Received: 2 December 2019 | Revised: 4 September 2020 | Accepted: 14 October 2020 | Published on: 20 November 2020

DOI: 10.1002/nav.21963

RESEARCH ARTICLE

WILEY

Walrasian equilibria from an optimization perspective: A guide to the literature

Martin Bichler | **Maximilian Fichtl** | **Gregor Schwarz**

Department of Informatics, Technical University of Munich, Munich, Germany

Correspondence
Martin Bichler, Department of Informatics, Technical University of Munich, Munich, Germany.
Email: bichler@in.tum.de

Funding information
Deutsche Forschungsgemeinschaft (DFG), Grant/Award Number: BI 1057/1-8.

History
Accepted by Saša Pekeč, algorithms, computation, and economics.

Abstract

An ideal market mechanism allocates resources efficiently such that welfare is maximized and sets prices in a way so that the outcome is in a competitive equilibrium and no participant wants to deviate. An important part of the literature discusses Walrasian equilibria and conditions for their existence. We use duality theory to investigate existence of Walrasian equilibria and optimization algorithms to describe auction designs for different market environments in a consistent mathematical framework that allows us to classify the key contributions in the literature and open problems. We focus on auctions with indivisible goods and prove that the relaxed dual winner determination problem is equivalent to the minimization of the Lyapunov function. This allows us to describe central auction designs from the literature in the framework of primal-dual algorithms. We cover important properties for existence of Walrasian equilibria derived from discrete convex analysis, and provide open research questions.

KEYWORDS

duality, primal–dual algorithms, Walrasian equilibrium

1 | INTRODUCTION

Many markets match supply and demand for multiple goods or services (which we also refer to as items) via optimization. Typically, the auctioneer computes an allocation and linear (i.e., item-level), anonymous prices. Linear and anonymous competitive equilibrium prices are often referred to as Walrasian prices in honor of Léon Walras, a French mathematical economist, who pioneered the development of general equilibrium theory. Prominent examples include financial markets (Klemperer, 2010), day-ahead electricity markets (Meeus et al., 2009; Triki et al., 2005), environmental markets (Bichler et al., 2019), logistics (Caplice & Sheffi, 2003; Bichler et al., 2006; Ağralı et al., 2008) or spectrum auctions (Bichler & Goeree, 2017). In some of these markets the auctioneer computes prices that are in a competitive equilibrium with linear and anonymous prices (aka. a Walrasian equilibrium),[1] in others Walrasian prices even lead to efficiency losses (Özer & Özturan, 2009; Lessan & Karabatı, 2018; Bichler et al., 2018; Meeus et al., 2009; Madani & Van Vyve, 2015). This raises the question, which market characteristics admit Walrasian equilibria.

While this is an established and central question in the economic sciences, there have been a number of significant contributions in computer science, economics, and operations research in recent years. The literature on auction algorithms initiated by Bertsekas (1988) is one of the early examples of the fruitful interplay between optimization and equilibrium

[1]There are also competitive equilibria with nonlinear prices (Bikhchandani & Ostroy, 2002). However, some authors only use competitive equilibrium to refer to one with linear and anonymous prices.

theory. In this paper, we survey the literature and describe established and more recent results. We primarily draw on convex analysis and linear programming duality, and provide a consistent mathematical optimization framework to position and explain the key results of this broad literature.

1.1 | Competitive equilibrium

Early in the study of markets, general equilibrium theory was used to understand how markets could be explained through the demand, supply, and prices of multiple commodities or objects. The Arrow–Debreu model shows that under convex preferences, perfect competition, and demand independence there must be a set of competitive equilibrium prices (Arrow & Debreu, 1954; McKenzie, 1959; Gale, 1963; Kaneko, 1976). Market participants are price-takers, and they sell or buy goods in order to maximize their value subject to their budget or initial wealth in this model. The results derived from the Arrow–Debreu model led to the well-known welfare theorems, important arguments for markets as efficient or welfare-maximizing ways to allocate resources. Stability in the form of competitive equilibria where each participant maximizes his utility at the prices is central to this theory. More specifically, the theory focuses on Walrasian equilibria where there is one equilibrium price per good (aka. linear prices) and the price is the same for all bidders (aka. anonymous prices). The first theorem states that any Walrasian equilibrium leads to a Pareto efficient allocation of resources. The second theorem states that any efficient allocation can be attained by a Walrasian equilibrium under the Arrow–Debreu model assumptions.

However, general equilibrium theory assumes divisible goods and convex preferences, and the results do not carry over to markets with indivisible goods and complex (nonconvex) preferences and constraints. Also, in general equilibrium models money does not have outside value and bidders maximize value subject to a budget constraint (Cole et al., 2016). More importantly, bidders are assumed to be nonstrategic price-takers. Based on the work by Vickrey (1961), attention in economics shifted to auction theory, which focuses on small and imperfectly competitive markets, where strategic players can influence prices. These bidders have a quasi-linear utility function, that is, they aim to maximize payoff (i.e., value minus price) (Krishna, 2009). Bayesian Nash equilibria (rather than competitive equilibria) are the central equilibrium solution concept in the auction literature, a branch of noncooperative and incomplete information game theory which led to remarkable results. Most importantly, the Vickrey–Clarke–Groves (VCG) mechanism was shown to be incentive-compatible, and truthful bidding to be a dominant strategy for bidders (Vickrey, 1961).

Many markets that have been implemented for trading financial products, electricity, or environmental access rights as discussed earlier are large markets involving many items and many market participants. Participants want to maximize payoff, but they might not be able to influence prices on such markets. As a consequence, much of the literature is based on a complete-information game-theoretical analysis where bidders are price-takers rather than an incomplete-information game (Baldwin & Klemperer, 2019). Competitive equilibria are the main design desideratum. Unfortunately, it is well known that in many of these markets linear (i.e., block-level) prices might not allow for a welfare-maximizing trade and that there might not be competitive equilibria (Meeus et al., 2009; Madani & Van Vyve, 2015b).

Such new markets have led to a renewed interest in the question of existence and computation of competitive equilibria (Kim, 1986; Bikhchandani & Mamer, 1997; Bikhchandani & Ostroy, 2002; Baldwin & Klemperer, 2019; Leme, 2017). The problem is fundamentally rooted in mathematical optimization, as we will show. In this survey, we will focus on central and recent results in competitive equilibrium theory and multiobject auction design and reformulate them in the language of optimization, specifically duality theory and primal-dual algorithms.

1.2 | Outline

There are various ways how surveys are written. Some articles collect and categorize a larger number of papers in a new and emerging field (Herroelen & Leus, 2005; Galindo & Batta, 2013; Olafsson et al., 2008), others provide a guide to a larger literature and introduce important concepts in a unified framework. Examples include a survey on bilevel programming by Colson et al. (2005) or a survey on the gross substitutes condition in economics by Leme (2017). We follow the latter path and discuss competitive equilibrium theory using duality theory and linear programming as a framework. While most of the literature on this subject is published in economics journals, key insights of this literature can be introduced conveniently using the mathematical framework of optimization. Fundamentally, auctions are algorithms for optimal resource allocation and there are plenty of questions where the OR community can contribute as we discuss in the last section.

The survey starts with markets for divisible goods and shows that the concave conjugate to the aggregate value function of all bidders yields prices, and that the minimizer of the Lyapunov function results in Walrasian prices if the aggregate value function is concave. A condition for concavity of the aggregate value function is concavity of the individual value functions, which is equivalent to diminishing marginal returns. The Lyapunov function is convex so that a simple subgradient algorithm finds the minimum efficiently. This algorithm has an interpretation as an auction.

We will next show that the same principles from duality theory carry over to markets with indivisible objects. For this, we describe the allocation problem as a binary program.

Whenever the linear programming relaxation of this binary program has integer solutions, then the dual variables of the capacity constraints have an interpretation as Walrasian prices for the respective resources. We prove that the dual of the linear programming relaxation of this binary program is equivalent to the Lyapunov function. Economic literature discusses conditions on individual value functions that allow for Walrasian equilibria. This is the case if the convolution of these individual functions results in a discrete concave aggregate value function.

As in the continuous case with divisible goods, we can use a steepest descent algorithm to find the minimizer of the Lyapunov function, which is equivalent to determining Walrasian prices for the market. This is exactly what the auction mechanism by Ausubel (2005) does, a central contribution to auction design. Primal-dual algorithms are well-known algorithms to solve linear programs, and they have a nice interpretation as a market with an auctioneer and the bidders optimizing alternatively. The steepest descent algorithm that minimizes the Lyapunov function is equivalent and we show the connections.

We contribute the equivalence of the Lyapunov function and the dual linear programming relaxation of the allocation problem in markets with indivisible goods, as well as the equivalence of primal-dual algorithms with central auction designs for selling multiple indivisible goods. These two results allow us to organize the material and use duality theory to discuss the literature on existence of Walrasian equilibria, and linear programming algorithms to discuss auction designs leading to Walrasian equilibria if it exists. The survey helps scholars with a background in mathematical optimization to understand central results in competitive equilibrium theory and draws important connections between competitive equilibrium theory, mathematical optimization, and discrete convexity.

In Section 2 we introduce the notation and standard assumptions in the economic literature for readers from operations research. Then we introduce important concepts for the understanding of Walrasian equilibria such as the Lyapunov function for markets with divisible goods in Section 3. The same concepts play a role for markets with indivisible goods and discrete value functions in Section 4. In Section 5 we use primal-dual algorithms and show that these are equivalent to important auction designs discussed in economics. Finally, we provide a research agenda and discuss open research problems for the operations research community.

2 | NOTATION AND ECONOMIC ENVIRONMENT

In the auction market, there are m types of items or goods, denoted by $k \in \mathcal{K} = \{1, \dots, m\}$, and n bidders $i \in \mathcal{I} = \{1, \dots, n\}$. In the multi-unit case, we have $s \in \mathbb{Z}^m_{\geq 0}$ units available, that is, $s(k)$ *homogeneous*units for each of the *heterogeneous* m items $k \in \mathcal{K}$. A bundle for bidder i is described by a vector $x_i \in \mathbb{Z}^m_{\geq 0}$. In case of single-unit supply the vector is binary, that is, $x_i \in \{0, 1\}^m$. We will sometimes omit the subscript i for convenience. Each bidder i has a value function $v_i : \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}_{\geq 0}$ over bundles of items or objects x_i. We assume integer-valued functions v_i as it will be more convenient to analyze the optimality of auction algorithms. Moreover, integer-valued functions v_i allow to use integral prices in ascending auctions without losing efficiency.

Unless stated otherwise this paper we assume that bidders have *preferences* described via a valuation function with the following properties:

- Pure private values: Bidder i's value $v_i(x_i)$ does not change when she learns other bidder's information.
- Quasilinearity: Bidder i's (direct) utility from bundle x_i is given by $\pi_i(x_i, p) = v_i(x_i) - \langle p, x_i \rangle$, where $\langle \cdot, \cdot \rangle$ is the dot product.
- Monotonicity: The function $v_i : \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is weakly increasing with $v_i(0) = 0$ and, if $x_i \geq x_i'$, then $v_i(x_i) \geq v_i(x_i')$.

An auctioneer wants to find an allocation of items to bidders. Such an allocation is *feasible* when the supply suffices to serve the aggregate demand of the bidders. Furthermore, the auctioneer aims for *allocative efficiency*. This means the auctioneer wants to maximize *social welfare* which is the sum of the utilities of all participants (the bidders and the auctioneer). Maximization of welfare is also referred to as a *utilitarian* welfare function. In case of quasilinear utility functions, prices cancel and the social welfare is defined as $\sum_{i \in \mathcal{I}} v_i(x_i)$.

For the remainder of this survey we assume that the auctioneer's valuation for all items is zero. As a consequence, the auctioneer would sell items to bidders for a price of zero. In some auction scenarios, however, the auctioneer may want to set *reserve prices* which are the minimum prices at which the auctioneer would be willing to sell the goods. Often these reserve prices can be implemented by introducing a dummy bidder who simply bids the reserve prices on behalf of the auctioneer in the auction. In case the dummy bidder wins any items in the auction, these items remain unsold.

The goal of the auctioneer is to find an efficient allocation that yields linear (i.e., item-level) and anonymous market clearing prices $p = \{p(k)\}_{k \in \mathcal{K}} \in \mathbb{R}^m$. The *linearity of prices* refers to the property that individual prices are set for each item $k \in \mathcal{K}$; the price for a bundle x is then simply the sum of the prices of its components, that is, it is given by the dot product $\langle p, x \rangle$. *Anonymity* means that the resulting prices p are the same for all bidders and there is no price differentiation. Furthermore, prices p are *market clearing* when the aggregate demand of all bidders at the given prices p meets the supply s.

