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Optimization — the basis of code-making and reliability verification

R. Rackwitz*

Technische Universität München, Arcisstr. 21, 80290 München, Germany

Abstract

As early as 1971 Rosenblueth and Mendoza (Rosenblueth E, Mendoza E. Reliability optimization in isostatic structures. J Eng Mech Div, ASCE 1971;97(EM6):1625–42) published a paper on structural optimization the concepts of which have been refined later by Hasofer in 1974 (Hasofer AM. Design for infrequent overloads. Earthquake Eng and Struct Dynamics 1974;2(4):387–8) and Rosenblueth in 1976 (Rosenblueth E. Optimum design for infrequent disturbances. J Struct Div, ASCE 1976;102(ST9):1807–25) particularly in the context of earthquake resistant design. In essence, these authors proposed to distinguish between structures that can fail upon completion or never and structures which can fail under rare 'disturbances'. Furthermore they distinguished between 'single mission structures' and structures which are systematically rebuilt after failure. The consequences of their findings for code making, especially for setting safety targets apparently have been overlooked since then. In fact, it is rather a yearly failure rate that has to be specified and verified and not a failure probability for an arbitrary reference time. The paper thoroughly reviews Rosenblueth's and Hasofer's developments and extends the concepts to failures including ultimate limit state failure under normal and extreme conditions, serviceability failure, fatigue and other deterioration and, finally, obsolescence. Some newly needed computational tools are addressed. Partial safety factors are derived for stationary failure processes and a new verification format for fatigue and other deterioration is proposed. Tools for optimization of structural components are presented.

Keywords: Structural optimization; Structural reliability; Safety targets; Failure rates; Outcrossing rates

1. Introduction

Since about 30 years the civil engineering profession is developing tools for quantifying the stochasticity in mechanics and evaluating structural reliability. Applications are widespread. Simultaneously, there is an ongoing debate on 'how safe is enough'. The late 1970s saw various national and international attempts to set up a code-type framework specifying targets, formats

* Tel.: +49-89-289-23050; fax: +49-89-289-23046.

E-mail address: rackwitz@massivban.bauwesen.tu-muenchen.de

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and methods [1–5]. More recently, European efforts concentrated on the so-called EUROCODES [6]. All of them essentially based their safety considerations on the newly developed FORM methodology implying that the targets are specified in terms of safety (reliability) indices. With the exception of ANS all related the target safety indices, essentially determined by calibration, to one year. ANS related it to a time period of 50 years. The EUROCODES recognized the necessity to distinguish between different reference times for different buildings. Being the latest the EUROCODES also recognized the necessity to handle fatigue and other types of deterioration by the same concept.

As early as 1971 Rosenblueth and Mendoza [7] proposed to use optimization for assessing targets with special reference to earthquake resistant design in a fundamental paper. The concepts developed therein were later refined by Hasofer [8] and again by Rosenblueth [9]. In particular, a distinction was made whether failure would occur upon construction or never and at "large random disturbances" only. A second distinction was made with respect to the reconstruction policy. In the extremes there are just two: no reconstruction after failure and systematic reconstruction or repair after failure, respectively. Whereas it is true that both types of failure should generally be considered depending on the type of loading on the structure the matter of reconstruction policy was apparently overlooked in the past. In fact, for almost all civil engineering structures systematic rebuilding after failure, be it caused by extreme loading, bad construction, fatigue, other deterioration, loss of serviceability, or by demolition after obsolesce, is mandatory, at least ideally because buildings serve the user and society. Optimization for one "mission" is thinkable for certain construction operations only.

In this paper the ideas of Rosenblueth and Hasofer will first be taken up again. On purpose all basic derivations are carried out with some detail, care and redundancy so that the interested reader can follow. To a certain extent those ideas will be generalized and concretized. It will be shown that it is possible to set up a consistent conceptual framework for optimal design and reliability verification. A suitable optimization scheme will be proposed. It will be demonstrated that failure rates rather than time-dependent failure probabilities are the basis for setting up safety targets. Important conclusions will be drawn with respect to code making, in particular when assessing safety targets and when deriving partial safety factors for practical use. The new philosophy will require new computational tools some of which will be discussed.

2. Rosenblueth and Hasofer's treatment

Assume that the objective function of a structural component is

$$Z(p) = B(p) - C(p) - D(p) \tag{1}$$

$B(p)$ is the benefit from the existence of the structure, $C(p)$ are the construction cost and $D(p)$ the expected damage cost. p generally is a design parameter vector. Without loss of generality all quantities will be measured in monetary units. A discussion on matters how and to what extent this is justified is beyond the scope of this paper. Statistical decision theory dictates that the expected values for $B(p)$, $C(p)$ and $D(p)$ have to be taken. $B(p)$, in general, will be unaffected or slightly decrease with each component of p but this will be neglected without substantial error so that $B = B(p)$. $C(p)$ increases with each component of p under normal circumstances. Frequently,

it can be approximated by $C(p) \approx C_0 + \Sigma c_i p_i$. C_0 are those costs which do not depend on p. In general, there is $C_0 >> \Sigma c_i p_i$. $D(p)$ decreases with p in some fashion. For each involved party, i.e. the builder, the user and the society, $Z(p)$ should be positive. Otherwise one should not undertake the realization of the structure. This is illustrated in Fig. 1. Benefits, cost and damages are not necessarily the same for all involved parties. Therefore, the intersection of the domains where $Z(p)$ is positive is the domain of p, which makes sense for all parties.

Furthermore, the decision about p has to be made at $t = 0$. This requires capitalization of all cost. In the following a continuous capitalization function is used.

$$\delta(t) = \exp(-\gamma t) \tag{2}$$

with γ the interest rate and t time in suitable time units. Usually, a yearly interest rate is defined and $\delta(t) = (1 + \gamma')^{-t}$ with γ' the yearly interest rate. Both forms are identical for $\gamma = \ln(1 + \gamma')$, where for $\gamma << 1$, we have in very good approximation $\gamma \approx \gamma'$. It will further be assumed that γ is corrected for de- and inflation and averaged over sufficiently long periods to account of fluctuations in time. For the moment it is also assumed that the time for construction is negligibly short as compared to the average lifetime of the structure. If necessary, adjustments for finite construction times can be made.

2.1. Failure upon construction

If the structure fails upon construction (or when it is put into service) or never and is abandoned after first failure Eq. (1) specializes to

$$Z(p) = B^*(1 - P_f(p)) - C(p) - H(p)P_f(p) = B^* - C(p) - (B^* + H(p))P_f(p) \tag{3}$$

$1 - P_f(p)$ and $P_f(p)$ are reliability and failure probability, respectively. $H(p)$ is the direct failure cost. In most cases $H(p)$ will be constant including the direct cost of direct physical damage and

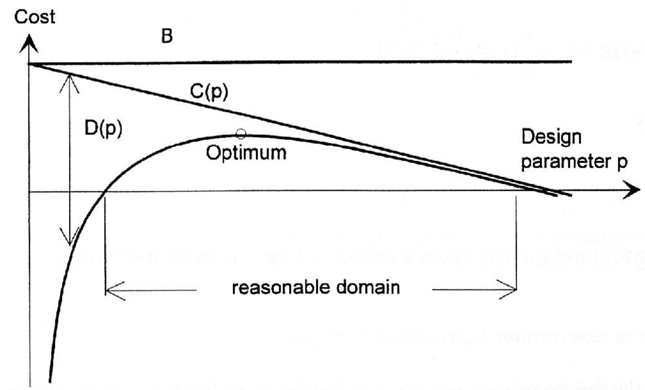

Fig. 1. Cost and benefit over design parameter p [10].

the cost of demolition and removal but also cost for human life and injury so that $H(p) = H$. B^* is the benefit derived from the existence of the structure. If the failure probability depends on an uncertain vector R an additional expectation operation is necessary and $P_f(p)$ is to be replaced by $E_R[P_f(p, R)]$. If the structure fails upon construction and is reconstructed immediately

$$Z(p) = B^* - C(p) - (C(p) + H(p))\sum_{n=1}^{\infty} nP_f(p)^n(1 - P_f(p))$$
$$= B^* - C(p) - (C(p) + H(p))\frac{P_f(p)}{1 - P_f(p)} \tag{4}$$

since

$$(1 - P_f(p))\sum_{n=0}^{\infty} nP_f(p)^n = (1 - P_f(p))\frac{P_f(p)}{(1 - P_f(p))^2} = \frac{P_f(p)}{1 - P_f(p)}$$

If the failure probability depends on an uncertain vector R an additional expectation operation is necessary as indicated previously. Rosenblueth and Mendoza discuss at length what is meant by reconstruction for the same reliability and independence of the failure events. Here, it is just assumed that reliability is already optimal so that there is no reason to modify the design rules after failure although, practically, the design itself will almost certainly be different from the previous one. For structures there is always $P_f(p)) << 1$, and the term $1/(1 - P_f(p))$ can be neglected. Then, assuming further $C(p) << H(p)$

$$Z(p) = B^* - C(p) - H(p)P_f(p) \tag{5}$$

This is the formulation most frequently used. Some ambiguity exists how to quantify the benefit B^* in Eqs. (3) to (5). We have for constant b and given reference time T

$$B(T) = \int_0^T b(t)\delta(t)\mathrm{d}t = \frac{b}{\gamma}[1 - \exp[-\gamma T]] \tag{6}$$

and for $T \to \infty$

$$B^* = \frac{b}{\gamma} \tag{7}$$

Unless the asymptotic value is taken a reference time has to be specified.

