Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 971 (2021) 115534

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Orbifolds from Sp(4, $\mathbb{Z}$) and their modular symmetries

Hans Peter Nilles [a], Saúl Ramos–Sánchez [b,*], Andreas Trautner [c], Patrick K.S. Vaudrevange [d]

[a] *Bethe Center for Theoretical Physics and Physikalisches Institut der Universität Bonn, Nussallee 12, 53115 Bonn, Germany*

[b] *Instituto de Física, Universidad Nacional Autónoma de México, POB 20-364, Cd.Mx. 01000, Mexico*

[c] *Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*

[d] *Physik Department T75, Technische Universität München, James-Franck-Straße 1, 85748 Garching, Germany*

Received 26 May 2021; received in revised form 11 August 2021; accepted 1 September 2021

Available online 6 September 2021

Editor: Stephan Stieberger

Abstract

The incorporation of Wilson lines leads to an extension of the modular symmetries of string compactification beyond SL(2, $\mathbb{Z}$). In the simplest case with one Wilson line Z, Kähler modulus T and complex structure modulus U, we are led to the Siegel modular group Sp(4, $\mathbb{Z}$). It includes SL(2, $\mathbb{Z})_T \times$ SL(2, $\mathbb{Z})_U$ as well as $\mathbb{Z}_2$ mirror symmetry, which interchanges T and U. Possible applications to flavor physics of the Standard Model require the study of orbifolds of Sp(4, $\mathbb{Z}$) to obtain chiral fermions. We identify the 13 possible orbifolds and determine their modular flavor symmetries as subgroups of Sp(4, $\mathbb{Z}$). Some cases correspond to symmetric orbifolds that extend previously discussed cases of SL(2, $\mathbb{Z}$). Others are based on asymmetric orbifold twists (including mirror symmetry) that do no longer allow for a simple intuitive geometrical interpretation and require further study. Sometimes they can be mapped back to symmetric orbifolds with quantized Wilson lines. The symmetries of Sp(4, $\mathbb{Z}$) reveal exciting new aspects of modular symmetries with promising applications to flavor model building.

* Corresponding author.

E-mail addresses: nilles@th.physik.uni-bonn.de (H.P. Nilles), ramos@fisica.unam.mx (S. Ramos–Sánchez), trautner@mpi-hd.mpg.de (A. Trautner), patrick.vaudrevange@tum.de (P.K.S. Vaudrevange).

https://doi.org/10.1016/j.nuclphysb.2021.115534

1. Introduction

Modular symmetries appear frequently in string theory. They might have applications in particle physics as discrete non-Abelian flavor symmetries. In the simplest case, these discrete modular symmetries descend from the modular group $\mathrm{SL}(2,\mathbb{Z})$ of a two-dimensional torus, on which two extra spatial dimensions have been compactified. The implementation within string theory requires some aspects of model building towards the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ Standard Model of particle physics. One of the key aspects is the desired presence of chiral fermions. This requires a twist of the torus. Then, chiral matter fields can be realized in the twisted sectors of the orbifold, located at the "fixed points" of the orbifold twist. Simplest examples correspond to the $\mathbb{Z}_K$ orbifolds $\mathbb{T}^2/\mathbb{Z}_K$ ($K=2,3,4,6$), where a full analysis has been performed recently [1–4]. They would correspond to six-dimensional string compactifications with an elliptic fibration.

In fact, in string theory a two-torus with background B-field is described by two moduli, the Kähler modulus T and the complex structure modulus U with modular symmetries $\mathrm{SL}(2,\mathbb{Z})_T\times\mathrm{SL}(2,\mathbb{Z})_U$. This is accompanied by mirror symmetry which interchanges T and U. In the $\mathbb{Z}_K$ orbifolds with ($K>2$), the complex structure modulus is frozen to allow for the orbifold twist and in these cases we have one unconstrained modulus T. This is, however, not the case for the $\mathbb{Z}_2$ orbifold, where we remain with two unconstrained moduli and manifest mirror symmetry.

In general, string theories have a much richer moduli structure as they require the compactification of six spatial dimensions. But even if we concentrate on a two-dimensional subsector, we have additional moduli in the form of gauge background fields (Wilson lines). The moduli structure of the simplest example of such a system is given by the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ with moduli T, U and one additional Wilson line modulus Z. This can be made manifest in the Narain lattice formulation [5]. In addition, this work is motivated by the recent bottom-up consideration of $\mathrm{Sp}(4,\mathbb{Z})$ flavor symmetries, see refs. [6,7].

$\mathrm{Sp}(4,\mathbb{Z})$ contains as subgroups $\mathrm{SL}(2,\mathbb{Z})_T\times\mathrm{SL}(2,\mathbb{Z})_U$ as well as mirror symmetry. From the string theory perspective, an application of $\mathrm{Sp}(4,\mathbb{Z})$ as modular flavor symmetry would again require some orbifolding to obtain chiral fermions. The first step in this direction is a classification of orbifolds from $\mathrm{Sp}(4,\mathbb{Z})$, which is the main purpose of the present paper. This is a generalization of the $\mathbb{Z}_K$ orbifolds mentioned earlier. To perform the classification of $\mathrm{Sp}(4,\mathbb{Z})$ orbifolds, we realize that each inequivalent fixed point in the string moduli space (T,U,Z) that is left invariant by a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ corresponds to an inequivalent orbifold (or, in general, to a set of orbifolds). Hence, the fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ correspond to string orbifolds. Then, chiral fermions could appear at the fixed points of these $\mathrm{Sp}(4,\mathbb{Z})$ orbifold actions on the extra-dimensional space.[1] A classification of the fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ has been given by Gottschling [8–10] long ago. There are altogether 13 different cases: two with complex dimension 2, five with complex dimension 1 and six of dimension 0.

The next step is the construction of those orbifolds that stabilize these fixed loci in moduli space. Our results are summarized in Table 1. We identify the conventional (geometrical) twists on the moduli and, as a new mechanism, twists via mirror symmetry. As a result of this, asymmetric orbifolds appear frequently (although some of them are dual to symmetric orbifolds with specifically transformed moduli). A direct intuitive geometrical interpretation is often not avail-

[1] Hence, there are two different kinds of fixed points: First, the fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ acting on the moduli space (T,U,Z) and, second, the fixed points of the orbifold action in extra-dimensional space, where chiral fermions can be localized.

Table 1
Summary of symmetric and asymmetric orbifold compactifications with Narain point groups classified by the inequivalent fixed points (T, U, Z) of $\mathrm{Sp}(4, \mathbb{Z})$. We use the definitions $\omega := \exp(2\pi \mathrm{i}/3)$, $\zeta := \exp(2\pi \mathrm{i}/5)$ and $\tilde{\eta} := \frac{1}{3}(1 + 2\sqrt{2}\mathrm{i})$. In the canonical basis the orbifold in section 3.1.2 appears to be asymmetric. We call it fake-asymmetric as it can be mapped to a symmetric orbifold via a basis change (as discussed in section 3.1.3). The remaining asymmetric orbifolds seem to be genuinely asymmetric. We are not aware of basis changes that allow us to map them to symmetric orbifolds.

Complex dimension of moduli space	Narain point group of orbifold	Type of orbifold	Moduli (T, U, Z)	Reference to section
2	$\mathbb{Z}_2$	symmetric	$(T, U, 0)$	3.1.1
2	$\mathbb{Z}_2$	fake-asymmetric	(T, T, Z)	3.1.2
2	$\mathbb{Z}_2$	symmetric (dual to 3.1.2)	$(T, U, 1/2)$	3.1.3
1	$\mathbb{Z}_4$	symmetric	$(T, \mathrm{i}, 0)$	3.2.1
1	$\mathbb{Z}_6$	symmetric	$(T, \omega, 0)$	3.2.2
1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	asymmetric	$(T, T, 0)$	3.2.3
1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	asymmetric	$(T, T, 1/2)$	3.2.4
1	S_3	asymmetric	$(T, T, T/2)$	3.2.5
0	$\mathbb{Z}_5$	asymmetric	$(-\zeta^{-1}, \zeta, \zeta + \zeta^{-2})$	3.3.1
0	S_4	asymmetric	$(\tilde{\eta}, \tilde{\eta}, \frac{1}{2}(\tilde{\eta} - 1))$	3.3.2
0	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	asymmetric	$(\mathrm{i}, \mathrm{i}, 0)$	3.3.3
0	$S_3 \times \mathbb{Z}_6$	asymmetric	$(\omega, \omega, 0)$	3.3.4
0	$S_3 \times \mathbb{Z}_2 \cong D_{12}$	asymmetric	$\frac{\mathrm{i}}{\sqrt{3}}(2, 2, 1)$	3.3.5
0	$\mathbb{Z}_{12}$	asymmetric	$(\mathrm{i}, \omega, 0)$	3.3.6

able as the presence of Wilson lines and asymmetric twists introduce some "non-geometrical" aspects.

If we set the Wilson lines to zero, we obtain the $\mathbb{T}^2/\mathbb{Z}_K$ examples discussed earlier: Section 3.1.1 in Table 1 represents the symmetric $\mathbb{Z}_2$ orbifold, section 3.2.1 the $\mathbb{Z}_4$ orbifold, and section 3.2.2 the symmetric $\mathbb{Z}_3$ orbifold (embedded in the $\mathbb{Z}_6$ case). The orbifold of section 3.1.2 corresponds to an apparently asymmetric $\mathbb{Z}_2$ orbifold. In this case, however, we can define a duality transformation that maps it to a symmetric $\mathbb{Z}_2$ orbifold with a quantized Wilson line (discussed in section 3.1.3).

The paper is organized as follows. In section 2, we introduce the modular transformations of $\mathrm{Sp}(4, \mathbb{Z})$ on the string moduli (see eq. (8)) within the framework of Narain orbifold compactifications. Section 3 discusses case by case the stabilization of moduli by $\mathrm{Sp}(4, \mathbb{Z})$ orbifolds that lead to the results summarized in Table 1. In addition, we explicitly construct for each orbifold the unbroken modular group $\mathcal{G}_{\text{modular}}$ as a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ plus a $\mathcal{CP}$-like transformation. While these are important steps towards applications to the flavor problem of the Standard Model of particle physics, there are still many open questions. These will be mentioned in section 4, devoted to conclusions and outlook. Finally, after a short discussion of the mirror symmetry in appendix A and the presentation of our notation in appendix B, we provide further explicit details of the $\mathrm{Sp}(4, \mathbb{Z})$ fixed points in appendix C.

2. (A)symmetric orbifolds and Sp(4, $\mathbb{Z}$)

2.1. Narain torus compactification and string moduli

To construct an orbifold in the Narain formulation of the heterotic string, we first discuss a general D-dimensional torus compactification with B-field and Wilson line backgrounds [11,12].

To do so, we have to impose torus boundary conditions on the D right- and $D+16$ left-moving (bosonic) string modes $(y_\mathrm{R}, y_\mathrm{L})$, respectively,

$$\begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix} \sim \begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix} + E\,\hat{N}\,, \quad \text{where} \quad \hat{N} = \begin{pmatrix} n \\ m \\ p \end{pmatrix} \in \mathbb{Z}^{2D+16}\,. \tag{1}$$

The $(2D+16)$-dimensional vector of integers $\hat{N}$ contains the winding numbers $n \in \mathbb{Z}^D$, the Kaluza–Klein numbers $m \in \mathbb{Z}^D$, and the gauge quantum numbers $p \in \mathbb{Z}^{16}$ corresponding to the $\mathrm{E}_8 \times \mathrm{E}_8$ (or SO(32)) gauge group of the supersymmetric heterotic string. Furthermore, E denotes the Narain vielbein. For the worldsheet one-loop string vacuum amplitude to be modular invariant, E has to satisfy

$$E^\mathrm{T}\,\eta\,E = \hat{\eta} := \begin{pmatrix} 0 & \mathbb{1}_D & 0 \\ \mathbb{1}_D & 0 & 0 \\ 0 & 0 & g \end{pmatrix}, \quad \text{where} \quad \eta := \begin{pmatrix} -\mathbb{1}_D & 0 & 0 \\ 0 & \mathbb{1}_D & 0 \\ 0 & 0 & \mathbb{1}_{16} \end{pmatrix}. \tag{2}$$

Here, η is the Narain metric of signature $(D, D+16)$, $g := \alpha_\mathrm{g}^\mathrm{T}\alpha_\mathrm{g}$ and the columns of the (16×16)-dimensional matrix α_g contain the simple roots of $\mathrm{E}_8 \times \mathrm{E}_8$ (or $\mathrm{Spin}(32)/\mathbb{Z}_2$). Consequently, E spans a so-called Narain lattice: an even, integer, self-dual lattice of signature $(D, D+16)$. The "rotational" symmetries of the Narain lattice give rise to the so-called modular group of the Narain lattice,

$$\mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z}) := \left\langle\, \hat{\Sigma} \,\middle|\, \hat{\Sigma} \in \mathrm{GL}(2D+16, \mathbb{Z}) \quad \text{with} \quad \hat{\Sigma}^\mathrm{T}\hat{\eta}\,\hat{\Sigma} = \hat{\eta} \,\right\rangle. \tag{3}$$

In order to understand the action of $\mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z})$, it is convenient to define the generalized metric of the Narain lattice,[2]

$$\begin{aligned} \mathcal{H} &:= E^\mathrm{T} E \\ &= \begin{pmatrix} \frac{1}{\alpha'}\left(G + \alpha' A^\mathrm{T} A + C^\mathrm{T} G^{-1} C\right) & -C^\mathrm{T} G^{-1} & (\mathbb{1}_D + C^\mathrm{T} G^{-1}) A^\mathrm{T} \alpha_\mathrm{g} \\ -G^{-1} C & \alpha' G^{-1} & -\alpha' G^{-1} A^\mathrm{T} \alpha_\mathrm{g} \\ \alpha_\mathrm{g}^\mathrm{T} A (\mathbb{1}_D + G^{-1} C) & -\alpha' \alpha_\mathrm{g}^\mathrm{T} A G^{-1} & \alpha_\mathrm{g}^\mathrm{T}\left(\mathbb{1}_{16} + \alpha' A G^{-1} A^\mathrm{T}\right)\alpha_\mathrm{g} \end{pmatrix} \end{aligned} \tag{4}$$

and $C := B + \frac{\alpha'}{2} A^\mathrm{T} A$. Here, α' is the Regge slope that renders E dimensionless, $G = e^\mathrm{T} e$ is the metric of the D-dimensional torus $\mathbb{T}^D$ defined by the torus basis vectors contained as the columns of the $D \times D$ vielbein matrix e. In addition, B is the anti-symmetric background B-field, while the $16 \times D$ matrix A gives rise to the Wilson lines along the D torus directions.

In the following, we will mainly set $D = 2$ and choose the two column vectors A_i of the Wilson line matrix A as

$$A_i = (a_i, -a_i, 0, \ldots, 0)^\mathrm{T} \quad \text{with} \quad a_i \in \mathbb{R} \quad \text{for} \quad i \in \{1, 2\}\,. \tag{5}$$

Then, we define the (dimensionless) string moduli as [15]

$$T := \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\sqrt{\det G}\right) + a_1\left(-a_2 + U\,a_1\right)\,, \tag{6a}$$

$$U := \frac{1}{G_{11}}\left(G_{12} + \mathrm{i}\sqrt{\det G}\right)\,, \tag{6b}$$

$$Z := -a_2 + U\,a_1\,. \tag{6c}$$

2 We use the conventions of refs. [13,14] with B replaced by $-B$.

The Kähler modulus T determines the B-field background, the overall size of the extra-dimensional two-torus, and is altered by the Wilson line parameters a_i. The complex structure modulus U parameterizes the shape of the two-torus $\mathbb{T}^2$, while the Wilson line modulus Z depends on the parameters a_i. The transformation of these moduli under a modular transformation $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ can be computed by considering the generalized metric

$$\mathcal{H}(T, U, Z) \xmapsto{\hat{\Sigma}} \mathcal{H}(T', U', Z') := \hat{\Sigma}^{-\mathrm{T}} \mathcal{H}(T, U, Z)\, \hat{\Sigma}^{-1}\,, \tag{7}$$

see for example ref. [16]. For the transformations $\hat{\Sigma} \in \{\hat{K}_\mathrm{S},\ \hat{K}_\mathrm{T},\ \hat{C}_\mathrm{S},\ \hat{C}_\mathrm{T},\ \hat{M},\ \hat{W}\binom{\ell}{m},\ \hat{\Sigma}_*\}$ given in appendix B this yields

$$T \xmapsto{\hat{K}_\mathrm{S}} -\frac{1}{T}\,, \qquad U \xmapsto{\hat{K}_\mathrm{S}} U - \frac{Z^2}{T}\,, \qquad Z \xmapsto{\hat{K}_\mathrm{S}} -\frac{Z}{T}\,, \tag{8a}$$

$$T \xmapsto{\hat{K}_\mathrm{T}} T + 1\,, \qquad U \xmapsto{\hat{K}_\mathrm{T}} U\,, \qquad Z \xmapsto{\hat{K}_\mathrm{T}} Z\,, \tag{8b}$$

$$T \xmapsto{\hat{C}_\mathrm{S}} T - \frac{Z^2}{U}\,, \qquad U \xmapsto{\hat{C}_\mathrm{S}} -\frac{1}{U}\,, \qquad Z \xmapsto{\hat{C}_\mathrm{S}} -\frac{Z}{U}\,, \tag{8c}$$

$$T \xmapsto{\hat{C}_\mathrm{T}} T\,, \qquad U \xmapsto{\hat{C}_\mathrm{T}} U + 1\,, \qquad Z \xmapsto{\hat{C}_\mathrm{T}} Z\,, \tag{8d}$$

$$T \xmapsto{\hat{M}} U\,, \qquad U \xmapsto{\hat{M}} T\,, \qquad Z \xmapsto{\hat{M}} Z\,, \tag{8e}$$

$$T \xmapsto{\hat{W}\binom{\ell}{m}} T + m(m\,U + 2\,Z - \ell)\,, \qquad U \xmapsto{\hat{W}\binom{\ell}{m}} U\,, \qquad Z \xmapsto{\hat{W}\binom{\ell}{m}} Z + m\,U - \ell, \tag{8f}$$

$$T \xmapsto{\hat{\Sigma}_*} -\bar{T}\,, \qquad U \xmapsto{\hat{\Sigma}_*} -\bar{U}\,, \qquad Z \xmapsto{\hat{\Sigma}_*} -\bar{Z}\,. \tag{8g}$$

2.2. Narain orbifold compactification and string moduli

We can extend the Narain torus boundary conditions (1) by an orbifold action [13,14,17–21]

$$\begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix} \sim \Theta \begin{pmatrix} y_\mathrm{R} \\ y_\mathrm{L} \end{pmatrix} + E\,\hat{N}\,, \quad \text{where} \quad \Theta := \begin{pmatrix} \theta_\mathrm{R} & 0 \\ 0 & \theta_\mathrm{L} \end{pmatrix} \in \mathrm{O}(D) \oplus \mathrm{O}(D+16)\,, \tag{9}$$

and $\hat{N} \in \mathbb{Z}^{2D+16}$. The rotation matrix Θ denotes the so-called Narain twist. The set of all Narain twists generates the so-called Narain point group P_Narain. If $\theta_\mathrm{L} = \theta_\mathrm{R} \oplus \mathbb{1}_{16}$ for all twists, the orbifold is called symmetric, otherwise it is called asymmetric. Moreover, the orbifold action (9) suggests to define transformations $(\Theta, E\,\hat{N})$, which generate the so-called Narain space group S_Narain. Then, an orbifold compactification (of worldsheet bosons) is fully specified by the choice of S_Narain. Due to its right-left structure θ_R and θ_L in eq. (9), a Narain twist Θ has to satisfy the conditions

$$\Theta^\mathrm{T}\,\Theta = \mathbb{1}_{2D+16} \qquad \text{and} \qquad \Theta^\mathrm{T}\,\eta\,\Theta = \eta\,. \tag{10}$$

