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Results in Mathematics

Places of Near-Fields: To Heinrich Wefelscheid

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Abstract. Wefelscheid (Untersuchungen über Fastkörper und Fastbereiche, Habilitationsschrift, Hamburg, 1971) generalised the well-known Theorem of Artin/Schreier about the characterization of formally real fields and the fundamental result of Baer/Krull to near-fields. In the last fifty years arose from the Theorem of Baer/Krull a theory, which analyses the entirety of the orderings of a field (E. Becker, L. Bröcker, M. Marshall et al.), as presented e.g. in the book by Lam (Orderings, valuations and quadratic forms, American Mathematical Society, Providence, 1983). At the centre of this theory are preorders and their compatibility with valuations or places. We develop some essential results of this theory for the near-field case. In particular, we derive the Brown/Marshall's inequalities and Bröcker's Theorem on the trivialisation of fans in the near-field case.

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1. Notations

By a *left-ordering* of a near-field F we understand a subset P of F^*[1] with the properties

(O1) $P \cup (-P) = F^*$, (O2) $P + P \subset P$ and (O3) $P \cdot P \subset P$.

The associated order relation $< = <_P$ ($x < y \Leftrightarrow y - x \in P$) is then linear and satisfies: From $x < y$ it follows

$$a + x < a + y \text{ for every } a \in F \text{ and } a\,x < a\,y \text{ for every } a \in P.$$

If $x < y$ and $a \in P$ also implies $x\,a < y\,a$, we call P an *ordering*.

[1]We set $A^* := A \backslash \{0\}$ for each subset A of a near-ring.

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Every left-ordering P of a near-field F contains the *square set* $F^{(2)} := \{x^2 \mid x \in F^*\}$. Every intersection T of left-orderings of F satisfies the conditions:

(PO1) $0 \notin T$, (PO2) $T + T \subset T$, (PO3) $T\,T \subset T$, (PO4) $F^{(2)} \subset T$.

A subset T of the near-field F with (PO1)–(PO4) is called an *preorder* of F.

A preorder S of a near-field F is called a *fan* if every subgroup P of index 2 of F^* containing S with $-1 \notin P$ is a left-ordering of F.

A near-ring $A \subseteq F$ is called a *valuation near-ring* of F, if $F^* = A^* \cup (A^*)^{-1}$. In this case $M_A := \{x \in F \mid x^{-1} \notin A\} \cup \{0\}$ is an ideal of A, which is the complement of the group of units U_A in A.

The set $\Gamma_A := \{x\,A \mid x \in F\}$ is linearly ordered by $< := \subsetneq$ with smallest element $0 := 0\,A$. And $v = v_A : x \to x\,A$ is a *valuation* on F, i.e.:

(B1) $v(x) = 0 \Leftrightarrow x = 0$,
(B2) $v(x+y) \leq \max\{v(x),\, v(y)\}$ for all $x, y \in F$,
(B3) $v(x) \leq v(y) \Rightarrow v(a\,x) \leq v(a\,y)$ for every $a \in F$.

If the ideal M_A has the property

$$x,\, y \in A,\, m \in M_A \Rightarrow (x+m)\,y - x\,y \in M_A\,,$$

the valuation near-ring A is called *strict*. In the case of a strict valuation near-ring by

$$(x + M_A) \cdot (y + M_A) := xy + M_A \ (x, y \in A)$$

a (well-defined) multiplication is given and $\overline{F}_A := A/M_A$ is then a near-field, the *residue near-field of* A.

Before we come to the connection between places, valuations and preorders, we introduce in a preparatory Section the Harrisson topology and some invariants for preorders that are essential for everything else, namely the degree of stability and the chain length—all this for the case of a given near-field.

From now on we consider a left-real near-field F.

2. Preparations

2.1. Harrison Topology

For a preorder T of the near-field F, let X_F/T denote the set of all left-orderings P with $P \supseteq T$.

For each $P \in X_F/T$ we denote by $\sigma_T(P)$ the (well-defined) multiplicative character $x\,T \to \operatorname{sgn}_P x$ of $G_T := G_F(T) = F^*/T$.

The mapping σ_T is an injection of X_F/T into the character group $\hat{G}_T$ of G_T, which can be interpreted as a subset of the Cartesian product $\Pi := \{1, -1\}^{G_T}$. Let $\mathfrak{T} = \mathfrak{T}_0^{G_T}$ be the product of the discrete topology $\mathfrak{T}_0$ of $\{1, -1\}$

on Π. The topology $\mathfrak{T}$ is Hausdorffian and, according to the Theorem of Tychonoff compact. A subbasis of $\mathfrak{T}$ is $\{R_1(a) \mid a \in F^*\} \cup \{R_{-1}(a) \mid a \in F^*\}$, where

$$R_i(a) := \{f : G_T \to \{1, -1\} \mid f(a\,T) = i\}\,.$$

Since

$$\sigma_T(P) \in R_1(a) \Leftrightarrow a \in P \Leftrightarrow -a \notin P \Leftrightarrow \sigma_T(P) \in R_{-1}(-a)$$

the sets

$$H'(a) := R_1(a) \cup \sigma_T(X_F/T) = \{\sigma_T(P) \mid a \in P \in X_F/T\}$$

form a subbasis of the trace topology of $\mathfrak{T}$ in $\sigma_T(X_F/T)$. Consequently, the system $\mathfrak{H}_T(F)$ of the *Harrison sets*

$$H_T(a) := \{P \in X_F/T \mid a \in P\}$$

forms a subbasis of the initial topology $\mathfrak{T}_H(T)$ of σ_T on X_F/T. Because of $H_T(-a) = X_F/T \backslash H_T(a)$ its elements $H_T(a)$ are open and closed. We call $\mathfrak{T}_H(T)$ the *Harrison topology* on X_F/T. It is obviously the trace topology of $\mathfrak{T}_H(F) := \mathfrak{T}_H(T(F))$ (and the coarsest topology on X_F/T for which all mappings $\varepsilon_a : P \to \operatorname{sgn}_P a$ are continuous).

