Sensitivity and importance measures in structural reliability

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In multivariate reliability problems, which depend on one or more parameters τ, a sensitivity factor $\alpha_E[\tau]$ is defined as the derivative of the equivalent reliability index $\beta_E(\tau) = -\phi^{-1}[P_f(\tau)]$ with $P_f(\tau)$ the failure probability. $\alpha_E[\tau]$ expresses the change of $\beta_E(\tau)$ due to small variations of τ. Since the numerical evaluation of $\alpha_E[\tau]$ is usually impractical, an approximation $\alpha_E[\tau] \approx \alpha[\tau]$ is derived, which is asymptotically exact for extreme reliability levels. Simple formulae for $\alpha[\tau]$ are given. The approximation $\alpha_E[\tau]:a[\tau]$ also provides the basis for a better understanding of the commonly used alpha values $\alpha_i = -1/\beta u_i^*$ as importance measures for stochastic variables.

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The so-called first- and second-order reliability methods (see, for example, references 1 and 2 and the references given there) are mainly concerned with approximations for the failure probability P_f or the equivalent (generalized) reliability index $\beta_E = -\phi^{-1}(P_f)$ of multivariable reliability problems, where $\phi(.)$ is the standard normal integral. The notion of a generalized safety index has been introduced in reference 3 in full rigour. It has been proposed as an alternative reliability measure to the failure probability. Given a vector $X = (X_1, \ldots, X_n)'$ of random variables, the failure event is formulated as a subset F_x of the n-dimensional real space. Failure occurs, if and only if X attains a value in F_x. Thus, $\{X \in F_x\}$ is the failure event, and $P_f = P[X \in F_x]$ the failure probability. For the special case that $X = U = (U_1, \ldots, U_n)'$ collects independent, standard normal variables U_i, the first-order approximation for $\beta_E[F] = -\phi^{-1}(P[U \in F])$ is simply:

$$\beta_E[F] \approx \beta[F] \tag{1}$$

where:

$$\beta = \beta[F] = \begin{cases} +\|u^*\|, & O \notin F \\ -\|u^*\|, & O \in F \end{cases}$$

is the safety index introduced first by Hasofer and Lind[4] and the beta-point u^* is the point on the boundary of F with minimal distance to the origin. Throughout the paper, it is assumed that only one beta-point exists, and that $\beta = \beta[F]$ is strictly positive. All results to be given also apply, nevertheless, to negative safety indices by passing over to the complement $S = R^n \setminus F$.

In addition to the safety-index β, the alpha-values:

$$\alpha_i = -\frac{1}{\beta} u_i^* \tag{2}$$

are considered to be the second important element in the first-order reliability method. They appear as coefficients of U_i in the first-order linearization H of F at the beta-point u^*:

$$\{U \in F\} \approx \{U \in H\} = \left\{\sum_{i=1}^{n} \alpha_i U_i + \beta < 0\right\} = \{Z + \beta < 0\} \tag{3}$$

and satisfy the normalizing condition:

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \tag{4}$$

They are interpreted as importance measures for the variables U_i; a variable with absolutely large alpha-value α_i is considered to be stochastically important. This interpretation appears to have been given first in reference 5.

Furthermore, it has been observed that α_i reflects the dependency of $\beta_E[H] = \beta[H]$ on a small variation of the mean value μ_i of the basic variable U_i. Replacing U_i by $U_i + \mu_i$, the reliability index of a linear failure event $H = \{u : \Sigma\, \alpha_i u_i + \beta < 0\}$ changes according to:

$$\beta_E(\mu) = \beta_E[\Sigma \alpha_i (U_i + \mu_i) + \beta < 0] = \beta + \Sigma\, \alpha_i \mu_i \tag{5}$$

or (for linear failure boundaries):

$$\frac{\partial \beta_E(\mu)}{\partial \mu_i} = \alpha_i \tag{6}$$

Finally, it is easily verified that, in the linear case, α_i is the correlation between the state variable Z in Eq (3) and U_i.

In the following, it is shown that the artificial restriction to a special parameter and to linear failure events is unnecessary. A more general and intuitively clear definition of alpha-values will be given, for which then first-order approximations are derived. The alpha-values α_i as described before will appear as special cases in that general framework.

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Sensitivity factors of parameters

Let the distribution of the basic variable vector $X(\tau) = (X_1(\tau), \ldots, X_n(\tau))'$ be arbitrary and depend on a vector $\tau = (\tau_1, \ldots, \tau_k)$ of parameters, which is briefly called parameter τ. In general, the dimension k of the parameter τ differs from the dimension n of X. Each of the parameters τ_j may be related to a single variable X_i, but also to some or even all of them. According to reference 6, $X(\tau)$ can always be written as a vector-valued function:

$$X(\tau) = T(U, \tau) = (T_1(U, \tau), \ldots, T_n(U, \tau))' \tag{7}$$

of a vector $U = (U_1, \ldots, U_n)'$ of independent, standard normal variables; since the distribution of $X(\tau)$ depends on the parameter, so does the transformation $T = T(\tau)$. Given now a failure domain F_x in the space of x-variables (failure occurs if and only if $X(\tau)$ attains a value in F_x), the failure event can also be expressed in the space of the U-variables (u-space):

$$\{X(\tau) \in F_x\} = \{T(U, \tau) \in F_x\} = \{U \in F(\tau)\} \tag{8}$$

with:

