Statistical nuclear properties (level densities, spin distributions)

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Abstract. A general overview is given on the phenomenological methods used to describe the level densities in nuclei. Two well-known two-parameter formulas of level densities, the Back-Shifted Fermi Gas (BSFG) model and the Constant Temperature (CT) model, were used. A common ingredient of both is the spin distribution function, which contains in Ericsons's parametrization the spin-cutoff parameter σ. A realistic description of the parameters of both spin distribution function and the two level density models has been obtained by fitting the experimental data of 310 nuclei between ^{18}F and ^{251}Cf, consisting of the complete level schemes at low excitation energies and the s-wave neutron resonance spacings at the neutron binding energy. We determine a simple formula for the spin-cutoff parameter as a function of mass number and excitation energy. Also, an even-odd spin staggering in the spin distribution of the even-even nuclei was observed, and described with a simple formula. Using this newly defined spin distribution function, an empirical set of parameters of the BSFG and CT models was determined by fitting both the low-energy levels and the neutron resonance spacings. For these parameters, simple formulas were proposed that involve only quantities available from the mass tables, and allow reasonable estimations of the level density parameters for nuclei far from stability. Both the BSFG and CT models describe equally well the level densities at energies up to at least the neutron binding energy. Finally, we discuss recent experimental evidence that the CT model is the more correct description of the nuclei in the low-excitation energy (pairing) regime.

1. Introduction

The nuclear level densities represent the basic statistical information on nuclei both at low and higher excitation energies. The formulas for the level density are usually separated in a part with the total level density $\rho(E)$ (which increases exponentially with the excitation energy E) and a function $f(J)$ for the spin distribution,

$$\rho(E, J) = f(J)\rho(E). \tag{1}$$

Here a possible dependence on the level parity is neglected.

Two formulas are frequently used for the description of the total level density, the back-shifted Fermi gas formula (BSFG) [1],

$$\rho_{BSFG}(E) = \frac{e^{2\sqrt{a(E-E_1)}}}{12\sqrt{2}\sigma a^{1/4}(E-E_1)^{5/4}} \tag{2}$$

with the free parameters a and E_1, and the constant temperature formula (CT) [1],

$$\rho_{CT}(E) = \frac{1}{T}e^{(E-E_0)/T} \tag{3}$$

with the free parameters T and E_0.

These formulas were found equivalent in describing the experimental data at excitation energies generally below 10 MeV [2, 3, 4]. The parameters of the level density formulas can be experimentally obtained by fitting known energy levels of complete level schemes at low excitation energies together with neutron resonances at the neutron binding energy. This approach was described in refs. [2, 3, 4]. In the present article we review the recent work along this line. The free parameters of both the spin distribution function and the two level density models were determined by a fit to updated experimental level scheme data. We use a database of complete low-energy level schemes for 310 nuclei between ^{19}F and ^{251}Cf. For most of these nuclei the neutron resonance densities were also known. A list of these nuclei is given in [4]. For the sets of empirical values for both the spin-cutoff parameter of the spin distribution function and the parameters of the BSFG and CT models, we propose simple formulas that can be easily calculated using only quantities from the mass tables. These formulas can be used to obtain reasonable estimations for the level densities of nuclei farther from stability. Finally, recent experimental evidence is reviewed that at low excitation energy the Constant Temperature model is the more correct description of nuclei.

2. Spin distribution function

The spin distribution function $f(J)$ in eq. (1) is described by the formula proposed by Ericson [6]

$$f(J,\sigma) = e^{-J^2/2\sigma^2} - e^{-(J+1)^2/2\sigma^2} \approx \frac{2J+1}{2\sigma^2}e^{-J(J+1/2)/2\sigma^2} \tag{4}$$

with a single free parameter σ. The spin-cutoff parameter σ is generally related to an effective moment of inertia. This parameter may depend on the nuclear mass A, the level density parameter a or the nuclear temperature T. Also, it is predicted to increase with the excitation energy E. Different formulas were proposed for these dependencies [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Because of these ambiguities and in addition due to the lack of systematic experimental information we performed studies of the spin distribution function, using the actual knowledge of the discrete levels at low excitation energies.

