SONDERFORSCHUNGSBEREICH 96

LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

BERICHTE zur ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

SOME ASYMPTOTIC RESULTS IN SECOND ORDER RELIABILITY

K. Breitung and M. Hohenbichler

Heft 69/1984

SONDERFORSCHUNGSBEREICH 96 (SFB 96)

Der SFB 96 "Zuverlässigkeit von Bauwerken" ist eine Einrichtung der Technischen Universität München und der Deutschen Forschungsgemeinschaft. Der SFB hat sich die Entwicklung baustoff- und bauartenübergreifender Sicherheitssysteme auf wahrscheinlichkeitstheoretischer Grundlage zum Ziel gesetzt.

LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)

Am LKI beteiligte Institute:

Institut für Bauingenieurwesen I

Baumechanik	Prof. Dr.-Ing. Grundmann
Baustatik	Prof. Dr.-Ing. Knittel

Institut für Bauingenieurwesen III

Baukonstruktion und Holzbau	Prof. Dr.-Ing. Heimeshoff
Baustoffkunde und Werkstoffprüfung	Prof. Dr.-techn. Springenschmid

Institut für Bauingenieurwesen III

Massivbau	Prof. Dr.-Ing. Kupfer
Stahlbau	Prof. Dipl.-Ing. Nather

Vorwort

Die in diesem Heft zusammengefaßten Arbeiten betreffen einige mathematische Aspekte der Zuverlässigkeitsmethode 1. bzw. 2. Ordnung. Insbesondere wird die Zuverlässigkeitsmethode 2. Ordnung als eine asymptotisch exakte Methode bestätigt.

Die Verfasser

Preface

This report collects 4 studies on mathematical aspects of the first and second order reliability methods. In particular, it is shown that the second order reliability method is asymptotically exact.

The authors

Table of Contents

Asymptotic Approximations for Multinormal Integrals

K. Breitung

Introduction

In many reliability problems it is necessary to calculate the probability content of part of a n-dimensional probability space. Especially, if the dimension of the space is large, the direct computation by numerical or Monte-Carlo integration is too time consuming. Therefore, approximation methods for this task were developed. (see [5], [7], [9], [12]) The basic idea of the proposed methods is to transform the random variables into a probability space, where they are mutually independent, standardized, normal distributed random variables and then to approximate the integral by simplifying the boundary of the integration domain. For independent random variables with continuous densities the transformation is quite simple (see [5], [7], [9]). In the case of dependent random variables a numerical procedure is put forward in [9] which transforms pointwise into the standardized space. But in this case no theoretical results about the performance of this algorithm are known to the author (for a discussion see [6]). Nevertheless, it can be assumed that in many cases the problem of integration can be reduced to the problem of integrating the n-dimensional standard normal density function over a part of the n-dimensional space.

In this space the so-called limit state function $g(\underline{x})$ ($\underline{x}=(x_1,\dots,x_n)$) is defined, it divides the space

into the safe domain $S=\{\underline{x}; g(\underline{x})>0\}$, the failure domain $F=\{\underline{x}; g(\underline{x})<0\}$ and the limit state surface $G=\{\underline{x}; g(\underline{x})=0\}$.
The failure occurs, if $g(\underline{x})<0$. The quantity to be computed is the probability of failure P

$$P = (2\pi)^{-n/2} \int\limits_{g(\underline{x})<0} \exp\left(-\frac{1}{2}|\underline{x}|^2\right) d\underline{x} \qquad (1)$$

($|\underline{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ the euclidean norm of the vector $\underline{x}$).

It is assumed in the following that the origin is not in F. First, all points $\underline{x}_1, \ldots, \underline{x}_k$ on the limit surface G with minimal distance to the origin are computed, i.e. the points with $|\underline{x}_i| = \min\limits_{g(\underline{x})=0} |\underline{x}|$. A globally convergent algorithm to minimize a function on hypersurfaces which can be used to find these points, is described in [11]. The approximations are now obtained by replacing $g(\underline{x})$ at $\underline{x}_1, \ldots, \underline{x}_n$ by its Taylor expansion $g^*(\underline{x})$ to the first or second order. Then, $P(g^*(\underline{X})<0)$ is taken approximation for $P(g(\underline{X})<0)$. The usual method which is described in [5], [9] is to take only the first derivatives of $g(\underline{x})$, i.e. to replace the limit surface by a hyperplane through the point with the same gradient as $g(\underline{x})$ at this point. In the cited papers it is stated that the quality of the obtained estimate is good and that it produces accurate approximations for the integral but no theoretical considerations or general error estimates to underline this proposition are given.

For a fixed failure domain the error of the approximation can be found only by comparing it with the exact result obtained by numerical integration. Since reliability

problems are considered the probability content of the failure region is in general small and the distance of the limit surface to the origin is usually large due to the standardization. It therefore appears reasonable to study the asymptotic behaviour of the approximations for sequences of increasing safe domains to obtain information on their quality. Methods of asymptotic analysis are used to do this.

The Asymptotic Behaviour of the Approximations

In the space of n independent, standard normal random variables the limit state function $g(\underline{x})$ is given as a twice continuously differentiable function. Further, it is assumed that $\min\limits_{g(\underline{x})=0} |\underline{x}| = 1$, $g(\underline{0})>0$ and that there is only one point $\underline{x}_0$ on the surface $G=\{\underline{x}; g(\underline{x})=0\}$ with $|\underline{x}_0|=1$.
The restriction that there is only one point on the limit surface is usually made since for the case of several minim distance points no general procedure to deal with them was developed so far, it was only proposed that polyhedrical approximations should be used (see [5], [9]).
Beginning with this one limit state function now a sequence of limit state functions depending on the parameter β is defined:

$$g(\underline{x};\beta) = g(\beta^{-1}\underline{x}) \qquad (2)$$

It is obvious that $g(\underline{x};1)=g(\underline{x})$ and that the limit surface $G(\beta)=\{\underline{x}; g(\underline{x};\beta)=0\}$ has only one point with minimal distance to the origin, namely $\beta\underline{x}_0$.

For these limit surfaces two approximations are considered. The first approximation is obtained by replacing $g(\underline{x};\beta)$ by its first order Taylor expansion $g_L(\underline{x};\beta)$ at $\beta\underline{x}_o$, i.e.:

$$g_L(\underline{x};\beta) = (\nabla g(\beta\underline{x}_o;\beta))^T(\underline{x}-\beta\underline{x}_o) = \tag{3}$$

($\nabla g(\underline{x};\beta)$ denotes the gradient of the function $g(\underline{x};\beta)$ at $\underline{x}$).
This is due to the definition of $g(\underline{x};\beta)$:

$$= (\nabla g(\underline{x}_o;1))^T(\beta^{-1}\underline{x}-\underline{x}_o) \tag{3a}$$

Then, instead of the hypersurface $G(\beta)$ the hyperplane defined by $g_L(\underline{x};\beta)=0$ is taken as approximating limit surface.

The second approximation is given by replacing $g(\underline{x};\beta)$ by its second order Taylor expansion $g_Q(\underline{x};\beta)$ at $\beta\underline{x}_o$, i.e.:

$$g_Q(\underline{x};\beta) = (\nabla g(\beta\underline{x}_o;\beta))^T(\underline{x}-\beta\underline{x}_o) + (\underline{x}-\beta\underline{x}_o)^T\,\underline{\underline{D}}(\beta)(\underline{x}-\beta\underline{x}_o)/2 \tag{4}$$

with $\underline{\underline{D}}(\beta)$ the matrix $\left.\dfrac{\partial^2 g(\underline{x};\beta)}{\partial x_i \partial x_j}\right|_{\underline{x}=\beta\underline{x}_o}$

of the second derivatives of $g(\underline{x};\beta)$ at $\beta\underline{x}_o$.
This is, due to the definition:

$$g_Q(\underline{x};\beta) = (\nabla g(\underline{x}_o;\beta))^T(\beta^{-1}\underline{x}-\underline{x}_o) + (\beta^{-1}\underline{x}-\underline{x}_o)^T\underline{\underline{D}}(1)(\beta^{-1}\underline{x}-\underline{x}_o)/2 \tag{4a}$$

The hypersurface $G(\beta)$ is replaced by the hypersurface given by $g_Q(\underline{x};\beta)=0$. Also, this new surface has only $\beta\underline{x}_o$ as point with minimal distance to the origin. (This can be shown by the Lagrange multiplier theorem, see [13], p. 19).
Then there are three probability contents to be compared:

$$P(\beta) = P(g(\underline{X};\beta)<0) \tag{5a}$$

$$L(\beta) = P(g_L(\underline{X};\beta)<0) \tag{5b}$$

$$Q(\beta) = P(g_Q(\underline{X};\beta)<0) \tag{5c}$$

The second and the third are approximations for the first one.

To evaluate $P(\beta)$ a substitution is made:

$$P(\beta) = (2\pi)^{-n/2}\int_{g(\underline{x};\beta)<0}\exp(-|\underline{x}|^2/2)\,d\underline{x} = \tag{6}$$

Substituting $(x_1,\ldots,x_n) \to (y_1,\ldots,y_n)$ with $y_i=\beta^{-1}x_i$:

$$= (2\pi)^{-n/2}\beta^n\int_{g(\underline{y};1)<0}\exp(-\beta^2|\underline{y}|^2/2)\,d\underline{y} \tag{7}$$

Define:

$$I(\beta) = \int_{g(\underline{y};1)<0}\exp(-\beta^2|\underline{y}|^2/2)\,d\underline{y} \tag{8}$$

$I(\beta)$ is an integral over a fixed domain whose integrand is an exponential function depending linearly on the parameter β^2.

These integrals are called Laplace type integrals. An extensive study of their asymptotic behaviour for β^2 can be found in [2], chapt. 8. Using the results given there the asymptotic form of $I(\beta)$ is (details see appendix I):

$$I(\beta) \sim (2\pi)^{(n-1)/2} \exp(-\beta^2/2)\,\beta^{-(n+1)}\,|J|^{-1/2} \quad (\beta\to\infty) \tag{9}$$

where J (defined in eq.(23a), appendix I) is a quantity independent of β depending only on the first and second derivatives of $g(\underline{x};1)$ at $\underline{x}_o$. This yields for $P(\beta)$:

$$P(\beta) \sim (2\pi)^{-1/2}\beta^{-1}\exp(-\beta^2/2)\,|J|^{-1/2} \quad (\beta\to\infty) \tag{10}$$

Using $\Phi(-x) \sim (2\pi)^{-1/2}\exp(-x^2/2)\,x^{-1} \ (x\to\infty)$ with $\Phi(.)$ denoting the standard normal integral:

$$P(\beta) \sim \Phi(-\beta)\,|J|^{-1/2} \quad (\beta\to\infty) \tag{11}$$

The asymptotic behaviour of $P(\beta)$ depends only on the value of β and the first and second derivatives of $g(\underline{x};1)$ at $\underline{x}_o$. For the second probability $Q(\beta)$ which is given by $Q(\beta)=P(g_Q(\underline{X};\beta)<0)$, the same result is obtained analogously since $g_Q(\underline{x};1)$ and $g(\underline{x};1)$ have the same first and second derivatives at $\underline{x}_o$ due to the definition of $g_Q(\underline{x};\beta)$:

$$Q(\beta) \sim \Phi(-\beta)\,|J|^{-1/2} \quad (\beta\to\infty) \tag{12}$$

This yields:

$$P(\beta) \sim Q(\beta) \quad (\beta\to\infty) \tag{12a}$$

The result shows that $Q(\beta)$ is an asymptotic approximation for $P(\beta)$ in the sense of asymptotic analysis, i.e. the relative error tends to zero for $\beta\to\infty$. For the linear approximation $L(\beta)$ the estimate is:

$$L(\beta) = \Phi(-\beta) \tag{13}$$

This gives, using (12) and (12a):

$$L(\beta)/P(\beta) \sim |J|^{1/2} \quad (\beta\to\infty) \tag{13a}$$

The relative error of the linear approximation given by $|L(\beta)/P(\beta)-1|$ converges therefore to $||J|^{1/2}-1|$, a quantity depending on the first and second derivatives of $g(\underline{x};1)$ at $\underline{x}_o$. Due to this, if an estimate for the failure probability $P(\beta)$ is needed, the quadratic approximation should be used. In particular, if further arithmetic operations with this estimate are performed, as in the case of decision problems, where the result is multiplied by the costs in case of a failure at least asymptotically correct results can be obtained using the outlined procedure of quadratic approximation.

The Generalized Reliability Index

In [4] Ditlevsen defines the generalized reliability ind which is a generalization of the reliability index of Has Lind [3]. For the failure domain $F(\beta)=\{\underline{x};g(\underline{x};\beta)<0\}$ the generalized reliability index β_G is given by:

$$\beta_G = -\Phi^{-1}(P(\beta)) \tag{14}$$

The Hasofer/Lind index for $F(\beta)$ is, due to the definition of $F(\beta)$, simply β. This is also the estimate for the generalized index which is obtained by a linearization of the limit surface at $\underline{x}_o$. For $\beta\to\infty$, for $P(\beta)$ the asymptotic approximation given in eq.(11) is valid; this yields:

$$\phi(-\beta_G) \sim \phi(-\beta)\,|J|^{-1/2} \quad (\beta\to\infty) \tag{14a}$$

By taking logarithms and retaining only the dominant terms:

$$-\beta_G^2/2-\ln(\beta_G) \sim \ln(|J|^{-1/2})-\beta^2/2-\ln(\beta) \tag{15}$$

Rewriting the equation, this gives:

$$-(\beta_G-\beta)(\beta_G+\beta)/2 \sim \ln(|J|^{-1/2})-\ln(\beta)+\ln(\beta_G) \tag{15a}$$

which yields, divided by $(\beta+\beta_G)$:

$$|\beta_G-\beta|\to 0 \quad (\beta\to\infty) \tag{15b}$$

The relative and the absolute error for estimating β_G by the usual Hasofer/Lind reliability index converges to zero for $\beta\to\infty$.

The Case of Several Minimal Distance Points

If there are k points $\underline{x}_1,\ldots\underline{x}_k$ on the surface G with $|\underline{x}_1|=\ldots=|\underline{x}_k|=1$, the quadratic approximation $Q(\beta)$ is derived (as demonstrated in [2]) by calculating the contribution given by equation (11) for each point $\underline{x}_i$ separately and then adding up the results, yielding:

$$Q(\beta) \sim \phi(-\beta)\sum_{i=1}^{k}|J_i|^{-1/2} \quad (\beta\to\infty) \tag{16}$$

with J_i denoting the value given in equation (23) computed for the point $\underline{x}_i$ instead of $\underline{x}_o$.

From the mathematical considerations given in appendix I it follows that it would be sufficient to postulate that $g(\underline{x})$ is twice continuously differentiable near the points on the surface G which have minimal distance to the origin and continuous elsewhere. The results are valid also in this case.

Calculation of the Quadratic Approximation for a Given Limit State

Let a limit state function $\tilde{g}(\underline{x})$ be given in the n-dimensional space of standard normal independent variables with $\tilde{g}(\underline{O})>0$. This function defines a limit surface $\tilde{G}=\{\underline{x};\tilde{g}(\underline{x})=0\}$. Define $g(\underline{x})$ by $g(\underline{x})=\tilde{g}(\beta_o\underline{x})$ with $\beta_o=\min_{z\in\tilde{G}}|\underline{z}|$. Then, $\tilde{g}(\underline{x})=g(\underline{x};\beta_o)$ with $g(\underline{x};\beta)$ defined as before. If β_o is large the asymptotic approximation $Q(\beta_o)$ can be used to approximate $P(\tilde{g}(\underline{X})<0)=P(g(\underline{X};\beta_o)<0)$. The approximation can be found according to the following procedure:

1) Calculate all points $\underline{z}_1,\dots,\underline{z}_k$ on $\tilde{G}$ with $|\underline{z}_1|=\dots=|\underline{z}_k|=\beta_o$ $(=\min\limits_{\underline{z}\in\tilde{G}}|\underline{z}|)$

2) For each point $\beta_o^{-1}\underline{z}_i$ calculate J_i as defined in equation (23).

3) The approximation for $P(\tilde{g}(\underline{X})<0)$ is then:

$$P(\tilde{g}(\underline{X})<0) \approx \Phi(-\beta_o)\left(\sum_{i=1}^{k}|J_i|^{-1/2}\right) \tag{17}$$

Using eq.(29) in appendix I, $|J_i|=\prod_{j=1}^{n-1}(1-\kappa_{i,j})$ with the $\kappa_{i,j}$ $(j=1,..,n-1)$ denoting the ordered main curvatures of the surface $G=\{\underline{x};g(\underline{x};1)=0\}$ at $\beta_o^{-1}\underline{z}_i$. But due to the definition of $g(\underline{x})$ the following relation is valid (see also /3/) :

$$\kappa_{i,j}=\beta_o\tilde{\kappa}_{i,j} \tag{17a}$$

with the $\tilde{\kappa}_{i,j}$ being the ordered main curvatures of the surface $\tilde{G}=\{\underline{x};\tilde{g}(\underline{x})=0\}$ at $\underline{z}_i$. This yields :

$$P(\tilde{g}(\underline{X})<0)\approx\Phi(-\beta_o)\left(\sum_{i=1}^{k}\left(\prod_{j=1}^{n-1}(1-\beta_o\tilde{\kappa}_{i,j})^{-1/2}\right)\right) \tag{17b}$$

This formula gives the estimate in the terms of the distance of the limit surface from the origin and the curvatures of the the surface at the minimal distance points.

Examples

a) Approximation of an Ellipse

In a two-dimensional space the limit state functions $g(\underline{x};\beta)=(x_1/a)^2+x_2^2-\beta^2$ $(a>1)$ are given. There are two points on the curve $g(\underline{x};1)=0$ with minimal distance to the origin $\underline{x}_1=(0,1)$ and $\underline{x}_2=(0,-1)$. This yields, using equation (16):

$$Q(\beta)=2\Phi(-\beta)(1-a^{-2})^{-1/2} \tag{18}$$

for the quadratic approximation $Q(\beta)$. This approximation is compared with the exact probability $P(\beta)$ (which is given in [10], p.164, equation (78)) in figure 1 for a=1.25,2.,4.. The relative error converges to zero for $\beta\to\infty$.

b) Sum of Exponentially Distributed Random Variables

Given are n independent, identically distributed random variables $Y_1,\dots Y_n$. Each has a standardized exponential distribution with cumulative distribution function $F(y)=1-\exp(-y)$.

Let the limit state function be given by $g(\underline{y})=n+\alpha\sqrt{n}-\sum_{i=1}^{n}y_i$ (where α is constant). The transformation in the space of standard normally distributed variables $X_1,\dots,X_n$ is given by $x_i=$ $=-\Phi^{-1}(\exp(-y_i))$ $(i=1,..,n)$. The limit state function in this space is $\tilde{g}(\underline{x})=\sum_{i=1}^{n}\ln(\Phi(-x_i))+n+\alpha\sqrt{n}$. Using the method of Lagrange multipliers (see [13], p. 19), the only point $\underline{z}_1$ on the limit surface with minimal distance to the origin is found:

$$\underline{z}_1=(z,\dots,z) \text{ with } z=-\Phi^{-1}(\exp(-\alpha/\sqrt{n}-1)) \tag{19}$$

The distance to the origin is $\sqrt{n}\,z$.

