SUBEXPONENTIAL TAIL ASYMPTOTICS FOR A RANDOM WALK WITH RANDOMLY PLACED ONE-WAY NODES

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ABSTRACT. – Let $p \in \,]0, 1/2]$ and assign to the integers random variables $\{\omega_x\}$, taking only the two values 1 and p, which serve as an environment. This environment defines a random walk $\{X_n\}$ which, when at x, moves one step to the right with probability ω_x, and one step to the left with probability $1 - \omega_x$. In particular, at the nodes, i.e., at the locations x with $\omega_x = 1$, no backtrack is possible. We will assume that the speed v_α of the random walk is positive. We then investigate, for $v < v_\alpha$, the decay of the probabilities $\mathbb{P}_\omega[X_n/n \leqslant v]$ (for fixed environment) and $\mathbb{P}[X_n/n \leqslant v]$ (averaged over the environment). These probabilities decay subexponentially and there is a wide range of possible normalizations, depending on the distribution of the lengths of the intervals without nodes. We show that in fact only the behaviour of the length of the largest interval without nodes (contained in $[0, n]$) matters. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Soit $p \in \,]0, 1/2]$ et soit $(\omega_x)_{x \in \mathbb{Z}}$ une suite de variables prenant les deux valeurs 1 et p, utilisée comme environnement. Dans cet environnement, on définit une marche aléatoire, qui va de x à $x + 1$ avec probabilité ω_x, et de x à $x - 1$ avec probabilité $1 - \omega_x$. Quand x est un noeud (un point où $\omega_x = 1$), la marche après avoir dépassé x ne peut plus jamais revenir en $x - 1$. Nous faisons l'hypothèse que cette marche a une vitesse strictement positive, que nous notons v_a. Nous étudions, pour $v < v_a$, la décroissance des probabilités $\mathbb{P}_\omega[X_n/n \leqslant v]$ (pour un environnement fixé) et $\mathbb{P}[X_n/n \leqslant v]$ (moyennée sur les environnements). Ces probabilités décroissent sous-exponentiellement et la normalisation dépend de la loi des longueurs des intervalles sans noeuds. Plus précisément, nous montrons que c'est seulement l'intervalle sans noeuds le plus long qui compte. © 2002 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction and statement of results

We will consider the following particular case of a one-dimensional random walk in random environment (RWRE). We first choose the environment, placing one-way nodes at certain random locations. The random walk in this environment is a nearest-neighbour random walk which is forced to go right at the nodes and has fixed (deterministic) probabilities to jump right or left on the intervals without nodes. Let $p \in]0, \frac{1}{2}]$ and $\Omega := \{1, p\}^{\mathbb{Z}}$. We denote the elements of Ω by $\omega = (\omega_x)_{x \in \mathbb{Z}}$. The sequence $(\omega_x)_{x \in \mathbb{Z}}$ will serve as an environment. We assume that the distribution α of $(\omega_x)_{x \in \mathbb{Z}}$ is stationary and ergodic with respect to the shift transformation. Let $\overline{\alpha}$ be the conditional distribution of α, given that $\omega_0 = 1$, i.e., $\overline{\alpha} := \alpha(\cdot \mid \omega_0 = 1)$. Let $\ell_1(\omega), \ell_2(\omega), \ldots$ be the lengths of the successive intervals without nodes, i.e.,

$$\ell_1(\omega) := \inf\{i \geqslant 1 \colon \omega_i = 1\}, \tag{1}$$

$$\ell_1(\omega) := \inf\{i \geqslant 1 \colon \omega_{\ell_1 + \cdots + \ell_{k-1} + i} = 1\}, \quad k \geqslant 2. \tag{2}$$

Then the random variables $\ell_1, \ell_2, \ldots$ form a stationary, ergodic sequence under $\overline{\alpha}$. For every fixed ω let $X = (X_n)_{n \geqslant 0}$ be the Markov chain on $\mathbb{Z}$ starting at $X_0 = 0$, with transition probabilities

$$\mathbf{P}_\omega(X_{n+1} = y \mid X_n = x) = \begin{cases} \omega_x & \text{if } y = x + 1, \\ 1 - \omega_x & \text{if } y = x - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We will denote the distribution of (X_n), given ω (or, equivalently, given $\ell_1, \ell_2, \ldots$), with $\mathbf{P}_\omega$, and the joint distribution of $(\omega, (X_n))$ under $\overline{\alpha} \times \mathbf{P}_\omega$ with $\mathbf{P}$. The process (X_n) was introduced in [13], in the case where α is a product measure, and studied further in [2], in the case where $\ell_1, \ell_2, \ldots$ are i.i.d. random variables. Limit laws for the distribution of X_n, suitably normalized, were investigated in [13] and in [2]. Let

$$\tau_{\ell_1} := \inf\{j \colon X_j = \ell_1\} \tag{4}$$

and

$$\tau_{\ell_{k+1}} := \inf\{j > \tau_{\ell_1} + \cdots + \tau_{\ell_k} \colon X_j = \ell_1 + \cdots + \ell_{k+1}\} - (\tau_{\ell_1} + \cdots + \tau_{\ell_k}), \quad k \geqslant 2. \tag{5}$$

Under appropriate assumptions, the random walk is moving to the right with a positive speed, see [1] and [2].

LEMMA 1. – *With*

$$v_\alpha := \frac{\mathbf{E}[\ell_1]}{\mathbf{E}[\tau_{\ell_1}]}, \tag{6}$$

where $v_\alpha = 0$ if $\mathbf{E}[\tau_{\ell_1}] = \infty$, we have

$$\frac{X_n}{n} \xrightarrow[n \to \infty]{} v_\alpha \quad \mathbf{P}\text{-}a.s., \tag{7}$$

i.e., $X_n/n \to v_\alpha$ $\mathbf{P}_\omega$-a.s., for $\overline{\alpha}$-a.a. ω.

