X}/S)$. In particular $\mathrm{Num}(\mathcal{X}/S)$ is constant of rank 2 and we get an isomorphism

$$\phi\colon \underline{U}_S \to \mathbf{Num}_{\mathcal{X}/S}.$$

We can now consider the following moduli problem for marked bielliptic surfaces.

Let $\mathcal{M}_U^{(a1)}$ denote the stack over $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ whose objects are marked families of bielliptic surfaces $(\pi\colon \mathcal{X} \to S, \phi)$ of type $(a1)$ over a scheme $S$.

A morphism

$$(\pi'\colon \mathcal{X}' \to S', \phi') \to (\pi\colon \mathcal{X} \to S, \phi)$$

is a pair $(a, b)$, where

$$a\colon S' \to S, \quad b\colon \mathcal{X}' \to \mathcal{X}$$

are morphism of algebraic spaces, such that the square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\;b\;} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\;a\;} & S \end{array}$$

is Cartesian and the following diagram commutes

$$\begin{array}{ccc} \underline{U}_S \times_S S' & \xrightarrow{a^*\phi := \phi \times_S S'} & \mathbf{Num}_{\mathcal{X}/S} \times_S S' \\ \downarrow & & \downarrow \\ \underline{U}_{S'} & \xrightarrow{\phi'} & \mathbf{Num}_{\mathcal{X}'/S'}. \end{array}$$

Note that the vertical arrows in the above diagram are isomorphisms.

We have shown that a family of bielliptic surfaces of type $(a1)$ coming from the product of modular curves is indeed a marked family of bielliptic surfaces (see Proposition 3.8). Thus, there is a well-defined morphism of stacks $\varphi_U$ fitting in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}_1(2) \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1} & \xrightarrow{\varphi_U} & \mathcal{M}_U^{(a1)} \\ & \searrow^{\varphi} & \downarrow {\scriptstyle forget} \\ & & \mathcal{M}^{(a1)}. \end{array}$$

The vertical arrow is in the diagram is the forgetful functor which forgets the marking.

**3.2. Getting a Numerical Polarization from a Marked Family.** We have seen that a bielliptic surface $X$ over an algebraically closed field $k$ comes naturally with a standard marking, as the Néron-Severi lattice is isomorphic to the hyperbolic plane $U$. Moreover, from the marking we get a unique numerical ample class $\eta \in \mathrm{Num}(X)$ of degree 2. The following is the analogous result for marked families of bielliptic surfaces.

PROPOSITION 3.11. *Let $(\pi\colon \mathcal{X} \to S, \phi\colon \underline{U}_S \to \mathbf{Num}_{\mathcal{X}/S})$ be a marked family of bielliptic surfaces over an irreducible base scheme $S$. Then there exist a unique numerical polarization $\lambda \in \mathrm{Num}(\mathcal{X}/S)$ of degree 2 on $\mathcal{X}/S$, that is, a global section $\lambda \in \mathrm{Num}(\mathcal{X}/S)$ with self-intersection $\lambda^2 = 2$, such that for every geometric point $\overline{s}$ of $S$ the section $\lambda_{\overline{s}} \in \mathrm{Num}(\mathcal{X}_{\overline{s}})$ is an ample numerical class.*

PROOF. From the marking $\phi$, we get an isometry $U \cong \mathrm{Num}(\mathcal{X}/S)$. Choosing the standard (hyperbolic) basis of $U$ we get elements

$$e, f \in \mathrm{Num}(\mathcal{X}/S) \ \text{ such that } \ e^2 = f^2 = 0, \ e \cdot f = 1.$$

Now set $\lambda := e+f \in \mathrm{Num}(\mathcal{X}/S)$. Then we have $\lambda^2 = 2 > 0$. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}/S) := \mathbf{Pic}_{\mathcal{X}/S}(S)$ be a global section with numerical class $[\mathcal{L}] = \lambda \in \mathrm{Num}(\mathcal{X}/S)$ and let $\eta \in S$ be the generic point of $S$.

Consider the restriction $\mathcal{L}_{\overline{\eta}} = \mathcal{L}|_{\mathcal{X}_{\overline{\eta}}} \in \mathrm{Num}(\mathcal{X}_{\overline{\eta}})$ of $\mathcal{L}$ to the geometric generic fiber $\mathcal{X}_{\overline{\eta}}$. We have $\mathcal{L}_{\overline{\eta}}^2 > 0$. From Lemma 3.10 of Chapter 1 it follows that either $\mathcal{L}_{\overline{\eta}}$ or $\mathcal{L}_{\overline{\eta}}^{-1}$ is effective. We may assume that $\mathcal{L}_{\overline{\eta}}$ is effective. Otherwise we just replace $e, f$ with $-e, -f$. In particular, $\mathcal{L}_{\overline{\eta}}$ is ample, since $\mathcal{L}_{\overline{\eta}}^2 > 0$.

Since $\mathcal{L}_{\overline{\eta}}$ is effective, we have $h^0(\mathcal{L}_{\overline{\eta}}) > 0$. By flat base change (cf. [**Sta19**, Tag 02KH]), it follows that $h^0(\mathcal{L}_\eta) > 0$, so $\mathcal{L}_\eta$ is also effective. Then, form the upper semi-continuity of the function $s \mapsto h^0(\mathcal{X}_s, \mathcal{L}_s)$ (cf. [**Mum08**, Chapter II, section 5]), it follows that for every $s \in S$ we have $h^0(\mathcal{L}_s) > 0$, i.e., $\mathcal{L}_s$ is effective for every $s \in S$.