For J_1 using eq. (23) and denoting the standard normal density by $\varphi(z)$ this yields:

$$J_1=(1-z(\varphi(z)/\Phi(-z)-z))^{n-1} \tag{20a}$$

The approximation is then:

$$P(\tilde{g}(\underline{X})<0)\approx\Phi(-\sqrt{n}\,z)\cdot(1-z(\varphi(z)/\Phi(-z)-z))^{-(n-1)/2} \tag{20b}$$

In figure 2 the exact probability (given by the incomplete gamma function), the approximation by eq.(20b) and the approximation $\Phi(-\sqrt{n}\,z)$ obtained by linearizing the limit surface are compared for $0.<\alpha<3.$ and n=10.

Summary and Conclusions

The problem of obtaining approximations for multinormal integrals is discussed. Two approximations, the linear and the quadratic, are examined. The result shows that only the quadratic approximation gives an asymptotic approximation for the integral in the sense of asymptotic analysis whereas the linear approximation produces an uncontrollable relative error. Only in the case that the generalized reliability index has to be computed the linear approach produces an asymptotic approximation. This implies that, if the failure probability has to be estimated only the quadratic approximation should be used.

Acknowledgement

The author thanks I. Hordan-Klaucke, M. Hohenbichler and especially N. Zorn for their help. The study was part of a project supported by the Deutsche Forschungsgemeinschaft.

Appendix I

Asymptotic Expansion of the Multinormal Integral

In [2], chapt. 8, asymptotic expansions are given for integrals of the form:

$$\hat{I}(\lambda) = \int_D \exp(\lambda f(\underline{x})) f_o(\underline{x}) d\underline{x} \quad (\lambda \to \infty) \tag{21}$$

D is a fixed domain in the n-dimensional space, $f(\underline{x})$ and $f_o(\underline{x})$ are at least two times continuously differentiable functions. Further, it is assumed that the boundary of D is given by the points $\underline{x}$ with $h(\underline{x})=0$, where $h(\underline{x})$ is also at least twice continuously differentiable. It is shown that, if $f(\underline{x})$ has no global maximum with respect to D at an interior point of D, the asymptotic behaviour of $\hat{I}(\lambda)$ depends on the points on the boundary where $f(\underline{x})$ attains its global maximum on the boundary. Due to the Lagrange multiplier theorem a necessary condition for these points is that $\nabla f(\underline{x}) = K \cdot \nabla h(\underline{x})$ (K is a constant). The contribution of one of these points to the asymptotic expansion of $\hat{I}(\lambda)$ is given in eq. (8.3.64), p.340, [2].
Defining $D=\{\underline{x};\ g(\underline{x};1)<0\}$, $\lambda=\beta^2$, $f(\underline{x}) = -|\underline{x}|^2/2$, $h(\underline{x})=g(\underline{x};1)$, the formula can be applied to obtain an asymptotic expansion for $I(\beta)$. Due to the assumption made at the beginning, there is only one point $\underline{x}_o$ on the surface $g(\underline{x};1)=0$ with minimal distance to the origin, i.e. only one point $\underline{x}_o$ where $-|\underline{x}|^2/2$ achieves its maximum. Eq. 8.3.64 yields then:

$$I(\beta) \sim (2\pi)^{(n-1)/2} \exp(-\beta^2 |\underline{x}_o|^2/2)\, \beta^{-(n+1)} |J|^{-1/2} \quad (\beta \to \infty) \tag{22}$$

with:

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} x_o^i\, x_o^j \operatorname{cof}[-\delta_{ij} - K g_{ij}] \tag{23}$$

$$x_o^i = \text{i-th component of } \underline{x}_o . \tag{23a}$$

$$\delta_{ij} = \text{delta Kronecker Symbol.} \tag{23b}$$

$$g_{ij} = \left. \frac{\partial^2 g(\underline{x}:1)}{\partial x_i \partial x_j} \right|_{\underline{x}=\underline{x}_o} \tag{23c}$$

$$K = |\nabla g(\underline{x}_o;1)|^{-1} \tag{23d}$$

cof $[-\delta_{ij} - Kg_{ij}]$ denotes the cofactor (see, e.g.[1], p. 372) of the element $(-\delta_{ij}-Kg_{ij})$ in the matrix $(-\delta_{ij}-Kg_{ij})_{i,j=1,\dots,n}$

Since $|\underline{x}_o| = 1$, the formula simplifies:

$$I(\beta) \sim (2\pi)^{(n-1)/2} \exp(-\beta^2/2)\beta^{-(n+1)}|J|^{-1/2} \tag{24}$$

Due to the rotational symmetry it can be assumed for the further considerations, that $\underline{x}_o=(0,..,0,1)$ (i.e. the unit vector in the direction of the x_n-axis) . Then, since $\underline{x}_o$ is parallel to the gradient of $g(\underline{x};1)$ at $\underline{x}_o$ due to the Lagrange multiplier theorem, the tangential space of the hypersurface G(1) at $\underline{x}_o$ is spanned by the unit vectors in the direction of the first n-1 axes. Then J is given (using the definition of the cofactor) :

$$|J| = |\text{cof}[-\delta_{nn}-Kg_{nn}]| = |\det(\underline{\underline{B}})| \text{ ,with } \underline{\underline{B}}=(\delta_{lm}+Kg_{lm})_{l,m=1,..,n-1} \tag{25}$$

Defining :

$$\underline{\underline{\tilde{D}}} = (-Kg_{lm})_{l,m=1,..,n-1} \tag{25a}$$

and denoting the unity matrix by $\underline{\underline{I}}$:

$$\det(\underline{\underline{B}}) = \det(\underline{\underline{I}} - \underline{\underline{\tilde{D}}}) \tag{26}$$

$\det(\underline{\underline{B}})$ is given by the product of the eigenvalues of $\underline{\underline{B}}$, which are the roots of $\det(\underline{\underline{B}}-\kappa\underline{\underline{I}}) = \det((1-\kappa)\underline{\underline{I}} - \underline{\underline{\tilde{D}}})$ But these roots are given by $1-\kappa_i$ $(i=1,.,n-1)$, where the κ_i's are the eigenvalues of the matrix $\underline{\underline{\tilde{D}}}$. This gives :

$$|J| = \left| \prod_{j=1}^{n-1} (1-\kappa_j) \right| \tag{27}$$

These eigenvalues are the main curvatures of the surface G(1) at $\underline{x}_o$ (see /13/,chap.12,ex. 12.1 and /7/, appendix II). The curvature is defined positive, if the surface is curved towards the origin, as in /13/.

The last formula shows, in which cases the approximation is not applicable. Since $\underline{x}_o$ is a point on the surface with minimal distance to the origin, the main curvatures at $\underline{x}_o$ must be not larger than unity. Elsewhere, consider a point $\underline{x}$ on the surface near $\underline{x}_o$ in the direction of a principal axis of curvature at $\underline{x}_o$ with curvature κ_i larger than unity.Due to the definition of the curvature, the curve on the surface connecting $\underline{x}_o$ and $\underline{x}$ is approximated by a part of a circle in the same direction through $\underline{x}_o$ with radius $1/\kappa_i$ and center $(0,.,0,1-1/\kappa_i)$. Using elementary trigonometric relations, for small distances $|\underline{x}-\underline{x}_o|$ the squared distance of $\underline{x}$ to the origin is approximately :

$$|x|^2 \approx 1 + (1-\kappa_i)|\underline{x}-\underline{x}_o|^2/2 < 1 \tag{28}$$

This contradicts the assumption, that $\underline{x}_o$ is a point on the surface with minimal distance to the origin with respect to the surface and therefore $\kappa_i \leq 1$. Due to this:

$$|J| = \prod_{j=1}^{n-1} (1-\kappa_j) \tag{29}$$

In the case, that one curvature is exactly equal to unity, the approximation can not be used. Then it becomes necessary to study higher derivatives of $g(\underline{x})$ and the global behaviour of the function, but no general results are available for this problem.

Appendix II - References

[1] Bellman, R., "Introduction to Matrix Analysis", 2nd Ed., McGraw Hill, New York, N.Y., 1970.

[2] Bleistein, N. and Handelsman, R.A., "Asymptotic Expansions of Integrals", Holt, Rinehart and Winston, New York, N.Y., 1975.

[3] Breitung, K., "An Asymptotic Formula for the Failure Probability", DIALOG 82-6, Department of Civil Engineering, Danmarks Ingeniørakademi, Lyngby, Denmark, 1982, (to appear).

[4] Ditlevsen, O., "Uncertainty Modelling", McGraw-Hill, New York, N.Y., 1981.

[5] Ditlevsen, O., "Principle of Normal Tail Approximation", Journal of the Engineering Mechanics Division, ASCE, Vol. 107, EM6, Dec. 1981, pp. 1191 - 1209.

[6] Dolinski, K., "First-Order Second Moment Approximation in Reliability of Structural Systems; Critical Review and an Alternative Approach", Structural Safety, (to appear).

[7] Fiessler, B., Neumann, H.-J. and Rackwitz, R., "Quadratic Limit States in Structural Reliability", Journal of the Engineering Mechanics Division, ASCE, Vol. 105, EM4, Aug. 1979 pp. 661 - 676.

[8] Hasofer, A.M. and Lind, N.C., "An Exact and Invariant First-Order Reliability Format", Journal of the Engineering Mechanics Division, ASCE, Vol. 100, EM1, Feb. 1974, pp. 111 - 121.

[9] Hohenbichler, M. and Rackwitz, R., "Non-Normal Dependent Vectors in Structural Safety", Journal of the Engineering Mechanics Division, ASCE, Vol. 107, EM6, Dec. 1981, pp. 1227 - 1241.

[10] Johnson, N.I. and Kotz, S., "Distributions in Statistics; Continuous Unvariate Distributions 2", Wiley, New York, N.Y., 1979.

[11] Pšeničnyj, B.N., "Algorithms for General Mathematical Programming Problems", Cybernetics, Vol. 6(5), 1970, pp. 120-125.

[12] Rackwitz, R. and Fiessler, B., "Structural Reliability under Combined Random Sequences", Computers and Structures, Vol. 9, 1978, pp. 489 - 494.

[13] Thorpe, J.A., "Elementary Topics in Differential Geometry", Springer, New-York, N.Y., 1979.

Appendix III - Notation

$\underline{\underline{B}}$ = matrix defined in eq. (25)

$\underline{\underline{D}}$ = matrix defined in eq. (25a)

$\underline{\underline{D}}(\beta)$ = matrix of second derivatives

$F, F(\beta)$ = failure domains

$g(\underline{x})$, $g^*(\underline{x})$, $\tilde{g}(\underline{x})$, $g(\underline{x};\beta)$ = limit state functions

$g_L(\underline{x};\beta)$ = linear approximation for the limit state function $g(\underline{x};\beta)$

$g_Q(\underline{x};\beta)$ = quadratic approximation for the limit state function $g(\underline{x};\beta)$

G, $G(\beta)$, $\tilde{G}(\beta)$ = limit surfaces

$\underline{\underline{I}}$ = unity matrix

$I(\beta)$ = integral defined by eq. (8)

J = quantity defined in eq. (23)

$L(\beta)$ = approximation for $P(\beta)$ using linear Taylor expansion

P, $P(\beta)$ = failure probability

$Q(\beta)$ = approximation for $P(\beta)$ using quadratic Taylor expansion

S = safe domain

β_G = generalized reliability index

$\nabla g(\underline{x}_o)$ = gradient of $g(\underline{x})$ at $\underline{x}_o$

$\kappa_i, \kappa_{i,j}, \tilde{\kappa}_{i,j}$ = main curvatures of surfaces

$\Phi(\underline{x})$ = standard normal integral

$\varphi(x)$ = standard normal density

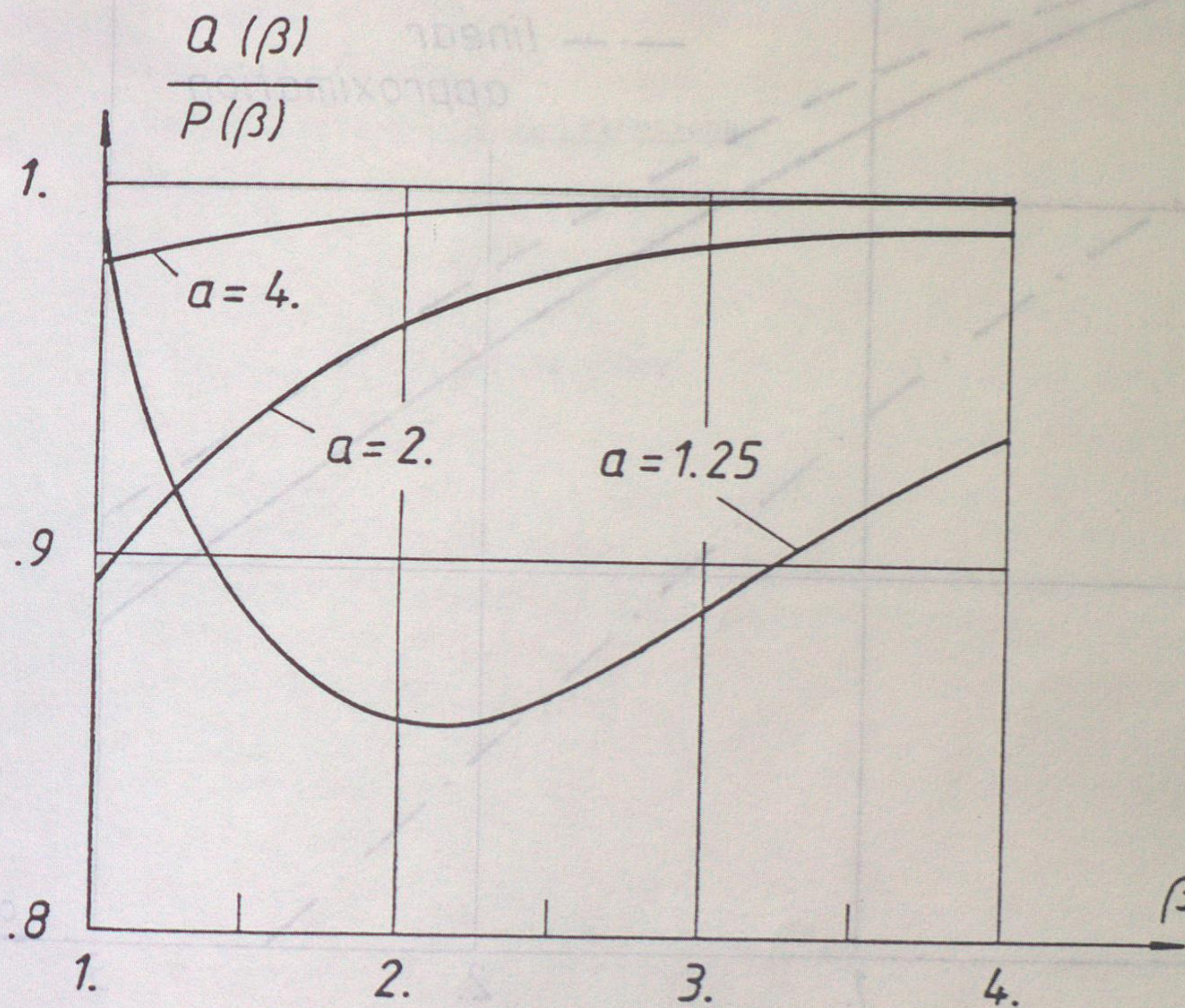


Figure 1: $Q(\beta)/P(\beta)$ versus β for a=1.25, 2., 4..

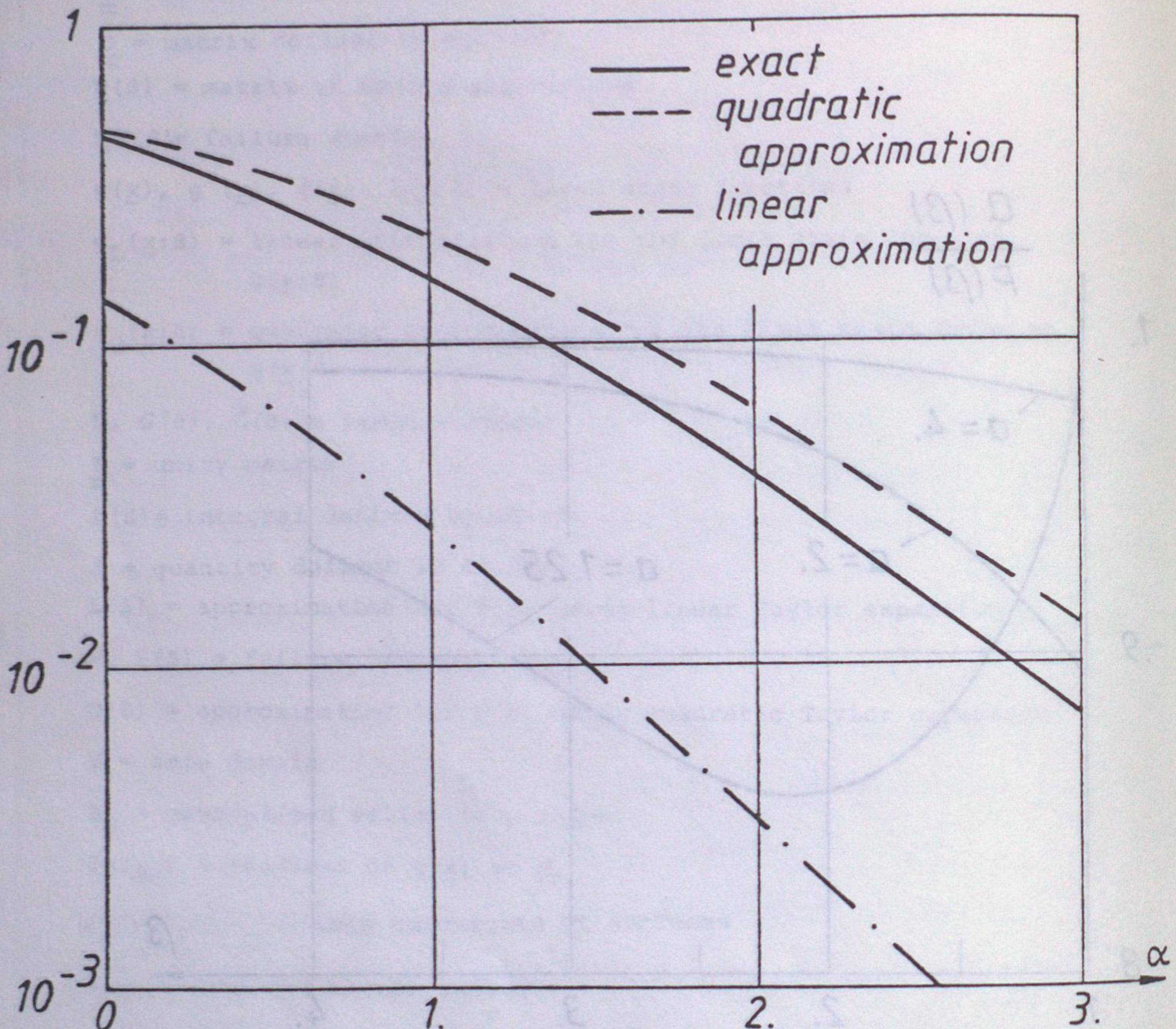


Figure 2: Sum of 10 exponentially distributed random variables.

An Asymptotic Formula for the Probability of Intersections

M. Hohenbichler

Let $\underline{U} = (U_1, \ldots, U_n)$ be a random vector, and F a measurable subset of $\mathbb{R}^n$. Without lack of generality [2] the variables U_i are assumed to be independent and standard normally distributed. Assume further that there exists a unique Beta-point $\underline{u}^*$ (which is a point $\underline{u}^*$ on F or its boundary with minimal distance to the origin) and that the reliability index ß of F is strictly positive:

$$\beta := \| \underline{u}^* \| > 0.$$

Specifically, $\underline{u}^*$ lies on the boundary of F, and the origin is not contained in F. If, however, F contains the origin, the following results can usually be applied to the complement $\mathbb{R}^n \setminus F$ of F.