Let

$$\rho := \frac{1-p}{p} \geqslant 1. \tag{8}$$

Later, see (45) and (56), we will give an explicit formula for $\mathbf{E}[\tau_{\ell_1}]$, leading to

$$\frac{1}{v_\alpha} = -\frac{\rho+1}{\rho-1} + \frac{1}{\mathbf{E}[\ell_1]} \left(\mathbf{E}[\rho^{\ell_1}] - 1\right) \frac{2\rho}{(\rho-1)^2} \quad \text{if } p < \frac{1}{2} \tag{9}$$

and

$$v_\alpha = \frac{\mathbf{E}[\ell_1]}{\mathbf{E}[\ell_1^2]} \quad \text{if } p = \frac{1}{2}. \tag{10}$$

In particular, if $p < \frac{1}{2}$, $v_\alpha > 0$ if and only if $\mathbf{E}[\rho^{\ell_1}] < \infty$ and, if $p = \frac{1}{2}$, $v_\alpha > 0$ if and only if $\mathbf{E}[\ell_1^2] < \infty$. We will always assume that $v_\alpha > 0$. Due to (7), for each $v < v_\alpha$, the probabilities $\mathbf{P}_\omega[X_n/n \leqslant v]$ and $\mathbf{P}[X_n/n \leqslant v]$ go to 0. The goal of this paper is to investigate the rate of this decay. For $v = 0$, one can show that $\mathbf{P}_\omega[X_n/n \leqslant 0] \leqslant \mathbf{P}_\omega[\tau_{\ell_1} > n]$ decays exponentially in n, for $v < 0$, we trivially have $\mathbf{P}_\omega[X_n/n \leqslant v] = 0$. Throughout the paper, we will only consider strictly positive values of v. In the general RWRE model, (ω_x) is an ergodic sequence of random variables with values in $[0, 1]$. Large deviations for this model were investigated in [8,5,3], see also the survey paper [7]. If $\omega_x > \frac{1}{2}$, for all x, it is known that $v_\alpha > 0$ and that for $v < v_\alpha$, $\mathbf{P}_\omega[X_n/n \leqslant v]$ and $\mathbf{P}[X_n/n \leqslant v]$ decay exponentially in n. The exponential rate of decay was identified in [8] and [3]. In contrast, we treat here an environment with "mixed drifts and positive speed". It was shown in [3] that for such environments, under mild assumptions on α, $\mathbf{P}_\omega[X_n/n \leqslant v]$ and $\mathbf{P}[X_n/n \leqslant v]$ decay subexponentially in n for $0 < v < v_\alpha$. For the i.i.d. environment case, i.e., if α is a product measure, subexponential asymptotics were derived in [5] and [12] for the decay of $\mathbf{P}[X_n/n \leqslant v]$, and in [6] and [11], for the decay of $\mathbf{P}_\omega[X_n/n \leqslant v]$. Strictly speaking, not all the above mentioned results apply in our situation, since some of them assume that the support of the distribution of ω_0 is contained in a compact subset of $]0, 1[$. In this paper, we prove subexponential asymptotics for the random walk with randomly placed one-way nodes. We show that under appropriate conditions (see Theorem 1), $\log \mathbf{P}_\omega[X_n/n \leqslant v]$ has "random fluctuations" and there is no deterministic normalization. The same behaviour is conjectured for the general RWRE model, see [7]), and was proved for a random walk with one-way nodes in an i.i.d. environment (see Theorem 9 in [7]). Let

$$N_\ell(n) := \max\{j\colon \ell_1 + \cdots + \ell_j \leqslant n\}, \qquad \ell_{\max}(n) := \max_{1 \leqslant j \leqslant N_\ell(n)} \ell_j(\omega), \tag{11}$$

and

$$N(n) := \min\{j\colon \ell_1 + \cdots + \ell_j \geqslant n\}, \qquad \ell_{\mathrm{MAX}}(n) := \max_{1 \leqslant j \leqslant N(n)} \ell_j(\omega). \tag{12}$$

Let

$$T_n := \inf\{j\colon X_j = n\}. \tag{13}$$

Throughout the paper, a_n, b_n, c_n and d_n are positive, increasing functions of n with $a_n \to \infty$, $b_n \to \infty$, $c_n \to \infty$, $d_n \to \infty$ and $b_n/a_n \to 0$, $c_n/a_n \to 0$ and $d_n/a_n \to 0$. We will further assume that a_n is slowly varying, i.e. that for each $k \geqslant 1$, $a_{nk}/a_n \to 1$ for $n \to \infty$. Our main results are the following.

THEOREM 1. – *Assume that $p < \frac{1}{2}$ and $\mathbf{E}[\rho^{\ell_1}] < \infty$. Assume further that a_n, b_n and c_n as above exist such that*

$$\mathbf{P}\big[\ell_{\max}(n) \geqslant a_n + b_n \text{ for infinitely many } n\big] = 1 \tag{14}$$

and

$$\mathbf{P}\big[\ell_{\max}(n) \leqslant a_n - c_n \text{ for infinitely many } n\big] = 1. \tag{15}$$

Then we have, for each $u > 1/v_\alpha$,

$$\liminf_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] = -\infty \quad \mathbf{P}\text{-a.s.}, \tag{16}$$

$$\limsup_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] = 0 \quad \mathbf{P}\text{-a.s.} \tag{17}$$

Assume in addition that for all $k \in \mathbb{N}$,

$$\limsup_{n \to \infty}(a_{kn} - a_n) < \infty. \tag{18}$$

Then, for $0 < v < v_\alpha$,

$$\liminf_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[X_n \leqslant nv] = -\infty \quad \mathbf{P}\text{-a.s.}, \tag{19}$$

$$\limsup_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[X_n \leqslant nv] = 0 \quad \mathbf{P}\text{-a.s.} \tag{20}$$

Remarks. – 1. In particular, (19) and (20) show that there is no deterministic function f such that,

$$-\infty < \liminf_{n \to \infty} \frac{1}{f(n)} \log \mathbf{P}_\omega[X_n \leqslant nv] \leqslant \limsup_{n \to \infty} \frac{1}{f(n)} \log \mathbf{P}_\omega[X_n \leqslant nv] < 0 \quad \mathbf{P}\text{-a.s.} \tag{21}$$

2. We conjecture that (18) is always satisfied under the previous assumptions of Theorem 1 (note that they imply (36) below!), but we were not able to show this.

THEOREM 2. – *Assume that $p = \frac{1}{2}$ and $\mathbf{E}[\ell_1^\gamma] < \infty$ for some $\gamma > 2$. Assume that for a_n and d_n,*

$$\lim_{n \to \infty} \frac{\ell_{\max}(n)}{a_n} = 1 \quad \mathbf{P}\text{-a.s.} \tag{22}$$

and

$$\mathbf{P}\big[\ell_{\max}(n) \geqslant a_n + d_n \text{ for infinitely many } n\big] = 0. \tag{23}$$

Then we have, for each $u > 1/v_\alpha$,

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] = -\frac{\pi^2}{8}\left(u - \frac{1}{v_\alpha}\right) \quad \mathbf{P}\text{-}a.s. \tag{24}$$

and, for $0 < v < v_\alpha$,

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}_\omega[X_n \leqslant nv] = -\frac{\pi^2}{8}\left(1 - \frac{v}{v_\alpha}\right) \quad \mathbf{P}\text{-}a.s. \tag{25}$$

The following theorem gives statements for the decay of $\mathbf{P}[T_n \geqslant nu]$ and $\mathbf{P}[X_n \leqslant nv]$. It is a general fact that the decay of these averaged probabilities is either slower or the same than for fixed environment.