Lemma 1. *$\sigma_T(X_F/T)$ is closed in* $(\Pi, \mathfrak{T})$.

Proof. Let $\chi \in \Pi \backslash \sigma_T(X_F/T)$. Then we have one of the following cases:

(1) $\chi(T) = -1$.
(2) $\chi(a\,T) = \chi(-a\,T)$ for an $a \in F^*$.
(3) $\chi(a\,T) = 1 = \chi(b\,T)$, and $\chi(a\,b\,T) = -1$ for certain $a, b \in F^*$.
(4) $\chi(a\,T) = 1 = \chi(b\,T)$, and $\chi((a+b)\,T) = -1$ for certain $a, b \in F^*$ with $a + b \in F^*$.

(Otherwise $P := \{x \in F^* \mid \chi(x\,T) = 1\}$ would be an element of X_F/T, and $\chi = \sigma_T(P)$.)

Then there exists a $\mathfrak{T}$-neighbourhood $U \subset \Pi \backslash \sigma_T(X_F/T)$ which contains χ:

$$U := R_{-1}(1) \text{ in case } (1)\,,$$
$$U := R_1(a) \cap R_1(-a) \text{ respectively } U := R_{-1}(a) \cap R_{-1}(-a) \text{ in case } (2)\,,$$
$$U := R_1(a) \cap R_1(b) \cap R_{-1}(a\,b) \text{ in case } (3) \text{ and}$$
$$U := R_1(a) \cap R_1(b) \cap R_{-1}(a+b) \text{ in case } (4)\,.$$

□

This gives us the important result:

Theorem 1. *For each preorder T of a near-field F, $(X_F/T, \mathfrak{T}_H(T))$ is a Boolean space (i.e. $(X_F/T, \mathfrak{T}_H(T))$ is Hausdorffian, completely disconnected and compact) with subbasis $\mathfrak{H}_T(F) = \{H_T(a) \mid a \in F^*\}$.*

Proof. For elements $P \neq P'$ from X_F/T there exists an $a \in P \backslash P'$ such that $P \in H_T(a)$ and $P' \in H_T(-a)$. Because of $H_T(a) \cap H_T(-a) = \emptyset$ and $H_T(a) \cup H_T(-a) = X_F/T$ the topology $\mathfrak{T}_H(T)$ is therefore completely disconnected (and Hausdorffian). And according to Lemma 1 $\mathfrak{T}_H(T)$ is compact. □

2.2. Degree of Stability and Chain Length

In this Section, two invariants for preorders are introduced and some of their properties are discussed.

Let a preorder T of a near-field F be given. The factor group $G_T = F^*/T$ is a vector space over F_2. Its dimension is denoted by $\delta(T)$. We further denote by

(A) $\mathrm{st}(T) := \sup\{\delta(S) - 1 \mid S \in X_F/T \text{ is a fan}\}$

the *degree of stability* of T.

Lemma 2. *For each preorder T of a near-field F we have:*

$$\mathrm{st}(T) = 0 \;\Leftrightarrow\; T \,\text{is a left-ordering.}$$

For a preorder T and each element $a \in F^*$, $T[a] = T \cup a\,T \cup (T + a\,T)$ is the smallest multiplicatively closed subgroup of $(F, +)$ containing T and a, therefore $T[a]$ is a preorder exactly if $0 \notin T[a]$; and that is equivalent to $a \notin -T$. The mapping

$$\varepsilon_T : \begin{cases} G_T \to \mathfrak{H}_T(F) \\ a\,T \to H_T(a) \end{cases}$$

is well-defined and surjective. From $H_T(a) = H_T(b)$ it follows $a^{-1}\,b \in P$ for every $P \in X_F/T$ and hence $a^{-1}\,b \in T$ according to the Theorem of Artin:

Lemma 3. *The mapping $\varepsilon_T : a\,T \to H_T(a)$ is a bijection of G_T onto $\mathfrak{H}_T(F)$.*

The mapping $H_T(a) \to T[a]$ is well-defined according to the Theorem of Artin, injective and inclusion-reversing. Consequently

(B) $a\,T < b\,T \;\Leftrightarrow\; T[a] \subsetneqq T[b] \Leftrightarrow H_T(a) \supsetneqq H_T(b)$

defines a (partial) order relation $<$ on G_T with smallest element $T[1] = T$ and largest element $T[-1] = F$.

Following Marshall, we call the element

(C) $\mathrm{cl}(T) := \sup\{k \in \mathbb{N} \mid a_0\,T < a_1\,T < \cdots < a_k\,T,\, a_i \in F^*,\, k \in \mathbb{N}\}$

f $\mathbb{N} \cup \{\infty\}$ the *chain length* of T. We note some simple facts:

Lemma 4. *For a preorder T of a near-field F we have:*

(a) $\mathrm{cl}(T) = 1 \Leftrightarrow T$ *is a left-ordering.*
(b) $\mathrm{cl}(T) \leq 2 \Leftrightarrow T$ *is a fan.*
(c) $\mathrm{cl}(T) \leq \delta(T)$.