2.2. Failure due to time-variant loads and/or resistances

Assume that the failure process can be modeled by an ordinary renewal process. According to renewal theory [11] a renewal process has independent and identically distributed, positive times between failures and subsequent renewals. The density function of the failure times is $g(t)$. For structures this means, for example; that each reconstruction realizes a new structure with properties independent from the previous ones. It is useful to distinguish between ordinary renewal processes where all times have density $g(t)$ and modified renewal processes where the time to first failure is $g_1(t)$ while all other failure times have density $g(t)$. For an equilibrium renewal process the time to first failure has the special form $g_1(t) = (1 - G(t))/\mu$ where $G(t)$ is the distribution function corresponding to $g(t)$ and $\mu = E[T]$. The modified renewal process may be important whenever the (structural) component is not "new" at $t = 0$ and the hazard function $r(t) = g(t)/(1 - G(t))$ is not constant, e.g. indicating some deterioration with age. Another situation in which a modified renewal process may be realistic is when failure is caused by random disturbances (earthquakes, storms, floods, etc.) which have been observed in the past. For some types of disturbances the likelihood of such disturbances is influenced by the time which has elapsed since the last observed disturbance. The equilibrium renewal process as a special case of the modified renewal process should be used if it assumed that the renewal process is running already for a long time and the time origin is placed randomly between two successive renewals. This model is to be used if the time of the last reconstruction or "disturbance" is not known. The refinements of the ordinary renewal process will be of importance only occasionally in the considerations to come. We recall some results from renewal theory. The renewal function is defined as

$$H(t) = E[N(t)] = \sum_{n=1}^{\infty} nP(N(t) = n) = \sum_{n=1}^{\infty} n(G_n(t) - G_{n+1}(t))$$
$$= \sum_{n=1}^{\infty} nG_n(t) - \sum_{n=2}^{\infty}(n - 1)G_n(t) = \sum_{n=1}^{\infty} G_n(t) = \sum_{n=1}^{\infty}\int_0^t g_n(\tau)\mathrm{d}\tau = \int_0^t h(\tau)\mathrm{d}\tau \tag{8}$$

where $N(t)$ is the number of renewals in $[0, t]$ and $G_n(t) = P\left(\sum_{j=1}^{n} T_j \leqslant t\right)$. The so-called renewal density as the derivative of the renewal function, also denoted as unconditional failure rate or failure intensity, is

$$h(t) = \lim_{\Delta \to 0^+} \frac{P(\text{one or more renewals in } [t, t + \Delta])}{\Delta} = \sum_{n=1}^{\infty} g_n(t) \tag{9}$$

Let

$$g_n(t) = \int_0^t g_{n-1}(t - \tau)g(\tau)\mathrm{d}\tau; \quad n = 2, 3, \ldots \tag{10}$$

be the density function of the time to the nth renewal written as a convolution integral. For convolution integrals as in Eq. (10) the Laplace transform can be used with advantage. Define by

$$g^*(\theta) = \int_0^{\infty} \exp[-\theta t]g(t)\mathrm{d}t \tag{11}$$

the Laplace transform of $g(t)$. If $g(t)$ is assumed to be a probability density we have $g^*(0) = 1$ and $0 < g^*(\theta) \leqslant 1$ for all $\theta > 0$.

For the important stationary Poisson process with intensity λ it is simply

$$g_1^*(\theta) = g^*(\theta) = \int_0^\infty \exp[-\theta t]\lambda \exp[-\lambda t]\mathrm{d}t = \frac{\lambda}{\theta + \lambda} \tag{12}$$

$g_1^*(\theta) = g^*(\theta)$ just expresses the "lack of memory" of the Poisson process. Further details about Laplace transforms for other failure time models are given in Appendix A. For convolutions we have

$$g_n^*(\theta) = g_1^*(\theta)g_{n-1}^*(\theta) = g_1^*(\theta)[g^*(\theta)]^{n-1} \tag{13}$$

If

$$g_1^*(\theta) = \int_0^\infty \exp[-\theta t]g_1(t)\mathrm{d}t \tag{14}$$

$$g^*(\theta) = \int_0^\infty \exp[-\theta t]g(t)\mathrm{d}t$$

and, therefore, for ordinary renewal processes

$$h^*(\theta) = \sum_{n=1}^\infty g_n^*(\theta) = \sum_{n=1}^\infty [g^*(\theta)]^n = \frac{g^*(\theta)}{1 - g^*(\theta)} \tag{15}$$

and

$$h_1^*(\theta) = \sum_{n=1}^\infty g_n^*(\theta) = \sum_{n=1}^\infty g_1^*(\theta)[g^*(\theta)]^{n-1} = \frac{g_1^*(\theta)}{1 - g^*(\theta)} \tag{16}$$

for modified renewal processes. For the special case of an equilibrium renewal processes there is in noting the Laplace transform of $g_1(t) = (1 - G(t))/\mu$ being

$$g_1^*(\theta) = \frac{1 - g^*(\theta)}{\theta\mu}$$

$$h_{1,e}^*(\theta) = \frac{1}{\mu\theta} \tag{17}$$

As shown later $h^*(\theta)$ is nothing else than the Laplace transform of the renewal density.

For *structural facilities given up after first failure* we then have with $\gamma = \theta$ and where the failure time density can be controlled by the design parameter vector p

$$B^*(p) = \int_0^\infty \int_0^t b\delta(\tau)\mathrm{d}\tau g_1(t,p)\mathrm{d}t = \frac{b}{\gamma}\int_0^\infty (1 - \exp[-\gamma t])g_1(t,p)\mathrm{d}t = \frac{b}{\gamma}(1 - g_1^*(\gamma, p)) \tag{18}$$

$$D(p) = \int_0^\infty g_1(t,p)\delta(t)H(p)\mathrm{d}t = g_1^*(\gamma, p)H(p) \tag{19}$$

and, therefore

$$Z(p) = \frac{b}{\gamma}(1 - g_1^*(\gamma, p)) - C(p) - H(p)g_1^*(\gamma, p) \tag{20}$$

The present value of the expected failure cost for *systematic reconstruction after failure* is for ordinary renewal processes

$$\begin{aligned} D(p) &= (C(p) + H(p))\sum_{n=1}^\infty \int_0^\infty \delta(t)g_n(t,p)\mathrm{d}t = (C(p) + H(p))\sum_{n=1}^\infty \int_0^\infty \exp[-\gamma t]g_n(t,p)\mathrm{d}t \\ &= (C(p) + H(p))\sum_{n=1}^\infty g^*(\gamma, p)^n = (C(p) + H(p))\frac{g^*(\gamma, p)}{1 - g^*(\gamma, p)} = (C(p) + H(p))h^*(\gamma, p) \end{aligned} \tag{21}$$

The benefit B^* is as in Eq. (7) in both cases. Therefore,

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))h^*(\gamma) \tag{22}$$

For modified renewal processes $h^*(\gamma, p)$ is replaced by $h_1^*(\gamma, p)$.

For Poissonian failure processes with exponential failure times with parameter $\lambda(p)$ which can be controlled by p it is with Eq. (12) for structures given up after first failure

$$Z(p) = \frac{b}{\gamma} - C(p) - \left(\frac{b}{\gamma} + H(p)\right)\frac{\lambda(p)}{\gamma + \lambda(p)} \tag{23}$$

and for structures being reconstructed systematically

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))\frac{\lambda(p)}{\gamma} \tag{24}$$

If $\lambda(p)$ depends on some random vector R the intensity $\lambda(p)$ has to be replaced by $E_R[\lambda(p, R)]$ as pointed out by Hasofer [8] and already used by Rosenblueth and Mendoza [7], i.e.

$$Z(p) = \frac{b}{\gamma} - C(p) - \left(\frac{b}{\gamma} + H(p)\right)E_R\left[\frac{\lambda(p, R)}{\gamma + \lambda(p, R)}\right] \tag{25}$$

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))\frac{E_R[\lambda(p, R)]}{\gamma} \tag{26}$$

2.3. Failure due to "extreme" overloads

In the original developments of Rosenblueth and Mendoza [7], Hasofer [8] and Rosenblueth [9] a slightly different case is considered. Assume that failure will be caused by random disturbances such as earthquakes or storms. If the disturbance occurs the failure probability is $P_f(p)$. The failure events are independent. The disturbance process is a renewal process with distribution function $F_1(t)$ for the time to the first event and $F(t)$ for the times between events. By definition, the times between the disturbances and failures are independent. Here, it sometimes makes sense to consider modified disturbance processes. The distribution functions of the time to first failure and the times between failures are still $G_1(t,p)$ and $G(t,p)$, respectively. Let

$$f_n(t) = \int_0^t f_{n-1}(t-\tau) f(\tau)\,d\tau; \quad n = 2, 3, \ldots \tag{27}$$

be the density function of the time to the nth arrival of a disturbance. Furthermore, let

$$\phi_1(t) = f(t)$$

$$\phi_n(t) = \int_0^t \phi_{n-1}(t-\tau)\phi(\tau)\,d\tau; \quad n = 2, 3, \ldots \tag{28}$$

Then,

$$g_1(t,p) = \sum_{n=1}^{\infty} P_f(p) f_n(t)(1-P_f(p))^{n-1}$$

$$g(t,p) = \sum_{n=1}^{\infty} P_f(p)\phi_n(t)(1-P_f(p))^{n-1} \tag{29}$$

These densities take account of all possible disturbances leading to failure, i.e. that failure occurs at the first disturbance or the structure survives the first disturbance but fails in the second, etc. It follows that with Eq. (14) and by reintroducing the parameter p

$$\begin{aligned} g_1^*(\theta,p) &= \sum_{n=1}^{\infty} f_1^*(\theta) f_{n-1}^*(\theta) P_f(p)(1-P_f(p))^{n-1} = \sum_{n=1}^{\infty} f_1^*(\theta)[f^*(\theta)]^{n-1} P_f(p)(1-P_f(p))^{n-1} \\ &= \frac{P_f(p) f_1^*(\theta)}{1-(1-P_f(p))f^*(\theta)} \end{aligned} \tag{30}$$

and with $\phi_n^*(\theta) = [f^*(\theta)]^n$

$$g^*(\theta,p) = \sum_{n=1}^{\infty} f^*(\theta) f_{n-1}^*(\theta) P_f(p)(1-P_f(p))^{n-1} = \frac{P_f(p) f^*(\theta)}{1-(1-P_f(p))f^*(\theta)} \tag{31}$$

Then, for *structural facilities given up after first failure*

$$\begin{aligned} B^*(p) &= \int_0^\infty \int_0^t b\delta(\tau)\,d\tau\, g_1(t,p)\,dt = \frac{b}{\gamma}\int_0^\infty (1-\exp[-\gamma t])g_1(t,p)\,dt = \frac{b}{\gamma}(1-g_1^*(\gamma,p)) \\ &= \frac{b}{\gamma}\left(1 - \frac{P_f(p) f_1^*(\gamma)}{1-(1-P_f(p))f^*(\gamma)}\right) \end{aligned} \tag{32}$$

$$D(p) = \int_0^\infty g_1(t,p)\delta(t)H(p)\,dt = g_1^*(\gamma,p)H(p) = \frac{H(p)P_f(p)f_1^*(\gamma)}{1-(1-P_f(p))f^*(\gamma)} \tag{33}$$

and, therefore

$$Z(p) = \frac{b}{\gamma}\left(1 - \frac{P_f(p) f_1^*(\gamma)}{1-(1-P_f(p))f^*(\gamma)}\right) - C(p) - \frac{H(p)P_f(p)f_1^*(\gamma)}{1-(1-P_f(p))f^*(\gamma)} \tag{34}$$

Interestingly, the benefit also depends on the parameters of the failure process in this case.

