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Heavy quark potentials in effective string theory

Sungmin Hwang

Physik-Department, Technische Universität München, James-Franck-Str.1, D-85747 Garching and Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany

E-mail: sungmin.hwang@tum.de

Abstract. We present our recent analytic calculation of the next-to-leading order contribution to the heavy quark-antiquark potentials, by utilizing the effective description of a long string. Discussions on the effective field theory power counting and the method for reducing a number of dimensionful parameters arising from the EST are followed. Furthermore, physical implications on the comparison to the lattice data are briefly discussed.

1. Introduction

Perturbative Quantum Chromodynamics (pQCD) [1] provides an analytic description of the interactions between quarks and gluons at high energy [2, 3, 4]. However, this approach is no longer valid as the coupling α_s becomes a large value at the scale around or below 200 MeV; this feature is called *color confinement* in QCD [5]. A possible solution to this theoretical challenge within a perturbative framework is still to be pursued to this day. Although a complete and consistent theory for the dynamics concerning color-charged particles from the deconfining to confining phase is absent, various kind of low-energy effective descriptions has been developed, and they differ depending on the particle interactions of one's interest.

Chiral perturbation theory (χPT) [6] provides a perturbative framework for the interactions among light mesons (π, σ, ρ, etc), which is valid below the chiral breaking scale $\sim$1 GeV. As for the heavy quarks (charm or bottom), heavy quark effective theory (HQET) [7] gives an effective description of the dynamics of a heavy-light quark (or antiquark) pair, while non-relativistic QCD (NRQCD) [8, 9] and potential NRQCD (pNRQCD) [10] are the effective field theories (EFTs) for a heavy quark-antiquark pair and its bound state, respectively.

On the other hand, lattice gauge theory (or lattice QCD-LQCD) provides numerical values of the mass spectra of quarks and hadrons. While LQCD is useful for extracting physical values at the non-perturbative regime, an analytic method for understanding the bound state between a quark-antiquark pair at long-distance has been developed. This method is called QCD *flux tube model* [11]. In this paper, Nambu proposed that if a quark-antiquark pair is separated to long distance, comparable to that of the confining scale, then the gluodynamics between the pair can be substituted by the dynamics of a fluctuating string. This model is verified when it is compared to the lattice simulation [12]. It shows that the energy density between the pair increases when they are separated, and this increase of density appears in the shape of a *flux tube*.

Lüscher et. al. developed this idea further [13, 14]. Based on the conjecture of the *Wilson loop-string partition function equivalence*, the static potential between the pair at long distance was calculated. Kogut and Parisi, shortly afterwards, derived the analytic expression of the spin-spin interaction part of the heavy quark potential by using the string picture [15].

During the last few decades, significant progress has been made in this line of research. Heavy quark-antiquark potential was shown to be equivalent to the Wilson loop and the gauge field insertions therein, at the large time limit [16, 17]. This was given by the matching calculation. Recently, Brambilla et. al. have calculated all of the heavy quark potentials by using the effective string theory (EST) up to leading order (LO) [18]. As was pointed out here, however, this LO calculation is not fully inclusive, because some of the next-to-leading order (NLO) contribution would contain terms which are of the same order of the expansion parameter. In other words, NLO terms can alter the leading order coefficients of the potentials. Thus, it is necessary for us to understand the proper EFT systematics of the string picture, so that not only the higher order suppression terms are derived, but all of the missing LO terms are also acquired.

This proceeding is organized as the following: In Sec. 2, we briefly introduce NRQCD and pNRQCD for a heavy quark-antiquark pair and its bound state. The equivalence between the Wilson loop expectation value (and gauge field insertions therein) from NRQCD and the potential terms from pNRQCD at the large time limit is shown. In Sec. 3, we introduce the EST, and through the Wilson loop-string partition function equivalence conjecture, the heavy quark potentials are analytically calculated up to NLO. We discuss the implication of this result in Sec. 4. Comparison of our result to the LQCD data as well as the extension of our method to other cases are given at the end.

2. Non-relativistic EFTs of QCD

NRQCD is a non-relativistic EFT for a heavy quark-antiquark pair [8, 9]. As the hierarchy of scales is realized, $M \gg Mv$ (M: heavy quark mass, v: relative velocity), the NRQCD Lagrangian is derived by integrating out M: it is organized by $1/M$ expansion. Then its dynamical degrees of freedom are Pauli spinor fields for the heavy quark and antiquark, SU(3) field strength terms for the soft gluons, and Dirac spinor fields for the light quarks. A series of operators concerning the heavy quark and antiquark fields carry Wilson coefficients. The coefficients contain information at the high-energy regime. They are determined by matching to the underlying theory, pQCD.

