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Maximal displacement for bridges of random walks in a random environment

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Abstract. It is well known that the distribution of simple random walks on $\mathbb{Z}$ conditioned on returning to the origin after $2n$ steps does not depend on $p = P(S_1 = 1)$, the probability of moving to the right. Moreover, conditioned on $\{S_{2n} = 0\}$ the maximal displacement $\max_{k\le 2n} |S_k|$ converges in distribution when scaled by $\sqrt{n}$ (diffusive scaling).

We consider the analogous problem for transient random walks in random environments on $\mathbb{Z}$. We show that under the quenched law P_ω (conditioned on the environment ω), the maximal displacement of the random walk when conditioned to return to the origin at time $2n$ is no longer necessarily of the order $\sqrt{n}$. If the environment is nestling (both positive and negative local drifts exist) then the maximal displacement conditioned on returning to the origin at time $2n$ is of order $n^{\kappa/(\kappa+1)}$, where the constant $\kappa > 0$ depends on the law on environments. On the other hand, if the environment is marginally nestling or non-nestling (only non-negative local drifts) then the maximal displacement conditioned on returning to the origin at time $2n$ is at least $n^{1-\varepsilon}$ and at most $n/(\ln n)^{2-\varepsilon}$ for any $\varepsilon > 0$.

As a consequence of our proofs, we obtain precise rates of decay for $P_\omega(X_{2n} = 0)$. In particular, for certain non-nestling environments we show that $P_\omega(X_{2n} = 0) = \exp\{-Cn - C'n/(\ln n)^2 + \mathrm{o}(n/(\ln n)^2)\}$ with explicit constants $C, C' > 0$.

Résumé. Il est bien connu que la distribution d'une marche aléatoire simple sur $\mathbb{Z}$, conditionée à retourner à l'origine au temps $2n$ est indépendante de $p = P(S_1 = 1)$, la probabilité d'un pas vers la droite. De plus, conditionellement à $\{S_{2n} = 0\}$, le déplacement maximum $\max_{k\le 2n} |S_k|$, divisé par $\sqrt{n}$, converge en distribution.

Nous considérons le mème problème pour les marches transientes en environnement aléatoire sur $\mathbb{Z}$. Nous montrons que sous la loi "quenched," le déplacement maximum pour la marche conditionnée à retourner à l'origine au temps $2n$ n'est pas toujours de l'ordre de $\sqrt{n}$. Si l'environement a des drifts locaux positifs et negatifs alors cet ordre de grandeur est $n^{\kappa/(\kappa+1)}$, où $\kappa > 0$ dépend de la loi de l'environement. Mais, si l'environement n'a que des drifts locaux positifs ou nuls, alors cet ordre de grandeur est proche de n.

Les preuves fournissent de plus l'ordre de grandeur de $P_\omega(X_{2n} = 0)$. Dans le cas où les drifts locaux sont tous positifs nous montrons que $P_\omega(X_{2n} = 0) = \exp\{-Cn - C'n/(\ln n)^2 + \mathrm{o}(n/(\ln n)^2)\}$.

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1. Introduction

Let $(S_n)_{n\ge 1}$ be a random walk with drift on the integers: let $p \in (0, 1)$ and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with $P(Y_1 = +1) = p = 1 - P(Y_1 = -1)$, and $S_n = \sum_{i=1}^n Y_i$, $n = 1, 2, 3, \ldots$. Consider the conditioned law

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of $(\frac{S_k}{\sqrt{2n}})_{1\le k\le 2n}$, conditioned on the event $\{S_{2n}=0\}$. It is easy to see that this conditioned laws converge to the law of a Brownian bridge for $n\to\infty$, for any value of $p\in(0,1)$. In fact, for $p=\frac{1}{2}$, this is a consequence of Donsker's invariance principle, but by symmetry, the conditioned laws do not depend on the value of p. In particular, the laws

$$P\left(\frac{1}{\sqrt{2n}}\max_{1\le k\le 2n}|S_k|\in\cdot\middle|S_{2n=0}\right) \tag{1}$$

converge, for $n\to\infty$, to the law of the maximum of the absolute value of a Brownian bridge.

In this paper, we address a question originally posed to us by Benjamini. The question is, what happens if we replace (S_n) with a random walk in random environment $(X_n)_{n\ge 1}$? In particular, what is the order of growth of $\max_{1\le k\le 2n}|X_k|$, conditioned on the event $\{X_{2n}=0\}$? Is it diffusive as in (1) or superdiffusive or subdiffusive, respectively?

The random walk in random environment (abbreviated RWRE) is defined as follows. Let $\omega=(\omega_x)_{x\in\mathbb{Z}}$ be a collection of random variables taking values in $(0,1)$ and let P be the distribution of ω. The random variable ω is called the "random environment." For each environment $\omega\in\Omega=(0,1)^{\mathbb{Z}}$ and $y\in\mathbb{Z}$, we define the RWRE starting at y as the time-homogeneous Markov chain (X_n) taking values in $\mathbb{Z}$, with $X_0=y$ and transition probabilities

$$P_\omega^y[X_{n+1}=x+1|X_n=x]=\omega_x=1-P_\omega^x[X_{n+1}=x-1|X_n=x],\quad n\ge 0.$$

We equip Ω with its Borel σ-field $\mathcal{F}$ and $\mathbb{Z}^{\mathbb{N}}$ with its Borel σ-field $\mathcal{G}$. The distribution of $(\omega,(X_n))$ is the probability measure $\mathbb{P}$ on $\Omega\times\mathbb{Z}^{\mathbb{N}}$ defined by

$$\mathbb{P}[F\times G]=\int_F P_\omega^0[G]P(\mathrm{d}\omega),\quad F\in\mathcal{F},\ G\in\mathcal{G}.$$

Since we will usually be concerned with random walks starting from the origin we will use P_ω and E_ω to denote P_ω^0 and E_ω^0, respectively. Expectations with respect to P, P_ω and $\mathbb{P}$ will be denoted by E_P, E_ω and $\mathbb{E}$, respectively. P_ω is referred to as the *quenched* law of the random walk, while $\mathbb{P}$ is referred to as the averaged (or annealed) law.

The goal of this paper is to study the magnitude of the maximal displacement $\max_{1\le k\le 2n}|X_k|$ of a RWRE under the quenched law P_ω, conditioned on the event $\{X_{2n}=0\}$. For a simple random walk, (1) tells us that the scaling is of order $\sqrt{n}$ (diffusive scaling). This is not the case for RWRE as we will show below. In fact, super-diffusive, sub-diffusive and diffusive scaling are possible (depending on the law P of the environment).

Throughout the paper we will make the following assumptions.

Assumption 1. *The environment is i.i.d. and uniformly elliptic. That is, the random variables $\{\omega_x\}_{x\in\mathbb{Z}}$ are i.i.d. under the measure P, and there exists a constant $c>0$ such that $P(\omega_x\in[c,1-c])=1$.*

Let $\rho_i=\rho_i(\omega):=(1-\omega_i)/\omega_i$, $i\in\mathbb{Z}$.

Assumption 2. $E_P\log\rho_0<0$.

As shown in [13], the second assumption implies that for P-almost all ω, the Markov chain (X_n) is transient and we have $X_n\to+\infty$, P_ω-a.s. A lot more is known about this one-dimensional model; we will not give background here, but we refer to the survey by Zeitouni [14] for limit theorems, large deviations results, and for further references.

Our main results are as follows. Let $\omega_{\min}:=\inf\{x:\ P(\omega_0\le x)>0\}$. We will distinguish between three different cases: $\omega_{\min}<\frac{1}{2}$ (nestling case), $\omega_{\min}=\frac{1}{2}$ (marginally nestling case) and $\omega_{\min}>\frac{1}{2}$ (non-nestling case). It turns out that in the nestling case, the magnitude of $\max_{1\le k\le 2n}|X_k|$ under the conditioned law is of order n^β, where the exponent $\beta=\kappa/(\kappa+1)$ for a parameter $\kappa>0$ which depends on the law of the environment (see Theorem 2.1 for the precise statement). The cases $\beta>\frac{1}{2}$, $\beta=\frac{1}{2}$ and $\beta<\frac{1}{2}$ are all possible, and we have $\beta>\frac{1}{2}$ if and only if (X_n) has a positive linear speed. In the marginally nestling and the non-nestling cases, we give additional assumptions excluding the deterministic case (i.e., the case of constant environment). We then show that the magnitude of $\max_{1\le k\le 2n}|X_k|$ under the conditioned law is between $n^{1-\varepsilon}$ and $n/(\ln n)^{2-\varepsilon}$ for any $\varepsilon>0$ (see Theorems 3.1 and 4.1 for precise statements). We conjecture in fact that the correct order of magnitude is $n/(\ln n)^2$ (see Conjecture 5.1).