Restricting to the geometric fiber $\mathcal{X}_{\overline{s}}$, we see that $\mathcal{L}_{\overline{s}}$ on $\mathcal{X}_{\overline{s}}$ is also effective, and since $\mathcal{L}_{\overline{s}}^2 > 0$ also holds for every geometric point $\overline{s} \in S$, we see that $\mathcal{L}_{\overline{s}}$ is ample for every geometric point $\overline{s} \in S$. Thus, $[\mathcal{L}_{\overline{s}}] = \lambda_{\overline{s}} \in \mathrm{Num}(\mathcal{X}_{\overline{s}})$ is an ample class for every geometric point $\overline{s}$ of $S$.

The degree of $\lambda$ can be checked on the fibers. Finally, the claim about the uniqueness is proved as in the proof of Proposition 3.2 and by restricting to the fibers.

□

**3.3. Numerically Polarized Bielliptic Surfaces.** Let us now consider numerically polarized bielliptic surfaces.

Let $\mathcal{M}^{(a1)}_{d,\mathrm{Num}}$ denote the stack over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{6}]$ whose objects are pairs $(\pi\colon \mathcal{X} \to S, \lambda)$, where $\pi\colon \mathcal{X} \to S$ is a family of bielliptic surfaces of type $(a1)$ over a scheme $S$, and $\lambda$ is a numerical class of a polarization $\mathcal{L}$ of degree $d$ on $\mathcal{X}/S$.

A morphism

$$(\pi'\colon \mathcal{X}' \to S', \lambda') \to (\pi\colon \mathcal{X} \to S, \lambda)$$

is a triple $(a, b, \epsilon)$, where

$$a\colon S' \to S, b\colon \mathcal{X}' \to \mathcal{X}$$

are morphism of algebraic spaces, such that the square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{b} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{a} & S \end{array}$$

is Cartesian and $\epsilon\colon b^*\lambda \to \lambda'$ is an isomorphism.

From Proposition 3.11 it follows that there is a well-defined morphism of stacks $\psi_{2,\mathrm{Num}}$ fitting in the following commuting diagram, where $\varphi_{2,\mathrm{Num}}$ denotes the composition $\psi_{2,\mathrm{Num}} \circ \varphi_U$:

$$\begin{array}{ccc}
\mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] & \xrightarrow{\varphi_U} & \mathcal{M}_U^{(a1)} \\
& \searrow^{\varphi_{2,\mathrm{Num}}} & \downarrow{\scriptstyle \psi_{2,\mathrm{Num}}} \\
& & \mathcal{M}_{2,\mathrm{Num}}^{(a1)} \\
& \searrow^{\varphi} & \downarrow{\scriptstyle forget} \\
& & \mathcal{M}^{(a1)}.
\end{array}$$

Both $\mathcal{M}_U^{(a1)}$ and $\mathcal{M}_{d,\mathrm{Num}}^{(a1)}$ are good candidates for being the "right" moduli space of bielliptic surfaces of type $(a1)$, since the extra structure required on them is naturally found in bielliptic surfaces. Moreover, we believe that the extra structure on these moduli stacks could be used to obtain more information about the morphisms $\varphi_U$ and $\varphi_{2,\mathrm{Num}}$, respectively. We, however, do not investigate this question further.

## 4. Outlook and Final Discussion

In this section we want to discuss some questions related to our main results. In order to do so, let us recall them.

(1) *Separation of families (Theorem 1.8)*: every family $\pi\colon \mathcal{X} \to S$ of bielliptic surfaces over a connected $\mathbb{Z}[\frac{1}{6}]$-scheme $S$ has as geometric fibers bielliptic surfaces of the same type. Consequently, the moduli stack $\mathcal{M}$ over $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ of bielliptic surfaces splits into seven disjoint components (see Proposition 2.2).

(2) *The epimorphism $\varphi$ (Theorem 2.15)*: by considering the moduli stack $\mathcal{M}^{(a1)}$ over $\mathbb{Z}[\frac{1}{6}]$, we constructed a morphism of stacks
$$\varphi\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}^{(a1)}$$
which is an epimorphism (cf. Theorem 2.15).

We want to discuss the following questions:

(1.1) What can be said about the separation of families in small characteristic?

(2.1) What would it follow, if $\varphi$ were an isomorphism of stacks? In other words, which nice properties does the product $\mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6]$ have, that would be preserved by $\varphi$, if it were an isomorphism?

(2.2) Can similar morphisms be constructed for each of the stacks $\mathcal{M}^{(i)}$ parameterizing bielliptic surfaces of type $(i)$? More precisely, can we relate each one of this stacks to (a product of) modular stacks in a similar way, as we have done for the stack $\mathcal{M}^{(a1)}$? What has to be taken into account in order to do so?

**4.1. Separation of Families in Small Characteristic.** In this section we discuss some partial results for classical bielliptic surfaces in small characteristic. We will work over a scheme $S = \operatorname{Spec} R$ with $R$ a complete discrete valuation ring.

Recall that the order of the canonical bundle of a bielliptic surface is given by the following table.

| Type | $\operatorname{char}(k) \neq 2,3$ | $\operatorname{char}(k) = 2$ | $\operatorname{char}(k) = 3$ |
|---|---|---|---|
| $(a)$ | 2 | 1 | 2 |
| $(b)$ | 3 | 3 | 1 |
| $(c)$ | 4 | 1 | 4 |
| $(d)$ | 6 | 3 | 2 |

TABLE 4.1. Order of $\omega_X$ in $\operatorname{Pic}(X)$.

DEFINITION 4.1. By a *classical* bielliptic surface we mean a bielliptic surfaces $X$ over an algebraically closed field $k$ of characteristic $p$ such that $\operatorname{ord}(\omega_X) \neq 1$.

