LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

BERICHTE
zur
ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

ON THE INTEGRATION OF MULTINORMAL DENSITIES OVER EQUALITIES

G. Schall, R. Rackwitz

UPDATING GENERAL FIRST-ORDER PROBABILITY INTEGRALS BY IMPORTANCE SAMPLING

S. Gollwitzer, R. Rackwitz

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ON THE INTEGRATION OF MULTINORMAL DENSITIES OVER EQUALITIES

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ABSTRACT: The available first–order techniques for the integration of multinormal densities on surfaces are reviewed. Asymptotic second–order corrections are then derived for general intersections of equalities and inequalities. The approximate results are compared with exact results for a simple but extreme example involving the non–central chi–square distribution. It is found that volume integrals over probability densities appear to be slightly more accurate than integrals over surfaces.

1. Introduction

Many reliability problems involve the computation of simple volume integrals of probabilities over certain domains in multidimensional spaces. This can conveniently be done by making use of first–order or asymptotic second–order concepts in the numerical analysis as described in [1,2]. Those domains must be given as inequalities.

However, the computation of outcrossing rates for Gaussian processes involves the determination of surface integrals which in some special cases involves simply an integration of probability densities over equalities [3]. In the reliability analysis of failure trees for systems the calculation of the probability of intersections of failure domains is required. But some further restrictions sometimes are given in the form of equalities, for example, if the initiating event is a discrete event. In elastic–plastic structures the analysis of successive yielding of hinges until collapse requires that the hinges except the temporary last one fulfill the yield conditions which again are given in terms of equalities. Discrete events depending on further random variables and to be introduced as equalities are also encountered in human error control as shown in [4]. Furthermore, the calculation of conditional probabilities involving equalities sometimes is required when updating structural reliability in view of special forms of inspection results [5].

In this note suitable computation schemes for integrals of the type

$$I(D) = \int_D \varphi_n(\mathbf{x})\, ds(\mathbf{x}) \tag{1}$$

where

$$D = E \cap F = \bigcap_{i=1}^{\ell} \{E_i\} \cap \bigcap_{j=\ell+1}^{m} \{F_j\} = \bigcap_{i=1}^{\ell} \{e_i(X) = 0\} \cap \bigcap_{j=\ell+1}^{m} \{g_j(X) \le 0\}$$

are developed. Herein, $\mathbf{X} = (X_1,...,X_n)^T$ is a n–dimensional random vector with given distribution function. $ds(\mathbf{x})$ means surface integration. Since the vector $\mathbf{X}$ can always be transformed into an independent standard normal vector $\mathbf{U} = (U_1,...,U_n)^T$ as shown in [6], it is sufficient to determine the integral in the so–called standard space. Of course, it is $\ell < n$. The surface integral in eq. (1) will be understood as a surface integral of the first kind. Thus, the results will be applicable precisely in the examples mentioned before. First, the available first–order results will briefly be reviewed also further highlighting the meaning of eq. (1). Then, asymptotic second–order results will be derived. Finally, an importance sampling scheme is discussed for updating volume and surface integrals over arbitrary domains.

2. Review of first–order results for integrals of multinormal densities over equalities

Let $\mathbf{U} = (U_1,...,U_n)^T$ be an independent, standard normal vector with density $\varphi_n(\mathbf{u}) = \Pi_{i=1}^{n} \varphi(u_i)$. A first–order approximation of integrals as in eq. (1) can be deduced from the considerations in [7]. In particular, let the events in eq. (1) be given by their linearizations

$$l_{e,i}(\mathbf{u}) = \boldsymbol{\alpha}_{e,i}^T \mathbf{u} - \boldsymbol{\alpha}_{e,i}^T \mathbf{u}^x = 0 \text{ and } l_{g,j}(\mathbf{u}) = \boldsymbol{\alpha}_{g,j}^T \mathbf{u} - \boldsymbol{\alpha}_{g,j}^T \mathbf{u}^x \le 0,$$

respectively, where $\mathbf{u}^x$ is the so–called joint β–point (see [2]) and $\boldsymbol{\alpha}_{.,.}$ the normalized gradient of these functions in $\mathbf{u}^x$. Alternatively, as in "crude" first–order reliability [2], the individual β–points $\mathbf{u}_i{}^x$ can be chosen with usually some loss of accuracy. Then, by viewing a volume integral as the limit of an integration of a sequence of appropriately chosen surface integrals the integral in eq. (1) can be given as:

$$\begin{aligned} I(E \cap F) &= I\Big(\bigcap_{i=1}^{\ell} \{e_i(\mathbf{u}) = 0\} \cap \bigcap_{j=\ell+1}^{m} \{g_j(\mathbf{u}) \le 0\}\Big) \\ &= \frac{\partial^\ell}{\partial c_1 \dots \partial c_\ell} P\Big(\bigcap_{i=1}^{\ell} \{l_{e,i}(\mathbf{u}) \le 0\} \cap \bigcap_{j=\ell+1}^{m} \{l_{g,j}(\mathbf{u}) \le 0\}\Big) \\ &= \frac{\partial^\ell}{\partial c_1 \dots \partial c_\ell} P\Big(\bigcap_{i=1}^{\ell+m} \{Z_i \le c_i\}\Big) \end{aligned} \tag{2}$$

with $c_i = -\boldsymbol{\alpha}_{e,i}{}^T \mathbf{u}_i{}^x$ $(i = 1,...,\ell)$ and $z_i = \boldsymbol{\alpha}_i{}^T \mathbf{u}$ and $z_i - c_i = 0$ the generic linear representation of the failure surfaces. It is worth noting that this interpretation does not cover all applications of surface integrals in probability theory. Evaluation of eq. (2) is best be performed by applying a recursive scheme. By conditioning on the first variable the first step of calculation is:

$$I(E_1 \cap F) = \frac{\partial}{\partial c_1}\Big[\int_{-\infty}^{c_1} P\Big(\bigcap_{i=2}^{\ell+m} (z_i \le c_i | z_1 = s)\Big)\, \varphi(s) ds\Big] \tag{3}$$

Introducing y_i as variables which are stochastically independent of z_1 for $i \ne 1$ with $z_i = \rho_{1i}\, z_1 + \sqrt{1-\rho_{1i}^2}\, y_i$ and $\rho_{1i} = \boldsymbol{\alpha}_1{}^T \boldsymbol{\alpha}_i / (\|\boldsymbol{\alpha}_i\|\, \|\boldsymbol{\alpha}_1\|)$ eq. (3) can be rewritten as:

$$I(E_1 \cap F) = \varphi(c_1)\, P\Big(\bigcap_{i=2}^{\ell+m} y_i \le \frac{c_i - \rho_{1i}\, c_1}{\sqrt{1-\rho_{1i}^2}}\Big) \tag{4}$$