$$F(\tau) = \{u : T(u, \tau) \in F_x\} \tag{9}$$

On the other hand, if $F_x = F_x(\tau)$ depends on a design parameter, which usually occurs as a parameter of a limit state function $g_x(\cdot, \tau) = g_x(x, \tau)$, and $X = T(U)$, the failure event transforms into the u-space according to:

$$\{X \in F_x(\tau)\} = \{T(U) \in F_x(\tau)\} = \{U \in F(\tau)\} \tag{10}$$

with:

$$F(\tau) = \{u : T(u) \in F_x(\tau)\} \tag{11}$$

Since no explicit difference between design parameters and distribution parameters is observed in the u-space, in general there will not be any discrimination between these two types of parameters. They are formally combined into a single parameter vector τ, on which both, the distribution of $X = X(\tau)$ and the failure domain $F_x = F_x(\tau)$ depend formally. Therefore, the failure event can generally be written as:

$$\{X(\tau) \in F_x(\tau)\} = \{T(U, \tau) \in F_x(\tau)\} = \{U \in F(\tau)\} \tag{12}$$

with:

$$F(\tau) = \{U : T(u, \tau) \in F_x(\tau)\} \tag{13}$$

It is now necessary to investigate the sensitivity of the failure probability:

$$P_f(\tau) = P[X(\tau) \in F_x(\tau)] = P[U \in F(\tau)] \tag{14}$$

or the generalized reliability index:

$$\beta_E(\tau) = -\phi^{-1}[P_f(\tau)] \tag{15}$$

against small variations of the parameter τ. In the first approximation this can be described by the partial derivatives $\frac{\partial}{\partial \tau_j} \beta_E(\tau)$, because, if $\beta_E(\tau)$ is continuously differentiable with respect to τ, there is:

$$\Delta\beta_E = \beta_E(\tau + \Delta\tau) - \beta_E(\tau) \approx \sum_{j=1}^{k} \frac{\partial \beta_E(\tau)}{\partial \tau_j} \Delta\tau_j \tag{16}$$

In analogy with Eq (6), the partial derivatives of $\beta_E(\tau)$ are denoted as (equivalent) alpha-values of the parameters τ_j or τ:

$$\alpha_E[\tau_j \mid \tau] = \frac{\partial}{\partial \tau_j} \beta_E(\tau) \tag{17}$$

$$\alpha[\tau] = \operatorname{grad} \beta_E(\tau) = \left(\frac{\partial}{\partial \tau_1} \beta_E(\tau), \ldots, \frac{\partial}{\partial \tau_k} \beta_E(\tau)\right)'$$

Here, the τ_j in $\alpha_E[\tau_j \mid \tau]$ is only a formal argument, comparable with the τ_j in the derivation symbol, $\frac{\partial}{\partial \tau_j}$.

Only in very fortunate cases, such as the one underlying Eq (5), are exact analytical or easy numerical solutions for α_E available. In view of the first-order approximation (Eq (1)), or, more precisely:

$$\beta_E(\tau) \approx \beta(\tau)$$

with:

$$\beta(\tau) = \beta[F(\tau)] \tag{18}$$

one might presume that $\frac{\partial}{\partial \tau_k} \beta(\tau)$ is also an approximation to $\frac{\partial}{\partial \tau_j} \beta_E(\tau)$. Therefore:

$$\alpha[\tau_j \mid \tau] = \frac{\partial}{\partial \tau_k} \beta(\tau) \tag{19}$$

$$\alpha[\tau] = \operatorname{grad} \beta(\tau) = \left(\frac{\partial}{\partial \tau_1} \beta(\tau), \ldots, \frac{\partial}{\partial \tau_k} \beta(\tau)\right)'$$

are introduced and simply denoted as alpha-values. As will be seen later, simple formulae for these alpha-values exist. Furthermore, they provide asymptotic approximations for α_E as described in reference 7.

A failure domain $F(\tau)$ in the u-space with large safety index $\beta(\tau) = \beta[F(\tau)]$ can be written as:

$$F(\tau) = b_0 G(\tau) = \{b_0 u : u \in G(\tau)\} \tag{20}$$

where b_0 is a large real number and $G(\tau) = \frac{1}{b_0} F(\tau)$ is a domain with moderate reliability index $\beta[G(\tau)]$. Therefore, an asymptotic relation like:

$$\lim_{b \to \infty} \frac{1}{b} \frac{\partial}{\partial \tau_j} \beta_E[bG(\tau)] = \frac{\partial}{\partial \tau_j} \beta[G(\tau)] \tag{21}$$

justifies the approximation:

$$\frac{1}{b_0} \frac{\partial}{\partial \tau_j} \beta_E[b_0 G(\tau)] \doteq \frac{1}{b_0} \frac{\partial}{\partial \tau_j} \beta_E[F(\tau)] \approx \frac{\partial}{\partial \tau_j} \beta[G(\tau)] \tag{22}$$

Since:

$$\beta[G(\tau)] = \frac{1}{b_0} \beta[F(\tau)]$$

then:

$$\frac{\partial}{\partial \tau_j} \beta_E[F(\tau)] \approx \frac{\partial}{\partial \tau_j} \beta[F(\tau)] \tag{23}$$

or, in other words:

$$\alpha_E[\tau_j \mid \tau] \approx \alpha[\tau_j \mid \tau] \tag{24}$$

$$\alpha_E[\tau] \approx \alpha[\tau]$$

These last approximations are supposed to be good, if the safety-index $\beta(\tau) = \beta[F\tau)]$ of $F(\tau)$ is large.