THEOREM 3. – *Assume that $\ell_1, \ell_2, \dots$ are i.i.d., $p < 1/2$ and $\mathbf{E}[\rho^{3\ell_1}] < \infty$. Assume $\varphi(n)$ is an increasing, slowly varying function such that $\varphi(n) \to \infty$ and for some continuous function $\kappa(\cdot) > 0$,*

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}\big[\ell_{\max}(n) \geqslant t \log n\big] = -\kappa(t) \quad \forall t. \tag{26}$$

Then we have, for each $u > 1/v_\alpha$,

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}[T_n \geqslant nu] = -\kappa\left(\frac{1}{\log \rho}\right), \tag{27}$$

and, for $0 < v < v_\alpha$,

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}[X_n \leqslant nv] = -\kappa\left(\frac{1}{\log \rho}\right). \tag{28}$$

Remark. – Our proof of Theorem 3 requires that $\ell_1, \ell_2, \dots$ are i.i.d. We don't know if this assumption is necessary for (27) and (28) to hold.

The main ingredients of our proofs are exit time asymptotics for the intervals without nodes, combined with the ergodic theorem applied to the environment sequence.

2. Proofs

Sketch of the proof of Lemma 1. – It is well-known that

$$\lim_{n\to\infty} \frac{X_n}{n} = \left(\lim_{n\to\infty} \frac{T_n}{n}\right)^{-1} \quad \mathbf{P}\text{-a.s.} \tag{29}$$

where both sides equal 0 if $T_n/n \to \infty$, see [13]. We will show that

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{} \frac{\mathbf{E}[\tau_{\ell_1}]}{\mathbf{E}[\ell_1]} \quad \mathbf{P}\text{-a.s.} \tag{30}$$

First note that due to the ergodic theorem,

$$\frac{N_\ell(n)}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}[\ell_1]} \quad \text{and} \quad \frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}[\ell_1]} \quad \mathbf{P}\text{-a.s.} \tag{31}$$

Further, the ergodic theorem implies that

$$\frac{1}{n}\sum_{j=1}^{n} \tau_{\ell_j} \underset{n\to\infty}{\longrightarrow} \mathbf{E}[\tau_{\ell_1}] \quad \mathbf{P}\text{-a.s.} \tag{32}$$

But

$$\sum_{j=1}^{N_\ell(n)} \tau_{\ell_j} \leqslant T_n \leqslant \sum_{j=1}^{N(n)} \tau_{\ell_j},$$

hence

$$\frac{N_\ell(n)}{n} \frac{1}{N_\ell(n)} \sum_{j=1}^{N_\ell(n)} \tau_{\ell_j} \leqslant \frac{T_n}{n} \leqslant \frac{N(n)}{n} \frac{1}{N(n)} \sum_{j=1}^{N(n)} \tau_{\ell_j}, \tag{33}$$

and (30) follows. $\quad\square$

We note for further reference that $\mathbf{E}[\ell_1] < \infty$ implies

$$\frac{\ell_{\mathrm{MAX}}(n)}{n} \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbf{P}\text{-a.s.} \tag{34}$$

To show (34), note that $\ell_{\mathrm{MAX}}(n) \leqslant \max_{1\leqslant j\leqslant n} \ell_j$, and $\ell_1, \ell_2, \dots$ are positive random variables with the same distribution and $\mathbf{E}[\ell_1] < \infty$. Let $c > 0$. Then,

$$\sum_{j=1}^{\infty} \mathbf{P}[\ell_j \geqslant cj] = \sum_{j=1}^{\infty} \mathbf{P}[\ell_1 \geqslant cj] = \mathbf{E}\left[\frac{\ell_1}{c}\right] < \infty \tag{35}$$

and the Borel–Cantelli lemma implies that $\mathbf{P}[\ell_j \geqslant cj$ for infinitely many $j] = 0$. We conclude that also $\mathbf{P}[\max_{1\leqslant j\leqslant n} \ell_j \geqslant cn$ for infinitely many $n] = 0$. We also note that under the hypothesis of Theorem 1, we have

$$\limsup_{n\to\infty} \frac{a_n}{\log n} < \infty. \tag{36}$$

To prove (36), taking into account (14), it is enough to show that, for some $c > 0$,

$$\mathbf{P}\big[\ell_{\max}(n) \geqslant c\log n \text{ for infinitely many } n\big] = 0. \tag{37}$$

But

$$\mathbf{P}\big[\ell_{\max}(n) \geqslant c\log n\big] \leqslant n\mathbf{P}[\ell_1 \geqslant c\log n] \leqslant n\mathbf{E}\big[\rho^{\ell_1}\big]\rho^{-c\log n} = n^{1-c\log\rho}\mathbf{E}\big[\rho^{\ell_1}\big] \tag{38}$$

and we see that, for $c > 2/\log\rho$, (37) follows with the Borel–Cantelli lemma.

Proof of Theorem 1. – The proof of (17) is the same as the proof of Theorem 9 in [7]. We will first give an estimate for the exit time of an interval without node. Let P_x be the distribution of the random walk (X_j), started at $x \in \{0, 1, \ldots, \ell\}$, with $\omega_0 = 1$, $\omega_1 = \omega_2 = \cdots = \omega_{\ell-1} = p$, let $\overline{\tau}_\ell := \inf\{j: X_j = \ell\}$ and $\overline{\tau}_0 := \inf\{j: X_j = 0\}$. Then, with ρ defined in (8),

$$P_1(\overline{\tau}_0 < \overline{\tau}_\ell) = \frac{\rho^\ell - \rho}{\rho^\ell - 1} \geqslant \frac{\rho^{\ell-1} - 1}{\rho^{\ell-1}} \quad \text{and} \quad P_1(\overline{\tau}_\ell \geqslant n) \geqslant P_1(\overline{\tau}_0 < \overline{\tau}_\ell)^n \geqslant \left(1 - \frac{1}{\rho^{\ell-1}}\right)^n.$$
(39)

We give a lower bound on $\mathbf{P}_\omega[T_n \geqslant nu]$ by simply picking the largest interval without nodes.

$$\mathbf{P}_\omega[T_n \geqslant nu] \geqslant P_1\big[\overline{\tau}_{\ell_{\max}(n)} \geqslant [nu]\big] \geqslant \left(1 - \frac{1}{\rho^{\ell_{\max}(n)-1}}\right)^{[nu]} \geqslant \left(1 - \frac{1}{\rho^{\ell_{\max}(n)-1}}\right)^{nu}$$
(40)

where $[nu]$ denotes the integer part of nu. We choose a (random) subsequence (n_k) such that $n_k \to \infty$ for $k \to \infty$ and

$$\ell_{\max}(n_k) \geqslant a_{n_k} + b_{n_k}$$

for all k. Then, due to (40), for each $\varepsilon > 0$,

$$\log \mathbf{P}_\omega[T_{n_k} \geqslant n_k u] \geqslant \log \left(1 - \frac{1}{\rho^{\ell_{\max}(n_k)-1}}\right)^{n_k u}$$

$$\geqslant -n_k u \rho^{-(a_{n_k}+b_{n_k})}(1 + \varepsilon) \quad \mathbf{P}\text{-a.s.}$$
(41)

for k large enough. Hence

$$\lim_{k\to\infty} \frac{\rho^{a_{n_k}}}{n_k} \log \mathbf{P}_\omega[T_{n_k} \geqslant n_k u] = 0 \quad \mathbf{P}\text{-a.s.}$$

and (17) follows. To show (16), we first prove the following formula for the exit time of a random walk with reflection at 0.