Breitung [1] recently derived a simple asymptotic formula for the probability

$$P[F] = P[\underline{U} \in F]$$

for the case, that the so-called "Beta-Point" $\underline{u}^*$ lies on a smooth portion of the boundary of F. More precisely, he assumed that there exists an environment $\mathcal{U}$ of $\underline{u}^*$ and a twice continuously differentiable function g (at least defined on $\mathcal{U}$) with

$$g(\underline{u}^*) = 0$$

$$F \cap \mathcal{U} = \{g < 0\} \cap \mathcal{U}$$

$$\text{grad } g(\underline{u}^*) := \left(\frac{\partial g}{\partial u_1}(\underline{u}^*), \ldots, \frac{g}{u_n}(\underline{u}^*)\right) \neq \underline{0}.$$

The following theorem generalizes Breitung's result and allows $\underline{u}^*$ also to lie on an edge of F:

Assumptions

(A1) F has an unique Beta-point $\underline{u}^*$ and it is $ß := \| \underline{u}^* \| > 0$.

There exist an environment $\mathcal{U}$ of $\underline{u}^*$ and twice continuously differentiable functions $g_1, g_2, \ldots, g_k$ $(k \geq 1)$, at least defined on $\mathcal{U}$, with

$$g_i(\underline{u}^*) = 0 \qquad (1 \leq i \leq k)$$

$$F \cap \mathcal{U} = \left(\bigcap_{i=1}^{k} \{g_i < 0\}\right) \cap \mathcal{U}$$

(A2) The gradients

$$\underline{a}_i := (a_{i1}, \ldots, a_{in}) := \text{grad } g_i(\underline{u}^*) \quad (1 \leq i \leq k)$$

are linearly independent.

Assumption (A2) implies $k \leq n$ and $\underline{a}_i \neq \underline{0}$ $(1 \leq i \leq k)$. Performing a suitable orthogonal transformation, it can further be assumed, without loss of generality, that

(A3) $\quad a_{ij} = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$

Condition (A3) is empty if k equals n; its only purpose is to simplify the formulation of the theorem.

Furthermore, we assume that $\underline{u}^*$ lies in the negative cone spanned by the vectors $\underline{a}_1, \ldots, \underline{a}_k$:

(A4) $\quad \underline{u}^* = \Sigma \gamma_i \, \underline{a}_i \quad$ with $\gamma_i < 0 \quad$ for $1 \leq i \leq k$.

The γ_i's are uniquely determined, since the a_i's are linearly independent. (A4) is, in fact, not very restrictive, since from theorems of optimization theory it follows that $\underline{u}^*$ is a linear combination of the $\underline{a}_i$'s

$$\underline{u}^* = \sum_{i=1}^{k} \gamma_i \, \underline{a}_i \quad \text{with } \gamma_i \leq 0.$$

The γ_i's are determined numerically solving the equations

$$u_j^* = \sum_{i=1}^{k} \gamma_i \, a_{ij} \quad \text{for } 1 \leq j \leq k,$$

while for $k + 1 \leq j \leq n$ both sides of that equation vanish.

It should also be noted here that F needs not as a whole, but only in an environment $\mathcal{U}$ of $\underline{u}^*$ be an intersection of events $\{g_i < 0\}$. Furthermore, if originally

$$F \cap \mathcal{U}_1 = \left(\bigcap_{i=1}^{k} \{g_i < 0\}\right) \cap \mathcal{U}_1$$

but

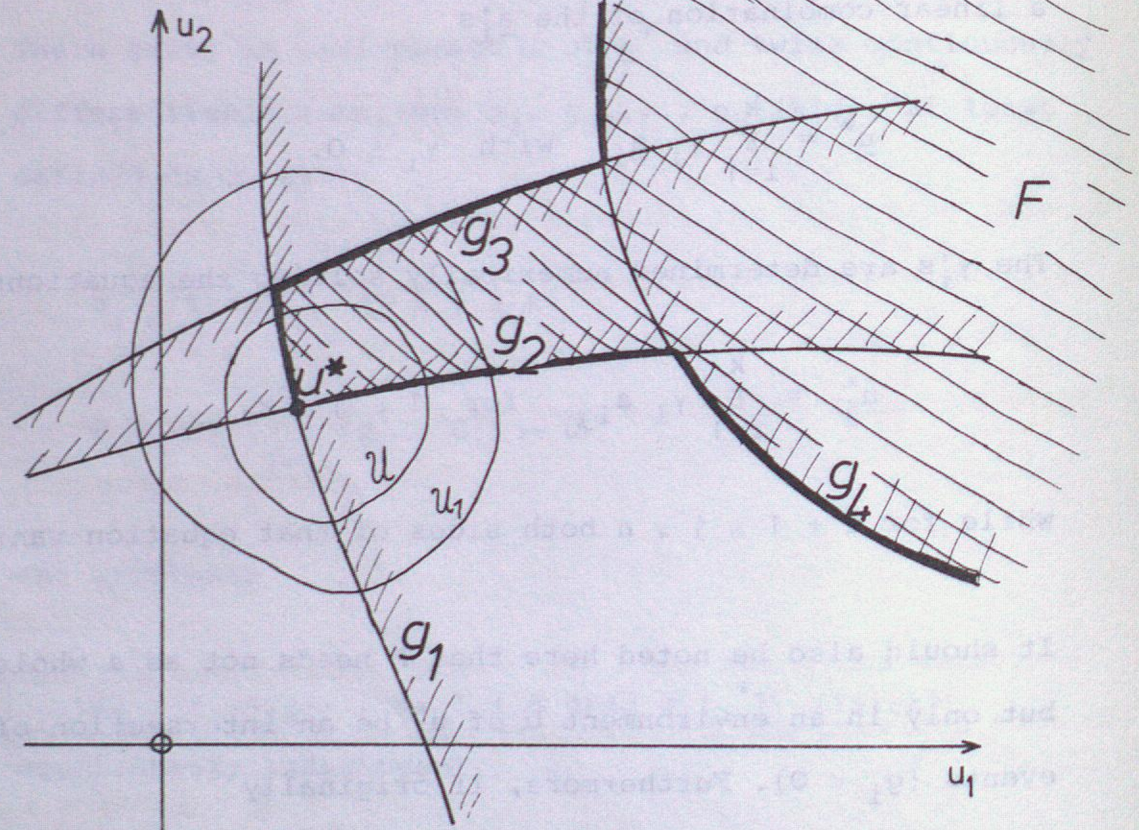
$$g_k(\underline{u}^*) < 0,$$

the function g_k can always be omitted in the representation

$$F \cap \mathcal{U} = \Big(\bigcap_{i=1}^{k-1} \{g_i < 0\}\Big) \cap \mathcal{U}$$

on passing to a smaller environment $\mathcal{U}$. For example, in the subsequent figure there is

$$F = [\{g_1<0\}\cap\{g_2<0\}\cap\{g_3<0\}]\cup\{g_4<0\}$$



$$F \cap \mathcal{U}_1 = [\{g_1<0\}\cap\{g_2<0\}\cap\{g_3<0\}]\cap\mathcal{U}_1$$

but

but $g_3(\underline{u}^*) < 0$, allowing for a representation

$$F \cap \mathcal{U} = (\{g_1 < 0\}\cap\{g_2 < 0\})\cap\mathcal{U}$$

with $g_1(\underline{u}^*) = g_2(\underline{u}^*) = 0$ as required in assumption (A1).

<u>Notations</u>

(N1) $\phi_k(\underline{x};\ \underline{\underline{S}})$ is the k-dimensional normal distribution function with covariance matrix $\underline{\underline{S}}$ and argument $\underline{x}$; i.e.

$$\phi_k(\underline{x};\underline{\underline{S}}) = P\Big[\bigcap_{i=1}^{k}\{X_i \le x_i\}\Big]$$

where $\underline{X} = (X_1,\ldots,X_2)$ is a normal random vector with

$$E[X_i] = 0 \qquad (1 \le i \le k)$$

$$\operatorname{cov}(X_i, X_j) = S_{ij} \qquad (1 \le i,j \le k)$$

(N2) $\varphi_k(\underline{x};\ \underline{\underline{S}})$ is the corresponding probability density.

$$\varphi_k(\underline{x};\ \underline{\underline{S}}) = \frac{\partial^k}{\partial x_1\ldots\partial x_k}\,\phi_k(\underline{x};\ \underline{\underline{S}}) = \frac{1}{\sqrt{(2\pi)^k\det(\underline{\underline{S}})}}\exp\{-\tfrac{1}{2}\underline{x}\ \underline{\underline{S}}^{-1}\underline{x}'\}$$

[det $(\underline{\underline{S}})$ is the determinant of $\underline{\underline{S}}$, and $\underline{x}'$ the transposed of $\underline{x}$]

(N3) $$Q_k(\underline{x};\ \underline{\underline{S}}) := \varphi_k(\underline{x};\ \underline{\underline{S}})\prod_{j=1}^{k}\frac{1}{(-\ \underline{x}\ \underline{\underline{S}}^{-1})_j}$$

[$(\underline{x}\,\underline{\underline{A}})_j$ is the j-th component of the vector $\underline{x}\,\underline{\underline{A}}$]

(N4) $c_i := \underline{u}^* \,\underline{a}_i'$

$\underline{c} := (c_1, \dots, c_k)$

$\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$ (see eq. (A4)

$\underline{\underline{R}} := (\underline{a}_i \,\underline{a}_j' : 1 \le i,j \le k)$

(N5) $\underline{\underline{I}} = (\delta_{ij} : k+1 \le i,j \le n) \in \mathbb{R}^{(n-k,n-k)}$ is the (n-k)-dimensional unit matrix

$\underline{\underline{D}} = (d_{ij} : k+1 \le i,j \le n) \in \mathbb{R}^{(n-k,n-k)}$

$$d_{ij} := \underline{\gamma}\,[\frac{\partial^2}{\partial u_i \partial u_j}\, \underline{g}(\underline{u}^*)]' = \sum_{s=1}^{k} \gamma_s \frac{\partial^2}{\partial u_i \partial u_j}\, g_s(\underline{u}^*)$$

In analogy with Breitung's result the following theorem describes the asymptotic behaviour of the probability $P[bF] = P\,[\underline{U} \in bF] = P[\frac{1}{b}\,\underline{U} \in F]$ of $bF = \{b\underline{u} : \underline{u} \in F\}$, as b and therefore also the reliability-index bß of bF become large:

Theorem

Under the assumptions (A1) - (A4), and using the notations (N1) - (N5), the following statements hold, provided that $d := \det(\underline{\underline{I}} - \underline{\underline{D}}) > 0$:

a) $$\lim_{b\to\infty} \frac{P[bF]}{\Phi_k(b\underline{c};\underline{\underline{R}})} = \frac{1}{\sqrt{d}}$$

where

$\det(\underline{\underline{I}} - \underline{\underline{D}}) := 1$ for $k = n$

b) $$\lim_{b\to\infty} \frac{\Phi_k(b\underline{c};\underline{\underline{R}})}{Q_k(b\underline{c};\underline{\underline{R}})} = 1$$

c) Consequently, there is also

$$\lim_{b\to\infty} \frac{P[bF]}{Q_k(b\underline{c};\underline{\underline{R}})} = \frac{1}{\sqrt{d}}$$

Remark 1

There is always $\det(\underline{\underline{I}} - \underline{\underline{D}}) \ge 0$

Remark 2

The evaluation of $Q_k(b\underline{c};\underline{\underline{R}})$ is somewhat simplified noting that due to (A3) and (A4) only the first k components of $\underline{a}_i$ and $\underline{u}^*$ are essential. Collecting those in the partial vectors

$$\hat{\underline{a}}_i = (a_{i1}, \dots, a_{ik})$$

and introducing

$$\underline{\underline{A}} = (\hat{\underline{a}}_1', \dots, \hat{\underline{a}}_k')$$

it is

$$Q_k(b\underline{c};\underline{\underline{R}}) = \frac{(-1)^k}{b^k\,|\det(\underline{\underline{A}})|} \prod_{i=1}^{k} [\frac{1}{\gamma_i}\, \varphi(bu_i^*)]$$

Denoting now by

$$H = \bigcap_{i=1}^{k} \{\underline{a}_i(\underline{U} - \underline{u}^*)' < 0\}$$

the linear approximation of F at the Beta-point $\underline{u}^*$, it is easily verified that

$$P[bH] = \phi_k(b\underline{c};\underline{\underline{R}}).$$

Since on the other hand a domain F with large safety-index ß can be represented as

$$F = bF_1, \quad b \text{ "large"}$$

where $F_1 = \frac{1}{b}F$ is a domain with "moderate"safety-index, and since the right hand sides of the asymptotic equations in the theorem are independent of b, the theorem justifies the asymptotic approximations

$$P[F] \sim P[H] \cdot [\det(\underline{\underline{I}} - \underline{\underline{D}})]^{-\frac{1}{2}} =$$

$$= \phi_k(\underline{c};\underline{\underline{R}}) \cdot [\det(\underline{\underline{I}} - \underline{\underline{D}})]^{-\frac{1}{2}} \sim$$

$$\sim Q_k(\underline{c};\underline{\underline{R}}) \cdot [\det(\underline{\underline{I}} - \underline{\underline{D}})]^{-\frac{1}{2}},$$

which exhibit "high" relative precision for "large" ß. For large b, the probability P[bH] of the linearization deviates from

P[bF] by a factor, which is independent of b and depends only on the second derivatives of the g_i's along the directions orthogonal to the gradients $\underline{a}_i$. In the special case k = n there exists no orthogonal direction, and one obtains the asymptotic approximation

$$P[F] \sim P[H] = \phi_k(\underline{c};\underline{\underline{R}}) \sim Q_k(\underline{c};\underline{\underline{R}}) \quad \text{for } k = n. \tag{17}$$

The last part $\phi_k \sim Q_k$ is due to Ruben [4]. In the other extremal case k=1 Breitung's formula

$$P[F] \sim \Phi(-ß)\ [\det(\underline{\underline{I}} - \underline{\underline{D}})]^{-\frac{1}{2}} \quad \text{for } k=1 \tag{18}$$

is regained, which is not surprising because the results of Breitung and Ruben are essential tools in the proof of the theorem.

A simple extension of the result is possible for domains with more than one, but finitely many beta-points $\underline{u}_1^*,\dots,\underline{u}_t^*$. Then, there exist environments $\mathcal{U}_s$ $(1 \le s \le t)$ of $\underline{u}_s^*$, which are mutually disjoint. Since the above approximation is an asymptotic approximation for $F \cap \mathcal{U}$ as well as for F, while

$$P[(F\cap\mathcal{U}_1)\cup\dots\cup(F\cap\mathcal{U}_t)] = P[F\cap\mathcal{U}_1]+\dots+P[F\cap\mathcal{U}_t], \tag{19}$$

an asymptotic approximation for P[F] is then obtained summing up the asymptotic contributions of $P[F\cap\mathcal{U}_s]$ at each of the different Beta-points. In the same way also the contributions of further "local Beta-points" can be taken into account, but those are asymptotically negligible.

Proof of Remark 2

It is $\underline{\underline{R}} = \underline{\underline{A}}' \underline{\underline{A}}, |\det(\underline{\underline{A}})| = \sqrt{\det(\underline{\underline{R}})}$

$$\underline{\hat{u}}^* := (u_1^*, \dots, u_k^*) = \underline{\gamma}\, \underline{\underline{A}}'$$

$$\underline{c} = \underline{\hat{u}}^* \underline{\underline{A}} = \underline{\gamma}\, \underline{\underline{A}}'\underline{\underline{A}} = \underline{\gamma}\, \underline{\underline{R}}$$

$$\underline{c}\, \underline{\underline{R}}^{-1} = \underline{\gamma}\, \underline{\underline{R}}\, \underline{\underline{R}}^{-1} = \underline{\gamma}$$

$$\underline{c}\, \underline{\underline{R}}^{-1} \underline{c}' = \underline{\gamma}\, \underline{\underline{R}}' \underline{\gamma}' = \underline{\gamma}\, \underline{\underline{A}}' \underline{\underline{A}}\, \underline{\gamma}' = (\underline{\hat{u}}^*)(\underline{\hat{u}}^*)'$$

$$\frac{1}{\sqrt{2\pi}^k} \exp\{-\frac{1}{2} b \underline{c}\, \underline{\underline{R}}^{-1} b \underline{c}'\} = \prod_{i=1}^{k} \varphi(b u_i^*)$$

Proof of the Theorem

In the following, there is always $\underline{u} \in \mathbb{R}^n$, $\underline{v} \in \mathbb{R}^k$, $\underline{\tau} \in \mathbb{R}^{n-k}$ $\underline{v}^* = (u_1^*, \dots, u_k^*)$, $F_{\underline{\tau}} = \{\underline{v} : (\underline{v}, \underline{\tau}) \in F\}$, $\| \cdot \|$ is the Euclidian norm, $K_\varepsilon(\underline{v}_0) = \{\underline{v}: \| \underline{v}-\underline{v}_0 \| < \varepsilon\}$. The environment $\mathcal{U}$ of $\underline{u}^*$ contains an open environment of the form $\mathcal{U} \supset \mathcal{V}_1 \times \mathcal{T}_1$ with $\mathcal{V}_1 \subset \mathbb{R}^k$, $\mathcal{T}_1 \subset \mathbb{R}^{n-k}$. Obviously, for $\underline{\tau} = \underline{0}$ the point $\underline{v}^*$ is the Beta-point of $F_{\underline{\tau}}$ and also of $F_{\underline{\tau}} \cap \mathcal{V}_1$. Investigating the mapping $\underline{T}^{-1} \circ \underline{S}$ with $\underline{T}(\underline{v},\underline{\tau}) = (g_1(\underline{v},\underline{\tau}), \dots, g_k(\underline{v},\underline{\tau}), \underline{\tau})$ and $\underline{S}(\underline{v},\underline{\tau}) = (g_1(\underline{v},\underline{0}), \dots, g_k(\underline{v},\underline{0}), \underline{\tau})$ it can be deduced that there exist open environments $\mathcal{V}_2$ and $\mathcal{V}_3$ of $\underline{v}^*$ and $\mathcal{T}_2$ of $\underline{0} \in \mathbb{R}^{n-k}$ and a twice continuously

differentiable mapping $\underline{G} : \mathcal{V}_2 \times \mathcal{T}_2 \to \mathcal{V}_1$ with $\underline{G}(\mathcal{V}_2, \underline{\tau}) \supset \mathcal{V}_3$ for all $\underline{\tau} \in \mathcal{T}_2$, and $g_i(\underline{G}(\underline{v},\underline{\tau}), \underline{\tau}) = g_i(\underline{v}, \underline{0})$ for all $1 \le i \le k$, $\underline{v} \in \mathcal{V}_2$, $\underline{\tau} \in \mathcal{T}_2$. Therefrom the following lemma, which connects $F_{\underline{\tau}}$ and $F_{\underline{0}}$ can be deduced by straightforward analysis.