2.1. Staggering of spin distribution function in even-even nuclei

In a first study [17], we concentrated on the general evolution of the spin-cutoff parameter σ at low excitation energies, with the mass number and possibly other quantities, without considering a dependence on the excitation energies. From our database with 310 nuclei, we selected complete level schemes in a given energy (usually up to $1-3$ MeV excitation) and spin range. This procedure yielded 8116 levels (with known spin value) in 1556 spin groups (levels with the same spin in a nucleus).

An important outcome of this study has been the observation of a spin staggering of the spin distribution function for the even-even nuclei. This is illustrated in Fig. 1 for the nucleus ^{112}Cd

where it can be seen that the spin distribution cannot be accurately described by the smooth function (4). This oscillation in spin has been observed for all even-even nuclei, but is absent
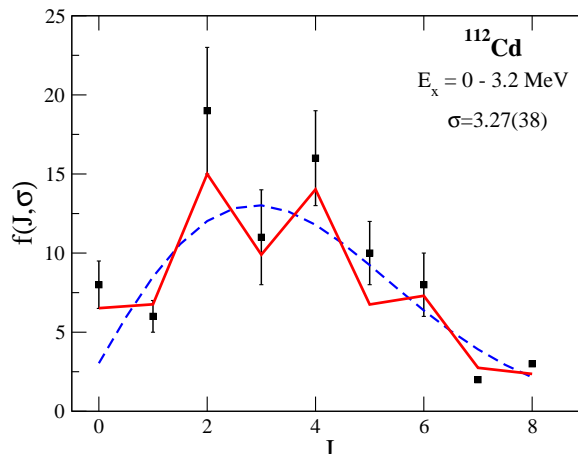
Figure 1. Experimental and calculated spin distribution for ^{112}Cd (symbols). Dashed line: fit with Ericson's formula (4). Full line: fit with the newly proposed formula (5).

in the odd-mass and the odd-odd nuclei. While the odd- and even-spin value deviations from the average description given by formula (4) are rather symmetrical, the deviation of the spin 0 value was found systematically larger. To describe this even-odd spin staggering, we proposed the following formula for the spin distribution of the even-even nuclei:

$$f_{ee}(J,\sigma) = f(J,\sigma)\cdot(1+x), \qquad (5)$$

where $f(J,\sigma)$ is given by eq. (4), and

$$x = \begin{cases} +0.227(14), & \text{for even spin values,} \\ -0.227(14), & \text{for odd spin values,} \\ +1.02(9), & \text{for zero spin levels.} \end{cases} \qquad (6)$$

The staggering parameter x was not found to depend on the mass number. The corresponding fit procedure of the experimental data (the number of levels of each spin in a nucleus) is described in ref. [17]. We investigated the dependence of σ on several quantities, like mass, level density parameter a, temperature T, moment of inertia, and deformation β, and found that only the mass dependence was important. The data could be well described by the simple formula $\sigma^2 = 2.61A^{0.28}$, which thus represents an average description of "low excitation energy" region.

The even-odd spin staggering of the spin distribution function in even-even nuclei was mentioned some time ago [2] but not systematically investigated and never discussed in connection with predictions of theoretical models. It appears, however, that, at least some theoretical models intrinsically contain this spin oscillation. This is shown in Fig. 2 for two models. In the left-side graph, experimental spin distribution functions for ^{168}Er in two excitation energy ranges ($0-1.5$ MeV and $1.5-2.3$ MeV) are compared with predictions of the IBA (Interacting Boson Approximation) model [18]. We have used the *spdf* version of this model, employing bosons of spin 0, 1, 2, and 3 [19]. The Hamiltonian parameters of the model were