As a consequence of the proofs of our main theorems we also obtain precise asymptotics on the rate of decay of the quenched probabilities $P_\omega(X_{2n}=0)$. In the nestling and marginally nestling cases the decay rates are $\exp\{-n^{\kappa/(\kappa+1)+o(1)}\}$ and $\exp\{-Cn/(\ln n)^2+o(n/(\ln n)^2)\}$ for an explicit constant $C>0$, respectively (see Lemmas 2.4 and 3.3). These decay rates are not surprising given the previous results on moderate and large deviations for RWRE in [5] and [11]. The non-nestling case turns out to be more interesting. It was known previously from large deviation results that $P_\omega(X_{2n}=0)=\exp\{-Cn+o(n)\}$ for an explicit constant $C>0$. For our results, however, we needed some sub-exponential corrections to this rate of decay. In Corollary 4.4 we show in fact that

$$P_\omega(X_{2n}=0)=\exp\bigl\{-Cn-C'n/(\ln n)^2+o\bigl(n/(\ln n)^2\bigr)\bigr\},\qquad P\text{-a.s.},$$

for explicit positive constants C and C' depending on the law on environments.

A brief remark is in order about what our main results tell us about what a RWRE bridge looks like. One of the dominating features of one-dimensional RWRE is the trapping effect of the environment. Informally, a "trap" is an atypical section of the environment for which the probability to stay confined to the interval for a long time is abnormally large. One way for the random walk to be back at the origin after $2n$ steps is for the random walk to go quickly to a certain large trap, stay in the trap until almost time $2n$, and then go quickly back to the origin at the end. For such a strategy, there is both a cost and a benefit for a larger maximal displacement. Travelling to and from a trap that is far from the origin requires backtracking a large distance, but travelling farther from the origin allows the random walk to reach a larger trap. Balancing these costs and benefits leads to a lower bound on the rate of decay of $P_\omega(X_{2n}=0)$. The difficulty in deriving the asymptotic decay of $P_\omega(X_{2n}=0)$ lies in proving a corresponding upper bound which suggests that indeed a RWRE bridge typically spends most of its time in a few large traps. Figure 1 shows the simulation of a RWRE bridge where this trapping behavior is clearly seen to occur.

The paper is organized as follows. In Sections 2, 3 and 4, respectively, the nestling, marginally nestling and non-nestling cases are treated. In Section 5 we state a conjecture on an improved lower bound for the maximal displacement in the marginally nestling and non-nestling cases.

We conclude this section with some notation that will be used throughout the rest of the paper. We will use θ to denote the standard shift operation on doubly infinite sequences. That is, $(\theta^x\omega)_y=\omega_{x+y}$. Also, we will use $T_k:=\inf\{n\ge 0\colon X_n=k\}$ to be the hitting time of the site k by the random walk. The law of a simple symmetric random walk (i.e., $\omega_x\equiv 1/2$) will be denoted by $P_{1/2}$ with corresponding expectations denoted $E_{1/2}$.

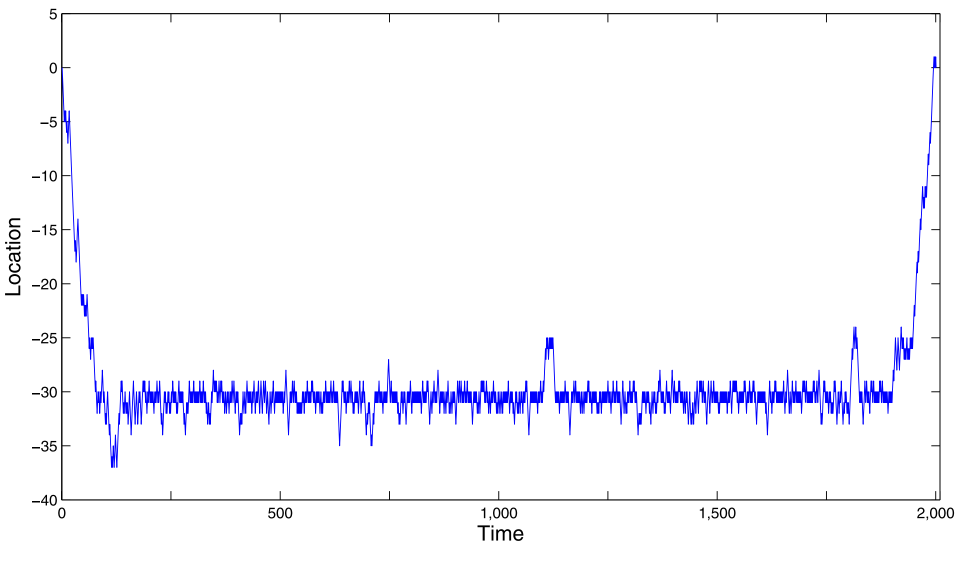

Fig. 1. A simulation of a RWRE bridge of 2000 steps. The distribution on the environment used was such that $P(\omega_0=3/4)=0.9$ and $P(\omega_0=1/4)=0.1$. Using the notation from Section 2, this distribution on environments has the parameter $\kappa=2$, and thus Theorem 2.1 implies that the maximal displacement of a bridge of length $2n$ in this case should be roughly of the order $n^{2/3}$ when n is large.

2. Case I: Nestling environment

Throughout this section we will make the following assumption on environments in addition to Assumptions 1 and 2.

Assumption 3 (Nestling). $P(\omega_0 < 1/2) > 0$ *and* $P(\omega_0 > 1/2) > 0$.

If Assumptions 1–3 are satisfied, then we can define a parameter $\kappa = \kappa(P)$ by

$$\kappa = \kappa(P) \text{ is the unique positive solution of } E_P \rho_0^\kappa = 1. \tag{2}$$

The parameter κ first appeared in [9] in relation to the scaling exponent for the averaged limiting distributions of transient one-dimensional RWRE. Moreover, the law of large numbers for transient RWRE derived by Solomon in [13] may be re-stated in terms of the parameter κ.

$$\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1-E_P\rho_0}{1+E_P\rho_0} =: \mathrm{v}_P > 0, & \kappa > 1, \\ 0, & \kappa \le 1. \end{cases} \tag{3}$$

If $\kappa < 1$, then the typical displacement of X_n is sub-linear and of the order n^κ (see [9] or [4]).

The main result of this section is that the the maximal displacement of bridges for RWRE under the above assumptions is approximately of the order $n^{\kappa/(\kappa+1)}$.

Theorem 2.1. *Let Assumptions 1–3 hold. Then, for P-a.e. environment ω,*

$$\lim_{n\to\infty} P_\omega\Big(\max_{k\le 2n} |X_k| \ge n^\beta \Big| X_{2n} = 0\Big) = \begin{cases} 1, & \beta < \frac{\kappa}{\kappa+1}, \\ 0, & \beta > \frac{\kappa}{\kappa+1}, \end{cases}$$

where $\kappa > 0$ is defined by (2).

Remark 2.2. *The cases $\frac{\kappa}{\kappa+1} > \frac{1}{2}$, $\frac{\kappa}{\kappa+1} = \frac{1}{2}$ and $\frac{\kappa}{\kappa+1} < \frac{1}{2}$ are possible. Note that, due to (3), $\frac{\kappa}{\kappa+1} > \frac{1}{2}$ if and only if (X_n) has a.s. positive linear speed.*

An important ingredient in the proof of Theorem 2.1 is the following lemma.

Lemma 2.3. *Let Assumptions 1–3 hold. If $\beta \in (0, 1 \wedge \kappa)$, then for P-a.e. environment ω,*

$$P_\omega\Big(\max_{k\le n} |X_k| < n^\beta\Big) = \exp\big\{-n^{1-\beta/\kappa + o(1)}\big\}.$$

Proof. For any environment ω, let ω^- and ω^+ be the modified environment by adding a reflection to the left or right, respectively, at the origin. That is,

$$(\omega^-)_x := \begin{cases} \omega_x, & x \ne 0, \\ 0, & x = 0, \end{cases} \quad \text{and} \quad (\omega^+)_x := \begin{cases} \omega_x, & x \ne 0, \\ 1, & x = 0. \end{cases} \tag{4}$$

The random walk starting at the origin stays in the interval $(-n^\beta, n^\beta)$ if both the left excursions and the right excursions from the origin take at least n steps to leave the interval. Therefore,

$$P_\omega\Big(\max_{k\le n} |X_k| < n^\beta\Big) \ge P_{\omega^-}(T_{-\lceil n^\beta\rceil} > n) P_{\omega^+}(T_{\lceil n^\beta\rceil} > n). \tag{5}$$

Explicit formulas for hitting probabilities (see Eq. (2.1.4) in [14]) imply that

$$P_{\omega^-}(T_{-\lceil n^\beta\rceil} > n) \ge P_\omega^{-1}(T_{-\lceil n^\beta\rceil} > T_0)^n \ge \Big(1 - \prod_{-\lceil n^\beta\rceil < i < 0} \rho_i\Big)^n.$$

The law of large numbers implies that $\prod_{-x<i<0}\rho_i = \exp\{x(E_P(\log\rho_0)+\mathrm{o}(1))\}$ as $x\to\infty$, P-a.s. Since $E_P\log\rho_0<0$, this implies that the first probability on the right-hand side of (5) tends to 1 as $n\to\infty$, P-a.s. In Theorem 1.2 in [5], it was shown that for any $\beta<\kappa$

$$P_{\omega^+}(T_{\lceil n^\beta\rceil}>n)=\exp\bigl\{-n^{1-\beta/\kappa+\mathrm{o}(1)}\bigr\},\quad P\text{-a.s.} \tag{6}$$

Recalling (5), this completes the proof of the lower bound.