LEMMA 2. – *Let $\omega_0 = 1$, $\omega_1 = \omega_2 = \cdots = \omega_{\ell-1} = p < \frac{1}{2}$, $X_0 = 0$, $\overline{\tau}_\ell := \inf\{j: X_j = \ell\}$ and $g(\ell) := g(\lambda, \ell) = E_0[\exp(\lambda \overline{\tau}_\ell)]$. Let*

$$\eta_1 = \eta_1(\lambda) = \frac{1}{2p}\left(e^{-\lambda} + \sqrt{e^{-2\lambda} - 4pq}\right),$$
(42)

$$\eta_2 = \eta_2(\lambda) = \frac{1}{2p}\left(e^{-\lambda} - \sqrt{e^{-2\lambda} - 4pq}\right)$$
(43)

with $q := 1 - p$, and

$$\lambda_{\mathrm{crit}} := \lambda_{\mathrm{crit}}(\ell) = \sup\{\lambda > 0: \ e^{-2\lambda} > 4pq, \ \eta_2^\ell(e^\lambda \eta_1 - 1) > \eta_1^\ell(e^\lambda \eta_2 - 1)\}.$$

Then, for $0 \leqslant \lambda < \lambda_{\text{crit}}$, we have

$$g(\ell) = \frac{e^\lambda (\eta_1 - \eta_2)}{\eta_2^\ell (e^\lambda \eta_1 - 1) - \eta_1^\ell (e^\lambda \eta_2 - 1)}. \tag{44}$$

As a consequence, we have

$$E_0[\overline{\tau}_\ell] = \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\ell)|_{\lambda=0} = -\ell \frac{\rho + 1}{\rho - 1} + (\rho^\ell - 1) \frac{2\rho}{(\rho - 1)^2}. \tag{45}$$

Proof. – See [7], proof of Lemma 6.

Remark. – We note for further reference that, using $E_0[(\overline{\tau}_\ell)^m] = \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} g(\ell)|_{\lambda=0}$, $m = 1, 2, \ldots$, one can check that $\mathbf{E}[(\tau_{\ell_1})^2] < \infty$ if and only if $\mathbf{E}[\rho^{2\ell_1}] < \infty$ and

$$\mathbf{E}\big[(\tau_{\ell_1})^3\big] < \infty \quad \text{if and only if} \quad \mathbf{E}\big[\rho^{3\ell_1}\big] < \infty. \tag{46}$$

As usual, we start the proof of the upper bound with Chebyshev's inequality. Fix $C > 0$ and let $\lambda_n = C\rho^{-a_n}$. We will specify a subsequence $(\tilde{n}_k)$ such that

$$\mathbf{P}_\omega[T_{\tilde{n}_k} \geqslant \tilde{n}_k u] \leqslant \mathbf{E}_\omega\big[e^{\lambda_{\tilde{n}_k} T_{\tilde{n}_k}}\big] e^{-\lambda_{\tilde{n}_k} \tilde{n}_k u} \leqslant \prod_{j=1}^{N(\tilde{n}_k)} g\big(\ell_j(\omega)\big) e^{-\lambda_{\tilde{n}_k} \tilde{n}_k u} \quad \mathbf{P}\text{-a.s.} \tag{47}$$

We can assume w.l.o.g. that (15) still holds true if we replace $\ell_{\max}(n)$ with $\ell_{\text{MAX}}(n)$. Define the (random) subsequence $(\tilde{n}_k)$ such that $\tilde{n}_k \to \infty$ and

$$\ell_{\text{MAX}}(\tilde{n}_k) \leqslant a_{\tilde{n}_k} - c_{\tilde{n}_k} \quad \mathbf{P}\text{-a.s.} \tag{48}$$

We have to show that, for k large enough, $\lambda_{\tilde{n}_k} < \lambda_{\text{crit}}$ is satisfied for our choice of λ_n and for all intervals along our subsequence. It suffices to show that for $\eta_1 = \eta_1(\lambda_{\tilde{n}_k})$ and $\eta_2 = \eta_2(\lambda_{\tilde{n}_k})$,

$$\left(\frac{\eta_1}{\eta_2}\right)^{\ell_{\text{MAX}}(\tilde{n}_k)} \big(e^{\lambda_{\tilde{n}_k}} \eta_2 - 1\big) \underset{k\to\infty}{\longrightarrow} 0 \quad \mathbf{P}\text{-a.s.} \tag{49}$$

We follow the proof of (62) in [7], and then, taking logarithms in (47), we proceed exactly as in the proof of (78) in [7], except that that there the explicit values of $\mathbf{E}[\ell_1]$ and $\mathbf{E}[\rho^{\ell_1}]$ were plugged in, and instead of (78) in [7], we end up with the statement that, $\mathbf{P}$-a.s.,

$$\liminf_{n\to\infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu]$$

$$\leqslant \limsup_{k\to\infty} \frac{\rho^{a_{\tilde{n}_k}}}{\tilde{n}_k} \log \mathbf{P}_\omega[T_{\tilde{n}_k} \geqslant \tilde{n}_k u]$$

$$\leqslant -\frac{C}{q-p} - C\frac{1}{\mathbf{E}[\ell_1]} \frac{2qp}{(q-p)^2} + C\frac{1}{\mathbf{E}[\ell_1]} \frac{2qp}{(q-p)^2} \mathbf{E}\big[\rho^{\ell_1}\big] - Cu$$

$$= C\left(-\frac{\rho+1}{\rho-1} - \frac{1}{\mathbf{E}[\ell_1]} \frac{2\rho}{(\rho-1)^2} + \frac{1}{\mathbf{E}[\ell_1]} \mathbf{E}\big[\rho^{\ell_1}\big] \frac{2\rho}{(\rho-1)^2}\right) - Cu$$

$$= C\left(\frac{1}{v_\alpha} - u\right) = -C\left(u - \frac{1}{v_\alpha}\right). \tag{50}$$