Furthermore, the Narain twist has to map the Narain lattice to itself. Hence, it is convenient to define the Narain twist in the Narain lattice basis

$$\hat{\Theta} := E^{-1}\,\Theta\,E \in \mathrm{GL}(2D+16, \mathbb{Z})\,, \tag{11}$$

such that $\hat{\Theta}$ is an integer matrix from $\mathrm{GL}(2D+16, \mathbb{Z})$. In the Narain lattice basis, we denote the Narain point group by $\hat{P}_\text{Narain}$ and the Narain space group by $\hat{S}_\text{Narain}$, where its elements are of the form $(\hat{\Theta}, \hat{N})$. Using the Narain lattice basis, the conditions (10) read

$$\hat{\Theta}^{\mathrm{T}}\,\mathcal{H}\,\hat{\Theta} \;=\; \mathcal{H} \qquad \text{and} \qquad \hat{\Theta}^{\mathrm{T}}\,\hat{\eta}\,\hat{\Theta} \;=\; \hat{\eta}\,. \tag{12}$$

Consequently, $\hat{\Theta} \in \mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z})$ has to be an element of the modular group of the Narain lattice that leaves the generalized metric $\mathcal{H}$ invariant, see eq. (7). In general, condition (12) fixes some of the moduli. In other words, the Narain twist $\hat{\Theta}$ is a symmetry of the Narain lattice only for some special values of the moduli. In this case, we say that some moduli are stabilized geometrically by the orbifold action. We denote the modular group after orbifolding by $\mathcal{G}_{\text{modular}}$. It is given by those elements $(\hat{\Sigma}, \hat{T}) \notin \hat{S}_{\text{Narain}}$ with $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z})$ and $\hat{T} \in \mathbb{Q}^{2D+16}$ that are outer automorphisms of the Narain space group $\hat{S}_{\text{Narain}}$, cf. refs. [16,22]. In the case where the Narain space group is generated only by elements of the form $(\hat{\Theta}, 0)$ and $(\mathbb{1}_{2D+16}, \hat{N})$ with $\hat{N} \in \mathbb{Z}^{2D+16}$, the modular group after orbifolding is given by[3]

$$\mathcal{G}_{\text{modular}} := \left\langle\, \hat{\Sigma} \;\middle|\; \hat{\Sigma} \,\in\, \mathrm{O}_{\hat{\eta}}(D, D+16, \mathbb{Z}) \text{ with } \hat{\Sigma}^{-1}\,\hat{\Theta}\,\hat{\Sigma} \in \hat{P}_{\text{Narain}} \text{ for all } \hat{\Theta} \in \hat{P}_{\text{Narain}} \,\right\rangle. \tag{13}$$

Then, one can compute the transformation of the moduli after orbifolding using the generalized metric eq. (7).

In addition to the modular group $\mathcal{G}_{\text{modular}}$ (which is in general an infinite, discrete group), there exists the closely related finite modular group $\mathcal{G}_{\text{fmg}}$, which can play an important role in flavor physics [23–25]. This group appears in string theory as follows: On orbifolds, there are so-called twisted strings that are localized in extra dimensions at the fixed points of the orbifold action. They transform in general under a modular transformation $\hat{\Sigma} \in \mathcal{G}_{\text{modular}}$ nontrivially with a unitary matrix representation $\rho_{\boldsymbol{r}}(\hat{\Sigma})$ of a finite modular group $\mathcal{G}_{\text{fmg}}$, for example $\mathcal{G}_{\text{fmg}} \cong T'$ for the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold [16,22,26–28] and $\mathcal{G}_{\text{fmg}} \cong (S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}$ for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold (without $\mathcal{CP}$) [3,4]. In addition, couplings Y in the superpotential become modular forms of the moduli. Hence, couplings Y also transform under a modular transformation $\hat{\Sigma} \in \mathcal{G}_{\text{modular}}$ in a unitary matrix representation $\rho_Y(\hat{\Sigma})$ of the finite modular group $\mathcal{G}_{\text{fmg}}$. However, in some cases ρ_Y is not a faithful representation of $\mathcal{G}_{\text{fmg}}$. Then, one can compute the finite modular group $\mathcal{G}_{\text{fmg of Y}}$ that is generated by the matrix representations $\rho_Y(\hat{\Sigma})$ of the modular forms such that $\mathcal{G}_{\text{fmg of Y}} \subset \mathcal{G}_{\text{fmg}}$. For example, for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold we have $\mathcal{G}_{\text{fmg of Y}} \cong (S_3^T \times S_3^U) \rtimes \mathbb{Z}_2^{\hat{M}}$, see refs. [3,4].

In this paper we will frequently utilize the correspondences

$$\begin{array}{cccccccc}
M_{(\mathrm{S},\mathbb{1}_2)}\,, & M_{(\mathrm{T},\mathbb{1}_2)}\,, & M_{(\mathbb{1}_2,\mathrm{S})}\,, & M_{(\mathbb{1}_2,\mathrm{T})}\,, & M_{\times}\,, & M\binom{\ell}{m} & \text{from} & \mathrm{Sp}(4,\mathbb{Z}) \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\
\hat{K}_{\mathrm{S}}\,, & \hat{K}_{\mathrm{T}}\,, & \hat{C}_{\mathrm{S}}\,, & \hat{C}_{\mathrm{T}}\,, & \hat{M}\,, & \hat{W}\binom{\ell}{m} & \text{from} & \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,,
\end{array} \tag{14}$$

and between $M_* \in \mathrm{GSp}(4, \mathbb{Z})$ and $\hat{\Sigma}_* \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ for $\mathcal{CP}$, see appendix B for notation and definition of these transformations (see also table 1 of ref. [5] and refs. [29–33]). Here, the generators S and T of the modular group $\mathrm{SL}(2, \mathbb{Z})$ are defined as

$$\mathrm{S} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{T} \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{15}$$

such that $M_{(\gamma_T,\gamma_U)} \in \mathrm{Sp}(4, \mathbb{Z})$ for $\gamma_T \in \mathrm{SL}(2, \mathbb{Z})_T$ and $\gamma_U \in \mathrm{SL}(2, \mathbb{Z})_U$. Note that our definition of the mirror symmetry generator $\hat{M} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ differs from the definition of ref. [5], as explained in appendix A.

[3] If the Narain space group contains generators that are nontrivial roto-translations $(\hat{\Theta}, \hat{N})$ with $\hat{N} \notin \mathbb{Z}^{2D+16}$, also the outer automorphisms of $\hat{S}_{\text{Narain}}$ can be roto-translations, see e.g. appendix B in ref. [3].

3. Stabilizing moduli by $\mathrm{Sp}(4,\mathbb{Z})$ orbifolds

In the following, we consider all inequivalent fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ in the Siegel upper half-plane $\mathcal{H}_2$, as listed in table 2 of ref. [6]. For each fixed point $\tau_\mathrm{f} \in \mathcal{H}_2$, we explicitly construct an orbifold compactification in the Narain formulation by specifying a Narain point group $\hat{P}_\mathrm{Narain}$. Then, the string moduli are stabilized geometrically by the orbifold action in agreement with the fixed point τ_f. To do so, we focus on the moduli (T, U, Z) of a $D = 2$ subsector of a full six-dimensional string compactification. In more detail, for each fixed point τ_f, we consider the stabilizer group $\bar{H} := H/\{\pm\mathbb{1}_4\}$ from appendix D of ref. [6], where

$$H := \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \gamma\,\tau_\mathrm{f} \;=\; \tau_\mathrm{f} \right\} \text{ and } \gamma\,\tau := (A\,\tau + B)\,(C\,\tau + D)^{-1}\;. \tag{16}$$

We take the generators of $\bar{H}$ and write them in terms of $\mathrm{Sp}(4,\mathbb{Z})$ basis elements using the notation of our appendix B. Then, we apply the dictionary eq. (14) between $\mathrm{Sp}(4,\mathbb{Z})$ and $\mathrm{O}_{\hat{\eta}}(2, 2+16,\mathbb{Z})$ in order to translate the stabilizer group $\bar{H}$ into a subgroup $\hat{P}_\mathrm{Narain}$ of $\mathrm{O}_{\hat{\eta}}(2, 2+16,\mathbb{Z})$. By construction, $\hat{P}_\mathrm{Narain}$ maps the Narain lattice in $D = 2$ to itself. Consequently, we can utilize $\hat{P}_\mathrm{Narain}$ as a Narain point group to define a Narain space group $\hat{S}_\mathrm{Narain}$ without roto-translations. It turns out that the moduli (T, U, Z) of the resulting (a)symmetric orbifold are fixed by the orbifold action due to condition (12). Hence, for each fixed point τ_f of $\mathrm{Sp}(4,\mathbb{Z})$ listed in ref. [6], we verify that the string moduli (T, U, Z) are fixed accordingly, i.e.

$$\tau_\mathrm{f} \;=\; \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \;\in\; \mathcal{H}_2 \quad\Leftrightarrow\quad \begin{pmatrix} U & Z \\ Z & T \end{pmatrix} \;\in\; \mathcal{H}_2\,, \tag{17}$$

using an appropriate orbifold compactification. In addition, we use the dictionary between $\mathrm{Sp}(4,\mathbb{Z})$ and $\mathrm{O}_{\hat{\eta}}(2, 2+16,\mathbb{Z})$ to translate both, the normalizer $N(H)$ from ref. [6] and the $\mathcal{CP}$ transformation from ref. [7] into $\mathrm{O}_{\hat{\eta}}(2, 2+16,\mathbb{Z})$. We check explicitly that the resulting transformations are outer automorphisms of the corresponding Narain space group. Hence, for each orbifold we identify the modular group $\mathcal{G}_\mathrm{modular}$, including a $\mathcal{CP}$-like transformation, and compute the transformations of the unfixed moduli with respect to the modular generators $\hat{\Sigma} \in \mathcal{G}_\mathrm{modular}$.

Finally, let us remark that the Narain point groups $\hat{P}_\mathrm{Narain}$ that we construct for each inequivalent fixed point of $\mathrm{Sp}(4,\mathbb{Z})$ are the "maximal" point groups that one can use for the given fixed point. In other words, one can also consider a subgroup of $\hat{P}_\mathrm{Narain}$ as the point group of an orbifold. For example, we will encounter a $\mathbb{Z}_6$ Narain point group for a specific fixed point of $\mathrm{Sp}(4,\mathbb{Z})$. In this case, also the $\mathbb{Z}_3$ subgroup of $\mathbb{Z}_6$ can serve as a point group that will geometrically stabilize the moduli in the same way as the $\mathbb{Z}_6$ orbifold.

In the remainder of this section we shall discuss the orbifolds of the 13 inequivalent fixed points separately and in detail. This is done for completeness of the presentation and as a basis for future research along these lines. Not all of these cases are directly relevant for the discussion of flavor structure, generalizing the orbifolds $\mathbb{T}^2/\mathbb{Z}_K$ $(K = 2, 3, 4, 6)$, including Wilson lines. The reader primarily interested in this flavor structure might thus concentrate on the subsections 3.1.1, 3.1.3, 3.2.1 and 3.2.2, and skip the remainder of this section. Some of the results of this section are summarized in appendix C.

3.1. Orbifolds with moduli spaces of dimension 2

According to ref. [6], there are two inequivalent subspaces of complex dimension 2 that are left invariant by subgroups of $\mathrm{Sp}(4,\mathbb{Z})$. As we show in the following, they can be implemented in string theory by the compactification on $\mathbb{Z}_2$ orbifolds.

3.1.1. Symmetric $\mathbb{Z}_2$ orbifold

Let us consider the point

$$\tau_{\mathrm{f}} \;=\; \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \;\in\; \mathcal{H}_2 \tag{18}$$

in the Siegel upper half-plane $\mathcal{H}_2$, i.e. a complex two-dimensional subspace of $\mathcal{H}_2$ given by $\tau_3 = 0$. In the following, we will explicitly construct a string compactification such that the string moduli are fixed accordingly. To do so, we first have to consider the stabilizer $\bar{H}$ of τ_{f}. For the point eq. (18) the stabilizer is $\bar{H} \cong \mathbb{Z}_2$, generated by

$$h \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;\in\; \mathrm{Sp}(4,\mathbb{Z}) \quad \text{such that} \quad h\,\tau_{\mathrm{f}} \;=\; \tau_{\mathrm{f}}\;, \tag{19}$$

see ref. [6, table 2 and appendix D]. Using the notation of appendix B, this element h is given by the square of a modular S transformation of the modulus τ_2,

$$h \;=\; M_{(\mathrm{S}^2,\mathbb{1}_2)} \;\in\; \mathrm{Sp}(4,\mathbb{Z})\;. \tag{20}$$

From the dictionary eq. (14) we know that this $\mathrm{Sp}(4,\mathbb{Z})$ element corresponds to the $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ element

$$\hat{\Theta} \;:=\; \left(\hat{K}_{\mathrm{S}}\right)^2 \;\in\; \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\;. \tag{21}$$

As a remark, in $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ we have $(\hat{K}_{\mathrm{S}})^2 = (\hat{C}_{\mathrm{S}})^2$. So, we could equally write $\hat{\Theta} = (\hat{C}_{\mathrm{S}})^2$. This Narain twist $\hat{\Theta}$ defines a $\mathbb{Z}_2$ Narain point group $\hat{P}_{\mathrm{Narain}}$ of a symmetric $\mathbb{Z}_2$ orbifold. The moduli (T,U,Z) in the generalized metric $\mathcal{H}$ are constrained by the invariance condition (12) under $\hat{\Theta}$. As a result, we find $a_1 = a_2 = 0$. Hence, the Wilson line modulus Z has to vanish, $Z = 0$. On the other hand, the moduli

$$T \;=\; \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\sqrt{\det G}\right) \quad \text{and} \quad U \;=\; \frac{1}{G_{11}}\left(G_{12} + \mathrm{i}\sqrt{\det G}\right) \tag{22}$$

are unconstrained, as it has been known from ref. [3,4], for example. In other words, the Narain lattice is mapped to itself under the Narain twist $\hat{\Theta}$ eq. (21) only if the Wilson line modulus is trivial $Z = 0$, while the Kähler modulus T and the complex structure modulus U can vary freely. One says that Z has been stabilized geometrically by the orbifold action. This is in agreement with τ_{f} under the identification $\tau_1 = U$, $\tau_2 = T$ and $\tau_3 = Z = 0$. In other words, the complex two-dimensional subspace with $\tau_3 = 0$ described in ref. [6] can be constructed in string theory by the compactification on a symmetric $\mathbb{Z}_2$ orbifold with vanishing Wilson line.

Next, we translate the normalizer

$$N(H) \;=\; \left\langle M_{(\mathrm{S},\mathbb{1}_2)}\;,\; M_{(\mathrm{T},\mathbb{1}_2)}\;,\; M_{(\mathbb{1}_2,\mathrm{S})}\;,\; M_{(\mathbb{1}_2,\mathrm{T})}\;,\; M_{\times} \right\rangle \;\subset\; \mathrm{Sp}(4,\mathbb{Z}) \tag{23}$$

from ref. [6] into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ and check that each generator is an outer automorphism of the Narain space group, see eq. (13). In this way, we identify the modular group (without $\mathcal{CP}$)

$$\mathcal{G}_{\text{modular}} = \left\langle \hat{K}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,,\ \hat{C}_\mathrm{S}\,,\ \hat{C}_\mathrm{T}\,,\ \hat{M} \right\rangle \cong \frac{\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U}{\mathbb{Z}_2} \rtimes \mathbb{Z}_2^{\hat{M}} \tag{24}$$

of the symmetric $\mathbb{Z}_2$ orbifold, where the $\mathbb{Z}_2$ quotient ensures the relation $(\hat{K}_\mathrm{S})^2 = (\hat{C}_\mathrm{S})^2$, see section 3 of ref. [16]. Then, we use the generalized metric (7) in order to identify the transformation of the moduli (T, U) and we obtain the transformations (8a)-(8e) with $Z = 0$, while $Z = 0$ is invariant under all of these transformations. As a remark, for the symmetric $\mathbb{Z}_2$ orbifold, the finite modular group $\mathcal{G}_{\text{fmg}}$ is known explicitly from the transformation matrices $\rho_{\boldsymbol{r}}(\hat{\Sigma})$ of twisted strings with respect to modular transformations $\hat{\Sigma}$. It is given by [3,4]

$$\mathcal{G}_{\text{fmg}} = \left(S_3^T \times S_3^U\right) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]\,, \tag{25}$$

where the $\mathbb{Z}_2^{\hat{M}}$ mirror symmetry of the moduli acts as a $\mathbb{Z}_4^{\hat{M}}$ symmetry on the twisted strings of the $\mathbb{Z}_2$ orbifold. Moreover, the four twisted strings localized at the four orbifold fixed points transform as $\boldsymbol{r} = \mathbf{4}_1$ of the finite modular group [144, 115].

Next, we consider table 1 of ref. [7] and translate their $\mathcal{CP}$ transformation into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. We obtain

$$\hat{\mathcal{CP}} = \hat{\Sigma}_*\,. \tag{26}$$

One can verify easily that $\hat{\mathcal{CP}}$ is an outer automorphism of the Narain space group, i.e.

$$\hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}^{-1}\,. \tag{27}$$

Furthermore, we use the generalized metric (7) and confirm the transformation of the string moduli (T, U),

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}\,, \qquad U \xmapsto{\hat{\mathcal{CP}}} -\bar{U}\,, \tag{28}$$

while $Z = 0$ is invariant under $\hat{\mathcal{CP}}$, see also ref. [34].

3.1.2. Fake-asymmetric $\mathbb{Z}_2$ orbifold

The second fixed point in the list of ref. [6] is given by

$$\tau_\mathrm{f} = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \in \mathcal{H}_2\,. \tag{29}$$

In this case, the stabilizer $\bar{H} \cong \mathbb{Z}_2$ is generated by the mirror element

$$h := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\times \in \mathrm{Sp}(4,\mathbb{Z})\,, \quad \text{such that} \quad h\,\tau_\mathrm{f} = \tau_\mathrm{f} \tag{30}$$

in the notation of appendix B. Since $\hat{M} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ denotes the corresponding mirror element in the string constructing, we choose a Narain twist

$$\hat{\Theta} := \hat{M}\,, \tag{31}$$

see eq. (157) in the appendix. Consequently, we construct an apparently asymmetric $\mathbb{Z}_2$ orbifold with mirror symmetry $\hat{M}$ as Narain twist. Then, eq. (12) yields

$$G_{11} \;=\; \alpha'(1-a_1^2) \quad \text{and} \quad B_{12} \;=\; a_1\, a_2\, \alpha' + G_{12}\,. \tag{32}$$

In this case, one can check that the expectations for the corresponding fixed point τ_{f} are fulfilled,

$$T \;=\; U\,, \quad \text{where} \quad U \;=\; \frac{1}{\alpha'\left(1-a_1^2\right)}\left(G_{12} + \mathrm{i}\sqrt{-G_{12}^2 + \alpha'\, G_{22}\left(1-a_1^2\right)}\right)\,, \tag{33}$$

and $Z = -a_2 + U a_1$. Hence, the two-dimensional subspace with $\tau_1 = \tau_2$ and an unconstrained τ_3, as described in the bottom-up construction of ref. [6], can be obtained in string theory from the compactification on an apparently asymmetric $\mathbb{Z}_2$ orbifold using mirror symmetry as orbifold twist. Note that a physical torus must satisfy $G_{11} > 0$. Thus, eq. (32) constrains the Wilson line to $a_1^2 < 1$. Let us remark that the Narain twist $\hat{\Theta}$ given by the mirror transformation $\hat{M}$ also acts on the 16 gauge degrees of freedom of the heterotic string. This will induce some gauge symmetry breaking, cf. refs. [35,36].