To prove the corresponding upper bound, note that $P_\omega(\max_{k\le n}|X_k|<n^\beta)$ depends only on the ω_x with $|x|<n^\beta$. Therefore, the probability is unchanged by modifying the environment so that $\omega_{-\lceil n^\beta\rceil}=1$. By the strong Markov property, the probability of staying confined to the interval $(-n^\beta,n^\beta)$ when starting at the origin in the original environment is less than the probability of taking more than n steps to reach $\lceil n^\beta\rceil$ when starting at $-\lceil n^\beta\rceil$ in the modified environment. That is,

$$P_\omega\Bigl(\max_{k\le n}|X_k|<n^\beta\Bigr)\le P_{(\theta^{-\lceil n^\beta\rceil}\omega)^+}(T_{2\lceil n^\beta\rceil}>n). \tag{7}$$

At this point, we would like to again apply (6) to the probability on the right in (7). However, the presence of the shifted environment in the quenched probability does not allow for a direct application. We claim that the proof of (6) in [5] may be easily modified to obtain that for any fixed sequence x_n,

$$P_{(\theta^{x_n}\omega)^+}(T_{\lceil n^\beta\rceil}>n)=\exp\bigl\{-n^{1-\beta/\kappa+\mathrm{o}(1)}\bigr\},\quad P\text{-a.s.} \tag{8}$$

Indeed, in [5], it was shown that there were collections of typical environments $\Omega_n\subset\Omega$ such that $\sum_n P(\Omega_n^c)<\infty$ so that the Borel–Cantelli lemma implied that $\omega\in\Omega_n$ for all n large enough, P-a.s. Then, upper and lower bounds were developed for the probabilities $P_{\omega^+}(T_{\lceil n^\beta\rceil}>n)$ which are uniform over $\omega\in\Omega_n$. These upper and lower bounds and the fact that $\omega\in\Omega_n$ for all n large enough imply (6). For any fixed sequence x_n the shift invariance of P implies that $P(\omega\in\Omega_n)=P(\theta^{x_n}\omega\in\Omega_n)$, and thus the Borel–Cantelli lemma implies that $\theta^{x_n}\omega\in\Omega_n$ for all n large enough, P-a.s. Therefore, the statement (8) follows from the uniform upper and lower bounds for environments in Ω_n. Applying (8) to (7) with the sequence $x_n=-\lceil n^\beta\rceil$ gives the needed upper bound. □

As a first step in computing the magnitude of displacement of bridges, we first calculate the asymptotic probability of being at the origin at time $2n$.

Lemma 2.4. *Let Assumptions 1–3 hold. Then, for P-a.e. environment ω,*

$$P_\omega(X_{2n}=0)=\exp\bigl\{-n^{\kappa/(\kappa+1)+\mathrm{o}(1)}\bigr\},$$

where $\kappa>0$ is defined by (2).

Proof. This lemma follows easily from Lemma 2.3 and the moderate deviation asymptotics derived in [5]. If $\nu\in(0,1\wedge\kappa)$, then Theorem 1.2 in [5] implies that

$$\lim_{n\to\infty}\frac{\ln(-\ln P_\omega(T_{n^\nu}>n))}{\ln n}=\lim_{n\to\infty}\frac{\ln(-\ln P_\omega(X_n<n^\nu))}{\ln n}=\left(1-\frac{\nu}{\kappa}\right)\wedge\frac{\kappa}{\kappa+1},\quad P\text{-a.s.}$$

Note that $1-\frac{\nu}{\kappa}<\frac{\kappa}{\kappa+1}$ if and only if $\nu>\frac{\kappa}{\kappa+1}$. Therefore, for $\nu\le\frac{\kappa}{\kappa+1}$,

$$P_\omega(X_{2n}=0)\le P_\omega\bigl(X_{2n}<(2n)^\nu\bigr)=\exp\bigl\{-n^{\kappa/(\kappa+1)+\mathrm{o}(1)}\bigr\}.$$

To get a corresponding lower bound, we will exhibit a strategy for obtaining $X_{2n}=0$. Force the random walk to stay in the interval $[-n^{\kappa/(\kappa+1)},n^{\kappa/(\kappa+1)}]$ for the first $2n-\lfloor n^{\kappa/(\kappa+1)}\rfloor$ steps, and then choose a deterministic path from the random walks current location to force the random walk to be at the origin at time $2n$. Uniform ellipticity (Assumption 1) and Lemma 2.3 imply

$$P_\omega(X_{2n}=0)\ge P_\omega\Bigl(\max_{k\le 2n}|X_k|\le n^{\kappa/(\kappa+1)}\Bigr)c^{2\lfloor n^{\kappa/(\kappa+1)}\rfloor}=\exp\bigl\{-n^{\kappa/(\kappa+1)+\mathrm{o}(1)}\bigr\}.$$ □

We are now ready to give the proof of the main result in this section.

Proof of Theorem 2.1. We first give a lower bound for $P_\omega(\max_{k\le n}|X_k|\ge n^\beta|X_{2n}=0)$.

$$P_\omega\Big(\max_{k\le 2n}|X_k|\ge n^\beta\big|X_{2n}=0\Big)=1-\frac{P_\omega(\max_{k\le 2n}|X_k|<n^\beta,X_{2n}=0)}{P_\omega(X_{2n}=0)}$$
$$\ge 1-\frac{P_\omega(\max_{k\le 2n}|X_k|<n^\beta)}{P_\omega(X_{2n}=0)}$$
$$=1-\frac{\exp\{-n^{1-\beta/\kappa+o(1)}\}}{\exp\{-n^{\kappa/(\kappa+1)+o(1)}\}},$$

where the last inequality is from Lemmas 2.3 and 2.4. Since $1-\frac{\beta}{\kappa}>\frac{\kappa}{\kappa+1}$ when $\beta<\frac{\kappa}{\kappa+1}$, we have thus proved Theorem 2.1 in the case $\beta<\frac{\kappa}{\kappa+1}$.

Next we turn to the case $\beta>\frac{\kappa}{\kappa+1}$. First of all, note that

$$P_\omega\Big(\max_{k\le 2n}|X_k|\ge n^\beta\big|X_{2n}=0\Big)=\frac{P_\omega(\max_{k\le 2n}|X_k|\ge n^\beta,X_{2n}=0)}{\exp\{-n^{\kappa/(\kappa+1)+o(1)}\}}.$$

Therefore, it is enough to show that, P-a.s., there exists a $\beta'\in(\frac{\kappa}{\kappa+1},\beta)$ such that for all n sufficiently large,

$$P_\omega\Big(\max_{k\le 2n}|X_k|\ge n^\beta,X_{2n}=0\Big)\le \mathrm{e}^{-n^{\beta'}}. \tag{9}$$

To this end, note that the event $\{\max_{k\le 2n}|X_k|\ge n^\beta,X_{2n}=0\}$ implies that either the random walk reaches $-\lfloor n^\beta\rfloor$ in less than $2n$ steps, or after first reaching $\lfloor n^\beta\rfloor$ the random walk returns to the origin in less than $2n$ steps. Thus, the strong Markov property implies that

$$P_\omega\Big(\max_{k\le 2n}|X_k|\ge n^\beta,X_{2n}=0\Big)\le P_\omega(T_{-\lceil n^\beta\rceil}<2n)+P_{\theta^{\lceil n^\beta\rceil}\omega}(T_{-\lceil n^\beta\rceil}<2n). \tag{10}$$

It was shown in Theorem 1.4 in [5] that for any $\beta\in(0,1)$,

$$\lim_{n\to\infty}\frac{\ln(-\ln\mathbb{P}(T_{-\lceil n^\beta\rceil}<2n))}{\ln n}=\lim_{n\to\infty}\frac{\ln(-\ln P_\omega(T_{-\lceil n^\beta\rceil}<2n))}{\ln n}=\beta,\quad P\text{-a.s.} \tag{11}$$

This implies that, P-a.s., the first term on the right-hand side of (10) is less than $\mathrm{e}^{-n^{\beta'}}$ for any $\beta'<\beta$ and all n large enough. To show that the same is true of the second term on the right-hand side of (10), note that Chebychev's inequality and the shift invariance of P imply that

$$P\big(P_{\theta^{\lceil n^\beta\rceil}\omega}(T_{-\lceil n^\beta\rceil}<2n)>\mathrm{e}^{-n^{\beta'}}\big)\le \mathrm{e}^{n^{\beta'}}\mathbb{P}(T_{-\lceil n^\beta\rceil}<2n)=\mathrm{e}^{n^{\beta'}}\mathrm{e}^{-n^{\beta+o(1)}}.$$

Since $\beta'<\beta$, this last term is summable, and thus the Borel–Cantelli lemma implies that, P-a.s., $P_{\theta^{\lceil n^\beta\rceil}\omega}(T_{-\lceil n^\beta\rceil}<2n)\le \mathrm{e}^{-n^{\beta'}}$ for all n large enough. Choosing $\beta'\in(\frac{\kappa}{\kappa+1},\beta)$ and recalling (10), we have that, P-a.s., (9) holds for all n large enough. This completes the proof of Theorem 2.1 in the case $\beta>\frac{\kappa}{\kappa+1}$. □

3. Case II: Marginally nestling environment

In this section we will assume the following condition on environments.