Lemma 1

There exist $\delta > 0, \varepsilon > 0$ and, for $\mathcal{V} = K_\varepsilon(\underline{v}^*) \subset \mathbb{R}^k$ and $\mathcal{T} = K_\delta(\underline{0}) \subset \mathbb{R}^{n-k}$, an open and twice continuously differentiable mapping $\underline{G}: \mathcal{V} \times \mathcal{T} \to \mathbb{R}^k$ with the following properties:

a) For each $\underline{\tau} \in \mathcal{T}$, $\underline{G}(\cdot,\underline{\tau})$ is injective

b) $g_i(\underline{G}(\underline{v},\underline{\tau}),\underline{\tau}) = g_i(\underline{v},\underline{0})$ for each $1 \le i \le k$, $\underline{v} \in \mathcal{V}$, $\underline{\tau} \in \mathcal{T}$

c) $\underline{v}^*(\underline{\tau}) := \underline{G}(\underline{v}^*,\underline{\tau})$ is the Beta-Point of $F_{\underline{\tau}} \cap \mathcal{V} \subset \mathbb{R}^k$

d) $\mathcal{V}_\varepsilon(\delta) := \{(\underline{v},\underline{\tau}): \| \underline{\tau} \| < \delta, \| \underline{v} - \underline{v}^*(\underline{\tau}) \| < \varepsilon\}$ is an environment of $\underline{u}^*$ (for each $\varepsilon > 0$, $\delta > 0$)

From Lemma 1, (b) [and (c)] it follows that $g_i(\underline{v}^*(\underline{\tau}),\underline{\tau}) = 0$ for $1 \le i \le k$, $\underline{\tau} \in \mathcal{T}$. Introducing now the linearization of $F_{\underline{\tau}}$ and $g_i(\cdot,\underline{\tau})$ at $\underline{v}^*(\underline{\tau})$, i.e.

$$\underline{b}_i(\underline{\tau}) := (\frac{\partial}{\partial v_1} g_i(\underline{v}^*(\underline{\tau}),\underline{\tau}), \dots, \frac{\partial}{\partial v_k} g_i(\underline{v}^*(\underline{\tau}),\underline{\tau}) \in \mathbb{R}^k$$

$$h_i(\underline{v},\underline{\tau}) := \underline{b}_i(\underline{\tau})\ (\underline{v} - \underline{v}^*(\underline{\tau}))'$$

$$H_{\underline{\tau}} := \bigcap_{i=1}^{k} \{h_i(\cdot,\underline{\tau}) < 0\} \subset \mathbb{R}^k$$

the following lemma, which connects $H_{\underline{\tau}}$ and $F_{\underline{\tau}}$, is proved by investigating the mapping $\underline{T}^{-1} \circ \underline{S}$ with $\underline{T}(\underline{v},\underline{\tau}) = (h_1(\underline{v},\underline{\tau}), \dots, h_k(\underline{v},\underline{\tau}), \underline{\tau})$ and $\underline{S}(\underline{v},\underline{\tau}) = (g_1(\underline{v},\underline{\tau}), \dots, g_k(\underline{v},\underline{\tau}), \underline{\tau})$:

Lemma 2

There exist $\varepsilon>0$, $\delta>0$ and an injective, open and twice continuously differentiable mapping $\underline{T} : \mathcal{V}_\varepsilon(\delta) \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with the following properties:

a) $\mathcal{V}_\varepsilon(\delta)$ and $\underline{T}(\mathcal{V}_\varepsilon(\delta))$ are open environments of $\underline{u}^*$.

b) $\underline{T}(\underline{v},\underline{\tau}) = (\hat{\underline{T}}(\underline{v},\underline{\tau}),\underline{\tau})$ with $\hat{\underline{T}}(\underline{v},\underline{\tau}) \in \mathbb{R}^k$

c) $h_i(\hat{\underline{T}}(\underline{v},\underline{\tau}),\underline{\tau}) = g_i(\underline{v},\underline{\tau})$ for all $(\underline{v},\underline{\tau}) \in \mathcal{V}_\varepsilon(\delta)$.

In particular, it is

$$\hat{\underline{T}}(F_{\underline{\tau}} \cap K_\varepsilon(\underline{v}^*(\underline{\tau})),\underline{\tau}) = H_{\underline{\tau}} \cap \hat{\underline{T}}(K_\varepsilon(\underline{v}^*(\underline{\tau})),\underline{\tau})$$

and

$$\hat{\underline{T}}(\underline{v}^*(\underline{\tau}),\underline{\tau}) = \underline{v}^*(\underline{\tau})$$

d) $\| \hat{\underline{T}}(\underline{v},\underline{\tau}) - \underline{v} \| = \sigma_\delta(\| \underline{v} - \underline{v}^*(\underline{\tau})\|^2)$, i.e.

$$\lim_{s\to 0} \sup \left\{ \frac{\| \hat{\underline{T}}(\underline{v},\underline{\tau})-\underline{v} \|}{\| \underline{v}-\underline{v}^*(\underline{\tau}) \|^2} : \| \underline{v}-\underline{v}^*(\underline{\tau}) \| = s, \| \underline{\tau} \| < \delta \right\} < \infty$$

e) For $\dfrac{\partial \hat{\underline{T}}(\underline{v},\underline{\tau})}{\partial \underline{v}} := \left(\dfrac{\partial \hat{T}_i(\underline{v},\underline{\tau})}{\partial v_j} : 1 \le i,j \le k\right)$ there is

$$\det\left(\frac{\partial \hat{\underline{T}}(\underline{v},\underline{\tau})}{\partial \underline{v}}\right) = 1 + \sigma_\delta(\| \underline{v} - \underline{v}^*(\underline{\tau}) \|), \text{ i.e.}$$

$$\lim_{s\to 0} \sup \left\{ \frac{\left|1-\det\left(\frac{\partial \hat{\underline{T}}(\underline{v},\underline{\tau})}{\partial \underline{v}}\right)\right|}{\| \underline{v}-\underline{v}^*(\underline{\tau}) \|} : \| \underline{v} - \underline{v}^*(\underline{\tau}) \| = s, \| \underline{\tau} \| < \delta \right\} < \infty$$

f) The inverse mapping $\underline{T}^{-1} : \underline{T}(\mathcal{V}_\varepsilon(\delta)) \to \mathcal{V}_\varepsilon(\delta)$ has the analogous properties.

The following lemma is a simple consequence from the definition of the Beta-point and properties of the normal density-function φ (see, e.g. [3])

Lemma 3

Let $\underline{u}^*$ be a unique Beta-point of F, and $\mathcal{U}$ an environment of $\underline{u}^*$. Then

a) $\inf\{ \| \underline{u} \| : \underline{u} \in F \setminus \mathcal{U} \} > \beta \; (= \| \underline{u}^* \|)$

(where $\inf \{ \| \underline{u} \| : \underline{u} \in \emptyset \} := +\infty$ for the empty set $\emptyset$)

b) If $P[F \cap \mathcal{W}] > 0$ for each environment $\mathcal{W}$ of $\underline{u}^*$, then

$$\lim_{b\to\infty} \frac{P[b(F \setminus \mathcal{U})]}{P[bF]} = 0$$

Lemma 4

There exists a $\delta > 0$, such that for each $\varepsilon > 0$

$$\lim_{b\to\infty} \sup_{\underline{\tau} \in K_\delta} \left| \frac{P[bF_{\underline{\tau}} \setminus K_\varepsilon(b\underline{v}^*(\underline{\tau}))]}{P[bF_{\underline{\tau}}]} \right| = 0,$$

and the same for $H_{\underline{\tau}}$ in place of $F_{\underline{\tau}}$.

Here, for $A \subset \mathbb{R}^k$,

$$P[A] = P[(U_1,\dots,U_k) \in A],$$

and

$$K_\delta = \{\underline{\tau} \in \mathbb{R}^{n-k} : \|\underline{\tau}\| < \delta\}$$

$$K_\varepsilon(\underline{v}_o) = \{\underline{v} \in \mathbb{R}^k : \|\underline{v} - \underline{v}_o\| < \varepsilon\}$$

Proof of Lemma 4:

Choose ε and δ in accordance with lemmas 1 and 2.

Since $\underline{v}^*(\underline{O}) = \sum \gamma_i \underline{b}_i(\underline{o})$ (cf. assumption (A4) and (A3)), and due to the continuity of $\underline{b}_i(\underline{\tau})$ and $\underline{v}^*(\underline{\tau})$ for $\|\underline{\tau}\| < \delta$ there is for some $\delta > 0$ and for all $\|\underline{\tau}\| < \delta$

$$\underline{v}^*(\underline{\tau}) = \sum \gamma_i(\underline{\tau}) \underline{b}_i(\underline{\tau}),$$

$$S_1 := \sup\{\gamma_i(\underline{\tau}) : 1 \le i \le k, \|\underline{\tau}\| < \delta\} < 0$$

$$S_2 := \sup\{|\gamma_i(\underline{\tau})| : 1 \le i \le k, \|\underline{\tau}\| < \delta\} < \infty$$

$\underline{b}_1(\underline{\tau}),\dots,\underline{b}_k(\underline{\tau})$ are linearly independent. (1)

This implies (for $\|\underline{\tau}\| < \delta$)

$$\max_{1 \le i \le k} |\underline{b}_i(\underline{\tau})\ \underline{v}'| > 0 \quad \text{for all } \underline{v} \in \mathbb{R}^k, \underline{v} \neq \underline{0}$$

and thus, due to continuity and compactness, for δ small enough

$$\inf\{\max_{1 \le i \le k} \frac{1}{\|\underline{v}\|} |\underline{b}_i(\underline{\tau})\ \underline{v}'| : \|\underline{\tau}\| \le \delta, \underline{v} \in \mathbb{R}^k \setminus \{0\}\} =$$

$$= \inf\{\max_{1 \le i \le k} |\underline{b}_i(\underline{\tau})\ \underline{v}'| : \|\underline{\tau}\| \le \delta, \|\underline{v}\| = 1\} =: c > 0 \tag{2}$$

Furthermore, there is $g_i(\underline{v},\underline{\tau}) = \underline{b}_i(\underline{\tau})(\underline{v} - \underline{v}^*(\underline{\tau}))' + O_\delta(\|\underline{v}-\underline{v}^*(\underline{\tau})\|^2) < 0$ for $\underline{v} \in F_{\underline{\tau}}$, $\underline{\tau} \in K_\delta$, and consequently for ε and δ small enough, each $\underline{v} \neq \underline{v}^*(\underline{\tau})$, $\|\underline{v}-\underline{v}^*(\underline{\tau})\| < \varepsilon$ and each $\underline{\tau} \in K_\delta$

$$\max_{1 \le i \le k} \{\frac{1}{\|\underline{v}-\underline{v}^*(\underline{\tau})\|} [b_i(\underline{\tau})(\underline{v}-\underline{v}^*(\underline{\tau}))']\} < O_\delta(\|\underline{v}-\underline{v}^*(\underline{\tau})\|) < \frac{c}{2} \tag{3}$$

Comparing this with (2), it follows that for each $\underline{v} \in F_{\underline{\tau}} \cap K_\varepsilon(\underline{v}^*(\underline{\tau}))$ $\underline{v} \neq \underline{v}^*(\underline{\tau})$ and $\underline{\tau} \in K_\delta$ there exists an integer $i_o = i_o(\underline{v},\underline{\tau})$ $(1 \le i \le k)$ with

$$\frac{1}{\|\underline{v}-\underline{v}^*(\underline{\tau})\|} [b_{io}(\underline{\tau})(\underline{v}-\underline{v}^*(\underline{\tau}))'] \leqq -c.$$

This implies again for $\underline{\tau} \in K_\delta$ and $\underline{v} \in F_{\underline{\tau}}$, $\|\underline{v} - \underline{v}^*(\underline{\tau})\| < \varepsilon$ (ε small enough, $\varepsilon > 0$) (see also (1) and (3), and note, that $\gamma_i(\underline{\tau}) < 0$)

$$\frac{1}{\|\underline{v}-\underline{v}^*(\underline{\tau})\|} [\underline{v}^*(\underline{\tau})(\underline{v}-\underline{v}^*(\underline{\tau}))'] = \frac{1}{\|\underline{v}-\underline{v}^*(\underline{\tau})\|} \sum \gamma_i(\underline{\tau}) \underline{b}_i(\underline{\tau})(\underline{v}-\underline{v}^*(\underline{\tau}))' \geq$$

$$\geq \gamma_{i_o}(\underline{\tau})(-c) - (k-1)S_2|O_\delta(\|\underline{v} - \underline{v}^*(\underline{\tau})\|)| \geq$$

$$\geq |S_1|c - \frac{1}{2}|S_1|c = \frac{1}{2}|S_1|c$$

and

$$\|\underline{v}\| \geq \frac{1}{\|\underline{v}^*(\underline{\tau})\|}\{\underline{v}\,\underline{v}^*(\underline{\tau})'\} = \frac{1}{\|\underline{v}^*(\underline{\tau})\|}[\{\underline{v}^*(\underline{\tau})+(\underline{v}-\underline{v}^*(\underline{\tau}))]\,\underline{v}^*(\underline{\tau})'\} =$$

$$= \|\underline{v}^*(\underline{\tau})\| + \frac{1}{\|\underline{v}-\underline{v}^*(\underline{\tau})\|}(\underline{v}-\underline{v}^*(\underline{\tau}))\underline{v}^*(\underline{\tau})' \,\frac{\|\underline{v}-\underline{v}^*(\underline{\tau})\|}{\|\underline{v}^*(\underline{\tau})\|} \geq$$

$$\geq \|\underline{v}^*(\tau)\| + \frac{1}{2}|S_1|c\,\frac{\|\underline{v}-\underline{v}^*(\underline{\tau})\|}{\|\underline{v}^*(\tau)\|}$$

From this, and lemmas 3a and 1c it can be concluded

$$\exists\,\delta > 0\ \exists\,\varepsilon > 0\ \exists\,\eta > 0\ \forall\,\lambda, 0 < \lambda < \varepsilon\ \forall\,\underline{\tau} \in K_\delta:$$

$$\underline{v} \in F_{\underline{\tau}},\ \|\underline{v}-\underline{v}^*(\underline{\tau})\| \geq \lambda \Rightarrow \|\underline{v}\| \geq \|\underline{v}^*(\underline{\tau})\| + \eta\lambda \tag{4}$$

Let now, for $\underline{v} \in \mathbb{R}^k$, $\varphi(\underline{v}) = \varphi(v_1)\ldots\varphi(v_k)$ and, for $A \subset \mathbb{R}^k$, $b \geq 1$

$$P_b[A] := P[bA] = P[(U_1,\ldots,U_k) \in bA] = b^k \int_A \varphi(b\underline{v})\,d\underline{v}.$$

Due to

$$\varphi(b\underline{v}) = \varphi(\underline{v})\exp\{-\frac{1}{2}(b^2-1)\|\underline{v}\|^2\}$$

we have for $\varepsilon > 0$, denoting by $\beta(\underline{\tau}) := \|\underline{v}^*(\underline{\tau})\|$

$$P_b[F_{\underline{\tau}}] \geq P_b[F_{\underline{\tau}} \cap K_{\beta(\tau)+\varepsilon/b}] \geq b^k P[F_{\underline{\tau}} \cap K_{\beta(\tau)+\varepsilon/b}]$$

$$\exp\{-\frac{1}{2}(b^2-1)[\beta(\underline{\tau}) + \frac{\varepsilon}{b}]^2\}$$

$$P_b[F_{\underline{\tau}} \setminus K_{\beta(\underline{\tau})+2\varepsilon/b}] \leq b^k \exp\{-\frac{1}{2}(b^2-1)[\beta(\underline{\tau}) + \frac{2\varepsilon}{b}]^2\}$$

and so for $b \geq 2$ $(\Rightarrow (b^2-1)/b^2 \geq 0$ and $(b^2-1)/b \geq b/2)$

$$0 \leq P_b[F_{\underline{\tau}} \setminus K_{\beta(\underline{\tau})+2\varepsilon/b}]/P_b[F_{\underline{\tau}}] \leq$$

$$\leq \exp\{-\frac{1}{2}(b^2-1)[\frac{2}{b}\varepsilon\beta(\underline{\tau}) + \frac{3}{b^2}\varepsilon^2]\}/P[F_{\underline{\tau}} \cap K_{\beta(\tau)+\varepsilon/b}] \leq$$

$$\leq \exp\{-\frac{1}{2}b\,\varepsilon\,\beta(\underline{\tau})\}/P[F_{\underline{\tau}} \cap K_{\beta(\underline{\tau})+\varepsilon/b}],$$

and if δ is small enough, for all $b \geq 2$ and $\underline{\tau} \in K_\delta$

$$0 \leq P_b[F_{\underline{\tau}} \setminus K_{\beta(\underline{\tau})+2\varepsilon/b}]/P_b[F_{\underline{\tau}}] \leq \exp\{-\frac{1}{4}b\,\varepsilon\,\beta\}/P[F_{\underline{\tau}} \cap K_{\beta(\underline{\tau})+\varepsilon/b}].$$

Now, transforming the integral $P[F_{\underline{\tau}} \cap K_{\beta(\underline{\tau})+\varepsilon/b}]$ into an integral over $H_{\underline{\tau}} \cap \hat{\underline{T}}(K_{\beta(\underline{\tau})+\varepsilon/b}, \underline{\tau})$ and employing lemma 2, it can be verified that

$$\exists\, s_o > 0\ \exists\, \delta > 0\ \exists\, c > 0\ \forall\, s,\ 0 \le s < s_o\ \forall\, \underline{\tau} \in K_\delta :$$

$$P[F_{\underline{\tau}} \cap K_{\beta(\underline{\tau})+s}] \ge c\, s^k ,$$

and therefore, for each $\varepsilon > 0$

$$\lim_{b\to\infty} \sup_{\underline{\tau}\in K_\delta} \{P_b[F_{\underline{\tau}} \setminus K_{\beta(\underline{\tau})+2\varepsilon/b}]/P_b[F_\tau]\} = 0$$

This together with (4) proves lemma 4.