Proof. (a) Only $\Rightarrow$ is to be verified. If T is not a left ordering, there exist $P \in X_F/T$ and $a \in P\backslash T$. Therefore, we have $T < aT < -T$.

(b) Let T be a fan and $a, b \in F^*$ with $T < aT < bT < -T$ be given. It follows $a \in T[a] \subset T[b] \subset T \cup bT$, and $a \notin T$. This shows $a \in bT$, i.e. $aT = bT$. Consequently, we have $\mathrm{cl}(T) \leq 2$.

If, on the other hand, T is not a fan, then there is an $a \in F^*\backslash(-T)$ with $T + aT \not\subset T \cup aT$. Obviously, we can assume $1 + a \notin T \cup aT$. Then $T < (1+a)T < aT < -T$ so that $\mathrm{cl}(T) \geq 3$.

(c) Any chain $a_0 T < a_1 T < \cdots < a_n T$ leads to $T[a_0] \subsetneqq T[a_1] \subsetneqq \cdots \subsetneqq T[a_n]$ and thus to a chain $T[a_0]/T \subsetneqq T[a_1]/T \subsetneqq \cdots T[a_n]/T$ of F_2-subspaces of F^*/T. □

We will also need the following property later [cf. for example [9] (8.13)]:

Theorem 2. *[10, (1.7)] Let $T \subset T_i$ $(i = 1, \ldots, n)$ be preorders of F with $X_F/T \subset \bigcup_{i=1}^n X_F/T_i$. Then $\mathrm{cl}(T) \leq \sum_{i=1}^n \mathrm{cl}(T_i)$.*

Proof. Let $<_i$ denote the order relation on F^*/T_i explained according to (B). From $aT \leq bT$, i.e. $H_T(b) \subset H_T(a)$, it follows $H_{T_i}(b) \subset H_{T_i}(a)$ thus $aT_i \leq_i bT_i$ for $i = 1, \ldots, n$. Furthermore:

$(*)$ *In the case $aT < bT$ we have $aT_i <_i bT_i$ for at least one $i \in \{1, \ldots, n\}$.*

Namely, let $aT_i = bT_i$, i.e. $T_i[a] = T_i[b]$ for $i = 1, \ldots, n$ be assumed. For each $c \in F^*$, we conclude $X_F/T[c] = \bigcup_{i=1}^n X_F/T_i[c]$ from the premise, so that $T[c] = \bigcap_{i=1}^n T_i[c]$ according to Artin's Theorem. It follows $T[a] = \bigcap_{i=1}^n T_i[a] = \bigcap_{i=1}^n T_i[b] = T[b]$ in contrary to the assumption.

Now let a chain $a_0 T < a_1 T < \cdots < a_k T$ be given in F^*/T. This induces chains $a_0 T_i \leq_i a_1 T_i \leq_i \cdots \leq_i a_k T_i$ in F^*/T_i for $i = 1, \ldots, n$. If in the i-th of these chains k_i strict inequalities $<_i$ occur, then with $(*)$ it follows obviously $k \leq k_1 + \cdots + k_n \leq \mathrm{cl}(T_1) + \cdots + \mathrm{cl}(T_n)$. This establishes $\mathrm{cl}(T) \leq \mathrm{cl}(T_1) + \cdots + \mathrm{cl}(T_n)$. □

3. The (Real) Place of a Left-Ordering

For each valuation near-ring A of F, let λ_A denote the extension

$$x \to \begin{cases} x + M_A & \text{if } x \in A \\ \infty & \text{if } x \in F\backslash A \end{cases}$$

of the projection $\pi_A : A \to A/M_A$ onto F. We call $\xi := \lambda_A$ the *place belonging to A*. It obviously has the following properties[2]:

(S1) $\xi(x) = \infty \Leftrightarrow x \neq 0$ and $\xi(x^{-1}) = 0$.
(S2) $\xi(x), \xi(y) \neq \infty \Rightarrow \xi(x \pm y) = \xi(x) \pm \xi(y)$.
(S3) $\xi(x), \xi(y) \neq \infty \Rightarrow \xi(xy) \neq \infty$.

[2] We write 0 for $M_A \in \overline{F}_A$.

If A is strict, the following stronger condition (S3') holds:

(S3') $\xi(x),\, \xi(y) \neq \infty \Rightarrow \xi(x\,y) = \xi(x)\,\xi(y)$.

On the other hand, if $F' = (F', +)$ is an abelian group, then a surjective mapping $\xi : F \to F' \cup \{\infty\}$ with the properties (S1), (S2), (S3) is a *F'-place of F*. If $F' = (F', +, \cdot)$ even is a near-field and if (S3') is satisfied, then we call ξ *multiplicative F'-place*. To ξ it belongs a valuation near-ring, from which ξ *essentially* arises in this manner:

Lemma 5. *Let $F' = (F', +)$ be an abelian group and ξ a F'-place of F. Then we have:*

(a) *$A_\xi := \xi^{-1}(F')$ is a valuation near-ring of F with maximal ideal $M_\xi := \xi^{-1}(\{0\})$.*
(b) *The mapping $\varepsilon_\xi : x + M_\xi \to \xi(x)$ is a group isomorphism of $\overline{F}_\xi := A_\xi / M_\xi$ onto F'.*
(c) *The valuation near-ring A_ξ is strict if and only if*

$$\text{(i)} \quad \xi(x) \cdot \xi(y) := \xi(x\,y) \quad (x,\, y \in A_\xi)$$

defines a well-defined operation $\cdot$ in F'.

