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Semicausality in Quantum Information Theory

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"The first principle is that you must not fool yourself — and you are the easiest person to fool. So you have to be very careful about that. After you've not fooled yourself, it's easy not to fool other scientists. You just have to be honest in a conventional way after that."

——— Richard P. Feynman

Zusammenfassung

Diese Dissertation befasst sich mit verschiedenen Fragestellungen in der Theorie der offenen Quantensysteme, welche alle auf der Idee des gerichteten Informationsflusses, welche im folgenden als Semikausalität bezeichnet wird, aufbauen.
In unserem ersten Beitrag benutzen wir Semikausalität um ein verallgemeinertes Modell für interaktionsfreie Messungen nach Elitzur-Vaidman herzuleiten. Der wesentliche Beitrag hierbei ist die vollständige Analyse des so entstandenen Modells. Das Hauptresultat ist die Antwort auf die Frage, welche Objekte man mittels komplett interaktionsfreier Messungen unterscheiden kann.
Der zweite Beitrag ist motivert durch die Frage, wie man den Verfallsprozess von Quantengeräten beschreiben kann. Als einfaches Modell untersuchen wir dynamische Halbgrupppen von sog. Superkanälen. Diese gehen mathematisch aus semikausalen Kanälen durch eine Ähnlichkeitstransformation hervor. Wir charakterisieren diese dynamischen Halbgruppen indem wir eine Normalform für die zugehörigen Generatoren ableiten. Diese ist analog zur GKSL Form von Generatoren gewöhnlicher quantendynamischer Halbgruppen.
Als letzten Beitrag verallgemeinern wir die Techniken, welche wir für Superkanäle entwickelt haben. Diese Verallgemeinerung führt zu einer Charakterisierung der Generatoren quantendynamischer Halbgruppen, welche eine atomare von Neumann Algebra invariant lassen.

Abstract

This dissertation treats several topics in the theory of open quantum systems, all of which are related to one-way information flow – a concept called semicausality.
In our first contribution, we use semicausality to derive a generalized model of “interaction-free" measurements in the spirit of the Elitzur-Vaidman bomb-tester experiment. We subsequently analyze that model, and provide a complete answer to the question of which objects can be discriminated in an “interaction-free" manner.
Motivated by the question “How can we describe the decay-process of quantum devices over time?", we study dynamical semigroups of superchannels, which are related to dynamical semigroups of semicausal channels by a similarity transform. We characterize these dynamical semigroups by providing a normal form for their generators – analogous to the GKLS-form for ordinary quantum dynamical semigroups.
Finally, we extend the scope of the techniques developed for superchannels to characterize the generators of quantum dynamical semigroups with an invariant atomic von Neumann algebra.

List of contributed articles

Core articles as principal author

I) [1] Markus Hasenöhrl and Michael M. Wolf
"Interaction-Free" Channel Discrimination
Published in Ann. Henri Poincaré (2022); `https://doi.org/10.1007/s00023-022-01175-z`

II) [2] Markus Hasenöhrl and Matthias C. Caro
Quantum and classical dynamical semigroups of superchannels and semicausal channels
Published in J. Math. Phys. 63, 072204 (2022); `https://doi.org/10.1063/5.0070635`

Further articles as principal author under review

III) [3] Markus Hasenöhrl and Matthias C. Caro
On the generators of quantum dynamical semigroups with invariant subalgebras
Accepted in *Open Syst. Inf. Dyn.*; `https://doi.org/10.48550/arXiv.2202.06812`

I, Markus Hasenöhrl, am the principal author of the core articles I, II, and III.

Contents

Chapter 1

Introduction

1.1 Outline

One-way processes are an important class of processes in many areas of science and technology. The most fundamental one-way process is certainly the flow of time (and the related notion of entropy). It is hard to imagine a world without the notion of cause and effect and without the inherent assumption that actions at a later time do not influence actions at an earlier time.

Another class of one-way processes (which is, as we will see, directly related to the direction of time) are one-communication processes. Quantum channels on a bipartite system that allow for one-way communication only are called semicausal channels. In this thesis, we investigate consequences of the structure of semicausal channels.

In our first contribution, semicausality is used to axiomatically derive a physical model that generalizes the notion of “interaction-free" measurements, as intruduced by Elitzur and Vaidman in their bomb-tester experiment [4]. In that work, we also use semicausality (in the form of superchannels) as a technical tool in the analysis of the model. This led to a full characterization for when totally “interaction-free" measurements are possible.

In our second contribution, we analyze the mathematical structure of semicausal channels and superchannels directly. Here, one notes that the set of semicausal channels as well as the set of superchannels becomes a semigroup under composition. This allows us to investigate the dynamics of these classes of objects – an endeavor that can be physically motivated by the question “How can we describe the decay-process of quantum devices over time?". The dynamics is investigated by studying (continuous) one-parameter semigroups of these objects. Our result is a characterization of the respective generators – analogous to the GKLS-form.

Finally, in our third contribution, we aim to find a more general mathematical setting such that the techniques developed for superchannels and semicausal channels can be applied. This leads us to the study of one-parameter semigroups of quantum channels that leave a given von Neumann algebra invariant. Our main finding is a normal form for the corresponding generators.

This thesis is structured as follows: In the remaining section of Chapter 1, we briefly outline the content of the contributed articles [1], [2] and [3]. In Chapter 2 we introduce the relevant machinery of quantum mechanics with emphasis on the axiomatic justification of completely positive maps as maps between quantum states. Chapter 3 formally introduces the notion of

semicausality as well as the related notion of CP-maps with an invariant von Neumann algebra, and superchannels. The later objects are motivated thoroughly, because they form the basis of (the interpretation of) Article [2].
Chapter 4 is an introduction to Article [1] and gives a high-level summary of the results therein. Chapter 5 is concerned with the theory of continuous one-parameter semigroups, also known as dynamical semigroups. After motivating and summarizing some of the fundamental results in this field, in particular the one-to-one relation between dynamical semigroups and their generators, we move on to quantum dynamical semigroups, which are of special interest of us. This sets the stage to discuss the results of Articles [2] and [3], where we characterize dynamical semigroups of superchannels and quantum dynamical semigroups with an invariant von Neumann algebra.

1.2 Summary of Results

As outlined in the last section, our results use and investigate semicausality in several ways. In the remainder of this section, we summarize our results.

- *Article [1]: "Interaction-free" channel discrimination*
  In this work, we propose and investigate a new generalization of "interaction-free" measurements. Namely, we view the Elitzur-Vaidman bomb-tester experiment as specific case of a quantum channel discrimination problem – with the additional requirement that the discrimination should be "interaction-free". Using semicausality, we derive a model that gives a general meaning to the term "interaction-free". Subsequently, we analyze this model thoroughly. Our main finding is a characterization for when totally "interaction-free" measurements are possible or impossible.

- *Article [2]: Quantum and classical dynamical semigroups of superchannels and semicausal channels*
  In this work, we investigate (the generators of) one-parameter semigroups of quantum superchannels, which are the most general admissible transformation between quantum channels. On a physical level, this investigation promises to shine light on the question "How do qunatum devices age?". On a mathematical level, we use the relation between superchannels and semicausal channels and a newly developed technique to arrive at a normal form for the generators of one-parameter semigroups – analogous to the famous GKSL-form for ordinary quantum dynamical semigroups.

- *Article [3]: On the generators of quantum dynamical semigroups with invariant subalgebras*
  In this work, we aim to generalize the techniques developed for superchannels and semicausal channels in Article [2]. We find that an appropriate generalization of quantum dynamical semigroups of semicausal channels are quantum dynamical semigroups that leave a given von Neumann algebra invariant. Our main finding is a refinement of the GKSL-form for the generators of quantum dynamical semigroups, if the invarinat von Neumann algebra is atomic.

Chapter 2

Elements of the Mathematical Structure of Quantum Theory

In this chapter, we will review some elements of the mathematical structure of quantum theory that are of particular relevance to our work. In terms of the level of abstraction and scope, we will follow Ref. [5].

2.1 Notation and Functional Analysis Basics

We start by fixing the notation. Throughout, $\mathcal{H}$ with some subscript denotes a separable complex Hilbert space, with an inner product denoted by $\langle\cdot|\cdot\rangle$, which is assumed to be linear in the second argument and anti-linear in the first one. We will use Dirac-notation throughout. If $\mathcal{X}, \mathcal{Y}$ are (complex) Banach spaces, then we denote by $\mathcal{B}(\mathcal{X}; \mathcal{Y})$ the Banach space of continuous linear operators $\mathcal{X} \to \mathcal{Y}$. We abbreviate $\mathcal{B}(\mathcal{X}, \mathcal{X})$ by $\mathcal{B}(\mathcal{X})$. We define the dual space $\mathcal{X}^* = \mathcal{B}(\mathcal{X}; \mathbb{C})$.
In addition to the norm-topology, we can consider the weak-* topology on $\mathcal{X}^*$, which is the weakest topology such that for all $x \in \mathcal{X}$, the linear functional $\ell_x : \mathcal{X}^* \to \mathbb{C}$, $x^* \mapsto x^*(x)$ is continuous. The weak-* topology has several nice properties. First, the closed unit ball in $\mathcal{X}^*$ is weak-* compact (this is the famous Banach-Alaoglu theorem). Moreover, if $\mathcal{X}$ is separable, then the weak-* topology is metrizable and hence the notions of compactness and sequential compactness coincide. A map $T : \mathcal{Y}^* \to \mathcal{X}^*$ is weak-* continuous if it is continuous when both $\mathcal{Y}^*$ and $\mathcal{X}^*$ are equipped with the weak-* topology. An important property of weak-* continuous linear maps $T \in \mathcal{B}(\mathcal{Y}^*; \mathcal{X}^*)$ that we will use repeatedly is that they have a predual map. More precisely, we have the following:

Lemma 2.1.1. ***[6, Proposition 2.4.12]*** $T \in \mathcal{B}(\mathcal{Y}^*; \mathcal{X}^*)$ *is weak-* continuous if and only if there exists a map* $T_* \in \mathcal{B}(\mathcal{X}; \mathcal{Y})$ *such that* $(y^* \circ T_*)(x) = T(y^*)(x)$ *for all* $x \in \mathcal{X}$ *and all* $y^* \in \mathcal{Y}^*$.

