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Research article

Duality properties of metric Sobolev spaces and capacity†

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Abstract: We study the properties of the dual Sobolev space $H^{-1,q}(\mathbb{X}) = (H^{1,p}(\mathbb{X}))'$ on a complete extended metric-topological measure space $\mathbb{X} = (X, \tau, \mathsf{d}, \mathfrak{m})$ for $p \in (1, \infty)$. We will show that a crucial role is played by the strong closure $H^{-1,q}_{\rm pd}(\mathbb{X})$ of $L^q(X, \mathfrak{m})$ in the dual $H^{-1,q}(\mathbb{X})$, which can be identified with the predual of $H^{1,p}(\mathbb{X})$. We will show that positive functionals in $H^{-1,q}(\mathbb{X})$ can be represented as a positive Radon measure and we will charaterize their dual norm in terms of a suitable energy functional on nonparametric dynamic plans. As a byproduct, we will show that for every Radon measure μ with finite dual Sobolev energy, Cap_p-negligible sets are also μ-negligible and good representatives of Sobolev functions belong to $L^1(X, \mu)$. We eventually show that the Newtonian-Sobolev capacity Cap_p admits a natural dual representation in terms of such a class of Radon measures.

Keywords: metric Sobolev spaces; capacity; modulus of a family of rectifiable curves; dynamic transport plans; dual Cheeger energy; capacitary measures

Dedicated to Sandro Salsa on the occasion of his 70th birthday.

1. Introduction

In this paper we investigate the properties of the duals of the metric Sobolev spaces $H^{1,p}(\mathbb{X})$, where $\mathbb{X} = (X, \tau, \mathsf{d}, \mathfrak{m})$ is an extended metric-topological measure space and $p \in (1, +\infty)$.

In the simpler case when (X, d) is a complete and separable metric space, τ is the topology induced by the metric and $\mathfrak{m}$ is a positive and finite Borel (thus Radon) measure on X, $H^{1,p}(\mathbb{X})$ can be defined as the natural domain of the $L^p(X, \mathfrak{m})$-relaxation of the pre-Cheeger energy form

$$\mathsf{pCE}_p(f) := \int_X (\operatorname{lip} f(x))^p \, \mathrm{d}\mathfrak{m}(x), \quad f \in \mathrm{Lip}_b(X),$$

initially defined only for bounded Lipschitz functions. Here $\operatorname{lip} f(x)$ defines the asymptotic Lipschitz constant

$$\operatorname{lip} f(x) = \limsup_{\substack{y,z\to x\\ y\neq z}} \frac{|f(y) - f(z)|}{\mathsf{d}(y, z)}.$$

For every function $f \in H^{1,p}(\mathbb{X})$ one can define the Cheeger energy

$$\mathsf{CE}_p(f) := \Big\{ \liminf_{n\to\infty} \mathsf{pCE}_p(f_n) : f_n \in \mathrm{Lip}_b(X),\ f_n \to f \text{ strongly in } L^p(X, \mathfrak{m})\Big\}$$

and the Sobolev norm

$$\|f\|_{H^{1,p}(\mathbb{X})} := \left(\|f\|^p_{L^p} + \mathsf{CE}_p(f)\right)^{1/p},$$

thus obtaining a Banach space. It is therefore quite natural to study its dual, which we will denote by $H^{-1,q}(\mathbb{X})$.

In such a general situation, however, when we do not assume any doubling and/or Poincaré assupmptions, $H^{1,p}(\mathbb{X})$ may fail to be reflexive or separable and it is not known if the generating class of bounded Lipschitz functions is strongly dense.

As a first contribution, we will show that it could be more convenient to consider the smaller subspace $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ of $H^{-1,q}(\mathbb{X})$ obtained by taking the strong closure of $L^q(X, \mathfrak{m})$. Linear functionals in $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ are characterized by their behaviour on $\mathrm{Lip}_b(X)$ (or on even smaller generating subalgebras) and their dual norm can also be computed by the formula

$$\|L\|_{H^{-1,q}(\mathbb{X})} = \sup\Big\{\langle L, f\rangle : f \in \mathrm{Lip}_b(X),\ \mathsf{pCE}_p(f) + \|f\|^p_{L^p} \le 1\Big\}, \tag{1.1}$$

which is well adapted to be applied to general Borel measures μ on X.

In Sections 3 and 4 we will show that $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ has three important properties:

(a) it can be identified with the predual of $H^{1,p}(\mathbb{X})$ (thus showing in particular that $H^{1,p}(\mathbb{X})$ is the dual of a separable Banach space);

(b) every positive Borel measure μ satisfying

$$\Big|\int_X f \, \mathrm{d}\mu\Big| \le C\Big(\mathsf{pCE}_p(f) + \|f\|^p_{L^p}\Big)^{1/p} \quad \text{for every } f \in \mathrm{Lip}_b(X)$$

can be extended in a unique way to a functional $L_\mu \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$;

(c) every positive functional on $\mathrm{Lip}_b(X)$ such that the supremum in (1.1) is finite can be represented by a positive Radon measure.

This last property relies on a representation formula of the dual of the Cheeger energy by (nonparametric) dynamic plans (Theorem 4.6) which is interesting by itself. As a further important application of this result, in the final section 5 we will show that negligible sets in E with respect to the Newtonian capacity Cap_p are also μ-negligible for every positive Borel measure with finite dual energy. As a byproduct, we can express the duality of μ with a function f in $H^{1,p}(\mathbb{X})$ in the integral form

$$\langle \mu, f\rangle = \int_X \tilde f \,\mathrm{d}\mu,$$

where $\tilde f$ is any good representative of f in the Newtonian space $N^{1,p}(\mathbb{X})$.

Our last application concerns the variational representation of the Newtonian capacity of a closed set $F \subset X$

$$\big(\mathrm{Cap}_p(F)\big)^{1/p} = \sup\Big\{\mu(F) : \mu \in \mathcal{M}_+(X),\ \mu(X \setminus F) = 0,\ \|L_\mu\|_{H^{-1,q}(\mathbb{X})} \le 1\Big\}.$$

Main notation

(X,τ)	Hausdorff topological space
(X,τ,d)	Extended metric-topological (e.m.t.) space, see §2.2 and Definition 2.2
$\mathbb{X} = (X,\tau,\mathsf{d},\mathfrak{m})$	Extended metric-topological measure (e.m.t.m.) space, see §2.2
$\mathcal{M}_+(X)$	Positive and finite Radon measures on a Hausdorff topological space X, § 2.1
$\mathscr{B}(X)$	Borel subsets of X
$f_\sharp\mu$	Push forward of $\mu \in \mathcal{M}(X)$ by a (Lusin μ-measurable) map $f : X \to Y$, (2.1)
$\mathrm{C}_b(X,\tau)$, $\mathrm{C}_b(X)$	τ-continuous and bounded real functions on X
$\mathrm{Lip}_b(X,\tau,\mathsf{d})$	Bounded, τ-continuous and d-Lipschitz real functions on X, (2.2)
$\mathrm{lip}\, f(x)$	Asymptotic Lipschitz constant of f at a point x, (2.4)
$\mathscr{A}$	Compatible unital sub-algebra of $\mathrm{Lip}_b(X,\tau,\mathsf{d})$, Definition 2.3
$\mathscr{L}^q(\sigma\|\mu)$	q-Entropy functionals on Radon measures, (4.2)
$\mathrm{C}([0,1];(X,\tau))$	τ-continuous curves defined in $[0,1]$ with values in X, § 2.4
τ_{C}, d_{C}	Compact open topology and extended distance on $\mathrm{C}([0,1];X)$, § 2.4
$\mathrm{BVC}([0,1];X)$	Continuous curves with d-bounded variation, § 2.4
$\mathrm{RA}(X)$	Continuous and rectifiable arcs, § 2.4
$\mathsf{e}(\cdot,t), \mathsf{e}_t(\cdot), \mathsf{e}[\cdot]$	Evaluation maps along curves and arcs, § 2.4
τ_{A}, d_{A}	Quotient topology and extended distance on $\mathrm{RA}(X)$, § 2.4
R_γ	Arc-length reparametrization of a rectifiable arc γ, § 2.4
$\int_\gamma f$	Integral of a function f along a rectifiable curve (or arc) γ, § 2.4
$\ell(\gamma)$	length of γ, § 2.4
ν_γ	Radon measure in $\mathcal{M}_+(X)$ induced by integration along a rectifiable arc γ, (2.9)
pCE_p, CE_p, $\mathsf{CE}_{p,\kappa}$	(pre)Cheeger energy, Definition 3.1
$H^{1,p}(\mathbb{X})$	Metric Sobolev space induced by the Cheeger energy, Definition 3.1
$\|\mathrm{D}f\|_\star$	Minimal p-relaxed gradient, (3.1)-(3.2)
$\mathrm{Bar}_q(\boldsymbol{\pi})$	q-barycentric entropy of a dynamic plan, Definition 4.2
$\mathcal{B}_q(\mathrm{RA}(X))$	Plans with barycenter in $L^q(X,\mathfrak{m})$, Definition 4.2
$\mathscr{D}_q(\mu_0,\mu_1)$	Dual dynamic cost, (4.5)
$\mathrm{Mod}_p(\Gamma)$	p-Modulus of a collection $\Gamma \subset \mathrm{RA}(X)$, Definition 5.1
$N^{1,p}(\mathbb{X})$	Newtonian space, Definition 5.3

Cap_p	Newtonian capacity, (5.2)

2. Preliminary results

2.1. Topological and measure theoretic notions

Let (Y, τ_Y) be a Hausdorff topological space. We will denote by $\mathrm{C}_b(Y, \tau_Y)$ the space of τ_Y-continuous and bounded real functions defined on Y; $\mathscr{B}(Y, \tau_Y)$ is the collection of the Borel subsets of Y; we will often omit the explicit indication of the topology τ_Y, when it will be clear from the context.

Definition 2.1 (Radon measures [22, Chap. I, Sect. 2])**.** *A finite Radon measure $\mu : \mathscr{B}(Y, \tau_Y) \to [0, +\infty)$ is a Borel nonnegative σ-additive finite measure satisfying the following inner regularity property:*

$$\forall\, B \in \mathscr{B}(Y, \tau_Y) : \quad \mu(B) = \sup\Big\{\mu(K) : K \subset B,\ K \text{ compact}\Big\}.$$

We will denote by $\mathcal{M}_+(Y)$ the collection of all the finite positive Radon measures on Y.

It is worth mentioning that every Borel measure in a Polish, Lusin, Souslin, or locally compact space with a countable base of open sets is Radon [22, Ch. II, Sect. 3]. In particular the notation of $\mathcal{M}_+(Y)$ is consistent with the standard one adopted e.g., in [4, 6, 24], where Polish or second countable locally compact spaces are considered.

If (Y, τ_Y) is completely regular, the weak (or narrow) topology $\tau_{\mathcal{M}_+}$ on $\mathcal{M}_+(Y)$ can be defined as the coarsest topology for which all maps

$$\mu \mapsto \int_Y h \,\mathrm{d}\mu \qquad \text{from } \mathcal{M}_+(Y) \text{ into } \mathbb{R}$$

are continuous as $h : Y \to \mathbb{R}$ varies in $\mathrm{C}_b(Y, \tau_Y)$ [22, p. 370, 371].

Recall that a set $A \subset Y$ is μ-measurable, $\mu \in \mathcal{M}_+(Y)$, if there exist Borel sets $B_1, B_2 \in \mathscr{B}(Y, \tau_Y)$ such that $B_1 \subset A \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$. A set is called *universally (Radon) measurable* if it is μ-measurable for every Radon measure $\mu \in \mathcal{M}_+(Y)$.

Let (Z, τ_Z) be another Hausdorff topological space. A map $f : Y \to Z$ is Borel (resp. μ-measurable) if for every $B \in \mathscr{B}(Z)$ $f^{-1}(B) \in \mathscr{B}(Y)$ (resp. $f^{-1}(B)$ is μ-measurable). f is Lusin μ-measurable if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset Y$ such that $\mu(Y \setminus K_\varepsilon) \le \varepsilon$ and the restriction of f to K_ε is continuous. A map $f : Y \to Z$ is called *universally Lusin measurable* if it is Lusin μ-measurable for every Radon measure $\mu \in \mathcal{M}_+(Y)$.

Every Lusin μ-measurable map is also μ-measurable. Whenever f is Lusin μ-measurable, its push-forward

$$f_\sharp\mu \in \mathcal{M}_+(Z), \quad f_\sharp\mu(B) := \mu(f^{-1}(B)) \quad \text{for every Borel subset } B \subset \mathscr{B}(Z) \tag{2.1}$$

induces a Radon measure in Z.

Given a power $p \in (1, \infty)$ and a Radon measure μ in (Y, τ_Y) we will denote by $L^p(Y, \mu)$ the usual Lebesgue space of class of p-summable μ-measurable functions defined up to μ-negligible sets.

2.2. Extended metric-topological (measure) spaces

Let (X, τ) be a Hausdorff topological space. An extended distance is a symmetric map $\mathsf{d} : X \times X \to [0, \infty]$ satisfying the triangle inequality and the property $\mathsf{d}(x, y) = 0$ iff $x = y$ in X: we call (X, d) an extended metric space. We will omit the adjective "extended" if d takes real values.

Let d be an extended distance on X. For every $f : X \to \mathbb{R}$ and $A \subset X$ we set

$$\mathrm{Lip}(f, A) := \inf\Big\{L \in [0, \infty] : |f(y) - f(z)| \le L\mathsf{d}(y, z) \quad \text{for every } y, z \in A\Big\}.$$

We adopt the convention to omit the set A when $A = X$. We consider the class of τ-continuous and d-Lipschitz functions

$$\mathrm{Lip}_b(X, \tau, \mathsf{d}) := \Big\{f \in \mathrm{C}_b(X, \tau) : \mathrm{Lip}(f) < \infty\Big\}, \tag{2.2}$$

and for every $\kappa > 0$ we will also consider the subsets

$$\mathrm{Lip}_{b,\kappa}(X, \tau, \mathsf{d}) := \Big\{f \in \mathrm{C}_b(X, \tau) : \mathrm{Lip}(f) \le \kappa\Big\}.$$

A particular role will be played by $\mathrm{Lip}_{b,1}(X, \tau, \mathsf{d})$. It is easy to check that $\mathrm{Lip}_b(X, \tau, \mathsf{d})$ is a real and commutative sub-algebras of $\mathrm{C}_b(X, \tau)$ with unit. According to [2, Definition 4.1], an extended metric-topological space (e.m.t. space) (X, τ, d) is characterized by a Hausdorff topology τ and an extended distance d satisfying a suitable compatibility condition.

Definition 2.2 (Extended metric-topological spaces)**.** *Let (X, d) be an extended metric space, let τ be a Hausdorff topology in X. We say that (X, τ, d) is an extended metric-topological (e.m.t.) space if:*

(X1) *the topology τ is generated by the family of functions* $\mathrm{Lip}_b(X, \tau, \mathsf{d})$*;*

(X2) *the distance d can be recovered by the functions in* $\mathrm{Lip}_{b,1}(X, \tau, \mathsf{d})$ *through the formula*

$$\mathsf{d}(x, y) = \sup_{f \in \mathrm{Lip}_{b,1}(X, \tau, \mathsf{d})} |f(x) - f(y)| \quad \textit{for every } x, y \in X. \tag{2.3}$$

We will say that (X, τ, d) is complete *if d-Cauchy sequences are d-convergent. All the other topological properties usually refer to (X, τ).*

The previous assumptions guarantee that (X, τ) is completely regular. When an e.m.t. space (X, τ, d) is provided by a positive Radon measure $\mathfrak{m} \in \mathcal{M}_+(X, \tau)$ we will say that

the system $\mathbb{X} = (X, \tau, \mathsf{d}, \mathfrak{m})$ is an extended metric-topological measure (e.m.t.m.) space.