With linear and anonymous prices $p = (p(1), \dots, p(k), \dots, p(m))$, the bidder's *indirect utility function* is defined as

$$u_i(p) = \max_{x \in \mathbb{Z}^m_{\geq 0}} \{v_i(x) - \langle p, x \rangle\}.$$

The indirect utility function is widely used in economics and returns the maximal utility that bidder i can obtain for any bundle at prices p. The *demand correspondence* $D_i(p)$ is the set of bundles that maximize the indirect utility function at prices p, that is,

$$D_i(p) = \arg \max_{x \in \mathbb{Z}^m_{\geq 0}} \{v_i(x) - \langle p, x\rangle\}.$$

If in an outcome (consisting of an allocation and prices) all bidders are allocated a bundle from their demand set, then the outcome is *envy-free*. No bidder would want to get another bundle, as a bidder cannot increase her utility at these prices. Envy-free prices always exist. For example, if prices were higher than the valuations, then every bidder would only want the empty set. If in addition to envy-freeness all items are allocated, $\sum_{i \in \mathcal{I}} x_i = s$, then the outcome is a competitive equilibrium.

Definition 1 (Competitive equilibrium, CE). A price vector p^* and a feasible allocation $(x_1, \ldots, x_n)$ form a *competitive equilibrium* if $\sum_{i \in \mathcal{I}} x_i = s$ and $x_i \in D_i(p^*)$ for every bidder $i \in \mathcal{I}$.

If there were unsold items, an auctioneer could always add unsold units to the allocation of a bidder without decreasing welfare as bidders are assumed to have monotone value functions v_i.

In our setting with linear and anonymous prices, a competitive equilibrium is also called a *Walrasian equilibrium*. If there exists a Walrasian price vector p^* such that $p^* \leq p'$ for any other Walrasian price vector p', then p^* is called the *bidder-optimal* Walrasian price vector. For Walrasian equilibria the well-known welfare theorems hold:

Theorem 1 *First and second welfare theorem (following* Blumrosen and Nisan *(2007)) Let $x = (x_1, \ldots, x_n)$ be an equilibrium allocation induced by a Walrasian equilibrium price vector p, then x yields the optimal social welfare. Conversely, if x is a Pareto efficient allocation, then it can be supported by a Walrasian price vector p so that the pair (p, x) forms a Walrasian equilibrium.*

3 | WALRASIAN EQUILIBRIA WITH DIVISIBLE GOODS AND CONJUGACY

In this article, we focus on markets with indivisible goods. However, for instructive purposes, we briefly consider the case of divisible goods to introduce relevant concepts. These can then be transferred to the indivisible case. Our aim is to give an intuitive graphical and analytical interpretation of how the aggregate valuation function is connected to the indirect utility function, the Lyapunov function and the market prices.

We consider a market with multiple bidders $i \in \mathcal{I}$ and multiple divisible goods $k \in \mathcal{K}$ with $|\mathcal{I}| = n$ and $|\mathcal{K}| = m$. The aggregate value function $v_\mathcal{I}$ is defined as the supremum convolution of concave functions $v_i : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ where v_i is the value function of the ith bidder.

$$v_\mathcal{I}(s) = \max_{\{x_i\}_{i \in \mathcal{I}}} \left\{ \sum_{i \in \mathcal{I}} v_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_{\geq 0} \text{ and } \sum_{i \in \mathcal{I}} x_i = s \right\}.$$

By compactness and continuity, the maximum exists. Concavity implies that $v_i((1-\alpha)x + \alpha y) \geq (1-\alpha)\ v_i(x) + \alpha\ v_i(y)$ with $x, y \in \mathbb{R}_{\geq 0}$ and $\alpha \in (0, 1)$. The economic interpretation of a concave valuation function is that it exhibits decreasing marginal valuations. Since every function v_i is concave, also their convolution $v_\mathcal{I}$ is concave.

The aggregate indirect utility is defined as $u_\mathcal{I}(p) = \sum_i u_i(p)$ and the aggregate demand set is given by the Minkowski sum $D_\mathcal{I}(p) = \sum_i D_i(p)$.

For the sake of simplicity of the following graphical interpretation of indirect utility and the concept of conjugacy, we consider a market with multiple bidders but only a single divisible good $x \in \mathbb{R}_{\geq 0}$. However, our explanations carry over directly to markets with multiple goods. It is also worth mentioning that in the presence of only a single bidder i the aggregate valuation function $v_\mathcal{I}$ becomes the individual valuation function v_i of the single bidder. Thus, even though the following example illustrates the aggregate valuation and indirect utility function of multiple bidders, it similarly applies to the valuation and indirect utility function of an individual bidder.

In our example, we assume $v_\mathcal{I}(x) = \ln(x+1)$. It is well known that for concave functions $v_\mathcal{I}$ local optimality implies global optimality and this yields efficient optimization algorithms.

At a given price, every rational bidder $i \in \mathcal{I}$ only demands a quantity of good x which maximizes her utility at this price. The utility of such a quantity is described by the indirect utility function $u_i(p) = \max_x \{v(x) - \langle p, x\rangle\}$, which is convex as it is the maximum of affine linear functions. As the aggregate indirect utility function $u_\mathcal{I}(p)$ is a sum of convex functions, it must also be convex.

A quantity x^* is demanded at prices p if and only if $v_\mathcal{I}(x^*) - \langle p, x^*\rangle \geq v_\mathcal{I}(x) - \langle p, x\rangle$ for all $x \in \mathbb{R}$. When rearranging terms to $v_\mathcal{I}(x^*) + \langle p,\ x - x^*\rangle \geq v_\mathcal{I}(x)$, it becomes clear that the left-hand side of the inequality describes the tangent at $v_\mathcal{I}(x^*)$ (see Figure 1). In other words, a quantity x^* is demanded at prices p whenever the slope of the tangent at $v_\mathcal{I}(x^*)$ equals the price p. The aggregate utility of quantity x^* is given by $\pi_\mathcal{I}(x^*, p) = v_\mathcal{I}(x^*) - \langle p,\ x^*\rangle$. As $x^* \in D_\mathcal{I}(p)$, the aggregate utility $\pi_\mathcal{I}(x^*, p)$ equals the aggregate indirect utility $u_\mathcal{I}(p)$. The graphical interpretation of the aggregate indirect utility function $u_\mathcal{I}(p)$ is the intercept of the tangent at $v_\mathcal{I}(x^*)$ with the ordinate.

We can now compute the quantity of good x that generates maximum utility at prices p. In our illustrative example with $v_\mathcal{I}(x) = \ln(x+1)$, the aggregate utility $\pi_\mathcal{I}(x, p) = \ln(x+1) - \langle p,\ x\rangle$ at given prices p is maximized when

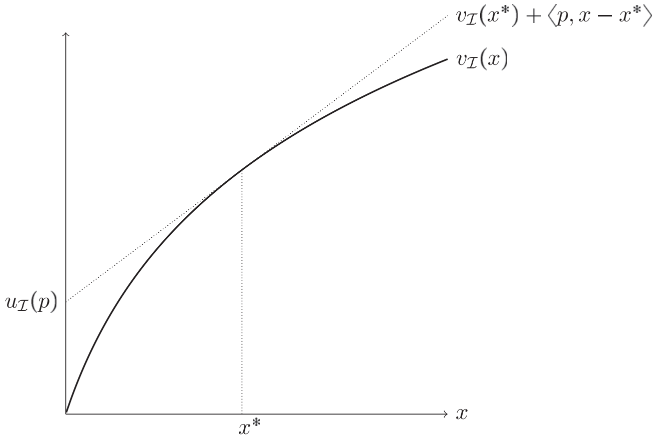

FIGURE 1 Graphical representation of $v_{\mathcal{I}}(x) = \ln(x+1)$ with tangent at $v_{\mathcal{I}}(x^*)$

$\partial\pi_{\mathcal{I}}/\partial x = 1/(x+1) - p = 0$. This means, at a price of $p = 1/3$ for example, the total utility $\pi_{\mathcal{I}}$ is maximized for a demand of $x^* = 2$. Thus, the aggregate indirect utility function at prices $p = 1/3$ equals $u_{\mathcal{I}}(1/3) = \pi_{\mathcal{I}}(2, 1/3) = \ln(3) - 2/3$. The concave conjugate (or Legendre transformation) of $v_{\mathcal{I}}$ is defined as $v_{\mathcal{I}}^{\bullet}(p) = \min_x\{\langle p, x\rangle - v_{\mathcal{I}}(x)\}$, which is the aggregate indirect utility function multiplied by -1. We also note that convex and concave conjugates are connected via $v_{\mathcal{I}}^{\bullet}(p) = -(-v_{\mathcal{I}})^*(-p)$, so $u_{\mathcal{I}}(p) = (-v_{\mathcal{I}})^*(-p)$. From these results, we can make the following connection: In order to construct the concave conjugate $v_{\mathcal{I}}^{\bullet}(p)$ of $v_{\mathcal{I}}(x) = \ln(x+1)$ for a fixed p, we must calculate the minimum of $\langle p, x\rangle - \ln(x+1)$. Taking the derivative, we see that a minimizing x must solve $x = 1/p - 1$, so we get $v_{\mathcal{I}}^{\bullet}(p) = 1 - p + \ln(p)$ and consequently $u_{\mathcal{I}}(p) = -v_{\mathcal{I}}^{\bullet}(p) = p - \ln(p) - 1$. For a given price of $p = 1/3$ the reader may verify that the bidders' aggregate indirect utility equals $u_{\mathcal{I}}(1/3) = 1/3 - \ln(1/3) - 1 = \ln(3) - 2/3$, which is in line with our calculations above.

Unlike in this single-item example, the price p is not known in an auction setting. Instead, the auctioneer tries to find a price vector p^* for which the supply s is a maximizer of the aggregate utility function $\pi_{\mathcal{I}}(x, p^*)$. Note that such a p^* is a Walrasian equilibrium price vector, because s maximizes $\pi_{\mathcal{I}}(s, p^*) = v_{\mathcal{I}}(s) - \langle p^*, s\rangle$ and the aggregate demand of the bidders equals the supply s.

We will now return to a market with multiple divisible goods $k \in \mathcal{K}$. First, we introduce important notions from convex analysis.

Definition 2 Let $f: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The *subdifferential* of f at x is the set of all tangents of f at x:

$$\partial f(x) = \{y \in \mathbb{R}^d | f(x') \geq f(x) + \langle y, x' - x\rangle \forall x' \in \mathbb{R}^d\}.$$

Any element of $\partial f(x)$ is called a *subgradient*. The *convex conjugate* or *Legendre transform* of f is the convex function

$$f^*(y) = \sup_{x \in \mathbb{R}^d} \langle y, x\rangle - f(x).$$

Under additional mild assumptions on the convex function f, the conjugate of the conjugate is again f, $f^{**} = f$, and subdifferentials of f and f^* are connected in the following way: $y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y)$. For more details, we refer to Rockafellar (2015). The concave conjugate defined above and the convex conjugate are related as follows: If g is concave, then $g^{\bullet}(y) = -(-g)^*(-y)$. In particular, we have for the indirect utility function $u_{\mathcal{I}}(p) = (-v_{\mathcal{I}})^*(-p)$. We make the following important observation: The bundle x is in the demand set $D_{\mathcal{I}}(p)$, if and only if $v_{\mathcal{I}}(x) - \langle p, x\rangle \geq v_{\mathcal{I}}(x') - \langle p, x'\rangle$ for all $x' \in \mathbb{R}^{|\mathcal{K}|}$. By rearranging terms we see that this is equivalent to $-v(x') \geq -v(x) + \langle -p, x' - x\rangle$ and thus to $-p \in \partial(-v_{\mathcal{I}})(x)$. Convex analysis tells us that this is equivalent to $x \in \partial(-v_{\mathcal{I}})^*(-p) = -\partial u_{\mathcal{I}}(p)$. Consequently, demand sets are equal to subdifferentials of the indirect utility function—a fact that allows us to interpret auctions as descent algorithms.

The Lyapunov function was a central concept already in the early literature on general equilibrium theory (Arrow & Hahn, 1971). The same function plays a central role in more recent auction designs for markets with indivisible goods (Ausubel, 2006). Since this function plays such a central role, we introduce it in detail for the continuous case.