Next, we consider the normalizer

$$N(H) \;=\; \left\langle\, M_{(\mathrm{S},\mathrm{S})}\,,\; M_{(\mathrm{T},\mathrm{T})}\,,\; M_{(\mathrm{S}^2,\mathbb{1}_2)}\,,\; M\begin{pmatrix}-1\\0\end{pmatrix}\right\rangle \subset \mathrm{Sp}(4,\mathbb{Z})\,, \tag{34}$$

given in ref. [6]. We use our dictionary eq. (14) and obtain the group

$$\mathcal{G}_{\text{modular}} \;=\; \left\langle\, \hat{K}_{\mathrm{S}}\,\hat{C}_{\mathrm{S}}\,,\; \hat{K}_{\mathrm{T}}\,\hat{C}_{\mathrm{T}}\,,\; \left(\hat{K}_{\mathrm{S}}\right)^2\,,\; \hat{W}\begin{pmatrix}-1\\0\end{pmatrix}\right\rangle. \tag{35}$$

We verify that this group gives rise to the rotational outer automorphisms of the Narain space group. Hence, $\mathcal{G}_{\text{modular}}$ is the modular group of this apparently asymmetric $\mathbb{Z}_2$ orbifold. Using eq. (7) we find that the two independent moduli (T, Z) transform as

$$T \xmapsto{\hat{K}_{\mathrm{S}}\,\hat{C}_{\mathrm{S}}} -\frac{T}{T^2-Z^2}\,, \qquad Z \xmapsto{\hat{K}_{\mathrm{S}}\,\hat{C}_{\mathrm{S}}} \frac{Z}{T^2-Z^2}\,, \tag{36a}$$

$$T \xmapsto{\hat{K}_{\mathrm{T}}\,\hat{C}_{\mathrm{T}}} T+1\,, \qquad Z \xmapsto{\hat{K}_{\mathrm{T}}\,\hat{C}_{\mathrm{T}}} Z\,, \tag{36b}$$

$$T \xmapsto{\left(\hat{K}_{\mathrm{S}}\right)^2} T\,, \qquad Z \xmapsto{\left(\hat{K}_{\mathrm{S}}\right)^2} -Z\,, \tag{36c}$$

$$T \xmapsto{\hat{W}\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)} T\,, \qquad Z \xmapsto{\hat{W}\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)} Z+1\,, \tag{36d}$$

under the generators of the modular symmetry $\mathcal{G}_{\text{modular}}$ of this apparently asymmetric orbifold.

Finally, we consider the $\mathcal{CP}$ transformation for the fixed point τ_{f} with $\tau_1 = \tau_2$ in table 1 of ref. [7]. Using our dictionary to $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$, this transformation corresponds to

$$\hat{\mathcal{CP}} \;=\; \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} \;=\; \hat{\Theta}^{-1}\,. \tag{37}$$

Hence, $\hat{\mathcal{CP}}$ is an outer automorphism of $\hat{S}_{\text{Narain}}$. From the generalized metric (7) we compute the transformation of the string moduli (T, Z), resulting in

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}\,, \qquad Z \xmapsto{\hat{\mathcal{CP}}} -\bar{Z}\,, \tag{38}$$

for this apparently asymmetric $\mathbb{Z}_2$ orbifold. In the canonical basis this orbifold appears to be asymmetric. As we will show in the next subsection, this orbifold can, however, be mapped to a symmetric orbifold via a basis change. This is why we called it fake-asymmetric in Table 1.

3.1.3. Symmetric $\mathbb{Z}_2$ orbifold with discrete Wilson line

Let us briefly show that the apparently asymmetric $\mathbb{Z}_2$ orbifold constructed in section 3.1.2 is equivalent to a symmetric $\mathbb{Z}_2$ orbifold with nontrivial discrete Wilson line [19,37]. According to appendix D.1 of ref. [6], the fixed point τ_f (with $\tau_1 = \tau_2$) is equivalent to a fixed point τ_f' using a $\mathrm{Sp}(4, \mathbb{Z})$ transformation

$$b := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = M_{(\mathrm{T}^{-1}, \mathbb{1}_2)}\, M\begin{pmatrix} 0 \\ 1 \end{pmatrix} M_{(\mathrm{TS}^3, \mathbb{1}_2)} \in \mathrm{Sp}(4, \mathbb{Z})\,, \tag{39}$$

such that

$$b\,\tau_\mathrm{f}' = \tau_\mathrm{f}\,, \quad \text{where} \quad \tau_\mathrm{f} := \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \quad \text{and} \quad \tau_\mathrm{f}' = \begin{pmatrix} \tau_1' & 1/2 \\ 1/2 & \tau_2' \end{pmatrix}. \tag{40}$$

Here, $\tau_1' := \frac{1}{2}(\tau_3 + \tau_1)$ and $\tau_2' := \frac{1}{2(\tau_3 - \tau_1)}$. Since τ_f is a fixed point of $h = M_\times \in \mathrm{Sp}(4, \mathbb{Z})$, see eq. (30), we find

$$\tau_\mathrm{f} = h\,\tau_\mathrm{f} \quad \Leftrightarrow \quad b\,\tau_\mathrm{f}' = h\,b\,\tau_\mathrm{f}' \quad \Leftrightarrow \quad \tau_\mathrm{f}' = \left(b^{-1} h\, b\right) \tau_\mathrm{f}'\,, \tag{41}$$

such that τ_f' is a fixed point of $\left(b^{-1} h\, b\right) \in \mathrm{Sp}(4, \mathbb{Z})$. Next, we map

$$h' := \left(b^{-1} h\, b\right) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = M_{(\mathrm{S}^2, \mathbb{1}_2)}\, M\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{42}$$

to the corresponding Narain twist $\hat{\Theta}' \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$,

$$\hat{\Theta}' := \hat{B}^{-1}\, \hat{\Theta}\, \hat{B}\,. \tag{43}$$

Here, $\hat{B} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ corresponds to $b \in \mathrm{Sp}(4, \mathbb{Z})$ from eq. (39), i.e.

$$\hat{B} := \left(\hat{K}_\mathrm{T}\right)^{-1} \hat{W}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{K}_\mathrm{T} \left(\hat{K}_\mathrm{S}\right)^3. \tag{44}$$

Since $\hat{\Theta}$ and $\hat{\Theta}'$ are related in eq. (43) by conjugation with $\hat{B} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$, the two Narain point groups belong to the same Narain $\mathbb{Z}$-class [13]: they describe the same physics but in different duality frames. Now, we use $\hat{\Theta} := \hat{M}$ and simplify the Narain twist $\hat{\Theta}'$ from eq. (43),

$$\hat{\Theta}' = \left(\hat{K}_\mathrm{S}\right)^2 \hat{W}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\hat{C}_\mathrm{S}\right)^2 \hat{W}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{45}$$

in agreement with eq. (42). As a result, we have shown that the apparently asymmetric $\mathbb{Z}_2$ orbifold with a Narain twist $\hat{\Theta}$ given by mirror symmetry is equivalent to a symmetric $\mathbb{Z}_2$ orbifold with Narain twist $\hat{\Theta}'$. Then, the new Narain twist $\hat{\Theta}'$ constrains the generalized metric eq. (12) such that the Wilson lines have to be fixed,

$$a_1 = 0 \quad \text{and} \quad a_2 = -1/2\,. \tag{46}$$

Consequently, the moduli (T, U, Z) of the symmetric $\mathbb{Z}_2$ orbifold read

$$Z = 1/2 \quad \text{and} \quad T,\ U \text{ unconstrained}\,, \tag{47}$$

as expected for τ_f'. The modular group of this orbifold can be obtained from eq. (35) using the basis change $\hat{B}$,

$$\mathcal{G}_\mathrm{modular} \;=\; \left\langle \hat{M}_1\,,\; \hat{M}_2\,,\; \hat{M}_3\,,\; \hat{M}_4 \right\rangle, \tag{48}$$

where we have defined

$$\hat{M}_1 \;:=\; \hat{B}^{-1}\,\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{B}\,, \qquad \hat{M}_2 \;:=\; \hat{B}^{-1}\,\hat{K}_\mathrm{T}\,\hat{C}_\mathrm{T}\,\hat{B}\,, \tag{49a}$$

$$\hat{M}_3 \;:=\; \hat{B}^{-1}\left(\hat{K}_\mathrm{S}\right)^2\,\hat{B}\,, \qquad \hat{M}_4 \;:=\; \hat{B}^{-1}\,\hat{W}\!\begin{pmatrix}-1\\0\end{pmatrix}\hat{B}\,. \tag{49b}$$

Note that these transformations satisfy the relations

$$\left(\hat{M}_1\right)^2 \;=\; \left(\hat{M}_3\right)^2 \;=\; \left(\hat{M}_1\,\hat{M}_2\right)^3 \;=\; \left(\hat{M}_1\,\hat{M}_4\right)^6 \;=\; \left(\hat{M}_3\,\hat{M}_4\right)^2 \;=\; \mathbb{1}_{20}\,. \tag{50}$$

Then, we use eq. (7) to compute the modular transformations of the moduli (T, U),

$$T \xmapsto{\hat{M}_1} -\frac{1}{4T}\,, \qquad U \xmapsto{\hat{M}_1} -\frac{1}{4U}\,, \tag{51a}$$

$$T \xmapsto{\hat{M}_2} -\frac{T}{2T-1}\,, \qquad U \xmapsto{\hat{M}_2} U+\frac{1}{2}\,, \tag{51b}$$

$$T \xmapsto{\hat{M}_3} -\frac{1}{4U}\,, \qquad U \xmapsto{\hat{M}_3} -\frac{1}{4T}\,, \tag{51c}$$

$$T \xmapsto{\hat{M}_4} \frac{T}{2T+1}\,, \qquad U \xmapsto{\hat{M}_4} U+\frac{1}{2}\,, \tag{51d}$$

while $Z = 1/2$ is invariant under all of these transformations. Using the relations

$$\hat{M}_1\,\hat{M}_2\left(\hat{M}_4\right)^{-1}\hat{M}_1 \;=\; \hat{K}_\mathrm{T} \quad , \quad \hat{M}_2\,\hat{M}_4 \;=\; \hat{C}_\mathrm{T} \quad \text{and} \quad \hat{M}_1\,\hat{M}_3 \;=\; \hat{M}\,\hat{\Theta}'\,, \tag{52}$$

we observe that the transformations

$$T \xmapsto{\hat{K}_\mathrm{T}} T+1\,, \qquad U \xmapsto{\hat{K}_\mathrm{T}} U\,, \tag{53a}$$

$$T \xmapsto{\hat{C}_\mathrm{T}} T\,, \qquad U \xmapsto{\hat{C}_\mathrm{T}} U+1\,, \tag{53b}$$

$$T \xmapsto{\hat{M}\,\hat{\Theta}'} U\,, \qquad U \xmapsto{\hat{M}\,\hat{\Theta}'} T\,, \tag{53c}$$

follow from eqs. (51). Thus, there exists an alternative set of generators of $\mathcal{G}_\mathrm{modular}$ given by $\hat{K}_\mathrm{T}$, $\hat{M}\,\hat{\Theta}'$, $\hat{M}_1$ and $\hat{M}_2$.

Finally, we analyze $\mathcal{CP}$ for the fixed point τ_f' with $\tau_3' = 1/2$. Using the basis change $\hat{B}$ and $\hat{\mathcal{CP}} = \hat{\Sigma}_*$ from eq. (37), we obtain

$$\hat{\mathcal{CP}}' \;=\; \hat{B}^{-1}\,\hat{\Sigma}_*\,\hat{B} \;=\; \left(\hat{K}_\mathrm{S}\right)^2\hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}'\,\hat{\Theta}'\,\hat{\mathcal{CP}}'^{-1} \;=\; \hat{\Theta}'^{-1}\,. \tag{54}$$

Thus, $\hat{\mathcal{CP}}'$ is an outer automorphism of the Narain space group of the symmetric $\mathbb{Z}_2$ orbifold with discrete Wilson line. The generalized metric (7) transforms under $\hat{\mathcal{CP}}'$ such that we find

$$T \xmapsto{\hat{\mathcal{CP}}'} -\bar{T}\,, \qquad U \xmapsto{\hat{\mathcal{CP}}'} -\bar{U}\,, \tag{55}$$

while $Z = 1/2$ is invariant under the transformation with $\hat{\mathcal{CP}}'$.

3.2. Orbifolds with moduli spaces of dimension 1

Following ref. [6], there are five inequivalent subspaces of complex dimension 1 that are left invariant by subgroups of $\mathrm{Sp}(4,\mathbb{Z})$. In the following, we implement them explicitly in string theory by the compactification on various symmetric and asymmetric orbifolds.

3.2.1. Symmetric $\mathbb{Z}_4$ orbifold

Consider the fixed point

$$\tau_\mathrm{f} \;=\; \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \tau_2 \end{pmatrix} \;\in\; \mathcal{H}_2\,. \tag{56}$$

In this case, the stabilizer $\bar{H} \cong \mathbb{Z}_4$ is generated by

$$h \;:=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;=\; M_{(\mathbb{1}_2,\mathrm{S})} \;\in\; \mathrm{Sp}(4,\mathbb{Z})\,, \tag{57}$$

using the notation of appendix B. This $\mathrm{Sp}(4,\mathbb{Z})$ element corresponds in string theory to a modular S transformation of the complex structure modulus. Thus, we choose a Narain twist

$$\hat{\Theta} \;:=\; \hat{C}_\mathrm{S} \;\in\; \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,. \tag{58}$$

As a result, we construct a symmetric $\mathbb{Z}_4$ orbifold with Narain twist $\hat{\Theta} = \hat{C}_\mathrm{S}$. In order to satisfy eq. (12) we have to set

$$G_{11} \;=\; G_{22} \quad , \quad G_{12} \;=\; 0 \quad \text{and} \quad a_1 \;=\; a_2 \;=\; 0\,. \tag{59}$$

Hence, we have to fix the Wilson lines to zero $Z = 0$, the complex structure modulus to $U = \mathrm{i}$, while the Kähler modulus T remains unconstrained,

$$T \;=\; \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\,G_{11}\right)\,, \tag{60}$$

as expected for the values of τ_f in this case.

The normalizer is given by

$$N(H) \;=\; \left\langle M_{(\mathrm{S},\mathbb{1}_2)}\,,\; M_{(\mathrm{T},\mathbb{1}_2)}\,,\; M_{(\mathbb{1}_2,\mathrm{S})} \right\rangle \;\subset\; \mathrm{Sp}(4,\mathbb{Z})\,, \tag{61}$$

see ref. [6]. Compared to ref. [6], we use $M_{(\mathrm{S},\mathbb{1}_2)}$ and $M_{(\mathbb{1}_2,\mathrm{S})}$ as generators of $N(H)$ instead of $M_{(\mathrm{S}^3,\mathbb{1}_2)}$ and $M_{(\mathbb{1}_2,\mathrm{S}^3)}$. The corresponding group in $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ reads

$$\mathcal{G}_\mathrm{modular} \;=\; \left\langle \hat{K}_\mathrm{S}\,,\; \hat{K}_\mathrm{T}\,,\; \hat{C}_\mathrm{S} \right\rangle \;\cong\; \left(\mathrm{SL}(2,\mathbb{Z})_T \;\times\; \mathbb{Z}_4^R\right)/\,\mathbb{Z}_2\,, \tag{62}$$

which is the modular group of the symmetric $\mathbb{Z}_4$ orbifold. Note that $\mathbb{Z}_4^R$ allows for a geometrical interpretation as a $\pi/2$ sublattice rotation, assuming that this $D = 2$ orbifold is a subsector of a full six-dimensional string compactification, see for example section 3 of ref. [2]. Consequently, $\mathbb{Z}_4^R$ is an R-symmetry, as our notation explicitly indicates. Note that the order of this geometrical $\mathbb{Z}_4$ sublattice rotation will generically be larger than four due to fractional R-charges of twisted strings, cf. refs. [1,2,38–42]. Using the generalized metric (7), we confirm that the Kähler modulus T transforms under the generators of the modular group $\mathcal{G}_\mathrm{modular}$ as expected, i.e.

$$T \overset{\hat{K}_\mathrm{S}}{\longmapsto} -\frac{1}{T} \quad , \quad T \overset{\hat{K}_\mathrm{T}}{\longmapsto} T+1 \quad \text{and} \quad T \overset{\hat{C}_\mathrm{S}}{\longmapsto} T\,. \tag{63}$$

Finally, we consider the $\mathcal{CP}$ transformation for the fixed point τ_f with $\tau_1 = \mathrm{i}$ and $\tau_3 = 0$ given in table 1 of ref. [7]. Using our dictionary eq. (14) from $\mathrm{GSp}(4,\mathbb{Z})$ to $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$, the corresponding $\mathcal{CP}$ transformation of this symmetric $\mathbb{Z}_4$ orbifold is given by

$$\hat{\mathcal{CP}} = \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}^{-1}\,. \tag{64}$$

Thus, it is an outer automorphism of $\hat{S}_\mathrm{Narain}$. We apply $\hat{\mathcal{CP}}$ to the generalized metric (7) and find that $U = \mathrm{i}$ and $Z = 0$ are invariant, while the Kähler modulus transforms as

$$T \overset{\hat{\mathcal{CP}}}{\longmapsto} -\bar{T}\,. \tag{65}$$

3.2.2. Symmetric $\mathbb{Z}_6$ orbifold

The next fixed point in the list of ref. [6] is given by

$$\tau_\mathrm{f} = \begin{pmatrix} \omega & 0 \\ 0 & \tau_1 \end{pmatrix} \in \mathcal{H}_2\,, \tag{66}$$

where $\omega := \exp(2\pi\mathrm{i}/3)$. In this case, the stabilizer $\bar{H} \cong \mathbb{Z}_6$ is generated by one element that we can write in the notation of appendix B as

$$h := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_{(\mathbb{1}_2,\mathrm{S}^3\mathrm{TST})} \in \mathrm{Sp}(4,\mathbb{Z})\,. \tag{67}$$

Hence, we choose a Narain twist

$$\hat{\Theta} := \left(\hat{C}_\mathrm{S}\right)^3 \hat{C}_\mathrm{T}\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T} \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,. \tag{68}$$

This Narain twist defines a symmetric $\mathbb{Z}_6$ orbifold. In order to satisfy eq. (12), we have to fix the string geometry as follows

$$G_{11} = -2\,G_{12} \quad , \quad G_{22} = G_{11} \quad \text{and} \quad a_1 = a_2 = 0\,. \tag{69}$$

This results in

$$U = \omega \quad \text{and} \quad Z = 0\,, \tag{70}$$

while the Kähler modulus T is unconstrained, as expected by comparing to the value of τ_f in the corresponding bottom-up construction of ref. [6].