(16) now follows since $C > 0$ was arbitrary. □

The statements for (X_n) follow from the statements for (T_n) by renewal duality. More precisely, we have

LEMMA 3. – *Under the assumptions of Theorems 1, 2 or 3, the following holds true. For **P**-a.a. ω, there is, for each $\varepsilon > 0$, $n_0(\omega)$ such that for $n \geqslant n_0(\omega)$,*

$$\mathbf{P}_\omega[T_n \geqslant [nu]] \leqslant \mathbf{P}_\omega[X_{[nu]} \leqslant n] \leqslant \mathbf{P}_\omega\big[T_{[n(1+\varepsilon)]} \geqslant [nu]\big]. \tag{51}$$

Proof. – The first inequality in (51) is obvious. To show the second inequality, note that

$$\begin{aligned}
&\mathbf{P}_\omega[X_{[nu]} \leqslant n] - \mathbf{P}_\omega\big[T_{[n(1+\varepsilon)]} \geqslant [nu]\big] \\
&\leqslant \sum_{j=[n(1+\varepsilon)]}^{[nu]} \mathbf{P}_\omega[T_{[n(1+\varepsilon)]} = j,\ X_{[nu]} \leqslant n] \\
&\leqslant [nu]\mathbf{P}_\omega\big[\exists j, m \text{ such that } X_j = [n(1+\varepsilon)],\ X_{j+m} \leqslant n\big] \\
&\leqslant [nu]I_{\{\ell_{\mathrm{MAX}}([n(1+\varepsilon)]) \geqslant [n\varepsilon]\}} \\
&= 0.
\end{aligned} \tag{52}$$

for n large enough, since $\ell_{\mathrm{MAX}}(n)/n \to 0$ **P**-a.s. due to (34). □

To prove (20), let $\varepsilon > 0$ and $u = \varepsilon + 1/v > 1/v_\alpha$. We have

$$\begin{aligned}
0 &\geqslant \limsup_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[X_n \leqslant nv] \\
&\geqslant \limsup_{n \to \infty} \frac{\rho^{a_{[nu]}}}{[nu]} \log \mathbf{P}_\omega\big[X_{[nu]} \leqslant [nu]v\big] \\
&\geqslant \limsup_{n \to \infty} \frac{\rho^{a_{[nu]}}}{[nu]} \log \mathbf{P}_\omega[X_{[nu]} \leqslant n] \\
&\geqslant \limsup_{n \to \infty} \frac{\rho^{a_{[nu]}}}{[nu]} \frac{n}{\rho^{a_n}} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] \\
&\geqslant -\liminf_{n \to \infty} \frac{\rho^{a_{[nu]}}}{[nu]} \frac{n}{\rho^{a_n}} \frac{-\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] \\
&\geqslant \limsup_{n \to \infty} \left(\frac{\rho^{a_{[nu]}}}{[nu]} \frac{n}{\rho^{a_n}}\right)\left(-\liminf_{n \to \infty} \frac{-\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu]\right) \\
&\geqslant \limsup_{n \to \infty} \left(\frac{\rho^{a_{[nu]}}}{[nu]} \frac{n}{\rho^{a_n}}\right) \limsup_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] \\
&\geqslant 0,
\end{aligned} \tag{53}$$

where the last inequality follows since $\limsup_{n \to \infty} \frac{\rho^{a_n}}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] = 0$ due to Lemma 3 and $\limsup_{n \to \infty} \left(\frac{\rho^{a_{[nu]}}}{[nu]} \frac{n}{\rho^{a_n}}\right) < \infty$ due to (18). The proof of (19) is similar. □

Proof of Theorem 2. – We have the following analogue of Lemma 2.

LEMMA 4. – *Let $\ell \in \mathbb{N}$, $\omega_0 = 1$, $\omega_1 = \omega_2 = \cdots = \omega_{\ell-1} = \frac{1}{2}$, $X_0 = 0$, $\overline{\tau}_\ell :=$ $\inf\{j\colon X_j = \ell\}$ and $\widetilde{g}(\ell) := \widetilde{g}(\ell, \lambda) = E_0[\exp(\lambda \overline{\tau}_\ell)]$. Let*

$$\widetilde{\lambda}_{\mathrm{crit}} := \widetilde{\lambda}_{\mathrm{crit}}(\ell) = -\log\left(\cos\frac{\pi}{2\ell}\right). \tag{54}$$

Then, for $0 \leqslant \lambda < \widetilde{\lambda}_{\mathrm{crit}}$, we have

$$\widetilde{g}(\ell) = \frac{1}{\cos(\ell \arccos \mathrm{e}^{-\lambda})}. \tag{55}$$

Proof. – Let $\omega_i = \frac{1}{2}$, $-\ell + 1 \leqslant i \leqslant \ell - 1$ and $\widetilde{\tau}_\ell := \inf\{j\colon |X_j| = \ell\}$. Then $\widetilde{\tau}_\ell$ has the same distribution as $\overline{\tau}_\ell$. Let $\widetilde{g}_x(\ell) := E_x[\exp(\lambda\widetilde{\tau}_\ell)]$. We have $\widetilde{g}_\ell(\ell) = \widetilde{g}_{-\ell}(\ell) = 1$ and $\widetilde{g}_x(\ell) = \frac{1}{2}\mathrm{e}^\lambda(\widetilde{g}_{x-1}(\ell) + \widetilde{g}_{x+1}(\ell))$, $-\ell + 1 \leqslant x \leqslant \ell - 1$. Solving this differential equation with boundary condition yields $\widetilde{g}_x(\ell) = \cos(c_\lambda x)/\cos(c_\lambda \ell)$, with $c_\lambda = \arccos(\exp(-\lambda))$, and, setting $x = 0$, (55) follows. $\square$

Remark. – Similarly, we get $E_x[\widetilde{\tau}_\ell] = \ell^2 - x^2$. In particular, $E_0[\widetilde{\tau}_{\ell_1}] = \mathbf{E}_\omega[\tau_{\ell_1}] = \ell_1^2$ and this implies

$$\mathbf{E}[\tau_{\ell_1}] = \mathbf{E}\big[\ell_1^2\big]. \tag{56}$$

We will also use the following exit time asymptotics for simple random walk.

LEMMA 5. – *In the setting of Lemma 4, for $\ell = \ell(n)$ increasing with $\ell(n)^2/n \to 0$ and $c > 0$, we have*

$$\lim_{n\to\infty} \frac{\ell(n)^2}{n} \log P_0\big[\overline{\tau}_{\ell(n)} \geqslant cn\big] = -c\frac{\pi^2}{8}. \tag{57}$$

The proof of Lemma 5 is standard, using the explicit formula in [14], p. 243.