We will often call weak-* continuous linear maps *normal*[1].
A Banach space of particular interest is the space of bounded linear operators on a Hilbert space, which has a rich structure. The adjoint $X^\dagger \in \mathcal{B}(\mathcal{H})$ of $X \in \mathcal{B}(\mathcal{H})$ is uniquely defined

[1] Normality is often only defined for positive linear maps – and then in terms of bounded increasing nets of operators. It can be shown that for positive maps, normality and weak-* continuity are equivalent properties. Hence, we feel justified to drop this distinction altogether and use weak-* continuity and normality synonymously.

by the relation $\langle\psi|X\phi\rangle = \langle X^\dagger\psi|\phi\rangle$, for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. The †-operation turns $\mathcal{B}(\mathcal{H})$ into a *-algebra (in fact a von Neumann algebra). An operator $X \in \mathcal{B}(\mathcal{H})$ is called self-adjoint (or hermitian) if $X^\dagger = X$. A self-adjoint $X \in \mathcal{B}(\mathcal{H})$ is called positive semidefinite, denoted by $X \geq 0$, if $\langle\psi|X\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$. The square-root lemma [7, Theorem VI.9] (or the functional calculus) says that for $X \in \mathcal{B}(\mathcal{H})$ with $X \geq 0$, there exists a unique positive semidefinite operator $\sqrt{X} \in \mathcal{B}(\mathcal{H})$ such that $X = \sqrt{X}\sqrt{X}$. For any operator $X \in \mathcal{B}(\mathcal{H})$, we define $|X| = \sqrt{X^\dagger X}$. For some fixed orthonormal basis $\{|e_n\rangle\}_{n\in\mathbb{N}}$ of $\mathcal{H}$, we define the set of trace-class operators $\mathcal{S}_1(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \,|\, \sum_{n\in\mathbb{N}}\langle e_n|\,|\rho|\,e_n\rangle < \infty\}$. In fact the definition of $\mathcal{S}_1(\mathcal{H})$ does not depend on the particular choice of the basis, nor does the linear functional $\mathrm{tr} : \mathcal{S}_1(\mathcal{H}) \to \mathbb{C}$, defined by $\rho \mapsto \sum_{n\in\mathbb{N}}\langle e_n|\rho e_n\rangle$. The trace-class operators become a Banach space with norm $\|\rho\| = \mathrm{tr}\,[|\rho|]$. The Riesz representation theorem tells us that for every $\ell \in (\mathcal{S}_1(\mathcal{H}))^* = \mathcal{B}(\mathcal{S}_1(\mathcal{H}); \mathbb{C})$ there exists a unique $L \in \mathcal{B}(\mathcal{H})$ such that $\ell(\rho) = \mathrm{tr}\,[L\rho]$ for all $\rho \in \mathcal{S}_1(\mathcal{H})$. The corresponding mapping $\ell \mapsto L$ is an isometric isomorphism $(\mathcal{S}_1(\mathcal{H}))^* \to \mathcal{B}(\mathcal{H})$. Thus we can identify $\mathcal{B}(\mathcal{H})$ as the dual of $\mathcal{S}_1(\mathcal{H})$. This identification allows us to define the weak-* topology on $\mathcal{B}(\mathcal{H})$ as the weakest topology such that the mappings $L \mapsto \mathrm{tr}\,[L\rho]$ are continuous for all $\rho \in \mathcal{S}_1(\mathcal{H})$. Moreover, for any $T_* \in \mathcal{B}(\mathcal{S}_1(\mathcal{H}_\mathtt{A}); \mathcal{S}_1(\mathcal{H}_\mathtt{B}))$ there is a normal map $T \in \mathcal{B}(\mathcal{B}(\mathcal{H}_\mathtt{B}); \mathcal{B}(\mathcal{H}_\mathtt{A}))$ such that

$$\mathrm{tr}\,[T(X_\mathtt{B})\rho_\mathtt{A}] = \mathrm{tr}\,[X_\mathtt{B} T_*(\rho_\mathtt{A})]$$

for all $\rho_\mathtt{A} \in \mathcal{S}_1(\mathcal{H})$ and $X_\mathtt{B} \in \mathcal{B}(\mathcal{H}_\mathtt{B})$. By Lemma 2.1.1, this is a one-to-one correspondence and the transition between Heisenberg and Schrödinger picture.

Let us further introduce two other topologies on $\mathcal{B}(\mathcal{H})$. The weak operator topology (WOT) is the weakest topology such that the map $X \mapsto \langle\psi|X\phi\rangle$ is continuous for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. The strong operator topology (SOT) is the weakest topology such that the map $X \mapsto X|\psi\rangle$ is continuous for all $|\psi\rangle \in \mathcal{H}$.

Finally, we define the commutator of a set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ as $\mathcal{S}' = \{X \in \mathcal{B}(\mathcal{H}) \,|\, [X, S] = 0 \text{ for all } S \in \mathcal{S}\}$.

2.2 Quantum Mechanics

We are now ready to introduce the mathematical framework of quantum mechanics.

2.2.1 States and Observables

In operational quantum mechanics, physical experiments are usually described by a two-part procedure consisting of a state preparation step and a measurement step. The goal here is to predict the probability of certain measurement outcomes (or, equivalently, the expectation of observables), given the description of the preparation and the measurement apparatus.

It is a postulate of quantum mechanics that for any experiment, the set of observables can be associated with the self-adjoint elements of a C^*-algebra $\mathcal{A}$. Since states and observables together are supposed to predict expectation values of observables, it is natural to associate states with continuous linear functionals on $\mathcal{A}$. To comply with the probabilistic interpretation, every state $\omega \in \mathcal{A}^*$ is then further required to be positive (in the sense that $\omega(A) \geq 0$ for all $A \geq 0$) and to

have unit norm.
In this thesis, we will only consider the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Moreover, we assume all states to be normal (in the sense of Section 2.1). Since $\mathcal{S}_1(\mathcal{H})^* \cong \mathcal{B}(\mathcal{H})$, Lemma 2.1.1 implies that for every normal state $\omega \in \mathcal{B}(\mathcal{H})^*$, there is a unique $\rho \in \mathcal{S}_1(\mathcal{H})$ such that $\omega(X) = \mathrm{tr}\,[\rho X]$ for all $X \in \mathcal{B}(\mathcal{H})$. Moreover, the positivity of ω translates to $\rho \geq 0$ and $\|\omega\| = 1$ translates to $\mathrm{tr}\,[\rho] = 1$. Thus, we identify the set of states with the positive semidefinite trace-class operators with unit trace, a.k.a. density operators:

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{S}_1(\mathcal{H}) \,|\, \rho \geq 0, \mathrm{tr}\,[\rho] = 1\}.$$

2.2.2 Transformations of States and Observables: Completely Positive Maps

In this part, we will introduce and motivate very carefully completely positive maps as physical transformations of states and observables. We do so in order to provide the analogy for the axiomatization and interpretation of superchannels in Section 3.3. In the last section, we have stated that in quantum mechanics, observables - associated to a measurement process - are described by operators on a Hilbert space and that the state of a system - associated with a preparation step - is described by a density operator. It is clear that the separation into these two parts is not unique. After all one could regard the whole experiment as a preparation step for reading a number off some display; or, equivalently, one can consider the whole process as part of the measurement procedure and the preparation step just consists of doing nothing. In this regard, it is essential to formalize the notion of "Doing something to a system", so that the whole process of an experiment can be decomposed and analyzed in parts.
Assume that we have access to a physical system, whose preparation is described by a density operator $\rho \in \mathcal{D}(\mathcal{H})$. Now suppose that we "do something" to the system. Certainly, the whole process (preparation + "do something") can be regarded as a preparation step (e.g. by a third party). Thus, the system will be described by a density operator $\rho' \in \mathcal{D}(\mathcal{H})$. Since this sort or reasoning holds independently of the initial preparation process, there must be a map $T_{\mathcal{D}} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ mapping ρ to ρ'. Not every such map is physical. To be physical it must comply with the structure of quantum mechanics. Suppose that the inner workings of the preparation process were such that with probability p, the system was prepared in state $\rho_1 \in \mathcal{D}(\mathcal{H})$ and with probability $(1-p)$ in state $\rho_2 \in \mathcal{D}(\mathcal{H})$. Then, for us not knowing the inner workings, we will describe the preparation step by the operator $\rho = p\rho_1 + (1-p)\rho_2$. If we now "do something", then the state will be $T_{\mathcal{D}}(\rho)$. On the other hand, if the state were ρ_1, then the final state will be $T_{\mathcal{D}}(\rho_1)$ and $T_{\mathcal{D}}(\rho_2)$ if the state were ρ_2. However, for us not knowing the inner workings, we will describe this situation by the state $pT_{\mathcal{D}}(\rho_1) + (1-p)T_{\mathcal{D}}(\rho_2)$. Thus, to reconcile these two points of view, we must demand that

$$T_{\mathcal{D}}(p\rho_1 + (1-p)\rho_2) = pT_{\mathcal{D}}(\rho_1) + (1-p)T_{\mathcal{D}}(\rho_2)$$

for all $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$ and $0 \leq p \leq 1$. In conclusion, the probabilistic structure of quantum mechanics implies that transformation maps are convex-linear. It can be shown that such map $T_{\mathcal{D}}$ can be uniquely extended to a (bounded) trace-preserving positive linear map $T_* : \mathcal{S}_1(\mathcal{H}) \to$

$\mathcal{S}_1(\mathcal{H})$ (by first extending to positive traceclass operators via $T_*(P_+) = \mathrm{tr}\,[P_+]\, T_{\mathcal{D}}(\frac{P_+}{\mathrm{tr}[P_+]})$ and then to all of $\mathcal{S}_1(\mathcal{H})$ by linearity - see e.g. [8, Page 175 and Proposition 2.30]). A further structure of quantum mechanics is that composite systems are described by tensor products. That is, if we have two systems $\mathtt{A}$ and $\mathtt{B}$ with associated Hilbert spaces $\mathcal{H}_\mathtt{A}$ and $\mathcal{H}_\mathtt{B}$ then the associated Hilbert space of the combined system is $\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}$. If we "do something" on system $\mathtt{B}$ and do nothing on system $\mathtt{A}$ then the transformation on the combined system is naturally given by $\mathrm{id}_\mathtt{A} \otimes T_* : \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$. Given that this map is associated with a transformation, it has to be a (bounded) trace-preserving positive linear map. Clearly the argument just given is valid for any choice of the system $\mathtt{A}$. It turns out that the property (positivity) holding for infinite-dimensional $\mathcal{H}_\mathtt{A}$ is already implied if positivity of $\mathrm{id}_\mathtt{A} \otimes T_*$ holds for all finite-dimensional $\mathcal{H}_\mathtt{A}$. We thus make the following definition:

Definition 2.2.1. (CP-map)
A linear map $T_* : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ is called completely positive (CP), if $(\mathrm{id}_n \otimes T_*) : \mathcal{S}_1(\mathbb{C}^n \otimes \mathcal{H}) \to \mathcal{S}_1(\mathbb{C}^n \otimes \mathcal{H})$ is a positive linear map for all $n \in \mathbb{N}_0$.
A linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called completely positive if $(\mathrm{id}_n \otimes T) : \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}) \to \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ is a positive linear map for all $n \in \mathbb{N}_0$.
A completely positive and trace-preserving (CPTP) linear map $T_* : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ is called (Schrödinger) quantum channel. A completely positive and unital (i.e., $T(\mathbb{1}) = \mathbb{1}$) linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called (Heisenberg) quantum channel.

Remark 2.2.2. It is immediate from the definition that a normal linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive if and only if the corresponding predual map $T_* : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ is completely positive. Moreover, $T(\mathbb{1}) = \mathbb{1}$ is equivalent to $\mathrm{tr}\,[T_*(\rho)] = \mathrm{tr}\,[\rho]$ for all $\rho \in \mathcal{S}_1(\mathcal{H})$. Thus Heisenberg quantum channels are precisely the dual maps of Schrödinger quantum channels.

Do we need further restrictions for physical "do something" maps? The answer to this question appears to be no, because the following theorem due to Stinespring [9] shows that completely positive maps can be dilated to a Hamiltonian evolution an a larger system - thus we recover ordinary quantum theory.