determined by fitting both positive- and negative-parity low-energy levels in this nucleus. In the lower energy range the model predicts rather well the experimental situation (as described by eq. (5)). At higher excitation energies the model predicts a lower number of states, as expected. The right-side graph shows predictions of Shell Model Monte Carlo (SMMC) calculations for ^{56}Fe, at several excitation energies, up to 11.5 MeV [15]. These calculations show that the spin staggering is strongest at low energy (and consistent with our formulas (5) and (6)), and gradually diminishes, being practically absent above 10 MeV excitation. Similar calculations for the neighboring nuclei ^{55}Fe and ^{60}Co do not show any spin oscillation (Fig. 5 in ref. [17]). Thus, realistic SMMC calculations correctly predict the spin staggering, very probably an effect of the pairing interactions [15], but also the fact that it decreases with excitation energy and disappears at a certain energy. Due to the lack of experimental data at higher excitation energies we could not investigate this behaviour.
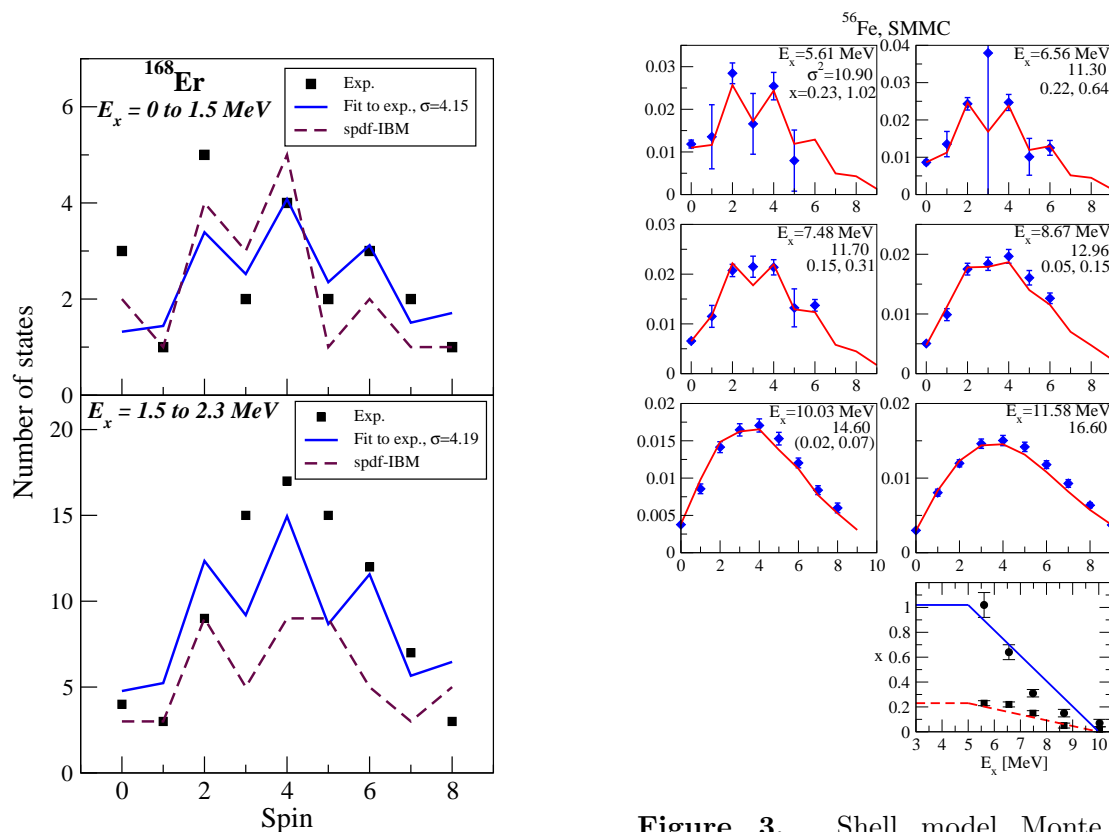

Figure 2. Spin distribution in ^{168}Er for two energy intervals. Symbols: experiment; full line: fits with eqs. (5)-(6); dashed line: predictions of the *spdf*-IBA model.

Figure 3. Shell model Monte Carlo predictions for ^{56}Fe, for different excitation energies. The solid lines are fits with eq. (5), yielding the displayed pairs of x values (eq. (5)). The lowest graph shows the evolution of x with the excitation energy (the straight lines are drawn to guide the eye). From [15].