Asymptotic relations for distribution parameters

Although a simple general formula for $\alpha[\tau]$ will be presented later, the subsequent discussion is devoted to the special case of distribution parameters, since their treatment allows for a solution which exhibits some numerical advantages. Also, the differentiability of $\beta_E(\tau)$ is verified here with more ease than in the general case, and with the exception of the uniqueness of the beta-point $\boldsymbol{u}^*$, no further assumptions about the failure domain $F(\tau)$ are required.

Therefore, let Eqs (8) and (9) hold, where F_x does not depend on a parameter. Assume also that the transformation $T(\cdot, \tau)$ (a a mapping from the $\boldsymbol{u}$-space to the $\boldsymbol{x}$-space) is, for each parameter τ in an (arbitrarily small) environment of the initial value τ_0 of τ, invertible, and with respect to both arguments $\boldsymbol{u}$ and τ, twice continuously differentiable. Abbreviating $(T(\cdot, \tau))^{-1}(x)$ by $T^{-1}(\boldsymbol{x}, \tau)$, the following theorem holds (for a proof, see reference 7).

Theorem 1

(a):

$$\alpha[\tau_j \mid \tau_0] = \frac{1}{\beta}\left[\boldsymbol{u}^* \frac{\partial}{\partial \tau_j} T^{-1}(\boldsymbol{x}^*, \tau_0)\right]$$

$$= \frac{1}{\beta} \sum_{i=1}^{n} u_i^* \left(\frac{\partial}{\partial \tau_j} T^{-1}(\boldsymbol{x}^*, \tau_0)\right)_i$$

where:

$$\beta = \beta(\tau_0) = \beta[F(\tau_0)]$$

$$\boldsymbol{x}^* = T(\boldsymbol{u}^*, \tau_0)$$

(b) Under the additional technical assumptions listed in appendix 1, $\beta_E(\tau) = \beta_E[F(\tau)]$ as a function of τ is continuously differentiable in an environment of the initial value τ_0 of τ, and:

$$\lim_{b \to \infty} \frac{1}{b} \frac{\partial}{\partial \tau_j} \beta_E[bF(\tau_0)] = \frac{\partial}{\partial \tau_j} \beta[F(\tau_0)]$$

$$= \alpha[\tau_j \mid \tau_0]$$

As pointed out in the previous section, part (b) of theorem 1 gives rise to the approximations:

$$\alpha_E[\tau_j \mid \tau] \approx \alpha[\tau_j \mid \tau] \qquad (25)$$

$$\boldsymbol{\alpha}_E[\tau] \approx \boldsymbol{\alpha}[\tau]$$

which are assumed to be good if the safety-index $\beta[F(\tau)]$ is large. Here, the evaluation of the right-hand side is rather simple in applying part (a) of the theorem. If the numerical evaluation of the direct transformation $T(\cdot, \tau)$ is easier than evaluation of the inverse $T^{-1}(\cdot, \tau)$, it is advantageous to use the relation:

$$\frac{\partial}{\partial \tau_j} T^{-1}(\boldsymbol{x}^*, \tau_0) = -A^{-1} d$$

with:

$$A = \begin{bmatrix} \frac{\partial T_1(\boldsymbol{u}^*, \tau_0)}{\partial u_1}, \ldots, \frac{\partial T_1(\boldsymbol{u}^*, \tau_0)}{\partial u_n} \\ \vdots \\ \frac{\partial T_n(\boldsymbol{u}^*, \tau_0)}{\partial u_1}, \ldots, \frac{\partial T_n(\boldsymbol{u}^*, \tau_0)}{\partial u_n} \end{bmatrix} \qquad (27)$$

$$d = \left(\frac{\partial T_1(\boldsymbol{u}^*, \tau_0)}{\partial \tau_j}, \ldots, \frac{\partial T_n(\boldsymbol{u}^*, \tau_0)}{\partial \tau_j}\right)' \qquad (28)$$

Until now it has been assumed that the vectors $X = X(\tau)$ and U have the same dimension, n. Although it is, in principle, always possible to find such a U and a corresponding transformation $X = T(U)$, it is often much easier in applications to find a transformation $X = T(U)$ where $U = (U_1, \ldots, U_m)'$, but $m > n$. Using a simple trick, theorem 1 also covers this more general situation.

Defining:

$$\tilde{X}(\tau) = (X_1(\tau), \ldots, X_n(\tau), U_{n+1}, \ldots, U_m)' \qquad (29)$$

and, with $\tilde{X}_i = X_i$ for $i \leqslant n$ and $\tilde{X}_i = U_i$ for $i \geqslant n+1$:

$$\tilde{F}_x = \{\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{x}_{n+1}, \ldots, \tilde{x}_m) : (\tilde{x}_1, \ldots, \tilde{x}_n) \in F_x\} \qquad (30)$$

$$\{X(\tau) \in F_x\} = \{\tilde{X}(\tau) \in \tilde{F}_x\} \qquad (31)$$

$$\tilde{X}(\tau) = \tilde{T}(U, \tau) = (T(U, \tau), U_{n+1}, \ldots, U_m) \qquad (32)$$

Hence, the equivalent reliability index and alpha-values for $\{\tilde{X}(\tau) \in \tilde{F}_x\}$ are the same as for $\{X(\tau) \in F_x\}$, while the transformation (32) of $\tilde{X}$ satisfies the dimensionality criterion $\dim(\tilde{X}) = \dim(U)$. Usually, $\tilde{T}$ is also invertible. If not, an invertible transformation might be obtained by renumbering the variables $U_1, \ldots, U_m$.