Definition 3 (Lyapunov function). The *Lyapunov function* is defined as $L(p) = \sum_{i \in \mathcal{I}} u_i(p) + \langle p, s\rangle$, where s is the supply and $u_i(p)$ is the indirect utility function of bidder $i \in \mathcal{I}$ at prices p.

The Lyapunov function has its roots in the dynamical systems literature (La Salle & Lefschetz, 2012). Since the indirect utility $u_i(p)$ is convex in p, also the Lyapunov function is convex, because it is the sum of convex functions. For convex functions such as $L(p)$ the vector p^* minimizes L iff 0 is a subgradient at p^*. The first-order condition for $L(p)$ yields $-\sum_{i \in \mathcal{I}} x_i + s = 0$, where $x_i \in D_i(p)$.

$\forall i \in \mathcal{I}$. In words, the prices are minimized when supply equals demand:

Proposition 1 *A vector $p^* \in \mathbb{R}^m$ is a Walrasian equilibrium price vector for supply s if and only if it is a minimizer of the Lyapunov function $L(p) = u_{\mathcal{I}}(p) + \langle p, s\rangle$.*

Proof If there is a Walrasian equilibrium, then $\sum_{i \in \mathcal{I}} x_i = s$ and $x_i \in D_i(p^*)$ need to hold. The minimizer p^* of $L(p)$ requires that $\partial L(p) = s - \sum_{i \in \mathcal{I}} x_i = 0$, which is equivalent to the first condition of a Walrasian equilibrium. Also, when $L(p) = \sum_{i \in \mathcal{I}} \max_{x_i}\{v_i(x_i) - \langle p, x_i\rangle\} + \langle p, s\rangle$ attains the minimum, then each bidder is assigned a bundle x_i that maximizes her utility $v_i(x_i) - \langle p, x_i\rangle$. This implies $x_i \in D_i(p^*)$ for all i, so that the second condition of a Walrasian equilibrium is fulfilled. Thus, if $L(p)$ is minimized then both conditions of a Walrasian equilibrium are satisfied. By reversing the argument it becomes evident that any price vector p^* supporting s in a Walrasian equilibrium is also a minimizer for $L(p)$. ■

Similar results can be found in Ausubel and Milgrom (2006) or later in Murota (2016). One way to find Walrasian equilibria is now to minimize the Lyapunov function. Since we can interpret the subdifferential of u_i at price p as the demand set at this price—for an auction setting it is natural to utilize standard subgradient methods for (approximately) minimizing $L(p)$—computing subgradients is then equivalent to asking bidders for their demand sets at a given price. Note that it is in general not possible to compute *exact* minimizers to general convex functions—algorithms for minimizing a convex function f can in general only provide complexity bounds for finding an ε-approximate solution x', in the sense that

$$f(x') \leq \varepsilon + \min_x f(x).$$

Note that in general x' does not even have to be close to the true minimizer x without additional assumption on f. Since the aim of our treatment of divisible economies is mainly to motivate the ideas in the indivisible case, we will not go into more detail here. If no additional regularity assumptions on L are imposed, it can be shown that finding ε-approximate solutions has a worst-case running time of $\Theta(1/\varepsilon^2)$ (Nesterov, 2018). Interestingly, for markets with indivisible goods where Walrasian equilibria exist, we will show that the Lyapunov function equals the dual of the allocation problem.

Central results of convex economic theory with divisible goods are reasonable approximations to large economies where nonconvexities vanish in the aggregate (Starr, 1969). However, most markets are such that indivisibilities and nonconvexities matter. As one would assume, the analysis of markets with indivisible items has proven much harder.

4 | EXISTENCE OF WALRASIAN EQUILIBRIA WITH INDIVISIBLE GOODS

In this section, we discuss sufficient and necessary conditions for the individual value functions of bidders such that Walrasian equilibria exist in markets with indivisible goods.

4.1 | Conditions on aggregate value functions

A simple multi-item market with remarkable properties is the assignment market by Shapley and Shubik (1971). In assignment markets each bidder can bid on multiple items but wants to win at most one (aka. *unit-demand*). As a consequence, the allocation problem reduces to an assignment problem, that is, the problem of finding a maximum weight matching in a weighted bipartite graph. On an aggregate level, the LP relaxation of the assignment problem is always integral. This is a consequence of the unit demand on an individual level and the resulting total unimodularity of the constraint matrix, and this is a sufficient condition for the existence of Walrasian prices. The environment of assignment markets allows for incentive-compatible auctions. Besides, simple ascending clock auctions yield bidder-optimal Walrasian prices (Demange et al., 1986).

4.1.1 | The allocation problem

Let us first extend the assignment market to a more general multi-item, multi-unit market which allows for package bids. Let $\mathcal{X}_i \subseteq \mathbb{Z}^m_{\geq 0}$ denote all bundles for which bidder i submitted a bid. For simplicity, we make the natural assumption that every bidder submits a bid with value 0 for the empty bundle. Let $z_i(x) \in \{0, 1\}$ be a binary decision variable denoting whether bidder i wins bundle $x \in \mathcal{X}_i$. The allocation or winner determination problem WDP can then be written as an integer program as follows:

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{I}} \sum_{x\in\mathcal{X}_i} v_i(x) z_i(x) && && \text{(WDP)} \\ \text{s.t.} \quad & \sum_{x\in\mathcal{X}_i} z_i(x) \leq 1 && \forall i \in \mathcal{I} && (\pi_i) \\ & \sum_{i\in\mathcal{I}} \sum_{x\in\mathcal{X}_i} x(k) z_i(x) \leq s(k) && \forall k \in \mathcal{K} && (p(k)) \\ & z_i(x) \in \{0,1\} && \forall i \in \mathcal{I}, \forall x \in \mathcal{X}_i && \end{aligned}$$

For a given supply s the WDP determines an allocation of bundles to bidders maximizing social welfare. The LP relaxation RWDP in standard form replaces $z_i(x) \in \{0, 1\}$ by $z_i(x) \geq 0$ and introduces additional slack variables. We use the standard form with slack variables (a_i, b_k) because it will be helpful in our algorithmic treatment of the subject.

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{I}} \sum_{x\in\mathcal{X}_i} v_i(x) z_i(x) && && \text{(RWDP)} \\ \text{s.t.} \quad & \sum_{x\in\mathcal{X}_i} z_i(x) + a_i = 1 && \forall i \in \mathcal{I} && (\pi_i) \\ & \sum_{i\in\mathcal{I}} \sum_{x\in\mathcal{X}_i} x(k) z_i(x) + b_k = s(k) && \forall k \in \mathcal{K} && (p(k)) \\ & z_i(x), a_i, b_k \geq 0 && \forall i \in \mathcal{I}, \forall x \in \mathcal{X}_i, \forall k \in \mathcal{K} && \end{aligned}$$

In contrast to the assignment problem where bidders have unit demand, the RWDP does not yield integer solutions in general.

Example 1 Consider a market with three items $\mathcal{K} = \{A, B, C\}$ and two bidders with valuations v_1 and v_2

	$x_\emptyset$	x_A	x_B	x_C	x_{AB}	x_{AC}	x_{BC}	x_{ABC}
x	(0, 0, 0)	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(1, 1, 0)	(1, 0, 1)	(0, 1, 1)	(1, 1, 1)
$v_1(x)$	0	1	2	1	2	2	2	2
$v_2(x)$	0	1	2	2	3	2	3	3

The optimal solution of the RWDP given these valuations is fractional:

$z_1(x_B) = z_1(x_{AC}) = z_2(x_C) = z_2(x_{AB}) = 0.5$ with all other decision variables set to 0. The optimal value of the RWDP with respect to this fractional solution is 4.5. An optimal integral solution (e.g., assigning bundle x_{AC} to the first and x_B to the second bidder) only leads to a social welfare of 4.

Let us also introduce the dual DRWDP of the RWDP.

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{I}} \pi_i + \sum_{k\in\mathcal{K}} s(k)p(k) && && \text{(DRWDP)} \\ \text{s.t.} \quad & \pi_i + \sum_{k\in\mathcal{K}} x(k)p(k) \geq v_i(x) && \forall i \in \mathcal{I}, \forall x \in \mathcal{X}_i && (z_i(x)) \\ & \pi_i \geq 0 && \forall i \in \mathcal{I} && (a_i) \\ & p(k) \geq 0 && \forall k \in \mathcal{K} && (b_k) \end{aligned}$$

We will draw on these models in the subsequent sections.

4.1.2 | Integrality of the linear program

Bikhchandani and Mamer (1997) describe a multi-item, single-unit market. Their central theorem shows that there exist clearing prices for the indivisible single-unit problem if and only if the RWDP has an integer solution. In this case, the set of equilibrium prices is the set of solutions to the dual LP projected to the price coordinates. The result can be proven via the strong duality theorem in linear programming (Blumrosen & Nisan, 2007). As was already noted by Bikhchandani and Mamer (1997), the result for multi-item, *multi*-unit markets also directly follows from their result, by considering each of the multiple units as separate items. As the proof is a particularly nice application of duality theory, we provide a direct proof in the Appendix. Note that this theorem proves the welfare theorems from general equilibrium theory (see Theorem 1).

Theorem 2 *Walrasian prices exist for the supply s if and only if the RWDP has an optimal integral solution.*

The proof can be found in Appendix A.

As indicated, the RWDP typically does not yield an integral solution, and there can be a significant integrality gap between the objective function value of the RWDP and that of the optimal integer program WDP. In the next sections, we will discuss conditions on the individual value functions, which yield integral solutions of the RWDP and Walrasian prices.

Before we do this, let us return to the Lyapunov function that has proven so helpful in our analysis of markets with divisible goods. A minimizer to this function yielded the Walrasian prices in Section 3, where we analyzed markets with divisible goods. It turns out that the Lyapunov function is actually equivalent to the DRWDP, as we show in the following proposition.

Proposition 2 *A vector $p^* \in \mathbb{R}^m$ minimizes the DRWDP if and only if it is a minimizer of the Lyapunov function $L(p) = u_{\mathcal{I}}(p) + \langle p, s\rangle$.*

Proof We can substitute the utilities π_i in the dual objective function $\min \sum_{i\in\mathcal{I}} \pi_i + \sum_{k\in\mathcal{K}} s(k)p(k)$ by the tight dual constraints $\pi_i = v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k)$ of the optimal DRWDP and get the following convex function:

$$\min_p \sum_{i\in\mathcal{I}} \max_{x\in\mathbb{Z}^m_{\geq 0}} \left[v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k) \right] + \sum_{k\in\mathcal{K}} s(k)p(k). \quad (4.1)$$

Note that this is equivalent to minimizing the Lyapunov function $L(p) = \sum_{i\in\mathcal{I}} u_i(p) + \langle p, s\rangle$. Obviously, $\langle p, s\rangle$ in $L(p)$ is equal to $\sum_{k\in\mathcal{K}} s(k)p(k)$, and $u_i(p)$ equals $\max_{x\in\mathbb{Z}^m_{\geq 0}} \left[v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k)\right]$ for every bidder i. Since the equivalence of the Lyapunov function and the DRWDP holds for any price vector p, minimizing prices of the Lyapunov function also constitutes a minimal solution to the DRWDP and vice versa. ■

In summary, both the Lyapunov function and the LP approach yield equilibrium prices, and such prices are minimizers of both problems. We will leverage this insight, when we analyze auction algorithms to solve the RWDP in Section 5.

4.2 | Conditions for individual value functions

In practical applications a market designer often wants to understand which assumptions on the individual value functions v_i allow for integer solutions of the LP relaxation and Walrasian prices. Discrete convex analysis identifies classes of convex functions defined on a subset of the discrete lattice $\mathbb{Z}^m$, which allow for integrality and efficient optimization algorithms.

First, we discuss single-unit, multi-item auctions. There are several classes of integrally convex functions such as separable-convex functions on $\mathbb{Z}^m$ or gross substitutes set functions on $\{0, 1\}^m$, which yield a discrete concave aggregate value function $v_{\mathcal{I}}$ and integral solutions of the RWDP, such that Walrasian equilibria exist.

4.2.1 | Single-unit multi-item auctions

Let us first define monotonicity and submodularity, two well-known properties of set functions that allow for efficient function minimization.

Definition 4 For a finite set $\mathcal{K}$ of items, the set function $v : 2^{\mathcal{K}} \to \mathbb{R}$ is

- *monotone* if $v(S) \leq v(T)$ for all $S, T \subseteq \mathcal{K}$ with $S \subseteq T$,

- *submodular* if $v(S \cup \{k\}) - v(S) \geq v(T \cup \{k\}) - v(T)$ for all $S, T \subseteq \mathcal{K}$ with $S \subseteq T$ and for all $k \notin T$.