An even lower-energy counterpart to NRQCD is pNRQCD [10], which is an effective framework of heavy quark-antiquark bound state. The hierarchy of scales for this system is realized as $M \gg Mv \gg Mv^2$, where Mv^2 is the relative kinetic energy between the quark-antiquark pair. By integrating out the relative momentum Mv, which scales like $\sim 1/r$ (r: relative distance), the pNRQCD Lagrangian is then organized by the multipole expansion in r, in addition to $1/M$ expansion from NRQCD. Then the dynamical degrees of freedom are SU(3) singlet and octet fields for the heavy quarkonium, SU(3) field strength term for ultra-soft gluons, Dirac spinor fields for the light quarks. Note that both the singlet and octet fields depend on time t, relative coordinate $\boldsymbol{r}$, and center-of-mass coordinate $\boldsymbol{R}$, $S(t, \boldsymbol{r}, \boldsymbol{R})$, $O^a(t, \boldsymbol{r}, \boldsymbol{R})$. A schematic form of the pNRQCD Lagrangian is given by (only the bilinear terms of singlet and octet are shown below)

$$\begin{aligned}\mathcal{L}_{\text{pNRQCD}} \ni \int d^3r \Big\{ &\text{Tr}\left[S^\dagger \left(i\partial_0 - V_S(r) + \dots\right) S + O^\dagger \left(iD_0 - V_O(r) + \dots\right) O\right] \\ &gV_A(r)\text{Tr}\left[O^\dagger \boldsymbol{r}\cdot\boldsymbol{E} + S^\dagger \boldsymbol{r}\cdot\boldsymbol{E}O\right] + g\frac{V_B(r)}{2}\text{Tr}\left[O^\dagger \boldsymbol{r}\cdot\boldsymbol{E}O + O^\dagger O\boldsymbol{r}\cdot\boldsymbol{E}\right]\Big\},\end{aligned} \tag{1}$$

in which $V(r)$'s are the heavy quark potentials. They are the Wilson coefficients of pNRQCD, which are to be determined by matching to NRQCD. In this report, we focus on the analytic expression of the singlet potential $V_S(r)$. The potential expanded up to $1/M^2$ is given by (from now on we omit the subscript S for the singlet sector) [10]

$$V(r) = V^{(0)}(r) + \frac{2}{M}V^{(1,0)}(r) + \frac{1}{M^2}\left[V^{(2,0)} + V^{(1,1)}\right] \tag{2}$$

where the superscript (a, b) denotes the order of $1/M_{1,2}$ expansion. Note that the mass of the heavy quark and the antiquark is identical, $M_1 = M_2 = M$. $V^{(2,0)}$ includes terms like $V^{(2,0)}_{L^2}, V^{(2,0)}_{p^2}, V^{(2,0)}_{LS}$, and $V^{(2,0)}_r$, and $V^{(1,1)}$ includes $V^{(1,1)}_{L^2}, V^{(1,1)}_{L_2S_1}, V^{(1,1)}_{S^2}, V^{(1,1)}_{S_{12}}, V^{(1,1)}_{p^2}$, and $V^{(1,1)}_r$.

By comparing the heavy quark-antiquark correlator of NRQCD to the singlet propagator of pNRQCD, the following relation holds at the zeroth order in $1/M$ expansion [16]

$$V^{(0)}(r) = \lim_{T\to\infty} \frac{i}{T} \ln\langle W_\square \rangle, \quad \text{for} \quad W_\square = P \exp\left\{ -ig \oint_{r\times T} dz^\mu A_\mu(z) \right\} . \tag{3}$$

The angular bracket around the rectangular Wilson loop $W_\square$ denotes the expectation value over the Yang-Mills action, and P stands for the path-ordering operator in color space. In the similar fashion, we obtain the following relation for the first order relativistic correction [17]:

$$V^{(1,0)}(r) = -\frac{1}{2} \lim_{T\to\infty} \int_0^T dt\, t \langle\langle g\boldsymbol{E}_1(t) \cdot g\boldsymbol{E}_1(0) \rangle\rangle_c , \tag{4}$$

where the double angular bracket denotes gauge fields insertions to the normalized Wilson loop expectation value, and the subscript c stands for the connected part, which is defined by $\langle\langle O_1(t_1)O_2(t_2)\rangle\rangle_c = \langle\langle O_1(t_1)O_2(t_2)\rangle\rangle - \langle\langle O_1(t_1)\rangle\rangle\langle\langle O_2(t_2)\rangle\rangle$, for $t_1 \geq t_2$. Also $\boldsymbol{E}_{1,2}(t) = \boldsymbol{E}(t, \pm r/2)$ as the quark and antiquark are located at $(0, 0, \pm r/2)$, respectively. The second order corrections are divided into spin-independent and spin-dependent parts. The spin-independent part is given by,

