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Parametric Amplification with Josephson Junctions

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For parametric amplifiers with Josephson junctions the Manley-Rowe relations, the transducer gain at ω_1, the transducer conversion gain from ω_1 to ω_2 and the gain-bandwidth products are derived. For the amplifiers described no pump frequency source is necessary because the pump frequency is generated by a DC voltage applied to the Josephson junction.

Parametrische Verstärkung mit Josephson-Kontakten

Für parametrische Verstärker mit Josephson-Tunnelkontakten werden die Manley-Rowe-Gleichungen abgeleitet. Die Gewinn-Bandbreite-Produkte werden für den Negativen-Widerstand-Verstärker und den Aufwärtsmischer berechnet. Für die beschriebenen Verstärker wird keine Quelle für die Pumpfrequenz benötigt, da die Pumpfrequenz durch eine an den Josephson-Kontakt angelegte Gleichspannung erzeugt wird.

1. Introduction

Parametric amplification with Josephson junctions has already been investigated experimentally [1]. The Josephson junction consists of two superconductors separated by a thin insulating layer. The thickness of the layer is in the order of 10 Å. The supercurrent across a Josephson junction is given by [2], [3]

$$i(t) = I_{\max} \sin\left[\frac{2e}{\hbar}\int v(t)\,\mathrm{d}t + \varphi_0\right]. \tag{1}$$

$I_{\max}$ is the maximum supercurrent which can be calculated from the normal resistance of the junction, the temperature and the energy gaps of the superconductors [4]. The applied voltage is v, and φ_0 denotes a phase constant.

If a constant voltage V_0 is applied, a current at a frequency

$$\omega_0 = 2e\,V_0/\hbar \tag{2}$$

is generated.

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2. The Small Signal Properties of the Josephson Junction Parametric Amplifier

The circuit model of the Josephson junction parametric amplifier (Fig. 1) consists of the signal current source i_{10}, the signal circuit Y_1, the idler circuit Y_2, the bias voltage source V_0, and the Josephson junction. The junction voltage is given by

$$v(t) = V_0 + |V_1|\cos(\omega_1 t + \varphi_1) + + |V_2|\cos(\omega_2 t + \varphi_2). \tag{3}$$

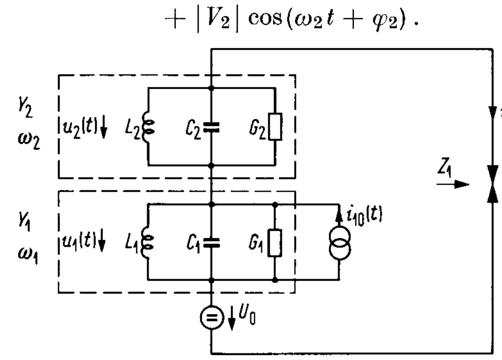

Fig. 1. Circuit model of the Josephson junction parametric amplifier.

Inserting into eq. (1) and using the expansion formula

$$e^{j a \sin \varphi} = \sum_{n=-\infty}^{+\infty} J_n(a)\, e^{j n \varphi} \tag{4}$$

we obtain [3]

$$i(t) = I_{max} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} J_m(a_1)\, J_n(a_2) \times \times \sin[(\omega_0 + m\,\omega_1 + n\,\omega_2)\,t + m\,\varphi_1 + n\,\varphi_2 + \varphi_0], \tag{5}$$

$$a_1 = \frac{\omega_0 |V_1|}{\omega_1 V_0}, \quad a_2 = \frac{\omega_0 |V_2|}{\omega_2 V_0}. \tag{6}$$

J_n are the ordinary Bessel functions of the nth order.