In the above definition, submodularity can be understood as diminishing marginal values. Alternatively, submodularity can be defined as $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ for all S, T. The vector notation $v : \{0, 1\}^m \rightarrow \mathbb{R}$ in the single-unit case maps a set S to a vector $x \in \{0, 1\}^m$ by setting $x(k) = 1$ whenever $k \in S$ and $x(k) = 0$ otherwise.

It is well-known that the minimization of unconstrained submodular functions can be done in polynomial time, for example via the ellipsoid method (Grötschel et al., 1981). The ellipsoid method is notoriously slow in practice. However, there are also more effective algorithms such as the Fujishige-Wolfe algorithm (Chakrabarty et al., 2014) and specialized subgradient methods (Chakrabarty et al., 2017). Unfortunately, even when submodularity and monotonicity are satisfied, this does not guarantee the integrality of a welfare maximization problem such as the RWDP.

Example 2 The reader may verify that the valuation functions of both bidders in example 1 satisfy monotonicity and submodularity. However, the optimal solution of the RWDP is not integral.

The subset of submodular valuations called gross substitutes valuations, however, has this desirable property. Gross substitutes roughly means that a bidder regards the items as substitute goods or independent goods but not complementary goods.

Definition 5 (Gross substitutes, GS). Let p denote the prices on all items, with item k demanded by bidder i if there is some bundle S, with $k \in S$, for which S maximizes the utility $v_i(S') - \sum_{j \in S'} p(j)$ across all bundles $S' \subseteq \mathcal{K}$. The gross substitutes condition requires that, for any prices $p' \geq p$ with $p'(k) = p(k)$, if item $k \in \mathcal{K}$ is demanded at the prices p then it is still demanded at p'.

The definition includes both substitute goods and independent goods, but rules out complementary goods.[2]

Example 3 Consider a market with three items $\mathcal{K} = \{A, B, C\}$ and a single bidder with a valuation function v fulfilling the gross substitutes condition

	$x_\emptyset$	x_A	x_B	x_C	x_{AB}	x_{AC}	x_{BC}	x_{ABC}
x	(0, 0, 0)	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(1, 1, 0)	(1, 0, 1)	(0, 1, 1)	(1, 1, 1)
$v(x)$	0	1	2	3	3	3	5	5

At prices $p = (0, 1, 2)$ the bidder's indirect utility is $u(p) = 2$ and the bidder's demand set is given by $D(p) = \{x_{AB}, x_{BC}, x_{ABC}\}$, that is, items A, B, and C are demanded as for each item there exists at least one bundle in the demand set containing the item. If the price for item A is raised to 1 but stays constant for items B and C, then the gross substitutes condition implies that items B and C must still be demanded at the new prices $p' = (1, 1, 2)$. This is obviously true as the demand set at the new prices p' is given by $D(p') = \{x_{BC}\}$. Note that price vectors p and p' were only chosen for illustrative purposes. In fact, valuation function v satisfies the gross substitutes condition for any price vectors $p, p' \in \mathbb{R}^3_{\geq 0}$ with $p' \geq p$.

Kelso and Crawford (1982) show that if all agents have GS valuations, then a Walrasian equilibrium always exists, which implies that the RWDP has an optimal integral solution. Ausubel and Milgrom (2002) prove that a bidder has GS valuations if and only if the indirect utility function u is submodular. Gross substitutes appear to be a rather restricted type of valuations, but it contains important subclasses such as unit-demand valuations (Shapley & Shubik, 1971) and additive valuations. Gul and Stacchetti (1999) show that GS excludes complementarity between goods and show equivalence with the so called single improvement property. The latter property states that whenever a bundle is not optimal at the given prices, then a better bundle can be found which is derived from the original one by performing any of the following operations: removing an item, adding an item, or doing both. Leme (2017) provides a survey of the extensive literature on the gross substitutes condition and its alternative definitions for multi-item, single-unit markets, and show that additive valuations ⊂ GS ⊂ submodular valuations ⊂ subadditive valuations. We also refer to Shioura and Tamura (2015) for an extensive survey of GS.

Sun and Yang (2006) identify the gross substitutes and complements (GSC) condition, which also guarantees for Walrasian equilibria in single-unit, multi-item markets. It describes an exchange economy with two classes of goods, where each class only contains substitutes, but there are complements across these classes of goods. Teytelboym (2014) generalizes the GSC condition in the sense that goods are partitioned into more than two classes. His generalized version of the GSC condition is satisfied if it is possible to partition goods into several classes so that whenever considering the bidders' valuations for items contained in only two of these classes in isolation, there exist some bidders for which these valuations satisfy the GSC condition.

4.2.2 | Multi-unit ulti-item auctions

Let us now concentrate on more general conditions for $x \in \mathbb{Z}^m_{\geq 0}$ rather than $x \in \{0, 1\}^m$. $A \subset \mathbb{Z}^m$ is *integrally convex* if

[2] Sometimes the word "gross" used by Kelso and Crawford (1982) is omitted, but it is useful to distinguish the single-unit case from substitutes valuations in other environments, such as the strong substitutes definition that we will introduce later.

$A = (conv\,A) \cap \mathbb{Z}^m$. First, we define the *convex closure* $\bar{f}$ of f as

$$\bar{f}(x) = \sup_{p \in \mathbb{R}^m, \alpha \in \mathbb{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \le f(y)\ \forall y \in \mathbb{Z}^m\}.$$

Geometrically, the epigraph of $\bar{f}$ is the convex hull of the epigraph of f. If the convex closure coincides with f on the set of integer vectors, that is, if $f(x) = \bar{f}(x)$ for all $x \in \mathbb{Z}^m$, f is called *convex-extensible*. In the same way, we can define the concave closure of f by $-\overline{(-f)}$. The definition can be restricted to the integral neighborhood of a bundle $x \in \mathbb{R}^m$ and is then referred to as a *local convex extension* $\tilde{f}$ of f (Murota, 2003, Chap. 3). Formally, set $N(x) = \{y \in \mathbb{Z}^m \mid \lfloor x(k) \rfloor \le y(k) \le \lceil x(k) \rceil \forall k = 1, \dots, m\}$. Then the local convex extension is given by

$$\tilde{f}(x) = \sup_{p \in \mathbb{R}^m, \alpha \in \mathbb{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \le f(y)\ \forall y \in N(x)\}.$$

Definition 6 A function $f : \mathbb{Z}^m \to \mathbb{R}$ is called *integrally convex* if the local convex extension of f is convex, or *integrally concave* if the function $-f$ is integrally convex.

Integrally convex functions share with convex functions the property that local minima are also global minima (Murota, 2016). We have already seen in the divisible case that concavity of the valuation functions is necessary for equilibrium prices to exist. We also want to make this connection here in the indivisible case, by explaining how convexity is related to integrality of the WDP—which is necessary and sufficient for the existence of equilibrium prices. To start with, consider the aggregate valuation function $v_{\mathcal{I}}(s)$, given by the value of the WDP for the supply s, and the "relaxed" aggregate valuation function $\tilde{v}_{\mathcal{I}}(s)$, given by the value of the RWDP at s. Note $\tilde{v}_{\mathcal{I}}$ is well-defined for all *real* supply vectors $s \ge 0$ and attains finite values at each such s. A central observation is the following: $\tilde{v}_{\mathcal{I}}$ is the concave extension of $v_{\mathcal{I}}$. This shows that $v_{\mathcal{I}}$ is concave-extensible, and thus $v_I = \tilde{v}_{\mathcal{I}}$ if and only if for every integral supply vector s, the RWDP has an integral solution, which—as we have seen—is equivalent to the existence of equilibrium prices. While the stronger assumption of integral concavity is not necessary for the existence of equilibrium prices, it is not hard to imagine, that this property is of importance for the algorithmic problem of computing equilibrium prices. Loosely speaking, since the value of the concave extension can then be evaluated at any point s by considering an easy to characterize neighborhood of s, the computation of subgradients of $v_{\mathcal{I}}$ gets much simpler. Unfortunately, concave extensibility, and even integral concavity of the individual valuation functions does not suffice to guarantee concave extensibility of the aggregate valuation function, or equivalently, existence of equilibrium prices. It is thus of central importance to identify conditions on the individual valuations that imply concave extensibility of the aggregate valuation, or equivalently integrality of the RWDP.

Definition 7 A function $f : \mathbb{Z}^m \to \mathbb{R} \cup \{\infty\}$ is said to be $M^\natural$-convex if for x, $y \in$ dom$f$ and $j \in supp^+(x - y)$

(i) $f(x) + f(y) \ge f(x - \mathbb{1}_j) + f(y + \mathbb{1}_j)$ or
(ii) $f(x) + f(y) \ge f(x - \mathbb{1}_j + \mathbb{1}_k) + f(y + \mathbb{1}_j - \mathbb{1}_k)$ for some $k \in supp^-(x - y)$.

A function f is $M^\natural$-concave if the function $-f$ is $M^\natural$-convex. A set $X \subseteq \mathbb{Z}^m$ is an $M^\natural$-convex set if its indicator function δ_X is $M^\natural$-convex.

Here $\mathbb{1}_j$ denotes the jth unit vector, whereas the positive and negative support are defined as $supp^+(x) = \{k \in \mathcal{K} | x(k) > 0\}$ and $supp^-(x) = \{k \in \mathcal{K} | x(k) < 0\}$, respectively. The effective domain is dom$f = \{z \in \mathbb{Z}^m | f(z) \ne \infty\}$. An $M^\natural$-convex function is integrally convex, and thus convex-extensible (Murota, 2003, Theorem 6.42). Since the exchange property (ii) is closely related to the exchange axiom of a matroid, the M stands for "matroid". It means that if we add the jth unit-vector to one point x and exchange it with the ith unit vector of another point y, then the function value decreases or stays the same. Fujishige & Yang, 2003 showed that for the single-unit case the GS condition is equivalent to $M^\natural$-concavity.

Theorem 3 (Fujishige and Yang (2003)). A value function $v : \{0, 1\}^m \to \mathbb{R}$ satisfies the GS condition if and only if it is an $M^\natural$-concave function.

This equivalence extends to multi-unit extensions of the gross substitutes property. Milgrom and Strulovici (2009) distinguish between *weak* and *strong* substitutes. The weak substitutes condition can be seen as the natural extension of the original gross substitutes property to the multi-unit setting by simply quantifying the demand for every item. Note however, that weak substitutes do not correspond to $M^\natural$ functions anymore (Shioura & Tamura, 2015). The strong substitutes condition, on the other hand, transforms a multi-unit to a single-unit valuation function by treating each copy of a good as an individual item. Whenever the corresponding single-unit valuation function fulfills the original gross substitutes property (as defined by Kelso and Crawford (1982)), the multi-unit valuation function satisfies the strong substitutes condition.

Definition 8 (Strong substitutes, SS). Let $\mathcal{K} = \{k_1, k_2, \dots, k_m\}$ be the set of items with $d_i \in \mathbb{N}$ denoting the number of units available of item k_i. Treating each copy of a good as an individual item leads to the definition of a set $\mathcal{K}_s = \{(k_i, z) | k_i \in \mathcal{K}, 1 \le z \le d_i\}$. A multi-unit valuation function $v : \mathbb{N}_0^m \to \mathbb{R}$ can then be transformed to a single-unit valuation function $v_s : \{0, 1\}^{\mathcal{K}_s} \to \mathbb{R}$ by setting $\mathrm{v_s}(x_s) = v(x)$

for $x_s \in \{0,1\}^{\mathcal{K}_s}$ where $x(i) = \sum_{z=1}^{d_i} x_s(k_i, z)$. The valuation v fulfills the strong substitutes condition if v_s is a gross substitutes valuation function.

There exist many equivalent definitions of the strong substitutes condition, among them the binary single-improvement property as shown by Milgrom and Strulovici (2009).

Danilov et al. (2001) and Milgrom and Strulovici (2009) show that a Walrasian equilibrium exists for every finite set of strong substitutes valuations. Ausubel (2006) shows that in case of strong substitutes valuations the Lyapunov function is submodular which ensures the existence of a bidder-optimal Walrasian price vector. While the strong substitutes property is a sufficient condition for the existence of Walrasian equilibria, it is not a necessary one and alternatives exist.

Shioura and Yang (2015) extend the gross substitutes and complements (GSC) condition to a multi-unit and multi-item economy with two classes of items, where units of the same type are substitutable, whereas goods across two classes are complementary. When there is only one class of indivisible goods, their generalized gross substitutes and complements (GGSC) condition becomes identical to the strong-substitute valuation of Milgrom and Strulovici (2009). Further, if each type of good has only one unit, it becomes the gross substitute condition of Kelso and Crawford (1982).