In this case, the normalizer reads

$$N(H) = \left\langle M_{(\mathrm{S},\mathbb{1}_2)}\,,\ M_{(\mathrm{T},\mathbb{1}_2)}\,,\ M_{(\mathbb{1}_2,\mathrm{S}^3\mathrm{T})} \right\rangle \subset \mathrm{Sp}(4,\mathbb{Z})\,, \tag{71}$$

see ref. [6]. Compared to ref. [6], we use $M_{(\mathrm{S},\mathbb{1}_2)}$ as generator of $N(H)$ instead of $M_{(\mathrm{S}^3,\mathbb{1}_2)}$. The corresponding group in $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ gives the modular group

$$\mathcal{G}_\mathrm{modular} = \left\langle \hat{K}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,,\ \left(\hat{C}_\mathrm{S}\right)^3 \hat{C}_\mathrm{T} \right\rangle \cong \left(\mathrm{SL}(2,\mathbb{Z})_T \times \mathbb{Z}_6^R\right) / \mathbb{Z}_2 \tag{72}$$

of the symmetric $\mathbb{Z}_6$ orbifold, where $(\hat{C}_\mathrm{S})^3\hat{C}_\mathrm{T}$ generates a geometrical $\mathbb{Z}_6$ rotation (and $(\hat{C}_\mathrm{S})^3\hat{C}_\mathrm{T} = \hat{\Theta}^{-1}$). As a sublattice rotation of a six-dimensional orbifold compactification this gives rise to an R-symmetry, whose order will generically be larger than six due to fractional R-charges of twisted strings [1,2,38–42]. Using the generalized metric (7), we compute the transformations of the Kähler modulus T under the generators of the modular group $\mathcal{G}_{\text{modular}}$ and obtain the expected results, i.e.

$$T \xmapsto{\hat{K}_{\mathrm{S}}} -\frac{1}{T} \quad , \quad T \xmapsto{\hat{K}_{\mathrm{T}}} T+1 \quad \text{and} \quad T \xmapsto{\left(\hat{C}_{\mathrm{S}}\right)^3 \hat{C}_{\mathrm{T}}} T\,. \tag{73}$$

Finally, we consider the $\mathcal{CP}$ transformation from table 1 of ref. [7] that leaves the fixed point τ_{f} with $\tau_1 = \omega$ and $\tau_3 = 0$ invariant. Using our dictionary eq. (14) from $\mathrm{GSp}(4,\mathbb{Z})$ to $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$, the corresponding $\mathcal{CP}$ transformation is given by

$$\hat{\mathcal{CP}} = \left(\hat{C}_{\mathrm{T}}\right)^{-1} \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}^{-1}\,. \tag{74}$$

Hence, $\hat{\mathcal{CP}}$ is an outer automorphism of the $\mathbb{Z}_6$ Narain space group. Using eq. (7) for the $\mathcal{CP}$-like transformation $\hat{\mathcal{CP}}$, we find that $U=\omega$ and $Z=0$ are invariant, while the Kähler modulus T transforms as

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}\,. \tag{75}$$

Let us briefly remark that one can define a symmetric $\mathbb{Z}_3$ orbifold by taking just the $\mathbb{Z}_3$ subgroup of the stabilizer $\bar{H} \cong \mathbb{Z}_6$ as Narain point group, i.e. consider a $\mathbb{Z}_3$ Narain point group $\hat{P}_{\text{Narain}}$ generated by the Narain twist $\hat{\Theta}^2$, where $\hat{\Theta}$ is given in eq. (68). This also fixes $U=\omega$ and $Z=0$, yields the same modular group $\mathcal{G}_{\text{modular}}$ and the same $\mathcal{CP}$ transformation.

3.2.3. *Asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold*

For the fixed point

$$\tau_{\mathrm{f}} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix} \in \mathcal{H}_2\,, \tag{76}$$

the stabilizer is given by $\bar{H} \cong D_8/\{\pm\mathbb{1}_4\}$, where D_8 is generated by two $\mathrm{Sp}(4,\mathbb{Z})$ elements,

$$h_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\times \quad \text{and} \quad h_2 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\times\, M_{(\mathrm{S}^2,\mathbb{1}_2)} \tag{77}$$

Consequently, we choose two associated Narain twists $\hat{\Theta}_1, \hat{\Theta}_2 \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ as

$$\hat{\Theta}_1 := \hat{M} \quad \text{and} \quad \hat{\Theta}_2 := \hat{M}\left(\hat{K}_{\mathrm{S}}\right)^2\,. \tag{78}$$

In $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$, the two Narain twists $\hat{\Theta}_1$ and $\hat{\Theta}_2$ generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Narain point group $\hat{P}_{\text{Narain}}$ of an asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Note that the Narain point group is not isomorphic to $D_8 \subset \mathrm{Sp}(4,\mathbb{Z})$ because in $\mathrm{Sp}(4,\mathbb{Z})$ we have $M_{(\mathrm{S}^2,\mathbb{1}_2)} \neq M_{(\mathbb{1}_2,\mathrm{S}^2)}$, while in $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ the identity $(\hat{K}_{\mathrm{S}})^2 = (\hat{C}_{\mathrm{S}})^2$ holds. In other words, the dictionary eq. (14) from $\mathrm{Sp}(4,\mathbb{Z})$ to $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ is two-to-one, such that $D_8/\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with a $\mathbb{Z}_2$ quotient that ensures $(\hat{K}_{\mathrm{S}})^2 = (\hat{C}_{\mathrm{S}})^2$.

In order to leave the generalized metric invariant eq. (12), the string geometry is constrained as follows

$$G_{11} = \alpha' \quad , \quad G_{12} = B_{12} \quad \text{and} \quad a_1 = a_2 = 0\,. \tag{79}$$

Hence, the moduli (T, U, Z) of this asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold read

$$Z = 0 \quad , \quad T = U\,, \quad \text{where} \quad U = \frac{1}{\alpha'}\left(G_{12} + \mathrm{i}\sqrt{\alpha' G_{22} - G_{12}^2}\right)\,, \tag{80}$$

so that only G_{12} and G_{22} are free, in agreement with $\tau_1 = \tau_2$ and $\tau_3 = 0$.

The normalizer $N(H)$ is generated by four elements given by

$$N(H) = \langle\, M_{(\mathrm{S},\mathrm{S})}\,,\; M_{(\mathrm{T},\mathrm{T})}\,,\; M_{(\mathrm{S}^2,\mathbb{1}_2)}\,,\; M_\times \,\rangle \subset \mathrm{Sp}(4,\mathbb{Z})\,. \tag{81}$$

Here, compared to ref. [6], we use $M_{(\mathrm{S},\mathrm{S})}$ as generator of $N(H)$ instead of $M_{(\mathrm{S}^3,\mathrm{S}^3)}$. Translated to $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ this yields the modular group

$$\mathcal{G}_{\text{modular}} = \langle\, \hat{K}_\mathrm{S}\hat{C}_\mathrm{S}\,,\; \hat{K}_\mathrm{T}\hat{C}_\mathrm{T}\,,\; \left(\hat{K}_\mathrm{S}\right)^2\,,\; \hat{M}\,\rangle \cong \mathrm{PSL}(2,\mathbb{Z}) \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\hat{M}}\,, \tag{82}$$

of the asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Note that we find $\mathrm{PSL}(2,\mathbb{Z})$ instead of $\mathrm{SL}(2,\mathbb{Z})$ because $\hat{K}_\mathrm{S}\hat{C}_\mathrm{S}$ is of order 2 (instead of 4). Then, the modulus T transforms nontrivially under $\mathrm{PSL}(2,\mathbb{Z})$,

$$T \xmapsto{\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}} -\frac{1}{T} \qquad \text{and} \qquad T \xmapsto{\hat{K}_\mathrm{T}\,\hat{C}_\mathrm{T}} T+1\,, \tag{83}$$

while it is invariant under $(\hat{K}_\mathrm{S})^2$ and $\hat{M}$.

According to table 1 of ref. [7] the fixed point τ_f with $\tau_1 = \tau_2$ and $\tau_3 = 0$ allows for a $\mathcal{CP}$ transformation given by

$$\hat{\mathcal{CP}} = \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}_i^{-1} \text{ for } i \in \{1,2\}\,. \tag{84}$$

Hence, $\hat{\mathcal{CP}}$ is an outer automorphism of this asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ Narain space group. Using eq. (7) for the $\mathcal{CP}$-like transformation $\hat{\mathcal{CP}}$, we find that $Z = 0$ is invariant, while the modulus $T = U$ transforms as

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}\,. \tag{85}$$

3.2.4. *Asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete Wilson line*

Consider the fixed point

$$\tau_\mathrm{f} = \begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_1 \end{pmatrix} \in \mathcal{H}_2\,. \tag{86}$$

In this case, the stabilizer is given by $\bar{H} \cong D_8/\{\pm\mathbb{1}_4\}$, where D_8 is generated by two $\mathrm{Sp}(4,\mathbb{Z})$ elements,

$$\begin{aligned} h_1 &:= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\times \quad \text{and} \\ h_2 &:= \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = M\begin{pmatrix} -1 \\ 0 \end{pmatrix} M_\times\, M_{(\mathbb{1}_2,\mathrm{S}^2)}\,. \end{aligned} \tag{87}$$

Using the dictionary eq. (14), we choose two associated Narain twists

$$\hat{\Theta}_1 := \hat{M} \quad \text{and} \quad \hat{\Theta}_2 := \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \hat{M} \left(\hat{C}_{\mathrm{S}}\right)^2 . \tag{88}$$

In $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$, the two Narain twists $\hat{\Theta}_1$ and $\hat{\Theta}_2$ generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Narain point group of an asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. In order to leave the generalized metric invariant eq. (12), we have to set

$$G_{11} = \alpha' \quad , \quad G_{12} = B_{12} \quad , \quad a_1 = 0 \quad \text{and} \quad a_2 = -1/2 . \tag{89}$$

This is similar to eq. (79) but with a non-trivial Wilson line $a_2 \neq 0$. Then, the moduli read

$$Z = 1/2 \quad , \quad T = U \, , \quad \text{where} \quad U = \frac{1}{\alpha'}\left(G_{12} + \mathrm{i}\sqrt{\alpha' G_{22} - G_{12}^2}\right) , \tag{90}$$

so that only G_{12} and G_{22} are free. This confirms the expectation for τ_{f} in this situation.

In order to identify the modular group $\mathcal{G}_{\mathrm{modular}}$ of this orbifold, we consider the normalizer

$$\begin{aligned} N(H) = \langle\, & M\begin{pmatrix}-1\\0\end{pmatrix} M_{(\mathrm{S}^2, \mathbb{1}_2)} \, , \; M_\times \, M\begin{pmatrix}0\\-2\end{pmatrix} M_\times \, M\begin{pmatrix}0\\1\end{pmatrix} M_{(\mathrm{ST}^{-1}\mathrm{S}^2, \mathrm{ST}^2\mathrm{ST})} \, M\begin{pmatrix}-1\\3\end{pmatrix} , \\ & M\begin{pmatrix}-1\\0\end{pmatrix} M_{(\mathrm{S}^2\mathrm{T}^{-1}, \mathrm{T}^{-1})} \,\rangle \subset \mathrm{Sp}(4, \mathbb{Z}) \, , \end{aligned} \tag{91}$$

see ref. [6]. We use the dictionary eq. (14) to translate this into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. We find

$$\mathcal{G}_{\mathrm{modular}} = \langle\, \hat{M}_1 \, , \; \hat{M}_2 \, , \; \hat{M}_3 \,\rangle \, , \tag{92}$$

where we have defined

$$\hat{M}_1 := \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \left(\hat{K}_{\mathrm{S}}\right)^2 = \hat{\Theta}_1 \, \hat{\Theta}_2 \, , \tag{93a}$$

$$\hat{M}_2 := \hat{M}\, \hat{W}\begin{pmatrix}0\\-2\end{pmatrix} \hat{M}\, \hat{W}\begin{pmatrix}0\\1\end{pmatrix} \hat{K}_{\mathrm{S}} \left(\hat{K}_{\mathrm{T}}\right)^{-1} \left(\hat{K}_{\mathrm{S}}\right)^2 \hat{C}_{\mathrm{S}} \left(\hat{C}_{\mathrm{T}}\right)^2 \hat{C}_{\mathrm{S}} \, \hat{C}_{\mathrm{T}} \, \hat{W}\begin{pmatrix}-1\\3\end{pmatrix} , \tag{93b}$$

$$\hat{M}_3 := \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \left(\hat{K}_{\mathrm{S}}\right)^2 \left(\hat{K}_{\mathrm{T}}\right)^{-1} \left(\hat{C}_{\mathrm{T}}\right)^{-1} . \tag{93c}$$

Since these transformations are outer automorphisms of the Narain space group, they give rise to the modular group of this asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete Wilson line. Using the generalized metric eq. (7), we identify their action on the modulus T and obtain

$$T \xmapsto{\hat{M}_2} -\frac{2T+3}{4T+2} \quad \text{and} \quad T \xmapsto{\hat{M}_3} T - 1 \, , \tag{94}$$

while $\hat{M}_1$ leaves T invariant.

Finally, for the fixed point τ_{f} with $\tau_1 = \tau_2$ and $\tau_3 = 1/2$ we identify $\mathcal{CP}$ from table 1 of ref. [7] as

$$\hat{\mathcal{CP}} = \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\, \hat{\Theta}_i \, \hat{\mathcal{CP}}^{-1} = \hat{\Theta}_i^{-1} \ \text{ for } i \in \{1, 2\} \, . \tag{95}$$

Hence, $\hat{\mathcal{CP}}$ is an outer automorphism of this asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ Narain space group. Using eq. (7), it leaves $Z = 1/2$ invariant, while the modulus $T = U$ transforms as

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T} \, . \tag{96}$$

3.2.5. Asymmetric S_3 orbifold

The next fixed point in the list of ref. [6] reads

$$\tau_\mathrm{f} \;=\; \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix} \;\in\; \mathcal{H}_2\,. \tag{97}$$

This point is stabilized by $\bar{H} \cong S_3$, generated by two elements

$$h_1 := \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \;=\; M_\times\, M\begin{pmatrix}0\\1\end{pmatrix} M_\times\, M_{(\mathbb{1}_2,\mathrm{S}^2)}\;, \tag{98a}$$

$$h_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;=\; M\begin{pmatrix}0\\1\end{pmatrix} M_{(\mathrm{S}^2,\mathbb{1}_2)}\;. \tag{98b}$$

As a consequence, we choose two corresponding $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ Narain twists

$$\hat{\Theta}_1 \;:=\; \hat{M}\,\hat{W}\begin{pmatrix}0\\1\end{pmatrix} \hat{M}\,\left(\hat{C}_\mathrm{S}\right)^2 \quad\text{and}\quad \hat{\Theta}_2 \;:=\; \hat{W}\begin{pmatrix}0\\1\end{pmatrix}\left(\hat{K}_\mathrm{S}\right)^2\,. \tag{99}$$

In $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$, the two Narain twists $\hat{\Theta}_1$ and $\hat{\Theta}_2$ satisfy the condition

$$\left(\hat{\Theta}_1\right)^2 \;=\; \left(\hat{\Theta}_2\right)^2 \;=\; \left(\hat{\Theta}_1\,\hat{\Theta}_2\right)^3 \;=\; \mathbb{1}_{20}\,. \tag{100}$$

Hence, they generate an S_3 Narain point group $\hat{P}_\mathrm{Narain}$ of an asymmetric S_3 orbifold. In order to leave the generalized metric invariant eq. (12), we have to set

$$G_{11} \;=\; \frac{3\,\alpha'}{4} \quad,\quad G_{12} \;=\; B_{12} \quad,\quad a_1 \;=\; 1/2 \quad\text{and}\quad a_2 \;=\; 0\,. \tag{101}$$

In this case, the moduli read

$$Z \;=\; T/2 \quad,\quad T \;=\; U\,, \tag{102}$$

as expected for the given values of τ_f.

According to ref. [6] and using the $\mathrm{Sp}(4,\mathbb{Z})$ generators defined in appendix B, the normalizer $N(H)$ can be generated by four elements

$$\begin{aligned} N(H) = \langle\; & M_{(\mathrm{S},\mathrm{S}^3)}\, M_\times\;,\; M_{(\mathrm{S}^3,\mathrm{S}^3)}\, M\begin{pmatrix}-1\\0\end{pmatrix} M_{(\mathrm{T}^{-2},\mathrm{T}^{-2})}\, M_{(\mathrm{S},\mathrm{S})}\;, \\ & M_\times\;,\; M\begin{pmatrix}0\\1\end{pmatrix} M_{(\mathrm{S}^2,\mathbb{1}_2)}\;\rangle \;\subset\; \mathrm{Sp}(4,\mathbb{Z})\,. \end{aligned} \tag{103}$$

Using the dictionary eq. (14), the corresponding group in $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ reads

$$\mathcal{G}_\mathrm{modular} \;=\; \left\langle\, \hat{M}_1\;,\; \hat{M}_2\;,\; \hat{M}\;,\; \hat{W}\begin{pmatrix}0\\1\end{pmatrix}\left(\hat{K}_\mathrm{S}\right)^2 \,\right\rangle, \tag{104}$$

where we defined

$$\hat{M}_1 := \hat{K}_{\mathrm{S}} \left(\hat{C}_{\mathrm{S}}\right)^3 \hat{M} \quad \text{and} \quad \hat{M}_2 := \left(\hat{K}_{\mathrm{S}}\right)^3 \left(\hat{C}_{\mathrm{S}}\right)^3 \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \left(\hat{K}_{\mathrm{T}}\right)^{-2} \left(\hat{C}_{\mathrm{T}}\right)^{-2} \hat{K}_{\mathrm{S}}\hat{C}_{\mathrm{S}}. \tag{105}$$

As the group $\mathcal{G}_{\text{modular}}$ consists of outer automorphisms of the asymmetric S_3 Narain space group, this yields the modular group of this orbifold. Then, using eq. (7) the Kähler modulus T transforms under the generators of $\mathcal{G}_{\text{modular}}$ as

$$T \xmapsto{\hat{M}_1} -\frac{4}{3T} \quad \text{and} \quad T \xmapsto{\hat{M}_2} \frac{2T}{3T+2}\,, \tag{106}$$

while T is invariant under $\hat{M}$ and $\hat{W}\binom{0}{1}(\hat{K}_{\mathrm{S}})^2$.