The bias voltage V_0 shall be such that the equation

$$\omega_1 + \omega_2 = \omega_0 \tag{7}$$

is satisfied. The DC component I_0 and the AC components $i_1(t)$ and $i_2(t)$ at the frequencies ω_1 and ω_2 are

$$I_0 = I_{max} J_1(a_1)\, J_1(a_2) \sin(\varphi_0 - \varphi_1 - \varphi_2), \tag{8}$$

$$i_1(t) = -I_{max} J_0(a_1)\, J_1(a_2) \sin(\omega_1 t + \varphi_0 - \varphi_2), \tag{9}$$

$$i_2(t) = -I_{max} J_1(a_1)\, J_0(a_2) \sin(\omega_2 t + \varphi_0 - \varphi_1). \tag{10}$$

Assuming that the AC voltages are much smaller than V_0 yielding $J_0(a) \doteq 1$, $J_1(a) \doteq a/2$ we get

$$I_0 = I_{max} \frac{\omega_0^2 |V_1|\,|V_2|}{4\,\omega_1 \omega_2 V_0^2} \sin(\varphi_0 - \varphi_1 - \varphi_2), \tag{11}$$

$$I_1 = j\, I_{max} \frac{\omega_0 V_2^*}{2\,\omega_2 V_0} e^{j\varphi_0}, \quad i_1 = \mathrm{Re}(I_1 e^{j\omega_1 t}), \quad V_2 = |V_2|\, e^{j\varphi_2}, \tag{12}$$

$$I_2 = j\, I_{max} \frac{\omega_0 V_1^*}{2\,\omega_1 V_0} e^{j\varphi_0}, \quad i_2 = \mathrm{Re}(I_2 e^{j\omega_2 t}), \quad V_1 = |V_1|\, e^{j\varphi_1}. \tag{13}$$

If the junction is terminated with a conductance Y_2 at the frequency ω_2, the impedance at a frequency ω_1 will be measured towards the junction:

$$Z_1 = -\frac{4\,\omega_1 \omega_2 V_0^2}{\omega_0^2 I_{max}^2} Y_2^*. \tag{14}$$

The real part of Z_1 is negative and therefore power gain is obtained in the signal circuit. Since the negative real part of Z_1 is a consequence of a positive real part of Y_2 the gain in the signal circuit is connected with losses in the idler circuit.

3. The Power Relations

The DC power is given by

$$P_0 = V_0 I_0 = I_{max} \frac{\omega_0^2 |V_1|\,|V_2|}{4\,\omega_1 \omega_2 V_0} \sin(\varphi_0 - \varphi_1 - \varphi_2). \tag{15}$$

The AC active powers are

$$P_1 = \frac{1}{2} \mathrm{Re}(V_1^* I_1) = -I_{max} \frac{\omega_0 |V_1|\,|V_2|}{4\,\omega_2 V_0} \sin(\varphi_0 - \varphi_1 - \varphi_2), \tag{16}$$

$$P_2 = \frac{1}{2} \mathrm{Re}(V_2^* I_2) = -I_{max} \frac{\omega_0 |V_1|\,|V_2|}{4\,\omega_1 V_0} \sin(\varphi_0 - \varphi_1 - \varphi_2). \tag{17}$$

If the active power has a negative sign, it flows from the contact to the outer circuit.

From the eqs. (15) to (17) we get the Manley-Rowe relations [5]:

$$P_1 = -\frac{\omega_1}{\omega_0} P_0, \quad P_2 = -\frac{\omega_2}{\omega_0} P_0, \quad P_2 = \frac{\omega_2}{\omega_1} P_1. \tag{18}$$

Since the output circuit is passive we have $P_2 \leqq 0$. From eqs. (18) we see that a power flow to the signal circuit is connected with a power flow to the idler circuit. Since the active powers are proportional to the frequencies a power gain ω_2/ω_1 can be obtained additional to the gain as a negative resistance amplifier if the amplified signal is coupled out at ω_2.

From the eqs. (18) we obtain

$$P_0 + P_1 + P_2 = 0. \tag{19}$$

The Josephson junction is non-dissipative.