Note that one could define classical bielliptic surfaces by one of the other equivalent conditions stated in Corollary 1.20 in Chapter 1.

The following facts still hold in small characteristic for classical bielliptic surfaces (cf. Proposition 1.7 and Lemma 1.9).

LEMMA 4.2. *Let $\pi \colon \mathcal{X} \to S$ be a family of bielliptic surfaces over $S = \operatorname{Spec} R$, where $R$ is a complete discrete valuation ring. Assume that the special geometric fiber $X_{\overline{k}}$ is a classical bielliptic surface. Then,*

*(1) The relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth.*
*(2) The identity component of the relative Picard scheme $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an elliptic curve over $S$.*
*(3) The torsion Néron-Severi scheme $\mathbf{NS}^{\tau}_{\mathcal{X}/S}$ is a finite and étale $S$-group scheme.*

PROOF. (1) Since $X_{\overline{k}}$ is a classical bielliptic surface, we have $H^2(X_k, \mathcal{O}_{X_k}) = 0$. From this, the claim can be proved following the argument in the proof of Proposition 1.7.

(2) Given that $X_{\overline{k}}$ is a classical bielliptic surface, the Picard scheme $\mathbf{Pic}^0_{X_{\overline{k}}/\overline{k}}$ is an elliptic curve, since it is reduced and of dimension 1. Then, one may argue as in the proof of Lemma 1.9 (a).

(3) This is a restatement of Lemma 1.9 (b). In the proof of that lemma we used the fact that the relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth, which in the present case is given by (1).

□

PROPOSITION 4.3. *Let $\mathcal{X} \to S$ be a family of bielliptic surfaces over $S = \operatorname{Spec} R$ where $R$ is a complete discrete valuation ring of characteristic $p \in \{2, 3\}$. Assume that the special geometric fiber $X_{\overline{k}}$ is a classical bielliptic surface of type $(i)$, then the geometric generic fiber $X_{\overline{K}}$ is a (classical) bielliptic surface of the same type.*

PROOF. First, we prove that the orders of the canonical bundles of $X_{\overline{K}}$ and $X_{\overline{k}}$, respectively, are the same. Indeed, by Lemma 4.2 (2), $\mathbf{Pic}^0_{\mathcal{X}/S}$ is an elliptic curve over $S$. Let $n$ be the least common multiple of the orders of the canonical bundles of the different types of bielliptic surfaces in the given characteristic (cf. Table 4.1). Thus, $n = 3$ if $\operatorname{char}(k) = 2$ and $n = 4$ if $\operatorname{char}(k) = 3$. In both cases $n$ is not divisible by the characteristic $p$. Hence, the kernel $\mathbf{Pic}^0_{\mathcal{X}/S}[n]$ of the multiplication by $n$ map on $\mathbf{Pic}^0_{\mathcal{X}/S}$ is a finite and étale $S$-group scheme (see, for example, [**KM85**, Theorem 2.3.1]). Then, by Hensel's lemma, the reduction map

$$\operatorname{Pic}^0(X_{\overline{K}})[n] \cong \mathbf{Pic}^0_{\mathcal{X}/S}[n](\overline{K}) \to \mathbf{Pic}^0_{\mathcal{X}/S}[n](\overline{k}) \cong \operatorname{Pic}^0(X_{\overline{k}})[n]$$

sending $\omega_{X_{\overline{K}}}$ to $\omega_{X_{\overline{k}}}$ is bijective. Thus, $\operatorname{ord}(\omega_{X_{\overline{K}}}) = \operatorname{ord}(\omega_{X_{\overline{k}}})$.

Second, we have $\operatorname{NS}(\omega_{X_{\overline{K}}}) = \operatorname{NS}(\omega_{X_{\overline{k}}})$. Indeed, since $\mathbf{NS}^{\tau}_{\mathcal{X}/S}$ is a finite and étale $S$-group scheme by Lemma 4.2(3), the reduction map

$$\operatorname{NS}^{\tau}(X_{\overline{K}}) \cong \mathbf{NS}^{\tau}_{\mathcal{X}/S}(\overline{K}) \to \mathbf{NS}^{\tau}_{\mathcal{X}/S}(\overline{k}) \cong \operatorname{NS}^{\tau}(X_{\overline{k}})$$

is bijective. Hence, $\operatorname{NS}^{\tau}(\omega_{X_{\overline{K}}}) = \operatorname{NS}^{\tau}(\omega_{X_{\overline{k}}})$. Then, it follows that $\operatorname{NS}(\omega_{X_{\overline{K}}}) = \operatorname{NS}(\omega_{X_{\overline{k}}})$, since both bielliptic surfaces have the same Picard number $\rho(X_{\overline{K}}) = \rho(X_{\overline{k}}) = 2$.

Finally, by arguing as in the case of bielliptic surfaces in characteristic different from 2 and 3 (see Proposition 1.14), the Néron-Severi groups can be computed either directly (in the case of the Jacobian bielliptic surfaces) or by lifting to characteristic zero (in the case of non-Jacobian bielliptic surfaces) [2]. Recall that, according to Theorem 5.12 of Chapter 1, bielliptic surfaces lift to characteristic zero, even in small characteristic. Moreover, the Néron-Severi group is invariant under deformation, again by Lemma 4.2(3).