This scheme can be applied ℓ times so that the first–order result finally becomes

$$I(E \cap F) = \prod_{i=1}^{\ell} \frac{\varphi(\tilde{c}_i)}{\sigma_i}\, P\Big(\bigcap_{j=\ell+1}^{m} \tilde{y}_j \le \tilde{c}_j\Big) \tag{5}$$

where the quantities $\tilde{c}_i$, $\sigma_i = (1 - \tilde{\rho}_{ij}^2)^{1/2}$ and $\tilde{y}_j$ indicate the modifications to be performed in each step. At most a m–ℓ–dimensional normal integral needs to be evaluated. Note that explicit surface integration is avoided, i.e. by a suitable parameterization of the equalities.

3. Asymptotic second–order results

a. One equality constraint

Let $\mathbf{U} = (U_1,...,U_n)^T$ be an independent, standard normal vector with density $\varphi_n(\mathbf{u}) = \Pi_{i=1}^n \varphi(u_i)$ and $E = \{e(\mathbf{u}) = 0\}$ the boundary of the domain $F = \{g(\mathbf{u}) \le 0\}$. The integral

$$I(E) = \int_{\mathbb{R}^n \cap E} \varphi_n(\mathbf{u})\, d\mathbf{u} \tag{6}$$

here is written as a surface integral of the first kind

$$I(E) = \int_E \varphi_n(\mathbf{u})\, ds(\mathbf{u}) \tag{7}$$

where $ds(\mathbf{u})$ means surface integration.

Define now the maximum point (β–point) of $\varphi_n(\mathbf{u})$ on E as usual by $\mathbf{u}^* = \min\{\|\mathbf{u}\|\}$ for $E = \{\mathbf{u}: e(\mathbf{u}) = 0\}$ with $\beta = \|\mathbf{u}^*\|$. Assume that by an orthogonal transformation the β–point lies on the negative u_n–axis, i.e. $\mathbf{u}^* = (0,0,...,-\|\mathbf{u}^*\|)^T$, the equality $e(\mathbf{u}) = 0$ is at least twice differentiable near the $\mathbf{u}^*$ and the mixed second derivatives of $e(\mathbf{u}^*)$ vanish which again can be achieved by a suitable orthogonal transformation [8]. The curvatures of $e(\mathbf{u})$ in $\mathbf{u}^*$ then are given by:

$$\kappa_i = -\frac{1}{\frac{\partial e(\mathbf{u}^*)}{\partial u_n}} \frac{\partial^2 e(\mathbf{u}^*)}{\partial u_i^2} \quad \text{for } i = 1,2,...,n-1 \tag{8}$$

In view of the results in [2,9] it is admissible to choose the following approximate parameterization for u_n around $\mathbf{u}^*$:

$$u_n = e^{-1}(u_1,...,u_{n-1}) = c(u_1,...,u_{n-1}) \approx -\beta + \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i u_i^2 \tag{9}$$

This is a special hyperparaboloid. The gradient of $e(\mathbf{u})$ is not vanishing in $\mathbf{u}^*$. The surface integral can then be calculated from the following volume integral:

$$I(E) = \int_{\mathbb{R}^{n-1}} \varphi_n(u_1,..,u_{n-1}, c(u_1,...,u_{n-1}))\, (1 + \sum_{i=1}^{n-1} (\frac{\partial c(\mathbf{u}^*)}{\partial u_i})^2)^{1/2} \prod_{i=1}^{n-1} du_i \tag{10}$$

Applying the usual scaling by the factor $b > 1$ as proposed in [2,9] and making the transformation $v_i = u_i/b$ yields:

$$I(bE) = b^{n-1} \int_{\mathbb{R}^{n-1}} \varphi_n(bv, c(bv))\, (1 + \sum_{i=1}^{n-1} (\frac{\partial c(bv^*)}{\partial v_i})^2)^{1/2} \prod_{i=1}^{n-1} dv_i \tag{11}$$

In writing

$$(2\pi)^{n/2}\, \varphi_n(bv, c(bv)) = \exp[-\frac{1}{2}(\sum_{i=1}^{n-1} (bv_i)^2 + (-\beta + \frac{b^2}{2}\sum_{i=1}^{n-1} (\kappa_i v_i^2))^2)]$$

and expanding the second term to first order

$$(-\beta + \frac{b^2}{2}\sum_{i=1}^{n-1} (\kappa_i v_i^2))^2 \approx \beta^2 - b^2\beta \sum_{i=1}^{n-1} (\kappa_i v_i^2) + ...$$

one can, since the last term in eq. (11) is unity, easily verify that it is:

$$I(bE) \sim b^{n-1}\, \varphi(\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-1/2} \prod_{i=1}^{n-1} \int_{-\infty}^{+\infty} (1-\beta\kappa_i)^{1/2} \varphi(bv_i\,(1-\beta\kappa_i)^{1/2})\, dv_i \tag{12}$$

The last product term in eq. (12) is also unity. Therefore, one has for large β ($b = 1$) the approximation:

$$I(E) \approx \varphi(\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-1/2} \tag{13}$$

This result was first given in [10] and is re–derived here on somewhat different lines.

b. Several equality constraints

Assume now that $E = \{\bigcap_{i=1}^{l} E_i\}$ with $l \le n-1$ and $E_i = \{e_i(\mathbf{u}) = 0\}$. The intersection E is a $n-l$–dimensional surface to which exactly the considerations of the foregoing section apply. The result is

$$I(E) \approx \prod_{i=1}^{l} \varphi(\alpha_i^T \mathbf{u}^*) \prod_{i=1}^{n-l} (1-\beta\kappa_i)^{-1/2} \tag{14}$$

where the α_i are the gradients of the equalities in $\mathbf{u}^*$

c. One equality constraint intersected with a domain F

In order to demonstrate the generalization for a mixture of equality and inequality constraints consider now the integral

$$I(E \cap F) = \int_{\mathbb{R}^n \cap E \cap F} \varphi_n(\mathbf{u})\, d\mathbf{u} \tag{15}$$

where $E = \{e_1(\mathbf{u}) = 0\}$ and $F = \cap_{i=2}^{m} F_i$ with $F_i = \{g_i(\mathbf{u}) \le 0\}$ and $g_i(\mathbf{0}) > 0$ for at least one i. Also, $e_1(\mathbf{u})$ and $g_i(\mathbf{u})$ are at least twice differentiable near the point $\mathbf{u}^*$ obtained from:

$$\mathbf{u}^* = \min\{\|\mathbf{u}\|\} \text{ for } \{u\colon e_1(\mathbf{u}) = 0 \cap \bigcap_{i=2}^{m} g_i(\mathbf{u}) \le 0\} \tag{16}$$

and there is $e_1(\mathbf{u}^*) = 0$ and $g_i(\mathbf{u}^*) = 0$ for i=2,...,k ≤ m.