4. The Gain-Bandwidth Product of the Parametric Amplifier

If ω_1 and ω_2 are sufficiently separated and therefore $Y_1(\omega_2) = Y_2(\omega_1) = \infty$, we can use the equivalent circuit diagram shown in Fig. 2. G_1 consists

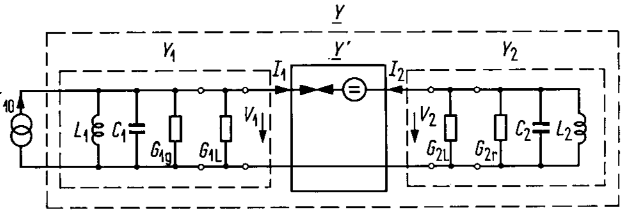

Fig. 2. Four terminal equivalent circuit diagram of the Josephson junction parametric amplifier.

of the generator conductance G_{1g} and the primary load conductance G_{1L}; G_2 of the secondary load conductance G_{2L} and the conductance G_{2r} due to the losses of the secondary resonant circuit:

$$G_{1g} = \alpha G_1, \quad G_{1L} = (1-\alpha)\, G_1, \quad G_{2r} = \beta G_2, \quad G_{2L} = (1-\beta)\, G_2. \tag{20}$$

I_{10} is the signal current source at the frequency ω_1. The resonant frequencies of Y_1 and Y_2 are ω_{10} and ω_{20}. The Q values and detuning parameters δ are defined in the usual way:

$$Q_1 = \frac{\omega_{01} C_1}{G_1}, \quad Q_2 = \frac{\omega_{02} C_2}{G_2}, \quad \delta_1 = \frac{\omega_1}{\omega_{01}} - \frac{\omega_{01}}{\omega_1}, \quad \delta_2 = \frac{\omega_2}{\omega_{02}} - \frac{\omega_{02}}{\omega_2}. \tag{21}$$

The admittances Y_1, Y_2 are given by

$$Y_1 = G_1(1 + j\, Q_1 \delta_1), \quad Y_2 = G_2(1 + j\, Q_2 \delta_2). \tag{22}$$

The frequencies ω_1 and ω_2 satisfy the condition (7). We also assume:

$$\omega_{01} + \omega_{02} = \omega_0 . \tag{23}$$

From eqs. (7) and (23) we get

$$\Delta\omega = \omega_1 - \omega_{01} = -(\omega_2 - \omega_{02}) . \tag{24}$$

For small detuning we may write

$$\delta_1 \approx 2\frac{\Delta\omega}{\omega_{01}}, \quad \delta_2 \approx -2\frac{\Delta\omega}{\omega_{02}} . \tag{25}$$

The Y-matrix of the Josephson junction is obtained from eqs. (2), (12), and (13):

$$\begin{pmatrix} I_1 \\ I_2^* \end{pmatrix} = \begin{pmatrix} 0 & Y_{12} \\ Y_{21}^* & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2^* \end{pmatrix}, \tag{26}$$

$$Y_{12} = \mathrm{j}\, I_{\max} \frac{e}{\hbar\,\omega_2} \mathrm{e}^{\mathrm{j}\varphi_0}, \qquad Y_{21} = \mathrm{j}\, I_{\max} \frac{e}{\hbar\,\omega_1} \mathrm{e}^{\mathrm{j}\varphi_0}. \tag{27}$$

The equation for the whole network is given by

$$\begin{pmatrix} I_{10} \\ 0 \end{pmatrix} = \begin{pmatrix} Y_1 & Y_{12} \\ Y_{21}^* & Y_2^* \end{pmatrix} \begin{pmatrix} V_1 \\ V_2^* \end{pmatrix} . \tag{28}$$

The power gain of the negative resistance parametric amplifier is $g_{\mathrm{p}1}$, the conversion power gain between ω_1 and ω_2 is $g_{\mathrm{p}2}$:

$$g_{\mathrm{p}1} = \frac{P_{\mathrm{L}1}}{P_{\mathrm{a}1}}, \quad g_{\mathrm{p}2} = \frac{P_{\mathrm{L}2}}{P_{\mathrm{a}1}} . \tag{29}$$