Assumption 4. $P(\omega_0\ge 1/2)=1$ *and* $\alpha=P(\omega_0=1/2)\in(0,1)$.

Note that Assumption 4 implies Assumption 2, and so in this section we will only assume Assumptions 1 and 4. Our main result in this section is that the displacement of bridges in this case is greater than $n^{1-\varepsilon}$ and less than $n/(\ln n)^{2-\varepsilon}$ for any $\varepsilon > 0$.

Theorem 3.1. *Let Assumptions 1 and 4 hold. Then for any $\beta < 2$,*

$$\lim_{n\to\infty} P_\omega\left(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}\,\middle|\, X_{2n}=0\right) = 0, \quad P\text{-a.s.}, \tag{12}$$

and for any $\gamma < 1$,

$$\lim_{n\to\infty} P_\omega\left(\max_{k\le 2n}|X_k| \ge n^\gamma \,\middle|\, X_{2n}=0\right) = 1, \quad P\text{-a.s.} \tag{13}$$

Remark 3.2. *Assumption 4 implies that $\omega_{\min} = 1/2$, which is sometimes referred to as the marginally nestling condition. The added condition of $\alpha \in (0,1)$ was also assumed previously in the quenched and averaged analysis of certain large deviations [2,7,11,12]. We suspect that if $\omega_{\min} = 1/2$ but $\alpha = 0$ then there exists a $\gamma < 2$ such that the displacement of bridges is bounded above by $n/(\ln n)^{\gamma-\varepsilon}$ for any $\varepsilon > 0$ (cf. Remark 1 on page 179 in [7]).*

The first step in the proof of Theorem 3.1 is a computation of the precise subexponential rate of decay of the quenched probability to be at the origin.

Lemma 3.3. *Let Assumptions 1 and 4 hold. Then,*

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_\omega(X_{2n}=0) = -\frac{|\pi \log\alpha|^2}{4}.$$

Proof. Theorem 1 in [11] implies that for any $v \in (0, \mathrm{v}_P)$

$$\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \log P_\omega(X_{2n}=0) \le \limsup_{n\to\infty} \frac{(\ln n)^2}{n} \log P_\omega(X_{2n} \le 2nv) = -\frac{|\pi \ln\alpha|^2}{4}\left(1 - \frac{v}{\mathrm{v}_P}\right).$$

Taking $v \to 0$ proves the upper bound needed.

For the corresponding lower bound we will force the random walk to stay in a portion of the environment where there is a long sequence of consecutive sites with $\omega_x = \frac{1}{2}$. We say x is a *fair* site if $\omega_x = \frac{1}{2}$. For any $n \ge 1$, let

$$L(n) := \max\left\{ j - i\colon\ 0 \le i < j \le n/(\ln n)^3, \omega_x = \frac{1}{2}, \forall i \le x < j \right\}$$

be the length of the longest interval of *fair* sites in $[0, n/(\ln n)^3)$, and let

$$i_n := \inf\bigl\{ i \in \bigl[0, n/(\ln n)^3 - L(n)\bigr]\colon\ \omega_x = 1/2, \forall i \le x < i + L(n) \bigr\}$$

be the leftmost endpoint of an interval of fair sites in $[0, n/(\ln n)^3]$ of maximal length $L(n)$. Theorem 3.2.1 in [3] implies that

$$\lim_{n\to\infty} \frac{L(n)}{\ln n} = \frac{1}{|\ln \alpha|}, \quad P\text{-a.s.} \tag{14}$$

Uniform ellipticity (Assumption 1) implies that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{(\ln n)^2}{n} \log P_\omega(X_{2n}=0) \\
&\quad \ge \liminf_{n\to\infty} \frac{(\ln n)^2}{n} \log P_\omega\Bigl(\max_{k\le 2n}|X_k| < n/(\ln n)^3\Bigr) \\
&\quad \ge \liminf_{n\to\infty} \frac{(\ln n)^2}{n} \log P_\omega^{i_n}\bigl(X_k \in \bigl(-n/(\ln n)^3, i_n + L(n)\bigr), \forall k \le 2n\bigr).
\end{aligned} \tag{15}$$

The random walk starting at i_n stays in the interval $(-n/(\ln n)^3, i_n + L(n))$ if both the left excursions and right excursions from i_n take more than $2n$ steps to leave the interval. Recalling the definitions of ω^- and ω^+ given in (4) we obtain that

$$P_\omega^{i_n}\big(X_k \in \big(-n/(\ln n)^3, i_n + L(n)\big), \forall k \le 2n\big) \\ \ge P_{(\theta^{i_n}\omega)^-}(T_{-n/(\ln n)^3 - i_n} > 2n) P_{(\theta^{i_n}\omega)^+}(T_{L(n)} > 2n). \tag{16}$$

As in the proof of Lemma 3.5, explicit formulas for hitting probabilities and Assumption 2 imply that the first probability on the right-hand side of (16) tends to 1 as $n \to \infty$, P-a.s. Due to the fact that $\omega_x = 1/2$ for all $i_n \le x < i_n + L(n)$, the second probability on the right-hand side of (16) is equal to the probability that a simple symmetric random walk stays in the interval $[-L(n), L(n)]$ for $2n$ steps. To this end, we recall the following small deviation asymptotics for a simple symmetric random walk (see Theorem 3 in [10]).

Lemma 3.4. *Let* $\lim_{n\to\infty} x(n) = \infty$ *and* $x(n) = \mathrm{o}(\sqrt{n})$. *Then,*

$$\lim_{n\to\infty} \frac{x(n)^2}{n} \ln P_{1/2}\Big(\max_{k\le n} |X_k| \le x(n)\Big) = -\frac{\pi^2}{8}.$$

Then, Lemma 3.4 and (14) imply that

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \log P_{(\theta^{i_n}\omega)^+}(T_{L(n)} \ge 2n) = -\frac{|\pi \ln \alpha|^2}{4}.$$

Recalling (15) and (16) implies the lower bound needed for the proof of the lemma. □

To prove Theorem 3.1 we need to compare the quenched probability to be at the origin at time $2n$ with the quenched probability to stay confined to the interval $[-n^\gamma, n^\gamma]$ for the first $2n$ steps of the random walk. The following proposition says that the exponential rate of the decay of the latter probability is also of the order $n/(\ln n)^2$ but with a larger constant than the probability to be at the origin.

Proposition 3.5. *Let Assumptions 1 and 4 hold. Then, for any* $\gamma \in (0, 1)$

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_\omega\Big(\max_{k\le 2n} |X_k| \le n^\gamma\Big) = -\frac{|\pi \ln \alpha|^2}{4\gamma^2}, \quad P\text{-a.s.}$$

Proof. As in the proof of Lemma 3.3, a lower bound is obtained by forcing the random walk to stay near a long stretch of fair sites (i.e., sites $x \in \mathbb{Z}$ with $\omega_x = 1/2$). However, Theorem 3.2.1 in [3] implies that for any $\gamma > 0$, the longest stretch of consecutive fair sites contained in $[0, n^\gamma]$ is $(\frac{\gamma}{|\ln \alpha|} + \mathrm{o}(1)) \ln n$. The remainder of the proof of the lower bound is the same as in the proof of Lemma 3.3 and is therefore omitted.

The proof of the upper bound is an adaptation of the upper bound for quenched large deviations of slowdowns given by Pisztora and Povel [11]. It turns out that an upper bound for $P_\omega(\max_{k\le 2n} |X_k| \le n^\gamma)$ with $\gamma < 1$ is substantially easier to prove than the upper bounds for large deviations of the form $P_\omega(X_n < nv)$ with $v \in (0, \mathrm{v}_P)$. (In fact the multi-scale analysis present in [11] can be avoided entirely.)