Lemma 5

There exists a $\delta > 0$ such, that

$$\lim_{b\to\infty} \sup_{\underline{\tau}\in K_\delta} \left| 1 - \frac{P[bF_{\underline{\tau}}]}{P[bH_{\underline{\tau}}]} \right| = 0$$

Proof of lemma 5

For small ε and δ, $\underline{T}(\mathcal{U}_\varepsilon(\delta))$ is defined and is an environment of $\underline{u}^*$ (lemma 2) and therefore contains another set of the form $\mathcal{U}_\varepsilon(\delta)$ (lemma 1). Denoting now by $A(\underline{\tau}, b, \varepsilon) = \hat{\underline{T}}(\cdot, \underline{\tau})^{-1}[H_{\underline{\tau}} \cap K_{\varepsilon/b}(\underline{v}^*(\underline{\tau}))]$ (lemma 2), we have for $b \ge 1$, small ε, δ and $\underline{\tau} \in K_\delta$

$$P_b[H_{\underline{\tau}} \cap K_{\varepsilon/b}(\underline{v}^*(\underline{\tau}))] = b^k \int_{A(\underline{\tau},b,\varepsilon)} \varphi(b\hat{\underline{T}}(\underline{v},\tau)) \left|\det\left(\frac{\partial \hat{\underline{T}}(\underline{v},\underline{\tau})}{\partial\, \underline{v}}\right)\right| d\underline{v}$$

and

$$A(\underline{\tau},b,\varepsilon) \subset F_{\underline{\tau}} \subset K_{2\varepsilon/b}(\underline{v}^*(\underline{\tau})) =: B(\underline{\tau},b,\varepsilon),$$

Further application of lemma 2 yields

$$P_b[H_{\underline{\tau}} \cap K_{\varepsilon/b}(\underline{v}^*(\underline{\tau}))] \le b^k \int_{B(\underline{\tau},b,\varepsilon)} \varphi(b\hat{\underline{T}}(v,\underline{\tau})) \left|\det\left(\frac{\partial \hat{\underline{T}}(\underline{v},\underline{\tau})}{\partial\, \underline{v}}\right)\right| d\underline{v} \le$$

$$\le b^k \int_{B(\underline{\tau},b,\varepsilon)} \sqrt{2\pi}^{\,-k} \exp\{-\tfrac{1}{2} b^2 \sum_{i=1}^{k} (v_i^2 + \mathcal{O}_\delta(\| \underline{v}-\underline{v}^*(\underline{\tau}) \|^2)\}$$

$$(1+\mathcal{O}_\delta(\| \underline{v}-\underline{v}^*(\underline{\tau}) \|))d\underline{v} =$$

$$= b^k \int_{B(\underline{\tau},b,\varepsilon)} \varphi(b\underline{v}) \exp\{ b^2 \mathcal{O}_\delta(\| \underline{v}-\underline{v}^*(\underline{\tau}) \|^2)\}$$

$$(1+\mathcal{O}_\delta(\| \underline{v}-\underline{v}^*(\underline{\tau}) \|)d\underline{v} \le$$

$$\le b^k \int_{B(\underline{\tau},b,\varepsilon)} \varphi(b\underline{v}) \exp\{\mathcal{O}_\delta(\varepsilon^2)\}(1+\mathcal{O}_\delta(\tfrac{\varepsilon}{b}))\, d\underline{v} =$$

$$= (1+\mathcal{O}_\delta(\varepsilon^2))(1+\mathcal{O}_\delta(\tfrac{\varepsilon}{b}))P_b[F_{\underline{\tau}} \cap K_{2\varepsilon/b}(\underline{v}^*(\underline{\tau}))].$$

Hence

$$\limsup_{b\to\infty}\ \sup_{\underline{\tau}\in K_\delta} \frac{P_b[H_{\underline{\tau}}\cap K_{\varepsilon/b}(\underline{v}^*(\underline{\tau}))]}{P_b[F_{\underline{\tau}}\cap K_{2\varepsilon/b}(\underline{v}^*(\underline{\tau}))]} \le 1 + \mathcal{O}_\delta(\varepsilon^2). \qquad (1)$$

The symmetric argument holds as well:

$$\limsup_{b\to\infty}\ \sup_{\underline{\tau}\in K_\delta} \frac{P_b[F_{\underline{\tau}}\cap K_{\varepsilon/b}(\underline{v}^*(\underline{\tau}))]}{P_b[H_{\underline{\tau}}\cap K_{2\varepsilon/b}(\underline{v}^*(\underline{\tau}))]} \le 1 + \mathcal{O}_\delta(\varepsilon^2) \qquad (2)$$

(1) and (2), together with lemma 4, imply lemma 5.

<u>Lemma 6</u>

Denoting by

$$c_i(\underline{\tau}) := \underline{b}_i(\underline{\tau})\underline{v}^*(\underline{\tau})',\ \underline{c}(\underline{\tau}) = (c_1(\underline{\tau}),\dots,c_k(\underline{\tau})),$$

$$\underline{\underline{R}}(\underline{\tau}) := (\underline{b}_i(\underline{\tau})\underline{b}_j(\underline{\tau})' : 1 \le i,j \le k),$$

there exists a $\delta > 0$ such, that

$$\lim_{b\to\infty}\ \sup_{\underline{\tau}\in K_\delta} \left| 1 - \frac{\Phi_k(b\underline{c}(\underline{\tau});\underline{\underline{R}}(\underline{\tau}))}{Q_k(b\underline{c}(\underline{\tau});\underline{\underline{R}}(\underline{\tau}))} \right| = 0.$$

<u>Proof of lemma 6</u>

Observing that $P[bH_{\underline{\tau}}] = P_b[H_{\underline{\tau}}] = \Phi_k(b\underline{c}(\underline{\tau});\underline{\underline{R}}(\underline{\tau}))$, lemma 6 is a simple consequence of the continuity of $\underline{c}(\underline{\tau})$ and $\underline{\underline{R}}(\underline{\tau})$ and Ruben's result [4] about the error term of the approximation $\Phi_k \sim Q_k$.

We are now ready to prove the theorem:

<u>Proof of the theorem and remark 1</u>

Let, for "small" $\underline{\tau}$,

$$\beta(\underline{\tau}) := \|\underline{v}^*(\underline{\tau})\|,\quad \beta := \beta(\underline{0}),$$

$\underline{c}(\underline{\tau})$ and $\underline{\underline{R}}(\underline{\tau})$ like in lemma 6

$$\gamma_i(\underline{\tau}) \text{ defined by } \underline{v}^*(\underline{\tau}) = \sum_{i=1}^{k} \gamma_i(\underline{\tau})\underline{b}_i(\underline{\tau})$$

$$\underline{\underline{B}}(\underline{\tau}) := (\underline{b}_1(\underline{\tau})',\dots,\underline{b}_k(\underline{\tau})')$$

$$Q_b(\underline{\tau}) := Q_k(b\underline{c}(\underline{\tau});\underline{\underline{R}}(\underline{\tau})) = \frac{(-1)^k\,\varphi(b\beta(\underline{\tau}))}{b^k\sqrt{2\pi}^{\,k-1}\,|\det(\underline{\underline{B}}(\underline{\tau}))|}\prod_{i=1}^{k}\frac{1}{\gamma_i(\underline{\tau})}.$$

(see remark 2). Define further

$$f_b^{(1)}(\underline{\tau}) := P[bF_{\underline{\tau}}]/Q_b(\underline{\tau})$$

$$f_b^{(2)}(\underline{\tau}) := \frac{|\det(\underline{\underline{B}}(\underline{0}))|}{|\det(\underline{\underline{B}}(\underline{\tau}))|}\prod_{i=1}^{k}\frac{\gamma_i(\underline{0})}{\gamma_i(\underline{\tau})}$$

$$f_b^{(3)}(\underline{\tau}) := \varphi(-b\beta(\underline{\tau}))/[b\beta(\underline{\tau})\Phi(-b\beta(\underline{\tau}))]$$

$$f_b^{(4)}(\underline{\tau}) := b\beta\Phi(-b\beta)/[\varphi(-b\beta)]$$

$$f_b^{(5)}(\underline{\tau}) := \beta(\underline{\tau})/\beta.$$

Then, it is

$$f_b(\underline{\tau}) := \frac{P[bF_{\underline{\tau}}]}{Q_b(\underline{O})}\,\frac{\Phi(-b\beta)}{\Phi(-b\beta(\underline{\tau}))} = \prod_{m=1}^{5} f_b^{(m)}(\underline{\tau}). \qquad (1)$$

Let further $0 < \varepsilon < \frac{1}{2}$. Then there exist $\delta > 0$ and $0 < b_o < \infty$ such that for each $\underline{\tau} \in K_\delta$ and each $b \geq b_o$ there is

$$|f_b^{(1)}(\underline{\tau}) - 1| < \varepsilon \qquad \text{(lemmas 5, 6)} \qquad (2a)$$

$$|f_b^{(2)}(\underline{\tau}) - 1| < \varepsilon \qquad \text{(continuity of } \underline{b}(\underline{\tau})) \qquad (2b)$$

$$|f_b^{(5)}(\underline{\tau}) - 1| < \varepsilon \qquad \text{(continuity of } \underline{v}^*(\underline{\tau})\text{; lemma 1)} \qquad (2c)$$

Since there is $\varepsilon < \frac{1}{2}$, the last equation implies also

$$\tfrac{1}{2}\beta < \beta(\underline{\tau}) < \tfrac{3}{2}\beta,$$

and due to

$$1 \leq \varphi(-bx)/[bx\Phi(-bx)] \leq 1 + 1/(b^2x^2) \quad \text{for } x>0$$

one obtains for $\underline{\tau} \in K_\delta$ and $b \geq b_o$

$$1 \leq \varphi(-b\beta(\underline{\tau}))/[b\beta(\underline{\tau})\Phi(-b\beta(\underline{\tau}))] \leq 1+4/(b_o^2\beta^2).$$

Therefore, if b_o is chosen large enough, we have for $\underline{\tau} \in K_\delta$ and $b \geq b_o$

$$|f_b^{(3)}(\underline{\tau}) - 1| < \varepsilon \qquad (2d)$$

$$|f_b^{(4)}(\underline{\tau}) - 1| < \varepsilon \qquad (2e).$$

From (1) and (2) it follows, that

$$|f_b(\underline{\tau}) - 1| \leq (1+\varepsilon)^5 - 1 \quad \text{for } b \geq b_o,\ \underline{\tau} \in K_\delta. \qquad (3)$$

Hence, for $b \geq b_o$ it is

$$\Big|\int_{K_\delta}\{P[bF_{\underline{\tau}}]/Q_b(\underline{O})\}\varphi(b\underline{\tau})\,d\underline{\tau} - \int_{K_\delta}\{\Phi[-b\beta(\underline{\tau})]/\Phi(-b\beta)\}\varphi(b\underline{\tau})\,d\underline{\tau}\Big| \leq \qquad (4)$$

$$\leq \int_{K_\delta}|f_b(\underline{\tau})-1|\{\Phi[-b\beta(\underline{\tau})]/\Phi(-b\beta)\}\varphi(b\underline{\tau})\,d\underline{\tau} \leq$$

$$\leq [(1+\varepsilon)^5-1]\int_{K_\delta}\{\Phi[-b\beta(\underline{\tau})]/\Phi(-b\beta)\}\varphi(b\underline{\tau})\,d\underline{\tau} =: r(\varepsilon)\ I_2(\delta,b)$$

Since due to lemma 1 the function $\underline{\tau} \to \beta(\underline{\tau})$ is twice continuously differentiable in K_δ, according to Breitung [1]

$$\lim_{b\to\infty} b^{n-k} I_2(\delta,b) = s_o := \sqrt{\det(\underline{\underline{I}}+\beta\underline{\underline{F}})}^{-1} \tag{5}$$

with $\underline{\underline{F}} = (F_{ij} : k+1 \le i,j \le n)$, $F_{ij} := \dfrac{\partial^2}{\partial\tau_i \partial\tau_j} \beta(\underline{O})$.

It will be shown later, that $0 < s_o < \infty$. Denoting by $I_1(\delta,b)$ the first and by $I_2(\delta,b)$ the second integral in equ. (4), it is thus proved:

$$\limsup_{b\to\infty} b^{n-k} \,|I_1(\delta,b) - I_2(\delta,b)| \le r(\varepsilon) s_o$$

for each $0 < \varepsilon < \frac{1}{2}$, or since $r(\varepsilon) \to 0$ for $\varepsilon \to 0$,

$$\lim_{b\to\infty} b^{n-k} |I_1(\delta,b) - I_2(\delta,b)| = 0. \tag{6}$$

Further, due to lemma 3b, applied with $\mathcal{U} = \mathbb{R}^k \times K_\delta$, we have

$$\lim_{b\to\infty} \frac{1}{I_1(\infty,b)} \,|I_1(\infty,b) - I_1(\delta,b)| = 0$$

and in connection with (5) and (6)

$$\lim_{b\to\infty} b^{n-k} \,|I_1(\infty,b) - I_1(\delta,b)| = 0. \tag{7}$$

(5), (6) and (7) imply

$$\lim_{b\to\infty} \{P[bF]/Q_b(0)\} = \lim_{b\to\infty} b^{n-k} \; I_1(\infty,b) = \sqrt{\det(\underline{\underline{I}}+\beta\underline{\underline{F}})}^{-1}. \tag{8}$$

It remains to evaluate s_o or $F_{ij} = \dfrac{\partial^2}{\partial\tau_i\partial\tau_j} \beta(\underline{\tau})$. In [3] (proof of theorem 3.3.5) it is shown that (for small $\underline{\tau}$)

$$\frac{\partial}{\partial\tau_j} \underline{v}^*(\underline{\tau}) = - \frac{\partial}{\partial\tau_j}\left[\underline{g}(\underline{v}^*,\underline{\tau})\right] \underline{\underline{B}}(\underline{\tau})^{-1} \qquad (\text{at } \underline{v}^* = \underline{v}^*(\underline{\tau})) \tag{9}$$

with $\underline{g} = (g_1,\ldots,g_k)$ and $\partial\tau_i = \partial u_{k+i}$. In particular (see (A3))

$$\frac{\partial}{\partial\tau_j} \underline{v}^*(0) = \underline{O}. \tag{10}$$

The second derivative is thus

$$\frac{\partial^2}{\partial\tau_i\partial\tau_j} \underline{v}^*(\underline{\tau}) = - \frac{d}{d\tau_i} \left[\frac{\partial}{\partial\tau_j} \underline{g}(\underline{v}^*(\underline{\tau}),\underline{\tau})\right]\underline{\underline{B}}(\underline{\tau})^{-1} -$$

$$- \frac{\partial}{\partial\tau_j} \underline{g}(\underline{v}^*(\underline{\tau}),\underline{\tau}) \frac{\partial}{\partial\tau_i} \left[\underline{\underline{B}}(\underline{\tau})^{-1}\right],$$

and due to (10), $(\underline{v}^*(\underline{O}),\underline{O}) = \underline{u}^*$ and (A3)

$$\frac{\partial^2}{\partial\tau_i\partial\tau_j} \underline{v}^*(\underline{O}) = - \frac{d}{d\tau_i} \left[\frac{\partial}{\partial\tau_j} \underline{g}(\underline{v}^*(\underline{\tau}),\underline{\tau})\right]_{\underline{\tau}=0} \underline{\underline{B}}(\underline{O})^{-1} =$$

$$= \{ - \sum_{m=1}^{k} \frac{\partial^2}{\partial v_m \partial\tau_j} \underline{g}(\underline{u}^*) \frac{\partial v_m^*(\underline{O})}{\partial\tau_i} - \frac{\partial^2}{\partial\tau_i\partial\tau_j} \underline{g}(\underline{u}^*) \} \; \underline{\underline{B}}(\underline{O})^{-1} =$$

$$= [\text{see } (10)]\ \{-\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{g}(\underline{u}^*)\}\ \underline{\underline{B}}(\underline{O})^{-1}.$$

Finally,

$$\frac{\partial}{\partial\tau_j}\ \beta(\underline{\tau}) = \frac{1}{\beta(\underline{\tau})}\ \{\underline{v}^*(\underline{\tau})[\frac{\partial}{\partial\tau_j}\ \underline{v}^*(\underline{\tau})]'\},$$

$$\frac{\partial}{\partial\tau_j}\ \beta(\underline{O}) = \underline{O},$$

$$\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \beta(\underline{O}) = -\frac{1}{\beta^2(\underline{O})}\ \{\frac{\partial}{\partial\tau_j}\ \beta(\underline{O})\}\{\underline{v}^*(\underline{\tau})[\frac{\partial}{\partial\tau_j}\ \underline{v}^*(\underline{\tau})]'\}\ +$$

$$+\ \frac{1}{\beta(\underline{O})}\ \{[\frac{\partial}{\partial\tau_i}\ \underline{v}^*(\underline{O})][\frac{\partial}{\partial\tau_j}\ \underline{v}^*(\underline{O})]'\} + \frac{1}{\beta(\underline{O})}\{\underline{v}^*(\underline{O})[\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{v}^*(\underline{O})]'\} =$$

$$= 0 + 0 + \frac{1}{\beta}\{\underline{v}^*(\underline{O})\ [\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{v}^*(\underline{O})]'\} =$$

$$= -\frac{1}{\beta}\ \underline{v}^*(\underline{O})[\underline{\underline{B}}(\underline{O})^{-1}]'\ [\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{g}(\underline{u}^*)]'$$

and due $\underline{v}^*(\underline{O}) = \underline{\gamma}(\underline{O})\underline{\underline{B}}(\underline{O})'$

$$\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \beta(\underline{O}) = -\frac{1}{\beta}\ \underline{\gamma}(\underline{O})[\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{g}(\underline{u}^*)]' = -\frac{1}{\beta}\ \underline{\gamma}\ [\frac{\partial^2}{\partial\tau_i\partial\tau_j}\ \underline{g}(\underline{u}^*)]'.$$

Therefore, $\underline{\underline{F}} = -\frac{1}{\beta}\ \underline{\underline{D}}$, and $\det(\underline{\underline{I}} + \beta F) = \det(\underline{\underline{I}} - \underline{\underline{D}})$. Since the matrix $\underline{\underline{I}} + \beta\underline{\underline{F}}$ is always positively semidefinite [1] it is always $\det(\underline{\underline{I}} - \underline{\underline{D}}) \geq 0$ (see remark 1 to the theorem). The supposition of the theorem implies even $\det(\underline{\underline{I}} - \underline{\underline{D}}) > 0$, whence $s_o > 0$; clearly, there is also $s_o < \infty$. This, in connection with (8) and lemma 6, proves the theorem.

References

[1] Breitung, K.: Asymptotic Approximations for Multinormal Integrals. Berichte zur Zuverlässigkeitstheorie der Bauwerke, Heft 69, TU-München (SFB 96), 1984, (also submitted to ASCE)

[2] Hohenbichler, M.; Rackwitz, R.: Non-normal Dependent Vectors in Structural Safety. Journal of the Engineering Mechanics Division, ASCE, Vol. 107, No. 6, 1981, S. 1227 - 1249.

[3] Hohenbichler, M.: Mathematische Grundlagen der Zuverlässigkeitsmethode erster Ordnung und einige Erweiterungen. Im November 1983 als Dissertation an der TU München eingereicht.

[4] Ruben, H.: An Asymptotic Expansion for the Multivariate Normal Distribution and Mills'Ratio. Journal of Research of the National Bureau of Standards, Vol. 68B, No. 1, 1964.

Numerical Evaluation of the Error Term in Breitung's Formula

M. Hohenbichler

Let $\underline{U} = (U_1, \ldots, U_n)$ be a random vector, which without loss of generality [4] is assumed to be standard normally distributed with independent components. Let further $g = g(\underline{u})$ be a continuous function twice differentiable near the Beta-point (Hasofer-Lind-point) $\underline{u}^*$ [3] of the domain $\{g < 0\} = \{\underline{u} : g(\underline{u}) < 0\}$, and satisfying $g(\underline{0}) > 0$. If originally there is $g(\underline{0}) < 0$, the results apply easily to the function $-g$.

At the ICASP-4 conference Harbitz [2] proposed an integral transformation combined with the importance sampling technique for the evaluation of the probability $P_f = P[g(\underline{U}) < 0]$, while Breitung [1] derived the asymptotic result

$$\lim_{b \to \infty} \frac{P[g(\frac{1}{b}\,\underline{U}) < 0]}{\Phi(-b\,\beta)} = \prod_{i=1}^{n-1} \frac{1}{\sqrt{1-\beta\kappa_i}} . \qquad (1)$$

Here Φ is the univariate standard normal integral, $\beta = \|\underline{u}^*\|$ is the Beta-index (Hasofer-Lind reliability index), and $\kappa_1, \ldots, \kappa_{n-1}$ are the main curvatures of the surface $\{g=0\}$ at $\underline{u}^*$. Performing a suitable orthogonal transformation, it can be assumed that $\underline{u}^*$ lies on the negative n-th coordinate axis, implying

$$\frac{\partial g}{\partial u_i}(\underline{u}^*) = 0 \quad \text{for } 1 \leq i \leq n-1, \qquad \frac{\partial g}{\partial u_n}(\underline{u}^*) > 0. \qquad (2)$$

A suitable rotation around the n-th axis yields further

$$\frac{\partial^2 g}{\partial u_i \partial u_j}(\underline{u}^*) = 0 \quad \text{for} \quad 1 \le i, j \le n-1, \; i \ne j, \tag{3}$$

and, using the abreviation $a = \frac{\partial g}{\partial u_n}(\underline{u}^*)$, there is

$$\kappa_i = -\frac{1}{a}\frac{\partial^2 g}{\partial u_i^2}(\underline{u}^*) \qquad (1 \le i \le n-1). \tag{4}$$

Setting now b = 1 in eq. (1), one obtains the asymptotic approximation

$$P[g(\underline{U}) < 0] \approx \phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1-\beta\kappa_i}} \tag{5}$$

which provides good results in a variety of examples [1], while one of the main objections is the lack of a numerically feasible estimate of the induced error. Combining however the main ideas of Breitung and Harbitz, a convenient expression for the error term can be found and also Harbitz's method is improved, which appears somewhat arbitrary in its present form.