Theorem 2.2.3. ***(Stinespring dilation theorem)*** *Let $T : \mathcal{B}(\mathcal{H}_\mathtt{I}) \to \mathcal{B}(\mathcal{H}_\mathtt{O})$ be a normal CP-map. Then there exists a Hilbert space $\mathcal{H}_\mathtt{E}$ and an operator $V \in \mathcal{B}(\mathcal{H}_\mathtt{O}; \mathcal{H}_\mathtt{I} \otimes \mathcal{H}_\mathtt{E})$ such that*

$$T(X) = V^\dagger (X \otimes \mathbb{1}_\mathtt{E}) V \tag{2.2.1}$$

for all $X \in \mathcal{B}(\mathcal{H}_\mathtt{I})$.
The pair $(V, \mathcal{H}_\mathtt{E})$ can be chosen such that $\mathrm{span}\{(X \otimes \mathbb{1}_\mathtt{E})V|\psi\rangle \,|\, X \in \mathcal{B}(\mathcal{H}_\mathtt{I}), \, |\psi\rangle \in \mathcal{H}_\mathtt{O}\}$ *is dense in $\mathcal{H}_\mathtt{I} \otimes \mathcal{H}_\mathtt{E}$. In that case, we call the pair $(V, \mathcal{H}_\mathtt{E})$ minimal. Moreover, in that case, we have the following uniqueness statement: If $(\tilde{V}, \mathcal{H}_{\tilde{\mathtt{E}}})$ is another pair satisfying* (2.2.1), *then there exists an isometry $U \in \mathcal{B}(\mathcal{H}_\mathtt{E}; \mathcal{H}_{\tilde{\mathtt{E}}})$ such that $\tilde{V} = (\mathbb{1}_\mathtt{I} \otimes U)V$.*

Remark 2.2.4. Stinespring proved his dilation theorem for CP-maps on arbitrary C^*-algebras, $\mathcal{C}$ [9]. More precisely he proved that there exists a Hilbert space $\mathcal{K}$ and a *-representation $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{K})$ such that $T(X) = V^\dagger \pi(X) V$ for all $X \in \mathcal{C}$. If $\mathcal{C}$ is a von Neuman algebra (or even W^*-algebra) and T is normal, then it can be shown that π can be chosen normal as well [10, P. 137]. Moreover, normal *-representations have a special form [11, Proposition 2.7.4]: namely, they can be written as a composition of an amplification, an induction and a spatial isomorphism. If $\mathcal{C} = \mathcal{B}(\mathcal{H}_\mathtt{I})$, then inductions are trivial and the spatial isomorphism can be absorbed into the operator V, leading to the specialized form (2.2.1) for *normal* CP-maps. A proof of the uniqueness statement can be found e.g. in [12, P. 46].

A well known corollary to the Stinespring dilation theorem is that normal CP-maps admit a Kraus representation.

Theorem 2.2.5. *Let $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a normal CP-map. Then there exists a family of operators $\{K_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H})$ such that*

$$T(X) = \sum_{n\in\mathbb{N}} K_n^\dagger X K_n,$$

where the series SOT-converges for all $X \in \mathcal{B}(\mathcal{X})$.

Proof. (sketch) Chose an orthonormal basis $\{|e_n\rangle\}_{n\in\mathbb{N}}$ of $\mathcal{H}_\mathtt{E}$ in Stinespring's theorem and define $K_n = (\mathbb{1} \otimes \langle e_n|)V$. The properties are then routinely verified. □

2.2.3 The Choi–Jamiołkowski isomorphism

An important tool (not only when dealing with CP-map) in finite-dimensional quantum information theory is the Choi–Jamiołkowski isomorphism (a.k.a. channel state duality). In this part (and whenever we deal with the Choi–Jamiołkowski isomorphism) we will only consider finite-dimensional Hilbert spaces.

Definition 2.2.6. (Choi–Jamiołkowski isomorphism) Let $\mathcal{H}_\mathtt{A}$ and $\mathcal{H}_\mathtt{B}$ be finite-dimensional Hilbert spaces and let $\{|a_n\rangle\}_{n=1}^{d_\mathtt{A}}$ be an orthonormal basis of $\mathcal{H}_\mathtt{A}$.
The Choi–Jamiołkowski isomorphism is the linear map $\mathfrak{C}_{\mathtt{A};\mathtt{B}} : \mathcal{B}(\mathcal{B}(\mathcal{H}_\mathtt{A}); \mathcal{B}(\mathcal{H}_\mathtt{B})) \to \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ defined by

$$\mathfrak{C}_{\mathtt{A};\mathtt{B}}(T) = (\mathrm{id}_\mathtt{A} \otimes T)(|\Omega\rangle\langle\Omega|),$$

where $|\Omega\rangle = \sum_{n=1}^{d_\mathtt{A}} |a_n\rangle \otimes |a_n\rangle$.

Remark 2.2.7. Note that the Choi–Jamiołkowski isomorphism depends on the choice of basis $\{|a_n\rangle\}_{n=1}^{d_\mathtt{A}}$. Throughout, we will assume some arbitrary but fixed choice.

The Choi–Jamiołkowski isomorphism is useful because of the following properties

Theorem 2.2.8. ***(Choi's theorem, [13])***

- $\mathfrak{C}_{\mathtt{A};\mathtt{B}}$ *is a bijective linear map with inverse given by*

$$\mathfrak{C}_{\mathtt{A};\mathtt{B}}^{-1}(\tau)(\rho) = \mathrm{tr}_\mathtt{A}\left[(\rho^T \otimes \mathbb{1}_\mathtt{B})\tau\right],$$

where ^{T} is the transpose w.r.t. $\{|a_n\rangle\}_{n=0}^{d_\mathtt{A}}$.

- *A linear map* $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{B}})$ *is hermiticity preserving (i.e.* $T(H)$ *is self-adjoint whenever* H *is self-adjoint) if and only if* $\mathfrak{C}_{\mathtt{A;B}}(T)$ *self-adjoint.*
- *A linear map* $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{B}})$ *is completely positive if and only if* $\mathfrak{C}_{\mathtt{A;B}}(T)$ *is positive semidefinite.*
- *A linear map* $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{B}})$ *is trace-preserving if and only if* $\mathrm{tr}_{\mathtt{B}}\left[\mathfrak{C}_{\mathtt{A;B}}(T)\right] = \mathbb{1}_{\mathtt{A}}$.
- *A linear map* $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{B}})$ *is unital if and only if* $\mathrm{tr}_{\mathtt{A}}\left[\mathfrak{C}_{\mathtt{A;B}}(T)\right] = \mathbb{1}_{\mathtt{B}}$.

Thus some relatively complex properties translate to much simpler properties when one looks at them through the lens of the Choi–Jamiołkowski isomorphism. In later applications, it is also useful to know how to do similarity transformations with the Choi–Jamiołkowski isomorphism. Here is a helpful lemma, proven in [2, Lemma V.20].

Lemma 2.2.9. ***(Translation Lemma, [2, Lemma V.20])*** *Let* $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ *be given by*

$$T(X) = \mathrm{tr}_{\mathtt{E}}\left[(\mathbb{1}_{\mathtt{A}} \otimes L_{\mathtt{B}})(L_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})X(R_{\mathtt{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{B}})(\mathbb{1}_{\mathtt{A}} \otimes R_{\mathtt{B}}^{\dagger})\right],$$

with Hilbert spaces $\mathcal{H}_{\mathtt{C}}$ *and* $\mathcal{H}_{\mathtt{E}}$*, operators* $L_{\mathtt{A}}, R_{\mathtt{A}} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{C}})$ *and* $L_{\mathtt{B}}, R_{\mathtt{B}} \in \mathcal{B}(\mathcal{H}_{\mathtt{C}} \otimes \mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$*. Then, for* $S \in \mathcal{B}(\mathcal{B}(\mathcal{H}_{\mathtt{A}}); \mathcal{B}(\mathcal{H}_{\mathtt{B}}))$ *and* $\rho \in \mathcal{B}(\mathcal{H}_{\mathtt{A}})$*,*

$$\left[\mathfrak{C}_{\mathtt{A;B}}^{-1} \circ T \circ \mathfrak{C}_{\mathtt{A;B}}\right](S)(\rho) = \mathrm{tr}_{\mathtt{E}}\left[V_L(S \otimes \mathrm{id}_{\mathtt{C}})\left(W_L \rho W_R^{\dagger}\right)V_R^{\dagger}\right],$$

with $V_L = L_{\mathtt{B}}\mathbb{F}_{\mathtt{B;C}}$*,* $V_R = R_{\mathtt{B}}\mathbb{F}_{\mathtt{B;C}}$*; and* $W_L = L_{\mathtt{A}}^{T_{\mathtt{A}}}$*,* $W_R = R_{\mathtt{A}}^{T_{\mathtt{A}}}$*. Here, the partial transpose on* $\mathcal{H}_{\mathtt{A}}$ *is taken w.r.t.* $\{|a_n\rangle\}_{n=0}^{d_{\mathtt{A}}}$ *and* $\mathbb{F}_{\mathtt{B;C}}$ *is the flip operator, flipping systems* B *and* C*.*

Chapter 3

Semicausality, Invariant Algebras and Superchannels

In this chapter, we introduce the notion of semicausality and its most important applications. Section 3.1 is devoted to the equivalence between semicausality and the more constructive notion of semilocalizability for CP-maps. In Section 3.3, we introduce superchannels [14], that is transformations between quantum channels. We study the relation between superchannels and semicausal CP-maps and discuss the fundamental results [14].
The notion of semicausality was introduced by Beckman et al. [15]. It captures the intuition of one-way communication $\mathtt{A} \to \mathtt{B}$, or, more precisely the impossibility of communication from $\mathtt{B}$ to $\mathtt{A}$. In order to ensure this, we demand that for a semicausal CPTP-map $T_* : \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$,

$$\mathrm{tr}_\mathtt{B}\left[T_*(\rho_{\mathtt{AB}})\right] = T_*^{\mathtt{A}}(\mathrm{tr}_\mathtt{B}\left[\rho_{\mathtt{AB}}\right])$$

holds for some CPTP-map $T_*^{\mathtt{A}} : \mathcal{S}_1(\mathcal{H}_\mathtt{A}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A})$ and all $\rho_{\mathtt{AB}} \in \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$. That is to say that the reduced evolution on system $\mathtt{A}$ does not depend on the $\mathtt{B}$-part of the system. Formally, we say that such a map is Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal.
We can also find the corresponding condition in the Heisenberg picture, which is useful for technical reasons. If T_* is a Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal map, with reduced map $T_*^{\mathtt{A}}$ and corresponding dual maps T and $T^{\mathtt{A}}$, then

$$\begin{aligned}
\mathrm{tr}\left[(T^{\mathtt{A}}(X_\mathtt{A}) \otimes \mathbb{1}_\mathtt{B})\rho_{\mathtt{AB}}\right] &= \mathrm{tr}\left[T^{\mathtt{A}}(X_\mathtt{A})\mathrm{tr}_\mathtt{B}\left[\rho_{\mathtt{AB}}\right]\right] \\
&= \mathrm{tr}\left[X_\mathtt{A} T_*^{\mathtt{A}}(\mathrm{tr}_\mathtt{B}\left[\rho_{\mathtt{AB}}\right])\right] \\
&= \mathrm{tr}\left[X_\mathtt{A}\mathrm{tr}_\mathtt{B}\left[T_*(\rho_{\mathtt{AB}})\right]\right] \\
&= \mathrm{tr}\left[(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{B})T_*(\rho_{\mathtt{AB}})\right] \\
\mathrm{tr}\left[T(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{B})\rho_{\mathtt{AB}}\right]
\end{aligned}$$

holds for all $X_{\mathtt{A}} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}})$ and all $\rho_{\mathtt{AB}} \in \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$. Thus, if T_* is Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal, then

$$T(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) = T^{\mathtt{A}}(X_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}. \tag{3.0.1}$$

We call a map satisfying (3.0.1) Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal. Let us summarize the previous discussion in the following definition, in which we replace that letter T by L, because we make no assumptions such as complete positivity in general.