$$\begin{aligned} V^{(2,0)}_{p^2}(r) &= \frac{i}{2} \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_1(0)\rangle\rangle_c , \\ V^{(1,1)}_{p^2}(r) &= i \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_2(0)\rangle\rangle_c , \\ &\dots \end{aligned} \tag{5}$$

where the ellipsis includes terms like V_{L^2}'s and V_r's. The spin-independent part is given by,

$$\begin{aligned} V^{(2,0)}_{LS} &= -\frac{c^{(1)}_F}{r^2} i\boldsymbol{r} \cdot \lim_{T\to\infty} \int_0^T dt\, t \langle\langle g\boldsymbol{B}_1(t) \times g\boldsymbol{E}_1(0)\rangle\rangle + \frac{c^{(1)}_s}{2r^2} \boldsymbol{r} \cdot \left(\nabla_r V^{(0)}\right) , \\ V^{(1,1)}_{L_2S_1}(r) &= \frac{c^{(1)}_F}{r^2} i\boldsymbol{r} \cdot \lim_{T\to\infty} \int_0^T dt\, t \langle\langle g\boldsymbol{B}_1(t) \times g\boldsymbol{E}_2(0)\rangle\rangle_c , \\ &\dots \end{aligned} \tag{6}$$

where the ellipsis includes the spin-spin interaction part of the potentials. Explicit form of these potentials are then derived by calculating the gauge field insertions to the Wilson loop expectation value. In the non-perturbative regime, LQCD provides numerical result [19, 20], which observes Poincaré invariance [21] with high accuracy, but it is limited only to the calculations of two-gauge field insertions. V_r contains three- and four-gauge field insertions, so it is necessary to look for some other ways to address this issue.

3. Effective string theory and heavy quark potentials

Analytic behavior of the potentials at long distance can be investigated by utilizing the effective string theory (EST) [11]. The EST is constructed based on the Wilson loop-string partition function equivalence conjecture [13]:

$$\lim_{T\to\infty} \langle W_\square \rangle = Z \int \mathcal{D}\xi^1 \mathcal{D}\xi^2\, e^{iS_{\text{String}}[\xi^1,\xi^2]} , \tag{7}$$

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where Z is the normalization constant and $\xi^{1,2}$ are the transversal vibrations of the string. With respect to the following hierarchy of scales: $r\Lambda_{\rm QCD} \gg 1$, and the power counting: $\partial_a \sim 1/r$ and $\xi^l \sim \Lambda_{\rm QCD}^{-1}$ (i.e., $\partial_a \xi^l \sim (r\Lambda_{\rm QCD})^{-1} \ll 1$), as well as the Dirichlet boundary condition: $\xi^l(t, \pm r/2) = 0$, the action of the EST in four-dimensional spacetime is derived from the Nambu-Goto action,

$$S_{\rm String} = -\sigma \int dtdz \sqrt{\det\left(\eta_{ab} + \partial_a \xi^l \partial_b \xi^l\right)} = -\sigma \int dtdz \left(1 - \frac{1}{2}\partial_a \xi^l \partial_b \xi^l + \dots\right) . \tag{8}$$

The ellipsis on the last equality contains higher order terms of the derivative expansion, and σ is the string tension ($\sim \Lambda_{\rm QCD}^2$), which is a fundamental parameter of the theory. Truncation has been made up to quadratic order. From this expression, it is clear that the LO potential is of linear order, $V^{(0)} \approx \sigma r$ [13]. Also, the Green's function is exactly computed by solving the equations of motion [22]:

$$G^{lm}(t, t'; z, z') = \frac{\delta^{lm}}{4\pi\sigma} \ln\left(\frac{\cosh\left[(t-t')\,\pi/r\right] + \cosh\left[(z+z')\,\pi/r\right]}{\cosh\left[(t-t')\,\pi/r\right] - \cosh\left[(z-z')\,\pi/r\right]}\right) . \tag{9}$$

Relativistic corrections to the potential at long distance can analytically be computed when a mapping between QCD and the effective string description is established [18]. In this report, we want to extend this idea, such that the subleading contribution to the potentials are acquired.