For example, for a stationary Poisson process with intensity λ of the disturbances and each disturbance associated with a failure probability $P_f(p)$, the failure process is also a Poisson process having interarrival time distribution $F(t,p) = 1 - \exp[-P_f(p)\lambda t]$. If $\lambda(p)$ depends on some random vector R the intensity $\lambda(p)$ has to be replaced by $E_R[\lambda P_f(p,R)]$. Then, it is for $T \to \infty$

$$\begin{aligned} Z(p) &= E_R\left[\frac{b - H(p)P_f(p,R)\lambda}{\gamma + P_f(p,R)\lambda}\right] - C(p) \approx \frac{b}{\gamma} - C(p) - E_R\left[\frac{H(p)P_f(p,R)\lambda}{\gamma + P_f(p,R)\lambda}\right] \\ &\approx \frac{b}{\gamma} - C(p) - H(p)\frac{E_R[P_f(p,R)\lambda}{\gamma} \end{aligned} \tag{35}$$

where the last approximation is acceptable for $P_f(p,R)\lambda << \gamma$.

For *systematic reconstruction after failure* the present value of the expected failure cost is

$$\begin{aligned} D(p) &= (C(p)+H(p))\frac{g_1^*(\gamma,p)}{1-g^*(\gamma,p)} = (C(p)+H(p))P_f(p)\frac{f_1^*(\gamma)}{1-f^*(\gamma)} \\ &= (C(p)+H(p))P_f(p)h^*(\gamma) \end{aligned} \tag{36}$$

The last factor in Eq. (36) is called discount factor by Hasofer [8]. Eq. (1) together with Eq. (7) then reads

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p)+H(p))P_f(p)h^*(\gamma) \tag{37}$$

For a Poissonian disturbance process with intensity λ

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))P_f(p)\frac{\lambda}{\gamma} \tag{38}$$

If the disturbance process is a Poisson process with intensity λ, the failure process is also a Poisson process with rate $\lambda(p) = P_f(p)\lambda$. If the failure intensity depends on some random vector R the term $P_f(p)\lambda$ in Eq. (38) has to be replaced by $E_R[\lambda P_f(p, R)]$

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))\frac{E_R[\lambda P_f(p, R)]}{\gamma} \tag{39}$$

It is noteworthy that with this modification Eq. (39) is very similar to the approximation of Eq. (35). The approximate Eq. (35) differs from Eq. (39) only by $C(p)$ in the failure cost term. The exact two cost factors are $f_1^*(\gamma)$ for structures given up after first failure and $f_1^*(\gamma)/(1 - f^*(\lambda))$ for structures systematically rebuilt, respectively. If $\gamma \to 0$ and $T \to \infty$, the damage cost (as well as the benefit) become finite in the first case but infinite in the second. Hence, no optimal solution to Eq. (1) can be found in the latter case. Therefore, if the second strategy is taken as the only reasonable one, a discount rate $\gamma > 0$ must be assumed. The same conclusion also holds for Eqs. (20) and (21). The results of Eqs. (20), (22), (34) and (37) are remarkable in that the objective functions contain only failure rates but no failure probability per reference time. The special results for Poissonian failure processes, for example Eq. (39), also suggest a different interpretation. The expected total failure costs are $1/\gamma$ times the cost for a single failure.

2.4. Finite reconstruction times

It is easy to adjust for finite reconstruction (repair) times. Let $E[T_f]$ be the mean time to failure and $E[T_r]$ the mean reconstruction time. Then, it can be shown that

$$A(\infty) = \frac{E[T_f]}{E[T_f] + E[T_r]} \tag{40}$$

is the asymptotic availability of the structure. Thus, the benefit term in Eq. (7) has to be multiplied by $A(\infty)$. The mean time between renewals now is $E[T_f] + E[T_r]$ and the density of failure times is obtained by convolution of the densities for T_f and T_r the result of which then has to be used instead of the failure time densities introduced before. Because usually $E[T_f] >> E[T_r]$, obvious simplifications are possible.

3. Applications

We are now going to use the foregoing results for general failure processes. Let (Schall et al., [12]):

- R be a vector of random variables which are used to model structural properties and possibly other (non-ergodic) uncertain variables like parameters of the loading variables,

- Q be a vector of stationary and ergodic random sequences which are used to model long term fluctuations in the parameters of the loading variables, for example traffic states, sea states, wind states (10 min regimes), etc.,
- S be a vector of sufficiently mixing, not necessarily stationary random processes,
- $g(r, q, s(t), t) > 0$ the safe state, $g(r, q, s(t), t) = 0$ the limit state and $g(r, q, s(t), t) \leqslant 0$ the failure state of a technical facility.

The conditional outcrossing rate is

$$\nu^+(V, \tau \mid r, q) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\Big(\{g(S(\tau), \tau) > 0 \mid r, q\} \bigcap \{g(S(\tau + \Delta), \tau + \Delta) \leqslant 0 \mid r, q\}\Big) \tag{41}$$

with $V = \{g(r, q, s(t), t) \leqslant 0\}$. If the process of outcrossings is a regular process (see, for example, [13,14]), the mean number of outcrossings in a given time interval $[t_1, t_2]$ is

$$E[N^+(t_1, t_2) \mid r, q] = \int_{t_1}^{t_2} \nu^+(V, \tau \mid r, q)d\tau \tag{42}$$

Asymptotically for small failure probabilities the first passage failure time distribution then is [13]

$$P_f(t_1, t_2) \approx 1 - E_{R,Q}[\exp[-E[N^+(t_1, t_2) \mid R, Q]]] \leqslant 1 - E_R[\exp[-E_Q[E[N^+(t_1, t_2) \mid R, Q]]]] \tag{43}$$

According to Schall et al. [12] the upper bound is a consequence of application of Jensen's inequality for expectations of convex functions. Eq. (43) is seen to correspond to a inhomogeneous Poisson process of conditional outcrossings. This has been achieved by classifying the random phenomena into R-, Q- and S-variables. In the stationary case Eq. (43) simplifies to

$$P_f(t_1, t_2) \leqslant 1 - E_R[\exp[-E_Q[\nu^+(V) \mid R, Q](t_2 - t_1)]] \tag{44}$$

It follows that the quantity $\lambda(p, R)$ in Eq. (26) or $\lambda P_f(p, R)$ in Eq. (39) can be replaced by the outcrossing rate. Eq. (26) now reads

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))\frac{E_{R,Q}[\nu^+(V, R, Q, p)]}{\gamma} \tag{45}$$

For Eq. (45) and the generalizations to come it is essential to recall that the failure process is a renewal process, i.e. must have independent failure times. This means that at each renewal also the non-ergodic vector R has independent realizations at each renewal.

In view of Eq. (4) an improvement can be made to account for failures just after reconstruction. If, further, there are more than one mode in which the structure can fail we can make use of the "crude" upper bound for unions of crossing events and obtain

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H(p))\frac{\sum_{i=1}^{m}\{E_{R,Q}[\nu^+(V_i, R, Q, p)](1 + P_{f,i}(0))\}}{\gamma} \quad (46)$$

by assuming that the failure cost is the same for all failure modes. $P_{f,i}(0)$ are the failure probabilities when the structure is put into service just after reconstruction. At the expense of some more numerical effort the "crude" upper bound for crossings into a union of failure mode domains can be replaced by an improved upper bound following the arguments in Schrupp and Rackwitz [15] and Ditlevsen [16].

In general, other cost components have to be taken into account in an overall economical consideration such as violations of the serviceability conditions and fatigue and other deterioration usually implying immediate repair but also obsolesce. Also, many structures are inspected more or less regularly. For simplicity the objective function now is written as

$$Z(p) = B^* - C(p) - I(p) - U(p) - M(p) - A(p) - D(p) \quad (47)$$

where $U(p)$ is associated with serviceability failures, $M(p)$ with fatigue and other aging failure, $A(p)$ with obsolesce whereas $D(p)$ still are the expected cost for ultimate limit state failure. It is assumed that different failure types do not interact. $I(p)$ are all inspection and resulting maintenance cost. This equation is studied numerically for a simple example in Appendix B.

Assume that the entrance into non-serviceability conditions also follows a Poisson process but with higher intensity. Then

$$U(p) = U\frac{\sum_{i=1}^{m_S} E_{R,Q}[\nu^+(V_{S,i}, R, Q, p]}{\gamma} \quad (48)$$

with U the repair cost and $V_{S,i}$ the serviceability failure domains.

Fatigue and other deterioration phenomena are more difficult to handle. Assume first that the mean number of cycles to failure is given by $N = \nu_0 t = C\Delta S^{-m}$. ν_0 is the cycling rate, C and m material properties and ΔS the stress range. In sufficient agreement with experiments the failure process can be assumed to be a Poisson process with exponential interarrival times having mean

$$E[T_{\text{fat}}] = \frac{C}{\nu_0 \Delta S^m} \quad (49)$$

Rackwitz and Faber [17], in fact, found at experiments for high strength steel wires the shape parameter of a Weibull distribution for lifetimes very close to unity, that is an exponential distribution. Thus, $1/E[T_{\text{fat}}]$ can be interpreted as the failure intensity and

$$M(p) = ME_R\left[\frac{\nu_0(\Delta S/p)^m}{C}\right]\frac{1}{\gamma} \quad (50)$$

is the expected cost for fatigue failures. ν_0 and C and possibly m are non-ergodic random R-variables. For random ergodic loading one has to use a damage-equivalent constant ΔS_{eq} in Eq. (49) which, however, may depend on other non-ergodic variables. The parameter p has been attached to ΔS (larger structural dimensions might reduce the stress ranges) but could, of course, be used, alternatively or in addition, at other variables. If more than one fatigue failure can occur one proceeds as for ultimate or serviceability failure. Since the ergodicity theorem is already used when assessing equivalent constant ΔS only the expectation over the R-variables has to be taken. The exponential failure time distribution may not be adequate for other types of deterioration. Then, one has to proceed using Laplace transforms (see Appendix A).

Frequently, the cumulative damage due to fatigue and other deterioration is assumed to be a smooth function of time, at least for larger times, and failure is assumed to occur if some limiting damage is exceeded. The failure time is deterministic for given parameters of the damage process. In the presence of uncertain parameters the failure probability for given t can be expressed as

$$P_f(t) = G(t) = P(g(R, Q, t) \leqslant 0) \quad (51)$$

and, thus, the failure time density is obtained by differentiating $P_f(t)$ with respect to time or by a parametric sensitivity of the failure probability, i.e. $f_T(t) = \frac{\partial}{\partial t}P(g(R, Q, t) \leqslant 0)$. The distribution function $G(t)$ generally must be determined pointwise by one of the well-known reliability methods and Eq. (14) must be applied for its Laplace transform. Alternatively, one can make use of the relationship between Laplace transforms of functions and their integrals, i.e. by determining the Laplace transform $G^*(\gamma)$ directly from $G(t)$ and then using $g^*(\gamma) = G^*(\gamma)\gamma$.