Definition 2.2 yields two important properties linking d and τ: first of all

$$\mathsf{d} \text{ is } \tau \times \tau\text{-lower semicontinuous in } X \times X,$$

since it is the supremum of a family of continuous maps by (2.3). On the other hand, every d-converging net $(x_j)_{j \in J}$ indexed by a directed set J is also τ-convergent:

$$\lim_{j \in J} \mathsf{d}(x_j, x) = 0 \quad \Rightarrow \quad \lim_{j \in J} x_j = x \quad \text{w.r.t. } \tau.$$

It is sufficient to observe that τ is the initial topology generated by $\mathrm{Lip}_b(X, \tau, \mathsf{d})$ so that a net (x_j) is convergent to a point x if and only if

$$\lim_{j \in J} f(x_j) = f(x) \quad \text{for every } f \in \mathrm{Lip}_b(X, \tau, \mathsf{d}).$$

In many situations it could be useful to consider smaller subalgebras which are however sufficiently rich to recover the metric properties of an extended metric topological space (X, τ, d).

Definition 2.3 (Compatible algebras of Lipschitz functions)**.** *Let $\mathscr{A}$ be a unital subalgebra of* $\mathrm{Lip}_b(X, \tau, \mathsf{d})$ *and let us set* $\mathscr{A}_\kappa := \mathscr{A} \cap \mathrm{Lip}_{b,\kappa}(X, \tau, \mathsf{d})$.

We say that $\mathscr{A}$ is compatible *with the metric-topological structure* (X, τ, d) *if*

$$\mathsf{d}(x, y) = \sup_{f \in \mathscr{A}_1} |f(x) - f(y)| \quad \textit{for every } x, y \in X.$$

In particular, $\mathscr{A}$ separates the points of X.

It is not difficult to show that any compatible algebra $\mathscr{A}$ is dense in $L^p(X, \mathfrak{m})$ [21, Lemma 2.27]. If we do not make a different explicit choice, we will always assume that an e.m.t.m. space $\mathbb{X}$ is endowed with the canonical algebra $\mathscr{A}(\mathbb{X}) := \mathrm{Lip}_b(X, \tau, \mathsf{d})$.

2.3. The asymptotic Lipschitz constant

For every $f : X \to \mathbb{R}$ and $x \in X$, denoting by $\mathscr{U}_x$ the directed set of all the τ-neighborhoods of x, we set

$$\mathrm{lip}\, f(x) := \lim_{U \in \mathscr{U}_x} \mathrm{Lip}(f, U) = \inf_{U \in \mathscr{U}_x} \mathrm{Lip}(f, U) \quad x \in X.$$

Notice that $\mathrm{Lip}(f, \{x\}) = 0$ and therefore $\mathrm{lip}\, f(x) = 0$ if x is an isolated point of X. We can also define $\mathrm{lip}\, f$ as

$$\mathrm{lip}\, f(x) = \limsup_{\substack{y,z \to x \\ y \neq z}} \frac{|f(y) - f(z)|}{\mathsf{d}(y, z)}, \tag{2.4}$$

where the convergence of y, z to x in (2.4) is intended with respect to the topology τ. In particular,

$$\mathrm{lip}\, f(x) \geq |\mathrm{D}f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(x, y)}. \tag{2.5}$$

It is not difficult to check that $x \mapsto \mathrm{lip}\, f(x)$ is a τ-upper semicontinuous map and f is locally d-Lipschitz in X iff $\mathrm{lip}\, f(x) < \infty$ for every $x \in X$. When (X, d) is a length space, $\mathrm{lip}\, f$ coincides with the upper τ-semicontinuous envelope of the local Lipschitz constant (2.5).

We collect in the next useful lemma the basic calculus properties of $\mathrm{lip}\, f$.

Lemma 2.4. *For every $f, g, \chi \in \mathrm{C}_b(X)$ with $\chi(X) \subset [0, 1]$ we have*

$$\begin{aligned} \mathrm{lip}(\alpha f + \beta g) &\leq |\alpha|\ \mathrm{lip}\, f + |\beta|\ \mathrm{lip}\, g \quad \textit{for every } \alpha, \beta \in \mathbb{R}, \\ \mathrm{lip}(fg) &\leq |f|\, \mathrm{lip}\, g + |g|\, \mathrm{lip}\, f, \\ \mathrm{lip}((1-\chi)f + \chi g) &\leq (1-\chi)\, \mathrm{lip}\, f + \chi\, \mathrm{lip}\, g + \mathrm{lip}\chi |f - g|. \end{aligned}$$

Moreover, whenever $\phi \in \mathrm{C}^1(\mathbb{R})$

$$\mathrm{lip}(\phi \circ f) = |\phi' \circ f|\, \mathrm{lip}\, f\,.$$

2.4. Continuous curves and rectifiable arcs

We briefly recap some useful results concerning the extended metric-topological structure of the space of rectifiable arcs in an e.m.t. space (X, τ, d). We refer to [21, § 3] for a more detailed discussion and for the related proofs.

For every $\gamma : [0,1] \to X$ and $t \in [0,1]$ we set

$$V_\gamma(t) := \sup\Big\{\sum_{j=1}^{N} \mathsf{d}(\gamma(t_j), \gamma(t_{j-1})) : 0 = t_0 < t_1 < \cdots < t_N = t\Big\}, \quad \ell(\gamma) := V_\gamma(1).$$

$\mathrm{BVC}([0,1]; X)$ will denote the space of d-continuous maps $\gamma : [0,1] \to X$ such that $\ell(\gamma) < \infty$; notice that if $\ell(\gamma) = 0$ then γ is constant. We will also consider the set of curves with constant velocity

$$\mathrm{BVC}_c([0,1]; X) := \{\gamma \in \mathrm{BVC}([0,1]; X) : V_\gamma(t) = \ell(\gamma)t\}. \tag{2.6}$$

Notice that for every $\gamma \in \mathrm{BVC}([0,1]; X)$ the map $V_\gamma : [0,1] \to [0, \ell(\gamma)]$ is continuous and surjective and whenever $\ell(\gamma) > 0$

$$\begin{gathered}\text{there exists a unique } \ell(\gamma)\text{-Lipschitz map } R_\gamma \in \mathrm{BVC}_c([0,1]; X) \text{ such that}\\ \gamma(t) = R_\gamma(\ell(\gamma)^{-1} V_\gamma(t)) \quad \text{for every } t \in [0,1],\end{gathered} \tag{2.7}$$

with $|R'_\gamma|(s) = \ell(\gamma)$ a.e.; when $\ell(\gamma) = 0$ then $R_\gamma(t) = \gamma(t)$ is constant. We can use R_γ to define the integral of a bounded or nonnegative Borel function $f : X \to \mathbb{R}$ along γ:

$$\int_\gamma f = \int_0^1 f(R_\gamma(s))|R'_\gamma|(s)\,\mathrm{d}s = \ell(\gamma)\int_0^1 f(R_\gamma(s))\,\mathrm{d}s. \tag{2.8}$$

We also notice that (2.8) yields

$$\int_\gamma f = \int_X f\,\mathrm{d}\nu_\gamma \quad \text{where} \quad \nu_\gamma := \ell(\gamma)(R_\gamma)_\sharp(\mathscr{L}^1 \llcorner [0,1]). \tag{2.9}$$

We will endow $\mathrm{BVC}([0,1]; X)$ with the compact-open topology τ_{C} induced by τ. By definition, a subbasis generating τ_C is given by the collection of sets

$$S(K, V) := \Big\{\gamma \in \mathrm{C}([0,1]; X) : \gamma(K) \subset V\Big\}, \quad K \subset [0,1] \text{ compact}, V\ \tau\text{-open in } X.$$

By [19, § 46, Thm. 46.8, 46.10] if the topology τ is induced by a distance δ, then the topology τ_{C} is induced by the uniform distance $\delta_{\mathrm{C}}(\gamma, \gamma') := \sup_{t\in[a,b]} \delta(\gamma(t), \gamma'(t))$ and convergence w.r.t. the compact-open topology coincides with the uniform convergence w.r.t. δ. If moreover τ is separable then also τ_{C} is separable [14, 4.2.18].

We will denote by $\mathsf{e} : \mathrm{BVC}([0,1]; X) \times [0,1] \to X$ the evaluation map, which is defined by $\mathsf{e}_t(\gamma) = \mathsf{e}(\gamma, t) := \gamma(t)$ for every $t \in [0,1]$; e is continuous. We will also adopt the notation $\mathsf{e}[\gamma] := \mathsf{e}(\{\gamma\} \times [0,1]) = \{\gamma(t) : t \in [0,1]\}$ for the image of γ in X.

The extended distance $\mathsf{d} : X \times X \to [0, \infty]$ induces the extended distance d_{C} in $\mathrm{BVC}([0,1]; X)$ by

$$\mathsf{d}_{\mathrm{C}}(\gamma_1, \gamma_2) := \sup_{t\in[a,b]} \mathsf{d}(\gamma_1(t), \gamma_2(t))$$

and $(\mathrm{C}([0,1];X),\tau_\mathrm{C},\mathsf{d}_\mathrm{C})$ is an extended metric-topological space [21, Prop. 3.2].

Let us denote by Σ the set of continuous, nondecreasing and surjective maps $\sigma:[0,1]\to[0,1]$. On $\mathrm{BVC}([0,1];X)$ we introduce the relation

$$\gamma_1\sim\gamma_2\quad\text{if}\quad \exists\,\sigma_i\in\Sigma:\ \gamma_1\circ\sigma_1=\gamma_2\circ\sigma_2,$$

and the function

$$\mathsf{d}_\mathrm{A}(\gamma_1,\gamma_2):=\inf_{\sigma_i\in\Sigma}\mathsf{d}_\mathrm{C}(\gamma_1\circ\sigma_1,\gamma_2\circ\sigma_2)\qquad\text{for every }\gamma_i\in\mathrm{BVC}([0,1];X).$$

It is possible to prove that $\sim$ is an equivalence relation [21, § 3.2, Cor. 3.5] and d_A satisfies

$$\mathsf{d}_\mathrm{A}(\gamma_1,\gamma_2)=\mathsf{d}_\mathrm{A}(R_{\gamma_1},R_{\gamma_2})=\inf_{\substack{\sigma\in\Sigma\\ \sigma\text{ injective}}}\mathsf{d}_\mathrm{C}(\gamma_1,\gamma_2\circ\sigma)=\inf_{\gamma_i'\sim\gamma_i}\mathsf{d}_\mathrm{C}(\gamma_1',\gamma_2').$$

In particular d_A satisfies the triangle inequality, is invariant with respect to $\sim$ and $\gamma\sim\gamma'$ if and only if $\mathsf{d}_\mathrm{A}(\gamma,\gamma')=0$. We collect a list of useful properties [21, § 3.2]:

Lemma 2.5. (a) *The space* $\mathrm{RA}(X):=\mathrm{BVC}([0,1];X)/\sim$ *endowed with the quotient topology* τ_A *is an Hausdorff space. We will denote by* $\mathfrak{q}:\mathrm{BVC}([0,1];X)\to\mathrm{RA}(X)$ *the quotient map.*

(b) *If the topology* τ *is induced by the distance* δ *then the quotient topology* τ_A *is induced by* δ_A *(considered as a distance between equivalence classes of curves).*

(c) $(\mathrm{RA}(X),\tau_\mathrm{A},\mathsf{d}_\mathrm{A})$ *is an extended metric-topological space.*

(d) *For every* $\gamma,\gamma'\in\mathrm{BVC}([0,1];X)$ *we have*

$$\gamma\sim\gamma'\quad\Leftrightarrow\quad R_\gamma=R_{\gamma'},$$

and all the curves γ' *equivalent to* γ *can be described as* $\gamma'=R_\gamma\circ\sigma$ *for some* $\sigma\in\Sigma$*. Moreover, if* $\gamma'\sim\gamma$ *then*

$$\ell(\gamma')=\ell(\gamma),\quad R_{\gamma'}=R_\gamma,\quad \int_\gamma f=\int_{\gamma'}f,$$

so that the functions R*,* ℓ*, the evaluation maps* $\mathsf{e}_0,\mathsf{e}_1$*, and the integral* $\int_\gamma f$ *are invariant w.r.t. reparametrizations. We will still denote them by the same symbols.*

(e) $\ell:\mathrm{RA}(X)\to[0,+\infty]$ *is* τ_A*-lower semicontinuous and* $\mathsf{e}_0,\mathsf{e}_1:\mathrm{RA}(X)\to X$ *are continuous. If* $f:X\to[0,+\infty]$ *is lower semicontinuous then the map* $\gamma\mapsto\int_\gamma f$ *is lower semicontinuous w.r.t.* τ_A *in* $\mathrm{RA}(X)$*.*

We conclude this section with a list of useful properties concerning the compactness in $\mathrm{RA}(X)$ and the measurability of some importants maps, see [21, Thm. 3.13].

Theorem 2.6. (a) *If* γ_i*,* $i\in I$*, is a converging net in* $\mathrm{RA}(X)$ *with* $\gamma=\lim_{i\in I}\gamma_i$ *and* $\lim_{i\in I}\ell(\gamma_i)=\ell(\gamma)$ *then*

$$\lim_{i\in I}R_{\gamma_i}=R_\gamma\quad\textit{w.r.t. }\tau_\mathrm{C},$$

and for every bounded and continuous function $f \in \mathrm{C}_b(X, \tau)$ *we have*

$$\lim_{i\in I} \int_{\gamma_i} f = \int_{\gamma} f.$$

In particular, we have

$$\lim_{i\in I} \nu_{\gamma_i} = \nu_\gamma \quad \textit{weakly in } \mathcal{M}_+(X).$$

(b) *The map* $\gamma \mapsto \nu_\gamma$ *from* $\mathrm{RA}(X)$ *to* $\mathcal{M}_+(X)$ *is universally Lusin measurable.*

(c) *The map* $\gamma \mapsto R_\gamma$ *is universally Lusin measurable from* $\mathrm{RA}(X)$ *to* $\mathrm{BVC}_c([0,1]; X)$ *endowed with the topology* τ_{C} *and it is also Borel if* (X, τ) *is Souslin.*

(d) *If* $f : X \to \mathbb{R}$ *is a bounded Borel function (or* $f : X \to [0, +\infty]$ *Borel) the map* $\gamma \mapsto \int_\gamma f$ *is Borel. In particular the family of measures* $\{\nu_\gamma\}_{\gamma\in\mathrm{RA}(X)}$ *is Borel.*

(e) *If* (X, τ) *is compact and* $\Gamma \subset \mathrm{RA}(X)$ *satisfies* $\sup_{\gamma\in\Gamma} \ell(\gamma) < +\infty$ *then* Γ *is relatively compact in* $\mathrm{RA}(X)$ *w.r.t. the* τ_{A} *topology.*

(f) *If* (X, d) *is complete and* $\Gamma \subset \mathrm{RA}(X)$ *satisfies the following conditions:*

1) $\sup_{\gamma\in\Gamma} \ell(\gamma) < +\infty$*;*
2) *there exists a* τ*-compact set* $K \subset X$ *such that* $\mathsf{e}[\gamma] \cap K \neq \emptyset$ *for every* $\gamma \in \Gamma$*;*
3) $\{\nu_\gamma : \gamma \in \Gamma\}$ *is equally tight, i.e. for every* $\varepsilon > 0$ *there exists a* τ*-compact set* $K_\varepsilon \subset X$ *such that* $\nu_\gamma(X \setminus K_\varepsilon) \le \varepsilon$ *for every* $\gamma \in \Gamma$*,*

then Γ *is relatively compact in* $\mathrm{RA}(X)$ *w.r.t. the* τ_{A} *topology.*

Notice that the third condition in the statement (f) of Theorem 2.6 implies the second one whenever $\inf_{\gamma\in\Gamma} \ell(\gamma) > 0$.