Proof of (24). – To show the lower bound, one proceeds as in the proof of (17), picking the largest interval without nodes. More precisely, let $m_n := \inf\{j\colon \ell_j = \ell_{\max}(n)\}$ and $\delta > 0$. We have

$$\mathbf{P}_\omega[T_n \geqslant nu] \geqslant \mathbf{P}_\omega\left[\tau_{m_n} \geqslant n\left(u - \frac{1}{v_\alpha} + \delta\right)\right] \cdot \mathbf{P}_\omega\left[\sum_{1 \leqslant j \leqslant N_\ell(n),\, j \neq m_n} \tau_{\ell_j} \geqslant n\left(\frac{1}{v_\alpha} - \delta\right)\right]. \tag{58}$$

The second term on the r.h.s. of (58) goes to 1, since due to (31) and (32),

$$\frac{1}{n}\sum_{j=1}^{N_\ell(n)} \tau_{\ell_j} \xrightarrow[n\to\infty]{} \frac{\mathbf{E}[\tau_{\ell_1}]}{\mathbf{E}[\ell_1]} = \frac{1}{v_\alpha} \quad \mathbf{P}\text{-a.s.} \tag{59}$$

For the first term on the r.h.s. of (58), (57) yields

$$\frac{\ell_{\max}(n)^2}{n} \log \mathbf{P}_\omega\left[\tau_{m_n} \geqslant n\left(u - \frac{1}{v_\alpha} + \delta\right)\right] \xrightarrow[n\to\infty]{} -\frac{\pi^2}{8}\left(u - \frac{1}{v_\alpha} + \delta\right) \quad \mathbf{P}\text{-a.s.} \tag{60}$$

and since $\lim_{n\to\infty} \ell_{\max}(n)/a_n = 1$, **P**-a.s., the lower bound in (24) follows by letting $\delta \to 0$. To show the upper bound in (24), let $\delta \in \,]0, 1[$ and

$$\lambda_n := -\log\left(\cos\frac{\pi(1-\delta)}{2(a_n+d_n)}\right).$$

Due to (55), we have

$$\mathbf{P}_\omega[T_n \geqslant nu] \leqslant \mathbf{E}_\omega\left[\mathrm{e}^{\lambda_n T_n}\right] \cdot \mathrm{e}^{-\lambda_n nu} = \prod_{j=1}^{N(n)} \frac{1}{\cos(\ell_j \arccos \mathrm{e}^{-\lambda_n})} \cdot \mathrm{e}^{-\lambda_n nu} \quad \mathbf{P}\text{-a.s.} \quad (61)$$

and, for **P**-a.a. ω, due to (54) and (23), the r.h.s. is well-defined for n large enough. Taking logarithms and multiplying with a_n^2/n yields

$$\frac{a_n^2}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] \leqslant \frac{a_n^2}{n} \sum_{j=1}^{N(n)} \log \frac{1}{\cos(\ell_j \frac{\pi(1-\delta)}{2(a_n+d_n)})} - a_n^2 \lambda_n u \quad \mathbf{P}\text{-a.s.} \quad (62)$$

We check that, w.l.o.g., (23) still holds true if we replace $\ell_{\max}(n)$ with $\ell_{\mathrm{MAX}}(n)$), and conclude that

$$d(\delta) := \cos\frac{\pi(1-\delta)\ell_{\mathrm{MAX}}(n)}{2(a_n+d_n)} > 0 \quad \mathbf{P}\text{-a.s.}$$

for n large enough, and the first term on the r.h.s of (62) is well-defined. Note that since $z^{-2}\log(\cos z) \to -1/2$ for $z \to 0$, we have for the second term on the r.h.s of (62), plugging in the value of λ_n,

$$a_n^2 \lambda_n u = -a_n^2 u \log\left(\cos\frac{\pi(1-\delta)}{2(a_n+d_n)}\right) \underset{n\to\infty}{\longrightarrow} \frac{\pi^2}{8} u(1-\delta)^2. \quad (63)$$

Let $\gamma > 2$ be as in Theorem 2. For each $d > 0$, there is $c = c(d)$ such that $-\log x \leqslant 1 - x + c(1-x)^{\gamma/2}$ for $d \leqslant x \leqslant 1$. Since $\cos x \geqslant 1 - x^2/2$, we also have

$$-\log(\cos x) \leqslant \frac{x^2}{2} + c\left(\frac{x^2}{2}\right)^{\gamma/2}$$

for $d \leqslant x \leqslant 1$. Therefore, for $c = c(\delta)$ large enough,

$$\frac{a_n^2}{n} \sum_{j=1}^{N(n)} \log \frac{1}{\cos(\ell_j \frac{\pi(1-\delta)}{2(a_n+d_n)})} \leqslant \frac{a_n^2}{n} \sum_{j=1}^{N(n)} \ell_j^2 \frac{\pi^2(1-\delta)^2}{8(a_n+d_n)^2} + c\frac{a_n^2}{n} \sum_{j=1}^{N(n)} \ell_j^\gamma \frac{\pi^\gamma(1-\delta)^\gamma}{8^{\gamma/2}(a_n+d_n)^\gamma}. \quad (64)$$

For the first term on the r.h.s of (64), note that

$$\lim_{n\to\infty} \frac{a_n^2}{n} \sum_{j=1}^{N(n)} \ell_j^2 \frac{\pi^2(1-\delta)^2}{8(a_n+d_n)^2} = \frac{\pi^2(1-\delta)^2}{8} \lim_{n\to\infty} \frac{a_n^2}{(a_n+d_n)^2} \frac{N(n)}{n} \frac{1}{N(n)} \sum_{j=1}^{N(n)} \ell_j^2$$

$$= \frac{\pi^2(1-\delta)^2}{8} \frac{1}{\mathbf{E}[\ell_1]} \mathbf{E}[\ell_1^2] = \frac{\pi^2(1-\delta)^2}{8} \frac{1}{v_\alpha} \quad \mathbf{P}\text{-a.s.} \quad (65)$$

where we used (31), the ergodic theorem and the convergence of d_n/a_n to 0 in the middle inequality and (10) in the last inequality. The same argument, using the assumption $\mathbf{E}[\ell_1^\gamma] < \infty$, shows that the second term on the r.h.s of (64) goes to 0. We proved

$$\limsup_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}_\omega[T_n \geqslant nu] \leqslant (1-\delta)^2 \frac{\pi^2}{8} \left(\frac{1}{v_\alpha} - u \right) \quad \mathbf{P}\text{-a.s.} \tag{66}$$

and since $\delta \in {]}0, 1{[}$ was arbitrary, the upper bound in (24) follows by letting $\delta \to 0$. $\quad\square$