Baldwin and Klemperer (2019) provide an innovative approach characterizing preferences where Walrasian equilibria exist. Instead of working with the value functions, their framework is based on properties of the geometric structure of the regions in the price space where a bidder demands different bundles. A demand type is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small price change. Intuitively, given a valuation v_i, consider the set $\mathcal{L}_i = \{p | D_i(p)| > 1\}$ of all prices at which more than one bundle is in the bidder's demand set. $\mathcal{L}_i$ can be shown to form a so-called *polyhedral complex*, and in particular is a union of hyperplanes, which splits price space into multiple full-dimensional regions where a unique bundle is demanded, which are called *unique demand regions (UDRs)*. Now given a set $\mathcal{D}$ of integral vectors, v_i is of the demand type defined by $\mathcal{D}$ if all normals of all hyperplanes in $\mathcal{L}_i$ are integral multiples of vectors in $\mathcal{D}$.[3] We say that the demand type defined by $\mathcal{D}$ is *unimodular* if any linear independent subset of vectors in $\mathcal{D}$ can be extended with integral vectors to a basis with determinant in $\{-1, 1\}$. It can be shown, that if participants' valuations are concave and all have the same unimodular demand type $\mathcal{D}$, then a Walrasian equilibrium exists. There are several proofs for the unimodularity theorem, see Baldwin and Klemperer (2019); Danilov et al. (2001); Tran and Yu (2015). The authors further show that an equilibrium is guaranteed for more classes of pure complements than of pure substitutes preferences. Note that while all agents being drawn from an equal certain valuation type (SS, GGSC, pure complements) allows for Walrasian equilibria, agent valuations drawn from a mixture of these types in general do not allow for one. Unimodularity of the demand types is a sufficient condition for the existence of Walrasian equilibria. Remarkably, it is also *necessary*: Given valuations of the agents, there exist equilibrium prices for *every* given supply if and only if the agents' demand types are unimodular. Again, whenever the unimodularity condition holds, the optimal solution to the RWDP is integral.

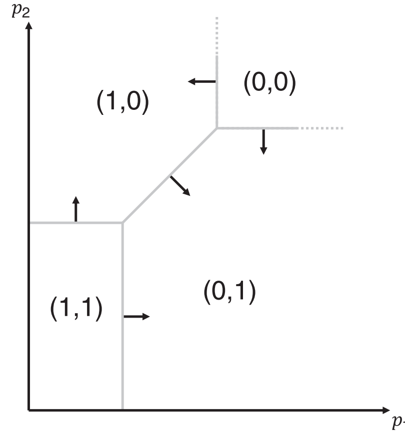

FIGURE 2 Illustration of $\mathcal{L}_i$ (gray). For each indifference hyperplane, we indicate one of the two normal vectors associated with this hyperplane. We can directly see that these normals all lie in $\mathcal{D}$ as defined in Example 4. The tuples (x_1, x_2) indicate the bundles that are demanded in the respective UDRs

Example 4 Consider a market with two items $\mathcal{K} = \{A, B\}$ and a single bidder with a valuation function v, given by the following table

	$x_\emptyset$	x_A	x_B	x_{AB}
x	(0, 0)	(1, 0)	(0, 1)	(1, 1)
$v(x)$	0	2	3	4

The set $\mathcal{L}$ is shown in Figure 2. We can see that v is of the demand type given by $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. It can be checked that $\mathcal{D}$ is actually unimodular.

4.2.3 | From individual to aggregate value functions

We now want to understand when we can expect individual value functions v_i to yield aggregate value functions $v_{\mathcal{I}}$ that are integrally concave. The aggregation of value functions is referred to as *convolution* (see Section 3).

[3]The normals of these hyperplanes have the following economic meaning: Consider a path in price space starting in some UDR. Each time the path crosses an indifference hyperplane, and thus entering another UDR, the demanded bundle changes by the normal vector of the crossed hyperplane, which points into the opposite direction of the price path. In Figure 2 for example, if the price path goes from the UDR (0, 1) to the UDR (1, 0) in a straight line, we cross the hyperplane with normal (1, −1), and of course (1, 0) = (0, 1) + (1, −1).

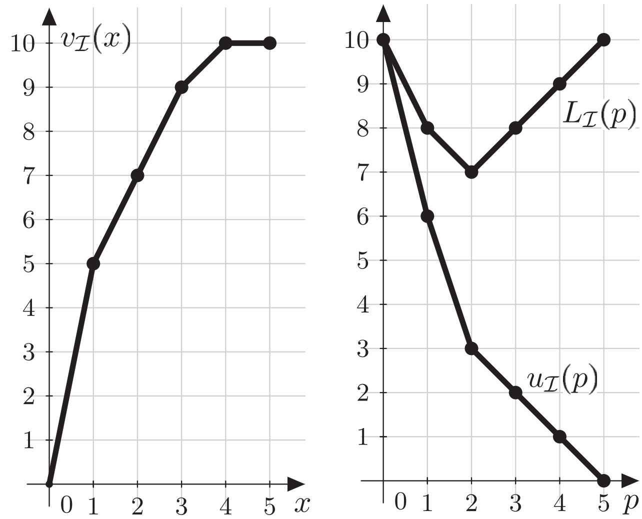

FIGURE 3 For a market with two units of a single indivisible item x, the figure shows the aggregate valuation function $v_{\mathcal{I}}(x)$, the aggregate utility function $u_{\mathcal{I}}$, and the Lyapunov function $L_{\mathcal{I}}(p)$. The Lyapunov function is minimized at $p = 2$, denoting the Walrasian equilibrium prices. Note that $p = 2$ is also the supergradient of $v_{\mathcal{I}}(x)$ at $x = 2$

Murota (2016)[p. 196] shows that if the individual value functions v_i of all bidders $i \in \mathcal{I}$ are $M^\natural$-concave, also their convolution is $M^\natural$-concave. Similarly, one can define the aggregate demand correspondence $D_{\mathcal{I}}(p)$, which is equal to the Minkowski sum $\sum_{i \in \mathcal{I}} D_i(p)$.

For $M^\natural$-concave functions there is a supergradient at any point that determines a Walrasian price p. To show this, let us consider an arbitrary bounded, integrally convex set $A \subset \mathbb{Z}^m_{\geq 0}$. Let $v_{\mathcal{I}} : A \to \mathbb{Z}$ be an $M^\natural$-concave valuation on this set. A bundle $x \in A$ is demanded at price $p \in \mathbb{R}^m$ iff $v_{\mathcal{I}}(x) - \langle p, x \rangle \geq v_{\mathcal{I}}(x') - \langle p, x' \rangle \ \forall x' \in A$, which is equivalent to $v_{\mathcal{I}}(x) + \langle p, x' - x \rangle \geq v_{\mathcal{I}}(x') \ \forall x' \in A$ (as for divisible goods in Section 3). Figure 3 now illustrates an integrally concave value function on the left and the resulting indirect utility function $u_{\mathcal{I}}(p)$ as well as the Lyapunov function $L_{\mathcal{I}}(p)$ for a single item on the right.

With indivisible items and an integrally concave aggregate value function $v_{\mathcal{I}}$, bundle x is demanded at p if and only if p is a supergradient of $v_{\mathcal{I}}$ at x. The superdifferential $\partial v_{\mathcal{I}}(x)$ of an integrally concave function $v_{\mathcal{I}} : \mathbb{Z}^m_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$ at $x \in \text{dom}\, v_{\mathcal{I}}$ is defined as

$$\partial v_{\mathcal{I}}(x) = \{p \in \mathbb{R}^m_{\geq 0} \mid v_{\mathcal{I}}(y) - v_{\mathcal{I}}(x) \leq \langle p, y - x \rangle \ \forall y \in \mathbb{Z}^m_{\geq 0}\}.$$

The individual and aggregate value functions are nondecreasing such that the gradient p^* of the superdifferential is $p^* \geq 0$. With an integrally concave value function $v_{\mathcal{I}}$ there exists an integral equilibrium price vector p^* (Murota et al., 2016). The integrality of the prices follows from the fact that an integer-valued $M^\natural$-concave function $v_{\mathcal{I}}$ on $\mathbb{Z}^m_{\geq 0}$ has an integral subgradient at every point x in dom $v_{\mathcal{I}}$. As both $v_{\mathcal{I}}(x)$ and the subgradient at x are integral, the tangent at $v_{\mathcal{I}}(x)$ has an integral slope p, which can be verified in Figure 3.

An underlying assumption in the study of competitive equilibria is that agents are price-takers, that is, agents honestly report their true demand in response to prices in each round of an auction. Mechanism design, a line of research initiated by Hurwicz (1972), wants to understand how such markets perform when agents are strategic about their demands. Unfortunately, Gul and Stacchetti (1999) showed that even if goods are substitutes, Walrasian markets are not incentive-compatible. The assignment market, where bidders have unit-demand is an exception where straightforward bidding is actually an ex post equilibrium (Shapley & Shubik, 1971; Demange et al., 1986).

5 | ALGORITHMIC AUCTION MODELS

Auctions can be understood as algorithms to solve a welfare maximization problem. Some algorithms provide models that allow us to understand when an auction can be expected to be efficient and when it yields a Walrasian equilibrium.

The auction proposed by Ausubel (2005) for strong substitutes valuations follows a greedy steepest descent algorithm to minimize the (integrally convex) Lyapunov function (Murota & Tamura, 2003). This algorithm has an intuitive interpretation as an ascending auction: subgradients of the Lyapunov function at p are oversupplies at this price: $\partial L(p) = s - D_{\mathcal{I}}(p)$.[4] Knowing that the Lyapunov function is equivalent to the DRWDP (see Proposition 1), the overall auction can now be described as a primal-dual algorithm to solve the RWDP. For the price minimization, both algorithms require all subgradients at each point, that is, the entire demand set needs to be revealed. A specific version of a primal-dual algorithm yields the same steps.

We focus on primal-dual algorithms as a consistent algorithmic framework to model Walrasian auction mechanisms. Let us first describe the auction by Ausubel (2005) as a steepest descent algorithm before we introduce the overall primal-dual auction framework.

5.1 | The auction by Ausubel (2005)

The auction algorithm starts with an arbitrary price vector p below the bidder-optimal Walrasian prices, possibly $p(k) = 0$ for all $k \in \mathcal{K}$. The algorithm then searches iteratively in each round $t \in T$ for a subset of goods $S \subseteq \mathcal{K}$ such that $L(p^t) - L(p^t + \mathbb{1}_S)$ is maximized. Here, p^t denotes the prices in round t. This is equivalent to determining the direction of steepest descent to find the global minimum of this function:

[4]Note that subgradient and steepest descent algorithms for convex minimization are equivalent for differential functions but not for the minimization of discrete functions as in the case of markets with indivisible goods. The difference between the two algorithms is that the steepest descent algorithm evaluates all subgradients at a point, while subgradient algorithms use only a single subgradient. This is equivalent to eliciting the entire demand demand correspondence or only a single bundle from the demand correspondence. As a result, the primal-dual algorithm needs fewer iterations to converge to the exact solution (de Vries et al., 2007).

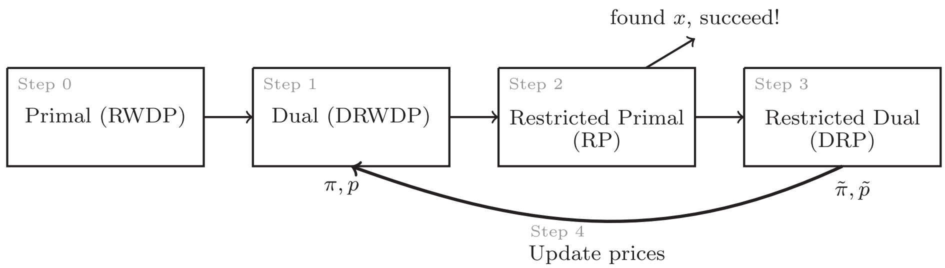

FIGURE 4 A primal-dual algorithm following Papadimitriou and Steiglitz (1998)

(i) At p^t the auctioneer asks each bidder $i \in \mathcal{I}$ for her entire demand set $D_i(p^t)$.
(ii) For all potential price update vectors $\widetilde{p} \in \{\mathbb{1}_S : S \subseteq \mathcal{K}\}$ the auctioneer determines each bidder's decrease of the indirect utility. The auctioneer chooses the price update $\widetilde{p} \in \{\mathbb{1}_S : S \subseteq \mathcal{K}\}$ such that the Lyapunov function is decreased the most, that is, $L(p^t) - L(p^t + \widetilde{p})$ is maximized. In case there are multiple such minimizers, the $\widetilde{p}$ with the smallest number of positive entries is selected. This price vector is referred to as the minimal minimizer and is guaranteed to be unique.
(iii) If no nonempty subset S can be found satisfying $L(p^t) - L(p^t + \mathbb{1}_S) > 0$, then the submodularity of the Lyapunov function guarantees that p^t is the bidder-optimal Walrasian price vector and the algorithm terminates. Otherwise the price p^{t+1} is set to $p^t + \widetilde{p}$ and the algorithm continues.