If fatigue or other deterioration phenomena are treated by the outcrossing approach one can use the general formulation in Eq. (43). The asymptotic first passage time distribution (all conditions and condition for failure at $t = 0$ dropped temporarily) is

$$G(t) = 1 - \exp[-\int_0^t \nu^+(\tau)\mathrm{d}\tau] \quad (52)$$

with density

$$g(t) = \nu^+(t)\exp[-\int_0^t \nu^+(\tau)\mathrm{d}\tau] \quad (53)$$

Its Laplace transform is

$$g^*(\gamma) = \int_0^\infty \exp[-\gamma t]\nu^+(t)\exp[-\int_0^t \nu^+(\tau)\mathrm{d}\tau]\mathrm{d}t = \int_0^\infty \nu^+(t)\exp\left[-\int_0^t(\gamma + \nu^+(\tau))\mathrm{d}\tau\right]\mathrm{d}t \quad (54)$$

Inspection and maintenance can be an important part of the total cost of a structure. A simple model for inspection and maintenance is to assume regular inspections and, depending on the result of the inspections, repairs with probability $P_R(p)$. As an example one could write

$$I(p) \approx I_0 \frac{1}{\tau_{\text{insp}}\gamma} + I_1(p) \frac{P_R(p)\exp[-\tau_{\text{insp}}\gamma]}{1 - \exp[-\tau_{\text{insp}}\gamma]} \tag{55}$$

where τ_{insp} is the time between inspections, I_0 the cost per inspection and $I_1(p)$ the repair cost. Here use has been made of Table A.1 for the Laplace transform of deterministic inspection times.

For completeness, the case of obsolescence is also considered. Structures become obsolete quite frequently after some time because they no longer fulfill their originally intended purpose. Let $\bar{\omega}$ be the rate at which a structure becomes obsolete and A be the demolition cost. After demolition there will be immediate reconstruction. Then

$$A(p) = (C(p) + A)\frac{\bar{\omega}}{\gamma} \tag{56}$$

assuming a Poisson process for the event of obsolesce. Also here it may be wise to switch to another lifetime distribution, for example to the normal and, thus, a "discount factor" as in Eq. (A.2).

4. Practical computation of outcrossing rates

In this section we review some typical examples for the computation of outcrossing rates, primarily for clarification of terminology. More specific results can be found in the literature. The formulations are presented in terms of first-order reliability methodology. The results are given conditional on R- and Q-variables. The necessary expectation operation can be performed together with the computations illustrated below as indicated by Schall et al. [12].

4.1. Rectangular wave renewal processes

Breitung and Rackwitz [18] have shown that under stationary conditions it is to first order:

$$\begin{aligned} \nu_J^+(V, r, q) &= \sum_{i=1}^{n_S} \lambda_i [P(\{S_i^- \in \bar{V}\} \cap \{S_i^+ \in V\})] = \sum_{i=1}^{n_S} \lambda_i [P(S_i^+ \in V) \\ &\quad - P(\{S_i^+ \in V\} \cap \{S_i^- \in V\})] \\ &\approx \sum_{i=1}^{n_S} \lambda_i [\Phi(-\beta) - \Phi_2(-\beta, -\beta; 1 - \alpha_i^2] \leqslant \sum_{i=1}^{n_S} \lambda_i \Phi(-\beta) \end{aligned} \tag{57}$$

where S_i^- is the vector of components of S before a jump of the ith component with jumping rate λ_i and S_i^+ the vector after the jump, β the geometrical safety index defined by $\beta = \min\{\|u\|\}$ for $g(u) \leqslant 0$ and $u = T^{-1}(r, q, s)$ a probability preserving transformation into the space of independent, standard normal variables (Hohenbichler and Rackwitz, [19]). α_i the mean value sensitivity of the ith component of S. V denotes the failure domain and $\bar{V}$ its complement. The second term in the brackets of the third line in Eq. (57) often is negligibly small. If second order methodology is applied a factor containing second order correction must be multiplied to Eq. (57). For further results see Ref. [20].

4.2. Differentiable Gaussian processes

The outcrossing rate of a stationary, standardized Gaussian process is to first order Veneziano et al. [21]

$$\nu_D^+(V, r, q) \approx \frac{\varphi(\beta)}{\sqrt{2\pi}} \omega_0 \tag{58}$$

where

$$\omega_0^2 = -\alpha^T \ddot{R} \alpha$$

and

$$\ddot{R} = \left\{ \frac{\partial^2}{\partial \tau_1 \partial \tau_2} \rho_{S,ij}(\tau_1, \tau_2)_{|\tau = \tau_1 = \tau_2}, \; i, j = 1, \ldots, n_S \right\}$$

the matrix of second derivatives of the matrix of correlation functions. α and β are defined as above. If second order methodology is applied essentially an additional correction factor must be multiplied to Eq. (58). For further results see Breitung [22]. For non-normal translation processes similar results are available (see, for example, Rackwitz [23]).

4.3. Combined rectangular wave renewal and Gaussian processes

Rectangular wave renewal and Gaussian processes can easily be combined because the regularity [14] of the outcrossing process enables simple addition [16]

$$\nu(V, r, q) = \nu_J(V, r, q) + \nu_D(V, r, q) \tag{59}$$

When computing $\nu_J(V, r, q)$ or $\nu_D(V, r, q)$ the other process is treated as a R-variable.

4.4. Non-stationary outcrossing rates

The results in Eqs. (57) and (58) are easily extended to the non-stationary case. For simplicity, only scalar processes are considered herein. Then, Eq. (57) is modified to

$$\nu_J^+(V, \tau, r, q) \leqslant \lambda(\tau)\Phi(-\beta(\tau))(1 - \Phi(-\beta(\tau))) \tag{60}$$

and Eq. (58) to

$$\nu_D^+(V, \tau, r, q) = \omega_0(\tau)(\beta(\tau))\Psi(\dot{\beta}(\tau)/\omega_0(\tau)) \quad (61)$$

with $\Psi(z) = \varphi(z) - z\Phi(-z)$ [36].

The case where the outcrossing rate increases monotonically with time as in fatigue or other deterioration is most important for our purpose The outcrossing rates, if plotted versus β, always are bell-shaped with peak around $\beta(\tau) = 0$ (see also Appendix D) because, for large β's, the outcrossing rate is small. If β becomes smaller the outcrossing rate increases. For very small or even negative β the likelihood of the process of being already in the failure domain becomes larger and, if there are crossings of the failure surface, those will be mostly incrossings and thus the outcrossing rate decays again down to zero.

5. Design for maximum admissible failure rates

The renewal density of a Poisson process is $h(t) = \lambda$. This result follows from

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!}\exp[-\lambda t]$$

and, therefore

$$H(t) = \sum_{i=1}^{n} nP(N(t) = n) = \lambda t \text{ and } h(t) = \frac{\mathrm{d}H(t)}{\mathrm{d}t} = \lambda$$

If the failure process is conditionally a stationary Poisson process it is now easy to impose some maximum admissible failure rate. For example, for failure in ultimate limit states the condition is

$$E_{R,Q}[\nu^+(V_U(R, Q, S, p)] \leqslant \nu_{\text{admissible}} \quad (62)$$

and similarly for serviceability failure.

For non-exponential failure times this condition must be modified. The failure rate then is not necessarily constant over time. In the limit it becomes infinitely large for deterministic failure times. Unfortunately, the relation between non-exponential failure time distributions and the corresponding outcrossing rates is not yet known unless the outcrossing approach is used as in Eq. (53). Even then, it is not obvious how and at which quantity to set reliability constraints. However, the following considerations will lead to a reasonable solution. Let $G(t)$ be the non-exponential distribution function of failure times and $g(t)$ its density. The process of failures and immediate reconstruction still is a renewal process. In renewal theory the following important result is proven [11]

$$\lim_{t\to\infty} h(t) = \frac{1}{\mu} \text{ provided that } f(t) \to 0 \text{ for } t \to \infty \quad (63)$$

with $\mu = E[T]$ the mean time between failures. It is valid for both ordinary and modified renewal processes. The condition for $f(t)$ in Eq. (63) is fulfilled for all failure time models of practical

interest. The limiting operation simply says that the renewal process is active already for a long time. The result in Eq. (63) also says that the renewal process becomes stationary for large t. This is in agreement with our basic assumptions, at least ideally. The unconditional failure rate or renewal density is asymptotically inversely proportional to the mean time between failures which must exist. No other detail of the particular distribution function of failure times is used. The renewal density of a Poisson renewal process with parameter λ is, as mentioned, $h(t) = \lambda$ giving Eq. (62) a new important interpretation. For the equilibrium renewal process the renewal density is independent of t and equals exactly $h(t) = 1/\mu$ which follows from Eq. (17).

Consequently, for reliability verification the mean failure time must be computed and the corresponding asymptotic renewal density must be checked against the admissible failure rate

$$\lim_{t\to\infty} h(t) = \frac{1}{E_{R,Q}[E[T(R, Q)]]} \leqslant \nu_{\text{admissible}} \quad (64)$$

with $E[T(R, Q)]$ the mean time to failure. Some tools for performing the expectation operation over R and Q are given in Appendix C. The same concept should be followed if fatigue or other deterioration is investigated by Eq. (52) or (53). Then, for Eq. (53) the limiting condition is, for example

$$\lim_{t\to\infty} h(t) = \frac{1}{[E_{R,Q}[\int_0^\infty t\nu^+(t, R, Q, p)\exp[-\int_0^t \nu^+(\tau, R, Q, p)\mathrm{d}\tau]\mathrm{d}\tau]]} \leqslant \nu_{\text{admissible}} \quad (65)$$

Unfortunately, this condition is difficult to verify. Eq. (65) will be illustrated at an example in Appendix D. If the outcrossing rate is decreasing with time, $E_R[E_Q[\nu^+(0, R, Q, p)]] \leqslant \nu_{\text{admissible}}$ clearly is a simple conservative alternative to Eq. (65).