Proof of Theorem 3. – Lower bound: Let h_n be an increasing function. We have

$$\mathbf{P}[T_n \geqslant nu] \geqslant \mathbf{P}\big[\ell_{\max}(n) \geqslant h_n\big] P_1[\overline{\tau}_{h_n} \geqslant nu]. \tag{67}$$

For the second term on the r.h.s. of (67), we have, using (39),

$$P_1[\overline{\tau}_{h_n} \geqslant nu] \geqslant \left(1 - \frac{1}{\rho^{h_n-1}} \right)^{nu}. \tag{68}$$

Hence, choosing $h_n = \log n / \log \rho$,

$$\frac{1}{\varphi(n)} \log \mathbf{P}[T_n \geqslant nu] \geqslant \frac{1}{\varphi(n)} \log \mathbf{P}\left[\ell_{\max}(n) \geqslant \frac{\log n}{\log \rho}\right] + \frac{1}{\varphi(n)} nu \log \left(1 - \frac{\rho}{n} \right) \tag{69}$$

and, using (26), (69) implies that

$$\liminf_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}[T_n \geqslant nu] \geqslant -\kappa \left(\frac{1}{\log \rho} \right). \tag{70}$$

Upper bound: Let $\varepsilon > 0$ and $h_n = h_n(\varepsilon) = (1-\varepsilon) \log n / \log \rho$. We have

$$\mathbf{P}[T_n \geqslant nu] \leqslant \mathbf{P}\big[\ell_{\mathrm{MAX}}(n) \geqslant h_n\big] + \mathbf{P}\big[T_n \geqslant nu, \ \ell_{\mathrm{MAX}}(n) < h_n\big]. \tag{71}$$

We will show that

$$\frac{1}{\varphi(n)} \log \mathbf{P}\big[T_n \geqslant nu, \ \ell_{\mathrm{MAX}}(n) < h_n\big] \underset{n\to\infty}{\longrightarrow} -\infty. \tag{72}$$

We check that (26) still holds true if we replace $\ell_{\max}(n)$ with $\ell_{\mathrm{MAX}}(n)$. Then, using (26), we see from (71) and (72) and the continuity of $\kappa(\cdot)$ that

$$\limsup_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}[T_n \geqslant nu] \leqslant \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}\big[\ell_{\mathrm{MAX}}(n) \geqslant h_n(\varepsilon)\big]$$

$$\leqslant -\kappa \left(\frac{1}{\log \rho} \right). \tag{73}$$

To show (72), let $\delta > 0$ and $M_\delta := (1 + \delta)/\mathbf{E}[\ell_1]$ and note that

$$\mathbf{P}\big[T_n \geqslant nu, \ \ell_{\mathrm{MAX}}(n) < h_n\big] \leqslant \mathbf{P}\big[N(n) \geqslant [nM_\delta]\big] + \mathbf{P}\Bigg[\sum_{j=1}^{[nM_\delta]} \overline{\tau}_{\ell_{j,n}} \geqslant nu\Bigg], \qquad (74)$$

where $\ell_{j,n}, \ j \in \mathbb{N}$, are i.i.d. with

$$\mathbf{P}[\ell_{j,n} = k] = \frac{\mathbf{P}[\ell_1 = k]}{\mathbf{P}[\ell_1 \leqslant [h_n]]} \quad \text{for } k = 1, 2, \ldots, [h_n]$$

and $\overline{\tau}_{\ell_{j,\,n}}, \ j \in \mathbb{N}$, are i.i.d. The distribution of $\overline{\tau}_{\ell_{j,n}}$, given $\ell_{j,n}$, is defined as in Lemma 2. For the first term on the r.h.s of (74), we have

$$\mathbf{P}\big[N(n) \geqslant [nM_\delta]\big] = \mathbf{P}\Bigg[\sum_{j=1}^{[nM_\delta]-1} \ell_j < n\Bigg] \leqslant \mathbf{P}\Bigg[\frac{1}{[nM_\delta]} \sum_{j=1}^{[nM_\delta]-1} \ell_j < \frac{n}{[nM_\delta]}\Bigg].$$

Since ℓ_j are i.i.d., nonnegative random variables with expectations strictly larger than $1/M_\delta$, the last term is decaying exponentially in n. In particular, since $\varphi(n)/n \to 0$, we have

$$\frac{1}{\varphi(n)} \log \mathbf{P}\big[N(n) \geqslant [nM_\delta]\big] \underset{n\to\infty}{\longrightarrow} -\infty. \qquad (75)$$

Let $C > 0$. We can estimate the second term on the r.h.s. of (74) as follows:

$$\mathbf{P}\Bigg[\sum_{j=1}^{[nM_\delta]} \overline{\tau}_{\ell_{j,n}} \geqslant nu\Bigg] \leqslant \mathbf{E}\Bigg[\exp\bigg(\frac{C\varphi(n)}{n}\overline{\tau}_{\ell_{1,n}}\bigg)\Bigg]^{[nM_\delta]} \exp\big(-C\varphi(n)u\big). \qquad (76)$$

We will show that

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)} \log \mathbf{E}\Bigg[\exp\bigg(\frac{C\varphi(n)}{n}\overline{\tau}_{\ell_{1,n}}\bigg)\Bigg] \leqslant C\mathbf{E}[\tau_{\ell_1}]. \qquad (77)$$

Together with (74)–(76), this implies that

$$\limsup_{n\to\infty} \frac{1}{\varphi(n)} \log \mathbf{P}\big[T_n \geqslant nu, \ \ell_{\mathrm{MAX}}(n) < h_n\big] \leqslant C\big(\mathbf{E}[\tau_{\ell_1}]M_\delta - u\big). \qquad (78)$$

But, recalling (6), for $u > 1/v_\alpha$ we can choose δ small enough such that the last term is negative, and since $C > 0$ was arbitrary, this proves (72). It remains to prove (77). Using the inequalities $\log x \leqslant x - 1$ and $\mathrm{e}^x - 1 \leqslant x + x^2\mathrm{e}^x$, we have

$$\frac{n}{\varphi(n)} \log \mathbf{E}\Bigg[\exp\bigg(\frac{C\varphi(n)}{n}\overline{\tau}_{\ell_{1,n}}\bigg)\Bigg]$$

$$\leqslant C\mathbf{E}[\overline{\tau}_{\ell_{1,n}}] + C^2 \frac{\varphi(n)}{n} \mathbf{E}\Bigg[(\overline{\tau}_{\ell_{1,n}})^2 \exp\bigg(\frac{C\varphi(n)}{n}\overline{\tau}_{\ell_{1,n}}\bigg)\Bigg] \qquad (79)$$

$$\leqslant C\mathbf{E}[\overline{\tau}_{\ell_1}] + C^2 \frac{\varphi(n)}{n} \mathbf{E}\big[(\overline{\tau}_{\ell_{1,n}})^3\big]^{2/3} \mathbf{E}\bigg[\exp\bigg(\frac{3C\varphi(n)}{n}\overline{\tau}_{\ell_{1,n}}\bigg)\bigg]^{1/3}. \tag{80}$$