We begin by dividing up space into *fair* and *biased* blocks as was done in [11]. Fix a $\delta \in (0, 1/3)$ and let $\varepsilon, \xi > 0$ be such that $0 < \varepsilon < P(\omega_0 \ge 1/2 + \xi)$ (ε and ξ will eventually be arbitrarily small). Divide $\mathbb{Z}$ into disjoint intervals (which we will call *blocks*)

$$B_j = B_j(n) := \big[j\lfloor(\ln n)^{1-\delta}\rfloor, (j+1)\lfloor(\ln n)^{1-\delta}\rfloor\big), \quad j \in \mathbb{Z}.$$

A block B_j is called *biased* if the proportion of sites $x \in B_j$ with $\omega_x \ge 1/2 + \xi$ is at least ε. Otherwise, the block is called *fair*. Let $J(n, \delta, \varepsilon, \xi) = J := \{j \in \mathbb{Z}\colon B_j \text{ is biased}\}$ be the indices of the biased blocks, and let $J = \bigcup_{k\in\mathbb{Z}}\{j_k\}$, with j_k increasing in k and $j_{-1} < 0 \le j_0$. Collect blocks into (overlapping) regions R_k defined by

$$R_k := \bigcup_{j=j_k}^{j_{k+1}} \overline{B}_j, \quad k \in \mathbb{Z},$$

where $\overline{B}_j = [j\lfloor(\ln n)^{1-\delta}\rfloor, (j+1)\lfloor(\ln n)^{1-\delta}\rfloor]$. Each region R_k begins and ends with a biased block and all interior blocks (possibly none) are fair blocks.

Note that with this construction the adjacent regions overlap, but each site $x \in \lfloor(\ln n)^{1-\delta}\rfloor\mathbb{Z}$ (which are the end-points of the blocks) is contained in the interior of exactly one region. The random walk may then be decomposed into excursions within the different regions. Let $T_0 = 0$ and let T_1 be the first time the random walk reaches the boundary of the region having $X_{T_0} = 0$ in its interior. For $k \geq 2$, let T_k be the first time after T_{k-1} that the random walk reaches the boundary of the region containing $X_{T_{k-1}}$ in its interior. Let $N := \max\{k\colon T_{k-1} < 2n\}$ be the number of excursions within regions started before time $2n$. The following lemma allows us to bound N from above.

Lemma 3.6. *For any environment* ω,

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_\omega\left(\max_{k\leq 2n}|X_k| \leq n^\gamma, N \geq \frac{5n}{(\ln n)^{3-2\delta}}\right) = -\infty.$$

Proof. As in [11], let $N_{2n}^{\leftarrow}$ denote the total number of instances before time $2n$ that the random walk crosses a biased block from right to left. Lemma 4 in [11] implies that for any environment ω,

$$\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_\omega\left(N_{2n}^{\leftarrow} \geq \frac{2n}{(\ln n)^{3-2\delta}}\right) = -\infty. \tag{17}$$

Also, note that on the event $\{\max_{k\leq n}|X_k| \leq n^\gamma\}$ (cf. (45) in [11]),

$$0 \leq N \leq \frac{2n^\gamma}{(\ln n)^{1-\delta}} + 2N_{2n}^{\leftarrow}.$$

Indeed, there are at most $2n^\gamma/(\ln n)^{1-\delta}$ regions intersecting $[-n^\gamma, n^\gamma]$ and each of these can be crossed once without any left crossings of biased blocks. Since each region begins and ends with a biased block, additional excursions within regions can only be accomplished by crossing a biased block from right to left and each such right-left crossing of a biased block can contribute to at most two more excursions within regions. Therefore, for n sufficiently large,

$$P_\omega\left(\max_{k\leq 2n}|X_k| \leq n^\gamma, N \geq \frac{5n}{(\ln n)^{3-2\delta}}\right) \leq P_\omega\left(N_{2n}^{\leftarrow} \geq \frac{2n}{(\ln n)^{3-2\delta}}\right).$$

Recalling (17) finishes the proof of Lemma 3.6. □

To finish the proof of Proposition 3.5, we need to show that

$$\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_\omega\left(\max_{k\leq 2n}|X_k| \leq n^\gamma, N < \frac{5n}{(\ln n)^{3-2\delta}}\right) \leq -\frac{|\pi \ln\alpha|^2}{4\gamma^2}, \quad P\text{-a.s.} \tag{18}$$

We next modify the hitting times T_k to account for exiting the interval $[-n^\gamma, n^\gamma]$ as well as reaching the boundary of a region. Let $T_0^* = 0$, and T_1^* be the first time the random walk either exits the interval $[-n^\gamma, n^\gamma]$ or reaches the boundary of the region containing $X_{T_0^*}$ in its interior. Similarly, for $k \geq 2$ let T_k^* be the first time after T_{k-1}^* that the random walk either exits the interval $[-n^\gamma, n^\gamma]$ or reaches the boundary of the region containing $X_{T_{k-1}^*}$ in its interior. Note that on the event $\{\max_{k\leq 2n}|X_k| \leq n^\gamma\}$ we have $T_k = T_k^*$ for all $k < N$. Then, for any $K \in \mathbb{N}$ and any $\lambda > 0$, the strong Markov property implies that

$$\begin{aligned}
&P_\omega\left(\max_{k\leq 2n}|X_k| \leq n^\gamma, N = K\right)\\
&\quad \leq P_\omega\left(\sum_{k=1}^{K}\left(T_k^* - T_{k-1}^*\right) > 2n; |X_{T_k^*}| \leq n^\gamma, \forall k < K\right)
\end{aligned}$$

$$\le \mathrm{e}^{-2\lambda n} E_\omega\left[\exp\left\{\lambda \sum_{k=1}^{K}\left(T_k^* - T_{k-1}^*\right)\right\}\mathbf{1}\left\{|X_{T_k^*}| \le n^\gamma, \forall k < K\right\}\right]$$

$$\le \mathrm{e}^{-2\lambda n} E_\omega\left[\exp\left\{\lambda \sum_{k=1}^{K-1}\left(T_k^* - T_{k-1}^*\right)\right\} E_\omega^{X_{T_{K-1}^*}}\left[\mathrm{e}^{\lambda T_1^*}\right]\mathbf{1}\left\{|X_{T_k^*}| \le n^\gamma, \forall k < K\right\}\right]. \tag{19}$$

The inner expectation in the last line above is of the form $E_\omega^x[\mathrm{e}^{\lambda T_I}]$, where $T_I = \min\{k \ge 0\colon\ X_k \notin I\}$ is the first time the random walk exits the interval I, $I = R_j^\circ \cap [-n^\gamma, n^\gamma]$, and R_j is the region with x in its interior R_j°. As was shown in [12] (see Eq. (47) through Lemma 3) we can choose an appropriate λ and bound such expectations uniformly in x, I, and all environments ω with $\omega_x \ge 1/2$ for all $x \in I$. Indeed, for any $\rho \in (0,1)$ there exists a constant $\chi(\rho) \in (1,\infty)$ such that

$$E_\omega^x\left[\mathrm{e}^{\lambda' T_I}\right] \le \chi(\rho), \quad \text{where } \lambda' = \frac{(1-\rho)\pi^2}{8(|I|-1)^2}.$$

Note that λ' decreases in the size of the interval I. Thus, choosing λ according to the largest region intersecting $[-n^\gamma, n^\gamma]$ and iterating the computation in (19) we obtain that

$$P_\omega\left(\max_{k\le 2n}|X_k| \le n^\gamma, N = K\right) \le \exp\left\{-n\frac{(1-\rho)\pi^2}{4(\max_j |R_j^\circ \cap [-n^\gamma, n^\gamma]|)^2}\right\}\chi(\rho)^K. \tag{20}$$

It remains to give an upper bound on $\max_j |R_j^\circ \cap [-n^\gamma, n^\gamma]|$. To this end, for any $p \in (0,1)$ and $x \in [0,1]$ let $\Lambda_p^*(x) = x\ln(x/p) + (1-x)\ln((1-x)/(1-p))$ be the large deviation rate function for a Bernoulli(p) random variable. Then, Lemma 3 in [11] implies that for any ε and ξ fixed as in the definition of the blocks, P-a.s., there exists an $n_1(\omega,\varepsilon,\xi)$ such that

$$\max_j\left|R_j^\circ \cap \left[-n^\gamma, n^\gamma\right]\right| \le \frac{(1+\varepsilon)\gamma}{\Lambda^*_{P(\omega_0\ge 1/2+\xi)}(\varepsilon)}\ln n \quad \forall n \ge n_1(\omega,\varepsilon,\xi).$$

Therefore, recalling (20),

$$\limsup_{n\to\infty}\frac{(\ln n)^2}{n}\ln P_\omega\left(\max_{k\le 2n}|X_k| \le n^\gamma, N < \frac{5n}{(\ln n)^{3-2\delta}}\right) \le -\frac{(1-\rho)\pi^2(\Lambda^*_{P(\omega_0\ge 1/2+\xi)}(\varepsilon))^2}{4(1+\varepsilon)^2\gamma^2}. \tag{21}$$

Note that the left-hand side does not depend on ρ, ε or ξ and that

$$\lim_{\xi\to 0}\lim_{\varepsilon\to 0}\Lambda^*_{P(\omega_0\ge 1/2+\xi)}(\varepsilon) = \lim_{\xi\to 0} -\ln P(\omega_0 < 1/2+\xi) = -\ln P(\omega_0 = 1/2) = -\ln\alpha.$$

Thus, taking ρ, ε, and then ξ to zero in (21) implies (18) and thus finishes the proof of Proposition 3.5. □