2.2. Energy dependence of the spin-cutoff parameter

The method mentioned in the preceding section, of counting levels in energy bins, is not adequate to deduce the dependence of σ on the excitation energy because there are not enough levels with known spin. In a subsequent paper [20] we propose the "moment method" which exploits both the energy and spin dependence of a given nuclear level scheme, as shown in Fig. 4.
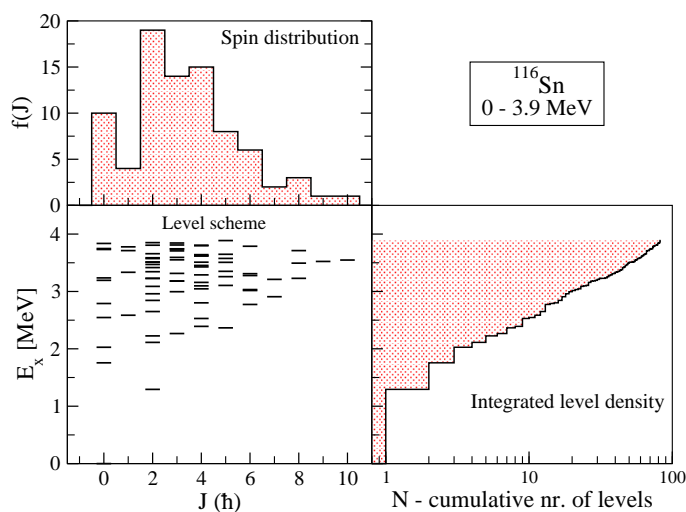
Figure 4. Example of the complete level scheme of ^{116}Sn up to $E_x = 3.9$ MeV, shown in two dimensions: spin and excitation energy. Projections on the two axes illustrates the spin distribution function and the level density, respectively.

Namely, we calculated individual moments in the (E, J) plane for each nucleus with the known levels (E_i, J_i) in the given energy and spin range, $M_{m,n}^{exp} = \sum_i (J_i^m \cdot E_i^n)$. The following nine moments were determined from the experimental values: J, J^2, J^3, $J \cdot E$, $J^2 \cdot E$, $J^3 \cdot E$, $J \cdot E^2$, $J^2 \cdot E^2$, $J^3 \cdot E^2$ and compared with the corresponding moments calculated with the CT model, eq. (3), for the level density. In the fit procedure, the dependence on both mass number A and excitation energy E were taken into account. The following result was obtained:

$$\sigma^2 = 0.391 \cdot A^{0.675}(E - 0.5 \cdot Pa')^{0.312}. \tag{7}$$

It is found that a backshift of the energy gives the best results. For this backshift we found that Pa', the so-called deuteron pairing energy, is very useful; this quantity will be discussed in detail in the next section. Formula (7) gives a good description to the existing (rather sparse) experimental determinations of σ ([20] and references therein). Because it is based mainly on the known low-lying levels, we think that the use of this formula up to energies of the order of the neutron binding energy is rather realistic. Formula (7) represents a practical alternative to the more classical formulas, such as those based on statistical mechanical calculations [21] or on the assumption of the nucleus as a rigid sphere [11], because both these formulas use the theoretical level density parameter a. Our formula depends only on mass and excitation energy, and through Pa' it implicitly distinguishes between isobars (types of nucleus).

3. Level density parameters

In our most recent work [20] we have determined the parameters of the BSFG model (eq. (2)) and of the CT model (eq. (3)) by fitting experimental level spacings and average neutron resonance spacings for the 310 nuclei in our data set (level density at the neutron binding energy is not known for 14 of these nuclei, but they were kept because their level scheme was complete up to rather high excitation energy). This new determination followed the procedure of ref. [4], but used the spin distribution eq. (7) described above with its experimentally determined dependence on mass and excitation energy, and taking into account the spin staggering for the even-even nuclei given by eq. (6). The first step was to determine for each nucleus the parameters a, E_1, and T, E_0 of the BSFG and CT models, respectively. These are shown in Figs. 4 and 5, respectively. A table with these parameters is given in [20]. The next step, also similar to the approach from [4], was to study the correlations between these parameters and different other quantities to be able to propose simple formulas (parametrizations) of these values. The quantities with which these parameters were found to be strongly correlated are the shell correction and the deuteron pairing [4, 20]. The correlation of a with the shell correction was observed in many works, starting with ref. [1]. The backshift parameters E_1 and E_0 turned out to be very similar to the deuteron pairing Pa'. The definitions of these two quantities are as follows. The shell correction S is the difference between the mass calculated with a liquid-drop Weizsäcker-type formula and the experimental atomic mass. For the liquid drop formula we use the parametrization of ref. [22], while the experimental masses are taken from [23]. The deuteron pairing energy is calculated from mass or mass excess values $M(A, Z)$ of the mass tables [23]:

$$Pa'_k = \frac{1}{2}[M(A+2, Z+1) - 2 \cdot M(A, Z) + M(A-2, Z-1)]. \tag{8}$$

It should be noted that this quantity has a different sign convention than Pa, the one used in our first work [4], which is also published in tables [24]: $Pa' = -(-1)^Z Pa$.