For a derivation of the formulae, the domain $\{g<0\}$ is expanded around the Beta-point $\underline{u}^*$. In an environment A of $\underline{u}^*$, the condition g = 0 defines a function $f = f(u_1,\dots,u_{n-1})$, i.e. for absolutely small $\underline{v} = (u_1,\dots,u_{n-1})$ and u_n such that

$(\underline{v},u_n) \in A$, there is

$$g(\underline{v},u_n) = 0 \quad \Longleftrightarrow \quad u_n = f(\underline{v}), \tag{6}$$

and further

$$[g(\underline{u}) < 0 \quad \Longleftrightarrow \quad u_n < f(\underline{v})] \quad \text{for} \quad \underline{u} = (\underline{v},u_n) \in A. \tag{7}$$

The second-order Taylor expansion of f at the point $\underline{u}^*$ is

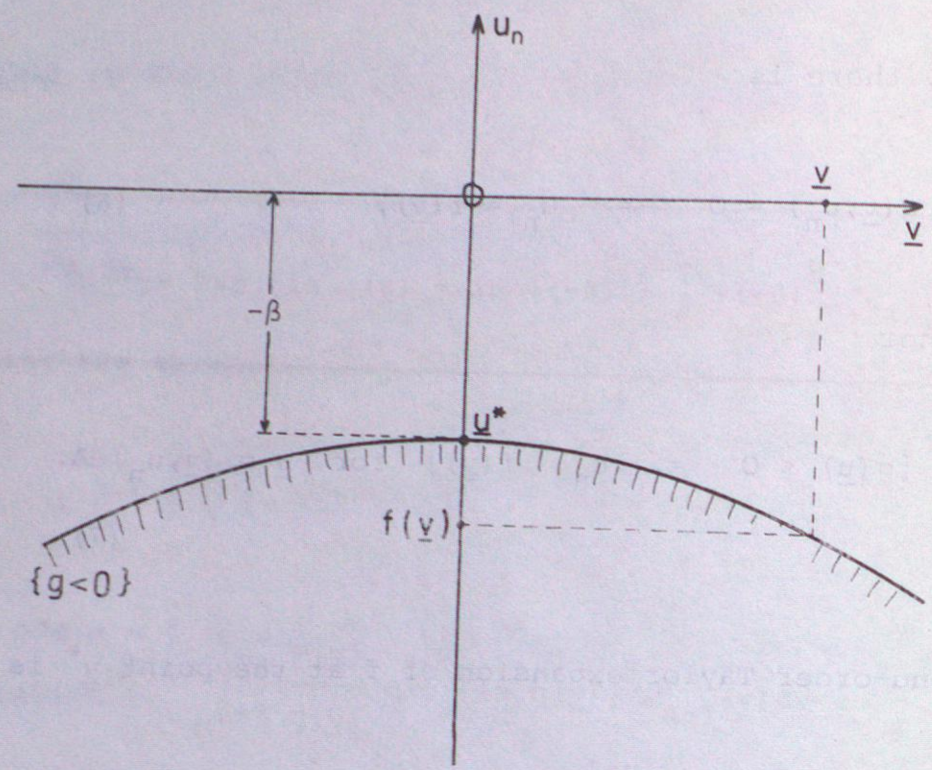
$$t(\underline{v}) = -\beta + \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i u_i^2. \tag{8}$$

Therefrom one obtains the quadratic approximation of $\{g<0\}$ at the Beta-point $\underline{u}^*$

$$A \cap \{g<0\} \approx A \cap \{u_n < t(\underline{v})\} \tag{9}$$

or more generally, since most of the mass of $\{g<0\}$ is concentrated around $\underline{u}^*$

$$\{g<0\} \approx \{u_n < t(\underline{v})\} = \{u_n < -\beta + \frac{1}{2}\sum_{i=1}^{n-1}\kappa_i u_i^2\} \tag{10}$$



Eq. (10) is a suitable basis for the application of the importance sampling technique. Following [3], and using Fubini's theorem, the probability P_f is decomposed as

$$P_f = P[g(\underline{U})<0] = \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} 1_{\{g<0\}}(\underline{v},u_n)\,\varphi(u_n)\,du_n\right] \varphi_{n-1}(\underline{v})\,d\underline{v} \; . \qquad (11)$$

φ is the univariate and φ_{n-1} the (n-1)-dimensional standard normal density and $1_{\{g<0\}}$ is the indicator function of the domain $\{g<0\}$ (for $u_n \in \mathbb{R}$ and $\underline{v} \in \mathbb{R}^{n-1}$ with $\underline{u} = (\underline{v}, u_n)$ there is $1_{\{g<0\}}(\underline{u}) = 1$ whenever $g(\underline{u})<0$, and $1_{\{g<0\}}(\underline{u})=0$ whenever $g(\underline{u}) \geq 0$.) Abbreviating

$$\boxed{p(\underline{v}) = \int_{\mathbb{R}} 1_{\{g<0\}}(\underline{v},u_n)\;\varphi(u_n)\;du_n \qquad (12)}$$

and introducing its approximation according to eq. (10)

$$\boxed{q(\underline{v}) = \int_{-\infty}^{t(\underline{v})} \varphi(u_n)\;du_n = \Phi(t(\underline{v})), \qquad (13)}$$

eq. (11) is rewritten as

$$P_f = \int_{\mathbb{R}^{n-1}} p(\underline{v})\;\varphi_{n-1}(\underline{v})\,d\underline{v} = \int_{\mathbb{R}^{n-1}} \frac{p(\underline{v})}{q(\underline{v})} q(\underline{v})\;\varphi_{n-1}(\underline{v})\,d\underline{v}. \qquad (14)$$

Further, observing that the Taylor expansion of $\ln \Phi$ is

$$\ln \Phi\;(-\beta + x) = \ln \Phi(-\beta) + x\,\Psi(-\beta)+\ldots \text{ with}$$

$$\Psi(-\beta) = \frac{\varphi(-\beta)}{\Phi(-\beta)}\,, \qquad (15)$$

the logarithm of $q(\underline{v})$ is expanded (see eqs. (8) and (13))

$$\ln q(\underline{v}) = \ln \Phi(-\beta) + \frac{1}{2}\,\Psi(-\beta) \sum_{i=1}^{n-1} \kappa_i\, u_i^2 + \ln(R(\underline{v}))$$

with

$$\boxed{R(\underline{v}) = \exp\{\ln q(\underline{v}) - \ln \phi(-\beta) - \frac{1}{2}\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i u_i^2\}\ .}$$

(17)

This leads to

$$P_f = \int_{\mathbb{R}^{n-1}} \frac{p(\underline{v})}{q(\underline{v})} \exp\{\ln q(\underline{v})\}\,\varphi_{n-1}(\underline{v})d\underline{v} =$$

$$= \int_{\mathbb{R}^{n-1}} \frac{p(\underline{v})}{q(\underline{v})}\,\phi(-\beta)\,\exp\{\frac{1}{2}\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i u_i^2\}$$

$$R(\underline{v})\varphi_{n-1}(\underline{v})d\underline{v} =$$

$$= \Phi(-\beta)\int_{\mathbb{R}^{n-1}} \frac{p(\underline{v})}{q(\underline{v})} R(\underline{v}) \prod_{i=1}^{n-1}\varphi(u_i\sqrt{1-\Psi(-\beta)\kappa_i})\,d\underline{v} =$$

$$= \Phi(-\beta)\prod_{i=1}^{n-1}\frac{1}{\sqrt{1-\Psi(-\beta)\kappa_i}}\int_{\mathbb{R}^{n-1}}\frac{p(\underline{v})}{q(\underline{v})}R(\underline{v})\,\hat{\varphi}(\underline{v})d\underline{v} \qquad (18)$$

where

$$\hat{\varphi}(\underline{v}) = \prod_{i=1}^{n-1}[\sqrt{1-\Psi(-\beta)\kappa_i}\ \varphi(u_i\ \sqrt{1-\Psi(-\beta)\kappa_i})] \qquad (19)$$

is the common density function of independent normal variables $\underline{V} = (V_1,\dots,V_{n-1})$ with mean and variance

$$E[V_i] = 0 \qquad (20)$$

$$\mathrm{var}\,[V_i] = \frac{1}{1-\Psi(-\beta)\kappa_i}$$

Finally, one obtains

$$\boxed{P_f = \Phi(-\beta)\prod_{i=1}^{n-1}\frac{1}{\sqrt{1-\Psi(-\beta)\kappa_i}}\,E[\frac{p(\underline{V})}{q(\underline{V})}R(\underline{V})]. \qquad (21)}$$

Observing that for large values of β there is $\Psi(-\beta) \approx \beta$, the first two factors are asymptotically (for large β) equivalent to Breitung's approximation eq. (5), while the third factor is the error term. If for small values of $\underline{v}$ there is (see eq. (7))

$$g(\underline{v},u_n) < 0 \quad \Longleftrightarrow \quad u_n < f(\underline{v}), \qquad (22)$$

it can easily be shown that the function

$$Z(\underline{v}) = \frac{p(\underline{v})}{q(\underline{v})} R(\underline{v}) \qquad (23)$$

satisfies the equations

$$Z(\underline{O}) = 1$$

$$\frac{\partial Z(\underline{O})}{\partial v_i} = 0 \qquad (1 \le i \le n-1)$$

$$\frac{\partial^2 Z(\underline{O})}{\partial v_i \partial v_j} = 0 \qquad (1 \le i,\ j \le n-1) \qquad (24)$$

Hence, $Z(\underline{v})$ is almost constant and close to one for small values of $\underline{v}$, and due to Breitung's result there is also $E[Z(\underline{V})]$ close to one. Therefore, the random variable $Z(\underline{V})$ usually has a small coefficient of variation, and $E[Z(\underline{V})]$ can effectively be determined by use of simulation techniques, the required number of simulation points being more or less independent of the failure probability P_f. This conclusion may hold even if the equivalence (22) is not strictly satisfied, since most of the probability content of $\{g<0\}$ is concentrated in the environment A of $\underline{u}^*$ (see eq. (7)).

Further Remarks

A) Since $\underline{u}^*$ is the Beta-point of $\{g<0\}$, there is $1-\beta\kappa_i \ge 0$ for $1 \le i \le n-1$. Therefore, there is usually $1-\Psi(-\beta)\kappa_i > 0$ for $1 \le i \le n-1$.

B) The derivations above are wrong if $1-\Psi(-\beta)\kappa_i \le 0$ for

some $1 \le i \le n-1$, since then the variance of V_i is infinite or negative. Then, the method must be modified. It is easily verified that the derivations also hold for any arbitrary value $\tilde{\kappa}_i$ in place of $\tilde{\kappa}_i$, with the only exception of the third line in eq. (24), provided that $1-\Psi(-\beta)\tilde{\kappa}_i > 0$ for $1 \le i \le n-1$. Therefore, one obtains the more general formula

$$P_f = \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1-\Psi(-\beta)\tilde{\kappa}_i}}\ E\left[\frac{p(\underline{V})}{\tilde{q}(\underline{V})}\ \tilde{R}(\underline{V})\right] \qquad (20^*)$$

where $\tilde{q}$ and $\tilde{R}$ are defined like q and R, but with $\tilde{\kappa}_i$ in place of κ_i. The value $\tilde{\kappa}_i$ should be chosen close to κ_i but with $1-\Psi(-\beta)\tilde{\kappa}_i > 0$.

C) The evaluation of $p(\underline{v})$ requires only one-dimensional integration. If the univariate normal integral Φ is available, it even suffices to determine, for each simulation point $\underline{v}$ of $\underline{V}$, the zeros of the function $h(u_n) = g(\underline{v}, u_n)$. If, for example, eq. (22) holds, there is $p(\underline{v}) = \Phi(f(\underline{v}))$.

References

[1] Breitung, K.: Asymptotic Approximations for Multivariate Domain and Surface Integrals. Proceedings of the 4th International Conference on Applications of Statistics and Probability in Soil and Structural Engineering (ICASP-4) Università di Firenze (Italy), 1983, pp. 755 - 769.

[2] Harbitz, A.: Efficient and Accurate Probability of Failure Calculation by Use of the Importance Sampling Technique. Proceedings of the 4th International Conference on Applications of Statistics and Probability in Soil and Structural Engineering (ICASP-4), Università di Firenze (Italy), 1983, pp. 825 - 836.

[3] Hasofer, A.M., Lind, N.C.: An Exact and Invariant First-Order Reliability Format. Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. 1, 1974, pp. 111 - 121.

[4] Hohenbichler, M.; Rackwitz, R.: Non-Normal Dependent Vectors in Structural Safety. Journal of the Engineering Mechanics Division, ASCE, Vol. 107, No. 6, 1981, pp. 1227 - 1249.

Asymptotic Approximations for the Maximum of the Sum of Poisson Square Wave Processes

by

K. Breitung

a) Introduction

In many reliability problems it is necessary to compute the distribution of the maximum of the sum of random processes. A survey of random processes used as models in load combination can be found in /6/. Here, a special model, the Poisson square wave process, will be examined and an approximation for the sum of these processes will be derived.

b) Definition of a Poisson Square Wave Process

A Poisson square wave process is defined by a homogeneous Poisson point process N(.) with intensity λ and a sequence Y_n (n=0,1,2,...) of independent, identically distributed random variables, which are independent of the point process. The Poisson square wave process x(t) is then defined by:

$$x(t) = Y_{N(0,t]} =$$

$$= Y_j, \text{ if } \quad N(0,t] = j.$$

The process is often used to describe loads which change in time (see [2], [6]). The model can be generalized to a vector process by taking random vectors instead of random variables.

c) The Model

Given is a n-dimensional Poisson square wave vector process $\underline{x}(t)=(x_1(t),\ldots,x_n(t))$. The i-th component is governed by a one-dimensional Poisson square wave process $x_i(t)$ with intensity λ_i and standard normally distributed amplitudes. All component processes are assumed to be independent of each other. The process $\underline{x}(t)$ changes its value at the jumps of the component processes $x_i(t)$.

Further, a limit state function $g(\underline{x})$ is given. Now, we consider the problem of determining the probability:

$$P(\ g(\underline{x}(t))>0,\quad 0\le t\le T).$$

If $g(\underline{x})$ describes the state of an engineering system with $g(\underline{x})<0$ defect and $g(\underline{x})>0$ intact, this is the probability that the system remains intact during the time period $[0,T]$.

To find approximations for this probability we consider the point process of the outcrossings out of the safe domain $S=\{\underline{x};\ g(\underline{x})>0\}$. First, the point process $N(A)$ of the jumps of $\underline{x}(t)$ is defined:

$$N(A):=\#\{t\in A;\ \begin{matrix}\text{There is a jump of one of the}\\ \text{component processes at t.}\end{matrix}\}\#$$

(where $\# A \#$ denotes the number of the elements of the set A). Further, the point process $U(A)$ of the outcrossings out of S is defined:

$$U(A):=\#\{t\in A;\ g(\underline{x}(t-0))>0>g(\underline{x}(t))\}\#$$

Now, since $\underline{x}(t)$ changes its value only at the jump times, we obtain:

$$1-P(g(\underline{x}(t))>0;\quad 0\le t\le T)=$$

$$=1-P(g(\underline{x}(0))>0,\quad U(0,T)=0)\le$$

$$\le P(g(\underline{x}(0))<0)+P(U(0,T)>0)$$

Since $U(0,T)\ge 0$, this yields:

$$\le P(g(\underline{x}(0))<0)+E(U(0,T))$$

Therefore, we obtain an upper bound for the probability of failure. In the following, we will show that under certain regularity conditions, $U(A)$ is approximately a Poisson point process, which yields:

$$P(U(0,T)=0)\approx e^{-\lambda_u\cdot T}$$

with λ_u denoting the intensity of the point process $U(A)$. Then:

$$1-P(g(\underline{x}(t))>0;0\le t\le T)\approx P(g(\underline{x}(0))<0)+(1-e^{-\lambda_u\cdot T})$$

Neglecting $P(g(\underline{x}(0))<0)$:

$$P(g(\underline{x}(t))>0;\ 0\le t\le T) \approx e^{-\lambda_u \cdot T}$$

This result means that the outcrossing point process for crossings out of a safe domain S is approximately a homogeneous Poisson point process.

d) The Asymptotic Behaviour of the Outcrossing Point Process

Consider a Poisson square wave vector process $\underline{x}(t)$. Given is further a limit state function $g(\underline{x})$ with $\min\limits_{g(\underline{x})=0} |\underline{x}| = 1$ and there is an unique beta point $\underline{x}_o$, (i.e. an unique point $\underline{x}_o$ with $|\underline{x}_o| = 1$ and $g(\underline{x}_o)=0$). The main curvatures of $g(\underline{x})$ at $\underline{x}_o$ are $\kappa_j (j=1,\dots,n-1)$. We define a sequence of limit state functions: $g_\beta(\underline{x}) = g(\beta^{-1}\underline{x})$. This gives a sequence of safe domains $S_\beta = \{\underline{x};\ g_\beta(\underline{x})>0\}$, failure domains $F_\beta = \{\underline{x};\ g_\beta(\underline{x})<0\}$ and limit state surfaces $G_\beta = \{\underline{x};\ g_\beta(\underline{x})=0\}$. For these domains the point processes $\overline{U}_\beta(A)$ of the outcrossings of the process $\underline{x}(t)$ out of the region S_β are defined:

$$\overline{U}_\beta(A) := \#\{t\epsilon A;\ g_\beta(\underline{x}(t-0))>0>g_\beta(\underline{x}(t))\}\#$$

We will examine the asymptotic behaviour of the standardized point processes $U_\beta(A)$, which are given by:

$$U_\beta(A) := \overline{U}_\beta(E_\beta^{-1}A) \quad \text{with } E_\beta = E(\overline{U}_\beta(0,1))$$

These processes converge towards a Poisson point process under some regularity conditions.

It will be assumed that

$$\underline{x}_o = \sum_{i=1}^{n} \alpha_i\, \underline{e}_i,\ \sum_{i=1}^{n} \alpha_i^2 = 1,\ |\alpha_i|>0 (i=1,\dots,n)$$

with $\underline{e}_i = (0,\dots,1,\dots,0)$
$\uparrow$ i-th component

This means that all direction cosines of $\underline{x}_o$ are not zero.

Let $\alpha_o = \min\limits_{i=1,\dots,n} |\alpha_i|$

Now, due to the definition of $\underline{x}_o$, it is clear, that there exists a $\delta>0$ and an $\varepsilon>0$, that for all $\underline{y}$ with $g(\underline{y})\le 0$ and $\sum_{i=1}^{n} y_i\, \alpha_i \le (1-\delta)|\underline{y}|$ (i.e. the cosine of the angle between $\underline{y}$ and $\underline{x}_o$ is less than $1-\delta$): $|\underline{y}| > (1+\varepsilon)|\underline{x}_o|$; elsewhere there would be another beta point. Let be defined:

$$\tilde{F}_\beta := \{\underline{x};\ g_\beta(\underline{x})<0,\ \sum_{i=1}^{n} x_i\, \alpha_i \le (1-\delta)|\underline{x}|\}.$$

$$F_\beta^* := F_\beta \setminus \tilde{F}_\beta, \text{ (see figure 1)}$$

The outcrossing point process $\overline{U}_\beta(A)$ is majorized by the point process $U_\beta^*(A)$:

$$U_\beta^*(A) := \#\{t\in A;\ \underline{x}(t-0)\neq\underline{x}(t),\ g_\beta(x(t))<0\}\#$$

i.e. $U_\beta^*(A) \ge \overline{U}_\beta(A)$.