With integer valuations, $L(p)$ decreases by at least 1 in each iteration and therefore converges after finitely many steps. Murota et al. (2016) analyze the convergence and number of iterations of this steepest descent algorithm. In particular, if the auction algorithm is initialized with $p(k) = 0$ for all $k \in \mathcal{K}$ and p^* is the minimal equilibrium price, the algorithm terminates in exactly $\|p^*\|_\infty = \max_{k \in \mathcal{K}} |p^*(k)|$ iterations. The price update step described in this subsection can now be interpreted as an operation in a primal-dual algorithm to solve the WDP, as we show next.

5.2 | The primal-dual auction framework

Let us now describe the auction by Ausubel (2005) in the context of the more general primal-dual framework. Primal-dual algorithms (Papadimitriou & Steiglitz, 1998) can be used to compute solutions of the RWDP and DRWDP (see Section 4.1.1). Based on a feasible solution of the DRWDP, one derives a restricted primal RP that determines whether supply equals demand at these prices or not. If this is not the case, the dual restricted primal DRP determines the price increment, which is then added to the current price vector of the dual DRWDP, before a new restricted primal is computed. The overall process is illustrated in Figure 4. There is some flexibility in choosing each iteration's direction of price adjustment. In this primal-dual auction framework, we compute the price update that yields the steepest descent of the DRWDP.

Instead of solving the RWDP and the DRWDP directly, the primal-dual algorithm replaces these linear programs by a series of other linear programs known as the restricted primal RP and the dual of the restricted primal DRP. As the primal dual algorithm follows the same price trajectory as Ausubel's auction as we will show below, exactly $\|p^*\|_\infty$ iterations must be executed where p^* is the minimal equilibrium price vector (Murota et al., 2016). In each iteration two linear programs (the RP and DRP) must be solved which both are of exponential size in the number of goods. Clearly, the primal dual algorithm does not give any runtime benefits over solving the RWDP and DRWDP directly. However, executing the primal-dual algorithm instead of solving the RWDP and DRWDP directly allows to interpret the auction by Ausubel (2005) in terms of a primal-dual framework. Moreover, unlike the solution obtained by solving the RWDP and DRWDP directly, the allocation and prices computed by the primal-dual algorithm are guaranteed to constitute the Walrasian equilibrium with bidder-optimal prices.

Let us discuss the algorithm in more detail. In an ascending auction the components of the initial price vector are set to $p(k) = 0$ for all $k \in \mathcal{K}$. To obtain an initial feasible dual solution, the dual is solved with these prices to find initial values for the indirect utility π_i of every bidder i.

With a feasible dual solution, one can exploit the complementary slackness conditions to derive an optimal primal solution which defines a welfare-maximizing allocation of bundles to bidders. Naturally, not every feasible dual solution allows for an optimal primal solution. To check this, one solves an optimization problem known as the restricted primal RP problem.

$$\begin{aligned} \max\ & -\sum_{i \in \mathcal{I}} \lambda_i c_i - \sum_{k \in \mathcal{K}} \mu_k d_k && \text{(RP)} \\ \text{s.t.}\ & \sum_{x \in \mathcal{X}_i} z_i(x) + a_i + c_i = 1 \quad \forall i \in \mathcal{I}(\widetilde{\pi}_i) \\ & \sum_{i \in \mathcal{I}} \sum_{x \in \mathcal{X}_i} x(k) z_i(x) + b_k + d_k = s(k) \quad \forall k \in \mathcal{K}(\widetilde{p}(k)) \\ & z_i(x), a_i, b_k \geq 0 \quad \forall z_i(x) \in \mathcal{J}_z, \forall a_i \in \mathcal{J}_a, \forall b_k \in \mathcal{J}_b \\ & z_i(x) = 0, a_i = 0, b_k = 0 \quad \forall z_i(x) \notin \mathcal{J}_z, \forall a_i \notin \mathcal{J}_a, \end{aligned}$$

$$\forall b_k \notin \mathcal{J}_b$$
$$c_i, d_k \geq 0 \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}$$

Given a feasible dual solution for the DRWDP, any tight dual constraint $\pi_i \geq v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k)$ corresponds to a bundle x that maximizes the utility of bidder i at prices p. Thus, the set of tight dual constraints $\mathcal{J}_z$ corresponds to the bidders' demand sets. In case the given dual solution is optimal, the complementary slackness conditions mandate that whenever the dual constraint has slack, that is, $\pi_i > v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k)$, the corresponding primal variable $z_i(x)$ defining whether bidder i is allocated bundle x equals zero. The interpretation of this is that a bidder is never allocated a bundle not being part of her demand set. Of course, if the given dual solution is not optimal, there might not exist an allocation such that each bidder receives a bundle from her demand set. Therefore, additional slack variables c_i and d_k are introduced to the RP that measure by how much the complementary slackness conditions are violated. A violation may either occur due to bidder i not being allocated a bundle from her demand set ($c_i > 0$) or an item k remaining (partially) unsold ($d_k > 0$). The restricted primal problem tries to find an allocation in which these violations are minimized. In fact, when the optimal solution of the RP equals 0, the complementary slackness conditions are fulfilled so that the current primal and dual solution constitute a Walrasian equilibrium. Otherwise, the price of some items needs to be raised.

Complementary slackness conditions must also hold for the dual constraints $\pi_i \geq 0$ and $p(k) \geq 0$. We denote the set of tight dual constraints by $\mathcal{J}_a$ and $\mathcal{J}_b$ respectively. Due to complementary slackness, the primal variable a_i must equal zero whenever the corresponding dual constraint $\pi_i \geq 0$ has slack. In other words this means that whenever a bidder's indirect utility is positive, she must be allocated a nonempty bundle from her demand set. Similarly, complementary slackness implies that when a price of an item $k \in \mathcal{K}$ is greater than zero, then slack variable b_k must equal zero, which guarantees that all units of item k are allocated in an optimal solution.

In the primal-dual framework of Papadimitriou and Steiglitz (1998) all coefficients λ_i and μ_k in the objective function of the restricted primal RP equal 1. Note that as long as λ_i and μ_k are chosen to be strictly positive, their specific values do not influence the termination criterion of the primal-dual algorithm as one only checks whether the objective equals zero. However, the particular choice of λ_i and μ_k affects the constraints in the dual of the restricted primal DRP, and we will take advantage of this to find a particular price update vector when solving the DRP.

In case the RP objective does not equal zero, the current dual solution of the DRWDP is updated using the solution to the dual of the restricted primal DRP. Solving the DRP essentially means computing a direction $\tilde{\pi}, \tilde{p}$ in which the dual objective function can be improved the most. We set $\tilde{\pi}, \tilde{p}$ such that it minimizes the function $\sum_{i\in\mathcal{I}}(\pi_i + \tilde{\pi}_i) + \sum_{k\in\mathcal{K}} s(k)(p(k) + \tilde{p}(k))$. This is equivalent to finding a subgradient to the Lyapunov function as we will show below.

As there may exist multiple potential directions $(\tilde{\pi}, \tilde{p})$ that minimize the Lyapunov function, we need to make small adaptions to the DRP such that the gradient found by the DRP is equivalent to the minimal minimizer in Ausubel's auction. For this purpose we introduce additional constraints $0 \leq \tilde{p}(k) \leq 1$ for all $k \in \mathcal{K}$. As proven in Ausubel (2005), the Lyapunov function restricted to the unit $|\mathcal{K}|$-dimensional cube $\{p + \tilde{p} \,:\, 0 \leq \tilde{p}(k) \leq 1 \; \forall k \in \mathcal{K}\}$ is minimized on the vertices of this cube. Thus, limiting price updates $\tilde{p}(k)$ to the interval [0, 1] for all $k \in \mathcal{K}$ ensures that the same potential price updates as in Ausubel's auction (i.e., $\{\mathbb{1}_S \,:\, S \subseteq \mathcal{K}\}$) are considered. Note that this also implies that in each iteration of our primal-dual auction framework the respective prices and price updates are integer valued.

Another adaption to be made is to chose λ_i suitably large for all $i \in \mathcal{I}$ so that the decrease of utility for each bidder i is unrestricted when raising prices. To guarantee that the gradient found by the DRP is not only a minimizer of the Lyapunov function but a minimal minimizer, price penalties $\tau_k > 0$ are added to the objective function that are small enough so that their impact on the objective value is negligible.

$$
\begin{aligned}
\min \quad & \sum_{i\in\mathcal{I}} \tilde{\pi}_i + \sum_{k\in\mathcal{K}} (s(k) + \tau_k)\tilde{p}(k) && \text{(DRP)} \\
\text{s.t.} \quad & \tilde{\pi}_i + \sum_{k\in\mathcal{K}} x(k)\tilde{p}(k) \geq 0 \qquad \forall i, x \,:\, z_i(x) \in \mathcal{J}_z && (z_i(x)) \\
& \tilde{\pi}_i \geq 0 \qquad \forall i \,:\, a_i \in \mathcal{J}_a && (a_i) \\
& \tilde{\pi}_i \geq -\lambda_i \qquad \forall i \,:\, a_i \notin \mathcal{J}_a && (c_i) \\
& \tilde{p}(k) \geq 0 \qquad \forall k \,:\, b_k \in \mathcal{J}_b && (b_k) \\
& \tilde{p}(k) \geq -\mu_k \qquad \forall k \,:\, b_k \notin \mathcal{J}_b && (d_k) \\
& 0 \leq \tilde{p}(k) \leq 1 \qquad \forall k \in \mathcal{K}
\end{aligned}
$$

In the following we make the connection between the DRP and the price update step of Ausubel's ascending auction explicit by demonstrating how to transform one approach into the other. Recall that in Ausubel (2006) the goal is to find a $\tilde{p} \in \{\mathbb{1}_S \,:\, S \subseteq \mathcal{K}\}$ leaving all entries of $p + \tilde{p}$ nonnegative and minimizing

$$L(p + \tilde{p}) - L(p).$$

Ausubel (2006) shows that for a fixed $\tilde{p}$ it holds that

$$L(p + \tilde{p}) - L(p) = \sum_{i\in\mathcal{I}} \max_{x\in D_i(p)} \left\{ -\sum_{k\in\mathcal{K}} x(k)\tilde{p}(k) \right\} + \sum_{k\in\mathcal{K}} s(k)\tilde{p}(k).$$

The term $\max_{x\in D_i(p)} \left\{-\sum_{k\in\mathcal{K}} x(k)\tilde{p}(k)\right\}$ is clearly equal to

$$
\begin{aligned}
\min \quad & \tilde{\pi}_i \\
\text{s.t.} \quad & \tilde{\pi}_i \geq -\sum_{k\in\mathcal{K}} x(k)\tilde{p}(k) \qquad \forall x \in D_i(p)
\end{aligned}
$$

Consequently, by adjusting notation and noting that $\mathcal{J}_z$ represents the demand set $D_i(p)$, we can rewrite the problem of minimizing $L(p+\widetilde{p})-L(p)$:

$$\begin{aligned}\min\ & \sum_{i\in\mathcal{I}}\widetilde{\pi}_i+\sum_{k\in\mathcal{K}}s(k)\widetilde{p}(k)\\ \text{s.t.}\ & \widetilde{\pi}_i+\sum_{k\in\mathcal{K}}x(k)\widetilde{p}(k)\geq 0 \qquad \forall i,x: z_i(x)\in\mathcal{J}_z\\ & p(k)+\widetilde{p}(k)\geq 0 \qquad \forall k\in\mathcal{K}\\ & 0\leq\widetilde{p}(k)\leq 1 \qquad \forall k\in\mathcal{K}\end{aligned}$$

As argued above, all price updates and consequently also the prices are integral in each step of our primal-dual auction framework. Hence, the second last set of inequalities can be replaced by

$$\widetilde{p}(k)\geq 0 \qquad \forall k: b_k\in\mathcal{J}_b$$

since $\mathcal{J}_b$ represents all indices where $p(k)$ equals 0.