3. Metric Sobolev spaces and their duals

In this section we will always assume that $\mathbb{X} = (X, \tau, \mathsf{d}, \mathfrak{m})$ is a *complete* e.m.t.m. space and $\mathscr{A}$ is a compatible sub-algebra of $\mathrm{Lip}_b(X, \tau, \mathsf{d})$. We also fix a summability exponent $p \in (1, \infty)$ with conjugate $q = p/(p-1)$.

3.1. The Cheeger energy

Let us first define the notion of Cheeger energy CE_p associated to $\mathbb{X}$, [3, 5, 6, 10, 21].

Definition 3.1 (Cheeger energy)**.** *For every* $\kappa \ge 0$ *and* $p \in (1, \infty)$ *we define the "pre-Cheeger" energy functionals*

$$\mathsf{pCE}_p(f) := \int_X (\operatorname{lip} f(x))^p \, \mathrm{d}\mathfrak{m}, \qquad \textit{for every } f \in \mathrm{Lip}_b(X, \tau, \mathsf{d}),$$

with $\mathsf{pCE}_p(f) = +\infty$ *if* $f \in L^p(X) \setminus \mathrm{Lip}_b(X, \tau, \mathsf{d})$*. The* L^p*-lower semicontinuous envelope of* pCE_p *is the "strong" Cheeger energy*

$$\mathsf{CE}_p(f) := \inf\Big\{ \liminf_{n\to\infty} \int_X (\operatorname{lip} f_n)^p \, \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X, \tau, \mathsf{d}),\ f_n \to f \textit{ in } L^p(X, \mathfrak{m}) \Big\}.$$

For every $k \geq 0$ and $f \in L^p(X, \mathfrak{m})$ we also set

$$\mathsf{pCE}_{p,\kappa}(f) := \mathsf{pCE}_p(f) + \kappa \|f\|^p_{L^p(X,\mathfrak{m})}, \qquad \mathsf{CE}_{p,\kappa}(f) := \mathsf{CE}_p(f) + \kappa \|f\|^p_{L^p(X,\mathfrak{m})}.$$

We denote by $H^{1,p}(\mathbb{X})$ the subset of $L^p(X, \mathfrak{m})$ whose elements f have finite Cheeger energy $\mathsf{CE}_p(f) < \infty$: it is a Banach space with norm $\|f\|_{H^{1,p}(\mathbb{X})} := \left(\mathsf{CE}_{p,1}(f)\right)^{1/p}$.

Remark 3.2 (The notation CE and $H^{1,p}$)**.** We used the symbol CE instead of Ch (introduced by [6]) in the previous definition to stress three differences:

- the dependence on the strongest $\operatorname{lip} f$ instead of $|\mathrm{D}f|$,
- the factor 1 instead of $1/p$ in front of the energy integral.

In this paper we will mainly adopt the "strong" approach to metric Sobolev spaces and we will use the notation $H^{1,p}(\mathbb{X})$ to stress this fact. We refer to [5, 6] for the equivalent weak definition of $W^{1,p}(\mathbb{X})$ by test plan. In the final section 5 we will also use a few properties related to the intermediate (but still equivalent) Newtonian point of view, see [8, 17].

It is not difficult to check that $\mathsf{CE}_p : L^p(X, \mathfrak{m}) \to [0, +\infty]$ is a convex, lower semicontinuous and p-homogeneous functional; it is the greatest L^p-lower semicontinuous functional "dominated" by pCE_p. Notice that when $\mathfrak{m}$ has not full support, two different elements $f_1, f_2 \in \mathrm{Lip}_b(X, \tau, \mathsf{d})$ may give rise to the same equivalence class in $L^p(X, \mathfrak{m})$. In this case, CE_p can be equivalently defined starting from the functional

$$\widetilde{\mathsf{pCE}}_p(f) := \inf\left\{\mathsf{pCE}_p(\tilde f) : \tilde f \in \mathrm{Lip}_b(X, \tau, \mathsf{d}),\ \tilde f = f\ \mathfrak{m}\text{-a.e.}\right\},$$

defined on the quotient space $\mathrm{Lip}_b(X, \tau, \mathsf{d})/\sim_{\mathfrak{m}}$.

Whenever $\mathsf{CE}_p(f) < \infty$ one can show [5, 6] that the closed convex set

$$S_p(f) := \left\{G \in L^p(X, \mathfrak{m}) : \exists f_n \in \mathrm{Lip}_b(X, \tau, \mathsf{d}) : f_n \to f,\ \operatorname{lip} f_n \rightharpoonup G \text{ in } L^p(X, \mathfrak{m})\right\} \tag{3.1}$$

admits a unique element of minimal norm, *the minimal relaxed gradient* denoted by $|\mathrm{D}f|_\star$. $|\mathrm{D}f|_\star$ is also minimal in $S_p(f)$ with respect to the natural order structure, i.e.,

$$|\mathrm{D}f|_\star \in S_p(f), \quad |\mathrm{D}f|_\star \leq G \quad \text{for every } G \in S_p(f). \tag{3.2}$$

The Cheeger energy CE_p admits an integral representation in terms of the minimal relaxed gradient:

$$\mathsf{CE}_p(f) = \int_X |\mathrm{D}f|^p_\star(x)\,\mathrm{d}\mathfrak{m}(x) \quad \text{for every } f \in H^{1,p}(\mathbb{X}),$$

and enjoys the following strong approximation result (see [5, 6] in the case of bounded Lipschitz functions, [7] for the "metric" algebra generated by truncated distance functions and [21, Thm. 12.1] for the general case):

Theorem 3.3 (Density in energy of compatible algebras)**.** *Let $\mathscr{A}$ be a compatible sub-algebra of* $\mathrm{Lip}_b(X, \tau, \mathsf{d})$ *and let I be a closed (possibly unbounded) interval of $\mathbb{R}$. For every $f \in H^{1,p}(\mathbb{X})$ taking values in I there exists a sequence $(f_n) \subset \mathscr{A}$ with values in I such that*

$$f_n \to f, \quad \operatorname{lip} f_n \to |\mathrm{D}f|_\star \quad \textit{strongly in } L^p(X, \mathfrak{m}).$$

We collect a list of useful properties [6] of the minimal p-relaxed gradient.

Theorem 3.4. *For every $f, g \in H^{1,p}(\mathbb{X})$ we have*

(a) *(Pointwise sublinearity) For $|\mathrm{D}(\alpha f + \beta g)|_\star \le \alpha |\mathrm{D}f|_\star + \beta |\mathrm{D}g|_\star$.*

(b) *(Leibniz rule) For every $f, g \in H^{1,p}(\mathbb{X}) \cap L^\infty(X, \mathfrak{m})$ we have $fg \in H^{1,p}(\mathbb{X})$ and*

$$|\mathrm{D}(fg)|_\star \le |f|\,|\mathrm{D}g|_\star + |g|\,|\mathrm{D}f|_\star. \tag{3.3}$$

(c) *(Locality) For any Borel set $N \subset \mathbb{R}$ with $\mathscr{L}^1(N) = 0$ we have*

$$|\mathrm{D}f|_\star = 0 \quad \mathfrak{m}\text{-a.e. on } f^{-1}(N).$$

In particular for every constant $c \in \mathbb{R}$

$$|\mathrm{D}f|_\star = |\mathrm{D}g|_\star \ \mathfrak{m}\text{-a.e. on } \{f - g = c\}.$$

(d) *(Chain rule) If $\phi \in \mathrm{Lip}(\mathbb{R})$ then $\phi \circ f \in H^{1,p}(\mathbb{X})$ with*

$$|\mathrm{D}(\phi \circ f)|_\star \le |\phi'(f)|\,|\mathrm{D}f|_\star. \tag{3.4}$$

Equality holds in (3.4) if ϕ is monotone or C^1.

3.2. Legendre transform of the Cheeger energy and the dual of the Sobolev space $H^{1,p}(\mathbb{X})$

Let us now study a few important properties of the Legendre transform of the p-Cheeger energy and its relation with the dual of the Sobolev space $H^{1,p}(\mathbb{X})$ when $p \in (1, \infty)$; recall that we denote by $q = p' = p/(p-1)$ the conjugate exponent of p. Let us first recall a simple property of p-homogeneous convex functionals (see e.g., [21, Lemma A.7].

Lemma 3.5 (Dual of positively p-homogeneous functionals)**.** *Let C be a convex cone of some vector space V, $p > 1$, and $\phi, \psi : C \to [0, \infty]$ with $\psi = \phi^{1/p}$, $\phi = \psi^p$. We have the following properties:*

(a) *ϕ is convex and positively p-homogeneous (i.e., $\phi(\kappa v) = \kappa^p \phi(v)$ for every $\kappa \ge 0$ and $v \in C$) in C if and only if ψ is convex and positively 1-homogeneous on C (a seminorm, if C is a vector space and ψ is finite and even).*

(b) *Under one of the above equivalent assumptions, setting for every linear functional $z : V \to \mathbb{R}$*

$$\frac{1}{q}\phi^*(z) := \sup_{v \in C} \langle z, v\rangle - \frac{1}{p}\phi(v), \quad \psi_*(z) := \sup\Big\{\langle z, v\rangle : v \in C,\ \psi(v) \le 1\Big\}, \tag{3.5}$$

we have

$$\psi_*(z) = \inf\Big\{c \ge 0 : \langle z, v\rangle \le c\,\psi(v) \quad \text{for every } v \in C\Big\}, \quad \phi^*(z) = (\psi_*(z))^q,$$

where in the first infimum we adopt the convention $\inf A = +\infty$ if A is empty.

(c) *An element $v \in C$ attains the first supremum in (3.5) if and only if*

$$\langle z, v\rangle = (\psi_*(z))^q = (\psi(v))^p.$$

We want to study the dual functionals related to $\mathsf{CE}_{p,\kappa}$ and $\mathsf{pCE}_{p,\kappa}$. The simplest situation is provided by $L^p - L^q$-duality:

$$\frac{1}{q}\mathsf{CE}^*_{p,\kappa}(w) := \sup_{u\in L^p} \int_X wu \,\mathrm{d}\mathfrak{m} - \frac{1}{p}\mathsf{CE}_{p,\kappa}(u) \quad \text{for every } w \in L^q(X,\mathfrak{m}),$$
$$\frac{1}{q}\mathsf{pCE}^*_{p,\kappa}(w) := \sup_{u\in \mathrm{Lip}_b(X,\tau,\mathsf{d})} \int_X wu \,\mathrm{d}\mathfrak{m} - \frac{1}{p}\mathsf{pCE}_{p,\kappa}(u) \quad \text{for every } w \in L^q(X,\mathfrak{m}).$$

By Fenchel-Moreau duality Theorem (see e.g., [9, Theorem 1.11], [13, Chap. IV]) it is immediate to check that

$$\mathsf{pCE}^*_{p,\kappa}(w) = \mathsf{CE}^*_{p,\kappa}(w) \qquad \text{for every } w \in L^q(X,\mathfrak{m}),$$
$$\frac{1}{p}\mathsf{CE}_{p,\kappa}(u) = \sup_{w\in L^q(X,\mathfrak{m})} \int_X uw \,\mathrm{d}\mathfrak{m} - \frac{1}{q}\mathsf{pCE}^*_{p,\kappa}(w) \qquad \text{for every } u \in L^p(X,\mathfrak{m}). \tag{3.6}$$

The situation is more complicated if one wants to study the dual of CE_p with respect to the Sobolev duality. Just to clarify all the possibilities we consider three normed vector spaces:

- The separable and reflexive Banach space $V := L^p(X,\mathfrak{m})$;
- The vector space $\mathscr{A}_p$ associated to a compatible algebra $\mathscr{A}$ endowed with the norm $\mathsf{pCE}^{1/p}_{p,1}$.
- The Banach space $W = H^{1,p}(\mathbb{X})$ with the norm $\mathsf{CE}^{1/p}_{p,1}$.

Notice that we do not know any information concerning the separability and the reflexivity of the Banach space $H^{1,p}(\mathbb{X})$ nor the (strong) density of $\mathscr{A}$ in W. Since both $\mathscr{A}$ and $W = H^{1,p}(\mathbb{X})$ are dense in $V = L^p(X,\mathfrak{m})$, if we identify V' with $L^q(X,\mathfrak{m})$ we clearly have

$$L^q(X,\mathfrak{m}) = V' \subset (\mathscr{A}_p)', \quad L^q(X,\mathfrak{m}) = V' \subset W' \quad \text{with continuous inclusions.}$$

On the other hand, every element $L \in W'$ can be considered as a bounded linear functional on $\mathscr{A}_p$ and thus induces an element L_{restr} of $(\mathscr{A}_p)'$ just by restriction, but it may happen that this identification map is not injective. Finally, since $\mathsf{pCE}_{p,1}$ may be strictly greater than $\mathsf{CE}_{p,1}$ on $\mathscr{A}_p$, in general not all the bounded linear functionals on $\mathscr{A}_p$ may admit an extension to W.

Taking all these facts into account, now we want to address the question of the unique extension of a given bounded linear functional L on $\mathscr{A}_p$ to an element of the dual Sobolev space W'. We begin with a precise definition.

Definition 3.6 (The spaces $H^{-1,q}(\mathbb{X})$, $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ and $\mathscr{A}'_q$)**.** *We define:*

- $H^{-1,q}(\mathbb{X})$ *as the dual W' of $H^{1,p}(\mathbb{X})$;*
- $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ *as the subset of $H^{-1,q}(\mathbb{X})$ whose elements L satisfy the following condition: for every choice of $f, f_n \in H^{1,p}(\mathbb{X})$, $n \in \mathbb{N}$, and every constant $C > 0$*

$$\mathsf{CE}_p(f_n) \le C, \quad \lim_{n\to\infty} \|f_n - f\|_{L^p(X,\mathfrak{m})} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \langle L, f_n\rangle = \langle L, f\rangle. \tag{3.7}$$

- $\mathscr{A}'_q$ *as the set of linear functionals* L *on* $\mathscr{A}$ *satisfying the following two conditions: there exists a constant* $D > 0$ *such that*

$$\left|\langle L, f\rangle\right| \le D\Big(\mathsf{pCE}_{p,1}(f)\Big)^{1/p} \quad \text{for every } f \in \mathscr{A}, \tag{3.8a}$$

and for every sequence $f_n \in \mathscr{A}$ *and every constant* $C > 0$

$$\mathsf{pCE}_p(f_n) \le C, \quad \lim_{n\to\infty} \|f_n\|_{L^p(X,\mathfrak{m})} = 0 \quad \Rightarrow \quad \liminf_{n\to\infty} \left|\langle L, f_n\rangle\right| = 0. \tag{3.8b}$$

When $\mathscr{A} = \mathscr{A}(\mathbb{X}) = \mathrm{Lip}_b(X, \tau, \mathsf{d})$ *we will write* $\mathscr{A}'_q = \mathscr{A}'_q(\mathbb{X})$.

It would not be difficult to check that if $H^{1,p}(\mathbb{X})$ is reflexive then $\mathscr{A}$ is strongly dense in $H^{1,p}(\mathbb{X})$ and $H^{-1,q}(\mathbb{X}) = H^{-1,q}_{\mathrm{pd}}(\mathbb{X}) \simeq \mathscr{A}'_q$. In the general case, only a partial result holds and we will show that $H^{1,p}(\mathbb{X})$ can be identified with the dual of $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$, i.e., $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ is a predual of $H^{1,p}(\mathbb{X})$ (this property justifies the index $_{\mathrm{pd}}$ in the notation). Let us start with a first identification:

Proposition 3.7 ($\mathscr{A}'_q \simeq H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$)**.** *The following properties hold:*

(a) $\mathscr{A}'_q$ *is a closed subspace of* $(\mathscr{A}_p)'$*: in particular, it is a Banach space with the norm*

$$\|L\|_{\mathscr{A}'_q} := \Big(\mathsf{pCE}^*_{p,1}(L)\Big)^{1/q} = \sup\Big\{\langle L, f\rangle : f \in \mathscr{A},\ \mathsf{pCE}_{p,1}(f) \le 1\Big\}.$$

(b) *A linear functional* L *on* $\mathscr{A}$ *belongs to* $\mathscr{A}'_q$ *if and only if for every* $\varepsilon > 0$ *there exists a constant* $\kappa > 0$ *such that*

$$\left|\langle L, f\rangle\right|^p \le \varepsilon\, \mathsf{pCE}_p(f) + \kappa\|f\|^p_{L^p} \quad \text{for every } f \in \mathscr{A}. \tag{3.9}$$

In this case (3.8b) *holds in the stronger form where* lim inf *is replaced by* lim sup.