Then $(F', +, \cdot)$ is a near-field and ε_ξ is a near-field isomorphism. (And ξ is a multiplicative F'-place.)

The proof is done as in the case of a field.

In analogy to the notations introduced for valuations we call A_ξ *valuation near-ring*, M_ξ the (maximal) *ideal*, $U_\xi := A_\xi \backslash M_\xi$ the *group of units*, $\overline{F}_\xi$ the *residual class group* (in case of strict A_ξ the *residual class near-field*), $v_\xi := v_{A_\xi} : x \to x\,A_\xi$ the *canonical valuation* of the place ξ, and we denote $U_\xi^{(1)} := 1 + M_\xi$. The place ξ is called *trivial* if A_ξ is trivial. Trivial places yield near-field isomorphisms.

Remark 1. In many cases (e.g. [1,5,11]) only multiplicative places ξ with the additional property

$$\xi(a\,x - b\,x) \neq \infty, \quad \xi(x) = \infty \;\Rightarrow\; \xi(a) = \xi(b)$$

are considered. These are those places whose valuation near-rings Kalhoff [7] denotes by *place-near-rings*.

The place ξ is called *compatible* with the left ordering $<$ or P, if v_ξ is compatible with $<$ or P respectively. Due to Karpfinger [8] (2.1)(8) and (5)(b) we have:

Lemma 6. *An F'-place ξ of F is compatible with a left-ordering $<$ (or P) if and only if by*

$$x \leq y \Leftrightarrow \xi(x) \leq_\xi \xi(y) \quad (\text{or } P_\xi := \xi(P \cap U_\xi))$$

a left order $<_\xi$ (or P_ξ) of $(F', +)$ is given.

If ξ is multiplicative, then $<_\xi$ (or P_ξ) is a near-field left-ordering. It is an ordering if $<$ is an ordering.

We call $<_\xi$ (resp. P_ξ) the *left-ordering induced* by $<$ and ξ (resp. P and ξ).

A place ξ is called *compatible* or *fully compatible* with a preorder T in F, if ξ is compatible with at least one or with every left-ordering of X_F/T.

It is herewith possible to make a particularly favourable formulation of Karpfinger [8] (2.5), which is going back to Dubois [4] and Brown [3]:

Theorem 3. *For every non-archimedean left-ordering $<$ of F there is exactly one isotonic (i.e. compatible with $<$) and multiplicative $\mathbb{R}$-place* [3] *$\lambda_<$ of F, namely*

$$\text{(ii)} \quad \lambda_< : x \to \begin{cases} \sup\{r \in \mathbb{Q} \,|\, r < x\}\,, & \text{if } x \in A_< \\ \infty\,, & \text{if } x \in F \backslash A_< \end{cases}.$$

We have $v_{\lambda_<} = v_<$ and $A_{\lambda_<} = A_<$.[4]

Proof. According to Karpfinger [8] (2.5) $v := v_<$ is strict and non-trivial with the archimedean left-ordered residue class near-field $(\overline{F}_v, <')$.[5] As every archimedian left-ordered near-field can be embedded into $\mathbb{R}$, there exists an isotonic near-field monomorphism $\varepsilon_<$ from $(\overline{F}_v, <')$ into $(\mathbb{R}, <)$. Consequently

$$\lambda_< : x \to \begin{cases} \varepsilon_<(x + M_v), & \text{if } x \in A \\ \infty, & \text{if } x \in F \backslash A \end{cases}$$

is a multiplicative and isotonic $\mathbb{R}$-place. For $x \in A_<$ and r, $s \in \mathbb{Q}$ we have

$$r < x < s \Rightarrow r = \lambda_<(r) \leq \lambda_<(x) \leq \lambda_<(s) = s.$$

This establishes the representation (ii).

If ξ_1, ξ_2 are two different multiplicative, isotonic $\mathbb{R}$-places of F, then there exists a $x \in F$ with $\xi_1(x) < \xi_2(x)$ and hence an $q \in \mathbb{Q}$ with $\xi_1(x) < q < \xi_2(x)$. Because of $\xi_1(q) = q = \xi_2(q)$ a contradiction arises. □

We call $\lambda_<$ the *real place* of $<$ and also write λ_P for $\lambda_<$. The mapping $P \to \lambda_P$ of $X(F)$ is denoted by λ. By Karpfinger [8] (3.7), (3.9)(b) and (2.1)(b) it follows:

Theorem 4. *For a multiplicative $\mathbb{R}$-place ξ of F the following statements are equivalent:*

(1) $\xi \in \lambda(X_F)$.
(2) $0 \notin \sum \xi(Q(F) \cap U_\xi)$.
(3) $U_\xi^{(1)} \cap (-T(F)) = \emptyset$.

Remark 2. If F is a field, then *each* multiplicative $\mathbb{R}$-place of F is in $\lambda(X_F)$. For a skew field F this is in general not true.

Remark 3. The ring $A_{\xi_<} = A_<$ is a place-ring.

[3] $\mathbb{R}$ is thereby provided with the usual ordering.
[4] For the natural valuation $v_<$ and $A_<$ see the definitions after Karpfinger [8] (2.5).
[5] Whereby $\overline{F}_v = A_</M_<$.