In closely following the arguments in [11], assume, without loss of generality, that the normalized gradient of $e_1(\mathbf{u}^*)$ is

$$\boldsymbol{\alpha}_1 = \mathbf{e}_1$$

with $\mathbf{e}_1$ the unit vector and $\mathbf{u}^* = \Sigma\, a_i\, \mathbf{e}_i$ with $a_i \neq 0$ for $i = 2$. Hence, the vectors $\mathbf{e}_1$, $\mathbf{u}^*$, $\mathbf{e}_3$,..., $\mathbf{e}_n$ are linearly independent. Introducing the scaling factor as before and rewriting eq.(15) as a surface integral gives:

$$I(b(E \cap F)) = b^{n-1} \int_{E \cap F} \varphi_n(b\mathbf{u})\, ds(\mathbf{u}) \tag{17}$$

Applying the mappings

$$S_1(\mathbf{u}) = (u_1 - u_1^*, \tfrac{1}{2}\sum_{i=1}^{n} u_i^2, u_3, \ldots, u_n)$$

$$S_2(\mathbf{u}) = (e(\mathbf{u}), \tfrac{1}{2}\sum_{i=1}^{n} u_i^2, u_3, \ldots, u_n)$$

one finds that for

$$T(\mathbf{u}) = (S_2(\mathbf{u})^{-1})^{T}\, S_1(\mathbf{u})$$

it is

$$T(\mathbf{u}) = \mathbf{u}^*$$

and since around $\mathbf{u}^*$, $S_1(\mathbf{u}) = S_2(T(\mathbf{u}))$

$$T_1(\mathbf{u}) = u_1 = u_1^*$$

$$T_j(\mathbf{u}) = u_j \text{ for } j \ge 3$$

$$\sum_{i=1}^{n} T_i(\mathbf{u})^2 = \sum_{i=1}^{n} u_i^2$$

A suitable parameterization of T around $\mathbf{u}^*$ then is

$$H = \{\mathbf{u}\colon u_1 = u_1^*\} \cap \bigcap_{j=2}^{k} \{g_j(T(\mathbf{u})) \le 0\} \tag{18}$$

and, therefore,

$$\int_{E \cap F} \varphi_n(b\mathbf{u})\, ds(\mathbf{u}) = \int_{H} \varphi_n(b\mathbf{u})\, r(\mathbf{u})\, d\mathbf{u} =$$

$$= \frac{\partial}{\partial u_1^*} \int_{-\infty}^{u_1^*} \int_{F^*} \varphi(b\mathbf{u} \mid u_1 = u_1^*)\, \varphi(u_1)\, r(\mathbf{u})\, d\mathbf{u}\, du_1$$

$$= \varphi(bu_1^*) \int_{F^*} \varphi(b\mathbf{v})\, r(u_1^*, \mathbf{v})\, d\mathbf{v} \tag{19}$$

where $\mathbf{v} = (u_2, \ldots, u_n)$ and $F^* = \cap_{j=2}^{k} F_j^*$ with $F_j^* = \{g_j(T(u_1^*, \mathbf{v})) \le 0\}$ and $r(u_1^*, \mathbf{v}) = (\det[J(T(\mathbf{u}^*))^{T} J(T(\mathbf{u}^*))])^{\frac{1}{2}} = 1$ with the Jacobian $J(T(\mathbf{u})) = \mathbf{I}$. Due to $r(u_1^*, \mathbf{v}) = 1$ the last integral is the usual probability integral for intersections [2]. Using $u_1^* = \mathbf{a}_1^{T}\mathbf{u}^*$ one obtains after some algebra the result

$$I(E \cap F) \approx \varphi(a_1^T u^*)\, \Phi_{k-1}(c;R)\, \frac{1}{(\det(I-D))^{\frac{1}{2}}} \qquad (20)$$

where

$$c = \{u^{*T} a_s - (u^{*T} a_1)\,(a_s^T a_1);\ s = 2,\ldots,k\}$$

$$R = \{a_s^T a_t - (a_s^T a_1)\,(a_t^T a_1);\ s,t = 2,\ldots,k\}$$

$$D = \{\sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(u^*)}{\partial u_i \partial u_j};\ i,j = k+1,\ldots,n\}$$

$$I = \{\delta_{ij};\ i,j = k+1,\ldots,n\}$$

a_r $(r = 1,\ldots,k)$ is the normalized gradient of constraint r at the point u^x and the $\gamma_s < 0$ are obtained as the solution of:

$$u^* = \sum_{s=1}^{k} \gamma_s a_s \quad \text{for } s = 1,2,\ldots,k$$

k is the total number of equality constraints and active inequality constraints in u^x. The first two factors represent the first–order result which could have been derived in a more elementary manner as illustrated before.

d. General case

The result is invariant under orthogonal transformations. Therefore, recursive application for the case of more than one equality constraint yields the general result. The forth expression in eq. (19) also suggests to condition on all equality constraints simultaneously so that the result eq. (20) obtains the form

$$I(\bigcap_{i=1}^{\ell} (e_i = 0) \cap \bigcap_{j=\ell+1}^{m} (g_j \le 0)) \approx \varphi_\ell(c_\ell : \Sigma_{\ell\ell})\, \Phi_q(c_{q|\ell} : \Sigma_{q|\ell})\, \frac{1}{(\det(I-D))^{\frac{1}{2}}} \qquad (21)$$

with

$$q = m - \ell;\ \ell \le k \le m$$

$$c = \begin{bmatrix} c_\ell \\ c_q \end{bmatrix};\ R = \begin{bmatrix} R_{\ell\ell} & R_{\ell q} \\ R_{q\ell} & R_{qq} \end{bmatrix}$$

$$\Sigma_{\ell\ell} = R_{\ell\ell}$$

$$c_{q|\ell} = R_{q\ell}^T R_{\ell\ell}^{-1} c_\ell$$

$$\Sigma_{q|\ell} = R_{qq} - R_{q\ell}^T R_{\ell\ell}^{-1} R_{\ell q}$$

D as in eq. (20) and

$$\varphi_\ell(c_\ell : \Sigma_{\ell\ell}) = (2\pi)^{\ell/2}\, (\det(\Sigma_{\ell\ell}))^{-1/2} \exp[-\tfrac{1}{2}\,(z^T \Sigma_{\ell\ell}^{-1} z)]$$

the ℓ–dimensional standard normal density.