Asymptotic relations for general parameters

Comparing Eqs (8) and (9) with (10) and (11) it is recognized that no essential difference between design and distribution parameters exists after transformation into the u-space. Both types of parameters occur as parameters of the transformed failure domain $F(\tau)$. On the other hand, in order to obtain a result as in theorem 1 also for design parameters, the failure domain $F(\tau)$ must now be specified more precisely as:

$$F(\tau) = \{g(\cdot, \tau) < 0\} = \{\boldsymbol{u} : g(\boldsymbol{u}, \tau) < 0\} \qquad (33)$$

where the function g is continuous in both variables. If, for instance, the failure domain F_x in the x-space is $F_x = \{g_x < 0\}$ where $g_x = g_x(x)$ contains no parameters, but $X = X(\tau) = T(U, \tau)$, then:

$$F(\tau) = \{\boldsymbol{u} : g_x(T(\boldsymbol{u}, \tau)) < 0\} = \{g(\cdot, \tau) < 0\} \qquad (34)$$

with:

$$g(\boldsymbol{u}, \tau) = g_x(T(\boldsymbol{u}, \tau)) \qquad (35)$$

Thus, a distributional parameter of the X-variables sppears as a parameter of the limit state function g in the u-space as well as a parameter of $g_x = g_x(\tau)$.

Eq (33) describes the behaviour of $F(\tau)$ on the whole u-space. In some (arbitrary small) environment E of the (unique!) beta-point u^* of $F(\tau_0)$, some more restrictive conditions are imposed on $F(\tau)$: it is required that there exist an environment E of u^*, an environment V of τ_0 (τ_0 being the initial value of τ), and functions $g_1(u, \tau), \ldots, g_t(u, \tau)$, which are defined and twice continuously differentiable for $u \in E$ and $\tau \in V$, with $g_i(u^*, \tau_0) = 0$ for $1 \leqslant i \leqslant t$, and such, that:

$$E \cap F(\tau) = E \cap \bigcap_{j=1}^{t} \{g_j(\cdot, \tau) < 0\} \quad \text{for} \quad \tau \in V \tag{36}$$

If $t = 1$, then g_1 equals the function g as in Eq (33).

Condition (36) is less restrictive than it might appear. If, for instance:

$$F(\tau) = \bigcap_{j=1}^{t+s} \{g_j(\cdot, \tau) < 0\} \tag{37}$$

where g_j is twice continuously differentiable and $g_j(u^*, \tau_0) = 0$ for $1 \leqslant j \leqslant t$, but $g_j(u^*, \tau_0) < 0$ for $t+1 \leqslant j \leqslant t+s$, then for small environments E of u^* and V of τ_0, Eq (36) holds, and $g_{t+1}, \ldots, g_{t+s}$ do not contribute locally. They must only be taken into account globally, i.e. for $u \notin E$ or $\tau \notin V$, and due to Eq (37), a possible choice of g in Eq (33) is:

$$g(u, \tau) = \max_{1 \leqslant j \leqslant t+s} g_j(u, \tau) \tag{38}$$

Note that in Eq (37) the case $g_j(u^*, \tau_0) > 0$ is impossible, since u^* would not lie on the boundary of $F(\tau_0)$.

In addition to Eq (36) it is also required that the gradients $a_1, \ldots, a_t$:

$$a_j = \text{grad } g_j(u^*, \tau_0)$$
$$= \left(\frac{\partial}{\partial u_1} g_j(u^*, \tau_0), \ldots, \frac{\partial}{\partial u_n} g_j(u^*, \tau_0)\right)' \tag{39}$$

are linearly independent. In particular, t cannot exceed n.

In contrast to theorem 1, the differentiability of $\beta_E(\tau)$, or the existance of the limits:

$$\alpha_E[\tau_j | \tau] = \lim_{h \to 0} \frac{1}{h} [\beta_E(\tau + he_j) - \beta_E(\tau)] \tag{40}$$

(e_j being the jth unit vector in R^k) cannot, to the author's knowledge, be guaranteed with practical conditions. Therefore, in place of α_E, upper and lower equivalent alpha-values are introduced, which are defined as the respective upper and lower limits:

$$\alpha_E^u[\tau_j | \tau] = \left(\frac{\partial}{\partial \tau_j}\right)_u \beta_E[F(\tau)] = \limsup_{h \to 0} \frac{1}{h} [\beta_E(\tau + he_j) - \beta_E(\tau)] \tag{41}$$

$$\alpha_E^l[\tau_j | \tau] = \left(\frac{\partial}{\partial \tau_j}\right)_l \beta_E[F(\tau)] = \liminf_{h \to 0} \frac{1}{h} [\beta_E(\tau + he_j) - \beta_E(\tau)] \tag{42}$$

Theorem 2 shows that asymptotically, for a large safety-index, α_E^u and α_E^l are close together and well-approximated by α for a proof, see reference 7):