Now if $<_1$ and $<_2$ are two non-archimedean left orderings of F with the same image under λ, then it follows by (3) $A := A_{<_1} = A_{<_2}$ and with the proof of (3) – and the terms there – $\varepsilon_{<_1} \pi_A = \lambda_{<_1} = \lambda_{<_2} = \varepsilon_{<_2} \pi_A$, such that $\varepsilon_{<_1} = \varepsilon_{<_2}$. And that has $(\overline{F}_{<_1}, <_1') = (\overline{F}_{<_2}, <_2')$ as a consequence. This yields:

Theorem 5. *For left-orderings $<_1$, $<_2$ of F we have $\lambda_{<_1} = \lambda_{<_2}$ if and only if the natural valuation near-rings $A_{<_1}$ and $A_{<_2}$ coincide and $<_1$, $<_2$ induce the same left-orderings in $\overline{F}_{<_1} = \overline{F}_{<_2}$.*

3.1. The Place-Topology

We are now following Lam again [9] §§9, 10, 11. The proofs given by Lam can often be applied literally to the nearfield case. However, because the literature quoted is not so easily accessible and we can occasionally give shorter proofs, we give in this Section all corresponding proofs.

For each preorder T of F we abbreviate $\lambda(X_F/T)$ with $L_F(T)$ and write L_F for $L_F(T(F))$ for short.

The quotient topology of the Harrison topology $\mathfrak{T}_H$ of $X(F)$ with respect to λ is denoted by $\mathfrak{T}_L$. Because of Theorem 1 we have:

Lemma 7. $(L_F, \mathfrak{T}_L)$ *is compact.*

To examine $\mathfrak{T}_L$ in more detail, we introduce for each preorder T of F and each element a of the Prüfer near-ring $A_T = \bigcap_{P \in X_F/T} A_P$ [8, Section 3.3] the *value function*:

$$\text{(iii)} \quad \varepsilon_a : \begin{cases} L_F(T) \to \mathbb{R} \\ \lambda_P \to \lambda_P(a) \end{cases}$$

Lemma 8. [4] *ε_a is continuous for every preorder T of F and $a \in A_T$.*

Here $\mathbb{R}$ is provided with the ordinary topology and $L_F(T)$ with the topology induced by $\mathfrak{T}_L$.

Proof. According to the definition of $\mathfrak{T}_L$ as quotient topology it suffices to show for $a \in A_T$ that $\mu_a := \varepsilon_a \lambda : X_F/T \to \mathbb{R}$ is continuous. This is the case if for each $r \in \mathbb{Q}$ the sets $\mu_a^{-1}(]r, +\infty[)$ and $\mu_a^{-1}(]-\infty, r[)$ are open in $(X_F/T, \mathfrak{T}_H(T))$. Because of (ii) in Theorem 3, we have for each $P \in X_F/T$:

$$\begin{aligned} P \in \mu_a^{-1}(]r, +\infty[) &\Leftrightarrow r < \mu_a(P) = \lambda_P(a) = \sup\{s \in \mathbb{Q} \mid s <_P a\} \\ &\Leftrightarrow r < s <_P a \text{ for a } s \in \mathbb{Q} \\ &\Leftrightarrow 0 < t <_P a - r \text{ for a } t \in \mathbb{Q}. \end{aligned}$$

Thus $\mu_a^{-1}(]r, +\infty[) = \bigcup_{0<t\in\mathbb{Q}} H_T(a - r - t)$ is open in $(X_F/T, \mathfrak{T}_F(T))$.

Similarly, one shows that $\mu_a^{-1}(]-\infty, r[)$ is open. □

We also need:

Lemma 9. *The set $\{\varepsilon_a \mid a \in A_{T(F)}\}$ separates different points in L_F.*

Proof. Let there be given different $\xi, \xi' \in L_F$. Then there exists an $x \in F$ with $\xi(x) < \xi'(x)$.

1. case: $\xi'(x) \neq \infty$. There is an $r \in \mathbb{Q}$ with $\xi(x) < \xi(r) = r = \xi'(r) < \xi'(x)$. For $y := x - r$ it follows $\xi(y) < 0 < \xi'(y)$. According to Karpfinger [8], (3.14), $a := (y + y^{-1})^{-1}$ is in $A_{T(F)}$, and $\xi(a) < 0 < \xi'(a)$.

2. case: $\xi'(x) = \infty$. We have $0 < \xi(x + n)$ for a suitable $n \in \mathbb{N}$. For $y := x + n$ it follows $0 < \xi(y) < \xi'(y) = \infty$ and hence $\xi'(y^{-1}) < \xi(y^{-1}) < \infty$, so that $\xi(-y^{-1}) < \xi'(-y^{-1}) < \infty$. Thus, we have again case 1 (for $x := -y^{-1}$). □

With these results we now get:

Lemma 10. [4]

(a) *The space $(L_F, \mathfrak{T}_L)$ is Hausdorffian.*
(b) *The mapping λ is continuous and closed.*
(c) *For every preorder T of F, $L_F(T)$ is a compact subset of L_F.*

Proof. (a) According to Lemma 9, for $\xi \neq \xi'$ in L_F there is an $a \in A_{T(F)}$ with $\xi(a) \neq \xi'(a)$. If I, I' are disjoint open intervals of $\mathbb{R}$ with $\xi(a) \in I$ and $\xi'(a) \in I'$, then, by Lemma 8, $\varepsilon_a^{-1}(I)$ and $\varepsilon_a^{-1}(I')$ are disjoint environments of ξ and ξ', respectively.

(b) follows from (a), Theorem 1 and the continuity of λ.