(c) *Every linear functional* $L \in \mathscr{A}'_q$ *admits a unique extension* $\tilde L$ *in* $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$*. The map* $L \mapsto \tilde L$ *is a surjective isometry between* $\mathscr{A}'_q$ *and* $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$*, which is therefore a closed subspace of* $H^{-1,q}(\mathbb{X})$*. In particular, if* $L, L' \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ *coincide on* $\mathscr{A}$ *then* $L = L'$.

Proof. **(a)** It is sufficient to prove that $\mathscr{A}'_q$ is closed in the Banach space $(\mathscr{A}_p)'$. Let L be an element of the closure and for every $\varepsilon > 0$ choose elements $L_\varepsilon \in \mathscr{A}'_q$ such that $\|L - L_\varepsilon\|_{(\mathscr{A}_p)'} \le \varepsilon$. For every sequence $f_n \in \mathscr{A}$ as in (3.8b) we have

$$\liminf_{n\to\infty} \left|\langle L, f_n\rangle\right| \le \limsup_{n\to\infty} \left|\langle L - L_\varepsilon, f_n\rangle\right| + \liminf_{n\to\infty} \left|\langle L_\varepsilon, f_n\rangle\right| \le C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain that $L \in \mathscr{A}'_q$.

(b) If L satisfies (3.9) and $f_n \in \mathscr{A}$ is a sequence as in (3.8b) we have

$$\limsup_{n\to\infty} \left|\langle L, f_n\rangle\right|^p \le \limsup_{n\to\infty} \Big(\varepsilon\, \mathsf{pCE}_p(f_n) + \kappa\|f_n\|^p_{L^p}\Big) \le \varepsilon\, C;$$

since ε is arbitrary we deduce that $\limsup_{n\to\infty} \left|\langle L, f_n\rangle\right| = 0$, thus (3.8b) in the stronger form.

In order to prove the converse implication, we argue by contradiction by assuming that there exists $\varepsilon > 0$ and a sequence $f_n \in \mathscr{A}$ such that

$$\left|\langle L, f_n\rangle\right|^p \ge \varepsilon \mathsf{pCE}_p(f_n) + n\|f_n\|^p_{L^p} > 0.$$

By possibly replacing f_n with $f_n(\mathsf{pCE}_{p,1}(f_n))^{-1/p}$, it is not restrictive to assume that $\mathsf{pCE}_{p,1}(f_n) = 1$; by (3.8a) we have for $n > \varepsilon$

$$\varepsilon \le \varepsilon \mathsf{pCE}_p(f_n) + n\|f_n\|_{L^p}^p \le \left|\langle L, f_n\rangle\right|^p \le D^p$$

so that $\lim_{n\to\infty} \|f_n\|_{L^p} = 0$ but $\liminf_{n\to\infty} \left|\langle L, f_n\rangle\right| \ge \varepsilon^{1/p} > 0$.

(c) In order to define $\tilde{L}$ we fix $f \in H^{1,p}(\mathbb{X})$ and any sequence $f_n \in \mathscr{A}$ such that $f_n \to f$ in $L^p(X, \mathfrak{m})$ with $E^p := \sup \mathsf{pCE}_p(f_n) < \infty$. By (3.9), for every $\varepsilon > 0$ there exists $\kappa > 0$ such that

$$|\langle L, f_n - f_m\rangle| \le 2\varepsilon^{1/p}E + \kappa^{1/p}\|f_n - f_m\|_{L^p}$$

which shows that the sequence $n \mapsto \langle L, f_n\rangle$ satisfies the Cauchy condition and thus admits a limit which we denote by $\langle \tilde{L}, f\rangle$. This notation is justified by the fact that the limit does not depend on the sequence f_n: in fact, if f_n' is another sequence converging to f in $L^p(X, \mathfrak{m})$ with equibounded energy, (3.7) shows that $\lim_{n\to\infty} \langle L, f_n - f_n'\rangle = 0$. It is also easy to check that the map $H^{1,p}(\mathbb{X}) \ni f \mapsto \langle \tilde{L}, f\rangle$ is a linear functional.

In order to show that $\tilde{L}$ is bounded, for every $f \in H^{1,p}(\mathbb{X})$ we select an optimal sequence f_n such that $\mathsf{CE}_p(f) = \lim_{n\to\infty} \mathsf{pCE}_p(f_n)$: by construction

$$\left|\langle \tilde{L}, f\rangle\right| = \lim_{n\to\infty} \left|\langle L, f_n\rangle\right| \le \limsup_{n\to\infty} \|L\|_{\mathscr{A}_q'}\Big(\mathsf{pCE}_{p,1}(f_n)\Big)^{1/p} = \|L\|_{\mathscr{A}_q'}\Big(\mathsf{CE}_{p,1}(f)\Big)^{1/p}$$

so that $\|\tilde{L}\|_{H^{-1,q}(\mathbb{X})} \le \|L\|_{\mathscr{A}_q'}$. On the other hand for every $f \in \mathscr{A}$ with $\mathsf{pCE}_p(f) \le 1$ by choosing the constant sequence $f_n \equiv f$ we get

$$\langle L, f\rangle = \langle \tilde{L}, f\rangle = \|\tilde{L}\|_{H^{-1,q}(\mathbb{X})}\Big(\mathsf{CE}_{p,1}(f)\Big)^{1/p} \le \|\tilde{L}\|_{H^{-1,q}(\mathbb{X})}$$

since $\mathsf{CE}_p(f) \le \mathsf{pCE}_p(f) \le 1$. It follows that $\|L\|_{\mathscr{A}_q'} \le \|\tilde{L}\|_{H^{-1,q}(\mathbb{X})}$ so that the extension map $\iota : L \mapsto \tilde{L}$ is an isometry.

It remains to prove that the image of ι coincides with $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$. Since it is clear that $H^{-1,q}_{\mathrm{pd}}(\mathbb{X}) \subset \iota(\mathscr{A}_q')$, it is sufficient to show the converse inclusion, i.e., that every element $\tilde{L} = \iota(L)$ satisfies (3.7). By linearity, it is not restrictive to check (3.7) for $f = 0$. If $f_n \in H^{1,p}(\mathbb{X})$ has equibounded Cheeger energy and $\lim_{n\to\infty} \|f_n\|_{L^p} = 0$, by the very definition of the Cheeger energy and the definition of $\tilde{L}$ we can find another sequence $g_n \in \mathscr{A}$ such that

$$\mathsf{pCE}_p(g_n) \le \mathsf{CE}_p(f_n) + \frac{1}{n}, \quad \|g_n - f_n\|_{L^p} \le \frac{1}{n}, \quad \left|\langle L, g_n\rangle - \langle \tilde{L}, f_n\rangle\right| \le \frac{1}{n}.$$

Since $L \in \mathscr{A}_q'$ and $\lim_{n\to\infty} \|g_n\|_{L^p} = 0$ we have $\lim_{n\to\infty} \langle L, g_n\rangle = 0$ so that $\lim_{n\to\infty} \langle \tilde{L}, f_n\rangle = 0$. □

Let us now express the dual functionals by a infimal convolution.

Lemma 3.8. *For every $L \in H^{-1,q}(\mathbb{X})$ and every $\alpha \ge 0$, $\beta > 0$ we have*

$$\begin{aligned} \frac{1}{q}\mathsf{CE}^*_{p,\alpha+\beta}(L) &= \sup_{g\in H^{1,p}(\mathbb{X})} \langle L, g\rangle - \frac{1}{p}\mathsf{CE}_{p,\alpha}(g) - \frac{\beta}{p}\|g\|_{L^p}^p \\ &= \min_{f\in L^q(X,\mathfrak{m})} \frac{1}{q}\mathsf{CE}^*_{p,\alpha}(L - f) + \frac{1}{q\beta^{q/p}}\|f\|_{L^q}^q. \end{aligned} \tag{3.10}$$

Proof. (3.10) is a particular case of the duality formula for the sum of two convex functions $\varphi, \psi : W \to (-\infty, +\infty]$

$$(\varphi + \psi)^*(L) = \min_{f \in W} \varphi^*(L - f) + \psi^*(f) \quad \text{for every } L \in W'$$

which holds in every Banach space W whenever there exists a point $w_0 \in W$ such that $\phi(w_0) < \infty$ and ψ is finite and continuous at w_0 by Fenchel-Rockafellar Theorem ([20], see also [9, Theorem 1.12]). Here $W = H^{1,p}(\mathbb{X})$, $\phi(g) := \frac{1}{p}\mathsf{CE}_{p,\alpha}(g)$, $\psi(g) := \frac{\beta}{p}\|g\|^p_{L^p}$. □

We collect in the next proposition a further list of useful properties. We will denote by $\mathrm{J}_p : L^p(X, \mathfrak{m}) \to L^q(X, \mathfrak{m})$ the duality map

$$(\mathrm{J}_p u)(x) := |u(x)|^{p-1} u(x), \qquad \int_X u\, \mathrm{J}_p u \,\mathrm{d}\mathfrak{m} = \|u\|^p_{L^p} = \|\mathrm{J}_p u\|^q_{L^q},$$

and by $\mathrm{A}_p : H^{1,p}(\mathbb{X}) \to \mathfrak{P}(H^{-1,q}(\mathbb{X}))$ the subdifferential of the Cheeger energy with respect to the Sobolev duality

$$L \in \mathrm{A}_p u \quad \Leftrightarrow \quad u \in H^{1,p}(\mathbb{X}), \quad \langle L, v - u\rangle \le \frac{1}{p}\mathsf{CE}_p(v) - \frac{1}{p}\mathsf{CE}_p(u) \quad \text{for every } v \in H^{1,p}(\mathbb{X}). \tag{3.11}$$

Since CE_p is continuous in $H^{1,p}(\mathbb{X})$, $\mathrm{A}_p u \neq \emptyset$ for every $u \in H^{1,p}(\mathbb{X})$ [13, Chap. 1, Prop. 5.3] (notice that A_p is different from the subdifferential of CE_p w.r.t. the L^p-L^q duality pair). The sum

$$\mathrm{Q}_{p,\kappa} := \mathrm{A}_p + \kappa \mathrm{J}_p \quad \text{is the subdifferential in } H^{1,p}(\mathbb{X}) \text{ of } \frac{1}{p}\mathsf{CE}_{p,\kappa}.$$

Proposition 3.9. *We have the following properties*

(a) *For every $L \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ and every $\kappa \ge 0$ we have*

$$\begin{aligned}\frac{1}{q}\mathsf{CE}^*_{p,\kappa}(L) &= \sup_{f \in H^{1,p}(\mathbb{X})} \langle L, f\rangle - \frac{1}{p}\mathsf{CE}_{p,\kappa}(f) \\ &= \sup_{f \in \mathscr{A}} \langle L, f\rangle - \frac{1}{p}\mathsf{CE}_{p,\kappa}(f) &(3.12)\\ &= \sup_{f \in \mathscr{A}} \langle L, f\rangle - \frac{1}{p}\mathsf{pCE}_{p,\kappa}(f) = \frac{1}{q}\mathsf{pCE}^*_{p,\kappa}(L). &(3.13)\end{aligned}$$

(b) *$H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ coincides with the (strong) closure of $V' = L^q(X, \mathfrak{m})$ in $W' = H^{-1,q}(\mathbb{X})$.*

(c) *For every $L \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ and $\kappa > 0$ there exists a unique solution $u_\kappa = \mathrm{Q}^{-1}_{p,\kappa}(L) \in H^{1,p}(\mathbb{X})$ of the problem*

$$\min_{u \in H^{1,p}(\mathbb{X})} \frac{1}{p}\mathsf{CE}_{p,\kappa}(u) - \langle L, u\rangle \tag{3.14}$$

which satisfies

$$\mathrm{Q}_{p,\kappa} u_\kappa = \mathrm{A}_p u_\kappa + \kappa \mathrm{J}_p u_\kappa \ni L, \quad \mathsf{CE}^*_{p,\kappa}(L) = \mathsf{CE}_{p,\kappa}(u_\kappa) = \langle L, u_\kappa\rangle. \tag{3.15}$$

(d) *For every* $L \in H^{-1,q}(\mathbb{X})$ *and* $\kappa > 0$ *there exists a unique function* $f_\kappa := \mathrm{R}_\kappa(L)$ *solving the minimum problem*

$$\frac{1}{q}\mathsf{CE}^*_{p,\kappa}(L) = \min_{f\in L^q(X,\mathfrak{m})} \frac{1}{q}\mathsf{CE}^*_p(L-f) + \frac{1}{q\kappa^{q/p}}\|f\|^q_{L^q} \tag{3.16}$$

The map $\mathrm{R}_\kappa : H^{-1,q}(\mathbb{X}) \to L^q(Z,\mathfrak{m})$ *is strongly continuous. Moreover, if* $L \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ *then* $f_\kappa = \mathrm{R}_\kappa(L) = \kappa \mathrm{J}_p u_\kappa = \kappa\mathrm{J}_p \circ \mathrm{Q}^{-1}_{p,\kappa}(L)$.

(e) *For every* $L \in H^{-1,q}(\mathbb{X})$ *we have*

$$\mathsf{CE}^*_p(L) = \lim_{\kappa\downarrow 0}\mathsf{CE}^*_{p,\kappa}(L) = \sup_{\kappa>0}\mathsf{CE}^*_{p,\kappa}(L).$$

Proof. **(a)** (3.13) (which implies (3.12)) follows by an easy approximation argument combining the definition of CE_p and the continuity property (3.7) and it follows by the same argument at the end of the proof of claim (c) of Proposition 3.7.

(b) Since $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ is a closed subspace of $H^{-1,q}(\mathbb{X})$ and clearly contains $L^q(X,\mathfrak{m})$, it is sufficient to prove that $L^q(X,\mathfrak{m})$ is dense in $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$. For every $n \in \mathbb{N}$ we consider the functional $\mathsf{G}_n := \mathsf{CE}^*_{p,1+n^p}$ and we want to show that

$$\limsup_{n\uparrow\infty} \mathsf{G}_n(L) = 0; \tag{3.17}$$

by using (3.10) (with $\alpha := 1, \beta := n^p$), (3.17) is in fact equivalent to the density of $L^q(X,\mathfrak{m})$ in $H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$. By the first formula of (3.10), for every $\varepsilon > 0$ we can find $g_n \in H^{1,p}(\mathbb{X})$ such that

$$\frac{1}{q}\mathsf{G}_n(L) \le \langle L, g_n\rangle - \frac{1}{p}\mathsf{CE}_{p,1}(g_n) - \frac{n^p}{p}\|g_n\|^p_{L^p} + \varepsilon. \tag{3.18}$$

Since

$$\langle L, g_n\rangle \le \frac{2^{q/p}}{q}\mathsf{CE}^*_{p,1}(L) + \frac{1}{2p}\mathsf{CE}_{p,1}(g_n)$$

and $\mathsf{G}_n(L) \ge 0$, we obtain

$$\frac{1}{2p}\mathsf{CE}_{p,1}(g_n) + \frac{n^p}{p}\|g_n\|^p_{L^p} \le \varepsilon + \frac{2^{q/p}}{q}\mathsf{CE}^*_{p,1}(L)$$

so that $\mathsf{CE}_{p,1}(g_n)$ is uniformly bounded and $\|g_n\|_{L^p} \to 0$ as $n \to \infty$. By (3.7) we conclude that $\lim_{n\to\infty}\langle L, g_n\rangle = 0$ and therefore (3.18) yields $\limsup_{n\to\infty}\mathsf{G}_n(L) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (3.17).