Using these facts a case by case study shows that the type of $X_{\overline{K}}$ and $X_{\overline{k}}$ must be the same. □

REMARKS 4.4. (1) The same result still holds if the discrete valuation ring $R$ is of mixed characteristic, provided the type of the geometric special fiber in question is not $(d)$. For this type the problem lies in the fact that the canonical bundle of

[2] In the case a bielliptic surface $X$ of type $(a3)$ (in characteristic 2) one has $\operatorname{NS}(X) \cong \mu_2$. This was directly computed by Lang in [**Lan79**, p. 497].

the geometric generic fiber could, in principle, have order 6, which is divisible by the residue characteristic $p \in \{2, 3\}$. Note that for the other classical types the canonical order is the same as in characteristic zero (cf. Table 4.1).

(2) The previous result is only about families of bielliptic surfaces in characteristic 2 and 3. In particular, we are assuming that the geometric generic fiber is a bielliptic surface (as are all geometric fibers of such a family). We are not considering families of (quasi-)bielliptic surfaces, whose geometric fibers can be either bielliptic or quasi-bielliptic surfaces. The situation for such families is much more delicate, since for a quasi-bielliptic surface $X$ the identity component of the Picard scheme $\mathbf{Pic}^0_{X/k}$ is non-reduced. Therefore, if $\pi\colon \mathcal{X} \to S$ is a family of (quasi-)bielliptic surfaces having some quasi-bielliptic surfaces as geometric fibers, the relative Picard scheme $\mathbf{Pic}^0_{\mathcal{X}/S}$ can not be an elliptic curves, for it would contain some non-reduced fibers.

**4.2. An Isomorphism of Stacks.** In what follows we shortly discuss a direct consequence that would follow, if one could find an isomorphism of stacks

$$\varphi_*\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}_*^{(a1)},$$

where $\mathcal{M}_*^{(a1)}$ denotes one of the moduli stacks $\mathcal{M}^{(a1)}, \mathcal{M}_U^{(a1)}$ or $\mathcal{M}_{2,\mathrm{Num}}^{(a1)}$. From the knowledge of certain properties of the stacks on the left-hand side, one could similarly conclude that the stack $\mathcal{M}_*^{(a1)}$ also possesses that properties.

As we saw at the beginning of Section 2.2, the stacks on the left-hand side have the following properties:

- The moduli stack of elliptic curves $\mathcal{M}_{1,1}$ is a smooth and separated Deligne-Mumford stack of finite type over $\mathbb{Z}$ (see, for example, [**Ols16**, Theorem 13.1.2]).
- The stack $\mathcal{Y}_1(2)[1/6]$ of elliptic curves with a $\Gamma_1$-structure over $\mathbb{Z}[\frac{1}{6}]$ is an algebraic stack (see, for example, [**DR73**, IV,(4.8)]).

Thus, we obtain the following conditional result.

Proposition 4.5. *Let*

$$\varphi_*\colon \mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6] \to \mathcal{M}_*^{(a1)}$$

*be a morphism of stacks, where $\mathcal{M}_*^{(a1)}$ denotes one of the moduli stacks $\mathcal{M}^{(a1)}, \mathcal{M}_U^{(a1)}$ or $\mathcal{M}_{2,\mathrm{Num}}^{(a1)}$. If $\varphi_*$ is an isomorphism of stacks, then $\mathcal{M}_*^{(a1)}$ is an algebraic stack.*

Proof. The product $\mathcal{Y}_1(2)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}[1/6]$ is an algebraic stack over $S$ by [**Sta19**, Tag 04TE] (see also [**LMB00**, Proposition (4.5)(i)]). Then, since $\varphi_*$ is an isomorphism of stacks, it follows that $\mathcal{M}_*^{(a1)}$ is an algebraic stack by [**Sta19**, Tag 03YQ]. □

Similarly, the stack $\mathcal{M}_*^{(a1)}$ will have any property shared by both stacks $\mathcal{Y}_1(2)[1/6]$ and $\mathcal{M}_{1,1}[1/6]$, which is preserved under the formation of the 2-categorical fiber product. We left to the reader the investigation of such further properties and refer to [**DR73**] and [**KM85**] for the theory and properties of the stacks $\mathcal{Y}_1(N)[1/N]$ and $\mathcal{M}_{1,1}$.

**4.3. Morphism Between Moduli for Other Types.** We now discuss the possibility of constructing morphisms of stacks $\varphi_{(i)}$ from (products of) modular stacks to the stacks $\mathcal{M}^{(i)}$ parameterizing families of bielliptic surfaces of type $(i)$, which are analogous to the morphism $\varphi$ constructed for the stack $\mathcal{M}^{(a1)}$. We will explain some of the difficulties we encountered when attempting to carry out the construction of such morphisms $\varphi_{(i)}$. Moreover, we will address the question of which of our results about $\varphi$ in the case $(a1)$ carry over to these other morphisms.

Let us first consider the case of the other Jacobian types, i.e., the types $(b1), (c1)$ and $(d)$. We start with the moduli stack $\mathcal{M}^{(b1)}$ parameterizing bielliptic surfaces of type $(b1)$.

4.3.1. *The Moduli Stack* $\mathcal{M}^{(b1)}$.

Consider the following data:

- $S \in (\mathrm{Sch}/\mathbb{Z}[\frac{1}{6}])$,
- $(f_1 \colon \mathcal{E}_1 \to S, e_1, P) \in \mathcal{Y}_1(3)[1/6](S)$, where $e_1 \colon S \to \mathcal{E}_1$ is a section of $f_1$, and $P$ a point of exact order 3 in $\mathcal{E}_1/S$, and
- $(f_0 \colon \mathcal{E}_0 \to S, e_0) \in \mathcal{M}_{1,1}[1/6](S)$, where $e_0 \colon S \to \mathcal{E}_0$ is a section of $f_0$, together with an automorphism $\omega \in \mathrm{Aut}(\mathcal{E}_0/S, e_0)$ of order 3 which fixes the identity section.