(c) By Theorem 1, X_F/T is a compact subset of $(X_F, \mathfrak{T}_H)$. Because of the continuity of λ, the assertion follows. □

A concrete description of $\mathfrak{T}_L$ is now possible:

Lemma 11. [4]

(a) *The topology $\mathfrak{T}_L$ is the initial topology of the mappings ε_a ($a \in A_{T(F)}$).*
(b) *The sets $H(a) := \{\xi \in L_F \mid \xi(a) > 0\}$ with $a \in A_{T(F)}$ form a subbasis of $\mathfrak{T}_L$.*

Proof. (a) Let $\mathfrak{T}'$ be the initial topology of the mappings ε_a ($a \in A_{T(F)}$), and $\mathfrak{T} := \mathfrak{T}_L$. According to Lemma 8, $\mathfrak{T}$ is finer than $\mathfrak{T}'$. Since $\mathfrak{T}$ is compact by Lemma 7, the topology $\mathfrak{T}'$ is compact, too. Since the mappings ε_a separate points in L_F and are continuous with respect to $(\mathfrak{T}_H, \mathfrak{T}')$, it follows literally as in the proof of Lemma 10(a), that $\mathfrak{T}'$ is also Hausdorffian. Consequently, Id : $(L_F, \mathfrak{T}) \to (L_F, \mathfrak{T}')$ is continuous and closed, i.e. a homeomorphism: $\mathfrak{T} = \mathfrak{T}'$.

(b) According to Lemma 8 the sets $H(a) = \varepsilon_a^{-1}(]0, +\infty[)$ are open with respect to $\mathfrak{T} = \mathfrak{T}_L$. On the other hand, the sets $\varepsilon_a^{-1}(]r, s[)$ with $a \in A_{T(F)}$ and $r < s$ in $\mathbb{Q}$ form a subbasis of $\mathfrak{T}' = \mathfrak{T}$ ($\mathfrak{T}'$ as in the proof of (a)). Furthermore,

$$\begin{aligned}\varepsilon_a^{-1}(]r, s[) =& \{\lambda_P \in \lambda(X(F)) \mid r < \lambda_P(a) < s\}\\ =& \{\lambda_P \in \lambda(X(F)) | \lambda_P(a - r) > 0\} \cap \{\lambda_P \in \lambda(X(F)) | \lambda_P(s - a) > 0\}\\ =& H(a - r) \cap H(s - a)\,,\end{aligned}$$

and $a - r,\ s - a \in A_{T(F)}$. This proves the assertion. □

Remark 4. Unlike $\mathfrak{T}_H$, $\mathfrak{T}_L$ need not be totally disconnected at all. There are even fields F for which $\mathfrak{T}_L$ is connected.

According to Lemma 8, for each preorder T of F the value functions ε_a $(a \in A_T)$ lie in the ring $C_T := C(L_F(T), \mathbb{R})$ of all continuous functions of $L_F(T)$ in $\mathbb{R}$. Since $L_F(T)$ is compact according to Lemma 10 (c), the supremum norm

$$\text{(iv)} \quad \|f\|_T := \sup\{|f(\lambda_P)| \,|\, \lambda_P \in L_F(T)\} \quad (f \in C_T)$$

is defined on C_T.

Lemma 12. *For each preorder T of F the set $\varepsilon(A_T) = \{\varepsilon_a \,|\, a \in A_T\}$ lies dense in C_T with respect to $\| \, \|_T$.*

Proof. Because of Lemma 9 $\varepsilon(A_T)$ separates points in $L_F(T)$; and the constant function $\varepsilon_1 : \xi \to 1$ lies in $\varepsilon(A_T)$. The assertion therefore follows with the Theorem of Stone/Weierstrass. □

This provides:

Theorem 6. (Separation criterion) *For each preorder T of F and disjoint closed subsets A, B of X_F/T are equivalent:*

(1) $\lambda(A) \cap \lambda(B) = \emptyset$.
(2) *There exists an $x \in A_T \cap \bigcap_{\xi \in \lambda(A \cup B)} U_\xi$ with $A \subset H_T(x)$ and $B \subset H_T(-x)$.*
(3) *There exists an $x \in A_T \cap \bigcap_{\xi \in \lambda(A)} U_\xi$ with $A \subset H_T(x)$ and $B \subset H_T(-x)$.*

Proof. (1) ⇒ (2): According to Lemma 10 $\lambda(A)$ and $\lambda(B)$ are closed. By Urysohn's lemma and due to $\lambda(A) \cap \lambda(B) = \emptyset$, there is a continuous function $f : X_F/T \to \mathbb{R}$ with $f(\xi) = 1$ for all $\xi \in \lambda(A)$ and $f(\zeta) = -1$ for all $\zeta \in \lambda(B)$. And according to Lemma 12 there exists an $x \in A_T$ with $\|\varepsilon_x - .f\|_T < 1$, i.e.

$$\text{(v)} \quad |\lambda_P(x) - f(\lambda_P)| < 1 \ \text{ for all } \lambda_P \in \lambda(X(F)).$$

For every $P \in A$ we have $\lambda_P \in \lambda(A)$ so that $|\lambda_P(x) - 1| < 1$ due to (v), and it follows $\lambda_P(x) > 0$. This justifies $x \in U_{\lambda_P}$ and $x \in P$ according to Theorem 3. Similarly, for any $P \in B$, i.e. $\lambda_P \in \lambda(B)$, we have $|\lambda_P(x) + 1| < 1$ due to (v) and therefore shows $\lambda_P(x) < 0$. This has $x \in U_{\lambda_P}$ and—again according to Theorem 3—$x \in -P$ as a consequence.

(2) ⇒ (3) is trivially correct.