4. Example

In order to demonstrate the validity and accuracy of the derived approximations they are compared with some exact "probabilities" on surfaces. One such exact result are the probability densities on non–central hyperspheres in multidimensional Gaussian spaces which are given by the derivative with respect to the radius r of the non–central chi–square distribution. For even n a suitable formula is

$$\frac{d}{dr} P_{\chi'^2}(r^2 | n, \lambda) =$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} e^{-\lambda/2} (\tfrac{\lambda}{2})^j \sum_{k=0}^{\frac{n}{2}+j-1} \frac{1}{k!} e^{-r^2/2} (\tfrac{1}{2})^k r\, (r^{2k-2}\, k\, r^{2(k-1)}) \qquad (22)$$

where r is the radius of the sphere, n the dimension of the standard space and $\lambda = \Sigma\, \delta_i^2$ the non–centrality parameter with the δ_i the offsets of the center of the sphere from the origin. In fig. 1 the ratio of the approximations according to the one dimensional versions of eq. 4 and eq. 20 and the exact result eq. 22 is presented. The curves are calculated for a constant ratio $r/\beta = 5$. For comparison, the corresponding results for the volume integral are given in fig. 2. It is seen that even in this extreme example (same curvatures in all directions) the ratios converge to unity for growing β for all n. The results for the volume integral are slightly better than for the surface integral.

The same example also allows to check the general case. In particular, if

$$E \cap F = \{(\sum_{i=1}^{n} (U_i + \delta_i)^2 - r^2 = 0) \cap \bigcap_{j=2}^{m} (U_j \geq 0)\}$$

it is easy to see that the exact probability density of this event for $\delta = (-\beta, -\epsilon, \ldots, -\epsilon)$ with $\epsilon \to 0$ is:

$$I(E \cap F) = \frac{\frac{d}{dr} P_{\chi'^2}(r^2 | n, \lambda)}{2^m}$$

On the other hand, the second order approximation must be

$$I(E \cap F) \approx \frac{\varphi(\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-\frac{1}{2}}}{2^m}$$

which can also be verified numerically by applying eq. (20). Therefore, fig. 1 is also valid in this case.

4. Conclusions

The asymptotic results for volume integrals as described in [2] can be generalized to general surface integrals in a straightforward manner. The asymptotic surface integrals represent a significant improvement of the corresponding first–order results. Convergence to the exact results with growing distance of $\|\mathbf{u}^*\|$ as defined in eq. (16) appears to be slightly slower for surface integrals than for volume integrals. However, arbitrarily exact results can always be achieved by a correction obtained by importance sampling in line of the investigations described in [12] at the expense of some more numerical effort.

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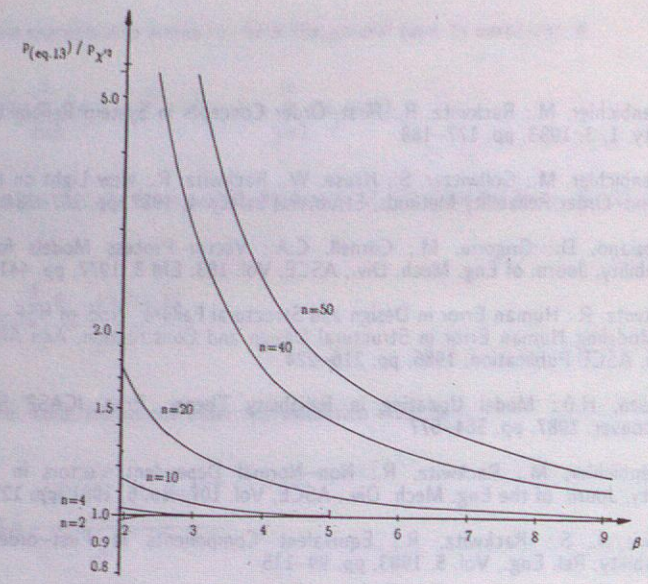

Fig 1: Ratio of denstiy functions $P_{(eq.13)} / P_{\chi'^2}$ with respect to β and n as parameter

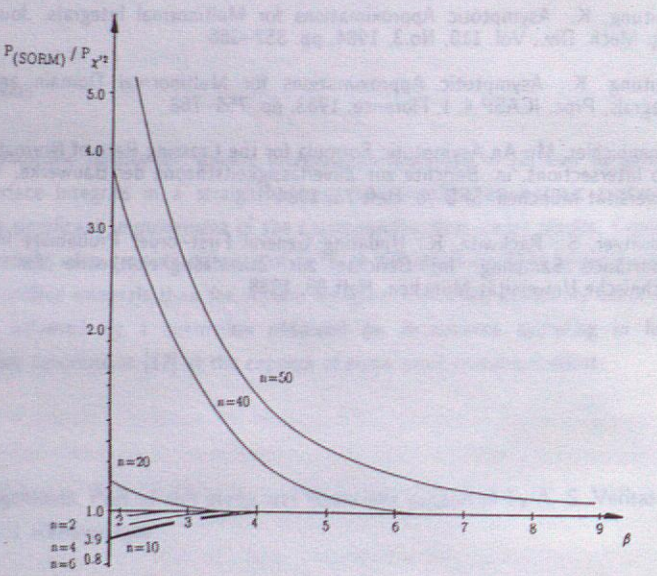

Fig 2: Ratio of CDF $P_{(SORM)} / P_{\chi'^2}$ with respect to β and n as parameter

UPDATING FIRST–ORDER PROBABILITY INTEGRALS BY IMPORTANCE SAMPLING

S. Gollwitzer and R. Rackwitz

ABSTRACT: An importance sampling scheme is designed for the determination of a correction factor for the probability of arbitrary domains in multidimensional spaces evaluated according to first–order reliability techniques. The correction factor essentially is independent of the magnitude of the probability to be estimated and independent of the shape of the integration domain. It is shown that the numerical effort grows by a factor of roughly ten if the coefficient of variation of the correction factor for the probability estimate is to be reduced below 5%. The importance sampling scheme is illustrated at two examples.