Let $\widetilde{\mathcal{X}} = \mathcal{E}_1 \times_S \mathcal{E}_0$ be the fiber product of the curves and $G_P \subset \mathcal{E}_1$ be the cyclic subgroup of rank 3 generated by $P$, i.e., $G_P = e_1(S) + P(S) + 2P(S) \subset \mathcal{E}_1$. Since 3 is invertible on $S$, $G_P$ is a finite and étale group scheme over $S$, which is isomorphic to the constant $S$-group scheme $(\mathbb{Z}/3\mathbb{Z})_S$ by [**KM85**, Lemma 1.4.4]. Moreover, we can define a free action

$$\sigma_P \colon G_P \times_S \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$$

of $G_P$ on $\widetilde{\mathcal{X}}$ as follows: for every $S$-scheme $T$ let $(\gamma_1 \colon T \to \mathcal{E}_1, \gamma_0 \colon T \to \mathcal{E}_0)$ be a $T$-point of $\widetilde{\mathcal{X}}$ and set

$$P_T \cdot (\gamma_1, \gamma_0) \coloneqq \sigma_P(P_T, (\gamma_1, \gamma_0)) = (\gamma_1 + P_T, \omega(\gamma_0) = \omega \circ \gamma_0),$$

where $P_T$ is the point on $\mathcal{E}_1(T)$ induced by the base change $T \to S$.

Similarly as in the case $(a1)$ (see Proposition 2.5), the quotient $\mathcal{X} \coloneqq \widetilde{\mathcal{X}}/G_P$ is a smooth and proper algebraic space over $S$, whose geometric fibers are bielliptic surfaces of type $(b1)$, i.e., $\mathcal{X} \to S \in \mathcal{M}^{(b1)}(S)$.

The first difficulty we encounter in generalizing our construction is the existence of the automorphism $\omega$ of $\mathcal{E}_0/S$ of order 3. We don't know which elliptic curves over $S$ admit such an automorphism. If $\mathcal{E}_0 = E_0 \times S$ is a trivial family defined by an elliptic curve $E_0$ of $j$-invariant $j(E_0) = 0$, then $\mathcal{E}/S$ admits such an automorphism $\omega$. Probably, an isotrivial family of elliptic curves, that is, a family whose fibers are all isomorphic to one elliptic curve, having as fibers elliptic curves of $j$-invariant zero, would also admit such an automorphism. Moreover, we don't know if there is a substack of $\mathcal{M}_{1,1}$ parameterizing elliptic curves $\mathcal{E}/S$ together with an automorphism of order 3.

Let us assume that there is a substack $\mathcal{M}_{1,1}^{(j=0)} \subset \mathcal{M}_{1,1}$ parameterizing elliptic curves $\mathcal{E}/S$ together with an automorphism $\omega$ of $\mathcal{E}/S$ of order 3. Then, we obtain a well-defined morphism of stacks

$$\varphi_{(b1)} \colon \mathcal{Y}_1(3)[1/6] \times_{\mathbb{Z}} [\tfrac{1}{6}]\mathcal{M}_{1,1}^{(j=0)}[1/6] \to \mathcal{M}^{(b1)}.$$

The second difficulty we encounter is the following: in the $(a1)$-case, we proved that the morphism $\varphi$ is an epimorphism (see, Theorem 2.15). Our proof uses the fact that a finite étale group scheme of rank 2 over a connected scheme has étale locally a *unique* generator. This is no longer true if the rank of the group scheme is 3, as it is in the $(b1)$-case we are currently considering. This rises the question of how important is the choice of the generator of the group $G_P$ one wants to quotient out. In the present case, the $S$-points $P$ and $2P$ generate the same group scheme $G_P$, but we can use each one of them to define two different actions $\sigma_P$ and $\sigma_{2P}$ on $\widetilde{\mathcal{X}}$. Does this two actions define the same quotient?

The results of Tsuchihashi on complex bielliptic surfaces, which we presented in Section 1.1, suggest that what is crucial for constructing a bielliptic surface $X = E \times F/G$ out of two elliptic curves $E, F$ with torsion data $G \subset E$ is the group $G$ itself and not necessarily the generator of the group $G$ one chooses to defined the action of $G$ on the product $E \times F$. Indeed, according to Tsuchihashi results (see, Theorem 1.3), a coarse moduli space $M^{(b1)}$ for bielliptic surfaces of type $(b1)$ is given by the modular curve $Y_0(3)$, which can be interpreted as the moduli space parameterizing (isomorphism classes of) pairs $(E, G)$, where $E$ is a complex elliptic curve and $G$ is a cyclic subgroup of $E$ of order 3 (see, for example, [**DS05**, Section 1.5]). This suggests that one should consider the modular stack $\mathcal{Y}_0(3)$ parameterizing elliptic curves $\mathcal{E}/S$ together with a $\Gamma_0(N)$-structure instead of the modular stack $\mathcal{Y}_1(3)$.

Let us recall the definition of a $\Gamma_0$-structure on a relative elliptic curve $\mathcal{E}/S$.

Definition 4.6. Let $N$ be an integer.

(1) A $\Gamma_0(N)$-*structure* on an elliptic curve $\mathcal{E}/S$ is an $N$-isogeny

$$\pi\colon \mathcal{E} \to \mathcal{E}'$$

which is *cyclic* in the sense that locally fppf on $S$, the kernel $\ker \pi$ admits a generator.