Theorem 2

(a) u^* can be represented as a linear combination of the gradients a_j:

$$u^* = \sum_{j=1}^{t} \lambda_j a_j$$

(b) If:

$$\beta = \beta(\tau_0) = \beta[F(\tau_0)]$$

$$c_j = \left(\frac{\partial}{\partial \tau_j} g_1(u^*, \tau_0), \ldots, \frac{\partial}{\partial \tau_j} g_t(u^*, \tau_0)\right)'$$

$$\lambda = (\lambda_1, \ldots, \lambda_t)'$$

then:

$$\alpha[\tau_j | \tau_0] = -\frac{1}{\beta} (\lambda' c_j)$$

(c) Under the assumptions listed in appendix 2 and using the notation introduced in Eqs (41) and (42):

$$\lim_{b \to \infty} \frac{1}{b} \left(\frac{\partial}{\partial \tau_j}\right)_u \beta_E[bF(\tau_0)] = \lim_{b \to \infty} \frac{1}{b} \left(\frac{\partial}{\partial \tau_j}\right)_l \beta_E[bF(\tau_0)] = \alpha[\tau_j | \tau_0]$$

For the special case that $t = 1$, the result of theorem 2(b) reduces to:

$$\alpha[\tau_j | \tau_0] = \frac{1}{\| a_1 \|} \frac{\partial}{\partial \tau_j} g_1(u^*, \tau_0) \tag{43}$$

As in the previous section, theorem 2 implies the asymptotic approximations:

$$\alpha_E^u[\tau_j | \tau] \approx \alpha_E^l[\tau_j | \tau] \approx \alpha[\tau_j | \tau] \tag{44}$$

where a formula for $\alpha[\tau_j | \tau]$ is given in part (b) of the theorem.

Comparison of theorems 1 and 2

For distributional parameters τ, theorem 1 applies as well as theorem 2, if the failure domain in the x-space is defined as:

$$F_x = \{g_x < 0\} \tag{45}$$

where g_x is a continuous function, and in some environment E of $x^* = T(u^*, \tau_0)$:

$$E \cap F_x = E \cap \bigcap_{j=1}^{t} \{g_{x,j} < 0\} \tag{46}$$

where the functions $g_{x,j}$ are twice continuously differentiable in E. Of course, the formulae for $\alpha[\tau_j | \tau_0]$ in both theorems are equivalent. Nevertheless, theorem 1 exhibits some numerical advantages, whenever the numerical evaluation of the partial derivatives of the function g_j becomes too involved, since theorem 1 requires only the derivatives of the transformation T and not those of g_j. It is also worth mentioning that, in contrast to theorem 2, theorem 1 requires no specific assumptions about the failure domain F, but implies the differentiability of the function $\beta_E(\tau)$. On the other hand, theorem 2 also relates to design parameters which are not covered by theorem 1.

Potential applications

The preceeding theorems allow estimation of the change of the equivalent reliability index $\beta_E(\tau)$ which is induced by a small variation $\Delta\gamma$ of the parameter γ. Under the assumptions of theorem 1, $\beta_E(\tau)$ is continuously differentiable, therefore:

$$\Delta\beta_E = \beta_E(\tau + \Delta\tau) - \beta_E(\tau) \approx \sum_{j=1}^{k} \frac{\partial \beta_E(\tau)}{\partial \tau_j} \Delta\tau_j \tag{47}$$
$$= \sum_{j=1}^{k} \alpha_j [\tau_j | \tau] \Delta\tau_j$$

Furthermore, since $\alpha_E[\tau_j|\tau]$ is asymptotically approximated by $\alpha[\tau_j|\tau]$, due to Eq (25), one obtains:

$$\Delta\beta_E \approx \sum_{j=1}^{k} \alpha[\tau_j|\tau]\,\Delta\tau_j \tag{48}$$

where a simple formula for $\alpha[\tau_j|\tau]$ is given in theorem 1. The same approximation (48) also follows from theorem 2 by applying it to the artificial parameter h defined by:

$$\tilde{g}(u, h) = g(u, \tau + h\Delta\tau).$$

The approximation (48) is especially useful in probabilistic structural design; in particular, it simplifies the reduction of cost under reliability constraints, or the improvement of reliability given a cost limit. Denoting the cost of a structure established with design parameters τ by $c(\tau)$, the change of cost due to a small variation $\Delta\tau$ of τ is:

$$\Delta c \approx \sum_{j=1}^{k} \frac{\partial c(\tau)}{\partial \tau_j}\,\Delta\tau_j =: \sum_{j=1}^{k} \gamma_j(\tau)\,\Delta\tau_j \tag{49}$$

Hence, in order to preserve the costs but increase the reliability, a reasonable choice of $\Delta\gamma$ is:

$$\Delta\tau = \varepsilon\pi_{(\gamma)} \tag{50}$$

Here, ε is a small, positive real number and $\pi_{(\gamma)}$ is the projection of:

$$\alpha^0 = \frac{1}{\|\alpha[\tau]\|}\,\alpha[\tau]$$

onto the hyperplane orthogonal to $\gamma(\tau) = (\gamma_1(\tau), \ldots, \gamma_k(\tau))'$:

$$\pi_{(\gamma)} = \alpha^0 - (\alpha^{0\prime}\gamma^0)\gamma^0 \tag{51}$$

where:

$$\gamma^0 = \frac{1}{\|\gamma[\tau]\|}\,\gamma[\tau]$$

Similarly, the costs are reduced but reliablity is preserved by taking:

$$\Delta\tau = -\varepsilon\pi_{(\alpha)} \tag{52}$$

where, again, ε is a small, positive number and:

$$\pi_{(\alpha)} = \gamma^0 - (\gamma^{0\prime}, \alpha^0)\alpha^0 \tag{53}$$

If $\gamma[\tau]$ and $\alpha[\tau]$ are parallel or, more precisely, α^0 equals γ^0, then both $\pi_{(\gamma)}$ and $\pi_{(\alpha)}$ vanish. In this case, provided that $\alpha_E[\tau]$ is well approximated by $\alpha[\tau]$ and only small variations $\Delta\tau$ of τ are taken into account, a substantial increase of reliability is always connected with additional cost, and each significant reduction of cost also implies a decrease in reliability. Inversely, the design can be improved as long as α^0 differs substantially from γ^0.