Definition 3.0.1. (Semicausal map)
A bounded linear map $L_* : \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ is called Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal, if there exists $L_*^{\mathtt{A}} : \mathcal{S}_1(\mathcal{H}_{\mathtt{A}}) \to \mathcal{S}_1(\mathcal{H}_{\mathtt{A}})$ such that $\mathrm{tr}_{\mathtt{B}}[L_*(\rho)] = L_*^{\mathtt{A}}(\mathrm{tr}_{\mathtt{B}}[\rho])$ for all $\rho \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$.
A bounded linear map $L : \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{A}})$ is called Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal if there exists $L^{\mathtt{A}} : \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}})$ such that $L(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) = L^{\mathtt{A}}(X_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$ for all $X_{\mathtt{A}} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}})$.

Remark 3.0.2. We have already seen that L_* being Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal implies that L is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal. Moreover, a converse is also true, i.e. the predual-map of a *normal* Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal map is Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal.

Remark 3.0.3. Requiring that a map L is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal is equivalent to requiring that L leaves the von-Neumann algebra $\mathcal{B}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$ invariant - this was already noted in [16]. This observation, together with the realization that the techniques developed for semicausal maps in Article 2 can be extended to capture the more general case of maps with an invariant atomic von-Neumann algebra led to the investigations in Article 3.

In the next part we discuss an operational characterization of semicausal CP-maps.

3.1 Semicausal CP-maps are semilocalizable

The usefulness of semicausal CP-maps comes from the fact that they have an operational interpretation in terms of a concept called semilocalizability (see Fig. 3.1 for a visual representation).

Definition 3.1.1. (Semilocalizable CP-map)
A CP-map $T_* : \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ is called Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semilocalizable if there exist Hilbert spaces $\mathcal{H}_{\mathtt{E}}$ and $\mathcal{H}_{\mathtt{F}}$, an operator $A \in \mathcal{B}(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{F}})$ and an isometry $U : \mathcal{B}(\mathcal{H}_{\mathtt{F}} \otimes \mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$ such that

$$T(\rho) = \mathrm{tr}_{\mathtt{E}}\left[(\mathbb{1}_{\mathtt{A}} \otimes U)(A \otimes \mathbb{1}_{\mathtt{B}})\rho(A \otimes \mathbb{1}_{\mathtt{B}})^\dagger(\mathbb{1}_{\mathtt{A}} \otimes U)^\dagger\right]$$

for all $\rho \in \mathcal{S}_1(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$.
A normal CP-map $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ is called Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semilocalizable if the predual map is Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semilocalizable. In particular, T is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semilocalizable if it is given by

$$T(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V, \quad \text{with} \quad V = (\mathbb{1}_{\mathtt{A}} \otimes U)(A \otimes \mathbb{1}_{\mathtt{B}}),$$

for all $X \in \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$.

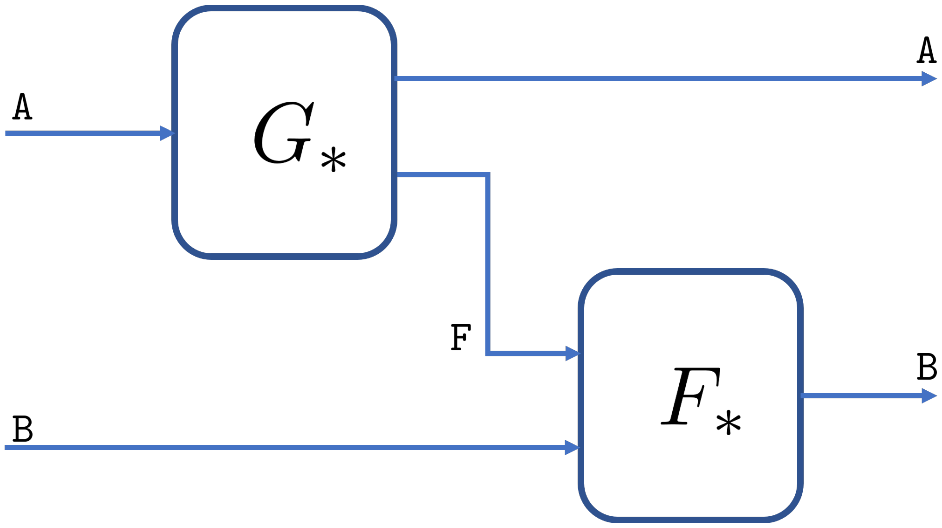

Figure 3.1: Interpretation of a Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semilocalizable CP-map in terms of two processes: 1) A CP-map $G_* : \mathcal{S}_1(\mathcal{H}_\mathtt{A}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{F})$, defined by $G_*(\rho_\mathtt{A}) = A\,\rho_\mathtt{A}\,A^\dagger$ that acts on Alice's side and sends a temporary system F to Bob; and 2) a *CPTP-map* $F_* : \mathcal{S}_1(\mathcal{H}_\mathtt{F} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{B})$, defined by $F_*(\rho_\mathtt{FB}) = \operatorname{tr}_\mathtt{E}\left[U\rho_\mathtt{FB}U^\dagger\right]$ that acts on Bob's side and processes the system F.

Remark 3.1.2. By using that U is an isometry, it is easy to see that semilocalizable maps are semicausal.

Remark 3.1.3. If T_* is trace-preserving, then A is an isometry. Hence the CP-maps F_* and G_*, defined in Fig. 3.1 are both trace-preserving.

Remark 3.1.4. A pivotal point in the definition above is that U is an isometry. In fact, if we drop this requirement then every CP-map $T_* : \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ (for finite-dimensional spaces) can be decomposed as

$$T_* = (\mathrm{id}_\mathtt{A} \otimes F_*) \circ (G \otimes \mathrm{id}_\mathtt{B}), \tag{3.1.1}$$

for CP-maps F_* and G_*, with signature defined in Fig. 3.1, as the following teleportation-inspired construction shows: Define $\mathcal{H}_\mathtt{F} = \mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{A}$ and $|\Omega\rangle = \sum_{i=1}^{d_\mathtt{A}} |a_i a_i\rangle \in \mathcal{H}_\mathtt{F}$, where $\{|a_i\rangle\}_i$ is some fixed orthonormal basis of $\mathcal{H}_\mathtt{A}$ and $d_\mathtt{A}$ is the dimension of $\mathcal{H}_\mathtt{A}$. We then define

$$F_*(\rho_\mathtt{AAB}) = (\langle\Omega| \otimes \mathbb{1}_\mathtt{B})\,[(\mathrm{id}_\mathtt{A} \otimes T)(\rho_\mathtt{AAB})]\,(|\Omega\rangle \otimes \mathbb{1}_\mathtt{B}), \quad \text{and} \quad G_*(\rho_\mathtt{A}) = |\Omega\rangle\langle\Omega| \otimes \rho_\mathtt{A}.$$

It is then easy to verify (3.1.1). Note also that the construction doesn't depend on T_* being completely positive - however, then G_* also wouldn't be completely positive.

While semicausality is an axiomatic notion for the impossibility of communication $\mathtt{B} \to \mathtt{A}$, semilocalizability explicitly gives a mechanism by which A communicates with B. The pleasing fact is that these two notions are equivalent. This equivalence was conjectured in [15], first proven by Eggeling et al. [16] and later rediscovered by Piani et al. [17]. Both proofs were in the finite-dimensional setting. An extension to the infinite-dimensional setting can be found in Ref. [18]. We state the theorem as follows:

Theorem 3.1.5. ***(Semicausal CP-maps are semilocalizable)*** *A CP-map* $T_* : \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ *is Schrödinger* $\mathtt{B} \not\to \mathtt{A}$ *semicausal if and only if it is Schrödinger* $\mathtt{B} \not\to \mathtt{A}$ *semilocalizable. It follows that a normal CP-map* $T : \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ *is Heisenberg* $\mathtt{B} \not\to \mathtt{A}$ *semicausal if and only if it is Heisenberg* $\mathtt{B} \not\to \mathtt{A}$ *semilocalizable.*

The proofs essential ingredient is the uniqueness of the Stinespring dilation up to unitaries. Since the techniques involved play an important role in Articles [2] and [3], we paraphrase the proof of references [16, 18] .

Proof. For technical convenience, we prove the statement in the Heisenberg picture. Let $T : \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ be a normal CP-map satisfying the semicausality condition $T(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{B}) = T^\mathtt{A}(X_\mathtt{A}) \otimes \mathbb{1}_\mathtt{B}$. Clearly, $T^\mathtt{A} : \mathcal{B}(\mathcal{H}_\mathtt{A}) \to \mathcal{B}(\mathcal{H}_\mathtt{A})$ is also a normal CP-map (take partial trace w.r.t. some state). Applying Stinespring's dilation theorem (Theorem 2.2.3) to both maps yields that there exist Hilbert spaces $\mathcal{H}_\mathtt{E}, \mathcal{H}_\mathtt{F}$ and operators $V \in \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}; \mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B} \otimes \mathcal{H}_\mathtt{E})$ and $A \in \mathcal{B}(\mathcal{H}_\mathtt{A}; \mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{E})$ such that

$$T(X) = V^\dagger(X \otimes \mathbb{1}_\mathtt{E})V, \quad \text{and} \quad T^\mathtt{A}(X_\mathtt{A}) = A^\dagger(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{F})A$$

for all $X \in \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ and all $X_\mathtt{A} \in \mathcal{B}(\mathcal{H}_\mathtt{A})$ and such that $S = \mathrm{span}\{(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{F})A|\psi_\mathtt{A}\rangle \mid X_\mathtt{A} \in \mathcal{B}(\mathcal{H}_\mathtt{A}), |\psi_\mathtt{A}\rangle \in \mathcal{H}_\mathtt{A}\}$ is dense in $\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{F}$. The semicausality relation then reads

$$\begin{aligned} V^\dagger(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{B} \otimes \mathbb{1}_\mathtt{E})V &= [A^\dagger(A_\mathtt{A} \otimes \mathbb{1}_\mathtt{F})A] \otimes \mathbb{1}_\mathtt{B} \\ &= (A \otimes \mathbb{1}_\mathtt{B})^\dagger(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{F} \otimes \mathbb{1}_\mathtt{B})(A \otimes \mathbb{1}_\mathtt{B}). \end{aligned}$$

It follows form the density of S that $\mathrm{span}\{(X_\mathtt{A} \otimes \mathbb{1}_\mathtt{F} \otimes \mathbb{1}_\mathtt{B})(A \otimes \mathbb{1}_\mathtt{B})|\psi_{\mathtt{AB}}\rangle \mid X_\mathtt{A} \in \mathcal{B}(\mathcal{H}_\mathtt{A}), |\psi_{\mathtt{AB}}\rangle \in \mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}\}$ is dense in $\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{F} \otimes \mathcal{H}_\mathtt{B}$. Hence $(A \otimes \mathbb{1}_\mathtt{B}, \mathcal{H}_\mathtt{F} \otimes \mathcal{H}_\mathtt{B})$ is a minimal dilation. Moreover, $(V, \mathcal{H}_\mathtt{B} \otimes \mathcal{H}_\mathtt{E})$ is another dilation of the same map $(X_\mathtt{A} \mapsto T^\mathtt{A}(X_\mathtt{A}) \otimes \mathbb{1}_\mathtt{B})$. Thus, by the uniqueness-part of Theorem 2.2.3, there exists and isometry $U \in \mathcal{B}(\mathcal{H}_\mathtt{F} \otimes \mathcal{H}_\mathtt{B}; \mathcal{H}_\mathtt{B} \otimes \mathcal{H}_\mathtt{E})$ such that $V = (\mathbb{1}_\mathtt{A} \otimes U)(A \otimes \mathbb{1}_\mathtt{B})$. This is the claim. □

In the next section, we generalize the notion of semicausality and state a generalized semicausality, semilocalizability equivalence theorem that is an important result in Article [3].