1. Introduction

In recent years the idea of importance sampling put forward first by Shinozuka [1] in the area of structural reliability has triggered an interesting development to replace or update first– and/or second–order estimates for the probabilities of failure domains as obtained by the methods described in [2] and [3]. For the considerations to come it is, without loss of generality, assumed that the probability space is a space in independent standard normal variables [4] and the failure domain is given in a suitable form, i.e. by $F = \{g(\mathbf{U}) \leq 0\}$ or by $F = \{\cup \cap F_{ij}\}$ with $F_{ij} = \{g_{ij}(\mathbf{U}) \leq 0\}$ and $\mathbf{U} = (U_1, U_2, \ldots, U_n)^T$ the standard normal vector. The basic importance sampling idea is to use information about the neighborhood of the most "interesting" region or, in terms of the terminology in [2,3], the most likely failure point(s) (or simply the β–point(s)) which contribute(s) most to the probability integral of interest. In the following it will be assumed that the "important" region(s) exist(s) and can be determined uniquely. The importance sampling methods proposed so far distinguish themselves by the degree of information they take from the properties of the failure surface around the β–point(s). In [1] and later, in slight improvement, in [5] it is suggested to sample around the β–point and count sample points in the failure domain. A valid point weighted with the ratio of the multinormal density and the sampling density produces a

rather efficient probability estimate. The method has the advantage to provide an error estimate which is difficult to obtain directly for the classical methods in [2] on [3]. Only information about the location of the β–point is used. According to the classification given in [6] this method will be denoted by method A in the sequel. A somewhat different, more efficient approach has been suggested in [7] where the potential error in first– or second–order methods is estimated by comparing point–wise the exact probability with the probability of an approximation of the failure domain yielding a correction factor. Convenient approximations of the failure domain are either linearizations or quadratic forms of the failure surfaces. In following [6], the use of linearizations will be denoted with method B while the use of quadratic approximations is associated with method C. In [6] a comparison is made between the various methods indicating that either importance sampling method can, in fact, be an efficient tool to update and even replace first– or second order methods provided that the β–point(s) and, if required, some further properties of the failure domains are known. The efficiency increases from method A to method C. In [6] it was found that the importance sampling updates increase the computation time by at least an order of magnitude as compared with the simpler estimates. Nevertheless and even if the probability estimates according to the methods described in [2] or [3] turn out to be sufficiently accurate from a practical point of view it is worthwhile to supplement those methods by methods which update and/or quantify their error.

The studies in [6] and [7] concentrated on simple differentiable failure surfaces. In this note we generalize the arguments in [6] to arbitrary failure domains that is to unions of intersections. We shall, in following the basic idea in [7], determine a correction factor together with its error in terms of the coefficient of variation of the correction factor. The general form of a probability estimate then is:

$$P(F) = P(A)\,\frac{P(F)}{P(A)} = P(A)\,C \qquad (1)$$

where P(A) is the probability of an approximation of the failure surface and the second factor is the correction factor to be determined by importance sampling. In general, this factor is estimated by:

$$C = \frac{P(F)}{P(A)} = \frac{1}{N}\sum_{i=1}^{N}\frac{H(\mathbf{u}_i \in F)}{H(\mathbf{u}_i \in A)} \qquad (2)$$

where N is the number of sampling points and the precise meaning of the quantities H in the numerator and the denominator, respectively, is explained below.

While the importance sampling method where only information about the location of the β–point is needed (method A), is shown in [6] to be the least efficient among the mentioned alternatives it must be considered as a robust method because it requires only simple function calls which determine whether a sampled point is in the failure domain or not. For given or only approximate location of the β–point and smooth failure surfaces one can expect "hits" with probability around 0.5 at each sampling. An even higher hit probability can be achieved if one samples only outside a sphere with radius β (see [8]). Unfortunately, the same argument no more holds if the failure domain is an intersection domain which, in extreme cases, can be a very small subdomain of $\mathbb{R}^n$. Then, the probability of "hits" in this method can reduce substantially and tends to zero in the mentioned extreme cases. The variance of the estimate then increases significantly. The method can become inefficient and, therefore, is discriminated as a suitable, general importance sampling method for cases where the failure domain is of a more complex nature. The probability of "hits" is substantially increased if a directional sampling scheme is used. For example in following [7], one can sample on a plane going through the origin with normal equal to the direction cosines of the β–point. In this case it is not only possible to integrate analytically in one dimension. It is also possible to use information about the curvature of the failure surface in the β–point. This scheme must be considered as the most efficient importance sampling method for smooth failure surfaces. It produces the smallest coefficient of variation of the probability estimate among the alternatives because the variability of the sampling density is adjusted to the actual curvatures as indicators for the "important" sampling domain. It has even been found that this adjustment is necessary in cases of extreme curvatures because not adjusted sampling densities most likely produce too many points outside the important region and there is no indication, e.g. by the magnitude of the coefficient of variation of the estimate, that the important region has not yet been sampled unless the number of sampled points is very large. The same phenomenon must be expected for arbitrary failure domains and especially for intersections with small probability content. Therefore, a sampling scheme must be designed with a large likelihood to hit the important region. Also, we require the scheme to be as robust as possible.

So far, method C where curvature information in the β–point is also used must be considered as the most efficient importance sampling method for smooth failure surfaces. It produces the smallest coefficient of variation of the probability estimate among the alternatives because the variability of the sampling density is adjusted to the actual curvatures as indicators for the "important" sampling domain. The same must be expected for arbitrary failure domains. However, if the probability of the approximation cannot be evaluated exactly, the final probability estimate necessarily will have a bias which cannot be removed even asymptotically for large N. If unbiasedness is considered a vital requirement for any importance sampling method it is, therefore, mandatory to choose an approximation A for the failure domain F for which the exact probability can be determined. If A is

approximated by a quadratic form as in [3] the probability can be computed only asymptotically exact ($P(.) \to 0$ or $P(.) \to 1$) with adequate numerical effort. For this reason the subsequent derivations for a correction factor will be concerned only with appropriately linearized failure domains for which a sufficiently accurate basis for the probabilities of the approximation exists in the form of the multinormal integral evaluated numerically according to the methods described in [9] or [10].