It is interesting to study the speed with which the renewal density approaches its asymptotic, stationary value. For non-exponential failure times the renewal density has a characteristic, damped oscillating graph around the asymptotic value with period approximately equal to twice the mean failure time μ for failure times with smaller coefficient of variation. The maxima occur at μ, 2μ, 3μ, ... where the first maximum is largest. Damping will increase with increasing standard deviation σ. For realistically large variability of failure times, say with coefficient of variation >0.2, the renewal density will reach its asymptotic value after a few oscillations. An intuitive interpretation of this behaviour is that for an ordinary renewal process the probability of renewals is larger at multiples of the mean than in between. For example, for a gamma distribution with density as in Table A.1 and $\mu = k/\lambda$ and $\sigma = \sqrt{k}/\lambda$ one finds

$$f_n(t) = \frac{\lambda^{nk} t^{nk-1}}{\Gamma(nk)}\exp[-\lambda t]; \quad h(t) = \sum_{n=1}^{\infty} \frac{\lambda^{nk} t^{nk-1}}{\Gamma(nk)}\exp[-\lambda t] \quad (66)$$

showing the described behavior. For the gamma distribution the infinite sum can be simplified for integer $k > 1$

$$h(t) = \frac{\lambda}{k}\sum_{j=1}^{k-1} \varepsilon(k)^j \exp[\lambda t(\varepsilon(k)^j - 1)] \quad (67)$$

with $\varepsilon(k) = \cos(2\pi/k) + i\sin(2\pi/k)$. With this model (integer k) only coefficients of variation of $V = 1/\sqrt{k}$ can be obtained. The renewal density is shown in Fig. 2 for three typical coefficients of variation.

The renewal density overshoots the asymptotic value by a factor of 2 to 4 for coefficients of variation between 0.2 to 0.1, respectively, and by much less for larger coefficients of variation.

Also, for normally distributed failure times one can find the exact renewal density.

$$f_n(t) = \frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{t-n\mu}{\sigma\sqrt{n}}\right);\ h(t) = \sum_{n=1}^{\infty}\frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{t-n\mu}{\sigma\sqrt{n}}\right) \tag{68}$$

It can be demonstrated that also for this distribution similar conclusions as for the gamma distribution hold. The renewal densities for the gamma and for the normal distribution with otherwise identical properties differ only very little. The compact results for the gamma and the normal distributions are, of course, due to the fact that for both distributions a convolution theorem holds. For other distributions that are not stable under convolutions the evaluation of renewal densities is very difficult and one usually must use the asymptotic value. Note, however, that the normal distribution extends to the negative axis and, therefore, is only an approximate failure time model, even if $\mu >> \sigma$. Practically, sufficient accuracy of the sum in Eq. (66) or (68) is reached after a few terms; convergence being the more rapid the larger the coefficient of variation.

One could argue that the maximum value, i.e. $\max\{h(t)\} \approx (h(\mu)$, must be used as a constraint instead of the asymptotic value $1/\mu$. For the user of a structure this requirement can make sense if and only if he/she knows its age and knows that failures do not occur totally at random. He/she then might not wish to be exposed to higher risk when the structures reaches ages of multiples of

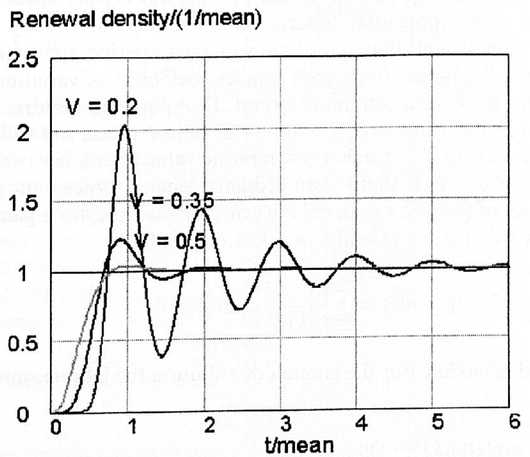

Fig. 2. Renewal density divided by asymptotic value versus time divided by mean failure time.

its mean failure time. If the conditions for a modified renewal process are fulfilled, possibly in regions with seismic activity where many structures will be affected by the same disturbance, the maximum renewal density may also be limited with some justification, also from a societal point of view. For society this policy, in general, would be somewhat doubtful. A steady, random stream of failures and subsequent reconstructions of many structures should be considered. But this corresponds precisely to the conditions of an equilibrium renewal process at any point in time and a constant limiting value should be used which is exactly $1/\mu$.

The determination of $\max\{h(t)\}$ for failure time densities known only pointwise unfortunately can involve heavy numerics. It is first necessary to take the Laplace transform numerically and then its inverse which is a notoriously difficult problem. It is mentioned that not only the Laplace integral needs to solved numerically but also the expectations with respect to the R- and Q-variables have to be taken. Relevant, more recent algorithms are described in de Hoog et. al. [24], Garbow et al. [25] and Murli and Rizzardi [26]. All of the algorithms proposed in these references have been found to work sufficiently well but further research is necessary. Whereas taking the Laplace transforms and their inverses of implicitly given failure time densities turned out to be relatively easy, the, at first sight, simpler problem of taking directly the Laplace transform for renewal densities and its inverse is rather difficult. The author will report about his numerical studies at another occasion.

One, therefore, concludes that generally the asymptotic value of the renewal density must be limited and that the maximum renewal density may be limited only in some rare cases.

6. Partial safety factors

It is possible to derive partial safety factors for practical use. We take as time unit one year and assume a target failure rate $<< 1$. Then, a definition of the time-variant partial safety factor is possible but suitably modified to account for the cycling velocity of time-variant loads. Time-invariant partial safety factors are defined as follows:

$$\gamma_i = \frac{x_i^*}{x_{c,i}} \text{ or } \gamma_i = \frac{x_{c,i}}{x_i^*} \tag{69}$$

depending on whether the variable is a loading variable or a resistance variable. For independent variables the design value (most likely failure point) is defined as

$$x_i^* = F_i^{-1}\Phi(-\alpha_i\beta)) \tag{70}$$

and the characteristic value by

$$x_{c,i} = F_i^{-1}(\Phi(u_{c,i})) \tag{71}$$

$F_i(.)$ is the distribution function of the basic variable. α_i is the mean value sensitivity of the ith variable defined by $\alpha_i = \partial\beta(u)/\partial u_{|u=u^*} \cdot u_{c,i}$ depends on the definition of the characteristic value. β is the geometrical safety index. If a better probability integration method is used, for example SORM, this index is to be replaced by the generalized safety index $\beta_{gen} = -\Phi^{-1}(P(g(X)\leqslant 0))$ with $P(g(X)\leqslant 0)$ some more accurate failure probability. The foregoing definition of sensitivities and partial safety factors makes sense only if the original basic variable vector X is independent. Then, each of the α-values in the standard space corresponds to the (normalized) sensitivity of changes in the median of the respective X-variable. However, relation (71) allows to define a representative $\alpha_{r,i}$, which is also valid for general dependent variables and reduces to the sensitivity given before for independent variables.

$$\alpha_{r,i} := \frac{\dfrac{-\Phi^{-1}[F_i(x_i^*)]}{\beta_{gen}}}{\|\alpha_r\|} \tag{72}$$

Re-normalization by $\|\alpha_r\|$ is only necessary if β_{gen} is used. x_i^* is obtained by transforming the most likely failure point u_i^* back into the original space. These representative $\alpha_{r,i}$-values do not correspond to the direction cosines of the β-point except in case of independent variables. Those representative α_r-values should be used in Eq. (70).

Recognizing the fact that the β-point is also the critical point in time-variant reliability and β in formula (70) is the geometrical safety index, another equivalent safety index β_{equ} can be defined, which takes account of time effects. Assuming stationarity it is according to Eq. (57)

$$\nu_J = \Sigma\lambda_i\Phi(-\beta_{equ})C_{SORM} \tag{73}$$

and according to Eq. (58)

$$\nu_D = \frac{\omega_0}{\sqrt{2\pi}}\varphi(\beta_{equ})C_{SORM} \tag{74}$$

and, therefore

$$\beta_{equ} = -\Phi^{-1}\left(\frac{\nu_J}{\Sigma\lambda_i C_{SORM}}\right) \tag{75}$$

$$\beta_{equ} = \sqrt{-2\ln\left(\frac{2\pi\nu_D}{\omega_0 C_{SORM}}\right)} \tag{76}$$

If both types of loading processes are acting β_{equ} has to be determined using Eq. (59) from

$$\nu_J + \nu_D = \Sigma\lambda_i\Phi(-\beta_{equ})C_{SORM} + \frac{\omega_0}{\sqrt{2\pi}}\varphi(\beta_{equ})C_{SORM} \tag{77}$$

by iteration. C_{SORM} stands for any correction of the first-order result. C_{SORM} is the same for both types of processes in Eq. (77) because it has to be determined in the space of all variables. In general, C_{SORM} is set to unity in first approximation. If the failure rates are given, $\nu_{admissible}$ replaces ν_J, ν_D or $\nu_J + \nu_D$, respectively. The equivalent safety index β_{equ} can then be used in Eq. (70), possibly together with a representative α_r. It is noted that these partial safety factors are much easier and more accurately to determine than those for time-variant failure probabilities as proposed by Rackwitz [27]. The distribution functions in which those factors are defined for time varying loading correspond to the point-in-time distribution. This might require a redefinition of the time reference of characteristic values for loads. The new definition of design values applies to both loading and resistance variables.

For example, let some occupancy load be Rayleigh distributed with location parameter $w = 0.5$ [kN/m²] and jump rate $\lambda = 0.1$ [1/year] and let $\nu_{admissible} = 10^{-5}$. The characteristic value is $x_c = 0.865$ [kN/m²] corresponding to the 95% quantile and $\alpha = -0.7$. From Eq. (75) one determines with $C_{SORM} = 1$

$$\beta_{equ} = -\Phi^{-1}(\nu_{admissible}/\lambda) = -\Phi^{-1}(10^{-5}/0.1) = 3.09$$

and from Eq. (70)

$$x^* = w\sqrt{-\ln(\Phi(\alpha\beta_{equ}))} = 0.5\sqrt{-\ln(\Phi(-0.7\cdot 3.72))} = 1.158.$$

The partial factor [Eq. (69)] is $\gamma = x^*/x_c = 1.158/0.865 = 1.34$.

The special case of fatigue and other deterioration phenomena requires additional considerations. If the outcrossing rate approach is used the same concept as outlined before can be applied. If the approach with assigned failure time distributions is used no equivalent to Eq. (69) exists and, consequently, no partial safety factors in the classical sense can be defined. The most natural verification format follows directly from Eq. (64). For example, for the simple model underlying Eq. (49) or any other life time distribution it is

$$E_R\left[\frac{\nu_0\Delta S^m}{C}\right] \approx \frac{\nu_0^*(\Delta S^*)^m}{C^*} \leqslant \nu_{admissible} \tag{78}$$

where now the "critical" R-variables denoted as ν_0^*, ΔS^* and C^* are obtained using the first order version of Eq. (C.3) in Appendix C. They can also be denoted as "critical" or "most likely" points. The ratio between these "most likely" values and the corresponding characteristic values may still be called "partial safety factors" but their definition differs from the one in Eq. (70).