Due to (46) and our assumption $\mathbf{E}[\rho^{3\ell_1}] < \infty$, we have

$$\limsup_{n\to\infty} \mathbf{E}\big[(\overline{\tau}_{\ell_{1,n}})^3\big] \leqslant \mathbf{E}\big[(\tau_{\ell_1})^3\big] < \infty. \tag{81}$$

Since $\ell_{1,n} \leqslant [h_n]$, we see from (80) that it suffices to show that for each $C > 0$,

$$\limsup_{n\to\infty} \mathbf{E}\bigg[\exp\bigg(\frac{C\varphi(n)}{n}\overline{\tau}_{[h_n]}\bigg)\bigg] < \infty. \tag{82}$$

But now we can use formula (44), plugging in $\ell = \ell_n = [h_n]$ and $\lambda = \lambda_n = C\varphi(n)/n$. Analysing the terms in (44), we see that $\lambda_n \to 0$, $\eta_2(\lambda_n) \to 1$ and $\eta_1(\lambda_n) \to \rho$. We show, with the same reasoning as in the proof of (72) in [7], that

$$\eta_2^{h_n} \underset{n\to\infty}{\longrightarrow} 1 \quad\text{and}\quad \eta_1^{h_n}\big(e^{\lambda_n}\eta_2 - 1\big) \underset{n\to\infty}{\longrightarrow} 0 \tag{83}$$

and this proves (82). For the proof of (28) from (27), note that $\mathbf{P}[X_n/n \leqslant v]$ is comparable to $\mathbf{P}[T_{[nv]} \geqslant n]$ by renewal duality, as in Lemma 3, and use the fact that φ is slowly varying; we refer to [3] for a more precise argument. $\quad\square$

3. Examples

Example 1. – Assume that α is a product measure, with $\alpha_1 := \alpha(\omega_0 = 1)$ and $\alpha(\omega_0 = p) = 1 - \alpha_1$. Then $\ell_1, \ell_2, \dots$ are i.i.d. and ℓ_1 has a geometric distribution with $\mathbf{P}[\ell_1 = k] = \alpha_1(1 - \alpha_1)^{k-1}$, $k = 1, 2, \dots$. In this case, (14), (15) and (23) are satisfied with

$$a_n = \frac{\log n}{-\log(1 - \alpha_1)}, \qquad b_n = \frac{\log_2 n}{-\log(1 - \alpha_1)},$$
$$c_n = \frac{\log_3 n}{-\log(1 - \alpha_1)}, \qquad d_n = \frac{2\log_2 n}{-\log(1 - \alpha_1)}, \tag{84}$$

and we have $\rho^{a_n} = n^{1/s}$ with $s := -\log(1 - \alpha_1)/\log\rho$. Here, a_n is given by the Erdős–Renyi law for longest runs, and (14), (15) and (23) follow from Theorem 2 in [4]. Hence, if $p < \frac{1}{2}$, we have for $v < v_\alpha$,

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega[X_n \leqslant nv] = -\infty \quad \mathbf{P}\text{-a.s.}, \tag{85}$$

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega[X_n \leqslant nv] = 0 \quad \mathbf{P}\text{-a.s.} \tag{86}$$

For the general RWRE in an i.i.d. environment with "mixed drifts and positive speed", with s defined by

$$\int \bigg(\frac{1 - \omega_0}{\omega_0}\bigg)^s \, \mathrm{d}\alpha = 1,$$

(86) was proved and (85) was conjectured in [6]. If $p < 1/2$ in our example, the hypotheses of Theorem 3 are satisfied with $\varphi(n) = \log n$ and $\kappa(t) = -1 - t \log(1 - \alpha_1)$, and we have

$$\lim_{n \to \infty} \frac{1}{\log n} \log \mathbf{P}[X_n \leqslant nv] = -(s - 1) \tag{87}$$

where $s = -\log(1 - \alpha_1)/\log \rho$ as above. (87) agrees with Theorem 1.1 in [5]. For $p = \frac{1}{2}$ and $\alpha_{1/2} := (1 - \alpha_1)$, Theorem 2 yields that for $v < v_\alpha$,

$$\lim_{n \to \infty} \frac{(\log n)^2}{n} \log \mathbf{P}_\omega[X_n \leqslant nv] = -\frac{\pi^2}{8} (\log \alpha_{1/2})^2 \left(1 - \frac{v}{v_\alpha} \right) \quad \mathbf{P}\text{-a.s.} \tag{88}$$

If $\ell_1, \ell_2, \ldots$ are i.i.d. and $\mathbf{E}[\ell_1^2] < \infty$, recall the integral criteria for the growth of maxima of i.i.d. random variables in [9] and [10] to find functions a_n, b_n, c_n and d_n such that (14), (15) and (23) and our assumptions on a_n are satisfied.

Example 2. – Assume that $\ell_1, \ell_2, \ldots$ are i.i.d. and $\mathbf{P}[\ell_1 = k] = c \exp(-bk^\gamma)$, $k = 1, 2, \ldots$, where $\gamma \geqslant 1$, $b > 0$ and

$$c = \left(\sum_{k=1}^\infty \exp(-bk^\gamma) \right)^{-1}.$$

Then (14), (15) and (23) are satisfied with

$$a_n = \left(\frac{1}{b} \log n \right)^{1/\gamma}, \qquad b_n = \left(\frac{1}{b} \log_2 n \right)^{1/\gamma},$$

$$c_n = \left(\frac{1}{b} \log_3 n \right)^{1/\gamma}, \qquad d_n = \left(\frac{2}{b} \log_2 n \right)^{1/\gamma}. \tag{89}$$

Note that $\gamma < 1$ would violate our assumption $\mathbf{E}[\rho^{\ell_1}] < \infty$. Hence, for $p < \frac{1}{2}$, (19), (20) and (25) are satisfied with a_n as above. If $\gamma > 1$ and $p < \frac{1}{2}$, Theorem 3 applies with $\varphi(n) = (\log n)^\gamma$ and $\kappa(t) = bt^\gamma$, and we have for each $u > \frac{1}{v_\alpha}$,

$$\lim_{n \to \infty} \frac{1}{(\log n)^\gamma} \log \mathbf{P}[T_n \geqslant nu] = -\frac{b}{(\log \rho)^\gamma} \tag{90}$$

and, for $v < v_\alpha$,

$$\lim_{n \to \infty} \frac{1}{(\log n)^\gamma} \log \mathbf{P}[X_n \leqslant nv] = -\frac{b}{(\log \rho)^\gamma}. \tag{91}$$

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