Proof of Theorem 3.1. We first prove (12) for $\beta < 2$ which gives an upper bound on the maximal displacement of a bridge. First, note that

$$P_\omega\left(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\right) \le P_\omega(T_{-n/(\ln n)^\beta} < n) + P_{\theta^{n/(\ln n)^\beta}\omega}(T_{-n/(\ln n)^\beta} < n). \tag{22}$$

The shift invariance of P and Chebychev's inequality imply that for any $\delta > 0$

$$P\left(P_\omega\left(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\right) \ge 2\mathrm{e}^{-\delta n/(\ln n)^\beta}\right)$$

$$\le 2P\left(P_\omega(T_{-n/(\ln n)^\beta} < n) > \mathrm{e}^{-\delta n/(\ln n)^\beta}\right)$$

$$\le 2\mathrm{e}^{\delta n/(\ln n)^\beta}\mathbb{P}(T_{-n/(\ln n)^\beta} < n). \tag{23}$$

Lemma 2.2 in [2] gives that $\mathbb{P}(T_{-x} < \infty) \le \frac{(E_P \rho_0)^x}{1 - E_P \rho_0}$ for any $x \ge 0$ (note that Assumption 4 implies that $E_P \rho_0 < 1$). Therefore, if $0 < \delta < -\ln(E_P \rho_0)$, then (23) and the Borel–Cantelli lemma imply that $P_\omega(\max_{k \le 2n} |X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0) \le 2\mathrm{e}^{-\delta n/(\ln n)^\beta}$ for all sufficiently large n, P-a.s. Therefore, Lemma 3.3 implies that, P-a.s., for $\delta < -\ln(E_P \rho_0)$, $C > \frac{|\pi \log \alpha|^2}{4}$, and n large enough,

$$P_\omega\left(\max_{k \le 2n} |X_k| \ge \frac{n}{(\ln n)^\beta} \Big| X_{2n} = 0\right) \le \frac{2\mathrm{e}^{-\delta n/(\ln n)^\beta}}{\mathrm{e}^{-Cn/(\ln n)^2}}.$$

If $\beta < 2$ the right-hand side vanishes as $n \to \infty$.

To get the lower bound on the maximal displacement fix $\gamma \in (0, 1)$. Then,

$$\begin{aligned} P_\omega\Big(\max_{k \le 2n} |X_k| > n^\gamma \Big| X_{2n} = 0\Big) &= 1 - P_\omega\Big(\max_{k \le 2n} |X_k| \le n^\gamma \Big| X_{2n} = 0\Big) \\ &\ge 1 - \frac{P_\omega(\max_{k \le 2n} |X_k| \le n^\gamma)}{P_\omega(X_{2n} = 0)}. \end{aligned}$$

Lemma 3.3 and Proposition 3.5 imply that the right-hand side tends to 1 as $n \to \infty$, P-a.s., thus completing the proof of (13). □

4. Case III: Non-nestling environment

In this section we will make the following assumptions on the environment.

Assumption 5. *$\omega_{\min} > \frac{1}{2}$, and $\alpha := P(\omega_0 = \omega_{\min}) \in (0, 1)$.*

Assumption 6. *There exists an $\eta > 0$ such that $P(\omega_0 \in (\omega_{\min}, \omega_{\min} + \eta)) = 0$.*

Assumption 5 is the crucial assumption needed for the main results. Assumption 6 is a technical assumption and all the main results should be true under only Assumption 5. Note that Assumption 5 implies Assumption 2.

The main result in this section is that the magnitude of displacement in bridges is the same in the non-nestling case as it is in the marginally nestling case.

Theorem 4.1. *Let Assumptions 1, 5, and 6 hold. Then for any $\beta < 2$,*

$$\lim_{n \to \infty} P_\omega\left(\max_{k \le 2n} |X_k| \ge \frac{n}{(\ln n)^\beta} \Big| X_{2n} = 0\right) = 0, \quad P\text{-a.s.}, \tag{24}$$

and for any $\gamma < 1$,

$$\lim_{n \to \infty} P_\omega\Big(\max_{k \le 2n} |X_k| \ge n^\gamma \Big| X_{2n} = 0\Big) = 1, \quad P\text{-a.s.} \tag{25}$$

The quenched large deviation principle for X_n/n, established in [1,8] implies that there exists a deterministic function $I(v)$ such that $P_\omega(X_n/n \approx v) \approx \mathrm{e}^{-nI(v)}$. While there is a probabilistic formula for $I(v)$, it is not computable in practice for most values of v. However, when $v = 0$, the value is known. If the environment is nestling or marginally nestling then $I(0) = 0$, but in the non-nestling case

$$I(0) = -\frac{1}{2} \ln\big(4\omega_{\min}(1 - \omega_{\min})\big) > 0.$$

As was the case for Theorem 3.1, the keys to the proof of Theorem 4.1 are the asymptotics of the quenched probabilities to be at the origin at time $2n$ or to stay confined to the interval $[-n^\gamma, n^\gamma]$ for the first $2n$ steps of the random walk. As will be shown below, both of these probabilities have the same exponential rate of decay:

$$\lim_{n\to\infty}\frac{1}{n}\ln P_\omega(X_{2n}=0)=\lim_{n\to\infty}\frac{1}{n}\ln P_\omega\Bigl(\max_{k\le 2n}|X_k|\le n^\gamma, X_{2n}=0\Bigr)=-2I(0),\quad P\text{-a.s.}$$

Thus, in order to prove Theorem 4.1, more precise asymptotics of the decays of these quenched probabilities will be needed.

A key tool of our analysis in the non-nestling case will be a transformation of the environment ω into an environment $\tilde\omega$ that is marginally nestling. For any environment ω, let $\tilde\omega$ be the environment defined by

$$\tilde\omega_x=\frac{\rho_{\max}}{\rho_x+\rho_{\max}},\quad x\in\mathbb{Z},\tag{26}$$

where $\rho_{\max}=\frac{1-\omega_{\min}}{\omega_{\min}}$. Note that under Assumption 5, $P(\tilde\omega_x\ge\frac12)=1$ and $P(\tilde\omega_x=\frac12)=P(\omega_x=\omega_{\min})=\alpha>0$. The usefulness of this transformation stems from the following lemma.

Lemma 4.2. *Let Assumptions 5 and 6 hold, and let*

$$B_n:=\#\{k<2n:\ \omega_{X_k}>\omega_{\min}\}.$$

Then there exist constants $c_1,c_2\in(0,1)$ (depending on $\omega_{\min}$ and η) such that for any event $A\in\sigma(X_0,X_1,\dots,X_{2n})$ depending only on the first $2n$ steps of the random walk such that $A\subset\{X_{2n}=0\}$,

$$e^{-2I(0)n}E_{\tilde\omega}\bigl[c_1^{B_n}\mathbf{1}_A\bigr]\le P_\omega(A)\le e^{-2I(0)n}E_{\tilde\omega}\bigl[c_2^{B_n}\mathbf{1}_A\bigr],\quad P\text{-a.s.}$$

Proof. Let $X_{[0,2n]}=(X_0,X_1,\dots,X_{2n})$ denote the path of the random walk in the time interval $[0,2n]$. Then,

$$P_\omega(A)=E_{\tilde\omega}\biggl[\frac{dP_\omega}{dP_{\tilde\omega}}(X_{[0,2n]})\mathbf{1}_A\biggr],\tag{27}$$

where for any $\mathbf{x}_{[0,2n]}=(x_0,x_1,\dots,x_{2n})$ we have

$$\frac{dP_\omega}{dP_{\tilde\omega}}(\mathbf{x}_{[0,2n]})=\frac{P_\omega(X_k=x_k,\forall k\in[0,2n])}{P_{\tilde\omega}(X_k=x_k,\forall k\in[0,2n])}=\prod_{k=0}^{2n-1}\frac{P_\omega^{x_k}(X_1=x_{k+1})}{P_{\tilde\omega}^{x_k}(X_1=x_{k+1})}.$$

Recalling the formula for $\tilde\omega_x$ in (26), we obtain that

$$\frac{P_\omega^{x_k}(X_1=x_{k+1})}{P_{\tilde\omega}^{x_k}(X_1=x_{k+1})}=\begin{cases}\omega_x(\rho_x+\rho_{\max})\rho_{\max}^{-1} & \text{if } x_{k+1}=x_k+1,\\ \omega_x(\rho_x+\rho_{\max}) & \text{if } x_{k+1}=x_k-1.\end{cases}$$

Then since $X_{2n}=0$ implies that exactly half of the first $2n$ steps of the random walk are to the right we obtain that,

$$\frac{dP_\omega}{dP_{\tilde\omega}}(X_{[0,2n]})=\rho_{\max}^{-n}\prod_{k=0}^{2n-1}\omega_{X_k}(\rho_{X_k}+\rho_{\max})\quad\forall X_{[0,2n]}\in\{X_{2n}=0\}.\tag{28}$$

If $\omega_x=\omega_{\min}$ then $\omega_x(\rho_x+\rho_{\max})=2(1-\omega_{\min})$, and since $\omega_x(\rho_x+\rho_{\max})$ is decreasing in ω_x there exist constants $c_1,c_2<1$ (depending on $\omega_{\min}$ and η) such that

$$c_1 2(1-\omega_{\min})\le\omega_x(\rho_x+\rho_{\max})\le c_2 2(1-\omega_{\min})\quad\forall\omega_x\in[\omega_{\min}+\eta,1].$$

Since Assumption 6 implies that $P(\omega_x\in\{\omega_{\min}\}\cup[\omega_{\min}+\eta,1])=1$, (28) implies that, P-a.s.,

$$\rho_{\max}^{-n}\bigl(2(1-\omega_{\min})\bigr)^{2n}c_1^{B_n}\le\frac{dP_\omega}{dP_{\tilde\omega}}(X_{[0,2n]})\le\rho_{\max}^{-n}\bigl(2(1-\omega_{\min})\bigr)^{2n}c_2^{B_n}\quad\forall X_{[0,2n]}\in\{X_{2n}=0\}.\tag{29}$$

Since $\rho_{\max}^{-n}(2(1-\omega_{\min}))^{2n} = (4\omega_{\min}(1-\omega_{\min}))^n = \mathrm{e}^{-2I(0)n}$, applying (29) to (27) completes the proof of the Lemma. □

We now apply Lemma 4.2 to prove precise decay rates of certain quenched probabilities.