Finally, for the fixed point τ_{f} with $\tau_1 = \tau_2$ and $\tau_3 = \tau_1/2$ we translate $\mathcal{CP}$ from table 1 of ref. [7] into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$, yielding

$$\hat{\mathcal{CP}} = \hat{W}\begin{pmatrix}-1\\0\end{pmatrix} \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}_i^{-1} \text{ for } i \in \{1,2\}\,. \tag{107}$$

Consequently, $\hat{\mathcal{CP}}$ is an outer automorphism of this asymmetric S_3 Narain space group. We use eq. (7) to show that $\hat{\mathcal{CP}}$ acts as

$$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}\,. \tag{108}$$

3.3. Orbifolds with moduli spaces of dimension 0

Finally, there are six inequivalent subspaces of complex dimension 0 that are left invariant by subgroups of $\mathrm{Sp}(4, \mathbb{Z})$ [6]. As before, for each fixed point the corresponding orbifold is constructed by embedding the stabilizer $\bar{H}$ from $\mathrm{Sp}(4, \mathbb{Z})$ into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. This yields a Narain point group $\hat{P}_{\text{Narain}}$ in $D = 2$. In all cases discussed here, it turns out that $\hat{P}_{\text{Narain}}$ gives rise to an asymmetric orbifold, whose moduli are stabilized geometrically, as expected. Note that the normalizers are given in terms of the stabilizers, $N(H) = H$, as there are no free moduli. Hence, embedding $N(H)$ into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ just gives $\mathcal{G}_{\text{modular}} \cong \hat{P}_{\text{Narain}}$. If the $D = 2$ orbifold constructed here is a subsector of a full $D = 6$ orbifold, $\mathcal{G}_{\text{modular}}$ yields a traditional flavor symmetry, as a transformation is called modular only if some modulus transforms nontrivially. In addition, we consider $\mathcal{CP}$ from ref. [7] and confirm that these transformations are also unbroken in the corresponding string constructions. As the string moduli are stabilized geometrically, one cannot move in moduli space away from the $\mathcal{CP}$-conserving point. Hence, $\mathcal{CP}$ cannot be broken spontaneously by the moduli in these cases.

3.3.1. Asymmetric $\mathbb{Z}_5$ orbifold

The next fixed point in the Siegel upper half-plane is given by

$$\tau_{\mathrm{f}} = \begin{pmatrix} \zeta & \zeta+\zeta^{-2} \\ \zeta+\zeta^{-2} & -\zeta^{-1} \end{pmatrix} \in \mathcal{H}_2\,, \tag{109}$$

where $\zeta := \exp(2\pi\mathrm{i}/5)$. In this case, the stabilizer $\bar{H} \cong \mathbb{Z}_5$ is generated by

$$h := \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = M_{(\mathbb{1}_2,\mathrm{T})}\, M_\times\, M\begin{pmatrix}0\\1\end{pmatrix} M_{(\mathrm{S}^2,\mathrm{S}^3)}\,, \tag{110}$$

using the $\mathrm{Sp}(4,\mathbb{Z})$ generators of appendix B. Then, we define the corresponding Narain twist $\hat{\Theta} \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$

$$\hat{\Theta} \;:=\; \hat{C}_{\mathrm{T}}\,\hat{M}\,\hat{W}\begin{pmatrix}0\\1\end{pmatrix}\,\left(\hat{K}_{\mathrm{S}}\right)^2\,\left(\hat{C}_{\mathrm{S}}\right)^3\,. \tag{111}$$

Note that $h^5 = -\mathbb{1}_4$, which is identified with $+\mathbb{1}_4$ in $\bar{H}$. Also the Narain twist $\hat{\Theta}$ is of order 5 such that $\hat{\Theta}$ defines a $\mathbb{Z}_5$ Narain point group of an asymmetric $\mathbb{Z}_5$ orbifold. This orbifold action is compatible with the generalized metric eq. (12) if

$$G_{11} \;=\; \frac{\alpha'}{2}\left(-5+3\sqrt{5}\right)\,,\qquad G_{22} \;=\; G_{11}\,,\qquad G_{12} \;=\; \frac{\alpha'}{2}\left(5-2\sqrt{5}\right)\,, \tag{112a}$$

$$B_{12} \;=\; \frac{\alpha'}{2}\left(2-\sqrt{5}\right)\,,\qquad a_1 \;=\; \frac{1}{2}\left(3-\sqrt{5}\right)\,,\qquad a_2 \;=\; \frac{1}{2}\left(-1+\sqrt{5}\right)\,. \tag{112b}$$

Consequently, all moduli are fixed by the orbifold action to the values

$$T \;=\; -\,\zeta^{-1}\quad,\quad U \;=\; \zeta\quad\text{and}\quad Z \;=\; \zeta+\zeta^{-2}\,, \tag{113}$$

in agreement with the fixed point $\tau_{\mathrm{f}} \in \mathcal{H}_2$ of $\mathrm{Sp}(4,\mathbb{Z})$.

Finally, we translate the $\mathcal{CP}$ generator of this case from table 1 of ref. [7] into $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ and obtain

$$\hat{\mathcal{CP}} \;=\; \left(\left(\hat{K}_{\mathrm{S}}\,\hat{C}_{\mathrm{S}}\,\hat{W}\begin{pmatrix}-1\\0\end{pmatrix}\right)^3\,\hat{W}\begin{pmatrix}-1\\0\end{pmatrix}\right)^{-1}\,\hat{\Sigma}_*\qquad\text{with}\qquad \hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}^{-1}\,. \tag{114}$$

Furthermore, the geometrically stabilized moduli (T,U,Z) are invariant under this $\mathcal{CP}$ transformation. Its a trivial fact that $\mathcal{CP}$ cannot be broken spontaneously by the moduli of this $\mathbb{Z}_5$ orbifold sector since all moduli are completely fixed according to eq. (113).

3.3.2. Asymmetric S_4 orbifold

Taking the fixed point

$$\tau_{\mathrm{f}} \;=\; \begin{pmatrix}\tilde{\eta} & \frac{1}{2}(\tilde{\eta}-1)\\ \frac{1}{2}(\tilde{\eta}-1) & \tilde{\eta}\end{pmatrix}\;\in\;\mathcal{H}_2\,, \tag{115}$$

where $\tilde{\eta} := \frac{1}{3}(1+2\sqrt{2}\,\mathrm{i})$, the stabilizer $\bar{H} \cong S_4$ is generated by two elements

$$h_1 \;:=\; \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad\text{and}\quad h_2 \;:=\; \begin{pmatrix}-1&1&1&0\\1&0&0&1\\-1&0&0&0\\1&-1&0&1\end{pmatrix}\,. \tag{116}$$

We write these $\mathrm{Sp}(4,\mathbb{Z})$ elements in terms of the generators given in appendix B

$$h_1 \;=\; M_\times\quad\text{and}\quad h_2 \;=\; M_{(\mathbb{1}_2,\mathrm{S}^3)}\,M_\times\,M\begin{pmatrix}0\\-1\end{pmatrix}\,M_{(\mathrm{S},\mathbb{1}_2)}M\begin{pmatrix}-1\\-1\end{pmatrix}\,. \tag{117}$$

Then, we define the corresponding Narain twists

$$\hat{\Theta}_1 \;:=\; \hat{M} \quad\text{and}\quad \hat{\Theta}_2 \;:=\; \left(\hat{C}_\mathrm{S}\right)^3 \hat{M}\,\hat{W}\begin{pmatrix}0\\-1\end{pmatrix}\hat{K}_\mathrm{S}\,\hat{W}\begin{pmatrix}-1\\-1\end{pmatrix} . \tag{118}$$

Note that $(h_2)^4 = -\mathbb{1}_4$ is of order 8 in H (and of order 4 in $\bar{H}$). The corresponding Narain twist $\hat{\Theta}_2$ is also of order 4. $\hat{\Theta}_1$ and $\hat{\Theta}_2$ obey the relations

$$\left(\hat{\Theta}_1\right)^2 \;=\; \left(\hat{\Theta}_2\right)^4 \;=\; \left(\hat{\Theta}_1\,\hat{\Theta}_2\right)^3 \;=\; \mathbb{1}_{20}\,. \tag{119}$$

Hence, $\hat{\Theta}_1$ and $\hat{\Theta}_2$ generate a S_4 Narain point group of an asymmetric S_4 orbifold. This orbifold action constrains the generalized metric eq. (12) such that

$$G_{11} \;=\; G_{22} \;=\; \frac{3\alpha'}{4}\;,\;\; G_{12} \;=\; \frac{\alpha'}{4}\;,\;\; B_{12} \;=\; \frac{\alpha'}{2}\;,\;\; a_1 \;=\; a_2 \;=\; 1/2\,. \tag{120}$$

Consequently, all moduli of this orbifold are stabilized geometrically,

$$T \;=\; U \;=\; \tilde{\eta} \quad\text{and}\quad Z \;=\; \frac{1}{2}\,(\tilde{\eta}-1)\;. \tag{121}$$

Hence, the subspace $\tau_1 = \tau_2 = \tilde{\eta}$ and $\tau_3 = \frac{1}{2}(\tilde{\eta}-1)$ can be implemented in string theory using an asymmetric S_4 orbifold.

We obtain a $\mathcal{CP}$ transformation for this asymmetric S_4 orbifold by translating the corresponding case from table 1 of ref. [7] into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. This yields

$$\hat{\mathcal{CP}} \;=\; \hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{W}\begin{pmatrix}-1\\0\end{pmatrix}\left(\hat{C}_\mathrm{T}\,\hat{K}_\mathrm{T}\right)^{-1}\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{\Sigma}_* \qquad\text{with}\qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} \;=\; \hat{\Theta}_i^{-1}\,, \tag{122}$$

for $i \in \{1, 2\}$. Then, we confirm that the geometrically stabilized string moduli (T, U, Z) given in eq. (121) are invariant under this $\mathcal{CP}$ transformation.

3.3.3. Asymmetric $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ orbifold

The fixed point

$$\tau_\mathrm{f} \;=\; \begin{pmatrix}\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix} \;\in\; \mathcal{H}_2\,, \tag{123}$$

is stabilized by $\bar{H} \cong (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, generated by three elements

$$h_1 := \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\,,\; h_2 := \begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix} \text{ and } h_3 := \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{124}$$

We write these $\mathrm{Sp}(4, \mathbb{Z})$ elements in terms of the generators of appendix B

$$h_1 \;=\; M_{(\mathrm{S}^2,\mathrm{S})} \quad\text{and}\quad h_2 \;=\; M_{(\mathrm{S},\mathrm{S}^3)} \quad\text{and}\quad h_3 \;=\; M_\times\,. \tag{125}$$

Then, we define the corresponding Narain twists

$$\hat{\Theta}_1 \;:=\; \left(\hat{K}_\mathrm{S}\right)^2 \hat{C}_\mathrm{S} \quad\text{and}\quad \hat{\Theta}_2 \;:=\; \hat{K}_\mathrm{S}\left(\hat{C}_\mathrm{S}\right)^3 \quad\text{and}\quad \hat{\Theta}_3 \;:=\; \hat{M}\,. \tag{126}$$

These Narain twists obey the relations

$$\left(\hat{\Theta}_1\right)^4 = \left(\hat{\Theta}_2\right)^2 = \left(\hat{\Theta}_3\right)^2 = \mathbb{1}_{20}\,, \tag{127a}$$

$$\hat{\Theta}_1\,\hat{\Theta}_2 = \hat{\Theta}_2\,\hat{\Theta}_1 \quad , \quad \hat{\Theta}_2\,\hat{\Theta}_3 = \hat{\Theta}_3\,\hat{\Theta}_2 \quad \text{and} \quad \hat{\Theta}_3\,\hat{\Theta}_1\,\hat{\Theta}_3 = \hat{\Theta}_1\,\hat{\Theta}_2\,. \tag{127b}$$

Hence, they define a $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ Narain point group of an asymmetric $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ orbifold. This orbifold action fixes all components of the generalized metric eq. (12) as follows

$$G_{11} = G_{22} = \alpha' \quad , \quad G_{12} = B_{12} = 0 \quad \text{and} \quad a_1 = a_2 = 0\,. \tag{128}$$

Consequently, all moduli are fixed

$$T = U = \mathrm{i} \quad \text{and} \quad Z = 0\,, \tag{129}$$

in agreement with $\tau_1 = \tau_2 = \mathrm{i}$ and $\tau_3 = 0$.

Finally, we consider the fixed point τ_{f} with $\tau_1 = \tau_2 = \mathrm{i}$ and $\tau_3 = 0$ in table 1 of ref. [7] and translate the corresponding $\mathcal{CP}$ transformation into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. Hence, we confirm explicitly that the outer automorphism of the Narain space group

$$\hat{\mathcal{CP}} = \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}_i^{-1}\,, \tag{130}$$

for $i \in \{1, 2, 3\}$, leaves the geometrically stabilized moduli (T, U, Z) given in eq. (129) invariant. Hence, we have identified a $\mathcal{CP}$-like transformation of this orbifold theory.

3.3.4. *Asymmetric $S_3 \times \mathbb{Z}_6$ orbifold*

Next, we consider the fixed point

$$\tau_{\mathrm{f}} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \in \mathcal{H}_2\,. \tag{131}$$

Its stabilizer $\bar{H} \cong S_3 \times \mathbb{Z}_6$ is generated by three elements

$$h_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; h_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } h_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{132}$$

h_1 and h_2 generate S_3, while h_3 is the generator of $\mathbb{Z}_6$. These $\mathrm{Sp}(4, \mathbb{Z})$ elements can be written in terms of the generators of appendix B and we obtain

$$h_1 = M_{(\mathrm{T}^{-1}\mathrm{S},\mathrm{S}^3\mathrm{T})}\,M_\times \quad , \quad h_2 = M_\times \quad \text{and} \quad h_3 = M_{(\mathrm{S}^3\mathrm{T},\mathrm{ST})}\,. \tag{133}$$

Using the dictionary eq. (14), we can define the corresponding Narain twists

$$\hat{\Theta}_1 := \left(\hat{K}_{\mathrm{T}}\right)^{-1} \hat{K}_{\mathrm{S}} \left(\hat{C}_{\mathrm{S}}\right)^3 \hat{C}_{\mathrm{T}}\,\hat{M} \quad , \quad \hat{\Theta}_2 := \hat{M} \quad \text{and} \quad \hat{\Theta}_3 := \left(\hat{K}_{\mathrm{S}}\right)^3 \hat{K}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}}\,\hat{C}_{\mathrm{T}}\,. \tag{134}$$

The Narain twists $\hat{\Theta}_1$ and $\hat{\Theta}_2$ satisfy the relations

$$\left(\hat{\Theta}_1\right)^2 = \left(\hat{\Theta}_2\right)^2 = \left(\hat{\Theta}_1\,\hat{\Theta}_2\right)^3 = \mathbb{1}_{20}\,. \tag{135}$$

Hence, they generate the permutation group S_3. Furthermore, the order 6 Narain twist $\hat{\Theta}_3$ commutes with both, $\hat{\Theta}_1$ and $\hat{\Theta}_2$. Consequently, the three Narain twists $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ generate an

$S_3 \times \mathbb{Z}_6$ Narain point group $\hat{P}_{\text{Narain}}$, which we use to define an asymmetric $S_3 \times \mathbb{Z}_6$ orbifold. Next, we construct the Narain lattice of this orbifold by demanding invariance of the generalized metric eq. (12) under the three Narain twists. This yields

$$G_{11} = G_{22} = \alpha' \;,\; G_{12} = B_{12} = -\frac{\alpha'}{2} \;,\; a_1 = a_2 = 0\,. \tag{136}$$

So, we see that all moduli (T, U, Z) are stabilized and their values read

$$T = U = \omega \quad \text{and} \quad Z = 0\,. \tag{137}$$

Thus, the invariant subspace $\tau_1 = \tau_2 = \omega$ and $\tau_3 = 0$ discussed in the bottom-up construction of ref. [6] can be constructed explicitly in string theory using an asymmetric orbifold compactification of $D = 2$ dimensions with Narain point group $\hat{P}_{\text{Narain}} \cong S_3 \times \mathbb{Z}_6$.

Using table 1 of ref. [7] we identify $\mathcal{CP}$ as

$$\hat{\mathcal{CP}} = \left(\hat{C}_{\text{T}}\,\hat{K}_{\text{T}}\right)^{-1}\,\hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} \in [\hat{\Theta}_i^{-1}]\,. \tag{138}$$

Hence, $\hat{\mathcal{CP}}$ is a class-inverting outer automorphism [43] of $\hat{S}_{\text{Narain}}$ that leaves the geometrically stabilized moduli eq. (137) invariant.

3.3.5. *Asymmetric $S_3 \times \mathbb{Z}_2$ orbifold*

The next fixed point in the Siegel upper half-plane from the list of ref. [6] is given by

$$\tau_{\text{f}} = \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathcal{H}_2\,. \tag{139}$$

In this case, the stabilizer $\bar{H} \cong S_3 \times \mathbb{Z}_2$ is generated by three elements

$$h_1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \;,\; h_2 := \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \text{ and}$$
$$h_3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,. \tag{140}$$

h_1 and h_2 generate S_3 (using that $(h_1)^2 = (h_2)^2 = -\mathbb{1}_4$ is identified with $+\mathbb{1}_4$ in $\bar{H}$), while the $\mathbb{Z}_2$ factor is generated by h_3. These $\text{Sp}(4,\mathbb{Z})$ elements can be decomposed in terms of the generators of appendix B and we find

$$h_1 = M_\times\, M_{(\text{S},\text{S})}\, M\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad,\quad h_2 = M_{(\text{S},\text{S})}\, M\begin{pmatrix} 0 \\ -1 \end{pmatrix} M_\times \quad \text{and} \quad h_3 = M_{(\text{S}^3,\text{S})}\, M_\times\,. \tag{141}$$

Next, we map these $\text{Sp}(4,\mathbb{Z})$ elements into the Narain construction using ref. [5] and define the following Narain twists

$$\hat{\Theta}_1 := \hat{M}\,\hat{K}_{\text{S}}\,\hat{C}_{\text{S}}\,\hat{W}\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad,\quad \hat{\Theta}_2 := \hat{K}_{\text{S}}\,\hat{C}_{\text{S}}\,\hat{W}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\hat{M} \quad \text{and} \quad \hat{\Theta}_3 := \left(\hat{K}_{\text{S}}\right)^3\hat{C}_{\text{S}}\,\hat{M}\,. \tag{142}$$

The Narain twists $\hat{\Theta}_1$ and $\hat{\Theta}_2$ satisfy the relations

$$\left(\hat{\Theta}_1\right)^2 = \left(\hat{\Theta}_2\right)^2 = \left(\hat{\Theta}_1\,\hat{\Theta}_2\right)^3 = \mathbb{1}_{20}\,. \tag{143}$$

Hence, they generate the permutation group S_3. Furthermore, the order 2 Narain twist $\hat{\Theta}_3$ commutes with both, $\hat{\Theta}_1$ and $\hat{\Theta}_2$. Thus, the Narain twists $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ generate an $S_3 \times \mathbb{Z}_2$ Narain point group. The resulting asymmetric $S_3 \times \mathbb{Z}_2$ orbifold restricts the generalized metric eq. (12) to a unique form, given by

$$G_{11} = \frac{3\,\alpha'}{4}\ ,\ G_{22} = \alpha'\ ,\ G_{12} = B_{12} = 0\ ,\ a_1 = 1/2\ ,\ a_2 = 0\,. \tag{144}$$

This orbifold is similar to the S_3 orbifold with moduli given in eq. (101), but with the additional constraints $G_{22} = \alpha'$ and $B_{12} = 0$. As a consequence, the moduli (T, U, Z) have to take the values

$$T = U = \frac{2\,\mathrm{i}}{\sqrt{3}} \quad \text{and} \quad Z = \frac{\mathrm{i}}{\sqrt{3}}\,, \tag{145}$$

as expected from the bottom-up discussion in ref. [6].