3.2 CP-maps with an invariant algebra

In the last sections we introducted semicausal CP-maps and characterized them in terms of semilocalizability. In this section, we are going to generalize the results of the previous sections. To this end, note that saying that $T : \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal is equivalent to stating that T leaves the type-I factor von Neuman algebra $\mathcal{B}(\mathcal{H}_\mathtt{A}) \otimes \mathbb{1}_\mathtt{B}$ invariant. In view of the explicit representation of semicausal CP-maps, this observation immediately suggests a generalized question: "How do (normal) CP-maps that leave a generic von Neuman algebra invariant look like?".

The partial answer to this question (we were only able the question for atomic von Neuman algebras – which does, however, include the important finite-dimensional case) is one of the main results of Article [3] (alongside answering the corresponding question for GKLS-generators). Before stating the result, we review some von Neumann algebra theory.

3.2.1 von Neumann algebras in a nutshell

The main aim of this interlude is to define the relavant notions from the theory of von Neuman algebras so that the results (which are presented in Sections 3.2.2 and 5.3.3) of Article [3] can be readily understood. We start with the fundamental definition.

Definition 3.2.1. A subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under adjoints and (topologically) closed w.r.t. the weak operator topology is called *weakly closed *-algebra*. If a weakly closed *-algebra contains the identity, then it is called a *von Neuman algebra*.

Qualitatively, there is not a big difference between weakly closed *-algebras and von Neumann algebras, since any weakly closed *-algebra is (unitarily equivalent to) the direct sum of the zero matrix and a von Neuman algebra. On a technical level, we have

Lemma 3.2.2. *[19, Proposition 5.1.8]*
*If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a weakly closed *-algebra, then there exists a projection $P \in \mathcal{A}$ such that $P \geq Q$ for all projections $Q \in \mathcal{A}$ and $P\mathcal{A}P = \mathcal{A}$.*

Let P be the projection given by the lemma above. Define $\mathcal{H}_{\overline{0}} = \mathrm{range}(P)$ and $\mathcal{H}_0 = \mathcal{H}_{\overline{0}}^{\perp}$. Then $\mathcal{H}$ is unitarily equivalent to $\mathcal{H}_{\oplus} = \mathcal{H}_0 \oplus \mathcal{H}_{\overline{0}}$, i.e. $\mathcal{H} = U_{\mathcal{A}}\mathcal{H}_{\oplus}$ for some unitary $U_{\mathcal{A}} \in \mathcal{B}(\mathcal{H}_{\oplus}; \mathcal{H})$. Let $P_0^{\oplus} \in \mathcal{B}(\mathcal{H}_{\oplus}; \mathcal{H}_0)$ and $P_{\overline{0}}^{\oplus} \in \mathcal{B}(\mathcal{H}_{\oplus}; \mathcal{H}_{\overline{0}})$ be the projections onto the corresponding summands and define $P_0 = P_0^{\oplus} U_{\mathcal{A}}^{\dagger}$ and $P_{\overline{0}} = P_{\overline{0}}^{\oplus} U_{\mathcal{A}}^{\dagger}$. Then, $\mathcal{A}_{\overline{0}} = P_0 \mathcal{A} P_0^{\dagger}$ is a von Neuman algebra and

$$\mathcal{A} = U_{\mathcal{A}}(0_0 \oplus \mathcal{A}_{\overline{0}})U_{\mathcal{A}}^{\dagger}, \tag{3.2.1}$$

where 0_0 is the zero operator in $\mathcal{B}(\mathcal{H}_0)$.
An important point that makes von Neuman algebras appealing is that there is a well-developed representation theory. The fundamental result in that domain is that every von Neuman algebra is unitarily equivalent to a direct integral of factors (see [20] Chapter 14 and in particular Theorems 14.2.1 and 14.2.2 for in an introduction to direct integrals and the corresponding representation theorems). Thus classifying von Neuman algebras can be reduced to the classification of factors. Those are defined and categorized as follows:

Definition 3.2.3. A von Neuman algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a factor if $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}\mathbb{1}$.

- A factor $\mathcal{A}$ is said to be of type-I, if it posesses a minimal non-zero projection $P \in \mathcal{A}$. That is, there is no projection $Q \in \mathcal{A}$, with $0 < Q < P$.
- A factor $\mathcal{A}$ is said to be of type-II, if it is not of type-I, but it posesses a non-zero finite projection $P \in \mathcal{A}$. That is, there is no projection $Q \in \mathcal{A}$ with $0 < Q < P$ such that $\mathrm{range}(Q) = V\mathrm{range}(P)$ for some partial isometry $V \in \mathcal{A}$.
- A factor $\mathcal{A}$ is said to be of type-III, if it does not posess any finite non-zero projection $P \in \mathcal{A}$. Hence, if and only if it is neither type-I nor type-II.

Remark 3.2.4. If $\mathcal{H}$ is finite-dimensional and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a factor, then it is of type-I. To see this, note that any non-zero projection $P_0 \in \mathcal{A}$ is either minimal or there exists a projection $P_1 \in \mathcal{A}$ with $0 < P_1 < P_0$. By repeating this conlcusion, we obtain a sequence of projections $P_0 > P_1 > P_2 > \cdots > 0$. This sequence has to terminate for some $n \in \mathbb{N}_0$. This is so, because $\dim(\text{range}(P_i)) \geq \dim(\text{range}(P_{i+1})) + 1$ and thus, if the sequence were infinite, then $\dim(\text{range}(P_0)) = \infty$. This contradicts that $\mathcal{H}$ is finite-dimensional.

Classifying factors is a quite complicated (and still not fully completed) endeavor, easily filling hundreds of pages [21–23]. However, for the case of type-I factors the theory is very clear-cut.

Theorem 3.2.5. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a type-I factor. Then there exist Hilbert spaces $\mathcal{H}_\mathtt{A}$ and $\mathcal{H}_\mathtt{B}$ and a unitary $U_\mathcal{A} \in \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}; \mathcal{H})$ such that*

$$\mathcal{A} = U_\mathcal{A}(\mathcal{B}(\mathcal{H}_\mathtt{A}) \otimes \mathbb{1}_\mathtt{B})U_\mathcal{A}^\dagger.$$

Remark 3.2.6. The previous theorem yields a concrete form for a type-I factor. It is easy to see that saying that a linear map T is semicausal is essentially equivalent to saying that $T(\mathcal{A}) \subseteq \mathcal{A}$ for some type-I factor $\mathcal{A}$.

In light of the previous remark and by taking into account that the representation theory for type-II and type-III factors is incomplete, it seems necessary to restrict our attention to von Neuman algebras whose direct integral decomposition consists only of type-I factors. However, even that case turns out to be tricky in general (due to measure-theoretic complications), so that we were unable (in Article [3]) to fully characterize CP-maps that leave a von Neumann algebra of that kind invariant. Thus, we had to restrict to so-called atomic algebras. Atomic von Neumann algebras are abstractly defined as follows.

Definition 3.2.7. [21, Definition 5.9]
A von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called *atomic* if for every non-zero projection $P \in \mathcal{A}$ there exists a non-zero minimal projection $Q \in \mathcal{A}$ such that $Q \leq P$.

Remark 3.2.8. If $\mathcal{H}$ is finite-dimensional, then every von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is atomic. Thus, our results cover the finite-dimensional case. This assertion can be seen by the same argument given in Remark 3.2.4.

The crucial point in using atomic von Neumann algebras is that they are exactly those von Neuman algebras that are unitarily equivalent to a direct sum of type-I factors. A proof of this fact can be found e.g. in the appendix of [24]. Combining (3.2.1) with the previous fact and Theorem 3.2.5, we arrive at the following (consistent and equivalent) definition for atomic weakly closed *-algebras.

Definition 3.2.9. A weakly closed *-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called *atomic* if

$$\mathcal{A} = U_\mathcal{A}\left(0_0 \oplus \bigoplus_{i \in I}(\mathcal{B}(\mathcal{H}_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{B}_i})\right)U_\mathcal{A}^\dagger,$$

for a Hilbert space $\mathcal{H}_0$, sequences of Hilbert spaces $\{\mathcal{H}_{A_i}\}_{i\in I}$ and $\{\mathcal{H}_{B_i}\}_{i\in I}$ indexed by a countable index set I, and a unitary $U_\mathcal{A} : \mathcal{H}_\oplus \to \mathcal{H}$, where $\mathcal{H}_\oplus = \mathcal{H}_0 \oplus \bigoplus_{i\in I}(\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i})$.

We further define for all $i \in I$ the Hilbert space $\mathcal{H}_i = \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{B}_i}$. For all $k \in I \cup \{0\}$, let $P_k^{\oplus} \in \mathcal{B}(\mathcal{H}_\oplus; \mathcal{H}_k)$ be the orthogonal projection onto $\mathcal{H}_k$ and let us define $P_k \in \mathcal{B}(\mathcal{H}; \mathcal{H}_k)$ as $P_k = P_k^{\oplus} U_{\mathcal{A}}^{\dagger}$.[1] Hence, an arbitrary element $X_{\mathcal{A}} \in \mathcal{A}$ can be written as SOT-convergent series $X_{\mathcal{A}} = \sum_{i\in I} P_i^{\dagger}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i$, for some operators $X_{\mathtt{A}_i} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}_i})$, with $\sup_{i\in I} \|X_{\mathtt{A}_i}\| < \infty$.