2. Intersections

An arbitrary intersection with failure domain

$$F = \left\{ \bigcap_{j=1}^{m} [g_j(U) \le 0] \right\} \tag{3}$$

is either a "small" intersection, i.e. with at least one $g_j(\mathbf{0}) \ge 0$ or a "large" intersection if $g_j(\mathbf{0}) < 0$ for all j. Small intersections and large intersections require separate treatment. For both cases it is supposed in the following that the first–order approximations of the active constraints exist. The linearized constraints are denoted as the "linear form" of the intersection in the sequel.

2.1. Small Intersection

If m denotes the number of constraints g_j of the intersection, $k \le m$ linearizations define the linear form. The linearizations are either the linearizations of the "active" constraints at the common β–point $\mathbf{u}_C^*$ (i.e. where $\cap_{j=1}^{k} g_j(\mathbf{u}_C^*) = 0$) and, possibly, also the linearizations of "inactive" constraints (i.e. where $g_j(\mathbf{u}_C^*) < 0$). Alternatively, individual linearizations of the constraints (corresponding to "crude" FORM as defined in [3]) can be used. In the latter case the common β–point $\mathbf{u}_I^*$ of the linear form has to be evaluated which also yields the number of active constraints in this case.

The linear form first is used for the multinormal integral which evaluates P(A) as defined in eq. (1), i.e.:

$$P(F) \approx \Phi_k(-\beta\,;\, R)\, C \tag{4}$$

with

$$\beta = \{-\mathbf{u}_{(j)}^{*\,T}\, \alpha_j\} \quad \text{and} \quad R = \{\rho_{i,j}\} = \{\alpha_i^T\, \alpha_j\}$$

and $P(A) = \Phi_k(-\beta\,;\, R)$. The α_j are the normalized ($\|\alpha_j\|=1$) gradients of the constraints at the point(s) $\mathbf{u}^*_{(j)}$.

The information supplied by the linear form is now used to define a sampling direction and the mean values and standard deviations of the sampling densities. An orthogonal transformation is performed such that the point $\mathbf{u}^*$ lies on the n–th axis, i.e. has coordinates $(0,\ldots,0,\|\mathbf{u}^*\|)^T$. Then, the integral for the correction factor is written as:

$$C = \int_F \frac{1}{P(A)}\, \varphi_n(\mathbf{v})\, d\mathbf{v} = \int_{\mathbb{R}^{n-1}} \frac{1}{P(A)} \int_{\mathbb{R}} 1_F(\tilde{\mathbf{v}}, v_n)\, \varphi(v_n)\, dv_n\, \varphi_{n-1}(\tilde{\mathbf{v}})\, d\tilde{\mathbf{v}} \tag{5}$$

where $1_F(\tilde{\mathbf{v}}, v_n)$ is the indicator function of the failure domain and $\tilde{\mathbf{v}} = (v_1,\ldots v_{n-1})^T$. The integration over v_n can be carried out analytically and by introducing a suitable sampling density $\psi(.)$ one obtains:

$$C = \int_{\mathbb{R}^{n-1}} \frac{1}{P(A)} \int_{\mathbb{R}} 1_F(\tilde{\mathbf{v}}, v_n)\, \varphi(v_n)\, dv_n\, \varphi_{n-1}(\tilde{\mathbf{v}})\, d\tilde{\mathbf{v}}$$

$$= \int_{\mathbb{R}^{n-1}} \frac{\Phi(-\lambda_F(\tilde{\mathbf{v}}))}{P(A)}\, \frac{\varphi_{n-1}(\tilde{\mathbf{v}})}{\psi_{n-1}(\tilde{\mathbf{v}})}\, \varphi_{n-1}(\tilde{\mathbf{v}})\, d\tilde{\mathbf{v}}$$

$$= E\left[\frac{\Phi(-\lambda_F(\tilde{\mathbf{v}}))}{P(A)}\, \frac{\varphi_{n-1}(\tilde{\mathbf{v}})}{\psi_{n-1}(\tilde{\mathbf{v}})}\right]$$

$$\sim \frac{1}{N} \sum_{i=1}^{N} \frac{\Phi(-\lambda_F(\tilde{\mathbf{v}}_i))}{P(A)}\, \frac{\varphi_{n-1}(\tilde{\mathbf{v}}_i)}{\psi_{n-1}(\tilde{\mathbf{v}}_i)} \tag{6}$$

where $\lambda_F(\tilde{\mathbf{v}})$ is the distance from the origin to the failure surface and the $\tilde{\mathbf{v}}_i$ are simulated points according to $\psi_{n-1}(.)$ (see also figure 3).

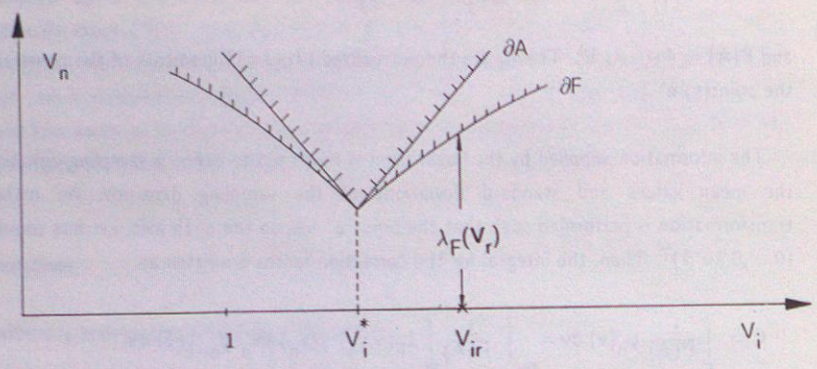

Figure 3: Illustration of sampling scheme for intersections

$\psi_{n-1}(.)$ now needs to be chosen such that the important region is sampled most. For example, points are sampled according to the normal distribution on the plane going through the point $\mathbf{u}^*$ and perpendicular to some representative value α_s of the gradients of the active constraints. More specifically, the sampling points are obtained from independent normal variables V_i with the coordinates of $\mathbf{u}^*$ as their mean values. Points on the failure surfaces are then sought in the direction of α_s. The sampling direction α_s is chosen as the mean of the gradients of the active constraints $\alpha_{j,a}$. The simplest choice for the standard deviation of the V_i is a value of one which also is chosen as the upper limit. The angles between α_s and the $\alpha_{j,a}$ however, allow a more efficient assessment of the standard deviations of the V_i. For example, the simple linear relationship

$$\sigma(V_i) = \vartheta_i / \pi \tag{7}$$

where $\sigma(V_i)$ is the standard deviation and ϑ_i the largest angle spanned by the i–th components of the gradients of the linear form. It concentrates sampling points around $\mathbf{u}^*$. The concentration becomes denser for decreasing angles. In this way it is assured that $0 \le \sigma \le 1$ for $0 \le \vartheta \le \pi$. Note that $\vartheta > \pi$ would violate the definition of $\mathbf{u}^*$ as the minimal distance of the intersection of the linearizations to the origin.