Equivalently, a $\Gamma_0(N)$-structure on $\mathcal{E}/S$ is a finite flat subgroup scheme $K \subset \mathcal{E}[N]$, locally free of rank $N$, which is *cyclic* in the sense that locally fppf on $S$, it admits a generator.

(2) The stack parameterizing elliptic curves $\mathcal{E}/S$ together with a $\Gamma_0(N)$-structure will be denoted by $\mathcal{Y}_0(N)$. According to [**DR73**, IV, (4.3)] the stack $\mathcal{Y}_0(N)[1/N] \coloneqq \mathcal{Y}_0(N) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N]$ is an algebraic stack.

Note that there is a natural morphism of stacks

$$\Phi_{forget}\colon \mathcal{Y}_1(N)[1/N] \to \mathcal{Y}_0(N)[1/N]$$

defined by forgetting the generator: let $S$ be a scheme over $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ and $\mathcal{E}/S$ an elliptic curve with a $\Gamma_1(N)$-structure, that is, together with a point $P \in \mathcal{E}[N](S)$ of exact order $N$. To $P$ we can associate the cyclic subgroup $G_P \coloneqq \langle P \rangle \subset \mathcal{E}$ generated by $P$. By associating to the pair $(\mathcal{E}/S, P)$ the pair $(\mathcal{E}/S, G_P)$ one obtains the desired morphism.

Coming back to the moduli stack $\mathcal{M}^{(b1)}$, note that the morphism of stacks $\varphi_{(b1)}$ factors as follows:

$$\begin{array}{ccc} \mathcal{Y}_1(3)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}^{(j=0)}[1/6] & \xrightarrow{(\Phi_{forget},\ \mathrm{id})} & \mathcal{Y}_0(3)[1/6] \times_{\mathbb{Z}[\frac{1}{6}]} \mathcal{M}_{1,1}^{(j=0)}[1/6] \\ & \searrow^{\varphi_{(b1)}} & \downarrow{\scriptstyle \psi_{(b1)}} \\ & & \mathcal{M}^{(b1)}. \end{array}$$

Using the same arguments as in the proof of Proposition 2.15, where we proved that the morphism $\varphi$ is an epimorphism, one can show that both $\varphi_{(b1)}$ and $\psi_{(b1)}$ are epimorphism. The main difference with the $(a1)$-case is that the cyclic subgroup scheme $\mathcal{G} \subset \mathcal{E}$ (in the notation of the proof) may admit several generators étale locally. This, however, is not crucial for proving that $\varphi_{(b1)}$ is an epimorphism. Nevertheless, it could be relevant in order to show the fully faithfulness of these morphisms. In particular, the morphism $\psi_{(b1)}$ have more chances to be an isomorphism of stacks than the morphism $\varphi_{(b1)}$.

4.3.2. *The Other Jacobian Types.* Similar as in the previous case, one can study the remaining Jacobian cases, that is, the cases $(c1)$ and $(d)$. In this cases we encounter the same difficulty regarding the existence of automorphism of order 4 and 3, respectively, for elliptic curves over a given scheme $S$. Moreover, under the assumption of the existence of substacks $\mathcal{M}_{1,1}^{(j=1728)}$ and $\mathcal{M}_{1,1}^{(j=0)}$ parameterizing elliptic curves $\mathcal{E}/S$ together with an automorphism $\omega$ of $\mathcal{E}/S$ of order 4 and 3, respectively, one may defined similar morphisms as we did for the case $(b1)$. We leave to the reader to carry out this explicitly.

4.3.3. *The Non-Jacobian Cases.* As for the non-Jacobian cases $(a2)$, $(b2)$ and $(c2)$ let us shortly mention that the main difficulty in generalizing the results from the $(a1)$-case to the case $(a2)$, for example, lies in the following fact: for a non-Jacobian bielliptic surface $X = E \times F/G$ the Abelian canonical cover $\widetilde{X}$ does not coincide with the product $E \times F$. Due to this fact, our strategy used to study the case $(a1)$ does not generalize directly to the non-Jacobian cases.

# Bibliography


[ABD$^+$66] M. Artin, J-E. Bertin, M. Demazure, A. Grothendieck, P. Gabriel, M. Raynaud, and J-P. Serre, *Schémas en groupes*, Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, Institut des Hautes Études Scientifiques, Paris, 1963/1966. 43

[Ant11] Marco Antei, *On the abelian fundamental group scheme of a family of varieties*, Israel J. Math. **186** (2011), 427–446. MR 2852331 60, 66, 70

[Băd01] Lucian Bădescu, *Algebraic surfaces*, Universitext, Springer-Verlag, New York, 2001, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. MR 1805816 3, 14, 15, 17, 20, 63

[Bd08] G. Bagnera and M. de Franchis, *Le superficie algebriche le quali ammettono una rappresentazione parametrica mediante funzioni iperellittiche di due argomenti.*, Mem. di Mat. e di Fis. Soc. It. Sc. (3) 15, 253-343 (1908)., 1908. 2, 5, 53

[Bea96] Arnaud Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. MR 1406314 2

[Ber63] Jean-Étienne Bertin, *Généralités sur les préschémas en groupes*, Schémas en Groupes (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1963/64), Fasc. 2a, Exposé 6b, Inst. Hautes Études Sci., Paris, 1963, p. 37. MR 0257095 61

[BFTV19] Jonas Bergstrm, Eugenia Ferrari, Sofia Tirabassi, and Magnus Vodrup, *On the brauer group of bielliptic surfaces*, 2019. 73

[BK86] Spencer Bloch and Kazuya Kato, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. (1986), no. 63, 107–152. MR 849653 33

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 24, 49, 59, 69

[BM76] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p. III*, Invent. Math. **35** (1976), 197–232. MR 0491720 2, 3, 4, 20