Also in this case, we can use table 1 of ref. [7] to identify a $\mathcal{CP}$-like transformation of the asymmetric $S_3 \times \mathbb{Z}_2$ orbifold,

$$\hat{\mathcal{CP}} = \hat{\Sigma}_* \qquad \text{with} \qquad \hat{\mathcal{CP}}\,\hat{\Theta}_i\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}_i^{-1}\,, \tag{146}$$

for all generators $i \in \{1, 2, 3\}$ of this $S_3 \times \mathbb{Z}_2$ Narain point group.

3.3.6. Asymmetric $\mathbb{Z}_{12}$ orbifold

Finally, we consider the fixed point

$$\tau_\mathrm{f} = \begin{pmatrix} \omega & 0 \\ 0 & \mathrm{i} \end{pmatrix} \in \mathcal{H}_2\,. \tag{147}$$

This point is stabilized by $\bar{H} \cong \mathbb{Z}_{12}$, which is generated by an element $h \in \mathrm{Sp}(4, \mathbb{Z})$ that we can write in terms of the generators defined in appendix B as follows

$$h := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = M_{(\mathrm{S},\mathrm{ST})}\,. \tag{148}$$

This $\mathrm{Sp}(4, \mathbb{Z})$ element can be mapped to a Narain twist $\hat{\Theta} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ using the dictionary eq. (14), and we obtain

$$\hat{\Theta} := \hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T}\,, \tag{149}$$

which is of order 12. Hence, $\hat{\Theta}$ generates a $\mathbb{Z}_{12}$ Narain point group and, consequently, an asymmetric $\mathbb{Z}_{12}$ orbifold in $D = 2$ dimensions, cf. ref. [44] and section 8 of ref. [13]. The generalized metric $\mathcal{H}$ needs to be invariant, see eq. (12), which fixes $\mathcal{H}$ to

$$G_{11} = G_{22} = \frac{2\,\alpha'}{\sqrt{3}}\ ,\ G_{12} = -\frac{\alpha'}{\sqrt{3}}\ ,\ B_{12} = 0\ ,\ a_1 = a_2 = 0\,. \tag{150}$$

As a consequence, the moduli (T, U, Z) have to take the values

$$T\ =\ \mathrm{i}\quad,\quad U\ =\ \omega\quad\text{and}\quad Z\ =\ 0\,. \tag{151}$$

Thus, we have found an explicit realization of the invariant subspace $\tau_1 = U = \omega$, $\tau_2 = T = \mathrm{i}$ and $\tau_3 = Z = 0$ in terms of an asymmetric $\mathbb{Z}_{12}$ orbifold, where the moduli $(T, U, Z) = (\mathrm{i}, \omega, 0)$ are frozen by the orbifold action.

Finally, we translate the $\mathcal{CP}$ transformation from table 1 of ref. [7] for the case $\tau_1 = U = \omega$, $\tau_2 = T = \mathrm{i}$ and $\tau_3 = Z = 0$ into $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$. We find

$$\hat{\mathcal{CP}}\ =\ \left(\hat{C}_{\mathrm{T}}\right)^{-1}\,\hat{\Sigma}_*\qquad\text{with}\qquad \hat{\mathcal{CP}}\,\hat{\Theta}\,\hat{\mathcal{CP}}^{-1} = \hat{\Theta}^{-1}\,. \tag{152}$$

Thus, $\hat{\mathcal{CP}}$ is an outer automorphism of the Narain space group of this asymmetric $\mathbb{Z}_{12}$ orbifold. As expected, it leaves the moduli eq. (151) invariant.

4. Conclusion and outlook

In the present work we have initiated the discussion of flavor symmetries of the Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ from a top-down perspective. In string theory, $\mathrm{Sp}(4, \mathbb{Z})$ describes properties of the moduli T and U of a two-torus compactification as well as a Wilson line Z, as can be derived from the Narain lattice construction. This can be visualized through the moduli of a Riemann surface of genus 2, which in the case of a vanishing Wilson line splits in two separate tori describing the T and U moduli independently.

The road to understand the relevance of $\mathrm{Sp}(4, \mathbb{Z})$ for flavor symmetries of the Standard Model requires several steps. In a first step, we have to insist on the presence of chiral matter fields, which can be achieved by an orbifold twist. In the case of the previously discussed two-torus with vanishing Wilson lines, we had identified the possible orbifolds as those with twists $\mathbb{Z}_K$ and $K = 2, 3, 4, 6$, with fixed points of the complex structure modulus U at the boundaries of the fundamental domain of $\mathrm{SL}(2, \mathbb{Z})_U$. The generalization to $\mathrm{Sp}(4, \mathbb{Z})$ then requires the classification of those orbifolds that lead to the fixed surfaces of $\mathrm{Sp}(4, \mathbb{Z})$ in the Siegel upper half plane. These include two surfaces of complex dimension 2, five of complex dimension 1 and six of dimension 0. We have identified the 13 corresponding orbifolds explicitly and summarize our results in Tables 1 and 2. In contrast to the previously discussed cases, we often find asymmetric orbifolds, which appear, for example, once we mod out the mirror symmetry (which interchanges T and U). For each orbifold, we obtain the unbroken modular group $\mathcal{G}_{\text{modular}}$ including $\mathcal{CP}$ and the associated moduli transformations. With these results, we have completed the first step towards the understanding of the flavor structure of $\mathrm{Sp}(4, \mathbb{Z})$.

In a second step, we would then have to analyze the properties of these orbifolds in detail. The symmetric orbifolds can be understood easily as they have a simple geometric interpretation. They extend the previously discussed cases $\mathbb{Z}_K$ with $K = 2, 3, 4, 6$. The construction in section 3.1.1 corresponds to the $\mathbb{Z}_2$ orbifold with complex moduli T an U and vanishing Wilson line. The cases with $K = 3, 4, 6$ require a fixing of the complex structure modulus U, which is addressed in section 3.2.1 for $\mathbb{Z}_4$ and section 3.2.2 for $\mathbb{Z}_3$ and $\mathbb{Z}_6$. The case discussed in section 3.1.2 corresponds to a fake-asymmetric orbifold with two complex moduli $T = U$ and Wilson line Z. As a direct geometric interpretation is lost in this case, an understanding of its properties requires further investigations. To regain the standard geometric picture it can be mapped to a symmetric $\mathbb{Z}_2$ orbifold with moduli T and U and a quantized Wilson line $Z = 1/2$ as shown in section 3.1.3. This reinterpretation of an apparently asymmetric orbifold as a symmetric orbifold with specifically transformed moduli is a very special case. It seems to be the only fake-asymmetric orbifold in Table 1. We are not aware of basis changes that allow the other asymmetric orbifolds to be mapped to symmetric orbifolds. Further studies should include the consideration of these asymmetric orbifolds towards the construction of models with the particle content of the Standard Model of particle physics.

In a third step, one would have to discuss for each orbifold the discrete flavor symmetries in the full eclectic picture of refs. [45,46]. This includes the traditional flavor group as well as the unbroken finite Siegel modular group that originates from the unbroken subgroup $\mathcal{G}_{\text{modular}}$ of $\mathrm{Sp}(4,\mathbb{Z})$ as given in section 3. The finite Siegel modular groups are denoted by $\Gamma_{2,N} = \mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(N)$, where $\Gamma_2(N)$ is the principal congruence subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ with genus 2 and level N. This generalizes the homogeneous finite modular groups $\Gamma_{1,N} = \Gamma'_N$ of $\mathrm{SL}(2,\mathbb{Z})$, discussed for example in ref. [47]. For $N = 2$ we have $\Gamma_{2,2} = \mathrm{Sp}(4,2) \cong S_6$ of order 720, while $\Gamma_{2,3} = \mathrm{Sp}(4,3)$ has already 51,840 elements [48]. These groups are huge, but they are usually not fully realized because of the orbifold twist that was introduced to get chiral matter from the torus. The task here would be to determine the unbroken finite Siegel modular groups for the 13 orbifold cases of Table 1 given the modular transformations determined in section 3. This is beyond the scope of the present paper. Some clues can be found from the previously discussed cases with vanishing Wilson lines. The symmetric $\mathbb{Z}_2$ orbifold from section 3.1.1 (with $Z = 0$) is known to have a finite modular group $(S_3 \times S_3) \rtimes \mathbb{Z}_4$ (from $\Gamma_2^T \times \Gamma_2^U$ combined with mirror symmetry [3,4]) which nicely fits into $\Gamma_{2,2} = S_6$ for level $N = 2$. It is not clear yet whether $\Gamma_{2,2}$ is also the relevant group for the fake-asymmetric $\mathbb{Z}_2$ orbifold discussed in section 3.1.2, although this seems plausible. On the other hand, in the case of the $\mathbb{Z}_3$ orbifold with vanishing Wilson line, the finite modular group was found to be $T' = \Gamma'_3$ of level $N = 3$ [16,22,26–28]. This leads to the conjecture that for the case described in section 3.2.2, the finite modular group would descend from the finite Siegel modular group $\Gamma_{2,3}$, where we have confirmed that $\Gamma_{2,3}$ contains T'. Thus, the result of this third step would be the determination of the finite Siegel modular flavor symmetry as well as the traditional flavor symmetry for each of the orbifolds given in Table 1.

Once this has been achieved, the ultimate step to establish the full connection to bottom-up constructions is to determine the representations of the matter fields with respect to the full eclectic flavor group $\mathcal{G}_{\text{efg}}$. Chiral fields tend to correspond to twisted fields located at the fixed points of the orbifold twist. In the $\mathbb{Z}_3$ case, for example, we have three fixed points and matter fields transform as triplet representations of the traditional flavor symmetry $\Delta(54) \subset \mathcal{G}_{\text{efg}}$ and as a $\mathbf{1} \oplus \mathbf{2}'$ of the finite modular group $T' \subset \mathcal{G}_{\text{efg}}$. This shows, among others, that twisted fields need not correspond to irreducible representations of the finite modular group. So far, determining the representations of matter fields under the discrete flavor symmetries requires explicit computations of string vertex operators and their associated operator product expansions. This or the identification of a simpler method remains to be explored in detail for each of the orbifolds under consideration. This underlines that top-down model building with modular flavor symmetries has just begun to unfold its various possibilities. More work is needed in order to finally bridge the gap to bottom-up constructions.

The consideration of the finite Siegel modular flavor symmetry from a bottom-up perspective has been pioneered recently in ref. [6]. They considered the case with two unconstrained moduli: $T = U$ and a Wilson line Z. Superficially, this would look like Case 3.1.2 in our Table 1, but this interpretation is not necessarily correct. By choosing the moduli at $T = U$ by hand, ref. [6] imposes $S_4 \times \mathbb{Z}_2$ as finite Siegel modular subgroup of $\Gamma_{2,2} \cong S_6$. In addition, some matter fields are postulated to build triplet representations of S_4. In our picture, the moduli can be stabilized at $T = U$ by the fake-asymmetric $\mathbb{Z}_2$ orbifold discussed in section 3.1.2. In principle, this orbifold also determines the unbroken finite Siegel modular group and the corresponding representations of matter fields. Unfortunately, their determination is technically involved and the results are currently not available. Hence, a detailed correspondence between ref. [6] and the top-down approach has yet to be clarified. To obtain a better geometric interpretation, one could reformulate this case as a symmetric orbifold with a quantized Wilson line as shown in section 3.1.3. This might help to make contact to the discussion in the bottom-up approach of ref. [6]. We hope to report on the resolution of these questions in a future publication.

Finally, we stress that the results from our present endeavor may have interesting applications also in the study of other top-down scenarios. For example, in the context of magnetized toroidal compactifications [49–55] one typically derives the flavor properties of the models from the modular properties associated with the complex structure of a two-torus, disregarding the modular behavior of the Kähler and Wilson line moduli also present in the construction. It would be interesting to study how our considerations change the conclusions in these cases.

CRediT authorship contribution statement

Hans Peter Nilles: Conceptualization, Methodology, Writing – original draft. **Saúl Ramos–Sánchez:** Conceptualization, Methodology, Writing – original draft. **Andreas Trautner:** Conceptualization, Methodology, Writing – original draft. **Patrick K.S. Vaudrevange:** Conceptualization, Methodology, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

P.V. is supported by the Deutsche Forschungsgemeinschaft (SFB1258).

Appendix A. Remark on mirror symmetry

Note that in $\mathrm{Sp}(4,\mathbb{Z})$ the mirror transformation $M_\times$ can be expressed as

$$M_\times \;=\; M\begin{pmatrix}0\\1\end{pmatrix} M_{(\mathrm{S},\mathbb{1}_2)}\, M\begin{pmatrix}1\\0\end{pmatrix} M_{(\mathrm{S},\mathbb{1}_2)}\, M\begin{pmatrix}0\\-1\end{pmatrix} \;\in\; \mathrm{Sp}(4,\mathbb{Z})\,. \tag{153}$$

Let us denote the original definition from ref. [5] of mirror symmetry by $\hat{M}'$. Then, we use the dictionary eq. (14) between $\mathrm{Sp}(4,\mathbb{Z})$ and $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ and map the right-hand side of eq. (153) into the modular group $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ of the string setup and define

$$\hat{M} \;:=\; \hat{W}\begin{pmatrix}0\\1\end{pmatrix} \hat{K}_\mathrm{S}\, \hat{W}\begin{pmatrix}1\\0\end{pmatrix} \hat{K}_\mathrm{S}\, \hat{W}\begin{pmatrix}0\\-1\end{pmatrix} \;\in\; \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})\,. \tag{154}$$

Crucially, the new mirror transformation $\hat{M}$ differs from the original definition of $\hat{M}'$,

$$\hat{M} \;\neq\; \hat{M}'\,. \tag{155}$$

This is contrary to our expectation from eq. (153), as one would associate $M_\times \in \mathrm{Sp}(4,\mathbb{Z})$ with $\hat{M}' \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ using the dictionary eq. (14) of ref. [5]. However, the generalized metric transforms identically under $\hat{M}$ and $\hat{M}'$, i.e. using e q. (7) we find

$$\mathcal{H}(T,U,Z) \overset{\hat{M}}{\longmapsto} \hat{M}^{-\mathrm{T}}\,\mathcal{H}(T,U,Z)\,\hat{M}^{-1} \;=\; \mathcal{H}(U,T,Z) \tag{156a}$$

$$\|$$

$$\mathcal{H}(T,U,Z) \overset{\hat{M}'}{\longmapsto} \hat{M}'^{-\mathrm{T}}\mathcal{H}(T,U,Z)\,\hat{M}'^{-1} \;=\; \mathcal{H}(U,T,Z)\,. \tag{156b}$$

Consequently, both transformations $\hat{M}$ and $\hat{M}'$ interchange T and U while leaving Z invariant. Hence, on the level of the moduli, both transformations define mirror symmetry in the string construction. In the following, we use the new transformation

$$\hat{M} \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{14} \end{pmatrix}, \tag{157}$$

defined in eq. (154) as the generator of mirror symmetry instead of $\hat{M}'$ that was defined in ref. [5]. Note that the new mirror transformation $\hat{M}$ also acts nontrivially on the 16 gauge degrees of freedom of the heterotic string as a $\mathbb{Z}_2$ reflection.

Appendix B. $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ and $\mathrm{GSp}(4, \mathbb{Z})$ transformations

This appendix provides the explicit expressions of the generators of the modular groups $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ and $\mathrm{GSp}(4, \mathbb{Z})$ that are used along this work. We also justify the correspondences among both groups, as given in eq. (14), following the discussion of ref. [5].

B.1. $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ *modular transformations*

As explained in section 2.1 and in ref. [5], the "rotational" outer automorphisms of the Narain lattice associated with a two-dimensional toroidal compactification of the heterotic string build the general group $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ of modular transformations of this space. The elements $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ act on the string moduli (T, U, Z), defined in eq. (6), according to eq. (8).

The matrix representations of a subset of $\mathrm{O}_{\hat{\eta}}(2, 2+16, \mathbb{Z})$ generators are given by

$$\hat{C}_{\mathrm{S}} \;=\; \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}, \qquad \hat{C}_{\mathrm{T}} \;=\; \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}, \tag{158a}$$

$$\hat{K}_{\mathrm{S}} \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}, \qquad \hat{K}_{\mathrm{T}} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{16} \end{pmatrix}, \tag{158b}$$

$$\hat{W}\begin{pmatrix}\ell\\ m\end{pmatrix} \;=\; \begin{pmatrix} \mathbb{1}_2 & 0 & 0\\ -\frac{1}{2}\Delta A^{\mathrm{T}} g\,\Delta A & \mathbb{1}_2 & \Delta A^{\mathrm{T}} g\\ -\Delta A & 0 & \mathbb{1}_{16}\end{pmatrix}\,,\qquad \hat{\Sigma}_* \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -\mathbb{1}_{16}\end{pmatrix}\,, \tag{158c}$$

and the mirror transformation $\hat{M}$ defined in eq. (157). Here g is defined as in eq. (2), and ΔA is given by the 16×2-dimensional matrix

$$\Delta A \;=\; \begin{pmatrix} m & 0 & \ldots & 0\\ \ell & 0 & \ldots & 0\end{pmatrix}^{\mathrm{T}} \qquad \text{with} \quad \ell, m \in \mathbb{Z}\,. \tag{159}$$

Further, the mirror transformation $\hat{M}$ that relates $U \leftrightarrow T$, see eq. (8e), establishes consequently a link between the transformations associated with these moduli: $\hat{K}_{\mathrm{S}} := \hat{M}\hat{C}_{\mathrm{S}}\hat{M}^{-1}$ and $\hat{K}_{\mathrm{T}} := \hat{M}\hat{C}_{\mathrm{T}}\hat{M}^{-1}$.

B.2. GSp(4, $\mathbb{Z}$) *elements*

The elements of Sp(4, $\mathbb{Z}$) can be expresed as [5]

$$M_{(\gamma_T,\gamma_U)} \;:=\; \begin{pmatrix} a_U & 0 & b_U & 0\\ 0 & a_T & 0 & b_T\\ c_U & 0 & d_U & 0\\ 0 & c_T & 0 & d_T\end{pmatrix}\,,\qquad M_\times \;:=\; \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\,, \tag{160a}$$

$$M\begin{pmatrix}\ell\\ m\end{pmatrix} \;:=\; \begin{pmatrix} 1 & 0 & 0 & -\ell\\ m & 1 & -\ell & 0\\ 0 & 0 & 1 & -m\\ 0 & 0 & 0 & 1\end{pmatrix}\,,\qquad \text{with} \quad \ell, m \in \mathbb{Z}\,, \tag{160b}$$

and products thereof. Here, the integers in $M_{(\gamma_T,\gamma_U)}$ satisfy $a_T d_T - b_T c_T = a_U d_U - b_U c_U = 1$, revealing that these elements are related to two SL(2, $\mathbb{Z}$) subgroups of Sp(4, $\mathbb{Z}$), whose elements are

$$\gamma_T \;:=\; \begin{pmatrix} a_T & b_T\\ c_T & d_T\end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_T \quad \text{and} \quad \gamma_U \;:=\; \begin{pmatrix} a_U & b_U\\ c_U & d_U\end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_U\,. \tag{161}$$

In general, an element $M \in \mathrm{Sp}(4,\mathbb{Z})$ can be decomposed in 2×2 blocks as $M = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)$, such that its action on a symmetric 2×2 matrix Ω reads

$$\Omega \;\overset{M}{\longmapsto}\; (A\,\Omega + B)\,(C\,\Omega + D)^{-1} \qquad \text{with} \quad \Omega \;=\; \begin{pmatrix} U & Z\\ Z & T\end{pmatrix}\,, \tag{162}$$

where U and T are the two complex moduli associated with SL(2, $\mathbb{Z}$)$_U$ and SL(2, $\mathbb{Z}$)$_T$, respectively, while Z is a complex modulus that relates both modular groups. The Sp(4, $\mathbb{Z}$) modular transformations of eq. (162) show, for example, that $M_\times$ acts on the moduli as $T \xleftrightarrow{M_\times} U$ and $Z \xmapsto{M_\times} Z$ and is, thus, a $\mathbb{Z}_2$ mirror transformation.