In the second extremal case that $\gamma[\tau]$ and $\alpha[\tau]$ are antiparallel or, $\alpha^0 = -\gamma^0$, an increase in reliablity is equivalent to a decrease of cost. More generally, if the scalar product of α and γ is negative by taking:

$$\Delta\tau = \varepsilon_1\alpha^0 - \varepsilon_2\gamma^0 \tag{54}$$

with $\varepsilon_1 \geqslant 0$ and $\varepsilon_2 \geqslant 0$ and $\varepsilon_1 + \varepsilon_2$ small but non-zero, reliability is increased while at the same time the costs are reduced. It would then usually be meaningless to use Eqs (50) or (52) instead of (54). If, for example, emphasis lies on reduction of cost, just take $\varepsilon_1 = 0$ in Eq (53). This fortunate case of negative $\alpha'\gamma$ will, however, not often occur in practical applications.

Alpha-values of variables

The aim of this section is to define an importance measure for stochastic variables, and to reveal its relation to the alpha-values α_i of variables. Consider a vector $X = (X_1, \ldots, X_n)$ of random variables, which are arbitrarily distributed with the exception of a single standard normal variable $X_i = U_i$. Provided that U_i is stochastically independent of the rest-vector $X_i = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, an intuitive measure for the stochastic importance of U_i is given by the increase $\Delta\beta_E$ of reliability, resulting from a complete reduction of its variance keeping the mean value constant. If $\Delta\beta_E$ is very small, then U_i could be modelled as a deterministic variable as well and the stochastic importance of U_i is negligible. If, however, $\Delta\beta_E$ is absolutely large the stochastic nature of U_i can be essential. For technical reasons, but also aiming at the well-known alpha-values α_i, only differentially small variations of the standard deviation σ_i of U_i are considered: analogously, small variations of the mean value μ_i are also investigated.

Therefore, let:

$$\tau = (\mu_i, \sigma_i) \quad \tau_0 = (0, 1)$$

$$X(\tau) = (X_1, \ldots, X_{i-1}, \mu_i + \sigma_i U_i, X_{i+1}, \ldots, X_n)$$

$$P_f(\tau) = P[X(\tau) \in F_x] \quad \beta_E(\tau) = -\phi^{-1}[P_f(\tau)]$$

Then, under the conditions described:

$$\alpha_\sigma[U_i] = \frac{\partial}{\partial \sigma_i}\,\beta_E(\tau_0) \tag{55}$$

is a measure for the stochastic importance of the variable $X_i = U_i$, and is called the alpha-value α_σ of the variable U_i. Analogously, the alpha-value α_μ of U_i is defined as:

$$\alpha_\mu[U_i] = \frac{\partial}{\partial \mu_i}\,\beta_E(\tau_0) \tag{56}$$

which reflects the effect of a small variation of the mean value.

Having defined the alpha-values of X_i for a standard normal variable X_i, which is stochastically independent of the rest-vector X_i, the general definition follows directly from a required scale-invariance: The alpha-values α_μ and α_σ of X_i must not depend on the specific scale to which X_i is related. For instance, X_i, given in feet or in meters, must have the same alpha-value, although the numerical values of X_i greatly depend on the respective scale. In general, any two variables X_i and Y_i, which are functionally related, such as:

$$Y_i = T_i(X_i) \tag{57}$$

where T_i is a strictly increasing function, can be considered as equivalent variables but measured on different scales. Therefore, if $Y = (Y_1, \ldots, Y_m)$ is a random vector and $T = (T_1, \ldots, T_m)$ a transformation with:

$$Y = T(X) = (T_1(X), \ldots, T_m(X)) \tag{58}$$

where $T_i(X) = T_i(X_i)$ depends only on the variable X_i and the function $T_i = T_i(X_i)$ is strictly increasing, then, the alpha-values of Y_i must be defined as:

$$\alpha_\sigma[Y_i] = \alpha_\sigma[X_i] \quad \text{and} \quad \alpha_\mu[Y_i] = \alpha_\mu[X_i] \tag{59}$$

In particular, $\alpha_\sigma[Y_i]$ is a measure for the stochastic importance of Y_i, based on a standardized variation of the variance of Y_i, or the corresponding variation of the standard normal variable X_i.

This definition already covers the general case of a non-degenerate variable, i.e. a variable Y_i with continuous distribution function, or equivalently:

$$P[Y_i = y] = 0 \tag{60}$$

for each real number y.

In reference 6, it is shown that under assumption (60) it is always possible to find a random vector X and a transformation $Y = T(X)$ which satisfies all the imposed conditions; X can even be chosen as a vector of independent, standard normal variables $X = U$. According to Ref 7, each admissible transformation $Y = T(X)$ leads to the same alpha-values α_σ and α_μ, so that the alpha-values of variables are thus well-defined.