As for the NLO calculation of the potential, it turns out that subleading part of the EST action contributes as the next-next-to-leading order. So we only consider the mapping at NLO:

$$\begin{aligned}
\langle\!\langle \dots \boldsymbol{E}^l_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \Lambda^2 \partial_z \xi^l (t, \pm r/2) + \overline{\Lambda}^2 \partial_z \xi^l \left(\partial\xi\right)^2 (t, \pm r/2) \dots \rangle \, , \\
\langle\!\langle \dots \boldsymbol{E}^3_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \Lambda''^2 + \overline{\Lambda''}^2 (\partial\xi)^2 (t, \pm r/2) \dots \rangle \, , \\
\langle\!\langle \dots \boldsymbol{B}^l_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \pm \Lambda' \epsilon^{lm} \partial_t \partial_z \xi^m (t, \pm r/2) \pm \overline{\Lambda'} \epsilon^{lm} \partial_t \partial_z \xi^m (\partial\xi)^2 (t, \pm r/2) \dots \rangle \, , \\
\langle\!\langle \dots \boldsymbol{B}^3_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \pm \Lambda''' \epsilon^{lm} \partial_t \partial_z \xi^l \partial_z \xi^m (t, \pm r/2) \pm \overline{\Lambda'''} \epsilon^{lm} \partial_t \partial_z \xi^l \partial_z \xi^m (\partial\xi)^2 (t, \pm r/2) \dots \rangle \, .
\end{aligned} \tag{10}$$

Note that the mapping for magnetic field insertion is derived by taking the duality transformation of the electric field. There are two types of dimensionful parameters, Λ's and $\overline{\Lambda}$'s, which are $\sim \Lambda_{\rm QCD}$, and they satisfy $\Lambda \geq \overline{\Lambda}$. Then we perform the similar calculation as in [18], but its difficulty lies on the divergence behavior of the string correlator, such as $\partial_z \partial_{z'} G$, $\partial_t \partial_z \partial_{z'} G$, and $\partial_t \partial_{t'} G$, defined at the same spacetime point. It turns out that this divergence stems from the derivation of the string correlator, which contains an infinite sum over the vibrational modes of the string and integral over the entire Fourier space. The infinite sum and integral are not physically compatible with the effective framework we started with, because the validity of theory does not encompass the entire energy scale. Thus, it is necessary to regulate this divergence and renormalize accordingly. We use *zeta-function regularization* for the infinite sum over the modes: $\sum_{n=0}^{\infty} n = -1/12$ and $\sum_{n=0}^{\infty} n^2 = 0$, and for the Fourier integral, we use *dimensional regularization*: $\int dy [y^2/(y^1+1)] = -2\pi$. In fact, these two regularization schemes are identical to each other: the zeta-function regularization is the discrete version of the dimensional regularization. By applying these schemes, we obtain the analytic result.

However, there are a number of free (dimensionful) parameters remaining in the potential. They are either from Eq. (10) or originate from the time integral for the potential. We exploit Poincaré invariance in QCD, which gives non-trivial relations between the potential terms.

Applying these constraints gives a significant result on the momentum-dependent potentials:

$$\begin{aligned} V_{p^2}^{(2,0)} &= \left(\frac{1}{12\pi}+\frac{\pi}{36}\right)\frac{1}{r}, \quad V_{p^2}^{(1,1)} = \left(\frac{1}{6\pi}-\frac{\pi}{36}\right)\frac{1}{r}, \\ V_{L^2}^{(2,0)} &= -\frac{\sigma r}{6}+\left(\frac{11}{36\pi}+\frac{2\pi}{27}\right)\frac{1}{r}, \quad V_{L^2}^{(1,1)} = \frac{\sigma r}{6}+\left(\frac{1}{9\pi}+\frac{5\pi}{216}\right)\frac{1}{r}. \end{aligned} \tag{11}$$

V_{p^2}'s were absent at LO in the EST [18], and here we observe non-trivial contributions arise at NLO. As for V_{L^2}'s, the linear part is from the LO calculation, which is now added by suppression terms. Note also that these NLO terms do not carry any free parameters due to constraint equations between the potentials (Poincaré invariance in QCD). The rest of the potentials are calculated in a similar fashion, and the full result will appear in our upcoming paper.

4. Outlook

Comparing the EST prediction to the LQCD data gives an interesting result. In [23], the comparison has been made, where the analytic result of V_b, V_c, V_d, and V_e (linear combinations of the momentum-dependent potentials) are given up to LO of the EST. It shows a significant discrepancy between lattice data and the analytic prediction, which implies that higher order terms are required for the improvement. Inclusion of NLO terms as in Sec. 3 shows decrease in the discrepancy, and this will be shown in our upcoming paper.

Method of the EST can be applied to the case of heavy hybrids and baryonic spectrum at long-distance regime. Eq. (7) might be different in this case due to the symmetry reason, but the procedure for deriving the analytic expression is similar. Currently, we are investigating these applications.

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