For example, if we use Eq. (49) with ΔS being normally distributed with $m_{\Delta S} = 200$ and standard deviation $\sigma_{\Delta S} = 100$ and C independently log-normally distributed with mean $m_C = 10^{13}$ and coefficient of variation of $V_C = 1(\delta_C^2 = \ln(1 + V_C^2))$, respectively, together with deterministic $\nu_0 = 100$ and $m = 3$, we obtain from Eq. (C.3) to first order

$$\begin{aligned} E_R\left[\frac{\nu_0\Delta S^m}{C}\right] &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{\nu_0\Delta S^m}{C}f_{\Delta S}(x)f_C(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\nu_0(u_1\sigma_{\Delta S} + \mu_{\Delta S})^m}{\exp(u_2\delta_C + \ln(m_C))}\varphi_2(u_1, u_2)\mathrm{d}u_1\mathrm{d}u_2 \\ &\approx \frac{\nu_0(u_1^*\sigma_{\Delta S} + \mu_{\Delta S})^m}{\exp(u_2^*\delta_C + \ln(m_C))}2\pi\varphi_2(u_1^*, u_2^*) = 3.23\cdot 10^{-4} \leqslant \nu_{admissible} \end{aligned}$$

where $\varphi_2(.)$ is the two-dimensional normal density. We find $(u_1^*, u_2^*) = (-0.833, 1)$ and, thus, $\Delta S^*, C^*) = (116.7, 1.63 \cdot 10^{13})$. The approximate, asymptotic renewal density is to be compared with the exact value of $2.84 \cdot 10^{-4}$ determined by numerical integration. The quality of the approximation Eq. (C.3) appears remarkable, especially in view of the unrealistically large, assumed variabilities.

For the model in Eq. (49), reliability verification by Eq. (78) is the same as designing a structural component such that the mean time to failure fulfills the equation

$$E[T_{\text{fat}}] = \frac{C^*}{\nu_0^*(\Delta S^*)^m} \geqslant T_{\text{admissible}} = \frac{1}{\nu_{\text{admissible}}}$$

7. Numerical optimization of Eq. (1)

Optimization of Eq. (1), or in a more general context, of Eq. (47) can be done by any suitable method — but not without difficulties. The difficulty simply is to design a scheme, which is efficient and reliable. Because expected failure intensities are best determined by FORM/SORM, which in turn involves optimization, and in the formulations above expectations over Laplace transforms, which again involves optimization as shown in Appendix C, optimization of Eq. (47) requires two algorithms on top of each other. From past experience it is known that this setting is somewhat delicate. Large numerical effort is required. But most important is that convergence proofs are rarely available. A general solution for optimizing $Z(p)$ has recently been proposed by Kuschel and Rackwitz [28] on the basis of a proposal made earlier by Friis Hansen and Madsen [29] and Kuschel and Rackwitz [30]. It requires a formulation of the reliability problem in standard space and twice differentiability of the limit state function. It rests on the observation that in the β-point the state function must be equal to zero and the normal vector to the limit state function must be perpendicular to the limit state function and parallel to the vector of direction cosines of the β-point. The advantage of the new approach is that only one algorithm is needed and convergence can be proven easily. The basic optimization task for stationary failure events in a specific failure mode can be written as:

$$\textit{Minimize}: -Z(p) = -\frac{b}{\gamma} + C(p) + (C(p) + H)\frac{E_{R,Q}[\nu^+(V_U(u_R, u_Q, U_S, p)]}{\gamma} \tag{79}$$

subject to:

$$\begin{aligned}
&E_{R,Q}[\nu^+(V_U(u_R, u_Q, u_S, p)] \leqslant \nu_{\text{admissible}}\\
&g(u_R, u_Q, u_S, p) = 0\\
&(u_R, u_Q, u_S)^T \nabla_u g(u_R, u_Q, u_S, p) + \|(u_R, u_Q, u_S)\| \, \|\nabla_u g(u_R, u_Q, u_S, p)\| = 0\\
&h_1(u_R, u_Q, u_S, p) = 0;\ i = 1, \ldots, k\\
&h_i(u_R, u_Q, u_S, p) \leqslant 0;\ i = k+1, \ldots, m\\
&(u_R, u_Q, u_S, p)_{\text{lower}} \leqslant (u_R, u_Q, u_S, p) \leqslant (u_R, u_Q, u_S, p)_{\text{upper}}
\end{aligned} \tag{80}$$

The first condition concerns a possible imposed upper value for the crossing rate. It may even be necessary to specify several such failure rate limits for different failure modes and types. The second and third conditions just formulate the Kuhn-Tucker condition for a valid β-point. The forth and fifth conditions describe the mathematical and physical admissibility of the design parameter vector. The sixth condition contains lower and upper bound for the transformed basic variable vector and the design vector. This task can conveniently be solved by a constrained sequential quadratic programming procedure [31]. The optimization scheme is a first order scheme. Second order corrections to the crossing rates can be introduced and the optimization problem is solved by iteration.

Eq. (79) is valid for Poissonian failure processes. For other than stationary Poissonian failure processes the first condition in Eq. (80) has to be modified according to Eq. (64).

If the additional cost terms for the extensions as in Eq. (47) are included care must be taken of the possibility of multiple minima.

Further technical details and some numerical results are described in Kuschel and Rackwitz [28]. The purpose of this section was merely to demonstrate that an explicit optimization of Eq. (1) and its variants is feasible and the new concepts outlined before are also operational from this point of view.

8. Discussion

The foregoing developments are based on more than 25 years old, but apparently since then overlooked findings by Rosenblueth and Hasofer. It is believed that the generalization put forward in this paper will to a certain extent affect the whole safety philosophy for structural facilities. They should, at least, change the philosophy for setting up reliability targets in codes.

First of all, the optimal solution for building facilities with or without a systematic rebuilding policy is based on failure intensities and not on time-dependent failure probabilities. It is neither necessary to define arbitrary reference times of intended use nor is it necessary to undertake the complicated task to compute first passage time distributions. No table of recommended reference times of usage of structures is needed. The same targets, in terms of failure rates, can be set for temporary structures and monumental buildings, given the same marginal cost for reliability and failure consequences. Nevertheless, it is necessary to define a time unit. For civil engineering facilities this is no doubt one year in consideration of the length of their life cycles. Other choices are possible provided that the failure intensities are small and much smaller than the interest rate. Also, the optimum design parameters are independent of assumed, highly variable lifetimes. This does not mean that lifetime aspects, especially in case of fatigue and other deterioration, are ignored. Here, design must be directly for mean failure times which are sufficient to derive the corresponding asymptotic renewal densities to be checked against target failure rates.

On the basis of optimization studies in Appendix B, an attempt is made to design a table of target values to be used in practice as an alternative to similar tables in codes such as Refs. [1–6] and as a substitute for direct optimization. Yearly safety indices $\beta_\nu = -\Phi^{-1}(\nu_{\text{admissible}})$ are given together with the yearly rates $\nu_{\text{admissible}}$ in parenthesis. Table 1

The grading for both the relative effort to achieve reliability and the expected failure consequences by an order of magnitude agrees well with the example calculations in Appendix B and other studies where several other stochastic models have been used. The proposed values agree

Table 1
Proposed target failure rates

	Ultimate limit state failure		
	Expected failure consequences		
Relative effort to achieve reliability	Insignificant	Normal	Large
High	2.3 (10^{-2})	3.1 (10^{-3})	3.7 (10^{-4})
Normal	3.1 (10^{-3})	3.7 (10^{-4})	4.3 (10^{-5})
Low	3.7 (10^{-4})	4.3 (10^{-5})	4.7 (10^{-6})
	Serviceability limit state failure		
High	1.3 (10^{-1})		
Normal	1.7 (5 10^{-2})		
Low	2.3 (10^{-2})		

with values proposed in the industry with high damage potential and elsewhere for accidental situations (for example, the so-called safe shut down earthquake has a return period of 10 000 years and the resulting return period of failure will be yet smaller). The "no safe shut down situation" would require a still smaller failure rate. For normal office, housing and industrial buildings the marginal cost for reliability are very small. One therefore could argue to lower the failure rate down to 10^{-5}. The targets for serviceability limit states appear rather conservative. Present practice appears to accept larger risks. The values are also compatible with the yearly risk for life and limb set in ISO 2394 as $P < 10^{-6}/P(D \mid F)$ where $P(D \mid F)$ is the probability of a person being killed given that the structure collapses.

The values in the table have been set for structural components (failure modes). For structural systems they are meant for the structural component dominating structural failure. If there are many components of the same importance or many equally likely failure modes with comparable failure consequences the values in the table have to be suitably lowered.

9. Summary and conclusions

Some early findings by Rosenblueth and Hasofer are reviewed and generalized. Others than failures by "rare disturbances" can also be handled. Those include serviceability failures, fatigue failures and ultimate limit state failures but also obsolesce. It is proposed that for almost all civil engineering facilities the only reasonable reconstruction policy is systematic rebuilding or repair. It is found that the failure rate (failure intensity, renewal density) is the decisive criterion for setting up safety/reliability targets for high reliability structures. This is approximately true also for "one mission" structures. Structures should be optimal. A general objective function is proposed. Tools for its maximization in a first order sense are presented. The principles of reliability verification but also of reliability-based design are illustrated. Given codified admissible failure rates, for example, related to one year, it is possible to derive partial safety factors so that a full probabilistic analysis is in complete correspondence with a deterministic partial safety factor design. An exception is fatigue and other deterioration where a more direct approach is necessary.

A number of new, primarily computational problems evolve. A few of them have been addressed. Others must be left to future development.

Actual optimization may not be practical in every day engineering work, however. For codes failure rates may still be assessed by optimization and, in parallel, by calibration at present practice using Lind's postulate that present practice is already "almost" optimal [35]. This postulate should primarily be used to determine the failure cost which also may contain certain "intangibles".

Acknowledgements

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Appendix A. Laplace transforms for failure time distributions and discount factors

Unfortunately, Laplace transforms are analytic only for a few stochastic models for interarrival times of failures. Table A.1 presents some results. with $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{exo}[-t^2]\mathrm{d}t$ the error function, $erfc(x) =, 1 - erf(x)$, $B(r, s)$ the complete betafunction, ${}_1F_1(c, d; x)$ the confluent hypergeometric function, $y = \frac{t-a}{b-a}$ with $a \leqslant t \leqslant b$, $z = t - b$, $Si(x) = \int_0^x \frac{\sin(u)}{u}\mathrm{d}u$ and $Ci(x) = -\int_x^\infty \frac{\cos(u)}{u}\mathrm{d}u$. For the Gaussian and Cauchy model it is important that the one-sided Laplace transform is taken implying a truncation of the distribution at $t = 0$. For other distribution models like the important lognormal or Weibull distribution the Laplace transform must be determined numerically. The result for the Cauchy distribution is given here because it is a distribution without finite moments and Eq. (63) fails. The result for the inverse Gaussian distribution is given in Hasofer and Rackwitz [32].