Including further the $\mathrm{Sp}(4,\mathbb{Z})$ outer automorphism

$$M_* := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{163}$$

enhances $\mathrm{Sp}(4,\mathbb{Z})$ to the generalized Siegel modular group $\mathrm{GSp}(4,\mathbb{Z})$. Elements $M \in \mathrm{GSp}(4,\mathbb{Z})$ containing M_* act on the modulus matrix Ω as

$$\Omega \xmapsto{M} \left(A\,\bar{\Omega} + B\right)\left(C\,\bar{\Omega} + D\right)^{-1}. \tag{164}$$

Note that $\Omega \xmapsto{M_*} -\bar{\Omega}$ because $A = -D = -\mathbb{1}_2$ and $B = C = 0$ for $M = M_*$.

Interestingly, by using the $\mathrm{GSp}(4,\mathbb{Z})$ transformations of the complex modulus matrix Ω, eqs. (162) and (164), one finds that the moduli transform following eq. (8), where the transformations are replaced according to the dictionary (14). This reveals that the $\mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ transformations presented in appendix B.1 are related to the Siegel modular group $\mathrm{GSp}(4,\mathbb{Z})$.

Appendix C. Summary of results

Table 2
Stabilizing the moduli (T,U,Z) at the fixed points of $\mathrm{Sp}(4,\mathbb{Z})$ by orbifold compactifications. We use the definitions $\omega := \exp(2\pi\mathrm{i}/3)$, $\zeta := \exp(2\pi\mathrm{i}/5)$ and $\tilde{\eta} := \frac{1}{3}(1+2\sqrt{2}\,\mathrm{i})$.

Sp(4, $\mathbb{Z}$)			String theory			
			symmetric $\mathbb{Z}_2$ orbifold			
τ_f	$=$	$\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$	$\mathcal{H}$	:	$a_1 = a_2 = 0$	
			moduli	:	$(T,U,0)$	
$\bar{H}$	$=$	$\left\langle M_{(\mathrm{S}^2,\mathbb{1}_2)}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \left(\hat{K}_\mathrm{S}\right)^2 \right\rangle \cong \mathbb{Z}_2$	
$N(H)$	$=$	$\left\langle M_{(\mathrm{S},\mathbb{1}_2)}\,,\ M_{(\mathrm{T},\mathbb{1}_2)},\ M_{(\mathbb{1}_2,\mathrm{S})}\,,\ M_{(\mathbb{1}_2,\mathrm{T})}\,,\ M_\times \right\rangle$	$\mathcal{G}_\mathrm{modular}$	$=$	$\left\langle \hat{K}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,,\ \hat{C}_\mathrm{S}\,,\ \hat{C}_\mathrm{T}\,,\ \hat{M} \right\rangle$	
				$\cong$	$\left(\left(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U\right)/\,\mathbb{Z}_2\right) \rtimes \mathbb{Z}_2^{\hat{M}}$	
					$T \xmapsto{\hat{K}_\mathrm{S}} -\frac{1}{T}$,	$U \xmapsto{\hat{K}_\mathrm{S}} U$,
					$T \xmapsto{\hat{K}_\mathrm{T}} T+1$,	$U \xmapsto{\hat{K}_\mathrm{T}} U$,
					$T \xmapsto{\hat{C}_\mathrm{S}} T$,	$U \xmapsto{\hat{C}_\mathrm{S}} -\frac{1}{U}$,
					$T \xmapsto{\hat{C}_\mathrm{T}} T$,	$U \xmapsto{\hat{C}_\mathrm{T}} U+1$,
					$T \xmapsto{\hat{M}} U$,	$U \xmapsto{\hat{M}} T$
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$	
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$,	$U \xmapsto{\hat{\mathcal{CP}}} -\bar{U}$
			fake-asymmetric $\mathbb{Z}_2$ orbifold			
τ_f	$=$	$\begin{pmatrix}\tau_1 & \tau_3\\ \tau_3 & \tau_1\end{pmatrix}$	$\mathcal{H}$	:	$G_{11} = \alpha'(1-a_1^2)$, $B_{12} = a_1\,a_2\,\alpha' + G_{12}$	
			moduli	:	(T,T,Z)	
$\bar{H}$	$=$	$\left\langle M_\times \right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \hat{M} \right\rangle \cong \mathbb{Z}_2$	

Table 2 (*continued*)

Sp(4, $\mathbb{Z}$)			String theory		
$N(H)$	$=$	$\left\langle M_{(\mathrm{S},\mathrm{S})}\,,\ M_{(\mathrm{T},\mathrm{T})},\ M_{(\mathrm{S}^2,\mathbb{1}_2)}\,,\ M\binom{-1}{0}\right\rangle$	$\mathcal{G}_{\text{modular}}$	$=$	$\left\langle \hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,\hat{C}_\mathrm{T}\,,\ \left(\hat{K}_\mathrm{S}\right)^2\,,\ \hat{W}\binom{-1}{0}\right\rangle$
					$T \xmapsto{\hat{K}_\mathrm{S}\hat{C}_\mathrm{S}} -\frac{T}{T^2-Z^2}$, $Z \xmapsto{\hat{K}_\mathrm{S}\hat{C}_\mathrm{S}} \frac{Z}{T^2-Z^2}$, $T \xmapsto{\hat{K}_\mathrm{T}\hat{C}_\mathrm{T}} T+1$, $Z \xmapsto{\hat{K}_\mathrm{T}\hat{C}_\mathrm{T}} Z$, $T \xmapsto{\left(\hat{K}_\mathrm{S}\right)^2} T$, $Z \xmapsto{\left(\hat{K}_\mathrm{S}\right)^2} -Z$, $T \xmapsto{\hat{W}\binom{-1}{0}} T$, $Z \xmapsto{\hat{W}\binom{-1}{0}} Z+1$
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$, $Z \xmapsto{\hat{\mathcal{CP}}} -\bar{Z}$
			symmetric $\mathbb{Z}_2$ orbifold with Wilson line		
τ_f'	$=$	$\begin{pmatrix}\tau_1' & 1/2\\ 1/2 & \tau_2'\end{pmatrix} = b^{-1}\begin{pmatrix}\tau_1 & \tau_3\\ \tau_3 & \tau_1\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$a_1=0,\ a_2=-1/2$ $(T, U, 1/2)$
$\bar{H}$	$=$	$\left\langle M_{(\mathrm{S}^2,\mathbb{1}_2)}\,M\binom{1}{0}\right\rangle$	$\hat{P}_{\text{Narain}}$	$=$	$\left\langle \left(\hat{C}_\mathrm{S}\right)^2\,\hat{W}\binom{1}{0}\right\rangle \cong \mathbb{Z}_2$
$N(H)$	$=$	$\left\langle b^{-1}M_{(\mathrm{S},\mathrm{S})}\,b\,,\ b^{-1}M_{(\mathrm{T},\mathrm{T})}\,b,\ b^{-1}M_{(\mathrm{S}^2,\mathbb{1}_2)}\,b\,,\ b^{-1}M\binom{-1}{0}\,b\right\rangle$ $b := M_{(\mathrm{T}^{-1},\mathbb{1}_2)}\,M\binom{0}{1}\,M_{(\mathrm{TS}^3,\mathbb{1}_2)}$	$\mathcal{G}_{\text{modular}}$	$=$	$\left\langle \hat{M}_1\,,\ \hat{M}_2\,,\ \hat{M}_3\,,\ \hat{M}_4\right\rangle$ $\hat{M}_1 := \hat{B}^{-1}\,\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{B}$, $\hat{M}_2 := \hat{B}^{-1}\,\hat{K}_\mathrm{T}\,\hat{C}_\mathrm{T}\,\hat{B}$, $\hat{M}_3 := \hat{B}^{-1}\left(\hat{K}_\mathrm{S}\right)^2\hat{B}$, $\hat{M}_4 := \hat{B}^{-1}\,\hat{W}\binom{-1}{0}\,\hat{B}$, $\hat{B} := \left(\hat{K}_\mathrm{T}\right)^{-1}\hat{W}\binom{0}{1}\,\hat{K}_\mathrm{T}\left(\hat{K}_\mathrm{S}\right)^3$
					$T \xmapsto{\hat{M}_1} -\frac{1}{4T}$, $U \xmapsto{\hat{M}_1} -\frac{1}{4U}$, $T \xmapsto{\hat{M}_2} -\frac{T}{2T-1}$, $U \xmapsto{\hat{M}_2} U+\frac{1}{2}$, $T \xmapsto{\hat{M}_3} -\frac{1}{4U}$, $U \xmapsto{\hat{M}_3} -\frac{1}{4T}$, $T \xmapsto{\hat{M}_4} \frac{T}{2T+1}$, $U \xmapsto{\hat{M}_4} U+\frac{1}{2}$
$\mathcal{CP}_s$	$=$	$M_{(\mathrm{S}^2,\mathbb{1}_2)}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\left(\hat{K}_\mathrm{S}\right)^2\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$, $U \xmapsto{\hat{\mathcal{CP}}} -\bar{U}$
			symmetric $\mathbb{Z}_4$ orbifold		
τ_f	$=$	$\begin{pmatrix}\mathrm{i} & 0\\ 0 & \tau_2\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=G_{22},\ G_{12}=0,\ a_1=a_2=0$ $(T, \mathrm{i}, 0)$
$\bar{H}$	$=$	$\left\langle M_{(\mathbb{1}_2,\mathrm{S})}\right\rangle$	$\hat{P}_{\text{Narain}}$	$=$	$\left\langle \hat{C}_\mathrm{S}\right\rangle \cong \mathbb{Z}_4$
$N(H)$	$=$	$\left\langle M_{(\mathrm{S},\mathbb{1}_2)}\,,\ M_{(\mathrm{T},\mathbb{1}_2)}\,,\ M_{(\mathbb{1}_2,\mathrm{S})}\right\rangle$	$\mathcal{G}_{\text{modular}}$	$=$ $\cong$	$\left\langle \hat{K}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,,\ \hat{C}_\mathrm{S}\right\rangle$ $\left(\mathrm{SL}(2,\mathbb{Z})_T \times \mathbb{Z}_4^R\right)/\,\mathbb{Z}_2$
					$T \xmapsto{\hat{K}_\mathrm{S}} -\frac{1}{T}$, $T \xmapsto{\hat{K}_\mathrm{T}} T+1$, $T \xmapsto{\hat{C}_\mathrm{S}} T$
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$

(*continued on next page*)

Table 2 (*continued*)

$\mathrm{Sp}(4,\mathbb{Z})$			String theory		
			symmetric $\mathbb{Z}_6$ orbifold		
τ_f	$=$	$\begin{pmatrix}\omega & 0\\ 0 & \tau_1\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=G_{22}=-2G_{12},\ a_1=a_2=0$ $(T,\omega,0)$
$\bar{H}$	$=$	$\left\langle M_{(\mathbb{1}_2,\mathrm{S}^3\mathrm{TST})}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \left(\hat{C}_\mathrm{S}\right)^3 \hat{C}_\mathrm{T}\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T}\right\rangle \cong \mathbb{Z}_6$
$N(H)$	$=$	$\left\langle M_{(\mathrm{S},\mathbb{1}_2)}\,,\ M_{(\mathrm{T},\mathbb{1}_2)}\,,\ M_{(\mathbb{1}_2,\mathrm{S}^3\mathrm{T})}\right\rangle$	$\mathcal{G}_\mathrm{modular}$	$=$ $\cong$	$\left\langle \hat{K}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\,,\ \left(\hat{C}_\mathrm{S}\right)^3\hat{C}_\mathrm{T}\right\rangle$ $\left(\mathrm{SL}(2,\mathbb{Z})_T\ \times\ \mathbb{Z}_6^R\right)/\,\mathbb{Z}_2$
					$T \xmapsto{\hat{K}_\mathrm{S}} -\frac{1}{T}, T \xmapsto{\hat{K}_\mathrm{T}} T+1,\ T \xmapsto{\left(\hat{C}_\mathrm{S}\right)^3\hat{C}_\mathrm{T}} T$
$\mathcal{CP}_s$	$=$	$M_{(\mathbb{1}_2,\mathrm{T}^{-1})}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\left(\hat{C}_\mathrm{T}\right)^{-1}\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$
			asymmetric $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold		
τ_f	$=$	$\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_1\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=\alpha',\ G_{12}=B_{12},\ a_1=a_2=0$ $(T,T,0)$
$\bar{H}$	$=$	$\left\langle M_\times\,,\ M_\times\, M_{(\mathrm{S}^2,\mathbb{1}_2)}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \hat{M}\,,\ \hat{M}\left(\hat{K}_\mathrm{S}\right)^2\right\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_2$
$N(H)$	$=$	$\left\langle M_{(\mathrm{S},\mathrm{S})}\,,\ M_{(\mathrm{T},\mathrm{T})},\right.$ $\left. M_{(\mathrm{S}^2,\mathbb{1}_2)}\,,\ M_\times\right\rangle$	$\mathcal{G}_\mathrm{modular}$	$=$ $\cong$	$\left\langle \hat{K}_\mathrm{S}\hat{C}_\mathrm{S}\,,\ \hat{K}_\mathrm{T}\hat{C}_\mathrm{T}\,,\ \left(\hat{K}_\mathrm{S}\right)^2\,,\ \hat{M}\right\rangle$ $\mathrm{PSL}(2,\mathbb{Z})\times\mathbb{Z}_2\times\mathbb{Z}_2^{\hat{M}}$
					$T \xmapsto{\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}} -\frac{1}{T},\qquad T \xmapsto{\hat{K}_\mathrm{T}\,\hat{C}_\mathrm{T}} T+1,$ $T \xmapsto{\left(\hat{K}_\mathrm{S}\right)^2} T,\qquad T \xmapsto{\hat{M}} T$
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$
			asymmetric $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold with Wilson line		
τ_f	$=$	$\begin{pmatrix}\tau_1 & 1/2\\ 1/2 & \tau_1\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=\alpha',\ G_{12}=B_{12},\ a_1=0,\ a_2=-1/2$ $(T,T,1/2)$
$\bar{H}$	$=$	$\left\langle M_\times\,,\ M\binom{-1}{0}\,M_\times\,M_{(\mathbb{1}_2,\mathrm{S}^2)}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \hat{M}\,,\ \hat{W}\binom{-1}{0}\,\hat{M}\,\left(\hat{C}_\mathrm{S}\right)^2\right\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_2$
$N(H)$	$=$	$\left\langle M\binom{-1}{0}\,M_{(\mathrm{S}^2,\mathbb{1}_2)},\right.$ $M_\times\,M\binom{0}{-2}\,M_\times\,M\binom{0}{1}\times$ $M_{(\mathrm{ST}^{-1}\mathrm{S}^2,\mathrm{ST}^2\mathrm{ST})}\,M\binom{-1}{3}\,,\ \ldots$	$\mathcal{G}_\mathrm{modular}$	$=$	$\left\langle \hat{M}_1\,,\ \hat{M}_2\,,\ \hat{M}_3\right\rangle$ $\hat{M}_1 := \hat{W}\binom{-1}{0}\,\left(\hat{K}_\mathrm{S}\right)^2 = \hat{\Theta}_1\,\hat{\Theta}_2,$ $\hat{M}_2 := \hat{M}\,\hat{W}\binom{0}{-2}\hat{M}\,\hat{W}\binom{0}{1}\hat{K}_\mathrm{S}\,\left(\hat{K}_\mathrm{T}\right)^{-1}\times\ldots$
		$\left.\ldots M\binom{-1}{0}\,M_{(\mathrm{S}^2\mathrm{T}^{-1},\mathrm{T}^{-1})}\right\rangle$			$\ldots\left(\hat{K}_\mathrm{S}\right)^2\,\hat{C}_\mathrm{S}\,\left(\hat{C}_\mathrm{T}\right)^2\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T}\,\hat{W}\binom{-1}{3},$ $\hat{M}_3 := \hat{W}\binom{-1}{0}\,\left(\hat{K}_\mathrm{S}\right)^2\,\left(\hat{K}_\mathrm{T}\right)^{-1}\,\left(\hat{C}_\mathrm{T}\right)^{-1}$
					$T \xmapsto{\hat{M}_2} -\frac{2T+3}{4T+2},\qquad T \xmapsto{\hat{M}_3} T-1,$ $T \xmapsto{\hat{M}_1} T$
$\mathcal{CP}_s$	$=$	$M\binom{-1}{0}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{W}\binom{-1}{0}\,\hat{\Sigma}_*$
					$T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$

Table 2 (*continued*)