Proposition 4.3. *Let Assumptions 1, 5, and 6 hold. Then, for any $\gamma \in (0,1]$,*

$$\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \Bigl\{ \ln P_\omega\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0\Bigr) + 2nI(0) \Bigr\} = -\frac{|\pi \ln\alpha|^2}{\gamma^2}, \quad P\text{-a.s.}$$

Before giving the proof of Proposition 4.3 we state the following corollary which is obtained by taking $\gamma = 1$.

Corollary 4.4. *Let Assumptions 1, 5 and 6 hold. Then,*

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} \bigl\{ \ln P_\omega(X_{2n} = 0) + 2nI(0) \bigr\} = -|\pi \ln\alpha|^2.$$

Remark 4.5. *Since $B_n \ge 0$ by definition and the environment $\tilde\omega$ is marginally nestling, a simple application of Lemma 4.2 and Proposition 3.5 implies that*

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \Bigl\{ \ln P_\omega\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0\Bigr) + 2nI(0) \Bigr\} \\
&\quad\le \limsup_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_{\tilde\omega}\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0\Bigr) \le -\frac{|\pi \ln\alpha|^2}{4\gamma^2}, \quad P\text{-a.s.}
\end{aligned}$$

This does not quite give the correct upper bound obtained in Proposition 4.3 and reflects a subtle but important difference between the way that a RWRE is confined to the interval $[-n^\gamma, n^\gamma]$ in marginally nestling and non-nestling environments. The proof of Proposition 3.5 suggests that under Assumption 4, B_n is typically of the order $n/\ln n$ on the event $\{\max_{k\le 2n} |X_k| \le n^\gamma\}$. In contrast, the proof of Proposition 4.3 below will suggest that under Assumptions 5 and 6, B_n is typically less than $n/(\ln n)^{2-\delta}$ for any $\delta > 0$ on the event $\{\max_{k\le 2n} |X_k| \le n^\gamma\}$.

Proof of Proposition 4.3. We first give a lower bound. Lemma 4.2 implies that, P-a.s.,

$$\begin{aligned}
&P_\omega\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0\Bigr) \\
&\quad\ge P_\omega\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0, B_n \le \frac{2n}{(\ln n)^3}\Bigr) \\
&\quad\ge \mathrm{e}^{-2I(0)n} c_1^{n/(\ln n)^3} P_{\tilde\omega}\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0, B_n \le \frac{2n}{(\ln n)^3}\Bigr),
\end{aligned} \tag{30}$$

where $B_n = \#\{k \le 2n\colon \tilde\omega_{X_k} > 1/2\}$ is the number of visits to biased sites in the environment $\tilde\omega$ before time $2n$. Since $\tilde\omega$ is a marginally nestling environment, we can obtain a lower bound for the last probability above in a similar manner as the lower bounds for Lemma 3.3 and Proposition 3.5. However, because of the added condition that $B_n \le \frac{2n}{(\ln n)^3}$, instead of adding a reflection at the left edge of the longest fair stretch in $[0, n^\gamma]$ (or $[0, n/(\ln n)^3]$ when $\gamma = 1$), we instead force the random walk to stay strictly inside the longest fair stretch for time $2n$. Since the longest fair stretch in $[0, n^\gamma]$ (or $[0, n/(\ln n)^3]$ when $\gamma = 1$) in the environment $\tilde\omega$ is of size $(\gamma/|\ln\alpha| + o(1))\ln n$, we obtain from uniform ellipticity that, P-a.s.,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_{\tilde\omega}\Bigl(\max_{k\le 2n} |X_k| \le n^\gamma, X_{2n} = 0, B_n \le \frac{2n}{(\ln n)^3}\Bigr) \\
&\quad\ge \liminf_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_{1/2}\Bigl(\max_{k\le 2n} |X_k| \le \frac{\gamma}{2|\ln\alpha|} \ln n\Bigr) = -\frac{|\pi \ln\alpha|^2}{\gamma^2}.
\end{aligned} \tag{31}$$

Note that the last equality is again obtained from Lemma 3.4. Combining (30) and (31) completes the proof of the needed lower bound.

To get a corresponding upper bound, first note that by Lemma 4.2, P-a.s., considering the events $\{B_n > \frac{n}{(\ln n)^{2-\delta}}\}$ and $\{B_n \le \frac{n}{(\ln n)^{2-\delta}}\}$,

$$P_\omega\Big(\max_{k\le 2n}|X_k|\le n^\gamma, X_{2n}=0\Big) \le \mathrm{e}^{-2I(0)n}\bigg\{c_2^{n/(\ln n)^{2-\delta}} + P_{\tilde\omega}\bigg(\max_{k\le 2n}|X_k|\le n^\gamma, B_n \le \frac{n}{(\ln n)^{2-\delta}}\bigg)\bigg\}.$$

Then, since $c_2 < 1$, it is enough to show that for some $\delta > 0$

$$\limsup_{n\to\infty}\frac{(\ln n)^2}{n}\ln P_{\tilde\omega}\bigg(\max_{k\le 2n}|X_k|\le n^\gamma, B_n\le\frac{n}{(\ln n)^{2-\delta}}\bigg)\le -\frac{|\pi\ln\alpha|^2}{\gamma^2}.$$

For any $\varepsilon>0$ and n large enough, $B_n\le n/(\ln n)^{2-\delta}<\varepsilon n$ implies that the random walk must spend at least $(2-\varepsilon)n$ steps at sites that are fair in $\tilde\omega$. Theorem 3.2.1 in [3] implies that, P-a.s., for all n large enough the length of the longest fair stretch in $[-n^\gamma, n^\gamma]$ in the environment $\tilde\omega$ is less than $\frac{(1+\varepsilon)\gamma}{|\ln\alpha|}\ln n$. Therefore, P-a.s., for n sufficently large,

$$P_{\tilde\omega}\bigg(\max_{k\le 2n}|X_k|\le n^\gamma, B_n\le\frac{n}{(\ln n)^{2-\delta}}\bigg) \le P_{\tilde\omega}(\tau\ge\varepsilon n) + P_{1/2}\bigg(\sum_{j\le n/(\ln n)^{2-\delta}}\sigma_{j,n}\ge 2n(1-\varepsilon)\bigg),$$

where $\tau := \min\{k\ge 0\colon \tilde\omega_{X_k}>1/2\}$ is the first time the random walk reaches a biased site in $\tilde\omega$ and the $\sigma_{j,n}$ are i.i.d. with common distribution equal to that of the first time a simple random walk started at $x=1$ exits the interval $[1,(1+\varepsilon)\gamma\ln n/|\ln\alpha|]$. For any fixed environment $\tilde\omega$, τ is the exit time of a simple symmetric random walk from a fixed bounded interval and thus $P_{\tilde\omega}(\tau>\varepsilon n)$ decays exponentially fast. Therefore, the proof of the upper bound is reduced to showing that for some $\delta>0$,

$$\limsup_{n\to\infty}\frac{(\ln n)^2}{n}\ln P_{1/2}\bigg(\sum_{j\le n/(\ln n)^{2-\delta}}\sigma_{j,n}\ge 2n(1-\varepsilon)\bigg)\le-\frac{|\pi\ln\alpha|^2}{\gamma^2}. \tag{32}$$

To this end, note that for any $\lambda>0$,

$$P_{1/2}\bigg(\sum_{j\le n/(\ln n)^{2-\delta}}\sigma_{j,n}\ge(1-\varepsilon)2n\bigg)\le \mathrm{e}^{-\lambda(1-\varepsilon)2n}\big\{E_{1/2}\big[\mathrm{e}^{\lambda\sigma_{1,n}}\big]\big\}^{n/(\ln n)^{2-\delta}}. \tag{33}$$

To complete the proof of the upper bound, we need the following lemma.