Hasofer [8] derives an approximation for $f^*(\theta)$ for small γ and σ/μ in terms of the cumulants of $f(t)$, which may be useful in some cases.

$$f^*(\theta) \approx \exp\left[-\theta\left(\mu - \frac{1}{2}\theta\sigma^2 - \ldots\right)\right] \tag{A.1}$$

This happens to be the two-sided Laplace transform of the normal distribution (i.e. without truncation). Eq. (A.1) is superior to an asymptotic formula resulting from expanding the term $\exp[-\theta t]$ into a Taylor series around $t = 0$ and integrating term by term [33]. If μ_1 and σ_1 are mean and standard deviation for the time to the first event and μ and σ are mean and standard deviation of the times between events, respectively, the discount factor is with $\gamma = \theta$

$$\frac{f_1^*(\gamma)}{1 - f^*(\gamma)} \approx \frac{1}{\exp\left[\gamma\left(\mu_1 - \frac{1}{2}\gamma\sigma_1^2\right)\right] - \exp\left[\gamma\left(\mu_1 - \mu - \frac{1}{2}\gamma\sigma_1^2 + \frac{1}{2}\gamma\sigma^2\right)\right]} \tag{A.2}$$

Table A.1
Analytic laplace transforms for some failure time models ($t \geqslant 0$)

Name	Density function $f(t)$	Laplace transform $f^*(\theta)$
δ-Spike	$\delta(a)$	$\exp[-\theta a]$
Exponential	$\lambda \exp[-\lambda t]$	$\frac{\lambda}{\theta + \lambda}$
Uniform	$\frac{1}{b-a}$	$\frac{\exp[-a\theta] - \exp[-b\theta]}{\theta(b-a)}$
Beta	$\frac{y^{r-1}(1-y)^{s-1}}{B(r,s)}$	${}_1F_1(r, r+s; -\theta)$
Cauchy	$\frac{\frac{a}{\pi(a^2+z^2)}}{\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{b}{a}\right)\right)}$	$\frac{\left[\sin(a\theta)Ci(a\theta) - \cos(a\theta)\left(Si(a\theta) - \frac{\pi}{2}\right)\right]}{a\pi\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{b}{a}\right)\right)}$
Rayleigh	$\frac{t}{w^2}\exp\left[-\left(\frac{t}{w}\right)^2\right]$	$1 - \theta w\sqrt{\frac{\pi}{2}}\exp\left(\frac{1}{2}\theta^2 w^2\right) erfc\left(\frac{\theta w}{\sqrt{2}}\right)$
Gamma	$\frac{\lambda^k}{\Gamma(k)} t^{k-1}\exp[-\lambda t]$	$\left(\frac{\lambda}{\theta+\lambda}\right)^k$
Truncated Gaussian	$\frac{\frac{1}{\sqrt{2\lambda}}\frac{1}{\sigma}\exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]}{\Phi\left(\frac{\mu}{\sigma}\right)}$	$\exp\left[-\theta\left(\mu - \frac{1}{2}\theta\sigma^2\right)\right]\frac{1 - erf\left(\frac{\theta\sigma^2-\mu}{\sqrt{2}\sigma}\right)}{2\Phi\left(\frac{m}{\sigma}\right)}$
Inverse Gaussian	$\frac{x_0}{\sigma\sqrt{2\pi t^3}}\exp\left[-\frac{a^2 t}{2\sigma^2} + \frac{a x_0}{\sigma^2} - \frac{x_0^2}{2\sigma_t^2}\right]$	$\exp\left[-x_0\left(\sqrt{\frac{a^2}{\sigma^4}+\frac{2\theta}{\sigma^2}} - \frac{a}{\sigma^2}\right)\right]$

If no distinction is made between the times to the first event and the times between events the second exponent-term in the denominator reduces to unity. However, it is found by numerical studies that it is better to compute the discount factor by using the Laplace transform of the truncated Gaussian distribution as in Table A.1.

Alternatively, the discount factor can be approximated by making use of the Laplace transform of the gamma-distribution specified by its first two moments

$$\frac{f^*(\gamma)}{1-f^*(\gamma)} = \frac{\left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}}{\gamma + \left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}}\left[1 - \frac{\left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}}{\gamma + \left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}}\right]^{-1} \approx \frac{\left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}}{\gamma + \left(\frac{m}{\sigma^2}\right)^{\left(\frac{m}{\sigma}\right)^2}} \quad \text{(A.3)}$$

Appendix B. Numerical study of Eq. (43)

In order to illustrate the various factors on optimal failure rates some numerical studies have been performed at a simple example. The loading process is assumed to be one-dimensional. A stationary rectangular wave renewal process with outcrossing rate $\nu^+(V, \tau) \approx \lambda\Phi(-\beta)$ is assumed. The state function is $g_U(x) = R - S$ for the ultimate limit state and $g_S(x) = R/a - S$ for the serviceability limit state. Both resistance and loading is assumed log-normally distributed. The design parameter is the central safety factor $p = m_R/m_S$. Fatigue, inspection and maintenance as well as failure upon reconstruction is not taken into account. Fig. B.1

Then, the objective function is

$$Z(p) \approx \frac{b}{\gamma} - (C_0 + C_1 p) - U\frac{\lambda\Phi\left(-\frac{\ln\left\{\frac{p}{a}\sqrt{\frac{1+V_R^2}{1+V_S^2}}\right\}}{\ln((1+V_R^2)(1+V_S^2))}\right)}{\gamma} - (C_0 + C_1 p + A)\frac{\bar{\omega}}{\gamma} - (C(p) + H)\frac{\lambda\Phi\left(-\frac{\ln\left\{p\sqrt{\frac{1+V_R^2}{1+V_S^2}}\right\}}{\sqrt{\ln((1+V_R^2)(1+V_S^2))}}\right)}{\gamma} \quad \text{(B.1)}$$

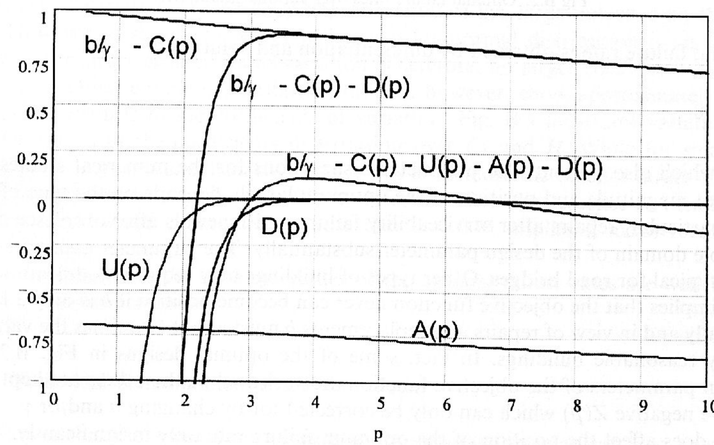

Fig. B.1. Cost over parameter p (R and S lognormal, $\lambda = 1$, $b = 2\gamma = 0.07$, $U/C_0 = A/C_0 = 0.2$, $C_1/C_0 = 0.03$, $H/C_0 = 3$, $\bar{\omega} = 0.02$, $a = R_U/R_S = 1.5$).

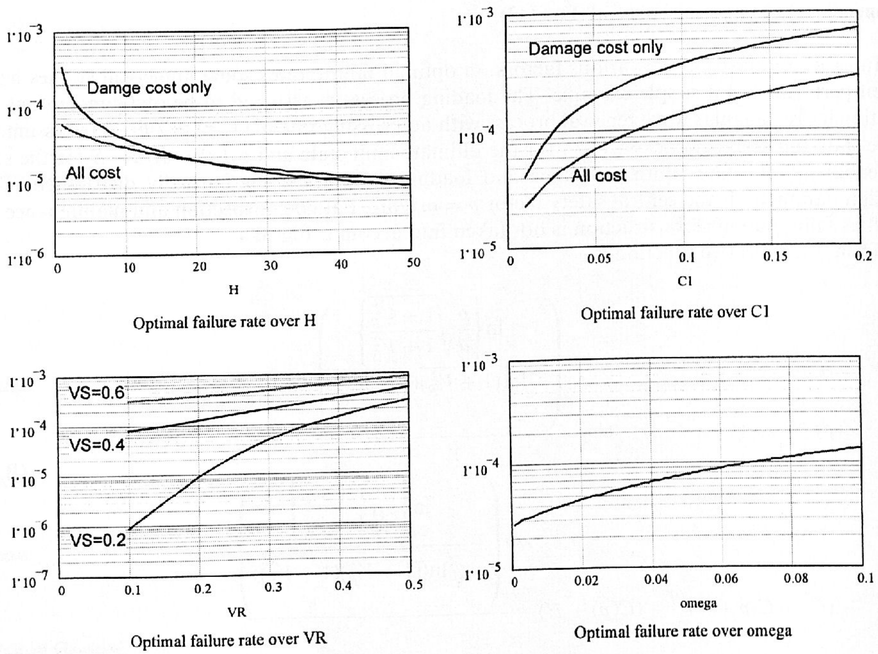

Fig B.2. Optimal failure rates over various parameters.

The optimal failure rate is obtained by differentiation and solution of

$$\frac{\partial Z(p)}{\partial p} = 0 \tag{B.2}$$

Fig. B.1, which also presents the parameter assumptions for the numerical studies to come, illustrates that magnitude and position of the optimum largely depends on the type of cost considered. In particular, repairs after serviceability failure and renewals after obsolesce can reduce the reasonable domain of the design parameter substantially. The parameter assumptions in Fig. B.1 may be typical for road bridges. Other types of buildings may require modification.

Eq. (B.1) implies that the objective function never can become positive if b is only a little larger than γ. Usually and in view of repairs and replacements b must about two times the value of γ for economically reasonable buildings. In fact, some of the optimal designs in Fig. B.2 and B.3, where certain parameters of the objective function are varied while the others are kept as in Fig. B.1, will have negative $Z(p)$ which can only be corrected for by changing b and/or γ.

Obsolesce does affect the position of the optimum failure rate only insignificantly. The larger the demolition cost and/or the obsolesce rate $\bar{\omega}$ the more narrow will be the reasonable domain as seen in Fig. B.1.