(c) The existence of a solution $u_\kappa \in H^{1,p}(\mathbb{X})$ to (3.14) follows by (3.7) and the Direct method of the Calculus of Variations. Let us take a minimizing sequence $f_n \in H^{1,p}(\mathbb{X})$ such that

$$\lim_{n\to\infty}\frac{1}{p}\mathsf{CE}_{p,\kappa}(f_n) - \langle L, f_n\rangle = M := \inf_{f\in H^{1,p}(\mathbb{X})}\frac{1}{p}\mathsf{CE}_{p,\kappa}(f_n) - \langle L, f_n\rangle. \tag{3.19}$$

Since f_n is uniformly bounded in $H^{1,p}(\mathbb{X})$, up to extracting a suitable subsequence (still denoted by f_n), it is not restrictive to assume that f_n is converging to a function $f \in H^{1,p}(\mathbb{X})$ weakly in $L^p(X, \mathfrak{m})$ and

$$S = \lim_{n\to\infty} \|f_n\|_{L^p} = \lim_{n\to\infty} \Big[\frac{p}{\kappa}\Big(M + \langle L, f_n\rangle - \frac{1}{p}\mathsf{CE}_p(f_n)\Big)\Big]^{1/p}. \tag{3.20}$$

We prove that f_n is a Cauchy sequence: by the uniform convexity of the $L^p(X, \mathfrak{m})$-norm, for every $\varepsilon > 0$ there exist $S' < S < S''$ such that for every $h_1, h_2 \in L^p(X, \mathfrak{m})$

$$\|h_1\|_{L^p} \le S'', \ \|h_2\|_{L^p} \le S'', \quad \Big\|\frac{h_1 + h_2}{2}\Big\|_{L^p} \ge S' \quad \Rightarrow \quad \|h_2 - h_1\| \le \varepsilon. \tag{3.21}$$

By (3.20) we can find $\bar n \in \mathbb{N}$ such that for every $n \ge \bar n$ and

$$\|f_n\|_{L^p} \le S'', \qquad M - \frac{1}{p}\mathsf{CE}_p(f_n) + \langle L, f_n\rangle \ge \frac{\kappa}{p}(S')^p.$$

For every $m, n \ge \bar n$ we thus get

$$\begin{aligned} M &\le \frac{1}{p}\mathsf{CE}_{p,\kappa}\Big(\frac{1}{2}(f_n + f_m)\Big) - \frac{1}{2}\langle L, f_n + f_m\rangle \\ &\le \frac{1}{2}\Big(\frac{1}{p}\mathsf{CE}_p(f_n) - \langle L, f_n\rangle + \frac{1}{p}\mathsf{CE}_p(f_m) - \langle L, f_m\rangle\Big) + \frac{\kappa}{p}\Big\|\frac{f_n + f_m}{2}\Big\|_{L^p}^p \\ &\le M - \frac{\kappa}{p}(S')^p + \frac{\kappa}{p}\Big\|\frac{f_n + f_m}{2}\Big\|_{L^p}^p, \end{aligned}$$

and therefore

$$\Big\|\frac{f_n + f_m}{2}\Big\|_{L^p} \ge S'$$

so that (3.21) yields $\|f_n - f_m\|_{L^p} \le \varepsilon$ for every $n, m \ge \bar n$. We deduce that $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$; since f_n is uniformly bounded in $H^{1,p}(\mathbb{X})$, (3.7) yields $\lim_{n\to\infty} \langle L, f_n\rangle = \langle L, f\rangle$ and the lower semicontinuity of the Cheeger energy yields $\mathsf{CE}_p(f) \le \liminf_{n\to\infty} \mathsf{CE}_p(f_n)$. By (3.19) we conclude that $\frac{1}{p}\mathsf{CE}_{p,\kappa}(f) - \langle L, f\rangle = M$ so that f is the unique minimizer of (3.14).

(d) (3.16) is an immediate consequence of (3.10) with $\alpha = 0$ and $\beta = \kappa$.

In order to prove the continuity of R_κ, let $L_n \in H^{-1,q}(\mathbb{X})$ be a sequence strongly converging to L and let $f_n = \mathrm{R}_\kappa(L_n) \in L^q(X, \mathfrak{m})$. Since $\mathsf{CE}^*_{p,\kappa}(L_n)$ is uniformly bounded, we obviously get a uniform bound for $\mathsf{CE}^*_{p,\kappa}(L_n - f_n)$ and $\|f_n\|_{L^q}$. Let $f \in L^q(X, \mathfrak{m})$ be any weak L^q limit point of f_n, e.g., attained along a subsequence $f_{n(j)}$. Since $\lim_{n\to\infty} \mathsf{CE}^*_{p,\kappa}(L_n) = \mathsf{CE}^*_{p,\kappa}(L)$ and

$$\liminf_{j\to\infty} \mathsf{CE}^*_{p,\kappa}(L_{n(j)} - f_{n(j)}) \ge \mathsf{CE}^*_{p,\kappa}(L - f), \qquad \liminf_{j\to\infty} \|f_{n(j)}\|^q_{L^q} \ge \|f\|^q_{L^q}$$

we deduce that

$$\mathsf{CE}^*_{p,\kappa}(L) \ge \mathsf{CE}^*_{p,\kappa}(L - f) + \frac{1}{q\kappa^{q/p}}\|f\|^q_{L^q}$$

so that $f = \mathrm{R}_\kappa(L)$. Since $\mathrm{R}_\kappa(L)$ is the unique weak limit point of the sequence f_n in L^q, we conclude that $f_n \rightharpoonup \mathrm{R}_\kappa(L)$ in $L^q(X, \mathfrak{m})$. The same variational argument also shows that $\limsup_{n\to\infty} \|f_n\|_{L^q} \le \|f\|_{L^q}$ so that the convergence is strong.

Finally, if $L \in H^{-1,q}_{\rm pd}(\mathbb{X})$, f_κ is the (unique) minimizer of (3.10) and u_κ is the (unique) minimizer of (3.14), we get

$$\frac{1}{q}\mathsf{CE}_p(L - f_\kappa) + \frac{1}{p}\mathsf{CE}_p(u_\kappa) - \langle L - f_\kappa, u_\kappa\rangle + \frac{1}{q\kappa^{q/p}}\|f_\kappa\|^q_{L^q} + \frac{\kappa}{p}\|u_\kappa\|^p_{L^p} - \langle f_\kappa, u_\kappa\rangle = 0$$

which yields

$$\mathrm{A}_p u_\kappa = L - f_\kappa, \quad f_\kappa = \kappa \mathrm{J}_p u_\kappa.$$

(e) Since the map $\kappa \mapsto \mathsf{CE}^*_{p,\kappa}(L)$ is nonincreasing, we have $\lim_{\kappa\downarrow 0}\mathsf{CE}^*_{p,\kappa}(L) = \sup_{\kappa>0}\mathsf{CE}^*_{p,\kappa}(L) \le \mathsf{CE}^*_p(L)$. On the other hand, for every $f \in H^{1,p}(\mathbb{X})$ and $\varepsilon > 0$, choosing $\kappa > 0$ sufficiently small so that $\frac{\kappa}{p}\|f\|^p_{L^p} \le \varepsilon$ we get

$$\begin{aligned}\langle L, f\rangle - \frac{1}{p}\mathsf{CE}_p(f) - \varepsilon &\le \langle L, f\rangle - \frac{1}{p}\mathsf{CE}_p(f) - \frac{\kappa}{p}\|f\|^p_{L^p} = \langle L, f\rangle - \frac{1}{p}\mathsf{CE}_{p,\kappa}(f) \\ &\le \frac{1}{q}\mathsf{CE}^*_{p,\kappa}(L) \le \frac{1}{q}\sup_{\kappa>0}\mathsf{CE}^*_{p,\kappa}(L)\end{aligned}$$

Since the inequality holds for every $\varepsilon > 0$ and every $f \in H^{1,p}(\mathbb{X})$, we obtain the converse inequality $\mathsf{CE}^*_p(L) \le \sup_{\kappa>0}\mathsf{CE}^*_{p,\kappa}(L)$. □

Proposition 3.9 yields the following interesting duality result, which is also related to the theory of derivations discussed in [12].

Corollary 3.10 ($H^{1,p}(\mathbb{X})$ is the dual of a Banach space)**.** *$H^{1,p}(\mathbb{X})$ can be isometrically identified with the dual of $H^{-1,q}_{\rm pd}(\mathbb{X})$. In particular, if $L^q(X, \mathfrak{m})$ is a separable space, $H^{1,p}(\mathbb{X})$ is the dual of a separable Banach space.*

Proof. Let z be a bounded linear functional on $H^{-1,q}_{\rm pd}(\mathbb{X})$. Since $L^q(X, \mathfrak{m})$ is continuously and densely imbedded in $H^{-1,q}_{\rm pd}(\mathbb{X})$, for every $f \in L^q(X, \mathfrak{m})$ $\langle z, f\rangle \le \|z\|_{H^{-1,q}_{\rm pd}(\mathbb{X})}\|f\|_{L^q}$, so that there exists a unique $u = \iota(\zeta) \in L^p(X, \mathfrak{m})$ such that

$$\langle z, f\rangle = \int_X u f \,\mathrm{d}\mathfrak{m} \quad \text{for every } f \in L^q(X, \mathfrak{m}).$$

By (3.6) and the strong density of $L^q(X, \mathfrak{m})$ in $H^{-1,q}_{\rm pd}(\mathbb{X})$

$$\begin{aligned}\frac{1}{p}\mathsf{CE}_{p,1}(u) &= \sup_{f\in L^q(X,\mathfrak{m})}\int_X u f \,\mathrm{d}\mathfrak{m} - \frac{1}{q}\mathsf{CE}^*_{p,1}(f) = \sup_{f\in L^q(X,\mathfrak{m})}\langle z, f\rangle - \frac{1}{q}\|f\|^q_{H^{-1,q}(\mathbb{X})} \\ &= \sup_{f\in H^{-1,q}_{\rm pd}(\mathbb{X})}\langle z, f\rangle - \frac{1}{q}\|f\|^q_{H^{-1,q}_{\rm pd}(\mathbb{X})} = \frac{1}{p}\|z\|^p_{(H^{-1,q}_{\rm pd}(\mathbb{X}))'}.\end{aligned}$$

It follows that ι is an isometry from the dual of $H^{-1,q}_{\rm pd}(\mathbb{X})$ and $H^{1,p}(\mathbb{X})$. Since ι is clearly surjective, we conclude. □

Remark 3.11 ($H^{1,p}(\mathbb{X})$ as Gagliardo completion [16])**.** Recall that if $(A, \|\cdot\|_A)$ is a normed vector space continuously imbedded in a Banach space $(V, \|\cdot\|_V)$, the *Gagliardo completion* $A^{V,c}$ is the Banach space defined by

$$W := \Big\{ v \in V : \exists (a_n)_n \subset A,\ \lim_{n\to\infty} \|a_n - v\|_V = 0,\ \sup_n \|a_n\|_A < \infty \Big\}$$

with norm

$$\|v\|_W := \inf\Big\{ \liminf_{n\to\infty} \|a_n\|_A : a_n \in A,\ \lim_{n\to\infty} \|a_n - v\|_V = 0 \Big\}.$$

When $\mathrm{supp}(\mathfrak{m}) = X$, we can identify $\mathscr{A}$ with a vector space A with the norm induced by pCE_p imbedded in $V := L^p(X, \mathfrak{m})$; it is immediate to check that $H^{1,p}(\mathbb{X})$ coincides with the Gagliardo completion of A in V. When A (and therefore W) is strongly dense in V, we can identify the dual V' of V as a subset of the dual W' of W and we can define the set W'_{pd} as the closure of V' in W'. If V is uniformly convex, the same statements and characterizations given in Propositions 3.7 and 3.9 hold in this more abstract setting. In particular, W can be isometrically identified with the dual of W'_{pd}.

3.3. Radon measures with finite (dual) energy

The following result provides a useful criterion to check if a linear functional on $\mathscr{A}$ belongs to $\mathscr{A}'_q$. Let us first recall that a subset $F \subset L^1(X, \mathfrak{m})$ is weakly relatively compact in $L^1(X, \mathfrak{m})$ if and only if it satisfies one of the following equivalent properties [13, Chap. VIII, Theorem 1.3]:

a) for all $\varepsilon > 0$ there exists $m \geq 0$ such that

$$\int_{|f(x)|\geq m} |f(x)| \,\mathrm{d}\mathfrak{m}(x) \leq \varepsilon \quad \text{for every } f \in F;$$

b) (Equiintegrability) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every Borel set $B \subset X$

$$\int_B |f(x)| \,\mathrm{d}\mathfrak{m}(x) \leq \varepsilon \quad \text{whenever } f \in F \text{ and } \mathfrak{m}(B) \leq \delta.$$

c) (Uniform superlinear estimate) There exists a positive, increasing, l.s.c. and convex function $\Phi : [0, \infty) \to [0, \infty)$ such that $\lim_{r\to\infty} \Phi(r)/r = +\infty$ and

$$\sup_{f\in F} \int_X \Phi(|f(x)|) \,\mathrm{d}\mathfrak{m}(x) < \infty.$$

Proposition 3.12. *Let L be a linear functional on $\mathscr{A}$ satisfying* (3.8a). *If for every sequence $f_n \in \mathscr{A}$ satisfying*

$$-1 \leq f_n \leq 1, \quad \lim_{n\to\infty} \|f_n\|_{L^p(X,\mathfrak{m})} = 0, \quad \{(\mathrm{lip}\, f_n)^p : n \in \mathbb{N}\} \quad \textit{is equiintegrable} \tag{3.22}$$

one has $\liminf_{n\to\infty} |\langle L, f_n\rangle| = 0$*, then* $L \in \mathscr{A}'_q$.

Proof. We split the proof in two steps.
Claim 1: *if L is a linear functional on $\mathscr{A}$ satisfying* (3.8a) *and for every sequence* $(f_n)_{n\in\mathbb{N}} \subset \mathscr{A}$

$$\lim_{n\to\infty} \|f_n\|_{L^p} = 0 \textit{ and } \{(\mathrm{lip}\, f_n)^p : n \in \mathbb{N}\} \textit{ equiintegrable} \quad \Rightarrow \quad \liminf_{n\to\infty} |\langle L, f_n\rangle| = 0 \tag{3.23}$$

then $L \in \mathscr{A}'_q$.

We argue by contradiction and we assume that there exists a sequence $f_n \in \mathscr{A}$ such that

$$\mathsf{pCE}_{p,1}(f_n) \le C, \quad \lim_{n\to\infty} \|f_n\|_{L^p} = 0 \quad \text{and} \quad \liminf_{n\to\infty} |\langle L, f_n\rangle| > 0.$$

By possibly changing the sign of f_n it is not restrictive to assume that $\langle L, f_n\rangle \ge c > 0$ for every $n \in \mathbb{N}$. Applying Mazur Lemma we find coefficients $\alpha_{n,m} \ge 0$, $n \in \mathbb{N}$, $0 \le m \le M(n)$ such that $g_n := \sum_{m=0}^{M(n)} \alpha_{n,m} \operatorname{lip} f_{n+m}$ is strongly converging in $L^p(X, \mathfrak{m})$. Thus $n \mapsto g_n^p$ is strongly converging in $L^1(X, \mathfrak{m})$ and it is therefore equi-integrable.