3.2.2 Normal Form

We are finally prepared to state our normal form for (normal) CP-maps with an *atomic* invariant weakly closed *-subalgebra - a central result in Article [3]. Slightly more generally, our result is a normal form for normal CP-maps T with the property that $T(\mathcal{A}) \subseteq \mathcal{C}$, for two atomic weakly closed *-algebras $\mathcal{A}$ and $\mathcal{C}$. Since we are now dealing with two algebras, we need to distinguish them in the notation in Definition 3.2.9. For the algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{A}})$: the index set is called I; the Hilbert spaces $\{\mathcal{H}_i\}_{i\in I\cup\{0\}}$ are denoted by $\mathcal{H}_{i:\mathcal{A}}$, with $\mathcal{H}_{i:\mathcal{A}} = \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{B}_i}$ $(i \in I)$; and the operators P_i are called $P_{i:\mathcal{A}} \in \mathcal{B}(\mathcal{H}_{\mathcal{A}}; \mathcal{H}_{i:\mathcal{A}})$. For the algebra $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{C}})$: the index set is called J; the Hilbert spaces $\{\mathcal{H}_j\}_{j\in J\cup\{0\}}$ are denoted by $\mathcal{H}_{j:\mathcal{C}}$, with $\mathcal{H}_{j:\mathcal{C}} = \mathcal{H}_{\mathtt{C}_j} \otimes \mathcal{H}_{\mathtt{D}_j}$ $(j \in J)$; and the operators P_j are called $P_{j:\mathcal{C}} \in \mathcal{B}(\mathcal{H}_{\mathcal{C}}; \mathcal{H}_{j:\mathcal{C}})$. With this notation in place, we can state our result:

Theorem 3.2.10. ***(Normal form for CP-maps, [3, Theorem 5])*** *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{A}})$ and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{C}})$ be two atomic weakly closed *-algebras. For a normal CP-map $T : \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \to \mathcal{B}(\mathcal{H}_{\mathcal{C}})$ defined by $T(X) = V^{\dagger}(X \otimes \mathbb{1}_{\mathtt{E}})V$, with $V \in \mathcal{B}(\mathcal{H}_{\mathcal{C}}; \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathtt{E}})$, the following are equivalent*

1. *$T(\mathcal{A}) \subseteq \mathcal{C}$.*
2. *There exist an operator $V_0 \in \mathcal{B}(\mathcal{H}_{\mathcal{C}}; \mathcal{H}_{0:\mathcal{A}} \otimes \mathcal{H}_{\mathtt{E}})$; and for all $i \in I$ and $j \in J$ Hilbert spaces $\mathcal{H}_{\mathtt{F}_{ij}}$, operators $A_{ij} \in \mathcal{B}(\mathcal{H}_{\mathtt{C}_j}; \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$, and isometries $U_{ij} \in \mathcal{B}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{D}_j}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$, such that*

   - *V can be decomposed as*

     $$V = (P_{0:\mathcal{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{E}})V_0 + \sum_{i\in I,\, j\in J} (P_{i:\mathcal{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{E}})V_{ij}P_{j:\mathcal{C}},$$

     with $V_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j})$, s.t. the series SOT-converges.
   - *The relation $U_{ik}^{\dagger}U_{il} = \delta_{kl}\mathbb{1}$ holds for all $i \in I$ and $k, l \in J$.*

The representation in 2 can be chosen such that $\operatorname{span}\{(X \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}|\psi\rangle \mid X \in \mathcal{B}(\mathcal{H}_{\mathtt{A}_i}), |\psi\rangle \in \mathcal{H}_{\mathtt{C}_j}\}$ *is dense in $\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}}$.*

3.3 Transformations of Quantum Channels: Superchannels

In this section, we introduce quantum superchannels. We will describe them axiomatically, describe their relation to semicausality and state the fundamental representation theorem. Historically, superchannels were the first step towards higher-order quantum theory [25]. They were introduced in [14], where also the fundamental representation theorem was proven. Although superchannels can be treated in the infinite-dimensional setting [26], we restrict ourselves to the

[1] Note that this definition is consistent with the one introduced in the first paragraph above.

finite-dimensional case.

In Chapter 2, we introduced quantum channels as the mathematical solution to the problem: Suppose you are handed a quantum system, what's is the most general thing that you can "do with it". Now, suppose you are handed an (unknown) apparatus – mathematically described as a quantum channel – what is the most general thing you can "do with it"? Superchannels are the mathematical answer to this question. It is interesting to note at this point that one doesn't have to stop asking this question on the level for superchannels (i.e. one can ask, what's the most general transformation on superchannels), but that one can repeat this indefinitely. This leads to the hierarchical framework of higher-order quantum theory [25]. To derive superchannels, we proceed as for quantum channels. Denote by $\mathrm{CPTP}(\mathcal{H})$ the set CPTP maps on $\mathcal{S}_1(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. To every "do something with a quantum channel" action there should be an associated map $S_{\mathrm{TP}} : \mathrm{CPTP}(\mathcal{H}) \to \mathrm{CPTP}(\mathcal{H})$. Similar to quantum channels, S_{TP} has to comply with the probabilistic structure of quantum theory - in that case with the probabilistic structure of quantum channels, not that of density matrices. The probabilistic structure of quantum channels is richer than just their convex structure. Suppose Eve implements a quantum channel by the following procedure: First she lets the system interact with her system $\mathtt{E}$, where the interaction is described by an isometry $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}}$ (which, of course can be extended to a unitary on a larger space). Then she perform an n-valued projective measurement on $\mathtt{E}$ where the associated projections are denoted by $P_0, P_2, \dots P_n$. The resulting channel is then given by

$$T(\rho) = \sum_{i=0}^{n} T_i(\rho),$$

where $T_i(\rho) = \mathrm{tr}_{\mathtt{E}}\left[(\mathbb{1} \otimes P_i) U \rho U (\mathbb{1} \otimes P_i)\right]$. Suppose this channel is subject to our "do something" action, then the resulting channel is $S_{\mathrm{TP}}(T)$. Suppose that many experiments (with different input states) are performed, using $S_{\mathrm{TP}}(T)$. Moreover, suppose that Eve keeps a list of outcomes of her measurement for each of these experiments. It should be possible – on physical grounds – to predict the outcome statistics of the experiment conditional on Eve's measurement outcome (if we know Eves outcome). However, in general, this can't be inferred from $S_{\mathrm{TP}}(T)$ alone. So, a full description of a "do something" operation must entail more than just a map on $\mathrm{CPTP}(\mathcal{H})$. To take this into account, note that the map T_i has the generic form of a completely positive trace non-increasing map, as a consequence of Ozawas theorem [27]. Thus in order to predict outcomes of arbitrary settings we should really have a map $S_{\downarrow} : \mathrm{CP}_{\downarrow}(\mathcal{H}) \to \mathrm{CP}_{\downarrow}(\mathcal{H})$, where $\mathrm{CP}_{\downarrow}(\mathcal{H})$ denotes the set of completely positive trace non-increasing maps. Moreover, if $\{T_i\}_{i=0}^{n}$ are trace non-increasing CP-maps such that $T = \sum_i T_i$ is trace-preserving, then

$$S_{\downarrow}(\sum_i T_i) = \sum_i S_{\downarrow}(T_i). \tag{3.3.1}$$

My claim is now that $S_\downarrow$ can be uniquely extended to a linear map: First note that (3.3.1) holds, even if the T_i's only add up to a trace non-increasing map. This is because, by Ozawa's theorem in that case there is always a CP-map Q such that $\sum_i T_i + Q$ is trace-preserving. Moreover, since

$$\sum_i S(T_i) + S(Q) = S(\sum_i T_i + Q) = S(\sum_i T_i) + S(Q),$$

it follows that $S(\sum_i T_i) = \sum_i S(T_i)$ whenever $T_i \in \mathsf{CP}_\downarrow(\mathcal{H})$. Since $\mathfrak{C}_{\mathcal{H};\mathcal{H}}(\mathsf{CP}_\downarrow(\mathcal{H})) = \{\rho \in \mathcal{H} \otimes \mathcal{H} \,|\, \rho \geq 0, \mathrm{tr}\,[\rho] \leq \dim(\mathcal{H})\}$, we can extend S uniquely to a bounded linear map, by the same arguments as for CPTP-maps. Thus our "do something" maps are bounded linear maps S such that $S(T)$ is CP whenever T is CP and $S(T)$ is CPTP whenever T is CPTP. As for the derivation of channels, we will also require that these maps are consistent with the tensor product structure. If we have a bipartite system $\mathtt{A} + \mathtt{B}$ and a channel $T : \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B}) \to \mathcal{S}_1(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathtt{B})$ and if we know how to do something to a channel acting only on $\mathtt{B}$ then it is natural to describe its action on T by $(\mathrm{id}_{\mathcal{S}_1(\mathcal{H}_\mathtt{A})} \otimes S)(T)$. By the same arguments as above, $(\mathrm{id}_{\mathcal{S}_1(\mathcal{H}_\mathtt{A})} \otimes S)$ should also map CP-maps to CP-maps and channels to channels. Thus we arrive at the definition of a superchannel.

Definition 3.3.1. A linear map $\hat{S} : \mathcal{B}(\mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{B}(\mathcal{H}))$ is called superchannel, if for all $n \in \mathbb{N}_0$, the map $(\mathrm{id}_{\mathcal{S}_1(\mathbb{C}^n)} \otimes \hat{S}) : \mathcal{B}(\mathcal{S}_1(\mathbb{C}^n \otimes \mathcal{H})) \to \mathcal{B}(\mathcal{S}_1(\mathbb{C}^n \otimes \mathcal{H}))$ has the property that $(\mathrm{id}_{\mathcal{S}_1(\mathbb{C}^n)} \otimes \hat{S})(T)$ is a CP-map whenever T is a CP-map and $(\mathrm{id}_{\mathcal{S}_1(\mathbb{C}^n)} \otimes \hat{S})(T)$ is CPTP whenever T is CPTP.

Remark 3.3.2. Note note that it is not enough to just require that Choi operators of channels are mapped to Choi operators of channels to ensure the positivity of the transformation map. To see this, let $0 < \alpha < 1$, let $\sigma_\mathrm{B} \in \mathcal{B}(\mathcal{H}_\mathrm{B})$ and $\tau_\mathrm{B} \in \mathcal{B}(\mathcal{H}_\mathrm{B})$ be different density operators, and let $|a_0\rangle, |a_1\rangle \in \mathcal{H}_\mathtt{A}$ be orthogonal unit vectors. Define the linear map $T : \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathrm{B}) \to \mathcal{B}(\mathcal{H}_\mathtt{A} \otimes \mathcal{H}_\mathrm{B})$ by

$$T(\rho_\mathtt{AB}) = \frac{\alpha}{d_\mathtt{A}} \mathrm{tr}\,[\rho_\mathtt{AB}]\,(\mathbb{1}_\mathtt{A} \otimes (\sigma_\mathrm{B} - \tau_\mathrm{B})) + \mathrm{tr}\,[|a_0\rangle\langle a_0| \mathrm{tr}_\mathtt{B}\,[\rho_\mathtt{AB}]]\,(\mathbb{1}_\mathtt{A} \otimes \tau_\mathrm{B}).$$

For Choi operators of channels, i.e. if $\mathrm{tr}_\mathtt{B}\,[\rho_\mathtt{AB}] = \mathbb{1}_\mathtt{A}$ and $\rho_\mathtt{AB} \geq 0$, we have

$$T(\rho_\mathtt{AB}) = \mathbb{1}_\mathtt{A} \otimes (\alpha \sigma_\mathrm{B} + (1 - \alpha)\tau_\mathrm{B})$$

This is a positive operator and $\mathrm{tr}_\mathtt{B}\,[T(\rho_\mathtt{AB})] = \mathbb{1}_\mathtt{A}$. Thus, T maps Choi operators of channels to Choi operators of channels. However, T is not positive since

$$T(|a_1\rangle\langle a_1| \otimes \tau_B) = \frac{\alpha}{d_\mathtt{A}} \mathbb{1}_\mathtt{A} \otimes (\sigma_\mathrm{B} - \tau_\mathrm{B})$$

is not a positive semidefinite operator.

Remark 3.3.3. The reader might rightfully be surprised about the linearity of superchannels. After all, what is wrong with the transformation $T \mapsto T \circ T$, which one might ascribe to the action of letting the system go through some device twice. The assumption excluding this transformation is that quantum channels provide a full description of the transformation. In particular, the exact implementation of a channel via its Stinespring dilation must not matter

– this is expressed in (3.3.1). But note that interpreting $T \circ T$ as invoking the same channel twice assumes that the channel is implemented such that there are no memory effects. Since knowledge about that fact is required, T alone does no longer describe the full behavior of the transformation. This subtlety of course already exists for transformations of quantum states (quantum channels), since there linearity is implied by the assumption that one cannot tell apart different convex combinations that lead to the same quantum state – which may not always be justified if one has a more refined knowledge of the preparation procedure.