Now denote $\tilde{\mathbf{v}}_s$ a sampled point on a plane through the origin with gradient α_s and λ_A the minimal distance from $\tilde{\mathbf{v}}_s$ to the intersection of the linear form in direction α_s. This distance can be computed analytically from the α_j, $\mathbf{v}^*$ and possibly the $\mathbf{v}^*{}_j$ from $\mathbf{v}_s$ and from α_s. The point $\mathbf{v}_s + \lambda_s\, \alpha_s$ lies on one of the linearizations, l_t say, and defines the starting solution for a Newton search in the direction α_s to the point $\tilde{\mathbf{v}}_t = \tilde{\mathbf{v}}_s + \lambda_s\, \alpha_s$ where $g_t(\mathbf{v}_t) = 0$. For this

point it is checked whether it is in the intersection of the m nonlinear constraints. If not, the Newton search is repeated with respect to the constraint (t') for which $g_t(\mathbf{v}_t) > 0$ to obtain a suitable point.

2.2. Large Intersection

For a "large" intersection (all $g_i(\mathbf{0}) < 0$) a slightly different sampling scheme is proposed. The linear form is given by the linearizations of all m constraints at their individual β–points $\mathbf{u}^*{}_j$ with

$$\beta = \{-\mathbf{u}_j\, \alpha_j\} = \{-\|\mathbf{u}_j\|\} \quad (\text{all } \beta_j < 0) \tag{8}$$

The complement of F in this case is interpreted as a "small" union:

$$\overline{F} = \left\{ \bigcup_{j=1}^{m} [g_j(\mathbf{U}) > 0] \right\} = \left\{ \bigcup_{j=1}^{m} [-g_j(\mathbf{U}) \le 0] \right\} \tag{9}$$

and P(F) is approximated by:

$$P(F) = 1 - P(\overline{F}) \approx 1 - [\Phi_m(+\beta\,;\, R)\, C] \ge 1 - \sum_{j=1}^{m} \Phi_1(+\beta_j) \tag{10}$$

Here, for each constraint the sampling density is obtained from the V_i with the coordinates of the $\mathbf{u}^*{}_j$ as mean values and standard deviations $\sigma(V_i)$ all set to one. Note that a better estimate of σ could be given according to [3,7] which requires the second–order derivatives of the g_j. For each constraint a number of points $\mathbf{u}_{j,s}$ are sampled on a plane through the origin with gradient α_j and, in analogy to the small intersection case, the $\lambda_{j,s}$ denote the distance from the sampled point to the nonlinear form in the direction $\alpha_s = \alpha_j$. The root of $g_j(\mathbf{u}_{j,s} + \lambda_{j,s}\, \alpha_j) = 0$ must again be found by a Newton search. As for the small intersection it is checked whether the point $\mathbf{u}_t = \mathbf{u}_{j,s} + \lambda_{j,s}\, \alpha_j$ is in the large intersection (or alternatively in the small union). If $g_i(\mathbf{u}_t) > 0$ for any $i \ne j$, the Newton search is repeated with respect to constraint number i (but still in direction α_j) which yields a point $\mathbf{u}_{t,i}$. Consequently $\lambda_{j,s}$ has to be updated. $\lambda_{j,s} = \alpha_i^T\, \mathbf{u}_{t,i}$ now is the distance of $\mathbf{u}_{t,i}$ to a plane through the origin with gradient α_i.

The right hand side of eq. (10) also allows to distribute the total computational effort according to the contribution of the constraints to the probability of the small union. For example, one could take

$$N_j = N \frac{\Phi_1(\beta_j)}{\sum_{j=1}^{m} \Phi_1(\beta_j)} \quad , \quad \sum_{j=1}^{m} N_j = N \tag{11}$$

with N the number of samples for the large intersection and N_j the number of sample points for each constraint provided that the denominator in eq. (11) is smaller than unity. Otherwise one has to normalize appropriately. The correction factor C for the case of a large intersection then is:

$$C = \frac{1}{N} \sum_{j=1}^{m} \sum_{s=1}^{N_j} \frac{\Phi(-\lambda_F(\tilde{\mathbf{v}}_{j,s}))}{P(A)} \frac{\varphi_{n-1}(\tilde{\mathbf{v}}_{j,s})}{\psi_{n-1}(\tilde{\mathbf{v}}_{i,s})} \tag{12}$$

3. Union of Small Intersections

If the failure set is given as a (minimal) cut set of small–probability intersection domains (cuts)

$$F = \left\{ \bigcup_{t=1}^{m} \bigcap_{j=1}^{k(t)} F_{t,j} \right\} \tag{13}$$

sufficiently narrow probability bounds can usually be derived on the basis of the material given in [2] and [3] provided that the cut set probabilities all are relatively small. Unfortunately, those bounds cannot be improved easily (see [2]). Therefore, a check of the accuracy of the results can be especially interesting for m very large and/or $P(F_{t,j})$ not small, i.e. when these bounds can become unsatisfactory wide. The sampling procedure is very similar to the one just described. The total number of sampling points first is distributed to the various cuts according to eq. (11) with obvious modifications. Then, the individual cuts are treated as before with the additional task to check whether projected points are also in other cuts and are closer to the plane on which the sampling points have been generated. If projected points are, in fact, also in other cuts one proceeds as described for the case of large intersections.