We now consider $\tilde{f}_n := \sum_{m=0}^{M(n)} \alpha_{n,m} f_{n+m}$. By construction

$$\langle L, \tilde{f}_n\rangle = \sum_{m=0}^{M(n)} \alpha_{n,m} \langle L, f_{n+m}\rangle \ge c > 0, \quad \lim_{n\to\infty} \|\tilde{f}_n\|_{L^p} = 0 \tag{3.24}$$

and

$$\operatorname{lip} \tilde{f}_n \le \sum_{m=0}^{M(n)} \alpha_{n,m} \operatorname{lip} f_{n+m} = g_n$$

so that (3.23) yields $\liminf_{n\to\infty} \langle L, \tilde{f}_n\rangle = 0$, which contradicts the first inequality of (3.24).

Claim 2: *it is sufficient to prove the implication* (3.23) *for sequences taking values in* $[-1, 1]$. Let $f_n \in \mathscr{A}$ as in (3.23), $m_n := \sup |f_n|$, $E^p := \sup_n \mathsf{pCE}_p(f_n) < \infty$, and let $\phi \in \mathrm{C}^1(\mathbb{R})$ be an odd function such that

$$\phi(r) = r \quad \text{if } |r| \le 1/2, \quad -1 \le \phi(r) \le 1, \quad 0 \le \phi'(r) \le 1 \quad \text{for every } r \in \mathbb{R}. \tag{3.25}$$

Let us fix $\varepsilon > 0$ and $\delta := \varepsilon/3E$ so that $E^p\delta^p \le \varepsilon^p/3$. For every choice of $n \in \mathbb{N}$ we can find an odd polynomial P_n such that (see e.g., [21, Lemma 2.23])

$$-1 \le P_n(r) \le 1, \quad 0 \le P_n'(r) \le 1, \quad |P_n'(r) - \phi'(r)| \le \delta \quad \text{for every } r \in [-m_n, m_n],$$

We set

$$g_n := P_n \circ f_n, \quad h_n := f_n - g_n = Q_n(f_n) \quad \text{where} \quad Q_n(r) = r - P_n(r);$$

notice that g_n and h_n belong to $\mathscr{A}$ as well. Since $\operatorname{lip} g_n \le \operatorname{lip} f_n$ and g_n takes values in $[-1, 1]$, by assumption we have $\liminf_{n\to\infty} |\langle L, g_n\rangle| = 0$. On the other hand, $\|h_n\|_{L^p} \le \|f_n\|_{L^p}$, $\operatorname{lip} h_n \le \operatorname{lip} f_n$, and $\operatorname{lip} h_n(x) \le Q_n'(f_n(x)) \operatorname{lip} f_n(x) \le \delta \operatorname{lip} f_n(x)$ whenever $|f_n(x)| < 1/2$. Since Chebichev inequality yields

$$\mathfrak{m}\{|f_n| \ge 1/2\} \le 2^p \|f_n\|^p_{L^p(X,\mathfrak{m})}, \quad \lim_{n\to\infty} \mathfrak{m}\{f_n \ge 1/2\} = 0,$$

we can choose n_0 sufficiently big so that for every $n \ge n_0$

$$\int_{\{|f_n|\ge 1/2\}} (\operatorname{lip} f_n)^p \, \mathrm{d}\mathfrak{m} \le \varepsilon^p/3, \quad \int_X |h_n|^p \, \mathrm{d}\mathfrak{m} \le \varepsilon^p/3,$$

and

$$\int_X (\operatorname{lip} h_n)^p \, \mathrm{d}\mathfrak{m} \le \delta^p \int_{\{|f_n|<1/2\}} (\operatorname{lip} f_n)^p \, \mathrm{d}\mathfrak{m} + \int_{\{|f_n|\ge 1/2\}} (\operatorname{lip} f_n)^p \, \mathrm{d}\mathfrak{m} \le \delta^p E^p + \frac{1}{3}\varepsilon^p \le \frac{2}{3}\varepsilon^p.$$

(3.8a) then yields $|\langle L, h_n\rangle| \le D\varepsilon$ for $n \ge n_0$ and therefore $\liminf_{n\to\infty} |\langle L, f_n\rangle| \le D\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude. □

Our last criterium concerns positive functionals, i.e., satisfying

$$\langle L, f\rangle \geq 0 \quad \text{whenever } f \in \mathscr{A},\ f \geq 0. \tag{3.26}$$

We will see in Theorem 4.7 that they are always induced by a Radon measure.

Theorem 3.13. *If L is a linear functional on $\mathscr{A}$ satisfying* (3.8a) *and* (3.26)*, then $L \in \mathscr{A}'_q$.*

Proof. We apply Proposition 3.12 and refine the last argument of its proof. Let $f_n \in \mathscr{A}$ as in (3.22) with $E^p := \sup_n \mathsf{CE}_p(f_n)$ We select strictly positive parameters $\varepsilon, \kappa > 0$, $\delta := \varepsilon/3E$, the odd function ϕ as in (3.25) with $\phi_\kappa(r) := \kappa\phi(r/\kappa)$, and odd polynomials $\tilde{P}_{\kappa,\varepsilon}$ satisfying

$$-1 \leq \tilde{P}_{\kappa,\varepsilon}(r) \leq 1, \quad 0 \leq \tilde{P}'_{\kappa,\varepsilon}(r) \leq 1, \quad |\tilde{P}'_{\kappa,\varepsilon}(r) - \phi'(r)| \leq \delta \quad \text{if } |r| \leq \frac{1}{\kappa}.$$

We also set $P_{\kappa,\varepsilon}(r) := \kappa\tilde{P}_{\kappa,\varepsilon}(r/\kappa)$, $Q_{\kappa,\varepsilon}(r) = r - P_{\kappa,\varepsilon}(r)$, $g_{n,\kappa} := P_{\kappa,\varepsilon}(f_n)$, $h_{n,\kappa} := f_n - g_{n,\kappa} = Q_{\kappa,\varepsilon}(f_n)$. By (3.26) and observing that $-\kappa \leq g_{n,\kappa} \leq \kappa$ and the constant function $1 \in \mathscr{A}$ has finite Cheeger energy, we have

$$-\kappa\langle L, 1\rangle = -\langle L, \kappa\rangle \leq \langle L, g_{n,\kappa}\rangle \leq \langle L, \kappa\rangle = \kappa\langle L, 1\rangle.$$

On the other hand, since $0 \leq Q'_{n,\kappa} \leq 1$ if $|r| \leq 1$ and $|Q'_{n,\kappa}(r)| \leq \delta$ if $|r| \leq \kappa/2$, we have

$$|h_{n,\kappa}| \leq |f_n|, \quad \operatorname{lip} h_{n,\kappa} \leq \operatorname{lip} f_n, \quad \operatorname{lip} h_{n,\kappa} \leq \delta \operatorname{lip} f_n \quad \text{if } |f_n| < \kappa/2.$$

Applying Chebychev inequality

$$\mathfrak{m}\{|f_n| \geq \kappa/2\} \leq \frac{2^p \|f_n\|^p_{L^p(X,\mathfrak{m})}}{\kappa^p},$$

we can find n_0 (depending on ε, κ) sufficiently big such that

$$\int_X |h_{n,\kappa}|^p \,\mathrm{d}\mathfrak{m} \leq \varepsilon^p/3, \quad \int_{\{f_n \geq \kappa/2\}} (\operatorname{lip} f_n)^p \,\mathrm{d}\mathfrak{m} \leq \varepsilon^p/3 \quad \text{for every } n \geq n_0,$$

so that

$$\int_X (\operatorname{lip} h_{n,\kappa})^p \,\mathrm{d}\mathfrak{m} \leq \delta^p \int_{\{|f_n| < \kappa/2\}} (\operatorname{lip} f_n)^p \,\mathrm{d}\mathfrak{m} + \int_{\{|f_n| \geq 1/2\}} (\operatorname{lip} f_n)^p \,\mathrm{d}\mathfrak{m} \leq \delta^p E^p + \frac{1}{3}\varepsilon^p \leq \frac{2}{3}\varepsilon^p.$$

By (3.8a) it follows that

$$\liminf_{n\to\infty} |\langle L, f_n\rangle| \leq \liminf_{n\to\infty} \big(|\langle L, g_{n,\kappa}\rangle| + |\langle L, h_{n,\kappa}\rangle|\big) \leq \kappa\langle L, 1\rangle + D\varepsilon.$$

Since ε, κ are arbitrary, we get $\liminf_{n\to\infty} |\langle L, f_n\rangle| = 0$. □

Definition 3.14 (Measure with finite dual energy)**.** *A Radon measure $\mu \in \mathcal{M}_+(X)$ has finite energy if there exists a constant $D > 0$ such that*

$$\int_X f \,\mathrm{d}\mu \leq D\Big(\mathsf{pCE}_{p,1}(f)\Big)^{1/p} \quad \textit{for every nonnegative } f \in \operatorname{Lip}_b(X, \tau, \mathsf{d}). \tag{3.27}$$

Corollary 3.15 (Measures with finite dual energy belong to $\mathscr{A}_q'(\mathbb{X})$)**.** *If $\mu \in \mathcal{M}_+(X)$ has finite energy then the linear functional $f \mapsto \int_X f\,\mathrm{d}\mu$ on $\mathscr{A}(\mathbb{X}) = \mathrm{Lip}_b(X,\tau,\mathsf{d})$ belongs to $\mathscr{A}_q'(\mathbb{X})$ and can be uniquely extended to a functional $L_\mu \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ satisfying*

$$\mathsf{CE}^*_{p,\kappa}(L_\mu) = \mathsf{pCE}^*_{p,\kappa}(\mu) \quad \textit{for every } \kappa > 0. \tag{3.28}$$

In particular there exists a unique element $u_\kappa = \mathrm{Q}^{-1}_{p,\kappa}(L_\mu) \in H^{1,p}(\mathbb{X})$ minimizing (3.14) *with $L = L_\mu$. u_κ satisfies the variational inequality*

$$\int_X f\,\mathrm{d}\mu - \kappa \int_X \mathrm{J}_p(u_\kappa) f\,\mathrm{d}\mathfrak{m} \le \frac{1}{p}\mathsf{CE}_p(u_\kappa + f) - \frac{1}{p}\mathsf{CE}(u_\kappa) \quad \textit{for every } f \in \mathrm{Lip}_b(X,\tau,\mathsf{d}). \tag{3.29}$$

Proof. We can apply Theorem 3.13 with $\mathscr{A} = \mathrm{Lip}_b(X,\tau,\mathsf{d})$. Clearly (3.26) holds; by decomposing $f \in \mathrm{Lip}_b(X,\tau,\mathsf{d})$ as the difference $f = f_+ - f_-$ of its positive and negative part and recalling that $\mathsf{pCE}_p(f_\pm) \le \mathsf{pCE}_p(f)$, (3.27) yields (3.8a) with constant $2D$. (3.29) follows by (3.15) and the definition of A_p given in (3.11). □

4. Dynamic representation of Radon measures with finite energy

4.1. Nonparametric dynamic plans and their barycentric entropy

Definition 4.1 ((Nonparametric) dynamic plans)**.** *A (nonparametric) dynamic plan is a Radon measure $\boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X))$ on $\mathrm{RA}(X)$ such that*

$$\boldsymbol{\pi}(\ell) := \int_{\mathrm{RA}(X)} \ell(\gamma)\,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty. \tag{4.1}$$

Using the universally Lusin-measurable map $R : \mathrm{RA}(X) \to \mathrm{BVC}_c([0,1];X)$ in (2.7) we can also lift $\boldsymbol{\pi}$ to a Radon measure $\tilde{\boldsymbol{\pi}} = R_\sharp\boldsymbol{\pi}$ on the subset $\mathrm{BVC}_c([0,1];X)$ of $\mathrm{BVC}([0,1];X)$ defined in (2.6). Conversely, any Radon measure $\tilde{\boldsymbol{\pi}}$ on $\mathrm{C}([0,1];(X,\tau))$ concentrated on $\mathrm{BVC}([0,1];X)$ yields the Radon measure $\boldsymbol{\pi} := \mathfrak{q}_\sharp\tilde{\boldsymbol{\pi}}$ on $\mathrm{RA}(X)$. Notice that $\mathfrak{q}_\sharp(R_\sharp\boldsymbol{\pi}) = \boldsymbol{\pi}$.

If $\boldsymbol{\pi}$ is a dynamic plan in $\mathcal{M}_+(\mathrm{RA}(X))$, thanks to Theorem 2.6(e) and Fubini's Theorem [11, Chap. II-14], we can define the Borel measure $\mu_{\boldsymbol{\pi}} := \mathrm{Proj}(\boldsymbol{\pi}) \in \mathcal{M}_+(X)$ by the formula

$$\int f\,\mathrm{d}\mu_{\boldsymbol{\pi}} := \int\!\!\int_\gamma f\,\mathrm{d}\boldsymbol{\pi}(\gamma) \quad \text{for every bounded Borel function } f : X \to \mathbb{R}.$$

$\mu_{\boldsymbol{\pi}}$ is a Radon measure with total mass $\boldsymbol{\pi}(\ell)$ given by (4.1) [21, § 8] and it can also be considered as the integral w.r.t. $\boldsymbol{\pi}$ of the Borel family of measures ν_γ, $\gamma \in \mathrm{RA}(X)$ [11, Chap. II-13], in the sense that

$$\int_X f\,\mathrm{d}\mu_{\boldsymbol{\pi}}(x) = \int_{\mathrm{RA}(X)} \Big(\int_X f\,\mathrm{d}\nu_\gamma\Big)\,\mathrm{d}\boldsymbol{\pi}(\gamma).$$

Recall that $p, q \in (1,\infty)$ is a fixed pair of conjugate exponents.

Definition 4.2. *We say that $\boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X))$ has barycenter in $L^q(X,\mathfrak{m})$ if there exists $h \in L^q(X,\mathfrak{m})$ such that $\mu_{\boldsymbol{\pi}} = h\mathfrak{m}$, or, equivalently, if*

$$\int\int_\gamma f\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int fh\,\mathrm{d}\mathfrak{m} \quad \textit{for every bounded Borel function } f : X \to \mathbb{R},$$

and we call $\mathrm{Bar}_q(\boldsymbol{\pi}) := \|h\|_{L^q(X,\mathfrak{m})}$ the barycentric q-entropy of $\boldsymbol{\pi}$. We will denote by $\mathcal{B}_q(\mathrm{RA}(X))$ the set of all plans with barycenter in $L^q(X,\mathfrak{m})$ and we will set $\mathrm{Bar}_q(\boldsymbol{\pi}) := +\infty$ if $\boldsymbol{\pi} \notin \mathcal{B}_q(\mathrm{RA}(X))$.

$\mathrm{Bar}_q : \mathcal{M}_+(\mathrm{RA}(X)) \to [0,+\infty]$ is a convex and positively 1-homogeneous functional, which is lower semicontinuous w.r.t. the weak topology of $\mathcal{M}_+(\mathrm{RA}(X))$, since it can also be characterized as the L^q entropy of the projected measure $\mu_{\boldsymbol{\pi}} = \mathrm{Proj}(\boldsymbol{\pi})$ with respect to $\mathfrak{m}$:

$$\mathrm{Bar}_q^q(\boldsymbol{\pi}) = \mathscr{L}_q(\mu_{\boldsymbol{\pi}}|\mathfrak{m}),$$

where for an arbitrary $\sigma \in \mathcal{M}_+(X)$

$$\mathscr{L}_q(\sigma|\mathfrak{m}) := \begin{cases} \displaystyle\int_X \Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\mathfrak{m}}\Big)^q \,\mathrm{d}\mathfrak{m} & \text{if } \sigma \ll \mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

Notice that $\mathrm{Bar}_q(\boldsymbol{\pi}) = 0$ iff $\boldsymbol{\pi}$ is concentrated on the set of constant arcs in $\mathrm{RA}(X)$.