Based on these correlations, we were able to propose simple formulas that describe the empirical values of the parameters of the two level density models [20] as function of mass, shell correction S, and deuteron pairing Pa'. Thus, for the BSFG model parameters, we propose the following formulas:

$$a = (p_1 + p_2 S')A^{p_3} = (0.199 + 0.0096 S')A^{0.869}, \tag{9}$$

with $S' = S + 0.5 \cdot Pa'$, and $p_1 = 0.199(7)$; $p_2 = 0.0096(4)$; $p_3 = 0.869(7)$.

$$E_1 = q_1 + 0.5 \cdot Pa' = -0.381 + 0.5 \cdot Pa', \tag{10}$$

with $q_1 = -0.381(14)$. The experimental values of a and E_1 and their fits with formulas (9) and (10) are shown in Fig. 5.

For the CT model parameters we obtain the following formulas:

$$T = A^{-2/3}/(p_1 + p_2 S') = A^{-2/3}/(0.0597 + 0.00198 S'), \tag{11}$$

with $S' = S + 0.5 \cdot Pa'$ and $p_1 = 0.0597(2)$; $p_2 = 0.00198(10)$.

$$E_0 = q_1 + 0.5 \cdot Pa' = -1.004 + 0.5 \cdot Pa', \tag{12}$$

with $q_1 = -1.004(21)$. The experimental values of T and E_0 and their fits with formulas (11) and (12) are shown in Fig. 6.

In ref. [25] it was observed that there are very good correlations between the parameters of the BSFG and CT models. In the case of the present values, these very compact correlations (see Fig. 13 of ref. [20]) are:
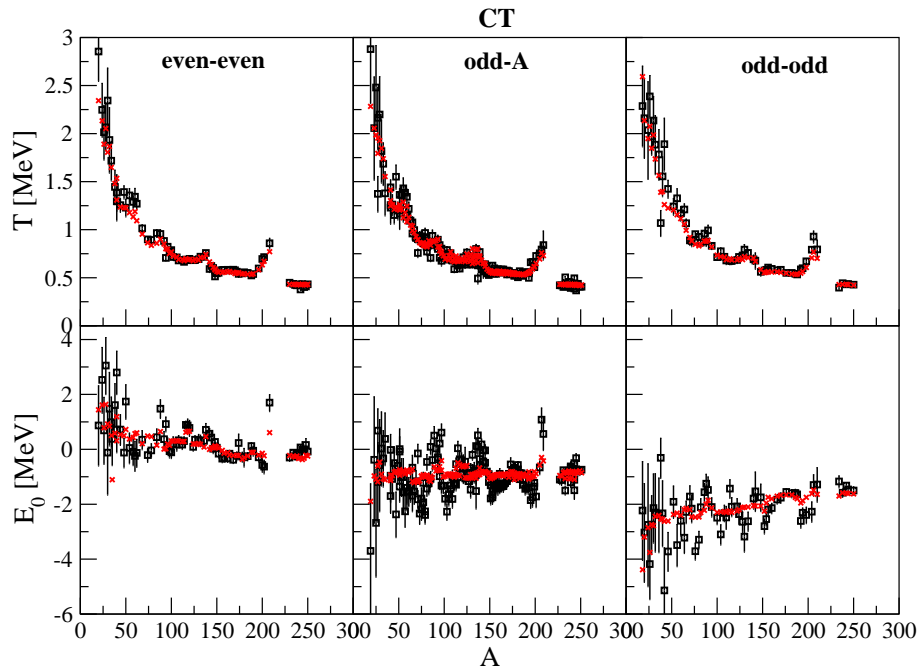
$$T = 5.164 \cdot a^{-0.791}, \tag{13}$$

while for the energy backshifts a linear relationship results from eqs. (10) and (12):