Further, for the point processes

$$N_\beta(A) := \#\{t \in A;\ g_\beta(\underline{x}(t-0)) > 0,\ \underline{x}(t) \in \tilde{F}_\beta\}\#$$

we see, that:

$$E(N_\beta(A)) \le \sum_{i=1}^{n} \lambda_i |A| P(g_\beta(\underline{x}(t-0)) > 0,\ \underline{x}(t) \in \tilde{F}_\beta) \le$$

($|A|$ denotes the Lebesgue measure of A)

$$\le |A| \sum_{i=1}^{n} \lambda_i\ P(\underline{x}(t) \in \tilde{F}_\beta).$$

$$\le |A|\ (\sum_{i=1}^{n} \lambda_i)\ P(|\underline{x}(t)| > \beta(1+\varepsilon)).$$

$$= o(\phi(-\beta)).$$

For the point processes U^*_β (A):

$$E(U^*_\beta(0,1)) = \sum_{i=1}^{n} \lambda_i\ P(g_\beta(\underline{x}(t)) < 0) \sim$$

Using the results of |1|:

$$\sim (\sum_{i=1}^{n} \lambda_i)\quad \phi(-\beta) \prod_{j=1}^{n-1} (1-\kappa_j)^{-1/2} \qquad (\beta \to \infty)$$

Since U^*_β (A) $\ge \overline{U}_\beta$(A), this yields an asymptotic upper bound for $E(\overline{U}_\beta(0,1))$. Now, we show, that this bound is in fact an asymptotic approximation for $E(\overline{U}_\beta(0,1))$. This is done by showing,

that $E(U^*_\beta(0,1)-\overline{U}_\beta(0,1))$ can be neglected asymptotically for $\beta\to\infty$. The point process $U^*_\beta(0,1)-\overline{U}_\beta(0,1)$ consists of all points t of $U^*_\beta(0,1)$ with $g_\beta(\underline{x}(t-0))<0$, i.e. of all points t where before the jump $g_\beta(\underline{x}(t-0))<0$ and afterwards again $g_\beta(\underline{x}(t))<0$. Now, the probability of obtaining a value in the region $\tilde{F}_\beta$ after the jump, is of order $o(\phi(-\beta))$, as shown before. If the jump results in a point in F^*_β, it is obvious that this is due to the occurence of a new component amplitude, which has a value larger than $\alpha_o\beta/2$. Therefore, the expected number $E(U^*_\beta(0,1)-\overline{U}_\beta(0,1))$ is less than the expected number of jumps multiplied with $P(g_\beta(\underline{x}(t))<0)$ and the probability of a new component amplitude larger than $\alpha_o\beta/2$ which is $\phi(-\alpha_o\beta/2)$, plus a term of order $o(\phi(-\beta))$.

Therefore:

$$E(U^*_\beta(0,1)-\overline{U}_\beta(0,1)) \le$$

$$\le (\sum_{i=1}^{n} \lambda_i)\ (P(g_\beta(\underline{x}(t-0))<0)\phi(-\alpha_o\beta/2)+o(\phi(-\beta)))$$

Asymptotically:

$$\sim (\sum_{i=1}^{n} \lambda_i)\ (\phi(-\beta)\phi(-\alpha_o\beta/2) \prod_{j=1}^{n-1} (1-\kappa_j)^{-1/2} + o(\phi(-\beta)) =$$

$$= o(\phi(-\beta)) \quad (\beta\to\infty).$$

This shows the asymptotic equivalence of the two point processes $U^*_\beta(0,1)$ and $\overline{U}_\beta(0,1)$ and yields finally:

$$E(\overline{U}_\beta(0,1)) \sim \phi(-\beta) \prod_{j=1}^{n-1} (1-\kappa_j)^{-1/2} \left(\sum_{i=1}^{n} \lambda_i \right)$$

If there is more than one beta point, the result is obtained by summing up the contributions of the several points. Let $\underline{x}_1,\dots,\underline{x}_n$ be beta points on the surface; then the result is:

$$E(\overline{U}_\beta(0,1)) \sim \left(\sum_{i=1}^{n} \lambda_i \right) \phi(-\beta) \left(\sum_{l=1}^{K} \prod_{j=1}^{n-1} (1-\kappa_{l,j})^{-1/2} \right)$$

with the $\kappa_{l,j}$ $(l=1,\dots,K;\ j=1,\dots,n-1)$ being the main curvatures of $g(\underline{x})=0$ at $\underline{x}_l$.

To prove the convergence to a Poisson point process for the standardized processes $U_\beta(A)$, we use the convergence theorem in appendix I. To this purpose we replace $U_\beta(A)$ by an approximating point process $\tilde{U}_\beta(A)$:

$$\tilde{U}_\beta(A) := \#\{t \in E_\beta^{-1} A;\ g(\underline{x}(t)) < 0,\ |\underline{x}(t-0)| \leq \beta - \log \beta\}\ \#$$

This is the point process of all outcrossings out of S_β, which start from a point in the sphere around the origin with radius β-log β. (see figure 2) Using similar arguments as before, where the asymptotic equivalence of $U^*_\beta(0,1)$ and $\overline{U}_\beta(0,1)$ was shown, it can be found that $U_\beta(0,1)$ is asymptotically equivalent to $U_\beta(0,1)$. Therefore,

if $\tilde{U}_\beta(A)$ converges to a Poisson point process for β , this is valid also for $U_\beta(A)$. In appendix II it is shown, that $\tilde{U}_\beta(A)$ satisfies the conditions of the convergence theorem in appendix I and so the convergence is obtained. This yields:

$$P(U_\beta(A)=0) \to e^{-|A|} \qquad (\beta \to \infty).$$

e) Local Outcrossing Rate

For a given Poisson square wave vector process and safe domains S_β a local outcrossing rate can be defined on the limit surface G_β. For simplicity, we assume that S_β is convex and contains the origin. The local outcrossing rate $\lambda_\beta(\underline{x})$ $(\underline{x} \in G_\beta))$ gives approximately the expected number of outcrossings out of S through the neighbourhood of $\underline{x}$, i.e. if ΔG_β is an infinitesimal part of G_β containing $\underline{x}$, $\lambda_\beta(\underline{x})\,|\Delta G_\beta|$ gives the expected number of outcrossing through ΔG_β. (for details of the definition see /2/). Under the stated conditions this gives:

$$\lambda_\beta(\underline{x}) = \sum_{i=1}^{n} \lambda_i \prod_{\substack{j=1 \\ j \neq 1}}^{n} \varphi(x_j)\,\phi(-|x_i|)\,(1-\phi(-|x_i|)-\phi(h_i(\underline{x})))\,|\alpha_i|$$

with:

$$h_i(\underline{x}) = \begin{cases} \min\limits_{z \in \mathbb{R}} \{z;\ (x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n) \in S\}, & x_i \geq 0 \\ \max\limits_{z \in \mathbb{R}} \{z;\ (x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n) \in S\}, & x_i < 0 \end{cases}$$

α_i is the i-th component of the normal vector of the surface G_β at $\underline{x}$.

For large $|x_i|$ $1-\phi(-|x_i|)-\phi(h_i(\underline{x})) \approx 1$ and near $\beta\underline{x}_o$ the normal to G_β at $\underline{x}$ is approximately equal to $\underline{x}_o$. This yields, since $\underline{x}_o = \sum_{i=1}^{n} \alpha_i \underline{e}_i$:

$$\phi(-|x_i|) \approx \frac{\varphi(-x_i)}{|\underline{x}|\alpha_i} \quad \text{for} \quad |x_i|>0$$

Then, for points near $\beta\underline{x}_o$:

$$\lambda_\beta(\underline{x}) \sim \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n} \varphi(x_j) \frac{\alpha_i}{|\underline{x}|\ \alpha_i} =$$

$$= \sum_{i=1}^{n} \lambda_i (2\pi)^{-n/2} e^{-|x|^2/2}/|\underline{x}| =$$

$$= ((2\pi)^{-(n-1)/2} (\sum_{i=1}^{n} \lambda_i)\phi(-|\underline{x}|).$$

Therefore, for large safe domains S_β, $\lambda_\beta(\underline{x})$ is near $\beta\underline{x}_o$ asymptotically proportional to $\phi(-|\underline{x}|)$. Therefore, since only the neighbourhood of $\beta\underline{x}_o$ is of interest asymptotically, maximizing $\lambda_\beta(\underline{x})$ is the same asymptotically as minimizing $|\underline{x}|$, i.e. searching the beta point. This result shows that the

method proposed in /2/ is asymptotically equivalent to the method of searching the beta points.

Appendix I

Point Processes

a) Definition of a Point Process

A point process is a stochastic process, whose realizations consist of a set of points on the real axis. These points describe events in time. For example, the instants of arrivals of vehicles at a bridge are a realisation of a point process.

b) Stationary Point Processes

A point process N(.) is called stationary, if its distributions are not changed by a time shift (see /3/, p. 24)

c) Orderly Point Processes

A point process N(.) is called orderly or regular, if for all t:

$$\lim_{h \to 0} h^{-1} \, P(N(t,t+h]>1)=0.$$

(see /3/, p. 25)

d) Moments and Product Densities of Point Processes

Given is N(.), a stationary and orderly point process. The product densities (see /3/, p. 36) of this process are defined by:

$$p_K(t_1,\dots,t_K) = \lim_{h \to o} h^{-K} P(N(t_1,t_1+h)=1,\dots,N(t_K,t_K+h)=1)$$

for $t_i \neq t_j \quad (i \neq j)$.

These product densities give approximately the probability that near the K points $t_1,\dots,t_K$ are points of the point process N(.). These densities are related to the factorial moments of the point process by:

$$E(N(A)(N(A)-1)\dots(N(A)-K+1)) =$$

$$= \underbrace{\int_A \dots \int_A}_{\text{K-fold integral}} p_K(t_1,\dots,t_K)\,dt_1\dots dt_K$$

This yields:

$$E(N(A)) = \int_A p_1(t)\,dt = p_1(O)\,|A|$$

(since N(.) is stationary, $p_1(t)$ is a constant.

$$E(N(A)(N(A)-1) = \int_A \int_A p_2(t_1,t_2)\,dt_1\,dt_2$$

$= E[N^2(A) - N(A)]$

$= E[N^2(A)] - E[N(A)]$

Further:

$$P(N(A) = O) =$$

$$= \sum_{s=1}^{\infty} \frac{(-1)}{s!} \underbrace{\int_A \dots \int_A}_{\text{s-fold}} p_s(t_1,\dots,t_s)\,dt_1\dots dt_s + 1$$

This is proven by using the generating function of N(A). N(A) is an integer valued random variable, its generating function g(x) is given by (see /4/):

$$g(x| = E(x^{N(A)}) = \sum_{j=o}^{\infty} P(N(A) = j)x^j.$$

This yields:

$$\frac{d^s}{dx^s} g(x)\Big|_{x=1} = E(N(A)(N(A)-1)\dots(N(A)-\overset{s}{\cancel{K}}+1))$$

$$g(O) = P(N(A) = O)$$

Since g(x) is an analytic function for $|x| \leq 1$, we obtain by Taylor expansion:

$$g(O) = \sum_{s=o}^{\infty} \frac{(-1)^s}{s!} \frac{d^s}{dx^s} g(x)\Big|_{x=1} =$$

$$= 1 + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} \int_A \dots \int_A p_s(t_1,\dots,t_s)\,dt_1\dots dt_s$$

Further, for all l>O due to the mean value theorem:

$$P(N(A)=O) = 1 + \sum_{s=1}^{l} \frac{(-1)^s}{s!} \int_A \dots \int_A p_s(t_1,\dots,t_s)\,dt_1\dots dt_2 +$$

$$+ (-1)^{l+1} \frac{d^{l+1}}{dx^{l+1}} g(x)\Big|_{x=\vartheta_o} \text{ (with } \vartheta_o \in [0,1]).$$

Due to the definition of $g(x)$ as a series with nonnegative coefficients, for $0 \le x \le 1$ $g^{(s)}(x)$ is always nonnegative.

This yields for l even:

$$1 + \sum_{s=1}^{l} \frac{(-1)^s}{s!} \int_A \cdots \int_A p_s(t_1,\ldots,t_s)\,dt_1\ldots dt_s \ge P(N(A)=0)$$

l odd:

$$1 + \sum_{s=1}^{l} \frac{(-1)^s}{s!} \int_A \cdots \int_A p_s(t_1,\ldots,t_s)\,dt_1\ldots dt_s \le P(N(A)=0)$$

For $l = 1,2$:

$$1 - E(N(A)) \le P(N(A)=0) \le 1 - E(N(A)) + \frac{1}{2} E(N(A)(N(A)-1))$$

e) Convergence Theorem

Given is a sequence $N_K(.)$ $(K \ge 1)$ of stationary and orderly point processes with the following properties:

a) There exists a function $H(K): N \to R$ with $H(K) \to \infty$ $(K \to \infty)$ such that:

1) $E(N\ (0,H(K))) \to 1$ $(K \to \infty)$.

2) $$\int_0^{H(K)^{1/2}} \int_0^{H(K)^{1/2}} p_2^K(t_1,t_2)\,dt_1 dt_2 = O(H(K)^{-1/2})$$

with $p_2^K(t_1,t_2)$ denoting the two dimensional product density of the point process $N_K(.)$.

b) There exists a function $\psi(u): R \to R$ with $u^n\ \psi(u) \to 0$ for $u \to \infty$ and all n N, such that for all events A_K and B_K, where A_K depends only on the behaviour of the point process $N_K(.)$ until the time t and B_K depend only on the behaviour of $N_K(.)$ after the time $t+uH(K)$:

$$|P(A_K\ B_K) - P(A_K)P(B_K)| \le \psi(uH(K))P(A_K)$$

(i.e. the normalized processes are uniformly mixing with mixing coefficient $\psi(.)$).

Under these conditions the normalized point processes $N_K(.)$ with $\tilde{N}_K(A) = N_K(H(K)\cdot A)$ converge to a homogeneous Poisson process with intensity 1.

The first condition says that, if the time axis is scaled by taking $H(K)$ as time unit for $N_K(.)$, the mean value of $N_K(.)$ remains constant for $K \to \infty$. The second condition states that the product densities $p_2^K(t_1,t_2)$ of the point processes decrease fast enough that the probability of obtaining more than one point of the process $N_K(.)$ in the time interval $[0,H(K)^{1/2}]$ can be neglected asymptotically. The third condition states, that for distant parts of the time axis the behaviour of the processes is asymptotically independent.

Proof:

Using theorem 2.5 /5/ it is sufficient to show:

$$P(\tilde{N}_K(\bigcup_{i=1}^{l} A_i) = 0) \to P(N(\bigcup_{i=1}^{l} A_i) = 0)$$

with the A_i's being disjoint intervals. Given are l disjoint intervals $A_i=(a_i,b_i)$. Due to condition b) then:

$$|P(\bigcap_{i=1}^{l} \{\tilde{N}_K(A_i)=0\}) - \prod_{i=1}^{l} P(\tilde{N}_K(A_i)=0)| \to 0$$

for $K\to\infty$. Therefore, it is sufficient to show for one interval $A=(a,b)$:

$$P(\tilde{N}_K(A)=0) \to P(N(A)=0)=e^{-(b-a)}$$

From the last equation in paragraph c) it follows:

$$E(N_K(A)) \geq P(N_K(A)>0) \geq E(N_K(A)) - E(N_K(A)(N_K(A)-1))/2$$

Since the factorial moments are given by integrals over product densities:

$$E(N_K(A)) \geq P(N_K(A)>0) \geq E(N_K(A)) - \frac{1}{2} \int_{H(K)\cdot A} \int_{H(K)\cdot A} p_K^{\,2}(t_1,t_2)\,dt_1 dt_2$$

Due to condition a) this yields:

$$E(N_K(A)) \sim P(N_K(A)>0) \qquad (K\to\infty)$$

Let $A = (0,1]$

The interval is divided into parts, depending on K:

Let $T_K := H(K)^{-1/2}$, $t_K := H(K)^{-3/4}$

$l_K := [H(K)^{1/2}]$ ([x] denotes the largest integer not larger than x).

Then, (0,1] is partioned into:

$$(0,1] = \bigcup_{j=1}^{l_K} \underbrace{((j-1)(T_K+t_K), jT_K+(j-1)t_K]}_{=:A_{j,K}} \cup$$

$$\cup \bigcup_{j=1}^{l_K} \underbrace{(jT_K+(j-1)t_K, j(T_K+t_K)]}_{=:B_{j,K}} \cup$$

$$\cup \underbrace{(l_K(T_K+t_K), 1]}_{=:R_K}$$

Further:

$$A_K := \bigcup_{j=1}^{l_K} A_{j,K}, \quad B_K := \bigcup_{j=1}^{l_K} B_{j,K} \quad R_K$$

Then:

$$P(N_K(B_K)>0) \leq E(N_K(B_K)) \leq$$

$$\leq (l_K+1)E(N_K(0,t_K)) = (l_K+1)H(K)^{-3/4} =$$

$$= (H(K)^{-1/2}+1)H(K)^{-3/4} = o(1) \qquad (K\to\infty)$$

Therefore:

$$P(N_K(0,1)>0) = P(N_K(A_K(A_K)>0)+o(1)$$

It has to be shown:

$$P(\bigcap_{j=1}^{l_K} \{N_K(A_{j,K})=0\}) = P(N_K(A_K)=0) \to P(N(A_K)=0) =$$

$$= P(N(0,1)=0) = e^{-1}.$$

Due to the uniform mixing condition:

$$|P(\bigcap_{j=1}^{l_K} \{N_K(A_{K,j})=0\}) - \prod_{j=1}^{l_K} P(N_K(A_{K,j})=0)| \to 0$$

Since $N_K(.)$ is stationary:

$$\prod_{j=1}^{l_K} P(N_K(A_{K,j})=0) = (P(N_K(A_{K,1})=0))^{l_K} =$$

This gives then, using $E(N_K(A_{K,1})) \sim P(N_K(A_{K,1})>0)$:

$$= (1-E(N_K(A_{K,1}))+o(H(K)^{-1/2}))^{H(K)^{1/2}} =$$

$$= (1-E(N_K(A_{K,1})))^{H(K)^{1/2}} + o(1) =$$

$$= (1-H(K)^{-1/2})^{H(K)^{1/2}} + o(1) =$$

$$= e^{-1}+o(1)$$

This yields the final proposition.

Appendix II

It will be shown that the point processes $\tilde{U}_\beta(A)$ converge towards a homogeneous Poisson point process with intensity 1. This is done by demonstrating that the unnormalized point processes $U^*_\beta(A)$ satisfy the conditions of the convergence theorem in appendix I. Let be defined $H(\beta)=E_\beta^{-1}$. Then, due to the asymptotic equivalence of $U_\beta(A)$ and $\tilde{U}_\beta(A)$, $E(U^*_\beta(O,H(\beta)))\to 1 \quad (\beta\to\infty)$. Therefore, the first part of condition a) is fulfilled.