For every $\mu_0, \mu_1 \in \mathcal{M}_+(X)$ we will consider the (possibly empty) set

$$\Pi(\mu_0,\mu_1) := \Big\{\boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X)) : \ (\mathsf{e}_i)_\sharp \boldsymbol{\pi} = \mu_i\Big\}$$

and we define

$$\mathscr{D}_q(\mu_0,\mu_1) := \inf\Big\{\mathrm{Bar}_q^q(\boldsymbol{\pi}) : \boldsymbol{\pi} \in \Pi(\mu_0,\mu_1)\Big\}, \tag{4.3}$$

with the usual convention $\mathscr{D}_q(\mu_0,\mu_1) := +\infty$ if $\Pi(\mu_0,\mu_1) = \emptyset$. Clearly $\mathscr{D}_q(\mu_0,\mu_1) = +\infty$ if $\mu_0(X) \neq \mu_1(X)$.

Assuming that (X,d) is complete, it is possible to show ([21, § 11]) that whenever $\mathscr{D}_q(\mu_0,\mu_1) < +\infty$ the infimum in (4.3) is attained and the set of optimal plans $\Pi_o(\mu_0,\mu_1)$ is a compact convex subset of $\mathcal{M}_+(\mathrm{RA}(X))$. Morever, for every optimal $\boldsymbol{\pi} \in \Pi_o(\mu_0,\mu_1)$ the measure $\mu_o = \mathrm{Proj}(\boldsymbol{\pi})$ is independent of $\boldsymbol{\pi}$.

$\mathscr{D}_q$ provides an important representation for the dual of the p-Cheeger energy.

Theorem 4.3 ([21, Thm. 11.8])**.** *For every $\mu_0, \mu_1 \in \mathcal{M}_+(X)$ we have*

$$\mathscr{D}_q(\mu_0,\mu_1) = \mathsf{pCE}_p^*(\mu_0 - \mu_1). \tag{4.4}$$

Remark 4.4. Let $\mu = \mu_0 - \mu_1$ with $\mu_i \in \mathcal{M}_+(X)$ and let $\mu_+, \mu_- \in \mathcal{M}_+(X)$ be mutually singular Radon measures providing the Jordan decomposition of μ as $\mu_+ - \mu_-$ with $\mu' = \mu_0 - \mu_+ = \mu_1 - \mu_- \in \mathcal{M}_+(X)$. (4.4) shows that

$$\mathscr{D}_q(\mu_0,\mu_1) = \mathsf{pCE}_p^*(\mu_0 - \mu_1) = \mathsf{pCE}_p^*(\mu_+ - \mu_-) = \mathscr{D}_q(\mu_+,\mu_-).$$

Denoting by $\mathfrak{c} : X \to \mathrm{RA}(X)$ the map that at every point x associates the constant curve taking values x, if $\boldsymbol{\pi}_o \in \Pi_o(\mu_+,\mu_-)$ and $\boldsymbol{\pi}' := \mathfrak{c}_\sharp \mu'$, it is easy to check that $\boldsymbol{\pi} := \boldsymbol{\pi}_o + \boldsymbol{\pi}' \in \Pi_o(\mu_0,\mu_1)$.

4.2. Dynamic representation of the dual energy

Definition 4.5. *For every nonparametric dynamic plan $\boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X))$ and $\kappa > 0$ we define*

$$\mathscr{E}_{q,\kappa}(\boldsymbol{\pi}) := \mathrm{Bar}_q^q(\boldsymbol{\pi}) + \kappa^{-q/p}\mathscr{L}_q((\mathsf{e}_1)_\sharp\boldsymbol{\pi}|\mathfrak{m}).$$

For every $\mu \in \mathcal{M}_+(X)$ we set

$$\mathscr{D}_{q,\kappa}(\mu) := \inf\Big\{\mathscr{E}_{q,\kappa}(\boldsymbol{\pi}) : \boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X)), \quad (\mathsf{e}_0)_\sharp\boldsymbol{\pi} = \mu\Big\}. \tag{4.5}$$

Theorem 4.6. *For every $\mu \in \mathcal{M}_+(X)$ we have*

$$\mathscr{D}_{q,\kappa}(\mu) = \mathsf{pCE}^*_{p,\kappa}(\mu).$$

Moreover, if one of the above quantities is finite

(a) *The infimum in* (4.5) *is attained and there exists a unique pair of functions $f_\kappa, g_\kappa \in L^q(X, \mathfrak{m})$ such that for every optimal plan $\boldsymbol{\pi}$*

$$g_\kappa \mathfrak{m} = \mathrm{Proj}(\boldsymbol{\pi}), \quad \mu_\kappa = f_\kappa \mathfrak{m} = (\mathsf{e}_1)_\sharp \boldsymbol{\pi}, \quad \boldsymbol{\pi} \in \Pi_o(\mu, \mu_\kappa).$$

(b) *There exists a unique solution $u_\kappa = \mathrm{Q}^{-1}_{p,\kappa}(L_\mu)$ of*

$$\mathrm{A}_p u + \kappa \mathrm{J}_p u \ni L_\mu$$

and it satisfies

$$\mathrm{J}_p(|\mathrm{D}u|_\star) = g_\kappa, \quad \kappa \mathrm{J}_p u = f_\kappa \tag{4.6}$$

$$\langle L_\mu, u\rangle = \mathsf{CE}_{p,\kappa}(u) = \mathsf{CE}^*_{p,\kappa}(L_\mu) = \mathsf{pCE}^*_{p,\kappa}(\mu).$$

Moreover, setting $\mu_\pm := (\mu - \mu_\kappa)_\pm$ and $\mu' := \mu - \mu_+ = \mu_\kappa - \mu_-$, we can always choose $\boldsymbol{\pi} = \boldsymbol{\pi}_o + \boldsymbol{\pi}'$ where $\boldsymbol{\pi}_o \in \Pi_o(\mu_+, \mu_-)$, $\boldsymbol{\pi}' = \mathfrak{c}_\sharp \mu'$, $\mathsf{pCE}^*_p(\mu - \mu_\kappa) = \mathrm{Bar}^q_q(\pi) = \mathrm{Bar}^q_q(\pi_o)$.

Proof. By Corollary 3.15 we can extend μ to a functional $L_\mu \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ satisfying (3.28). We can then apply Proposition 3.9 and find nonnegative $f_\kappa \in L^q(X, \mathfrak{m})$ and $u_\kappa \in L^p(X, \mathfrak{m})$ such that

$$\mathsf{CE}^*_{p,\kappa}(L_\mu) = \mathsf{CE}^*_p(L_\mu - f_\kappa) + \kappa^{-q/p}\|f_\kappa\|^q_{L^q(X,\mathfrak{m})} = \mathsf{pCE}^*_p(\mu - f_\kappa\mathfrak{m}) + \kappa^{-q/p}\mathscr{L}_q(f_\kappa \mathfrak{m}|\mathfrak{m})$$

and u_κ satisfies $f_\kappa = \kappa \mathrm{J}_p u_\kappa$ and (3.15) with $L = L_\mu$. Setting $\mu_\kappa := f_\kappa \mathfrak{m}$ and selecting $\boldsymbol{\pi} \in \Pi_o(\mu, \mu_\kappa)$ according to Remark 4.4, (4.4) yields

$$\mathsf{CE}^*_{p,\kappa}(L_\mu) = \mathrm{Bar}^q_q(\boldsymbol{\pi}) + \kappa^{-q/p}\mathscr{L}_q((\mathsf{e}_1)_\sharp\boldsymbol{\pi}|\mathfrak{m}) = \mathscr{E}_{q,\kappa}(\boldsymbol{\pi}) \geq \mathscr{D}_{q,\kappa}(\mu).$$

On the other hand, it is easy to check that $\mathscr{D}_{q,\kappa}(\mu) \geq \mathsf{pCE}^*_{p,\kappa}(\mu)$, since for every plan $\boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X))$ as in (4.5) and every $f \in \mathrm{Lip}_b(X, \tau, \mathsf{d})$

$$\begin{aligned}
\int_X f\,\mathrm{d}\mu &\leq \int_X f\,\mathrm{d}(\mathsf{e}_0)_\sharp\boldsymbol{\pi} - \int_X f\,\mathrm{d}(\mathsf{e}_1)_\sharp\boldsymbol{\pi} + \int_X f\,\mathrm{d}(\mathsf{e}_1)_\sharp\boldsymbol{\pi} \\
&\leq \int_{\mathrm{RA}(X)} \int_\gamma \mathrm{lip}\, f\,\mathrm{d}\boldsymbol{\pi}(\gamma) + \|f\|_{L^p}\Big(\mathscr{L}_q((\mathsf{e}_1)_\sharp\boldsymbol{\pi}|\mathfrak{m})\Big)^{1/q} \\
&\leq \big\|\mathrm{lip}\, f\big\|_{L^p}\mathrm{Bar}_q(\boldsymbol{\pi}) + \|f\|_{L^p}\Big(\mathscr{L}_q((\mathsf{e}_1)_\sharp\boldsymbol{\pi}|\mathfrak{m})\Big)^{1/q} \leq \Big(\mathsf{pCE}_{p,\kappa}(f)\Big)^{1/p}\Big(\mathscr{E}_{q,\kappa}(\boldsymbol{\pi})\Big)^{1/q}.
\end{aligned}$$

Using now the fact that $\langle L_\mu - f_\kappa, u\rangle = \mathrm{Bar}^q_q(\boldsymbol{\pi}) = \mathsf{CE}_p(u_\kappa)$ we get

$$\int_X g^q_\kappa\,\mathrm{d}\mathfrak{m} = \int_X |\mathrm{D}u_\kappa|^p_\star\,\mathrm{d}\mathfrak{m} = \langle L_\mu - f_\kappa, u_\kappa\rangle. \tag{4.7}$$

We can also select a sequence $w_n \in \mathrm{Lip}_b(X,\tau,\mathsf{d})$ such that $w_n \to u_\kappa$, $\mathrm{lip}\, w_n \to |\mathrm{D}u_\kappa|_\star$ strongly in $L^p(X,\mathfrak{m})$, so that

$$\begin{aligned}\langle L_\mu - f_\kappa, u_\kappa\rangle &= \lim_{n\to\infty} \langle L_\mu - f_\kappa, w_n\rangle = \lim_{n\to\infty}\Big(\int_X w_n\,\mathrm{d}\mu - \int_X w_n\,\mathrm{d}\mu_1\Big)\\ &= \lim_{n\to\infty}\Big(\int_{\mathrm{RA}(X)} w_n(\mathsf{e}_0(\gamma)) - w_n(\mathsf{e}_1(\gamma))\Big)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\le \limsup_{n\to\infty}\int_{\mathrm{RA}(X)}\int_\gamma \mathrm{lip}\, w_n\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \limsup_{n\to\infty}\int_X g_\kappa\,\mathrm{lip}\, w_n\,\mathrm{d}\mathfrak{m} = \int_X g_\kappa |\mathrm{D}u|_\star\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Inserting this inquality in (4.7) we obtain the first identity of (4.6). □

Let us give a first important application of the above result to the representation of positive functionals.

Theorem 4.7. *Let $\mathscr{A}$ be a compatible subalgebra of* $\mathrm{Lip}_b(X,\tau,\mathsf{d})$ *and let L be functional on $\mathscr{A}$ satisfying*

$$|\langle L, f\rangle| \le D\Big(\mathsf{pCE}_{1,p}(f)\Big)^{1/p} \quad \textit{for every } f\in\mathscr{A}, \quad \langle L, f\rangle \ge 0 \quad \textit{for every positive } f\in\mathscr{A}.$$

Then L admits a unique extension $\tilde L \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ and there exists a unique $\mu\in\mathcal{M}_+(X)$ representing L as

$$\langle L, f\rangle = \int_X f\,\mathrm{d}\mu \quad \textit{for every } f\in\mathscr{A}. \tag{4.8}$$

Proof. By Theorem 3.13 and Proposition 3.7(c) we know that L is the restriction to $\mathscr{A}$ of a unique functional $\tilde L\in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$. It is easy to check that $\tilde L$ is also positive on $H^{1,p}(\mathbb{X})$ and applying Proposition 3.9 we can find a sequence $w_n\in L^q(X,\mathfrak{m})$, $w_n\ge 0$, strongly converging to $\tilde L$ in $H^{-1,q}(\mathbb{X})$. Let $\mu_n := w_n\mathfrak{m}$ and $\nu_n := \mathrm{R}_1(w_n)\mathfrak{m}$; applying Theorem 4.6 we can find optimal dynamic plans $\boldsymbol{\pi}_n\in\Pi_o(\mu_n,\nu_n)$ such that $\mathrm{Bar}_q(\boldsymbol{\pi}_n) = \mathsf{pCE}^*_{1,p}(\mu_n-\nu_n)\le C$. Since $\mathscr{L}_q(\nu_n|\mathfrak{m})$ is also uniformly bounded, the sequence $\boldsymbol{\pi}_n$ satisfies the tightness criterium of [21, Lemma 8.5], so that it admits a subsequence (still denoted by $\boldsymbol{\pi}_n$) weakly converging to $\boldsymbol{\pi}\in\mathcal{B}_q(\mathrm{RA}(X))\subset\mathcal{M}_+(\mathrm{RA}(X))$.

The Radon measure $\mu := (\mathsf{e}_0)_\sharp\boldsymbol{\pi}$ is the weak limit of μ_n: in particular, for every $f\in\mathscr{A}$

$$\langle L, f\rangle = \langle\tilde L, f\rangle = \lim_{n\to\infty}\int_X f\,\mathrm{d}\mu_n = \int_X f\,\mathrm{d}\mu.$$ □

5. Measures with finite energy and Newtonian capacity

In this last section we apply the previous result to prove new properties of the Newtonian capacity. We first recall the basic facts about the Newtonian approach [18,23], based on the notion of p-Modulus which has been introduced by Fuglede [15] in the natural framework of collection of positive measures, as in [1]. We refer to [8, 17] for a comprehensive presentation of this topic. As usual, $p,q\in(1,\infty)$ denote a pair of conjugate exponents.

5.1. p-Modulus of a family of arcs and Newtonian Sobolev spaces

Definition 5.1 (p-Modulus of a family of rectifiable arcs)**.** *The p-Modulus of a collection* $\Gamma \subset \mathrm{RA}(X)$ *is defined by*

$$\mathrm{Mod}_p(\Gamma) := \inf\left\{\int_X f^p \,\mathrm{d}\mathfrak{m} : f : X \to [0,\infty] \text{ is Borel}, \int_\gamma f \ge 1 \quad \text{for all } \gamma \in \Gamma\right\}.$$

Γ *is said to be* Mod_p*-negligible if* $\mathrm{Mod}_p(\Gamma) = 0$*. We say that a property P on* $\mathrm{RA}(X)$ *holds* Mod_p*-a.e. if the set of arcs where P fails is* Mod_p*-negligible. We say that a property P on X holds* quasi everywhere *(q.e.) if the set of points E where P fails is* $\mathfrak{m}$*-negligible and satisfies*

$$\mathrm{Mod}_p(\Gamma_E) = 0 \quad \text{where} \quad \Gamma_E := \left\{\gamma \in \mathrm{RA}(X) : \ell(\gamma) > 0,\ \mathsf{e}[\gamma] \cap E \neq \emptyset\right\}.$$

Notice that if E is $\mathfrak{m}$-negligible then for Mod_p-a.e. arc γ the set $\{t \in [0,1] : R_\gamma(t) \in E\}$ is $\mathscr{L}^1$-negligible. It can be shown (see e.g., [8]) that Mod_p is an increasing and subadditive functional which is continuous along increasing sequences. In fact, by [1] and [21, § 7.2], Mod_p is also continuous along decreasing sequence of compact sets, so that it is a Choquet capacity for the compact paving in $\mathrm{RA}(X)$ [11, Chap. III, 28].