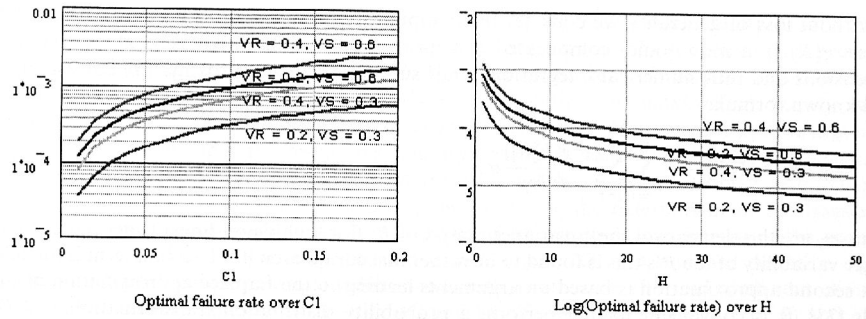

Fig. B.3. Optimal failure rate for various coefficients of variation of R and S.

As already indicated in Fig. B.1 the optimum total costs are by a factor of 2 or more when all cost components are considered than for failure cost alone. The corresponding failure rate has the same tendency. In Fig B.1 this is primarily due to the maintenance cost after serviceability failure. The failure rate for serviceability failure at optimum is around 0.01. It changes as the failure rate for ultimate limit state failure. This is, of course, a consequence of how serviceability failure was defined (see parameter p/a in Eq. B.1).

Fig. B.2 shows a number of interesting features. If C_1 is increased by an order of magnitude the optimal failure rate also increases by an order of magnitude. If the failure costs H are decreased by an order of magnitude the optimal failure rate also decreases by an order of magnitude, but by less than an order of magnitude for very high failure cost H.

The coefficient of variation of either loading or resistance has large influence on the optimal failure rate. However, it should be mentioned that a log-normal distribution is an inadequate model for loads for larger coefficients of variation. Therefore, no larger coefficients of variation than 0.6 are used. Other common distribution models, however, show approximately the same sensitivity against changes in the coefficients of variation. Fig. B.3 illustrates variations of the optimal failure rate with the coefficients of variation over C_1 and H. While for small V_S the optimal failure rate increases significantly with V_R this is no more the case for larger V_S.

The variation of $\bar{\omega}$ shows that shorter renewal intervals after obsolescence can lead to larger optimal failure rates, which makes sense. Note that $\bar{\omega} = 0.01$ implies a mean reconstruction interval of 100 years.

Finally it is observed that the absolute values of the optimal failure rates are well within the range of rates proposed previously.

Appendix C. Numerical determination of $E_R[f^*(\gamma, R, p)]$

It will be shown that the expectation operations in previous formulae can also be performed accurately by a FORM/SORM-like procedure without resorting to multidimensional numerical integration.

Without loss of generality we consider the computation of a general expectation like $E[g(R)]$ where $g(R)$ is a monotonic, complicated function of the vector R. In general, R has many dimensions and thus numerical integration is not suitable. A first approximation consists of the well-known formula

$$E[g(R)] \approx (E[R]) + \frac{1}{2}\sum_{i=1}^{n_R}\sum_{j=1}^{n_R}\frac{\partial^2 g(E[R])}{\partial r_i \partial r_j}\sigma_{ij} \tag{C.1}$$

where σ_{ij} are the elements of the covariance matrix of R. For highly non-linear functions $g(R)$ and larger variability of the R's this is found to be rather inaccurate even if the second term is included.

A second approximation is based on arguments leading to the Laplace approximation of integrals [33]. It is convenient to first perform a probability distribution transformation $x = T(u)$ where u is a standard normal vector with independent components [19]. Then, given that the integrand in

$$E[g(T(u))] = \int_{R^{n_R}} g(T(u))\varphi(u)\mathrm{d}u$$

has a unique maximum the expectation can be written as:

$$E[g(T(u))] = \int_{R^{n_R}} (2\pi)^{-n_R/2}\exp\left[-\frac{1}{2}k(u)\right]\mathrm{d}u \tag{C.2}$$

$k(u)$ is expanded to second order

$$k(u) = \|u\|^2 - 2\ln(g(T(u))) \approx k(u^*) + \frac{1}{2}(u-u^*)^T S(u^*)(u-u^*) + \ldots \tag{C.3}$$

with the matrix of second derivatives of $k(u)$

$$S(u^*) = \left\{\left(\delta_{ij} - \frac{1}{g(T(u^*))}\frac{\partial^2 g(T(u^*))}{\partial u_i \partial u_j} + \frac{1}{g(T(u^*))^2}\frac{\partial g(T(u^*))}{\partial u_i}\frac{\partial g(T(u^*))}{\partial u_j}\right); i,j = 1,\ldots,n_R\right\} \tag{C.4}$$

and u^* the solution of $\max\{g(T(u))\varphi(u)\}$ or of $\min\{k(u)\}$. Making use of Aitken's integral

$$(2\pi)^{-n/2}\int_{R^n}\exp\left[-\frac{1}{2}x^T A x\right]\mathrm{d}x = |\det(A)|^{-1/2}$$

with A a positive definite matrix the result is

$$E[g(T(u))] \approx g(T(u^*))(2\pi)^{n_R/2}\varphi(\|u^*\|)|\det(S(u^*))|^{-1/2} \tag{C.5}$$

with excellent numerical accuracy. Quite frequently, $|\det(S)|^{-1/2}$ is close to unity. If it is set to unity it makes sense to speak of a first order approximation. In Eq. (65) an additional expectation operation over the failure density is required. The same formulae apply also for this case except that now $g(R,t)$ replaces $g(R)$ and (u^*, t^*) is the solution of $\max\{g(T(u),t)\varphi(u)\}$ or of $\min\{k(u,t)\}$. All other quantities have to be evaluated for $t = t^*$. Here, no asymptotic argument can be applied except that the results become asymptotically exact for all standard deviations approaching homogeneously to zero. It is important to use Eq. (C.3), i.e. to take $\ln(g(T(u)))$ in the exponent of $k(u)$. Otherwise the critical point is $u^* = 0$, the determinant of the Hessian is unity and a trivial result is obtained. For example, if we use the same data as in the example following Eq. (78), i.e. ΔS normally distributed with $m_{\Delta S} = 200$ and standard deviation $\sigma_{\Delta S} = 100$ and C independently log-normally distributed with mean $m_C = 10^{13}$ and coefficient of variation of 1, respectively, together with $\nu_0 = 100$ and $m = 3$, we obtain from Eq. (C.1) and exact numerical integration $E[R]/\text{exact}\approx 0.41$. From Eq. (C.3) we obtain to first order $E[R]/\text{exact} \approx 1.14$ and to second order $E[R]/\text{exact} \approx 1.01$, respectively. The assumed large variability makes the function $g(R)$ highly non-linear.

Alternatively, the expectations may be taken by Monte Carlo. An adaptive importance sampling scheme adjusting the sampling density iteratively appears especially appropriate.

Further comparisons between methods and some other developments are given in Ref. [34].

Appendix D. Example for Eq. (54)

If the failure time distribution is given as in Eq. (54), for example, the renewal density or mean failure time can easily be computed for most failure time models, possibly using Appendix C for the expectation operations. If the model Eq. (54) is used the failure time distribution is not known explicitly. Assume a structural component whose resistance is decaying according to

$$b(t) = b_0 + b_1 t^m \tag{D.1}$$

It is loaded by either a stationary Gaussian or a stationary rectangular wave renewal process with zero mean and unit standard deviation. Their outcrossing rates are given by

$$\nu^+(b(\tau), r) \leqslant \lambda(1 - \Phi(-b(\tau)))\Phi(-b(\tau)) \tag{D.2}$$

for the rectangular wave renewal process [see Eq. (57)] and by

$$\nu^+(b(\tau), r) = \omega_0\varphi(b(\tau)\Psi(\dot{b}(\tau)/\omega_0) \tag{D.3}$$

for the Gaussian process [see Eq. (58)]. The deterministic or random vector r collects, for example, the quantities b_0, b_1 and m. Let $b_0 = 5$, $b_1 = -0.000005$, $m = 2$, $\lambda = \omega_0 = 200$ be deterministic implying that the outcrossing rate is largest for $t = 1000$ time units and the failure time density is largest for $t \approx 500$ time units (see Fig. D.1). The Laplace transform of the failure time density to be used in Eq. (14) becomes

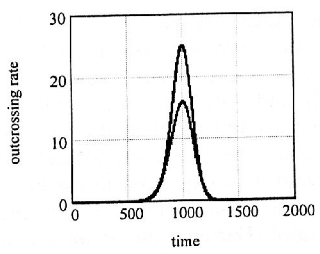

Fig. D.1. Outcrossing rate.

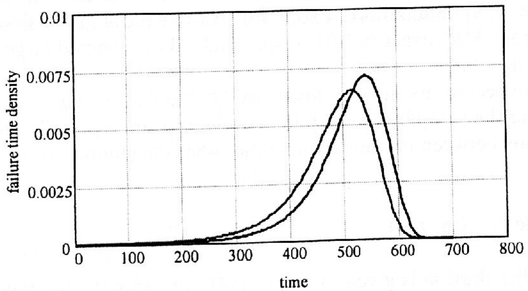

Fig. D.2. Failure time density.

$$f^*(\gamma, r) = \int_0^\infty \nu^+(b(t), r)\exp[-\int_0^t \nu^+(b(\tau), r) + \gamma)\mathrm{d}\tau]\mathrm{d}t \tag{D.4}$$

For $\gamma = 0.05$ one determines by numerical integration $f^*(0.05, r) = 1.17 \times 10^{-3}$ for the rectangular wave renewal process and $f^*(0.05, r) = 2.42 \times 10^{-3}$ for the Gaussian process. The failure density is (see Fig. D.2)

$$f(t, r) = \nu^+(b(t), r)\exp[-\int_0^t \nu^+(b(\tau), r)\mathrm{d}\tau] \tag{D.5}$$

with maximum density $\max\{f(t^*, r\} = 5.63 \times 10^{-3}$ at $t^* = 542$ and $\max\{f(t^*, r\} = 7.15 \times 10^{-3}$ at $t^* = 516$, respectively. Mean and standard deviation are $E[T] = 504$ and $D[T] = 77.5$ for the rectangular wave renewal process and $E[T] = 472$ and $D[T] = 86.5$ for the Gaussian process. The failure rate or renewal density is thus $1/E[T] \approx 0.002$, which can be compared with an admissible failure rate according to Eq. (65). Note that the coefficient of variation of the failure time distribution is only about 15%. Note further that the time for the maximum of the failure time density is approximately one half of the maximum of the outcrossing rate, which is due to the exponent term in Eq. (D.5). Also, due to the same reason, the densities are skewed to the left.

If the vector R has uncertain components the computational problem is substantially more complicated and numerically involved. The considerations in Appendix C can facilitate numerical calculations. Further research is necessary.

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