4. Presence of Equalities

If equality constraints are present the importance sampling scheme becomes even simpler. A sampled point then must lie on the intersection of the equality constraints and must fulfill the inequality constraints. The integral is written as:

$$I(D) = I(A) \frac{I(D)}{I(A)} = I(A) \int \mathbf{1}_D(\mathbf{u}) \frac{\varphi_l(\mathbf{u})}{I(A)} \frac{\varphi_{n-l}(\mathbf{u})}{\psi_{n-l}(\mathbf{u})} \psi_{n-l}(\mathbf{u})\, ds(\mathbf{u}) \tag{14}$$

If there are ℓ equality constraints $n-\ell$ independent sample points are generated according to some normal sampling density which, of course, is centered around the joint β–point. The rest of the coordinates is found by solving the system of equations for the equality constraints. A suitable starting point for the non–linear equation solver is the solution of the corresponding linear form. If no detailed information about the behavior of the intersection of the equalities in the neighborhood of the β–point is available the obvious choice for the standard deviation of the sampling densities is unity. This essentially is a sampling according to method *B*. It certainly can be improved but we will make no attempt to do so herein because some numerical investigations have shown that very little extra efficiency can be achieved.

5. Examples

In Ref. [3] the reliability of a simple life–line system is studied. Its failure domain is given by a minimal cut set consisting of five small intersections in the union and the exact failure probability is $P_f = 2.25 \cdot 10^{-6}$ (for details see [3]). This example was recalculated with N=500 sample points (100 points per intersection on the average) with the following results (C.o.V. = Coefficient of Variation in %):

Crude FORM (individual linearizations of the constraints)

without importance sampling: $3.84 \cdot 10^{-6} \le P_f \le 3.91 \cdot 10^{-6}$

with importance sampling: $P_f = 1.90 \cdot 10^{-6}$ (C.o.V. = 5%)

FORM (linearizations of the constraints at $\mathbf{u}_c^*$)

without importance sampling: $2.62 \cdot 10^{-6} \le P_f \le 2.63 \cdot 10^{-6}$

with importance sampling: $P_f = 2.16 \cdot 10^{-6}$ (C.o.V. = 2%)

It can be seen that the importance sampling scheme gives a fairly good approximation of the exact result. SORM reproduces the exact result (see [3]). It remains to compare the computational effort in terms of the number n_g of evaluations of the constraint functions.

Crude FORM without/with importance sampling: $n_g = 125 \;/\; 3621$

FORM without/with importance sampling: $n_g = 246 \;/\; 3202$

SORM: $n_g = 396$

Due to the small number of U–space variables in this example (n = 6) the second–order method uses less computation time. For FORM with importance sampling n_g is smaller than for crude FORM because the linearizations of the constraints at $\mathbf{u}_C^*$ provide better starting solutions for the Newton iterations.

As a second example the (small) intersection of the failure domain of the "Daniels"–system with elastic–brittle elements as studied in [3] is recalculated. It should be emphasized that this test example is particularly awkward in the sense that first–order methods produce unusual bad results and even second–order methods are not completely satisfying in the non–asymptotic case because the failure surfaces have significant curvatures also outside the neighborhood of the β–point. The following table 1 repeats the values given in [3] and compares them with the results of the importance sampling scheme in terms of the equivalent safety index $\beta_E = -\Phi^{-1}(P_f)$. N=100 sample points were taken in all cases for comparison.

In table 1, *I* denotes individual linearization (crude FORM). *II* stands for a linearization of the active constraints at $\mathbf{u}_C^*$ and linearizations of the inactive constraints. In this case quadratic approximations (SORM) of the active constraints do not improve the FORM–result. II_s is the importance sampling improvement of *II*. The values in brackets are the numbers n_g of evaluations of the constraint functions.

Table 1: Results for Intersection Problem ("Daniels"–system)

m	$\lvert\mathbf{u}_C^*\rvert$	*I*	*II*	II_s (N = 100)	C.o.V.(II_s)	exact
1	2.00	2.00	2.00 (9)	2.00 (209)	0.0%	2.00
3	1.50	1.63	1.82 (36)	1.83 (440)	0.9%	1.82
5	1.34	1.59	1.83 (84)	1.90 (763)	3.4%	1.87
10	1.19	1.63	1.84 (275)	2.04 (1225)	6.3%	2.03
15	1.18	1.69	2.05 (756)	2.13 (2021)	7.0%	2.19
20	1.16	1.73	2.20 (1322)	2.25 (2891)	8.7%	2.32

The importance sampling scheme reduces the error of the classical methods which here is primarily due to the non–asymptotic conditions (low reliability level). If the equivalent safety index were larger than a value of around 4, say, SORM would reproduce the exact results. Crude FORM–results, however, are quite inaccurate for higher dimensions m. The coefficient of variation of the probability estimate increases with the dimension m of the intersection. One can recognize that the exact results always are within the 2σ–bounds. A substantial increase of the number of sample points, of course, reproduces the exact result. Finally, it is noted that the number of function calls for the importance sampling updates is roughly ten times larger than for the classical methods if the same quality of estimates is required. For very large m the importance sampling scheme would require comparatively less function calls.

6. Summary and concluding remarks

The importance sampling scheme proposed in the foregoing requires the β–point(s) together with at least the active gradients of the constraints to be known. If these are known or at least known approximately a correction factor can be determined updating the semi–analytical results obtained by first–order reliability techniques. The correction factor is an unbiased estimate. Its coefficient of variation can be made arbitrarily small. As a rule the correction requires roughly ten times the effort of simple first–order estimates to reduce the coefficient of variation below 5% which parallels the findings in [6] for smoothly bounded failure domains. It is presumed that by choosing a slightly modified sampling density which uses more information of the failure surfaces in the neighborhood of the β–point(s), the scheme can still be made slightly more efficient. But any such further improvement must be judged in view of the additional effort required for the importance sampling update. For any further refinement of the method there also appears to be some loss of robustness in the sense that then the β–point(s) must be located with great accuracy.

The proposed importance sampling update of probability estimates by the inexpensive first–order reliability methods thus not only supplements and/or provides a check against the exact result. It also can be used as a check of the results produced by asymptotic second–order methods where those appear to be not adequate. Numerical experience in a number of further applications suggests to prefer the importance sampling updates of first–order results to the more "expensive" asymptotic second–order results in high dimensional problems sometimes even in the asymptotic case (β large) for numerical reasons because the computation of the second–order derivatives of the failure surfaces in the β–point occasionally is less reliable. This will certainly be true when there are only a few active constraints in the β–point and the search for the higher–order β–points in a cut set is difficult. What is probably most important in some practical applications is that the inherent

but not easily improved weakness of the Ditlevsen–bounds for not high–reliability problems can be overcome easily.

Finally, a warning appears appropriate. The correction factor is by its very nature of how it is computed not differentiable. This implies that the determination of sensitivity and/or importance factors is not possible. Their evaluation would require special schemes which are not discussed herein.

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