An exact numerical evaluation of those alpha-values is, however, impractical in most applications, and approximations are highly desirable. As a consequence of definitions (54)-(59), it suffices here to derive approximations for $\alpha_\mu[U_i]$ and $\alpha_\sigma[U_i]$, where $U = (U_1, \ldots, U_n)'$ is a vector of independent, standard normal variables. The (transformed) failure domain in the u-space is again denoted by F.

If:

$$\tau = (\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n)'$$

$$\tau_0 = (0, \ldots, 0, 1, \ldots, 1)'$$

$$U(\tau) := T(U, \tau) = (\mu_1 + \sigma_1 U_1, \ldots, \mu_n + \sigma_n U_n)'$$

then (compare with Eq (25)):

$$\alpha_\sigma[U_i] = \frac{\partial}{\partial \sigma_i} \beta_E[U(\tau_0) \in F]$$

$$= \alpha_E[\sigma_i \mid \tau_0] \approx \alpha[\sigma_i \mid \tau_0] \tag{61}$$

$$\alpha_\mu[U_i] = \frac{\partial}{\partial \mu_i} \beta_E[U(\tau_0) \in F]$$

$$= \alpha_E[\mu_i \mid \tau_0] \approx \alpha[\mu_i \mid \tau_0]$$

and from theorem 1(a), it is easily derived that:

$$\alpha[\sigma_i \mid \tau_0] = \frac{1}{\beta}\,[(u^*)'\,(0, \ldots, 0, -u_i^*, 0, \ldots, 0)']$$

$$= -\frac{1}{\beta}\,(u_i^*)^2 = -\beta\alpha_i^2 \tag{62}$$

$$\alpha[\mu_i \mid \tau_0] = \frac{1}{\beta}\,[(u^*)'\,(0, \ldots, 0, -1, 0, \ldots, 0)']$$

$$= -\frac{1}{\beta}\,u_i^* = \alpha_i \tag{63}$$

or, finally:

$$\alpha_\sigma[U_i] \approx -\beta\alpha_i^2 \tag{64}$$

$$\alpha_\mu[U_i] \approx \alpha_i \tag{65}$$

In particular, α_i^2 is an approximate measure for the stochastic importance of the variable U_i (or equivalent variable Y_i), while α_i indicates the sensitivity of β_E due to a standardized variation of the mean value.

A more rigorous derivation of Eqs (64) and (65) is given in reference 7. In the same reference, it is also shown that $\alpha_\mu[U_i]$ is related to the conditional mean value of U_i in case that failure occurs:

$$E[U_i \mid U \in F] = -\alpha_\mu[U_i]\,\frac{\psi(-\beta_E)}{\phi[-\beta_E]} \tag{66}$$

or:

$$E[\phi^{-1}G(Y_i) \mid Y \in F_y] = -\alpha_\mu[Y_i]\,\frac{\psi(-\beta_E)}{\phi[-\beta_E]} \tag{67}$$

Here, ϕ is the distribution function of U_i, ψ the standard normal density, and G the distribution function of Y_i.

Conclusions

If the failure probability $P_f = P_f(\tau)$ or the equivalent reliability index $\beta_E = \beta_E(\tau)$ depend on a vector $\tau = (\tau_1, \ldots, \tau_k)$ of parameters, the partial derivatives $\alpha_E[\tau_i \mid \tau_o] = \frac{\partial}{\partial \tau_i}\beta_E[\tau_0]$ express the dependency of $\beta_E[\tau]$ against small variations of τ_i, and can, therefore, be interpreted as sensitivity factors of parameters. While their numerical evaluation is often impractical, simple formulae for the partial derivatives $\alpha[\tau_i \mid \tau_0]$ of the Hasofer–Lind reliability index $\beta(\tau)$ are presented[4], which are furthermore shown to be asymptotic approximations for $\alpha_E[\tau_i]$. They therefore provide a useful tool for the improvement of structural designs under reliability aspects. Finally, a unique importance measure $\alpha_\sigma[Y]$ for random variables has been defined as the derivative of β_E with respect to the standard deviation of Y, measured on the intrinsic standard normal scale of Y. This importance measure is asymptotically approximated by the square of the Paloheimo–Hannus alpha-value α_i[5].

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Nomenclature

F, $F(\tau)$	failure domain in u-space
F_x, $F_x(\tau)$	failure domain in x-space
g, g_i	limit state function
grad	gradient
P_f, $P[F] = P[U \in F]$, $P[F_x] = P[X \in F_x]$	failure probabilities
R^n	n-dimensional real space
T, $T(u, \tau)$	transformation from u-space to x-space
lim, lim inf, lim sup	limit, lower limit, upper limit
$U = (U_1, \ldots, U_n)$	vector of independent, standard-normal random variables
u-space	space of U-variables
$u^* = (u_1^*, \ldots, u_n^*)'$	beta-point[4]
$X = (X_1, \ldots, X_n)$	vector of arbitrarily distributed random variables
x-space	space of X-variables
$\alpha_i = -\frac{1}{\beta} u_i^*$	alpha-values[5]
$\alpha[\tau_i]$, $\alpha[\tau_i \mid \tau_0] = \frac{\partial}{\partial \tau_i}\beta(\tau_0)$	alpha-value of parameter (sensitivity factor)
$\alpha_E[\tau_i]$, $\alpha_E[\tau_i \mid \tau_0] = \frac{\partial}{\partial \tau_i}\beta_E(\tau_0)$	equivalent alpha-value of parameter (sensitivity factor)