Lemma 4.6. *Let 2ℓ be an integer greater than* 1 *and let* $\varepsilon\in(0,1)$. *Then, there exists a constant $C_1<\infty$ depending only on ε such that for* $\lambda(\varepsilon,\ell) := \frac{(1-\varepsilon)^2\pi^2}{8\ell^2}$,

$$E_{1/2}\big[\mathrm{e}^{\lambda(\varepsilon,\ell)\sigma}\big]<1+\frac{C_1}{\ell},$$

where σ is the first time a simple random walk started at $x=1$ exits the interval $[1,2\ell-1]$.

Using Lemma 4.6, we complete the proof of the upper bound in Proposition 4.3. Let $\lambda = \lambda(\varepsilon, \ell(n)) = \frac{(1-\varepsilon)^2\pi^2}{8\ell(n)^2}$ in (33), where $2\ell(n) - 1 = \lfloor (1+\varepsilon)\gamma \ln n/|\ln\alpha| \rfloor$. Then Lemma 4.6 implies that for any $\delta \in (0,1)$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{(\ln n)^2}{n} \ln P_{1/2}\biggl(\sum_{j\le n/(\ln n)^{2-\delta}} \sigma_{j,n} \ge (1-\varepsilon)2n\biggr) \\
&\quad\le \limsup_{n\to\infty} -\frac{(1-\varepsilon)^3\pi^2(\ln n)^2}{4\ell(n)^2} + (\ln n)^\delta \ln\biggl(1+\frac{C_1}{\ell(n)}\biggr) \\
&\quad= -\frac{(1-\varepsilon)^3|\pi\ln\alpha|^2}{(1+\varepsilon)^2\gamma^2}.
\end{aligned}$$

Taking $\varepsilon \to 0$ finishes the proof of (32). It remains only to give the proof of Lemma 4.6.

Proof of Lemma 4.6. As was shown in the proof of Lemma 4 in [6], the moment generating function for the first exit time of a simple random walk from an interval may be solved explicitly. In particular, for $0 \le \lambda < \lambda_{\text{crit}}(\ell) = -\ln(\cos\frac{\pi}{2\ell})$ we obtain that

$$E_{1/2}\bigl[e^{\lambda\sigma}\bigr] = \frac{\cos(c_\lambda(\ell-1))}{\cos(c_\lambda\ell)} = e^{-\lambda} + \sqrt{1-e^{-2\lambda}}\tan(c_\lambda\ell),$$

where $c_\lambda = \arccos(e^{-\lambda})$. In particular, we get the simple upper bound

$$E_{1/2}\bigl[e^{\lambda\sigma}\bigr] < 1 + \sqrt{2\lambda}\tan(c_\lambda\ell). \tag{34}$$

Recalling the definitions of $\lambda(\varepsilon,\ell)$ and c_λ, and using the fact that $\arccos(e^{-x}) < \sqrt{2x}$ for any $x > 0$ we obtain that

$$c_{\lambda(\varepsilon,\ell)}\ell = \arccos\biggl(\exp\biggl\{-\frac{(1-\varepsilon)^2\pi^2}{8\ell^2}\biggr\}\biggr)\ell < \frac{(1-\varepsilon)\pi}{2}.$$

Therefore, $\tan(c_{\lambda(\varepsilon,\ell)}\ell)$ is bounded above, uniformly in ℓ. Recalling (34) we obtain

$$E_{1/2}\bigl[e^{\lambda(\varepsilon,\ell)\sigma}\bigr] < 1 + \sqrt{2\lambda(\varepsilon,\ell)}\tan\biggl(\frac{(1-\varepsilon)\pi}{2}\biggr) = 1 + \frac{(1-\varepsilon)\pi}{2\ell}\tan\biggl(\frac{(1-\varepsilon)\pi}{2}\biggr).$$

This completes the proof of Lemma 4.6 with $C_1 = \frac{(1-\varepsilon)\pi}{2}\tan(\frac{(1-\varepsilon)\pi}{2})$. □

We now are ready to give the proof of the main result in this section.

Proof of Theorem 4.1. Let $\tilde\omega$ be the marginally nestling environment obtained from the non-nestling environment ω as defined in (26). Lemma 4.2 implies that

$$\begin{aligned}
P_\omega\Bigl(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\Bigr) &= E_{\tilde\omega}\biggl[\frac{dP_\omega}{dP_{\tilde\omega}}(X_{[0,2n]})\mathbf{1}\Bigl\{\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\Bigr\}\biggr] \\
&\le e^{-2nI(0)} P_{\tilde\omega}\Bigl(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\Bigr).
\end{aligned}$$

Then, since $\tilde\omega$ is a marginally nestling environment, the proof of Theorem 3.1 implies that for δ sufficiently small, P-a.s., for all n sufficiently large,

$$P_\omega\Bigl(\max_{k\le 2n}|X_k| \ge \frac{n}{(\ln n)^\beta}, X_{2n} = 0\Bigr) \le e^{-2nI(0)-\delta n/(\ln n)^\beta}.$$

This combined with Corollary 4.4 completes the proof of (24). Equation (25) follows directly from Proposition 4.3 and Corollary 4.4. □

5. Further refinements

We believe that in the marginally nestling and non-nestling regimes, the maximal displacement of bridges is really of the order $n/(\ln n)^2$. The proofs of Theorems 3.1 and 4.1 suggest that the most likely way for the random walk to be back at the origin after $2n$ steps is to go quickly to a long interval I with $\omega_x = \omega_{\min}$ for all $x \in I$ and then stay in the interval I for almost $2n$ steps before returning quickly to the origin. However, the longest such interval $I \subset [-n/(\ln n)^\beta, n/(\ln n)^\beta]$ has length of order $\ln n/|\ln \alpha|$ for any $\beta > 0$. Thus, it is difficult to show that the maximal displacement is at least $n/(\ln n)^\beta$ for any $\beta > 2$ when conditioned on $\{X_{2n} = 0\}$. Nevertheless, for any fixed $\beta > 2$ the longest interval $I \subset [-n/(\ln n)^2, n/(\ln n)^2]$ with $\omega_x = \omega_{\min}$ for all $x \in I$ is with high probability not contained in $[-n/(\ln n)^\beta, n/(\ln n)^\beta]$. This leads us to the following conjecture.

Conjecture 5.1. *Let Assumption 1 hold, and let $\omega_{\min} \geq 1/2$ and $P(\omega_0 = \omega_{\min}) = \alpha > 0$. Then, for any $\beta > 2$,*

$$\lim_{n\to\infty} P_\omega\left(\max_{k\leq 2n} |X_k| \geq \frac{n}{(\ln n)^\beta} \middle| X_{2n} = 0\right) = 1, \quad \textit{in } P\textit{-probability.}$$

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References

[1] F. Comets, N. Gantert and O. Zeitouni. Quenched, annealed and functional large deviations for one-dimensional random walk in random environment. *Probab. Theory Related Fields* **118** (2000) 65–114. MR1785454
[2] A. Dembo, Y. Peres and O. Zeitouni. Tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.* **181** (1996) 667–683. MR1414305
[3] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, 2nd edition. *Applications of Mathematics* **38**. Springer, New York, 1998. MR1619036
[4] N. Enriquez, C. Sabot and O. Zindy. Limit laws for transient random walks in random environments on $\mathbb{Z}$. *Ann. Institut Fourier* **59** (2009) 2469–2508. MR2640927
[5] A. Fribergh, N. Gantert and S. Popov. On slowdown and speedup of transient random walks in random environment. *Probab. Theory Related Fields* **147** (2010) 43–88. MR2594347
[6] N. Gantert. Subexponential tail asymptotics for a random walk with randomly placed one-way nodes. *Ann. Inst. H. Poincaré Probab. Statist.* **38** (2002) 1–16. MR1899228
[7] N. Gantert and O. Zeitouni. Quenched sub-exponential tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.* **194** (1998) 177–190. MR1628294
[8] A. Greven and F. den Hollander. Large deviations for a random walk in random environment. *Ann. Probab.* **22** (1994) 1381–1428. MR1303649
[9] H. Kesten, M. V. Kozlov and F. Spitzer. A limit law for random walk in a random environment. *Compos. Math.* **30** (1975) 145–168. MR0380998
[10] A. A. Mogul'skiĭ. Small deviations in the space of trajectories. *Teor. Verojatn. Primen.* **19** (1974) 755–765. MR0370701
[11] A. Pisztora and T. Povel. Large deviation principle for random walk in a quenched random environment in the low speed regime. *Ann. Probab.* **27** (1999) 1389–1413. MR1733154
[12] A. Pisztora, T. Povel and O. Zeitouni. Precise large deviation estimates for a one-dimensional random walk in a random environment. *Probab. Theory Related Fields* **113** (1999) 191–219. MR1676839
[13] F. Solomon. Random walks in a random environment. *Ann. Probab.* **3** (1975) 1–31. MR0362503
[14] O. Zeitouni. Random walks in random environment. In *Lectures on Probability Theory and Statistics* 189–312. *Lecture Notes in Math.* **1837**. Springer, Berlin, 2004. MR2071631