(3) ⇒ (1): Suppose $\lambda(P) = \lambda(Q)$ for certain $P \in A$ and $Q \in B$. And let x be chosen as in (3). Since $x \in U_{\lambda_P} = U_{\lambda_Q}$ and $x \in P$ and $x \in -Q$ we get a contradiction $\lambda_P(x) > 0$ to Theorem 3 and $\lambda_Q(x) < 0$. □

An interesting consequence is:

Corollary 1. *For each two preorders T, T' of F we have*

$$\lambda(X_F/(T \cap T')) = \lambda(X_F/T) \cup \lambda(X_F/T').$$

Proof. The inclusion $\lambda(X_F/T) \cup \lambda(X_F/T') \subset \lambda(X_F/(T \cap T'))$ is trivially correct. Suppose there is an $P \in X_F/(T \cap T')$ with $\lambda(P) \notin \lambda(X_F/T) \cup \lambda(X_F/T') = \lambda(X_F/T \cup X_F/T')$. According to Theorem 6 [(1) $\Rightarrow$ (2)], $\{P\}$ can be separated from the set $B := X_F/T \cup X_F/T'$ which is closed by Theorem 1: There exists an $x \in P$ with $-x \in Q$ for each $Q \in B$. According to Artin's Theorem it follows $-x \in T \cap T'$ in contradiction to $T \cap T' \subset P$. $\square$

We derive a corollary for natural evaluation near-rings and Prüfer near-rings:

Corollary 2. *For each two preorders T, T' of F we have*

$$A_{T\cap T'} = A_T \cap A_{T'} \quad \text{and} \quad A^{T\cap T'} = .A^T\, A^{T'}.$$ [6]

Proof. The inclusions $A_{T\cap T'} \subset A_T \cap A_{T'}$ and $A^T\, A^{T'} \subset A^{T\cap T'}$ are clear. On the other hand, according to Corollary 1 for each $P \in X_F/(T \cap T')$ A_P coincides with $A_{P'}$ for any $P' \in X_F/T \cup X_F/T$. From this the reverse inclusions follow. $\square$

If T is a preorder of F, then $\{X_\xi \,|\, \xi \in L_F(T)\}$ with

$$\text{(vi)} \quad X_\xi := \{P \in X_F/T \,|\, \lambda_P = \xi\}$$

forms a decomposition of X_F/T according to Theorem 3. We now show that $X_\xi = X_F/T_\xi$ for some fan T_ξ with $T_\xi \supseteq T$.

Let $P \in X_F/T$ be a left-ordering and $\xi := \lambda_P$. It follows from Theorem 3 and the fact that $\xi = \varepsilon_\xi\, \pi_{v_\xi}$ [cf. Lemma 5 (b)] and [8], (3.2) that:

$$T_\xi := T\,\xi^{-1}(\overline{F}^*_\xi \cap \mathbb{R}^{(2)}) = T\,\xi^{-1}(\xi(U_\xi \cap P)) = T\,\pi_{v_\xi}^{-1}(\pi_{v_\xi}(U_\xi \cap P)) = T \wedge \pi_{v_\xi}(P).$$ [7]

And according to Karpfinger [8], (3.2), the preorder T_ξ of F is fully compatible with π_{v_ξ} (i.e. with ξ) and $\pi_{v_\xi}(T_\xi) = \pi_{v_\xi}(P)$, i.e. due to Theorem 3: $\xi(T_\xi \cap U_\xi) = \overline{F}^*_\xi \cap \mathbb{R}^{(2)}$. According to Karpfinger [8], (2.3) (b) $\pi_{v_\xi}(P)$ is a left-ordering of $\overline{F}_\xi$; and by Karpfinger [8], (3.10) T_ξ is a fan of F. This shows the first of the following statements:

Lemma 13. *For each preorder T of F and each place $\xi \in L_F(T)$ we have:*

- (a) $T_\xi := T\,\xi^{-1}(\overline{F}^*_\xi \cap \mathbb{R}^{(2)})$ *is a fan of F fully compatible with ξ; and* $\xi(T_\xi \cap U_\xi) = \xi(P \cap U_\xi) = \overline{F}^*_\xi \cap \mathbb{R}^{(2)}$ *for each $P \in X_F/T$ with $\xi = \lambda_P$.*
- (b) $X_\xi = X_F/T_\xi$, *i.e. for $P \in X_F/T$ we have: $T_\xi \subset P \Leftrightarrow \lambda_P = \xi$.*
- (c) X_F/T *is the disjoint union of the sets X_F/T_ξ with $\xi \in L_F(T)$.*

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[6] $A^T\, A^{T'}$ denotes the nearring generated by A^T and $A^{T'}$.

[7] With the repeatedly noted agreement $\pi_{v_\xi}(A) := \pi_{v_\xi}(A \cap U_\xi)$. And $\mathbb{R}^{(2)} = \{x^2 \,|\, x \in \mathbb{R}^*\} = \{x \in \mathbb{R} \,|\, x > 0\}$.

Proof. (b) From $P \in X_F/T$ and $\lambda_P = \xi$ it follows with (a):

$$T_\xi = T\,\xi^{-1}(\overline{F}_\xi^* \cap \mathbb{R}^{(2)}) = T\,\xi^{-1}(\xi(P \cap U_\xi)) \subset T\,P = P$$

(since $\xi(P \cap U_\xi) \subset \mathbb{R}^{(2)}$).