$$E_0 = E_1 - 0.623, \tag{14}$$
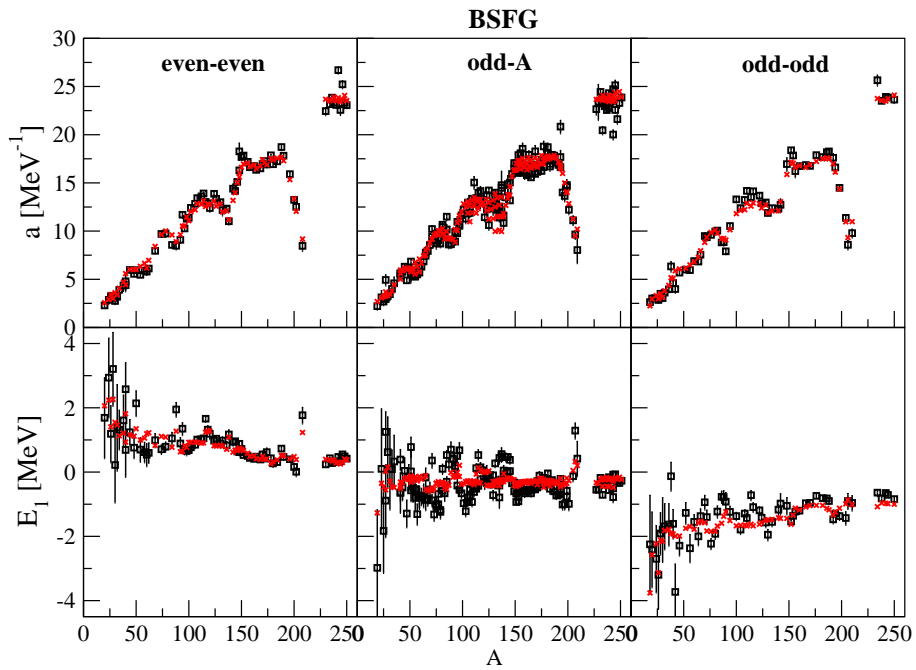
Figure 5. Level density parameters for the BSFG model. Rectangles with error bars: experimentally determined values; red x symbols: values provided by formulas (9), (10).


Figure 6. Level density parameters for the CT model. Rectangles with error bars: experimentally determined values; red x symbols: values provided by formulas (11), (12).

4. Constant temperature in nuclei

As remarked in our previous papers [4, 20, 25], both the BSFG model and the CT model can be successfully applied in an empirical way to low-energy levels and neutron resonance spacings, therefore both appear as convenient ways to parametrize the level densities at least up to the neutron binding energy. Figure 7 illustrates, by several cases of nuclei from light to heavy, and of different types (even-even, odd-A, and odd-odd) that the two models offer an equally good description to the cumulative number of levels.
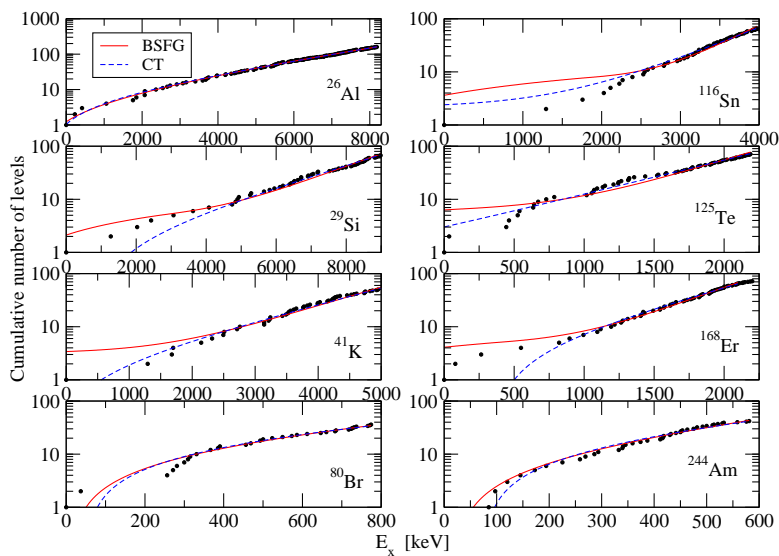
Figure 7. Examples of fits of the experimental cumulative number of levels of different nuclei from our set with the BSFG and CT model formulas. The fit also includes the level spacing at the neutron resonance energy, which is not shown in the graphs.