At present, superchannels are abstract objects. The following theorem tells us that up to a similarity transformation, superchannels are just a certain kind of CP-map.

Theorem 3.3.4. ***(Superchannels are similar to semicausal CP-maps, [14, Lemma 3])*** *For a finite-dimensional space $\mathcal{H}_{\mathtt{A}} = \mathcal{H}_{\mathtt{B}}$, let $\hat{S} : \mathcal{B}(\mathcal{S}_1(\mathcal{H}_{\mathtt{A}}); \mathcal{S}_1(\mathcal{H}_{\mathtt{B}})) \to \mathcal{B}(\mathcal{S}_1(\mathcal{H}_{\mathtt{A}}); \mathcal{S}_1(\mathcal{H}_{\mathtt{B}}))$ be a linear map and define $S = \mathfrak{C}_{\mathtt{A;B}} \circ \hat{S} \circ \mathfrak{C}_{\mathtt{A;B}}^{-1}$. Then $\hat{S}$ is a superchannel if and only if S is a Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal CP-map such that the reduced map $S^{\mathtt{A}}$ satisfies $S^{\mathtt{A}}(\mathbb{1}_{\mathtt{A}}) = \mathbb{1}_{\mathtt{A}}$.*

Remark 3.3.5. In the previous theorem, complete positivity of S follows from the requirement that $(\mathrm{id}_{\mathcal{S}_1(\mathbb{C}^n)} \otimes \hat{S})(T)$ is a CP-map whenever T is a CP-map. Semicausality follows from the requirement to map trace-preserving CP-maps to trace-preserving CP-maps.
Intuitively, this equivalence comes about because the Choi–Jamiołkowski isomorphism transforms temporal properties (here, trace-preservation) into spatial properties. Slightly more precisely, since $S[\mathfrak{C}_{\mathtt{A;B}}(T_2 \circ T_1)] = S[(\mathrm{id}_{\mathtt{A}} \otimes T_2)(\mathfrak{C}_{\mathtt{A;B}}(T_1))]$, the translated trace-preserving property (Theorem 2.2.8) implies $\mathrm{tr}_{\mathtt{B}}\left[S[(\mathrm{id}_{\mathtt{A}} \otimes T_2)(\mathfrak{C}_{\mathtt{A;B}}(T_1))]\right] = \mathbb{1}_{\mathtt{A}}$ for all CPTP-maps T_1 and T_2. Thus the reduced state on $\mathtt{A}$ is invariant under operations on $\mathtt{B}$, which should tell us that we cannot communicate from $\mathtt{B} \to \mathtt{A}$.

Theorem 3.3.4 tells us that superchannels are related to Schrödinger $\mathtt{B} \not\to \mathtt{A}$ semicausal CP-maps. But those maps are well understood in terms of semilocalizability (Theorem 3.1.5). Moreover, Lemma 2.2.9 helps us with performing the similarity transformation. This leads to a very convenient, circuital form for superchannels:

Theorem 3.3.6. ***(Circuit form of superchannels, [14, Theorem 1])*** *A $\hat{S} : \mathcal{B}(\mathcal{S}_1(\mathcal{H})) \to \mathcal{B}(\mathcal{S}_1(\mathcal{H}))$ is a superchannel, if and only if there exists a (finite-dimensional) Hilbert sapce $\mathcal{H}_{\mathtt{E}}$ and two CPTP-maps $E : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $D : \mathcal{S}_1(\mathcal{H} \otimes \mathcal{H}_{\mathtt{E}}) \to \mathcal{S}_1(\mathcal{H})$ such that*

$$\hat{S}(T) = D \circ (S \otimes \mathrm{id}_{\mathtt{E}}) \circ E.$$

The theorem above tells us that superchannels are exactly those maps on channels that can be implemented by a quantum circuit. Thus, similarly to the Stinespring dilation theorem for CP-maps, the constructive and axiomatic approach yield the same results and the basic theory is thus complete.

Chapter 4

“Interaction-Free" Measurements

The purpose of this section[1] is to give a high-level overview of the concepts and results of Article [1]. It assumes some familiarity with the works of Elitzur and Vaidman [4] and of Kwiat et al. [28]. For a comprehensive introduction to “interaction-free" measurements we refer the reader to the introduction in Article [1].

Let's start with a short recap: The notion “interaction-free" measurement was coined by Elitzur and Vaidman in their bomb-tester experiment [4]. It is now commonly associated with the act of inferring the presence or absence of an object without altering it. For an opaque object (usually imagined to be a bomb), Kwiat et al. [28] showed that if the object can be tested multiple times, then there is a protocol such that the probability for a successful “interaction-free" measurement approaches one as the number of tests approaches infinity. The main idea behind the Kwiat et al. protocol is that an opaque object can be interpreted as a measurement device and that one can then take advantage of the quantum Zeno effect [29–33].

Since those two fundamental works, “interaction-free" measurements gained a lot of attention. The theoretical literature (as opposed to experimental) can be divided into two categories. First, since “interaction-free" measurements exploit one of the most fundamental properties of quantum mechanics - superposition - it shouldn't come as a surprise that this an interesting field for researchers working on the foundations of quantum mechanics. Hence the focus of the first category is on interpretational issues [34–37]. The second category concerns itself with the practical applications of “interaction-free" measurements. These applications include applications include “interaction-free" imaging [38, 39], counterfactual quantum computation [40–42], counterfactual communication [43] and cryptography [44], and complexity theory [45]. An important result (in particular for our purposes) is that it is possible to infer the presence or absence of on optically semi-transparent object in a totally “interaction-free" manner (i.e. with probability approaching one), but it is impossible to infer the transparency of such an object in that way [46, 47].

Despite all those different applications, there was no general, practically oriented framework that pinpoints which information can or cannot be inferred in a totally “interaction-free" manner. The main purpose of our work is to address this problem.

[1]This section is based on an unpublished extended abstract written by the author and previously submitted to various conferences.

4.1 Concepts

An important observation leading to our generalized model is that we can interpret the task of inferring the presence or absence of an object as a quantum channel discrimination problem, where one channel models the object, and the identity channel models 'no object'. The generalization then follows naturally, as we can now ask: Which channels can be discriminated in a totally "interaction-free" manner? Note that an even more general question would be about "interaction-free" channel tomography. However, as our no-go results will imply, "interaction-free" channel tomography is mostly hopeless.

The biggest conceptual challenge now is to define what "interaction-free" should mean for quantum channels. Before we look at that, let us be more precise about our setting for the quantum channel discrimination problem. So, let us consider first the usual (i.e. non "interaction-free") problem of discriminating between two channels T_A and T_B acting on a system I. The most general (causally ordered) discrimination strategy is given by the sequential scheme, aka quantum comb [48], depicted in Figure 4.1. Such a strategy is completely defined by specifying an initial state $s_0 \in \mathcal{B}(\mathcal{H}_I \otimes \mathcal{H}_Z)$ and intermediate channels $\Lambda_0, \Lambda_1, \ldots, \Lambda_N$, acting on the bipartite system $I + Z$, where Z denotes an arbitrarily large ancillary system. The output of such a protocol is a quantum state ρ_N^T, recursively defined via $\rho_0^T := \Lambda_0(s_0)$; $\rho_n^T := \Lambda_n((T \otimes \mathrm{id}_Z)(\rho_{n-1}^T))$, which depends on the unknown channel T. Hence, for a fixed strategy D, the channel discrimination problem reduces to the state discrimination problem. This allows us to define our first quantity of merit, the *error probability* $P_e(D, \Pi) := \frac{1}{2}\left(\mathrm{tr}\left[\pi_B \rho_N^{T_A}\right] + \mathrm{tr}\left[\pi_A \rho_N^{T_B}\right]\right)$, which is defined relative to the two-valued POVM $\Pi := \{\pi_A, \pi_B\}$.

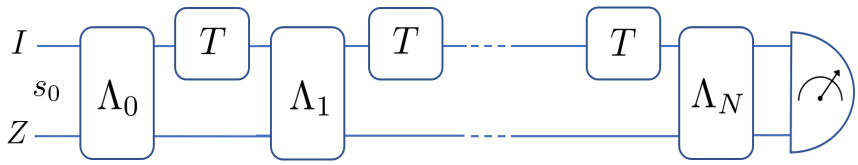

Figure 4.1: Sequential scheme for quantum channel discrimination, with unknown channel T.

Let us now get to defining the term "interaction-free" for a given discrimination strategy D and a channel T. In order to make an "interaction-free" discrimination not trivially impossible, we need to assume that there is a way to use the channel such that no "interaction" occurs. To this end, we assume that there exists a distinguished unit vector $v \in \mathcal{H}_I$, the *vacuum vector*, such that no interaction occurs if the channel is applied to the *vacuum state* $|v\rangle\langle v|$. We will also assume that $T(|v\rangle\langle v|)$ is a pure state, because if $T(|v\rangle\langle v|)$ were mixed then this would mean that the testing system I became entangled with the tested object, and hence an interaction must have occurred. If T satisfies the latter condition, we will call T a *channel with vacuum* v. To finally define the term "interaction-free", we imagine a Demon who has (for each of the individual channel uses) access to the output of the conjugate channel [49] of T and to an arbitrarily large quantum memory (see Figure 4.2). The Deamon's goal is to determine, by a final measurement on his quantum memory, if during the execution of the protocol, T was ever applied[2] to any state different from $|v\rangle\langle v|$. If the Demon concludes that T was applied to a state different from $|v\rangle\langle v|$,

[2]The states to which T is applied to are given by $\mathrm{tr}_Z\left[\rho_0^T\right], \mathrm{tr}_Z\left[\rho_1^T\right], \ldots, \mathrm{tr}_Z\left[\rho_{N-1}^T\right]$.

then we say that an "interaction" occurred. The probability that the Demon concludes that an "interaction" occurred depends on the Demon's strategy to store and processes the information he gets and on his final measurement. We define our second quantity of merit, the *"interaction" probability* $P_I^T(D)$, by maximizing this probability over all possible strategies, with the constraint that the Demon must not conclude that an "interaction" occurred, if the channel really was only applied to $|v\rangle\langle v|$. We can now properly define "interaction-free" channel discrimination: We say that two channels with vacuum T_A and T_B can be discriminated in an "interaction-free" manner, if for every $\epsilon, \delta > 0$, there exists a discrimination strategy D and a two-valued POVM Π such that $\max(P_I^{T_A}(D), P_I^{T_B}(D)) < \epsilon$ and $P_e(D, \Pi) < \delta$.
It might be that you found the description above a little dense. Let us thus remark that the model we have just described can be justified axiomatically by using, among other things, the equivalence between semicausal and semilocalizable channels. The details of this derivation are layed out in Section 3 of Article [1].

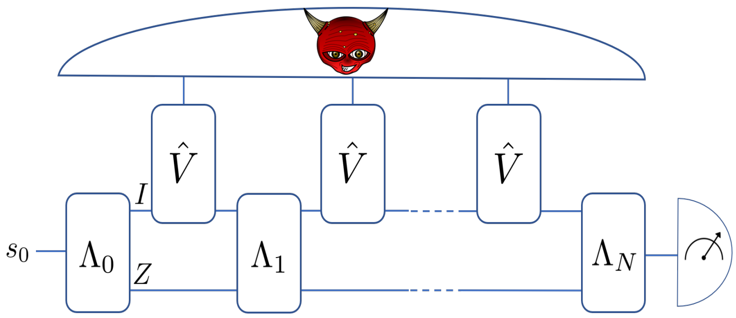

Figure 4.2: Model to define $P_I^T(D)$, where $\hat{V}$ is any isometric channel such that $T(\cdot) = \mathrm{tr}_E\left[\hat{V}(\cdot)\right]$.