$P_{\mathrm{a}1}$ is the available input power. $P_{\mathrm{L}1}$ and $P_{\mathrm{L}2}$ are the powers dissipated in $G_{1\mathrm{L}}$ and $G_{2\mathrm{L}}$, respectively:

$$P_{\mathrm{a}1} = \frac{|I_{10}|^2}{8 G_{1\mathrm{g}}}, \qquad P_{\mathrm{L}1} = \frac{1}{2}|V_1|^2 G_{1\mathrm{L}}, \quad P_{\mathrm{L}2} = \frac{1}{2}|V_2|^2 G_{2\mathrm{L}} . \tag{30}$$

From eqs. (28) to (30) we obtain

$$g_{\mathrm{p}1} = \frac{4|Y_2|^2 G_1^2 \alpha(1-\alpha)}{|Y_1 Y_2^* - Y_{12} Y_{21}^*|^2}, \tag{31}$$

$$g_{\mathrm{p}2} = \frac{4|Y_{21}|^2 G_1 G_2 \alpha(1-\beta)}{|Y_1 Y_2^* - Y_{12} Y_{21}^*|^2} . \tag{32}$$

We define the gain degradation factors ξ_1, ξ_2 and the damping reduction constant η:

$$\xi_1 = 2 Q_1 \frac{\Delta\omega}{\omega_{01}}, \quad \xi_2 = 2 Q_2 \frac{\Delta\omega}{\omega_{02}}, \tag{33}$$

$$\eta = \frac{Y_{12} Y_{21}^*}{G_1 G_2} . \tag{34}$$

Inserting the eqs. (22), (25), (33), and (34) into the eqs. (31) and (32) we obtain

$$g_{\mathrm{p}1} = \frac{4\alpha(1-\alpha)}{\left(1 - \frac{\eta}{1-\xi_2^2}\right)^2 + \left(\xi_1 + \frac{\xi_2}{1-\xi_2^2}\right)^2}, \tag{35}$$

$$g_{\mathrm{p}2} = \frac{4\frac{\omega_2}{\omega_1}\alpha(1-\beta)\eta}{(1-\xi_1\xi_2-\eta)^2 + (\xi_1+\xi_2)^2} . \tag{36}$$

For zero detuning the maximum power gains are given by

$$g_{\mathrm{p}1\,\max} = \frac{4\alpha(1-\alpha)}{(1-\eta)^2}, \tag{37}$$

$$g_{\mathrm{p}2\,\max} = \frac{4\frac{\omega_2}{\omega_1}\alpha(1-\beta)\eta}{(1-\eta)^2} . \tag{38}$$

The maximum power gains depend on the damping reduction constant. The amplifier is only stable for $\eta < 1$. For a large η the bandwidth is small, because in this case the gain is decisively reduced by a small detuning. For narrow band amplifiers ($1 - \eta \ll 1$, ξ_1, ξ_2 small) the negative resistance gain $g_{\mathrm{p}1}$ and the transducer gain $g_{\mathrm{p}2}$ can be approximated by

$$g_{\mathrm{p}1} = \frac{4\alpha(1-\alpha)}{(1-\eta)^2 + (\xi_1+\xi_2)^2}, \tag{39}$$

$$g_{\mathrm{p}2} = \frac{4\frac{\omega_2}{\omega_1}\alpha(1-\beta)\eta}{(1-\eta)^2 + (\xi_1+\xi_2)^2} . \tag{40}$$

For $1 - \eta = \xi_1 + \xi_2$ the gain decreases to the half of its maximum value. The products of the bandwidth B and the root of the gain are

$$B\sqrt{g_{\mathrm{p}1\,\max}} = \frac{1}{\pi}\frac{\sqrt{\alpha(1-\alpha)}}{\frac{Q_1}{\omega_1}+\frac{Q_2}{\omega_2}}, \tag{41}$$

$$B\sqrt{g_{\mathrm{p}2\,\max}} = \frac{1}{\pi}\frac{\sqrt{\frac{\omega_2}{\omega_1}\alpha(1-\beta)\eta}}{\frac{Q_1}{\omega_1}+\frac{Q_2}{\omega_2}} \tag{42}$$

with $B = |\Delta\omega/\pi|$.