$\alpha_u[U_i], \alpha_u[X_i]$, $\alpha_\sigma[U_i], \alpha_\sigma[X_i]$	alpha-values of random variables (importance measure)
β	reliability index[4]
β_E	generalized or equivalent reliability index[3]
ψ	standardnormal density function
ϕ	(cumulative) standardnormal distribution function
$\{\dots\}$	set
$\lVert\cdot\rVert$	Euclidian norm of vector
$\in$	'element of', 'contained in'
$\frac{\partial}{\partial\tau}$	partial derivative
$\left(\frac{\partial}{\partial\tau}\right)_u, \left(\frac{\partial}{\partial\tau}\right)_l$	see Eqs (41) and (42)
Σ	sum
$'$ (e.g. $(x_1, \dots, x_n)'$)	transposition of vector
$\cap, \bigcap$	intersection of sets
$\cup$	union of sets
$\backslash$	difference of sets
$\times$	Cartesian product of sets

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Appendix 1 Conditions for theorem 1

(1) For an environment W of τ_0, $T(u, \tau)$ is defined for each $u \in R^n$ and $\tau \in W$, and $T(\cdot, \tau)$ is invertible for each $\tau \in W$.

(2) If $T^{-1}(u, \tau) = [T(\cdot, \tau)]^{-1}(u)$ and $G(u, \tau) = T^{-1}[T(u, \tau_0), \tau]$, then G is twice continuously differentiable in $R^n \times W$.

(3) For $1 \leqslant i \leqslant k$, the functions:

$$f_i(u) = \sup_{\tau \in W} \left| \psi[G(u, \tau)] \frac{\partial G(u, \tau)}{\partial u} \left[G(u, \tau) \frac{\partial}{\partial \tau_i} G(u, \tau) \right] \right|$$

$$h_i(u) = \sup_{\tau \in W} \left| \psi[G(u, \tau)] \frac{\partial}{\partial \tau_i} \frac{\partial G(u, \tau)}{\partial u} \right|$$

with $\psi[G(u, \tau)] = \psi[G_1(u, \tau)] \dots \psi[G_n(u, \tau)]$, ψ being the standard normal density, and $\frac{\partial G}{\partial u}$ being the functional determinant, are Lebesgue-integrable over $F = F(\tau_0)$:

$$\int_F f_i(u)\, du < \infty \qquad \int_F h_i(u)\, du < \infty$$

(4) For each $\tau \in W$:

$$P\left[\frac{\partial G(U, \tau)}{\partial u} = 0\right] = 0$$

(5) F has a unique beta-point which is $\beta[F] > 0$

Appendix 2 Conditions for theorem 2

(1) F has a unique beta-point $\mathbf{u}^*$ which is $\beta[F] > 0$.

(2) For an environment W of τ_0, there exists a continuous function $g = g(u, \tau)$ on $R^n \times W$, such that:

$$\{g(\cdot, \tau) < 0 \subset F(\tau) \subset \{g(\cdot, \tau) \leqslant 0\}$$

for each $\tau \in W$, and:

$$\partial F(\tau_0) = \{g(\cdot, \tau_0) = 0\}$$

where $\partial F(\tau_0)$ is the boundary of $F(\tau_0)$.

(3) For an environment E of $\mathbf{u}^*$, there exist functions $g_j = g_j(\mathbf{u}, \tau)$ on $E \times W (1 \leqslant j \leqslant t)$, which are twice continuously differentiable on $E \times W$ and such that:

$$E \cap F(\tau) = E \cap \bigcap_{j=1}^{t} \{g_i(\cdot, \tau) < 0\} \quad \text{for} \quad \tau \in W$$

(4) The gradients $\mathbf{a}_1, \dots, \mathbf{a}_t$:

$$a_j = \text{grad}\, g_j(u^*, \tau_0) = \left(\frac{\partial}{\partial u_1} g_j(u^*, \tau_0), \dots, \frac{\partial}{\partial u_n} g_j(u^*, \tau_0)\right)'$$

are linearly independent.

(5) For $1 \leqslant j \leqslant t$:

$$\limsup_{h \to 0} \frac{1}{|h|} P[F(\tau_0 + he_j)\, \Delta F(\tau_0)] < \infty$$

where $A \Delta B = (A \cup B) \backslash (A \cap B)$.

Furthermore, a sufficient condition for (5) is made up of the following:

(1) For $\tau \in W$, $F(\tau)$ can be represented finitely by many unions and intersections of sets $F_j(\tau)$, with:

$$\{f_j(\cdot, \tau) < 0\} \subset F_j(\tau) \subset \{f_j(\cdot, \tau) \leqslant 0\}$$

where the functions $f_j = f_j(u, \tau)$ are continuously differentiable on $R^n \times W$.

(2) The distribution of the random variables $f_j(U, \tau_0)$ has a probability density $\psi_j = \psi_j(s)$, which is continuous at $s = 0$.

(3) For some $\varepsilon > 0$, the random variables:

$$H_j = \sup_{|h| < \varepsilon} \left| \frac{d}{dh} f_i(U, \tau_0 + he_j) \right|$$

have finite variance.

(4) The conditional mean values:

$$M_j(s) = E[H_j^2 \mid f_j(U, \tau_0) = s]$$

are continuous at $s = 0$.