[BM77] ———, *Enriques' classification of surfaces in char. p. II*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 23–42. MR 0491719 2, 3, 15, 16, 17, 18, 19, 20, 22, 39, 67, 82, 83

[BM90a] Curtis Bennett and Rick Miranda, *The automorphism groups of the hyperelliptic surfaces*, Rocky Mountain J. Math. **20** (1990), no. 1, 31–37. MR 1057972 42, 44, 45

[BM90b] ———, *The automorphism groups of the hyperelliptic surfaces*, Rocky Mountain J. Math. **20** (1990), no. 1, 31–37. MR 1057972 44

[CD89] François R. Cossec and Igor V. Dolgachev, *Enriques surfaces. I*, Progress in Mathematics, vol. 76, Birkhäuser Boston, Inc., Boston, MA, 1989. MR 986969 24, 25

[CDL20] Franois Cossec, Igor Dolgachev, and Christian Liedtke, *Enriques Surfaces I*, 2020, http://www.math.lsa.umich.edu/ idolga/EnriquesOne.pdf, p. 587. 25, 29, 34, 36, 74

[CL98] Antoine Chambert-Loir, *Cohomologie cristalline: un survol*, Exposition. Math. **16** (1998), no. 4, 333–382. MR 1654786 31

[DG80] Michel Demazure and Peter Gabriel, *Introduction to algebraic geometry and algebraic groups*, North-Holland Mathematics Studies, vol. 39, North-Holland Publishing Co., Amsterdam-New York, 1980, Translated from the French by J. Bell. MR 563524 61

[dJ05] Aise Johan de Jong, *A result of Gabber*, Preprint, `http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf`, 2005. 73

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 143–316. Lecture Notes in Math., Vol. 349. MR 0337993 8, 78, 79, 102, 104

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 55, 104

[EHS12] T. Ekedahl, J. M. E. Hyland, and N. I. Shepherd-Barron, *Moduli and periods of simply connected Enriques surfaces*, arXiv e-prints (2012), arXiv:1210.0342. 93

[ES09] Federigo Enriques and Francesco Severi, *Mémoire sur les surfaces hyperelliptiques*, Acta Math. **32** (1909), no. 1, 283–392. MR 1555054 2, 5, 53

[ES10] ———, *Mémoire sur les surfaces hyperelliptiques*, Acta Math. **33** (1910), no. 1, 321–403. MR 1555061 2, 5, 53

[EV92] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR 1193913 39

[FC90a] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. MR 1083353 1

[FC90b] ———, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. MR 1083353 66, 72

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. 19

[Gab63] Pierre Gabriel, *Construction de préschémas quotient*, Schémas en Groupes (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1963/64), Fasc. 2a, Exposé 5, Inst. Hautes Études Sci., Paris, 1963, p. 37. MR 0257095 61

[Gro61] Alexander Grothendieck, *éléments de géométrie algébrique : II. étude globale élémentaire de quelques classes de morphismes*, Publications Mathématiques de l'IHÉS **8** (1961), 5–222 (fr). 60

[Gro63] ———, *éléments de géométrie algébrique : III. étude cohomologique des faisceaux cohérents, seconde partie*, Publications Mathématiques de l'IHÉS **17** (1963), 5–91 (fr). MR 163911 59

[Gro67] ———, *éléments de géométrie algébrique : IV. étude locale des schémas et des morphismes de schémas, quatrième partie*, Publications Mathématiques de l'IHÉS **32** (1967), 5–361 (fr). 37, 60, 72, 89, 90

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises. MR 2675155 61, 64

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 17, 19, 31, 69

[Har10] ———, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010. MR 2583634 51

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 26, 27

[HKW93] Klaus Hulek, Constantin Kahn, and Steven H. Weintraub, *Moduli spaces of abelian surfaces: compactification, degenerations, and theta functions*, De Gruyter Expositions in Mathematics, vol. 12, Walter de Gruyter & Co., Berlin, 1993. MR 1257185 1

[Igu60] Jun-ichi Igusa, *Betti and Picard numbers of abstract algebraic surfaces*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 724–726. MR 118725 16

[Iit69] Shigeru Iitaka, *Deformations of compact complex surfaces*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 267–272. 54

[Iit70] ———, *Deformations of compact complex surfaces. II*, J. Math. Soc. Japan **22** (1970), 247–261. 54

[Ill79] Luc Illusie, *Complexe de De Rham-Witt et cohomologie cristalline.*, Ann. Sci. Éc. Norm. Supér. (4) **12** (1979), 501–661 (French). 29, 32, 34, 36, 75

[Ill83] L. Illusie, *Finiteness, duality, and Künneth theorems in the cohomology of the de Rham-Witt complex*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 20–72. MR 726420 32, 76

[Kle80] Steven L. Kleiman, *Relative duality for quasicoherent sheaves*, Compositio Math. **41** (1980), no. 1, 39–60. MR 578050 68

[Kle05] ———, *The Picard scheme*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 235–321. MR 2223410 24, 60, 61, 66, 69

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of mathematics studies ; 108, Princeton Univ. Press, Princeton, NJ, 1985. 8, 50, 61, 78, 80, 100, 102, 103

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin-New York, 1971. MR 0302647 60

[Lan79] William E. Lang, *Quasi-elliptic surfaces in characteristic three*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 473–500. MR 565468 4, 30, 31, 32, 63, 75, 101

[Lan87] Serge Lang, *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987, With an appendix by J. Tate. MR 890960 20

[Lan95] William E. Lang, *Examples of liftings of surfaces and a problem in de Rham cohomology*, vol. 97, 1995, Special issue in honour of Frans Oort, pp. 157–160. MR 1355122 52