The parameters α and β depend on the matching. The negative resistance amplifier is matched at $\alpha = 1/2$. For the negative resistance frequency converter β shall be as small as possible and $\alpha = 1$. Setting $\eta \doteq 1$ for narrow band amplifiers we obtain

$$B\sqrt{g_{\mathrm{p}1\,\max}} = \frac{1}{2\pi}\frac{1}{\frac{Q_1}{\omega_1}+\frac{Q_2}{\omega_2}}, \tag{43}$$

$$B\sqrt{g_{\mathrm{p}2\,\max}} = \frac{1}{\pi}\sqrt{\frac{\omega_2}{\omega_1}}\frac{1}{\frac{Q_1}{\omega_1}+\frac{Q_2}{\omega_2}} . \tag{44}$$

These equations are well known for other parametric amplifiers [6], [7], [8].

5. HF Properties of the Josephson Junction Parametric Amplifier

Josephson junctions can be used up to 1000 GHz. The upper frequency limit is reached when the superconducting Cooper pairs are broken up by the microwave photons. For 0° K the pairing energy 2ε is approximately given by

$$2\varepsilon \doteq 3.5\, k_{\mathrm{B}} T_{\mathrm{c}} . \tag{45}$$

T_c is the critical temperature of the superconductor. From this we obtain the upper frequency limit

$$f_g = 73\,\mathrm{GHz}/{}^\circ\mathrm{K}\,. \tag{46}$$

For higher temperatures the pairing energy and the frequency limit decrease (Fig. 3).

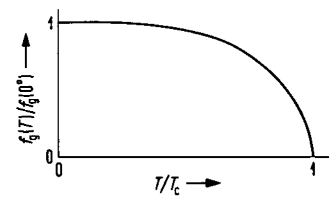

Fig. 3. Temperature dependence of the frequency limit.

In addition to the lossless Josephson current a dissipative normal electron tunneling current flows across the junction [4], [9]. In Fig. 4 (5) this tunneling current is plotted for two equal (different) superconductors forming a tunneling junction.

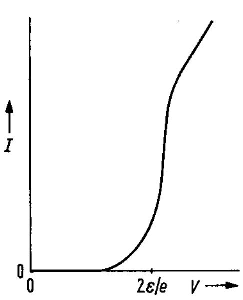

Fig. 4.
Tunneling current between equal superconductors.

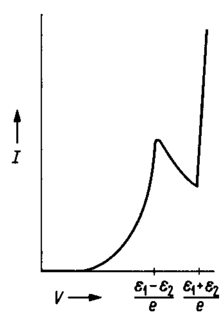

Fig. 5.
Tunneling current between different superconductors.

Since the applied DC voltage V_0 is proportional to the frequency ω_0, these losses increase with frequency. In the range $V_0 < 2\,\varepsilon/e$ and $V_0 < (\varepsilon_1 - \varepsilon_2)/e$, respectively, the normal current decreases for increasing purity of the superconductor, where $2\,\varepsilon$ is the energy gap of the superconductor. As a consequence of photon assisted tunneling [10] additional losses may occur. In this case the current voltage characteristic has echos at the distances $n\hbar\omega_0/e$. As the wave functions of the paired electrons are coherent, no noise contribution will come from the Josephson current. Thermal noise will only be caused by the normal tunneling current [11].

In order to get a high upper frequency limit and a low noise figure it is necessary to use pure superconductors with a high critical temperature and to operate the contact sufficiently far below the critical temperature.

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