The only remaining difference to the DRP is that we are missing the inequalities $\widetilde{\pi}_i \geq 0$ for $a_i \in \mathcal{J}_a$. From the definition we see, however, that $a_i\in\mathcal{J}_a$ if and only if the utility of bidder i at price p is 0. But this means that the empty bundle is in her demand set. Hence, $\widetilde{\pi}_i\geq 0$ is actually one of the constraints $\widetilde{\pi}_i+\sum_{k\in\mathcal{K}}x(k)\widetilde{p}(k)\geq 0$. As a result we get that one step of the Lyapunov minimization approach is exactly the same as one step of the primal-dual algorithm.

We restricted our attention so far on explaining the relationship between the primal-dual algorithm and the ascending version of the tâtonnement process described by Ausubel (2005). However, similar observations can also be made for the descending version. The only adaptions to be made in our argument concern the formulation of the DRP. Instead of applying positive price penalties τ_k in the objective function, negative ones have to be used to ensure that a maximal minimizer is found in each iteration. Furthermore, the price updates $\widetilde{p}(k)$ need to be bounded to the interval $[-1,0]$ instead of [0, 1]. Of course, this also implies that μ_k must be chosen suitably large, that is, $\mu_k\geq 1$, in order to allow for price updates of -1.

While the auction described by Ausubel (2005) requires the bidders' valuations to satisfy the strong substitutes condition, the primal-dual algorithm also works for other environments, in particular for economies where the preferences of the bidders fulfill the more general GGSC condition. Sun and Yang (2006) propose the *dynamic double-track auction* (DDT) that terminates in a Walrasian equilibrium if bidders bid straightforwardly and have GSC valuations. Given two sets S_1 and S_2 describing two classes of goods, the auctioneer announces start prices of zero for items in S_1 and suitable high start prices in S_2 such that items in S_1 are overdemanded while items in S_2 are underdemanded. In the course of the auction the auctioneer simultaneously adjusts prices of items S_1 upwards but those of items in S_2 downwards.

Shioura and Yang (2015) introduce the *generalized double-track auction* which is an extension of the DDT to multi-item multi-unit economies where bidders' valuations satisfy the GGSC condition. Their auction starts with an arbitrary integral price vector and then proceeds in two phases. While in the first phase the auctioneer adjusts prices of items in S_1 upwards and prices in S_2 downwards, the price update directions are reversed in the second phase.

Similar to the auction proposed by Ausubel (2005), the price updates in the generalized double-track auction correspond to the steepest descent direction of the Lyapunov function, which can be embedded into a primal-dual algorithm. Essentially, the primal-dual algorithm for the generalized double-track auction combines the DRP adaptions for the ascending and descending version of the auction by Ausubel (2005) as described above. Let the set S_1 and S_2 denote the set of items with an upward and downward moving price trajectory, respectively. While price updates for items in S_1 are bounded to the interval [0, 1], they are restricted to interval $[-1, 0]$ for items in S_2. Similarly, the price penalties in the objective of the DRP are positive for items in S_1 and negative for items in S_2. Once the generalized double-track auction moves from the first to the second phase, the price trajectories of items in S_1 and S_2 are inverted so that the adaptions made to the DRP for items in S_1 now apply for items in S_2 and vice versa.

5.3 | Allocation of items

While our paper focuses on the process of determining equilibrium prices, of course, the auctioneer must determine an equilibrium allocation as well. That is, given a target supply s and an equilibrium price vector p^*, we must find allocations $x_i\in D_i(p^*)$ for every bidder, such that $\sum_{i\in\mathcal{I}}x_i=s$. Since we assume access to demand oracles, that is, each bidder i reports her whole demand set $D_i(p^*)$ in each iteration of the auction, and as demand sets only contain integer points, we could just try every of the finitely many combinations of allocations $x_i\in D_i(p^*)$ in order to match the target supply. This approach is however not very efficient: the number of combinations we possibly have to check is $\Pi_{i\in\mathcal{I}}\,|D_i(p^*)|$, which can clearly be exponential.

The allocation problem can also be interpreted as a flow problem: Consider the directed graph $G=(V,A)$ consisting of $|\mathcal{I}|\cdot|\mathcal{K}|$ vertices $b_i(k)$, describing bidder i's demand of good k, and $|\mathcal{K}|$ vertices $t(k)$, describing the total supply of good k. For each $i\in\mathcal{I}$ and $k\in\mathcal{K}$, there is an arc pointing from $t(k)$ to $b_i(k)$. Now consider a flow x on this graph, where $x_i(k)$ denotes the amount of flow from vertex $b_i(k)$ to vertex $t(k)$. We interpret $x_i(k)$ as the number of units of good k bidder i receives. As usually, given a flow x, and a node v in the graph, the *excess* at node v is the difference of the flow entering the node and the flow leaving the node:

$$\partial x(v)=\sum_{(w,v)\in A}x(w,v)-\sum_{(v,w)\in A}x(v,w).$$

We call the vector ∂x the *boundary* of x. In our above defined graph, we have $\partial x(b_i(k))=x_i(k)$ and $\partial x(t(k))=-\sum_{i\in\mathcal{I}}x_i(k)$. The total number of goods of type k should be equal to the supply of good k. Hence, we have the constraint $\partial x(t(k))=-s(k)$.

Also, each bidder should receive an allocation in her demand set $D_i(p^*)$, so $(\partial x(b_i(1)), \dots, \partial x(b_i(|\mathcal{K}|)) \in D_i(p^*)$ should hold. Thus, the allocation problem can be interpreted as finding a feasible flow with respect to these constraints on the boundary. In the case of strong-substitutes valuations, the demand sets $D_i(p^*)$ are all $M^\natural$-convex, so this is an instance of the *M-convex submodular flow problem*. Polynomial-time algorithms have been developed for this problem, many of them are based on well-known algorithms for min-cost flows. For an overview, see for example (Murota, 2003, Ch. 10).

6 | SUMMARY AND RESEARCH AGENDA

A number of assumptions are crucial for the existence of Walrasian equilibria. Apart from (a) *integral concavity of the aggregate value function*, (b) the *bidders' valuations need to be independent* of each other, and all bidders need to be pure payoff maximizers, that is, have a (c) *quasilinear utility function*. Also, we assume that (d) the bidders are *price-takers and truthfully reveal their demand correspondence* in each round. With these assumptions we can guarantee Walrasian equilibria. However, these are strong assumptions, which might not hold in the field.

(i) Bidder valuations in real-world auctions include complements and substitutes such that Walrasian equilibria might not even exist. Competitive equilibria with nonlinear and personalized prices always exist in ascending auctions under the assumptions above.[5]

(ii) Quasilinearity is not always given as there might exist budget constraints, spitefulness, or market-power effects. For example, if bidders have financial constraints, quasilinearity is violated, and ascending auctions with budget constrained bidders have only been analyzed recently (Gerard van der Laan, 2016; Yang et al., 2018). Even if one tries to set budget constraints endogenously for bidders, it might not always be possible to implement an efficient outcome via an auction (Bichler & Paulsen, 2018).

(iii) Finally, bidders might not bid straightforward in a simple clock auction and behave strategically. A number of papers discusses variations or extensions of simple clock auctions, which yield incentive compatibility (Ausubel, 2006). These are, however, quite different from the simple clock auctions we see in the field.

The assumptions (i)–(iii) above also lead to corresponding research challenges for the operations research community.

1. Most resource allocation problems analyzed in operations research (e.g., scheduling or packing problems) do not satisfy the assumptions that allow for Walrasian equilibria. Duality breaks for nonconvex integer programming problems and new concepts for competitive equilibrium prices need to be derived. The literature on integer programming duality can provide useful insights and guidance how to derive equilibrium prices for such nonconvex allocation problems (Wolsey, 1981).
2. Budget constraints play a major role in many markets. We need to understand equilibria in markets where bidders maximize payoff, but are financially restricted. Very recent results suggest that budget constraints have a substantial impact on the computational complexity of the allocation and pricing problem and require bilevel integer programs which are known to be Σ_2^p-hard (Bichler & Waldherr, 2019). Overall, it will be useful to analyze utility models different from the standard quasi-linear utility function as they have been observed in advertising and other domains where bidders might not maximize payoff but their net present value or return on investment (Fadaei & Bichler, 2017; Baisa, 2017; Baldwin et al., 2020). Effective ways to compute market equilibria in such an environment still need to be developed.
3. Finally, incentive-compatibility plays an important role in small markets where participants can influence the price. Recent research tries to design simple ascending auction and pricing rules that are incentive-compatible (Baranov, 2018). Incentive-compatibility is very restrictive in most environments. For example, in markets with purely quasilinear utilities, the Vickrey–Clarke–Goves mechanism is unique (Green & Laffont, 1979). For larger markets it can also be useful to understand weaker notions of robustness against strategic manipulation (Azevedo & Budish, 2018).

[5]For example, Sun and Yang (2014) introduces an ascending and incentive-compatible auction in markets with only complements using non-linear and anonymous prices. Ausubel and Milgrom (2002), Parkes and Ungar (2000) and de Vries et al. (2007) discuss ascending auctions for markets where bidders have substitutes and complements and allow for discriminatory and non-linear prices. These auctions are incentive-compatible if the bidders' valuations were gross substitutes.

Overall, competitive equilibrium theory is closely related to mathematical optimization and it provides a rich field for operations research to contribute.

ACKNOWLEDGMENT

The financial support from the Deutsche Forschungsgemeinschaft (DFG) (BI 1057/1-8) is gratefully acknowledged. Open access funding enabled and organized by Projekt DEAL.

ORCID

Martin Bichler https://orcid.org/0000-0001-5491-2935

REFERENCES

Ağralı, S., Tan, B., & Karaesmen, F. (2008). Modeling and analysis of an auction-based logistics market. *European Journal of Operational Research*, *191*(1), 272–294.

Arrow, K. J., & Debreu, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, *22*, 265–290.

Arrow, K. J., & Hahn, F. H. (1971). General competitive equilibrium. *Advanced Textbooks in Economics*, *12*.

Ausubel, L. (2005). *Walrasian tatonnement for discrete goods* (Unpublished paper).

Ausubel, L., & Milgrom, P. (2006). *The lovely but lonely vickrey auction*. In P. Cramton, Y. Shoham, & R. Steinberg (Eds.), Combinatorial auctions (pp. 17–40). Cambridge, MA, Ch. 1: MIT Press.

Ausubel, L. M. (2006). An efficient dynamic auction for heterogeneous commodities. *American Economic Review*, *96*(3), 602–629.

Ausubel, L. M., & Milgrom, P. R. (2002). Ascending auctions with package bidding. *Advances in Theoretical Economics*, *1*(1).

Azevedo, E. M., & Budish, E. (2018). Strategy-proofness in the large. *The Review of Economic Studies*, *86*(1), 81–116.

Baisa, B. (2017). Auction design without quasilinear preferences. *Theoretical Economics*, *12*(1), 53–78.

Baldwin, E., Edhan, O., Jagadeesan, R., Klemperer, P., & Teytelboym, A. (2020). *The equilibrium existence duality: Equilibrium with indivisibilities & income effects*. arXiv preprint arXiv:2006.16939.

Baldwin, E., & Klemperer, P. (2019). Understanding preferences: "Demand types," and the existence of equilibrium with indivisibilities. *Econometrica*, *87*(3), 867–932.

Baranov, O. (2018). An efficient ascending auction for private valuations. *Journal of Economic Theory*, *177*, 495–517.

Bertsekas, D. P. (1988). The auction algorithm: A distributed relaxation method for the assignment problem. *Annals of Operations Research*, *14*(1), 105–123.

Bichler, M., Davenport, A., Hohner, G., & Kalagnanam, J. (2006). Industrial procurement auctions. *Combinatorial Auctions*, *1*, 593–612.

Bichler, M., Fux, V., & Goeree, J. (2018). A matter of equality: Linear pricing in combinatorial exchanges. *Information Systems Research*, *29*(4), 1024–1043.

Bichler, M., Fux, V., & Goeree, J. K. (2019). Designing combinatorial exchanges for the reallocation of resource rights. *Proceedings of the National Academy of Sciences of the United States of America*, *116*(3), 786–791.

Bichler, M., & Goeree, J. K. (2017). Handbook of spectrum auction design. Cambridge, MA: Cambridge University Press.

Bichler, M., & Paulsen, P. (2018). A principal-agent model of bidding firms in multi-unit auctions. *Games and Economic Behavior*, *111*, 20–40.