It is not difficult to check that for every dynamic plan $\boldsymbol{\pi} \in \mathcal{B}_q(\mathrm{RA}(X))$ and every $\boldsymbol{\pi}$-measurable set $\Gamma \subset \mathrm{RA}(X)$

$$\boldsymbol{\pi}(\Gamma) \le \mathrm{Bar}_q(\boldsymbol{\pi})\,\mathrm{Mod}_p^{1/p}(\Gamma),$$

which in particular shows that Borel Mod_p-negligible sets are also $\boldsymbol{\pi}$-negligible for every $\boldsymbol{\pi} \in \mathcal{B}_q(\mathrm{RA}(X))$. In fact, a much more refined result holds [1,21]:

Theorem 5.2. *If* $\mathbb{X}$ *is a complete e.m.t.m. space and* τ *is a Souslin topology for X, then every Borel or Souslin set* Γ *in* $\mathrm{RA}(X)$ *is* Mod_p*-capacitable and satisfies*

$$\left(\mathrm{Mod}_p(\Gamma)\right)^{1/p} = \sup\left\{\boldsymbol{\pi}(\Gamma) : \boldsymbol{\pi} \in \mathcal{M}_+(\mathrm{RA}(X)),\ \mathrm{Bar}_q(\boldsymbol{\pi}) \le 1\right\}.$$

In particular, Γ *is* Mod_p*-negligible if and only if* $\boldsymbol{\pi}(\Gamma) = 0$ *for every* $\boldsymbol{\pi} \in \mathcal{B}_q(\mathrm{RA}(X))$.

Recall that $\mathsf{e}_i(\gamma)$, $i = 0, 1$, denote the initial and final points of a rectifiable arc $\gamma \in \mathrm{RA}(X)$.

Definition 5.3 (Newtonian weak upper gradient)**.** *Let* $f : X \to \mathbb{R}$ *be* $\mathfrak{m}$*-measurable and p-summable function. We say that f belongs to the Newtonian space* $N^{1,p}(\mathbb{X})$ *if there exists a nonnegative* $g \in L^p(X, \mathfrak{m})$ *such that*

$$\left|f(\mathsf{e}_1(\gamma)) - f(\mathsf{e}_0(\gamma))\right| \le \int_\gamma g \quad \text{for } \mathrm{Mod}_p\text{-a.e. arc } \gamma \in \mathrm{RA}(X). \tag{5.1}$$

In this case, we say that g is a $N^{1,p}$*-weak upper gradient of f.*

Functions with Mod_p-weak upper gradient have the important Beppo-Levi property of being absolutely continuous along Mod_p-a.e. arc γ (more precisely, this means $f \circ R_\gamma$ is absolutely continuous, see [23, Proposition 3.1]). Notice that functions in $N^{1,p}(\mathbb{X})$ are everywhere defined. We say that $\tilde{f} \in N^{1,p}(\mathbb{X})$ is a *good representative* of a function $f \in L^p(X, \mathfrak{m})$ if $\mathfrak{m}(\{\tilde{f} \neq f\}) = 0$.

Weak upper gradient enjoys a strong stability property [8, Prop. 2.3], resulting from a refined version of Fuglede's Lemma:

Theorem 5.4. *Let $f_n \in N^{1,p}(\mathbb{X})$, $g_n \in L^p(X,\mathfrak{m})$ be two sequences strongly converging to $f, g \in L^p(X,\mathfrak{m})$ respectively in $L^p(X,\mathfrak{m})$. If g_n is a $N^{1,p}$-weak upper gradient of f then there exists a* good representative *$\tilde{f} \in N^{1,p}(\mathbb{X})$ of f and a subsequence $k \mapsto n(k)$ such that $f_{n(k)} \to \tilde{f}$ quasi everywhere; moreover g is a $N^{1,p}$-weak upper gradient of $\tilde{f}$.*

It is clear that a function $f \in \mathrm{Lip}_b(X,\tau,\mathsf{d})$ belongs to $N^{1,p}(\mathbb{X})$ and $\mathrm{lip}\, f$ is a $N^{1,p}$-weak upper gradient (it is in fact an upper gradient). By Theorems 5.4 and 3.3 one can easily get that also every $f \in H^{1,p}(\mathbb{X})$ admits a good representative $\tilde{f} \in N^{1,p}(\mathbb{X})$ and $|\mathrm{D}f|_\star$ is a $N^{1,p}$-weak upper gradient of $\tilde{f}$. Equivalently, $\tilde{f}$ is absolutely continuous along Mod_p-a.e. arc and g satisfies (5.1) Mod_p-a.e.

In fact these two notions are essentially equivalent modulo the choice of a representative in the equivalence class [1, 5, 6, 21]:

Theorem 5.5. *Let us suppose that $\mathbb{X}$ is a complete e.m.t.m. space. Every function $f \in N^{1,p}(\mathbb{X})$ also belongs to $H^{1,p}(\mathbb{X})$ and every $N^{1,p}$-weak upper gradient g satisfies $g \geq |\mathrm{D}f|_\star$ $\mathfrak{m}$-a.e., so that $|\mathrm{D}f|_\star$ is also the minimal $N^{1,p}$-weak upper gradient of f.*

5.2. Applications to the Newtonian capacity

The Newtonian capacity $\mathrm{Cap}_p(E)$ of a subset $E \subset X$ can be defined as

$$\mathrm{Cap}_p(E) := \inf\Big\{\mathsf{CE}_{p,1}(u) : u \in N^{1,p}(\mathbb{X}),\ u \geq 1 \text{ on } E\Big\}. \tag{5.2}$$

By choosing u as the function taking the constant value 1 it is immediate to check that in our setting the capacity of a set is always finite and

$$\mathrm{Cap}_p(E) \leq \mathfrak{m}(X) < \infty \quad \text{for every } E \subset X.$$

It can be proved [8, Prop. 1.48] that

$$E \subset X \text{ has 0 capacity if and only if } E \text{ is } \mathfrak{m}\text{-negligible and } \mathrm{Mod}_p(\Gamma_E) = 0, \tag{5.3}$$

so that a property P on X holds *quasi everywhere* if the set where P fails has 0 capacity. Notice that if $\tilde{f}_i \in N^{1,p}(\mathbb{X})$ coincide $\mathfrak{m}$-a.e., then $\tilde{f}_1 = \tilde{f}_2$ q.e. in X. Notice moreover that we can use q.e. inequality in (5.2), i.e.,

$$u \in N^{1,p}(\mathbb{X}), \quad u \geq 1 \text{ q.e. on } E \quad \Rightarrow \quad \mathrm{Cap}_p(E) \leq \mathsf{CE}_{p,1}(u).$$

We also recall that the capacity satisfies the following properties [8, Thm 1.27, Prop. 2.22, Thm. 6.4]:

$$\begin{aligned}
&\mathrm{Cap}_p(\emptyset) = 0\\
&\mathfrak{m}(E) \leq \mathrm{Cap}_p(E)\\
&E_1 \subset E_2 \quad \Rightarrow \quad \mathrm{Cap}_p(E_1) \leq \mathrm{Cap}_p(E_2)\\
&\mathrm{Cap}_p(E_1 \cup E_2) + \mathrm{Cap}_p(E_1 \cap E_2) \leq \mathrm{Cap}_p(E_1) + \mathrm{Cap}_p(E_2)\\
&\mathrm{Cap}_p\Big(\bigcup_{n=1}^\infty E_n\Big) \leq \sum_{n=1}^\infty \mathrm{Cap}_p(E_n)\\
&E_n \uparrow E \quad \Rightarrow \quad \mathrm{Cap}_p(E) = \lim_{n\to\infty} \mathrm{Cap}_p(E_n) = \sup_{n>0} \mathrm{Cap}_p(E_n).
\end{aligned}$$

We want now to study the relation between the Newtonian capacity and measures $\mu \in \mathcal{M}_+(X)$ with finite energy, according to Definition 3.14. We will denote by

$$\mu = \mu^a + \mu^\perp, \quad \mu^a = \varrho\mathfrak{m} \ll \mathfrak{m}, \quad \mu^\perp \perp \mathfrak{m} \tag{5.4}$$

the canonical Lebesgue decomposition of μ w.r.t. $\mathfrak{m}$. Since by a simple truncation argument it is easy to check that

$$\frac{1}{q}\mathsf{pCE}^*_{p,1}(\mu) = \sup\Big\{\int_X f\,\mathrm{d}\mu - \frac{1}{p}\mathsf{pCE}_{p,1}(f) : f \in \mathrm{Lip}_b(X,\tau,\mathsf{d}),\ f \geq 0\Big\}$$

we obtain that

$$\mu \leq \nu \quad \Rightarrow \quad \mathsf{pCE}^*_{p,1}(\mu) \leq \mathsf{pCE}^*_{p,1}(\nu).$$

In particular $\mathsf{pCE}^*_{p,1}(\mu^\perp) \leq \mathsf{pCE}^*_{p,1}(\mu) < \infty$.

Theorem 5.6. *Let $\mu \in \mathcal{M}_+(X)$ be a measure with finite energy and let $L_\mu \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ be the linear functional associated to μ according to Corollary 3.15.*

(a) *If E is a universally measurable subset of X with* 0 *capacity then E is μ-negligible.*

(b) *If $u \in H^{1,p}(\mathbb{X})$ is nonnegative and $\tilde{u} \in N^{1,p}(\mathbb{X})$ is a good representative of u, then $\tilde{u} \in L^1(X,\mu)$ and*

$$\langle L_\mu, u\rangle = \int_X \tilde{u}\,\mathrm{d}\mu \leq \Big(\mathsf{pCE}^*_{p,1}(\mu)\Big)^{1/q}\Big(\mathsf{CE}_{p,1}(u)\Big)^{1/p}. \tag{5.5}$$

Proof. **(a)** Let E be a set with 0 capacity according to (5.3); since $\mathfrak{m}(E) = 0$, by considering the Lebesgue decomposition (5.4) it is sufficient to show that $\mu^\perp(E) = 0$. It is not restrictive to assume $\mu^\perp(X) > 0$; by Theorem 4.6 we can find a plan $\boldsymbol{\pi} \in \mathcal{B}_q(\mathrm{RA}(X))$ such that

$$\mu^\perp = (\mathsf{e}_0)_\sharp\boldsymbol{\pi}.$$

It follows that

$$\mu^\perp(E) = \boldsymbol{\pi}\{\gamma \in \mathrm{RA}(X) : \mathsf{e}_0(\gamma) \in E\} \leq \boldsymbol{\pi}(\Gamma_E) \leq \mathrm{Mod}_p(\Gamma_E)\,\mathrm{Bar}_q(\boldsymbol{\pi}) = 0.$$

(b) Let us first assume that $\tilde{u} \leq M$ for some constant $M > 0$. We can find a sequence $u_n \in \mathrm{Lip}_b(X,\tau,\mathsf{d})$ taking values in $[0, M]$, converging to $\tilde{u}$ q.e. and with $u_n \to u$, $\mathrm{lip}\,u_n \to |\mathrm{D}u|_\star$ in $L^2(X,\mathfrak{m})$. The uniform bound, the q.e. convergence and the fact that $L_\mu \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ yield

$$\lim_{n\to\infty}\int_X u_n\,\mathrm{d}\mu = \int_X \tilde{u}\,\mathrm{d}\mu = \langle L_\mu, \tilde{u}\rangle = \langle L_\mu, u\rangle \leq \Big(\mathsf{pCE}^*_{p,1}(\mu)\Big)^{1/q}\Big(\mathsf{CE}_{p,1}(u)\Big)^{1/p}$$

The case of a general nonnegative u follows by passing to the limit in the sequence $u_M := u \wedge M$ as $M \uparrow \infty$ and using monotone convergence. □

Theorem 5.7. *For every Borel set $E \subset X$ and every measure $\mu \in \mathcal{M}_+(X)$ with finite energy we have*

$$\mu(E) \leq \Big(\mathrm{Cap}_p(E)\Big)^{1/p}\Big(\mathsf{pCE}^*_{p,1}(\mu)\Big)^{1/q}. \tag{5.6}$$

If $F \subset X$ is a closed set then there exists $\mu \in \mathcal{M}_+(X)$ supported in F with

$$\mu(F) = \mathrm{Cap}_p(F) = \mathsf{pCE}^*_{p,1}(\mu) = \mathsf{CE}_{p,1}(u)$$

where $u \in \mathbb{N}^{1,p}(\mathbb{X})$ realizes the infimum of (5.2) *and*

$$L_\mu = \mathrm{J}_p u + \mathrm{A}_p u \quad \text{in } H^{-1,q}_{\mathrm{pd}}(\mathbb{X}).$$

Equivalently, for every closed set $F \subset X$

$$\big(\mathrm{Cap}_p(F)\big)^{1/p} = \max\Big\{\mu(F) : \mu \in \mathcal{M}_+(X),\ \mathsf{pCE}^*_{p,1}(\mu) \le 1\Big\}. \tag{5.7}$$

Proof. (5.6) follows easily by (5.5).

Let us now consider the case when F is closed and let us set $\mathcal{K} := \{u \in N^{1,2}(\mathbb{X}) :\ u \ge 1 \text{ q.e. on } F\}$; $\mathcal{K}$ can be identified with a convex subset of $H^{1,p}(\mathbb{X})$. Let $u_n \in \mathcal{K}$ be a sequence satisfying $\lim_{n\to\infty} \mathsf{CE}_{p,1}(u_n) = \mathrm{Cap}_p(F)$. By a truncation argument, it is not restrictive to assume that $0 \le u_n \le 1$.

By applying Mazur's Theorem and Theorem 5.4 it is not restrictive to assume that there exists $u \in N^{1,p}(\mathbb{X})$ such that

$$u_n \to u \quad \text{q.e.,} \quad \big\|u_n - u\big\|_{L^2} + \big\||\mathrm{D}u_n|_\star - |\mathrm{D}u|_\star\big\|_{L^2} \to 0$$

Up to modifying u in a set of 0 capacity, we can thus suppose that $u \in \mathcal{K}$, $0 \le u \le 1$, and $\mathsf{CE}_{p,1}(u) = \mathrm{Cap}_p(F)$. The minimality yields that there exists $L \in \mathrm{A}_p(u) + \mathrm{J}_p(u) \subset H^{-1,p}(\mathbb{X})$ such that

$$\langle L, v - u\rangle \ge 0 \quad \text{for every } v \in \mathcal{K}.$$

In particular, choosing $v = u + \phi$ with ϕ nonnegative we get

$$\langle L, \phi\rangle \ge 0 \quad \text{for every } \phi \in N^{1,p}(\mathbb{X}),\ \phi \ge 0,$$

so that $L \in H^{-1,q}_{\mathrm{pd}}(\mathbb{X})$ and its action on bounded Lipschitz functions can be represented by a positive Radon measure μ according to (4.8) thanks to Theorem 4.7

Choosing now $\phi \in \mathrm{Lip}_b(X, \tau, \mathsf{d})$ such that $\phi \equiv 0$ on F we get

$$\langle L, \phi\rangle = \int_X \phi \,\mathrm{d}\mu = 0,$$

so that $\mu(X \setminus F) = 0$ and μ is concentrated on F (recall that $\mathrm{Lip}_b(X, \tau, \mathsf{d})$ generates the τ topology of X). Since L has finite energy $\mathsf{pCE}^*_{p,1}(\mu) = \mathsf{CE}^*_{p,1}(L)$. Since $u \in N^{1,p}(\mathbb{X})$ is nonnegative it follows that

$$\mathrm{Cap}_p(F) = \mathsf{CE}_{p,1}(u) = \mathsf{CE}^*_{p,1}(L) = \langle L, u\rangle = \int_X u \,\mathrm{d}\mu = \mu(F).$$

The renormalization $\tilde\mu := \mu\big(\mathsf{pCE}^*_{p,1}(\mu)\big)^{-1/q}$ provides (5.7). □

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Conflict of interest

The authors declare no conflict of interest.

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