Thus, $X_\xi \subset X_F/T_\xi$. On the other hand, any left-ordering P of X_F/T_ξ is compatible with ξ by (a). Therefore, due to Theorem 3 we have $\lambda_P = \xi$, i.e. $P \in X_\xi$.

(c) follows directly from (b). □

We now prove the *Brown/Marshall's inequalities* (cf. for example Lam [9] 10.10).

Theorem 7. *For each preorder T of F we have*

$$|L_F(T)| \leq \mathrm{cl}(T) \leq 2\,|L_F(T)|.$$

Proof. To prove the first inequality, let $P_1, \ldots, P_n \in X_F/T$ with different real places $\lambda_{P_1}, \ldots, \lambda_{P_n}$ be given. We construct inductively elements $a_0, \ldots, a_n \in A_T$ with the properties:

(a) $a_0 = -1$.
(b) $a_i \in U_{P_1} \cap \ldots \cap U_{P_n}$.
(c) For each $i = 1, \ldots, n$ a_i is positive with respect to the left-orderings in $H_T(a_{i-1}) \cup \{P_i\}$ and negative with respect to $P_{i+1}, \ldots, , P_n$.

It then follows $\emptyset = H_T(a_0) \subsetneqq H_T(a_1) \subsetneqq \cdots \subsetneqq H_T(a_n)$. According to Sect. 2.2, (B) and (C), this results in $n \leq \mathrm{cl}(T)$, which proves the first inequality.

Now, if $a_0, \ldots, a_{i-1}$ $(i \geq 0)$ with the properties (a)–(c) are already constructed, then it follows with Theorem 6 [(3) $\Rightarrow$ (1)]: $\lambda(H_T(a_{i-1})) \cap \lambda(\{P_{i+1}, \ldots, P_n\}) = \emptyset$, since a_{i-1} separates the disjoint and closed sets $A := \{P_{i+1}, \ldots, P_n\}$ and $B := H_T(a_{i-1})$ and lies in $U_{P_{i+1}} \cap \ldots \cap U_{P_n}$. Then also $\lambda(A) \cap \lambda(B') = \emptyset$ for $B' := H_T(a_{i-1}) \cup \{P_i\}$. With Theorem 6 [(1) $\Rightarrow$ (2)]—for B' instead of B—it follows that there exists an element a_i with properties (b) and (c).

To prove $\mathrm{cl}(T) \leq 2\,|L_F(T)|$ we may assume $L_F(T)$ as finite. According to Lemma 13 X_F/T is a (disjoint) union $X_F/T = X_F/T_{\xi_1} \cup \cdots \cup X_F/T_{\xi_n}$ with places $\xi_i \in L_F(T)$; and each T_{ξ_i} is a (T covering) fan. Due to Theorem 2 and Lemma 4(b) we have

$$\mathrm{cl}(T) \leq 2\,(\mathrm{cl}(T_{\xi_1}) + \cdots + \mathrm{cl}(T_{\xi_n})) \leq 2\,n.$$

□

From Theorem 7 and Lemma 4(b) one obtains directly:

Corollary 3. *For each fan S of F we have $|L_F(S)| \leq 2$.*

We can now prove Bröcker's Theorem [2] about the *trivialisation of fans* (c.f. Kalhoff [6]):

Theorem 8. *For each fan S of F for which X_F/S contains at least one ordering, there is a place $\xi : F \to F' \cup \{\infty\}$ fully compatible with S for which $\xi(S \cap U_\xi)$ is a trivial fan*[8] *of F'.*

Proof. If S is a trivial fan, we choose the trivial place ξ. Let S therefore be nontrivial. According to corollary 3, two cases are possible:

1. Case: $L_F(S) = \{\xi\}$ with non-trivial ξ. In this case, any $P \in X_F/S$ is compatible with ξ and by Lemma 13 (and the Theorem of Artin) it follows $S = T_\xi$ and $\xi(S \cap U_\xi) = \overline{F}_\xi^* \cap \mathbb{R}^{(2)}$; and that is a left-order of $\overline{F}_\xi$.

2. Case: $L_F(S) = \{\eta, \zeta\}$ with different places η, ζ, and without loss of generality $\zeta = \lambda_Q$ for some ordering Q of X_F/S. All $P \in X_F/S$ are non-archimedean, i.e. η, ζ are non-trivial, and topologically equivalent. By Karpfinger [8], (2.9) it follows that $A := A_\eta \, A_\zeta$ is a strict valuation near-ring $\neq F$ compatible with any $P \in X_F/S$. Now $\xi = \lambda_A$ is nontrivial. According to Karpfinger [8], (3.10) and Lemma 5, $S' := \xi(S \cap U_\xi)$ is a ξ-fan of the near-field $\overline{F}_\xi$. If S' were non-trivial, then a non-trivial place $\xi' : \overline{F}_\xi \to F' \cup \{\infty\}$ fully compatible with S' exists. This is shown in the last part of the proof applied to $\overline{F}_\xi$, since $\xi(Q \cap U_\xi)$ is an ordering of $\overline{F}_\xi$ due to Karpfinger [8], (2.3) (b). Then—with $\xi'(\infty) := \infty$—$\xi' \, \xi : F \to F' \cup \{\infty\}$ is a place of F fully compatible with S. Since ξ' is nontrivial, we have $A_{\xi' \xi} \subsetneqq A_\xi$. Since $\xi' \, \xi$ is compatible with all $P \in X_F/S$, we have on the other hand $A_\eta, A_\zeta \subset A_{\xi' \xi}$ and hence we get the contradiction $A_\xi = A = A_\eta \, A_\zeta \subset A_{\xi' \xi}$. □

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