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR 2095471 40, 60

[Lic68] Stephen Lichtenbaum, *Curves over discrete valuation rings*, Amer. J. Math. **90** (1968), 380–405. MR 0230724 80

[Lie15a] Christian Liedtke, *Arithmetic moduli and lifting of Enriques surfaces*, J. Reine Angew. Math. **706** (2015), 35–65. MR 3393362 1

[Lie15b] ———, *Arithmetic moduli and lifting of Enriques surfaces*, J. Reine Angew. Math. **706** (2015), 35–65. MR 3393362 93

[Lie16] ———, *Lectures on supersingular K3 surfaces and the crystalline Torelli theorem*, K3 surfaces and their moduli, Progr. Math., vol. 315, Birkhäuser/Springer, [Cham], 2016, pp. 171–235. MR 3524169 25, 31

[LLR04] Qing Liu, Dino Lorenzini, and Michel Raynaud, *Néron models, Lie algebras, and reduction of curves of genus one*, Invent. Math. **157** (2004), no. 3, 455–518. MR 2092767 24

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000. MR 1771927 77, 78, 86, 102

[Mar20] Gebhard Martin, *Automorphism schemes of (quasi-)bielliptic surfaces*, 2020. 9, 42, 43, 44, 45

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid. MR 879273 67

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906 1, 24, 72

[Mil80] James S Milne, *étale cohomology*, Princeton mathematical series, vol. 33, Princeton Univ. Press, Princeton, NJ, 1980 (eng). 23, 27, 28, 29, 30, 36, 62, 69, 71, 73

[Mil13] James S. Milne, *Lectures on etale cohomology (v2.21)*, 2013, Available at www.jmilne.org/math/, p. 202. 87

[MO67] Hideyuki Matsumura and Frans Oort, *Representability of group functors, and automorphisms of algebraic schemes*, Invent. Math. **4** (1967), 1–25. MR 217090 43

[MP12] Davesh Maulik and Bjorn Poonen, *Néron-Severi groups under specialization*, Duke Math. J. **161** (2012), no. 11, 2167–2206. MR 2957700 37, 62, 64

[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. 14

[Mum69] ———, *Enriques' classification of surfaces in* char $p$. *I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 325–339. MR 0254053 2, 67

[Mum08] ———, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. MR 2514037 83, 85, 97

[Ols08] Martin C. Olsson, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics, vol. 1958, Springer-Verlag, Berlin, 2008. MR 2446415 1

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications, vol. 62, American Mathematical Society, Providence, RI, 2016. MR 3495343 79, 102

[Par13] Holger Partsch, *Deformations of elliptic fiber bundles in positive characteristic*, Nagoya Math. J. **211** (2013), 79–108. 46, 49, 50, 51

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR 3729254 73

[Ray70] M. Raynaud, *Spécialisation du foncteur de Picard*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 27–76. MR 282993 69

[Ray78] ———, *Contre-exemple au "vanishing theorem" en caractéristique* $p > 0$, C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, pp. 273–278. MR 541027 39

[Riz06] Jordan Rizov, *Moduli stacks of polarized* $K3$ *surfaces in mixed characteristic*, Serdica Math. J. **32** (2006), no. 2-3, 131–178. MR 2263236 1

[Ryd13] David Rydh, *Existence and properties of geometric quotients*, J. Algebraic Geom. **22** (2013), no. 4, 629–669. MR 3084720 81

[Sei87a] Wolfgang K. Seiler, *Global moduli for elliptic surfaces with a section*, Compositio Math. **62** (1987), no. 2, 169–185. MR 898732 6, 56

[Sei87b] ———, *Global moduli for polarized elliptic surfaces*, Compositio Math. **62** (1987), no. 2, 187–213. MR 898733 6, 56, 57, 67, 68

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York-Heidelberg, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216 36

[Ser90] Fernando Serrano, *Divisors of bielliptic surfaces and embeddings in* $\mathbb{P}^4$, Math. Z. **203** (1990), no. 3, 527–533. MR 1038716 10, 35, 39, 40

[Ser91] ———, *The picard group of a quasi-bundle*, manuscripta mathematica **73** (1991), no. 1, 63–82. 26, 40

[Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR 2247603 48, 49, 51

[SGA71] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. MR 0354655 37, 62, 95

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. 20

[Siu98] Yum-Tong Siu, *Invariance of plurigenera*, Invent. Math. **134** (1998), no. 3, 661–673. MR 1660941 54, 57

[Siu02] ———, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 223–277. MR 1922108 54, 57

[Sta19] The Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu`, 2019. 59, 61, 64, 67, 69, 72, 80, 81, 83, 88, 89, 94, 97, 102

[Suw69] Tatsuo Suwa, *On hyperelliptic surfaces*, J. Fac. Sci. Univ. Tokyo Sect. I **16** (1969), 469–476 (1970). 5, 7, 53, 54

[Suw83] Noriyuki Suwa, *de Rham cohomology of algebraic surfaces with* $q = -p_a$ *in* char. $p$, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 73–85. MR 726421 30

[Tsu79] Hiroyasu Tsuchihashi, *Compactifications of the moduli spaces of hyperelliptic surfaces*, Tôhoku Math. J. (2) **31** (1979), no. 3, 319–347. 5, 7, 8, 53, 55

[Tsu02] Hajime Tsuji, *Deformation invariance of plurigenera*, Nagoya Math. J. **166** (2002), 117–134. MR 1908576 54, 57

[Vak] Ravi Vakil, *Foundations of Algebraic Geometry*. 69