Sp(4, $\mathbb{Z}$)			String theory		
			asymmetric S_3 orbifold		
τ_f	$=$	$\begin{pmatrix}\tau_1 & \tau_1/2\\ \tau_1/2 & \tau_1\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=\frac{3\alpha'}{4}$, $G_{12}=B_{12}$, $a_1=1/2$, $a_2=0$ $(T, T, T/2)$
$\bar{H}$	$=$	$\Big\langle M_\times\, M\binom{0}{1}\, M_\times\, M_{(\mathbb{1}_2,\mathrm{S}^2)},$ $M\binom{0}{1}\, M_{(\mathrm{S}^2,\mathbb{1}_2)}\Big\rangle$	$\hat{P}_\mathrm{Narain}$	$=$ $\cong$	$\Big\langle\, \hat{M}\,\hat{W}\binom{0}{1}\,\hat{M}\,\big(\hat{C}_\mathrm{S}\big)^2\,,\ \hat{W}\binom{0}{1}\,\big(\hat{K}_\mathrm{S}\big)^2\,\Big\rangle$ S_3
$N(H)$	$=$	$\Big\langle M_{(\mathrm{S},\mathrm{S}^3)}\, M_\times,$ $M_{(\mathrm{S}^3,\mathrm{S}^3)}\, M\binom{-1}{0} M_{(\mathrm{T}^{-2},\mathrm{T}^{-2})} M_{(\mathrm{S},\mathrm{S})},$ $M_\times\,,\ M\binom{0}{1}\, M_{(\mathrm{S}^2,\mathbb{1}_2)}\Big\rangle$	$\mathcal{G}_\mathrm{modular}$	$=$	$\Big\langle\, \hat{M}_1\,,\ \hat{M}_2\,,\ \hat{M}\,,\ \hat{W}\binom{0}{1}\big(\hat{K}_\mathrm{S}\big)^2\,\Big\rangle$ $\hat{M}_1 := \hat{K}_\mathrm{S}\,\big(\hat{C}_\mathrm{S}\big)^3\,\hat{M}$ $\hat{M}_2 := \big(\hat{K}_\mathrm{S}\big)^3\big(\hat{C}_\mathrm{S}\big)^3\,\hat{W}\binom{-1}{0}\big(\hat{K}_\mathrm{T}\big)^{-2}\times$ $\big(\hat{C}_\mathrm{T}\big)^{-2}\,\hat{K}_\mathrm{S}\hat{C}_\mathrm{S}$
					$T \xmapsto{\hat{M}_1} -\frac{4}{3T}$, $\quad T \xmapsto{\hat{M}_2} \frac{2T}{3T+2}$, $T \xmapsto{\hat{M}} T$, $\quad T \xmapsto{\hat{W}\binom{0}{1}\left(\hat{K}_\mathrm{S}\right)^2} T$
$\mathcal{CP}_s$	$=$	$M\binom{-1}{0}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{W}\binom{-1}{0}\,\hat{\Sigma}_*$ $T \xmapsto{\hat{\mathcal{CP}}} -\bar{T}$
			asymmetric $\mathbb{Z}_5$ orbifold		
τ_f	$=$	$\begin{pmatrix}\zeta & \zeta+\zeta^{-2}\\ \zeta+\zeta^{-2} & -\zeta^{-1}\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=\frac{\alpha'}{2}\big(-5+3\sqrt{5}\big)$, $\quad B_{12}=\frac{\alpha'}{2}\big(2-\sqrt{5}\big)$, $G_{12}=\frac{\alpha'}{2}\big(5-2\sqrt{5}\big)$, $\quad a_1=\frac{1}{2}\big(3-\sqrt{5}\big)$, $G_{22}=G_{11}$, $\quad a_2=\frac{1}{2}\big(-1+\sqrt{5}\big)$ $(-\zeta^{-1}, \zeta, \zeta+\zeta^{-2})$
$\bar{H}$	$=$	$\Big\langle M_{(\mathbb{1}_2,\mathrm{T})}\, M_\times\, M\binom{0}{1}\, M_{(\mathrm{S}^2,\mathrm{S}^3)}\Big\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\Big\langle\, \hat{C}_\mathrm{T}\,\hat{M}\,\hat{W}\binom{0}{1}\,\big(\hat{K}_\mathrm{S}\big)^2\,\big(\hat{C}_\mathrm{S}\big)^3\,\Big\rangle \cong \mathbb{Z}_5$
$\mathcal{CP}_s$	$=$	$\Big(\Big(M_{(\mathrm{S},\mathrm{S})}\, M\binom{-1}{0}\Big)^3\, M\binom{-1}{0}\Big)^{-1}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\Big(\Big(\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{W}\binom{-1}{0}\Big)^3\,\hat{W}\binom{-1}{0}\Big)^{-1}\,\hat{\Sigma}_*$
			asymmetric S_4 orbifold		
τ_f	$=$	$\begin{pmatrix}\tilde{\eta} & \frac{1}{2}(\tilde{\eta}-1)\\ \frac{1}{2}(\tilde{\eta}-1) & \tilde{\eta}\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=G_{22}=\frac{3\alpha'}{4}$, $G_{12}=\frac{\alpha'}{4}$, $B_{12}=\frac{\alpha'}{2}$, $a_1=a_2=1/2$ $(\tilde{\eta}, \tilde{\eta}, \frac{1}{2}(\tilde{\eta}-1))$
$\bar{H}$	$=$	$\Big\langle M_\times\,,\ M_{(\mathbb{1}_2,\mathrm{S}^3)}\, M_\times\, M\binom{0}{-1}\times$ $M_{(\mathrm{S},\mathbb{1}_2)}\, M\binom{-1}{-1}\Big\rangle$	$\hat{P}_\mathrm{Narain}$	$=$ $\cong$	$\Big\langle\, \hat{M}\,,\ \big(\hat{C}_\mathrm{S}\big)^3\,\hat{M}\,\hat{W}\binom{0}{-1}\,\hat{K}_\mathrm{S}\,\hat{W}\binom{-1}{-1}\Big\rangle$ S_4
$\mathcal{CP}_s$	$=$	$M_{(\mathrm{S},\mathrm{S})}\, M\binom{-1}{0} M_{(\mathrm{T}^{-1},\mathrm{T}^{-1})} M_{(\mathrm{S},\mathrm{S})}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{W}\binom{-1}{0}\,\big(\hat{C}_\mathrm{T}\,\hat{K}_\mathrm{T}\big)^{-1}\,\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{\Sigma}_*$
			asymmetric $(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$ orbifold		
τ_f	$=$	$\begin{pmatrix}\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}$	$\mathcal{H}$ moduli	: :	$G_{11}=G_{22}=\alpha'$, $G_{12}=B_{12}=0$, $a_1=a_2=0$ $(\mathrm{i}, \mathrm{i}, 0)$
$\bar{H}$	$=$	$\Big\langle M_{(\mathrm{S}^2,\mathrm{S})}\,,\ M_{(\mathrm{S},\mathrm{S}^3)}\,,\ M_\times\Big\rangle$	$\hat{P}_\mathrm{Narain}$	$=$ $\cong$	$\Big\langle\, \big(\hat{K}_\mathrm{S}\big)^2\,\hat{C}_\mathrm{S}\,,\ \hat{K}_\mathrm{S}\,\big(\hat{C}_\mathrm{S}\big)^3\,,\ \hat{M}\Big\rangle$ $(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$

(*continued on next page*)

Table 2 (*continued*)

Sp(4, $\mathbb{Z}$)			String theory		
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$
			asymmetric $S_3 \times \mathbb{Z}_6$ orbifold		
τ_f	$=$	$\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}$	$\mathcal{H}$	:	$G_{11}=G_{22}=\alpha'$, $G_{12}=B_{12}=-\frac{\alpha'}{2}$, $a_1=a_2=0$
			moduli	:	$(\omega,\omega,0)$
$\bar{H}$	$=$	$\left\langle M_{(\mathrm{T}^{-1}\mathrm{S},\mathrm{S}^3\mathrm{T})}\,M_\times\,,\;M_\times,\; M_{(\mathrm{S}^3\mathrm{T},\mathrm{ST})}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \left(\hat{K}_\mathrm{T}\right)^{-1}\hat{K}_\mathrm{S}\left(\hat{C}_\mathrm{S}\right)^3\hat{C}_\mathrm{T}\,\hat{M}\,,\;\hat{M},\; \left(\hat{K}_\mathrm{S}\right)^3\hat{K}_\mathrm{T}\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T}\right\rangle \cong S_3\times\mathbb{Z}_6$
$\mathcal{CP}_s$	$=$	$M_{(\mathrm{T}^{-1},\mathrm{T}^{-1})}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\left(\hat{C}_\mathrm{T}\,\hat{K}_\mathrm{T}\right)^{-1}\hat{\Sigma}_*$
			asymmetric $S_3 \times \mathbb{Z}_2$ orbifold		
τ_f	$=$	$\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$	$\mathcal{H}$	:	$G_{11}=\frac{3\alpha'}{4}$, $G_{22}=\alpha'$, $G_{12}=B_{12}=0$, $a_1=1/2$, $a_2=0$
			moduli	:	$(\frac{2\mathrm{i}}{\sqrt{3}},\frac{2\mathrm{i}}{\sqrt{3}},\frac{\mathrm{i}}{\sqrt{3}})$
$\bar{H}$	$=$	$\left\langle M_\times\,M_{(\mathrm{S},\mathrm{S})}\,M\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right),\; M_{(\mathrm{S},\mathrm{S})}\,M\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)M_\times\,,\;M_{(\mathrm{S}^3,\mathrm{S})}\,M_\times\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \hat{M}\,\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{W}\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right),\;\hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{W}\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)\hat{M},\; \left(\hat{K}_\mathrm{S}\right)^3\hat{C}_\mathrm{S}\,\hat{M}\right\rangle \cong S_3\times\mathbb{Z}_2$
$\mathcal{CP}_s$	$=$	$\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\hat{\Sigma}_*$
			asymmetric $\mathbb{Z}_{12}$ orbifold		
τ_f	$=$	$\begin{pmatrix}\omega & 0\\ 0 & \mathrm{i}\end{pmatrix}$	$\mathcal{H}$	:	$G_{11}=G_{22}=\frac{2\alpha'}{\sqrt{3}}$, $G_{12}=-\frac{\alpha'}{\sqrt{3}}$, $B_{12}=0$, $a_1=a_2=0$
			moduli	:	$(\mathrm{i},\omega,0)$
$\bar{H}$	$=$	$\left\langle M_{(\mathrm{S},\mathrm{ST})}\right\rangle$	$\hat{P}_\mathrm{Narain}$	$=$	$\left\langle \hat{K}_\mathrm{S}\,\hat{C}_\mathrm{S}\,\hat{C}_\mathrm{T}\right\rangle \cong \mathbb{Z}_{12}$
$\mathcal{CP}_s$	$=$	$M_{(\mathbb{1}_2,\mathrm{T}^{-1})}\,\mathcal{CP}$	$\hat{\mathcal{CP}}$	$=$	$\left(\hat{C}_\mathrm{T}\right)^{-1}\hat{\Sigma}_*$

References

[1] H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, Eclectic flavor scheme from ten-dimensional string theory – I. Basic results, Phys. Lett. B 808 (2020) 135615, arXiv:2006.03059 [hep-th].

[2] H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, Eclectic flavor scheme from ten-dimensional string theory – II. Detailed technical analysis, Nucl. Phys. B 966 (2021) 115367, arXiv:2010.13798 [hep-th].

[3] A. Baur, M. Kade, H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold, J. High Energy Phys. 02 (2021) 018, arXiv:2008.07534 [hep-th].

[4] A. Baur, M. Kade, H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange, Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold, J. High Energy Phys. 06 (2021) 110, arXiv:2104.03981 [hep-th].

[5] A. Baur, M. Kade, H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, Siegel modular flavor group and CP from string theory, Phys. Lett. B 816 (2021) 136176, arXiv:2012.09586 [hep-th].

[6] G.-J. Ding, F. Feruglio, X.-G. Liu, Automorphic forms and fermion masses, J. High Energy Phys. 01 (2021) 037, arXiv:2010.07952 [hep-th].

[7] G.-J. Ding, F. Feruglio, X.-G. Liu, CP symmetry and symplectic modular invariance, SciPost Phys. 10 (2021) 133, arXiv:2102.06716 [hep-ph].

[8] E. Gottschling, Über die Fixpunkte der Siegelschen Modulgruppe, Math. Ann. 143 (2) (1961) 111–149.

[9] E. Gottschling, Über die Fixpunktuntergruppen der Siegelschen Modulgruppe, Math. Ann. 143 (5) (1961) 399–430.

[10] E. Gottschling, Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen, Math. Ann. 169 (1) (1967) 26–54.

[11] K.S. Narain, New heterotic string theories in uncompactified dimensions < 10, Phys. Lett. B 169 (1986) 41–46.

[12] K.S. Narain, M.H. Sarmadi, E. Witten, A note on toroidal compactification of heterotic string theory, Nucl. Phys. B 279 (1987) 369–379.
[13] S. Groot Nibbelink, P.K.S. Vaudrevange, T-duality orbifolds of heterotic Narain compactifications, J. High Energy Phys. 04 (2017) 030, arXiv:1703.05323 [hep-th].
[14] S. Groot Nibbelink, A worldsheet perspective on heterotic T-duality orbifolds, J. High Energy Phys. 04 (2021) 190, arXiv:2012.02778 [hep-th].
[15] P. Mayr, S. Stieberger, Moduli dependence of one loop gauge couplings in (0, 2) compactifications, Phys. Lett. B 355 (1995) 107–116, arXiv:hep-th/9504129.
[16] A. Baur, H.P. Nilles, A. Trautner, P.K.S. Vaudrevange, A string theory of flavor and $\mathcal{CP}$, Nucl. Phys. B 947 (2019) 114737, arXiv:1908.00805 [hep-th].
[17] L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds, Nucl. Phys. B 261 (1985) 678–686.
[18] L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds. 2, Nucl. Phys. B 274 (1986) 285–314.
[19] L.E. Ibáñez, H.P. Nilles, F. Quevedo, Orbifolds and Wilson lines, Phys. Lett. B 187 (1987) 25–32.
[20] K.S. Narain, M.H. Sarmadi, C. Vafa, Asymmetric orbifolds, Nucl. Phys. B 288 (1987) 551.
[21] K.S. Narain, M.H. Sarmadi, C. Vafa, Asymmetric orbifolds: path integral and operator formulations, Nucl. Phys. B 356 (1991) 163–207.
[22] A. Baur, H.P. Nilles, A. Trautner, P.K.S. Vaudrevange, Unification of flavor, CP, and modular symmetries, Phys. Lett. B 795 (2019) 7–14, arXiv:1901.03251 [hep-th].
[23] F. Feruglio, Are neutrino masses modular forms?, in: A. Levy, S. Forte, G. Ridolfi (Eds.), From My Vast Repertoire...: Guido Altarelli's Legacy, 2019, pp. 227–266, arXiv:1706.08749 [hep-ph].
[24] J.C. Criado, F. Feruglio, Modular invariance faces precision neutrino data, SciPost Phys. 5 (5) (2018) 042, arXiv: 1807.01125 [hep-ph].
[25] F. Feruglio, A. Romanino, Lepton flavor symmetries, Rev. Mod. Phys. 93 (1) (2021) 015007, arXiv:1912.06028 [hep-ph].
[26] J. Lauer, J. Mas, H.P. Nilles, Duality and the role of nonperturbative effects on the world sheet, Phys. Lett. B 226 (1989) 251–256.
[27] W. Lerche, D. Lüst, N.P. Warner, Duality symmetries in $N = 2$ Landau-Ginzburg models, Phys. Lett. B 231 (1989) 417–424.
[28] J. Lauer, J. Mas, H.P. Nilles, Twisted sector representations of discrete background symmetries for two-dimensional orbifolds, Nucl. Phys. B 351 (1991) 353–424.
[29] G. Lopes Cardoso, D. Lüst, T. Mohaupt, Moduli spaces and target space duality symmetries in (0, 2) Z(N) orbifold theories with continuous Wilson lines, Nucl. Phys. B 432 (1994) 68–108, arXiv:hep-th/9405002.
[30] D. Bailin, G. Kraniotis, A. Love, The effect of Wilson line moduli on CP violation by soft supersymmetry breaking terms, Phys. Lett. B 432 (1998) 343–352, arXiv:hep-th/9804135.
[31] A. Malmendier, D.R. Morrison, K3 surfaces, modular forms, and non-geometric heterotic compactifications, Lett. Math. Phys. 105 (8) (2015) 1085–1118, arXiv:1406.4873 [hep-th].
[32] A. Font, I. García-Etxebarria, D. Lüst, S. Massai, C. Mayrhofer, Heterotic T-fects, 6D SCFTs, and F-theory, J. High Energy Phys. 08 (2016) 175, arXiv:1603.09361 [hep-th].
[33] A. Font, B. Fraiman, M. Graña, C.A. Núñez, H.P. De Freitas, Exploring the landscape of heterotic strings on T^d, J. High Energy Phys. 10 (2020) 194, arXiv:2007.10358 [hep-th].
[34] P.P. Novichkov, J.T. Penedo, S.T. Petcov, A.V. Titov, Generalised CP symmetry in modular-invariant models of flavour, J. High Energy Phys. 07 (2019) 165, arXiv:1905.11970 [hep-ph].
[35] L.E. Ibáñez, H.P. Nilles, F. Quevedo, Reducing the rank of the gauge group in orbifold compactifications of the heterotic string, Phys. Lett. B 192 (1987) 332–338.
[36] S. Förste, H.P. Nilles, A. Wingerter, Geometry of rank reduction, Phys. Rev. D 72 (2005) 026001, arXiv:hep-th/0504117.
[37] J. Erler, D. Jungnickel, H.P. Nilles, Space duality and quantized Wilson lines, Phys. Lett. B 276 (1992) 303–310.
[38] T. Kobayashi, S. Raby, R.-J. Zhang, Searching for realistic 4d string models with a Pati-Salam symmetry: orbifold grand unified theories from heterotic string compactification on a Z(6) orbifold, Nucl. Phys. B 704 (2005) 3–55, arXiv:hep-ph/0409098.
[39] N.G. Cabo Bizet, T. Kobayashi, D.K. Mayorga Peña, S.L. Parameswaran, M. Schmitz, I. Zavala, R-charge conservation and more in factorizable and non-factorizable orbifolds, J. High Energy Phys. 05 (2013) 076, arXiv:1301.2322 [hep-th].
[40] H.P. Nilles, S. Ramos-Sánchez, M. Ratz, P.K.S. Vaudrevange, A note on discrete R symmetries in $\mathbb{Z}_6$-II orbifolds with Wilson lines, Phys. Lett. B 726 (2013) 876–881, arXiv:1308.3435 [hep-th].

[41] N.G. Cabo Bizet, T. Kobayashi, D.K. Mayorga Peña, S.L. Parameswaran, M. Schmitz, I. Zavala, Discrete R-symmetries and anomaly universality in heterotic orbifolds, J. High Energy Phys. 02 (2014) 098, arXiv:1308.5669 [hep-th].
[42] H.P. Nilles, Stringy origin of discrete R-symmetries, PoS CORFU2016 (2017) 017, arXiv:1705.01798 [hep-ph].
[43] M.-C. Chen, M. Fallbacher, K.T. Mahanthappa, M. Ratz, A. Trautner, CP violation from finite groups, Nucl. Phys. B 883 (2014) 267–305, arXiv:1402.0507 [hep-ph].
[44] J.A. Harvey, G.W. Moore, C. Vafa, Quasicrystalline compactification, Nucl. Phys. B 304 (1988) 269–290.
[45] H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, Eclectic flavor groups, J. High Energy Phys. 02 (2020) 045, arXiv:2001.01736 [hep-ph].
[46] H.P. Nilles, S. Ramos-Sánchez, P.K.S. Vaudrevange, Lessons from eclectic flavor symmetries, Nucl. Phys. B 957 (2020) 115098, arXiv:2004.05200 [hep-ph].
[47] X.-G. Liu, G.-J. Ding, Neutrino masses and mixing from double covering of finite modular groups, J. High Energy Phys. 08 (2019) 134, arXiv:1907.01488 [hep-ph].
[48] J.H. Conway, et al., Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups, Oxford University Press, New York, 1985.
[49] T. Kobayashi, S. Tamba, Modular forms of finite modular subgroups from magnetized D-brane models, Phys. Rev. D 99 (4) (2019) 046001, arXiv:1811.11384 [hep-th].
[50] H. Ohki, S. Uemura, R. Watanabe, Modular flavor symmetry on a magnetized torus, Phys. Rev. D 102 (8) (2020) 085008, arXiv:2003.04174 [hep-th].
[51] S. Kikuchi, T. Kobayashi, S. Takada, T.H. Tatsuishi, H. Uchida, Revisiting modular symmetry in magnetized torus and orbifold compactifications, Phys. Rev. D 102 (10) (2020) 105010, arXiv:2005.12642 [hep-th].
[52] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, H. Uchida, Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of Γ_N, J. High Energy Phys. 11 (2020) 101, arXiv:2007.06188 [hep-th].
[53] K. Hoshiya, S. Kikuchi, T. Kobayashi, Y. Ogawa, H. Uchida, Classification of three-generation models by orbifolding magnetized $T^2 \times T^2$, PTEP 2021 (3) (2021) 033B05, arXiv:2012.00751 [hep-th].
[54] S. Kikuchi, T. Kobayashi, H. Uchida, Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds, arXiv:2101.00826 [hep-th], 2021.
[55] Y. Almumin, M.-C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, S. Shukla, Metaplectic flavor symmetries from magnetized tori, J. High Energy Phys. 05 (2021) 078, arXiv:2102.11286 [hep-th].