Bichler, M., & Waldherr, S. (2019). *Computing core-stable outcomes in combinatorial exchanges with financially constrained bidders*. In Proceedings of the 2019 ACM Conference on Economics and Computation (pp. 747–747). New York, NY: ACM.

Bikhchandani, S., & Mamer, J. W. (1997). Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory*, *74*(2), 385–413.

Bikhchandani, S., & Ostroy, J. M. (2002). The package assignment model. *Journal of Economic Theory*, *107*(2), 377–406.

Blumrosen, L., & Nisan, N. (2007). Combinatorial auctions. *Algorithmic Game Theory*, *267*, 300.

Caplice, C., & Sheffi, Y. (2003). Optimization-based procurement for transportation services. *Journal of Business Logistics*, *24*(2), 109–128.

Chakrabarty, D., Jain, P., & Kothari, P. (2014). *Provable submodular minimization using Wolfe's algorithm*. In Proceedings of the 27th International Conference on Neural Information Processing Systems—Volume 1. NIPS'14 (pp. 802–809). Cambridge, MA: MIT Press.

Chakrabarty, D., Lee, Y. T., Sidford, A., & Wong, S. C.-w. (2017). *Subquadratic submodular function minimization*. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing. STOC 2017 (pp. 1220–1231). New York, NY: Association for Computing Machinery. https://doi.org/10.1145/3055399.3055419.

Cole, R., Devanur, N. R., Gkatzelis, V., Jain, K., Mai, T., Vazirani, V. V., & Yazdanbod, S. (2016). *Convex program duality, fisher markets, and nash social welfare*. arXiv preprint arXiv:1609.06654.

Colson, B., Marcotte, P., & Savard, G. (2005). Bilevel programming: A survey. *4OR*, *3*(2), 87–107.

Danilov, V., Koshevoy, G., & Murota, K. (2001). Discrete convexity and equilibria in economies with indivisible goods and money. *Mathematical Social Sciences*, *41*(3), 251–273.

de Vries, S., Schummer, J., & Vohra, R. V. (2007). On ascending vickrey auctions for heterogeneous objects. *Journal of Economic Theory*, *132*(1), 95–118.

Demange, G., Gale, D., & Sotomayor, M. (1986). Multi-item auctions. *Journal of Political Economy*, *94*(4), 863–872.

Fadaei, S., & Bichler, M. (2017). Truthfulness with value-maximizing bidders: On the limits of approximation in combinatorial markets. *European Journal of Operational Research*, *260*(2), 767–777.

Fujishige, S., & Yang, Z. (2003). A note on Kelso and Crawford's gross substitutes condition. *Mathematics of Operations Research*, *28*(3), 463–469.

Gale, D. (1963). A note on global instability of competitive equilibrium. *Naval Research Logistics Quarterly*, *10*(1), 81–87.

Galindo, G., & Batta, R. (2013). Review of recent developments in or/ms research in disaster operations management. *European Journal of Operational Research*, *230*(2), 201–211.

Gerard van der Laan, Z. Y. (2016). An ascending multi-item auction with financially constrained bidders. *Journal of Mechanism and Institution Design*, *1*, 109–149.

Green, J., & Laffont, J. (1979). Incentives in public decision making. Amsterdam, Netherlands: North Holland.

Grötschel, M., Lovász, L., & Schrijver, A. (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, *1*(2), 169–197.

Gul, F., & Stacchetti, E. (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, *87*, 95–124.

Herroelen, W., & Leus, R. (2005). Project scheduling under uncertainty: Survey and research potentials. *European Journal of Operational Research*, *165*(2), 289–306.

Hurwicz, L. (1972). On informationally decentralized systems. *Decision and Organization*, *12*.

Kaneko, M. (1976). On the core and competitive equilibria of a market with indivisible goods. *Naval Research Logistics Quarterly*, *23*(2), 321–337.

Kelso, A. S., & Crawford, V. P. (1982). Job matching, coalition formation, and gross substitute. *Econometrica*, *50*, 1483–1504.

Kim, S. (1986). Computation of a large-scale competitive equilibrium through optimization. *Computers and Operations Research*, *13*(4), 507–515.

Klemperer, P. (2010). The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association*, *8*(2–3), 526–536.

Krishna, V. (2009). Auction theory. Boston, MA: Academic Press.

La Salle, J., & Lefschetz, S. (2012). Stability by Liapunov's direct method with applications by Joseph L Salle and Solomon Lefschetz (Vol. *4*). Amsterdam, Netherlands: Elsevier.

Leme, R. P. (2017). Gross substitutability: An algorithmic survey. *Games and Economic Behavior*, *106*, 294–316.

Lessan, J., & Karabatı, S. (2018). A preference-based, multi-unit auction for pricing and capacity allocation. *Computers and Operations Research*, *91*, 237–246.

Madani, M., & Van Vyve, M. (2015). Computationally efficient mip formulation and algorithms for european day-ahead electricity market auctions. *European Journal of Operational Research*, *242*(2), 580–593.

McKenzie, L. W. (1959). On the existence of general equilibrium for a competitive market. *Econometrica: Journal of the Econometric Society*, *27*, 54–71.

Meeus, L., Verhaegen, K., & Belmans, R. (2009). Block order restrictions in combinatorial electric energy auctions. *European Journal of Operational Research*, *196*, 1202–1206.

Milgrom, P., & Strulovici, B. (2009). Substitute goods, auctions, and equilibrium. *Journal of Economic Theory*, *144*(1), 212–247.

Murota, K. (2003). *Discrete convex analysis*. Society for Industrial and Applied Mathematics. Retrieved from https://epubs.siam.org/doi/abs/10.1137/1.9780898718508.

Murota, K. (2016). Discrete convex analysis: A tool for economics and game theory. *Journal of Mechanism and Institution Design*, *1*(1), 151–273.

Murota, K., Shioura, A., & Yang, Z. (2016). Time bounds for iterative auctions: A unified approach by discrete convex analysis. *Discrete Optimization*, *19*, 36–62.

Murota, K., & Tamura, A. (2003). New characterizations of m-convex functions and their applications to economic equilibrium models with indivisibilities. *Discrete Applied Mathematics*, *131*(2), 495–512.

Nesterov, Y. (2018). Lectures on convex optimization. Berlin, Heidelberg: Springer.

Olafsson, S., Li, X., & Wu, S. (2008). Operations research and data mining. *European Journal of Operational Research*, *187*(3), 1429–1448.

Özer, A. H., & Özturan, C. (2009). A model and heuristic algorithms for multi-unit nondiscriminatory combinatorial auction. *Computers and Operations Research*, *36*(1), 196–208 Part Special Issue: Operations Research Approaches for Disaster Recovery Planning.

Papadimitriou, C. H., & Steiglitz, K. (1998). Combinatorial optimization: Algorithms and complexity. Chelmsford, MA: Courier Corporation.

Parkes, D. C., & Ungar, L. H. (2000). Iterative combinatorial auctions: Theory and practice (p. 7481). AAAI/IAAI.

Rockafellar, R. T. (2015). Convex analysis. Princeton, NJ: Princeton University Press.

Shapley, L. S., & Shubik, M. (1971). The assignment game I: The core. *International Journal of Game Theory*, *1*(1), 111–130.

Shioura, A., & Tamura, A. (2015). Gross substitutes condition and discrete concavity for multi-unit valuations: A survey. *Journal of the Operations Research Society of Japan*, *58*(1), 61–103.

Shioura, A., & Yang, Z. (2015). Equilibrium, auction, and generalized gross substitutes and complements. *Journal of the Operations Research Society of Japan*, *58*(4), 410–435.

Starr, R. M. (1969). Quasi-equilibria in markets with non-convex preferences. *Econometrica: Journal of the Econometric Society*, *37*, 25–38.

Sun, N., & Yang, Z. (2006). Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica*, *74*(5), 1385–1402.

Sun, N., & Yang, Z. (2014). An efficient and incentive compatible dynamic auction for multiple complements. *Journal of Political Economy*, *122*(2), 422–466.

Teytelboym, A. (2014). Gross substitutes and complements: A simple generalization. *Economics Letters*, *123*(2), 135–138.

Tran, N. M. & Yu, J. (2015). *Product-mix auctions and tropical geometry*.

Triki, C., Beraldi, P., & Gross, G. (2005). Optimal capacity allocation in multi-auction electricity markets under uncertainty. *Computers and Operations Research*, *32*(2), 201–217.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, *16*(1), 8–37.

Wolsey, L. A. (1981). Integer programming duality: Price functions and sensitivity analysis. *Mathematical Programming*, *20*(1), 173–195.

Yang, Z., Yu, J., et al. (2018). *Efficient ascending menu auctions with budget constrained bidders* (Tech. rep).

How to cite this article: Bichler M, Fichtl M, Schwarz G. Walrasian equilibria from an optimization perspective: A guide to the literature. *Naval Research Logistics* 2021;68:496–513. https://doi.org/10.1002/nav.21963

APPENDIXA.

Proof of Theorem 2. First, let $\{z_i^*(x)\}_{i\in\mathcal{I},x\in\mathcal{X}_i}$ be an optimal solution to the RWDP and $(\{\pi_i^*\}_{i\in\mathcal{I}}, \{p^*(k)\}_{k\in\mathcal{K}})$ be an optimal solution to the DRWDP. By assumption, the optimal value of the WDP is equal to the one of the RWDP, so we may assume that all $z_i^*(x)$ are in $\{0, 1\}$. We may further assume without loss of generality that for each bidder i, there exists exactly one x with $z_i^*(x) = 1$: If $z_i^*(x) = 0$ for all $x \in \mathcal{X}_i$, we can just set $z_i^*(\mathbf{0}) = 1$, where $\mathbf{0}$ is the empty bundle, without altering the value of the WDP, since $v_i(\mathbf{0}) = 0$. Similarly, if for some $k \in \mathcal{K}$, $\sum_{i\in\mathcal{I}}\sum_{x\in\mathcal{X}_i} x(k)z_i^*(x) < s(k)$, we may distribute the remaining items of type k arbitrarily among the agents. This does not decrease the value of the WDP because of monotonicity of the agents' valuations. The (possibly altered) variables $z_i^*(x)$ thus constitute an allocation where the whole supply is distributed among the agents—so the first criterion of a Walrasian equilibrium is satisfied. Let us now check that every bidder receives a bundle in her demand set: If $z_i^*(\bar{x}) = 1$, that is, bidder i receives bundle $\bar{x}$, we have by complementary slackness $\pi_i = v_i(\bar{x}) - \sum_{k\in\mathcal{K}} \bar{x}(k)p^*(k)$. Since π_i^* is part of an optimal solution,

$$\pi_i^* = \max_{x\in\mathcal{X}_i} v_i(x) - \sum_{k\in\mathcal{K}} x(k)p^*(k).$$

Otherwise, we could decrease π_i^*, making the value of the DRWDP smaller. Consequently, $v_i(\bar{x}) - \sum_{k\in\mathcal{K}} \bar{x}(k)p^*(k) = \max_{x\in\mathcal{X}_i} v_i(x) - \sum_{k\in\mathcal{K}} x(k)p^*(k)$, so $\bar{x}$ is in the demand set of bidder i at prices $\{p^*(k)\}_{k\in\mathcal{K}}$. The second condition of a Walrasian equilibrium is thus satisfied, and $\{p^*(k)\}_{k\in\mathcal{K}}$ are equilibrium prices.

For the other direction, let $\{p^*(k)\}_{k\in\mathcal{K}}$ be equilibrium prices together with an allocation, described by binary variables $\{z_i^*(x)\}_{i\in\mathcal{I},x\in\mathcal{X}_i}$. Let $\bar{x}$ be the bundle with $z_i^*(\bar{x}) = 1$. Set $\pi_i^* = v_i(\bar{x}) - \sum_{k\in\mathcal{K}} \bar{x}(k)p(k)$. Since $\bar{x}$ is in the demand set of bidder i, $\pi_i^* \geq v_i(x) - \sum_{k\in\mathcal{K}} x(k)p(k)$ for all bundles x, so $(\{p^*(k)\}, \{\pi_i^*\})$ is feasible for the DRWDP ($\pi_i^* \geq 0$ follows from choosing $x = \mathbf{0}$ in the above inequality). By definition of the Walrasian equilibrium, $\{z_i^*(x)\}$ is also feasible for the (R)WDP. All inequalities in the WDP actually hold with equality—so complementary slackness of the primal problem is trivially fulfilled. From the choice of π_i^* we also directly see, that complementary slackness is satisfied for the dual problem. It follows that the optimal value of the WDP equals the optimal value of the DRWDP.