Now, to study the behaviour of the two dimensional product density $f_\beta(t_1,t_2)$ of $U^*_\beta(A)$, we separate it into two parts:

$$f_\beta(t_1,t_2) = P_1 f^1_\beta(t_1,t_2)+P_2 f^2_\beta(t_1,t_2)$$

with:

$$P_1 = P\underbrace{\begin{pmatrix}\text{All point processes } N_K(A) \;\; (K-1,\dots,n) \\ \text{have at least one point in } (t_1,t_2)\end{pmatrix}}_{=A}$$

$$P_2 = 1 - P_1 = P(\bar{A})$$

$f^1_\beta(t_1,t_2)$ conditional intensity of $U^*_\beta(A)$ under condition A

$f^2_\beta(t_1,t_2)$ conditional intensity of $U^*_\beta(A)$ under condition $\bar{A}$.

In the first case all oint processes $N_K(A)$ have at least one point between t_1 and t_2. The probability P_1 for this is larger than $1 - \sum_{i=1}^{n} e^{-\lambda_i(t_2-t_1)}$. The condition means that all loads have changed and the processes at these two points are independent. In this case, the conditional intensity $f^1_\beta(t_1,t_2)$ is, due to the independence, just the square of the one dimensional intensity $\lambda_\beta \sim E_\beta^{-1}$.

Therefore, asymptotically:

$$P_1 f^1_\beta(t_1,t_2) \le 2E_\beta^{-2} \qquad (\beta\to\infty)$$

In the second case at least one process did not change during (t_1,t_2). The probability for this is less than $\sum_{i=1}^{n} e^{-\lambda_i(t_2-t_1)}$. The conditional intensity in this case is less than the intensity of the occurence of a component larger than $\log(\beta)$ at t_2, since only in this case a jump of the point process $U^*_\beta(A)$ takes place, multiplied with the one dimensional intensity of $U^*_\beta(A)$ which is asymptotically equal to E_β^{-1}.

Therefore, for $\beta\to\infty$:

$$P_2 f_\beta{}^1(t_1,t_2) \le 2E_\beta^{-1}\Big(\sum_{i=1}^{n} e^{-\lambda_i(t_2-t_1)}\Big)\Big(\sum_{i=1}^{n}\lambda_i\Big)\phi(-\log\beta)$$

Together, this yields:

$$f_\beta(t_1,t_2) \le 2\Big(E_\beta^{-2}+E_\beta^{-1}\Big(\sum_{i=1}^{n} e^{-\lambda_i(t_2-t_1)}\Big)\Big(\sum_{i=1}^{n}\lambda_i\Big)\phi(-\log\beta)\Big)$$

For the integral:

$$\int_0^{H(\beta)^{1/2}}\int_0^{H(\beta)^{1/2}} f_\beta(t_1,t_2)\, dt_1\, dt_2 \le$$

$$\leq 2(E_\beta^{-1} + E_\beta^{-1/2}\ \phi(-\log(\beta)) =$$

since $E_\beta = H(\beta)^{-1}$:

$$= o(H(\beta)^{-1/2}) \qquad (\beta\to\infty).$$

This shows that the second part of condition a) is fulfilled. To check the second condition b), put $\Psi(u) = \sum_{i=1}^{n} e^{-\lambda_i u}$. Then, the condition is fulfilled.

Appendix III: References

/1/ Breitung, K.: Asymptotic Approximations for Multinormal Integrals, in this report.

/2/ Breitung, K. and Rackwitz, R.: Non-linear Combination of Load Processes, J. Struct. Mech., 10(2), 145 - 166, 1982.

/3/ Cox, D.R. and Isham, V.: Point Processes, Chapman and Hall, London, 1980.

/4/ Johnson, N.L. and Kotz, S.: Discrete Distributions, Wiley, New York, 1969.

/5/ Kallenberg, O.: Characterization and Convergence of Random Measures and Point Processes, Z. für Wahrscheinlichkeitstheorie u. verw. Gebiete, Vol. 27, 9-21, 1973.

/6/ Madsen, H.O.: Load Models and Load Combinations, Report R. 113, Structural Research Laboratory, Technical University of Denmark, Lyngby, Denmark, 1979.

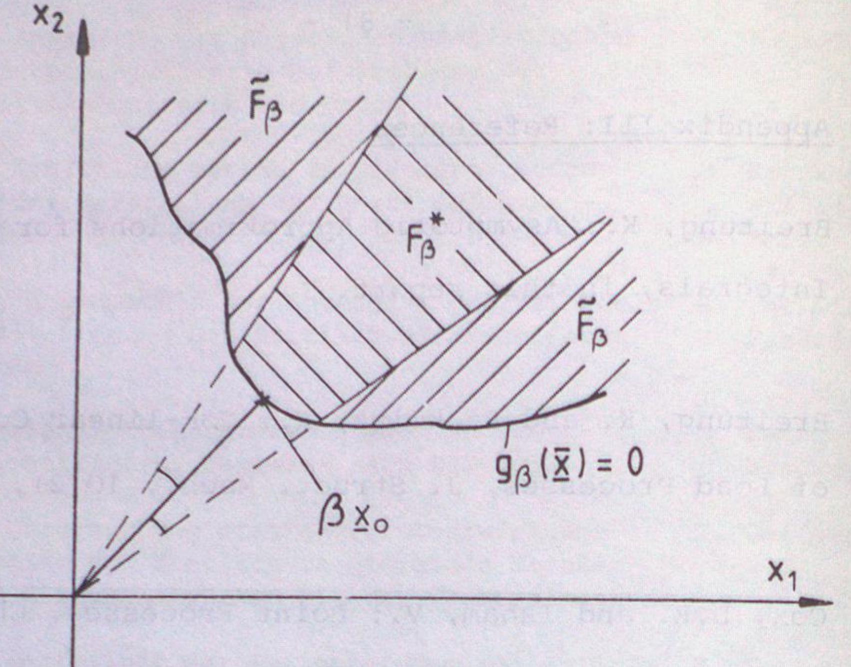


Figure 1

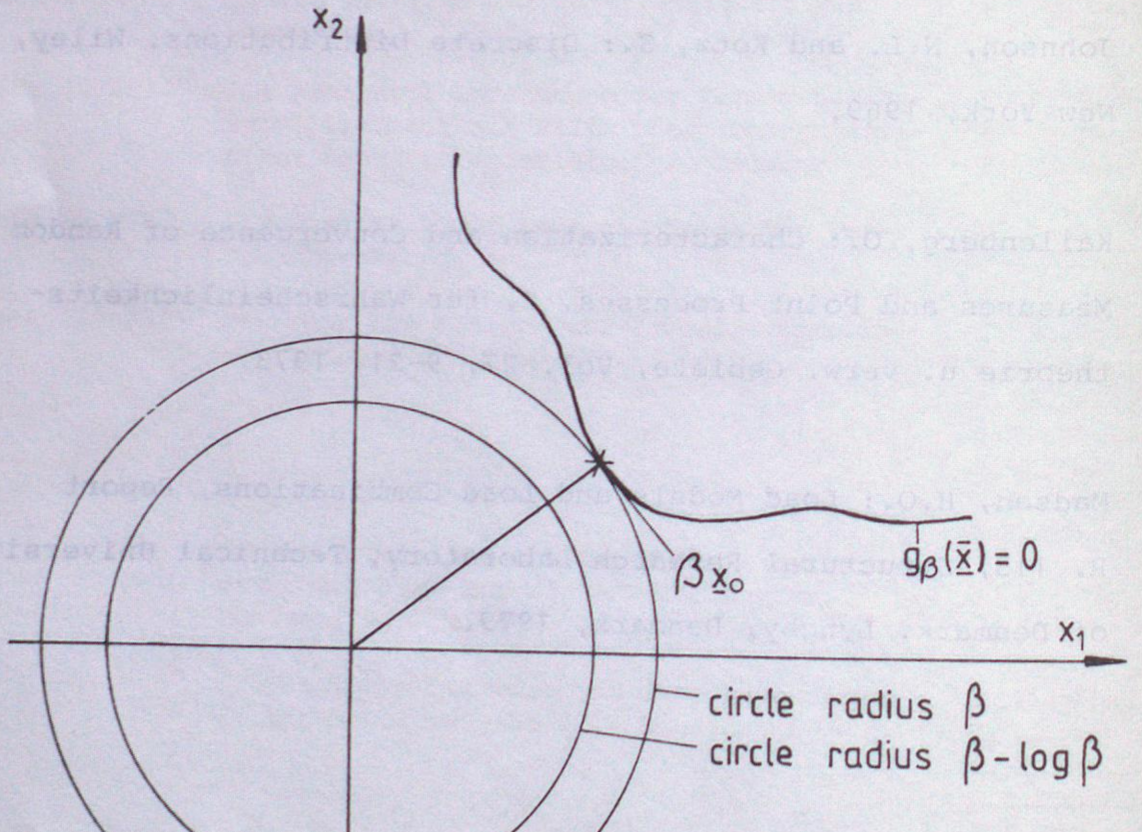


Figure 2

<u>In der Berichtsreihe sind bisher erschienen:</u>

Heft 1/1972*)	Mathematische Hilfsmittel zur Sicherheitstheorie	(7 Beiträge)
Heft 2/1973*)	Seminarvorträge zur Sicherheitstheorie	(7 Beiträge)
Heft 3/1973*)	Beiträge zur Zuverlässigkeit von Betonbauwerken	(11 Beiträge)
Heft 4/1973*)	Wahrscheinlichkeitstheoretische Untersuchung der Knicksicherheit von schlanken Stahlbetonstützen	O. Knappe R. Rackwitz
Heft 5/1974*)	Zur Sicherheitstheorie im konstruktiven Stahlbau	Chr. Petersen R. Hawranek
Heft 6/1975	Monte Carlo-Studie zur Zuverlässigkeit von durchlaufenden Stahlbetondecken in Bürogebäuden	U. Kraemer R. Rackwitz E. Grasser
Heft 7/1973*)	Festigkeitsverhalten von Fichtenbrettschichtholz; Teil 1: Versuchseinrichtung für Kurzzeit-Druckversuche	P. Glos W. Maier U. Weigle
Heft 8/1975*)	Sicherheit gedrückter Stahlstützen, Teil I: Grundlagenvergleich mit den Versuchen der Europäischen Konvention der Stahlbauverbände am Profil IPE 160	R. Hawranek Chr. Petersen
Heft 9/1975	Zur Sicherheit von statisch beanspruchten HV-Verbindungen unter besonderem Bezug auf die DASt-Richtlinien der Jahre 1956, 1963 und 1974	R. Hawranek
Heft 10/1975*)	Deterministische und stochastische Analyse des Tragverhaltens von Stahlbetonbauteilen unter Last- und Zwangbeanspruchungen	G. Thielen
Heft 11/1976*)	Statistische Untersuchungen von geometrischen Abweichungen an ausgeführten Stahlbetonbauteilen, Teil 1: Geometrische Imperfektionen bei Stahlbetonstützen	G. Maaß R. Rackwitz
Heft 12/1976*)	Wahrscheinlichkeitstheoretische Analyse der Lebensdauerverteilung nach Freudenthal et al.	B. Krzykacz M. Kersken-Bradley
Heft 13/1976*)	Studien für ein stochastisches Modell der Betondruckfestigkeit, Teil 1: Untersuchung zur Betondruckfestigkeit im Bauwerk sowie zum Qualitätsangebot von Beton	R. Rackwitz K.F. Müller G. Maaß
Heft 14/1976	Numerische Methoden probabilistischer Bemessungsverfahren und Sicherheitsnachweise	B. Fießler H. Hawranek R. Rackwitz

Heft 15/1976	Die Anwendung der Bayesschen statistischen Entscheidungstheorie auf Probleme der Qualitätskontrolle von Beton	R. Rackwitz
Heft 16/1977*)	Zur Ermittlung optimaler Sortiermethoden bei der Herstellung von Brettschichtbauteilen	M. Kersken-Bradley W. Maier
Heft 17/1977*)	Zwei Anwendungen der Zuverlässigkeitstheorie erster Ordnung bei zeitlich veränderlichen Lasten	R. Rackwitz B. Fießler
Heft 18/1977*)	Zuverlässigkeitsuntersuchungen an Brettschichtträgern, bemessen nach DIN 1052	M. Kersken-Bradley
Heft 19/1977*)	Zur Untersuchung stationärer Lastwirkungsprozesse von statisch reagierenden Straßen- und Eisenbahnbrücken mit der Spektralmethode	T. Geidner
Heft 20/1977*)	Zur Verteilung der Parameter der Wöhlerlinie für St 37 und St 52	R. Quel
Heft 21/1977*)	Einige Beiträge zur Zuverlässigkeit von Bauwerken	R. Rackwitz
Heft 22/1977*)	Die genäherte Berechnung der Versagenswahrscheinlichkeit mit Hilfe rotationssymmetrischer Grenzzustandsflächen 2. Ordnung	B. Fießler H.-J. Neumann R. Rackwitz
Heft 23/1978	Zur Statistik der Lage und Größe der Vorspannbewehrung	G. Maaß
Heft 24/1978*)	Beiträge zur Risiko- und Zuverlässigkeitsbeurteilung von Tragwerken	J. Bauer, H.S.Choi H.Kappler,O.W.Knappe H.-J.Melzer H.Panggabean K.-H.Reichmann G.I.Schuëller R.F.Schwarz
Heft 25/1978*)	Neuere Ergebnisse aus der Theorie der Normung, Beiträge zum 43. SFB-Kolloquium, München, 1.März 1978	
Heft 26/1978*)	Statistische Methoden und ihre Anwendungen im Ingenieurbau - Teil I: Grundlagen	H. Kappler O.W.Knappe K.-H.Reichmann G.I.Schuëller R.F.Schwarz
Heft 27/1978*)	Statistische Methoden und ihre Anwendungen im Ingenieurbau - Teil II: Anwendungen	H.-J.Melzer H.-J.Niemann G.I.Schuëller
Heft 28/1978*)	Statistische Untersuchungen von geometrischen Abweichungen an ausgeführten Stahlbetonbauteilen, Teil II: Meßergebnisse geometrischer Abweichungen bei Stützen, Wänden, Balken und Decken des Stahlbetonhochbaus	G. Maaß

Heft 29/1878*)	Zuverlässigkeitstheorie 1. Ordnung und nichtnormale Vektoren, quadratische Grenzzustandsflächen, Tragsysteme bzw. außergewöhnliche Einwirkungen	B. Fießler B. Krzykacz H.-J. Neumann R. Rackwitz
Heft 30/1978*)	Beitrag zu einem Zuverlässigkeitsmodell für Dackbalken aus Brettschichtholz unter besonderer Berücksichtigung seines Festigkeitsverhaltens	W. Maier
Heft 31/1978*)	Darstellung und Auswertung von Schneehöhenmessungen in der Bundesrepublik Deutschland	H. Luy R. Rackwitz
Heft 32/1978*)	Niveauüberschreitungen von Summen von Zufallsprozessen, deren Pfade Sprungfunktionen sind	K. Breitung
Heft 34/1978*)	Optimierung von Sicherheitsnachweisen mit besonderem Bezug auf den Tragsicherheitsnachweis von Stützen aus Formstahl	R. Hawranek
Heft 35/1978*)	Zur Bestimmung von Festigkeitsverhalten von Brettschichtholz bei Druckbeanspruchung aus Werkstoff- und Einwirkungskenngrößen	P. Glos
Heft 36/1978*)	Zur Zuverlässigkeit von ermüdungsbeanspruchten Konstruktionselementen in Stahl	R. Quel
Heft 37/1978*)	Zur Anwendung der Spektralmethode auf Lasten und Beanspruchungen bei Straßen- und Eisenbahnbrücken	T. Geidner
Heft 38/1979*)	Theoretische Grundlagen für die Bestimmung des Bemessungswertes von Bauteilwiderständen aus Versuchen	R. Rac
Heft 39/1979*)	Neuere Erkenntnisse in der Risiko- und Zuverlässigkeitsbeurteilung von Tragwerken	J. Baue H. S. C H. Kapp H.-J. M H. Panc K.-H. R G. I. Schueller R. F. Schwarz
Heft 42/1979*)	Überschreitungsrate für Lastprozesse mit rechteckförmigen Impulsen	K. Breitung R. Rackwitz
Heft 43/1979*)	Das Programmsystem FORM zur Berechnung der Versagenswahrscheinlichkeit von Komponenten von Tragsystemen	B. Fießler

Heft 44/1979*)	Beitrag zur Bestimmung der Zuververlässigkeit nichtlinearer Strukturen unter Berücksichtigung kombinierter stochastischer Einwirkungen	R. F. Schwarz
Heft 45/1979*)	Nichtlineare Lastkombination von Poissonschen Erneuerungsprozessen	K. Breitung R. Rackwitz
Heft 46/1980	Überlegungen zur Zuverlässigkeit von Tragsystemen	U. Kraemer
Heft 47/1980*)	Ausgewählte Beispeile der probabilistischen Behandlung von Probelmen der Bruchmechanik, Lastkombination und Strukturdynamik	P. Kafka H. Krawinkler H.-J. Melzer G.F. Oswald W. Schmitt G.I. Schuëller Y.K. Wen
Heft 48/1980	Zur zuverlässigkeitstheoretischen Untersuchung von Seriensystemen	M. Hohenbichler
Heft 49/1980*)	Seminar über stochastische Tragwerksmechanik und Zuverlässigkeit	
Heft 51/190 *)	Benchmark Study on Methods to Determine Collaps failure probability of Redundant Structures	M. Grimmelt G.I. Schuëller
Heft 52/1980	Numerische Unsicherheitsanlyse von Böschungen	B. Peintinger R. Rackwitz
Heft 53/1980 *)	Zwei Schätzprobleme bei Gauß'schen Folgen	S. Pöhlmann R. Rackwitz
Heft 54/1980	Tragwerksschwingungen unter Zufallslast mit nicht-Gaußischer Wahrscheinlichkeitsverteilung	H.-J. Melzer
Heft 55/1981	Zwei Anwendungen der Bayesschen Statistik	K. Breitung
Heft 56/1981	Beanspruchbarkeit von Bauteilquerschnitten bei streuenden Kenngrößen des Kraftverformungsverhaltens innerhalb des Querschnitts	M. Kersken-Bradley
Heft 57/1981	Zufällige Systemeigenschaften bei dynamisch belasteten Tragwerken mit einem Freiheitsgrad	E. Großmann R. Juli
Heft 58/1981	Studien zur Zuverlässigkeit von redundanten Tragsystemen	R. Rackwitz M. Hohenbichler S. Gollwitzer B. Peintinger
Heft 60/1981	Die Genauigkeit und Leistungsfähigkeit der Spektralmethode	R.J. Scherer G.I. Schuëller
Heft 61/1981	Zur Modellierung des Festigkeitsverhaltens von Bauholz bei Druck-, Zug- und Biegebeanspruchung	P. Glos

Heft 62/1982	Festigkeitsverhalten von Brettschichtholz unter zweiachsiger Beanspruchung Teil 1: Ermittlung des Festigkeitsverhaltens von Brettelementen aus Fichte durch Versuche	R. Spengler
Heft 63/1982	First Order Second Moment Approximation in Reliability of Structural Systems; Critical Review and Some Reliability Bounds	K. Dolinski
Heft 64/1982	Konzepte zur Berechnung der Versagenswahrscheinlichkeit bei Ermüdungsbeanspruchung	K. Reppermund
Heft 65/1983	Ein Konzept zur Zuverlässigkeitsanalyse von Tragwerken unter Berücksichtigung der Werkstoffermüdung	G.F. Oswald
Heft 66/1982	Predictive Distribution of Strength under Control	R. Rackwitz
Heft 67/1982	Response Surfaces in Structural Reliability	R. Rackwitz
Heft 68/1983	Entwicklung von Regeln zur Kombination stochastischer Lasten für die Tragwerksbemessung	B. Fießler

*) Diese Hefte sind bereits vergriffen.