The BSFG model, based on the original Fermi-gas model of Bethe [26], calculates the level densities by counting the number of possibilities in which the excitation energy can be distributed among the single-particle states, a combinatorial problem that can use methods of statistical mechanics [27]. On the other hand, the much simpler CT model is based on the definition of the nuclear temperature T as [6, 27]

$$\frac{1}{T} = \frac{d[\ln \rho(E)]}{dE}$$

where $\rho(E)$ is the level density at the excitation energy E. If one assumes that T is constant, one obtains the CT model expression of eq. (3). Therefore, according to the CT model in a certain energy range, if we increase the excitation energy of the nuclei their temperature remains constant. This behavior is similar to that of systems with a first-order critical phase transition (like solid-liquid or liquid-gas). The nuclei behave like a water/ice mixture what means that their number of degrees of freedom increases with the excitation energy in such a way that the temperature stays constant; this effect may be explained by the destruction of the nuclear Cooper pairs with the excitation energy ("melting" of the Cooper pairs)[28], thus leaving the mean energy constant per excited nucleon according to the formula $T = E/n_{ex}$ [27], where n_{ex} is the number of excited nucleons. The early model of Gilbert and Cameron [1] used a composite

model, with a CT level density below the neutron binding energy, and a BSFG model above that.

There is increasing experimental evidence that the CT model is the right description of the nuclei at even higher excitation energies. A strong such evidence was borne out by a Hauser-Feshbach approach analysis of neutron and proton evaporation spectra from the ^{6}Li + ^{55}Mn and d + ^{59}Co reactions [29]. Using a Fermi-gas like temperature dependence failed to reproduce the experimental data, while assuming the CT model described the spectra up to about 20 MeV excitation, indicating that for mass ≈ 60 nuclei the CT model works up to that energy. This agrees with earlier experiments that revealed also that the CT formula represents well data up to about 10 MeV [30]. Another evidence came from a different field: the behavior of fragments from low-energy neutron induced fission [31]. It concerns the average number of neutrons emitted by the fission fragments as a function of their mass measured in neutron-induced fission of ^{237}Np at two neutron energies, 0.8 and 5.55 MeV, where a very interesting feature has been noted: higher incident neutron energy, that is higher excitation energy of the system, leads to an increase of the number of evaporated neutrons only for the heavy fragments. Thus, it takes place a peculiar sharing of the excitation energy between the two fragments, the heavy fragments taking practically the whole energy. This is inconsistent with a Fermi-gas behaviour, that would lead to the division of the excitation energy between the two fragments in their mass ratio (as it happens in many binary reactions involving high excitation energies). On the other hand, it is fully consistent with the Constant Temperature model. Indeed, according to eq. (11), the temperature of the heavy fragment is generally lower than that of the light fragment (except for special cases, as an effect of shell closures). If the two fragments at the scission point, therefore in thermal contact, are in the pairing regime where they have different temperatures, then the cooler heavy fragments will absorb the entire available excitation energy and therefore will emit more neutrons. This particular sorting of the available energy is a consequence of the two different temperatures of the fragments, and shows that a correct description of the behavior of the nuclei at low and moderate excitation energies by the CT model leads to an understanding of these features of the low-energy neutron induced fission.

5. Conclusions

We reviewed the description of the nuclear level density by simple models whose parameters were empirically determined from existing experimental data.

An empirical mass and energy description has been proposed for the spin-cutoff parameter of the spin distribution function. Also, for the observed spin staggering of the spin distribution function of the even-even nuclei a simple formula was proposed, and comparison with predictions of two structure models was presented.

Using this description for the spin distribution function, the parameters of the BSFG and CT models for the level density were determined for 310 nuclei between ^{19}F and ^{251}Cf by fitting both the completely known low-energy discrete levels and the level spacings at the neutron binding energy. Simple formulas were proposed that reasonably describe these sets of parameters as a function of mass number. The formulas proposed for both spin-cutoff paramater and the BSFG and CT model parameters, respectively, contain only quantities that can be extracted from mass tables, and are thought to be reasonable approximations up to at least the neutron binding energy. They can also be used as a practical way to extrapolate level density estimations for nuclei far from stability, for which such quantities cannot be experimentally determined.

Both the BSFG and CT models give equally good description to experimental level densities at excitation energies at least up to the neutron binding energy. Recent experimental evidences are reviewed that at least up to this energy (but, in ceratin mass regions, even at higher energies) the correct description of nuclei is given by the constant temperature model.

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