Getting back to the model: Note that a vanishing "interaction" probability does not necessarily mean that T was only applied to $|v\rangle\langle v|$. For example, if T is a unitary channel, then the output of the conjugate channel (what the Demon sees) is independent of the state to which T is applied to. Hence, if T is unitary, then no "interaction" occurs, independently of the discrimination strategy. Mathematically, this insight can be expressed by a slight variation of the concept of a decoherence-free subspace. Let $\mathcal{V}$ be a subspace of the Hilbert space $\mathcal{H}$. We say that a channel T is *isometric on* $\mathcal{V}$ if there exists an isometry $V : \mathcal{V} \to \mathcal{H}$, such that $T|_{\mathcal{B}(\mathcal{V})}(\cdot) = V \cdot V^\dagger$.[3] These are the concepts needed to understand our results.

4.2 Results

Our main result is a characterization for when it is possible to discriminate two channels in a totally "interaction-free" manner. The details are as follows:

Theorem 4.2.1. *Two channels $T_A, T_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with vacuum $v \in \mathcal{H}$ can be discriminated in an "interaction-free" manner, if and only if there exists a subspace $\mathcal{V} \subseteq \mathcal{H}$ such that $v \in \mathcal{V}$, at least one of the two channels is isometric on $\mathcal{V}$ and $T_A|_{\mathcal{B}(\mathcal{V})} \neq T_B|_{\mathcal{B}(\mathcal{V})}$.*

[3] In a slight abuse of notation, $\mathcal{B}(\mathcal{V})$ denotes the operators in $\mathcal{B}(\mathcal{H})$ with range and support in $\mathcal{V}$.

It turns out that for a channel T with vacuum $v \in \mathcal{H}$, the union $\mathcal{V}_T$ of all subspaces $\mathcal{V} \subseteq \mathcal{H}$ such that $v \in \mathcal{V}$ and T is isometric on $\mathcal{V}$, is again a subspace with these two properties. Furthermore, $\mathcal{V}_T$ can be determined efficiently by linear algebraic methods. Since it is clearly enough to check the conditions of Theorem 4.2.1 on $\mathcal{V}_{T_A}$ and on $\mathcal{V}_{T_B}$, Theorem 4.2.1 provides an efficiently verifiable criterion for the possibility of an "interaction-free" channel discrimination. For later reference, we call $\mathcal{V}_T$ the *maximal vacuum subspace*.

Theorem 4.2.1 is the qualitative combination of two quantitative results. For the case where "interaction-free" channel discrimination is possible, it turns out that one does not need complete information about the two channels to perform the discrimination task. To account for this, we consider the more general task, where we want to know to which one of two known, disjoint sets of channels the unknown channel belongs. Specifically we consider the following: Given a channel T with vacuum $v \in \mathcal{V}$ that is isometric on $\mathcal{V}$, we take as our first set (a subset of) the set of channels that equal T, if we restrict their domains to $\mathcal{B}(\mathcal{V})$. The second set is less restricted in that we only assume that all channels must be channels with (the same) vacuum vector v and that the restrictions to $\mathcal{B}(\mathcal{V})$ must not equal $T|_{\mathcal{B}(\mathcal{V})}$. We obtain the following result:

Theorem 4.2.2 (Discrimination strategy). *Let $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{S}_1(\mathcal{H}))$ be two closed sets of channels and $\mathcal{V}$ be a subspace of $\mathcal{H}$ such that: for all $T \in \mathcal{C}_A \cup \mathcal{C}_B$, T is a channel with vacuum $v \in \mathcal{V}$; for all $T \in \mathcal{C}_A$, T is isometric on $\mathcal{V}$; the set $\mathcal{C}_A|_{\mathcal{S}_1(\mathcal{V})} := \{T|_{\mathcal{S}_1(\mathcal{V})} \,|\, T \in \mathcal{C}_A\}$ contains exactly one element; and $\mathcal{C}_A|_{\mathcal{S}_1(\mathcal{V})}$ and $\mathcal{C}_B|_{\mathcal{S}_1(\mathcal{V})} := \{T|_{\mathcal{S}_1(\mathcal{V})} \,|\, T \in \mathcal{C}_B\}$ are disjoint. Then there exist a constant $C > 0$, and for every $N \in \mathbb{N}$, an N-step discrimination strategy D and a two-valued POVM Π, such that*

$$P_I^{T_A}(D) = 0, \quad P_I^{T_B}(D) \leq \frac{C}{N} \quad \text{and} \quad P_e(D, \Pi) \leq \frac{C}{N^2},$$

for all $T_A \in \mathcal{C}_A$ and all $T_B \in \mathcal{C}_B$. The error probability $P_e(D, \Pi)$ is taken to be the maximum over all choices of channels from $\mathcal{C}_A$ and $\mathcal{C}_B$.

There are two major steps in proving Theorem 4.2.2: Firstly, one establishes, using techniques from perturbation theory, that a strategy similar to the one used in the original paper by Kwiat et al. works if $\mathcal{C}_A = \{\mathrm{id}\}$ and if the channels in $\mathcal{C}_B$ jointly satisfy a certain spectral gap condition. We establish this even for infinite-dimensional systems. Secondly, one reduces the general case to the restricted case above by showing that there is a superchannel R [14] such that $R(\mathcal{C}_A) = \{\mathrm{id}\}$ and such that the channels in $R(\mathcal{C}_B)$ satisfy the spectral gap condition. We propose an implementation for the superchannel R that requires only one ancillary qubit and might thus be implementable in near-term experiments. Furthermore, we show that a superchannel with the required properties cannot be implemented without an ancillary system.

The performance of the protocol underlying Theorem 4.2.2 is limited by the $\frac{1}{N}$ decay of the "interaction" probability. A natural question is, if there are better protocols. That this is not the case, except in very special circumstances, is shown by our next result:

Theorem 4.2.3 (Rate limit theorem). *Let $T_A, T_B : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ be two channels with vacuum $v \in \mathcal{H}$ and $\mathcal{V}_{T_A}$ and $\mathcal{V}_{T_B}$ be their maximal vacuum subspaces. Suppose that $T_A|_{\mathcal{S}_1(\mathcal{V})} = T_B|_{\mathcal{S}_1(\mathcal{V})}$*

and that $\mathcal{V}^\perp \cap \mathcal{V}_{T_A}$ *and* $\mathcal{V}^\perp \cap \mathcal{V}_{T_B}$ *are orthogonal, where* $\mathcal{V} := \mathcal{V}_{T_A} \cap \mathcal{V}_{T_B}$. *Then there exists a constant* $C > 0$ *such that*

$$\max(P_I^{T_A}(D), P_I^{T_B}(D)) \geq C\,\frac{(1-2P_e(D,\Pi))^4}{N},$$

for all N*-step discrimination strategies* D*, and any two-valued POVM,* Π.

If one could remove the condition that $\mathcal{V}^\perp \cap \mathcal{V}_A$ and $\mathcal{V}^\perp \cap \mathcal{V}_B$ must be orthogonal, then Theorem 4.2.3 would tell us that the best achievable decay rate of the "interaction-probability" is $\frac{1}{N}$, except in the trivial case, where the problem reduces to a discrimination between two isometries, in which case a finite number of channel uses suffices [50]. If it is possible to remove this condition, however, is an open problem.
Theorem 4.2.3 can be proven by an iterative application of a new kind of data processing inequality for the fidelity. A similar technique yields the converse for Theorem 4.2.1:

Theorem 4.2.4 (No-go). *Let* $T_A, T_B : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ *be two channels with vacuum* $v \in \mathcal{H}$ *such that no subspace satisfies the conditions in Theorem 4.2.1. Then there exists a constant* $C > 0$*, such that*

$$(1-2P_e(D,\Pi))^2 \leq C \max(P_I^{T_A}(D), P_I^{T_B}(D)),$$

for all discrimination strategies D *and all two-valued POVMs,* Π.

To summarize, our work generalizes the notion of "interaction-free" measurements and we found the exact conditions for when totally "interaction-free" discrimination is possible.

Chapter 5

Dynamical Semigroups

5.1 Motivation

If we approach quantum theory through the operational path that we described in Chapter 2, then there is no explicit reference to time. We have seen before that a transformation on density matrices that respects the probabilistic structure of quantum theory and that is compatible with tensor products must necessarily be a CPTP-map. Thus we should be able to describe the time-evolution by a family of CPTP-maps $\{T_t\}_{t\geq 0} \subset \mathsf{CP}(\mathcal{H})$ (one for each point in time). If a family of CPTP-maps describes an uninterrupted physical evolution, then the evolution should be continuous in time (in some sense). It would be very nice to specify the type of continuity that needs to be imposed such that the resulting evolution is physical. This, however, appears to be an open problem. We might remark at this point that an old result by Davies [51] tells us that if the map $t \mapsto T_t$ is strongly continuous, then T_t emerges out of a limiting process of strongly continuous unitary evolutions on a larger system (by tracing out the environment). It is unclear if one can get rid of the limiting process: see also [52] for a nice overview of related results.

In the general setting we have just discussed, the channel T_{t_1} describes the evolution from time $t = 0$ to time $t = t_1$. However, for times $t_2 > t_1$ there is generally no CPTP-map, Q_{t_2,t_1}, such that

$$T_{t_2} = Q_{t_2,t_1} \circ T_{t_1} \tag{5.1.1}$$

Thus, the system at time t is not solely described by its state $\rho_t = T_t(\rho_0)$, where $\rho_0 \in \mathcal{S}_1(\mathcal{H})$ is the initial state, but the history of the evolution also needs to be taken into account. Physically, this is due to “memory-effects". The general setting is therefore very challenging to analyze. Luckily there are many physical systems (for example an atom in an excited state or any closed system), whose time-evolution is described to a good approximation (see [53, Section 3.3] for when) by a family of CPTP-maps such that for all $t_2 > t_1 \geq 0$, Eq. (5.1.1) holds. Moreover, more often than not, Q_{t_2,t_1} only depends on the difference $\Delta t = t_2 - t_1$: That is, the evolution is time-homogenous and $Q_{t_2,t_1} = \hat{Q}_{t_2-t_1}$. In that case, if we assume that $T_0 = \mathrm{id}$, we can conclude that $T_{t+s} = \hat{Q}_{(t+s)-s} \circ T_s = T_t \circ T_s$ (set $t_2 = t + s$ and $t_1 = s$) for all $s, t \geq 0$. This is what is called the semigroup property. The study of families of maps with the semigroup property yields a very fruitful theory. The following section is devoted to the general theory of dynamical semigroups.

5.2 General Theory

In this section we summarize the most important definitions and results in the theory of dynamical semigroups, a.k.a. one-parameter semigroups. This section follows [54] and we start with the definition of a dynamical semigroup.

Definition 5.2.1. (One-parameter semigroup) Let $\mathcal{X}$ be a Banach space. A family $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{X})$ is called dynamical semigroup, or more precisely, one-parameter semigroup if

$$\begin{aligned} T_0 &= \mathrm{id} \\ T_{t+s} &= T_t \circ T_s \text{ for all } s,t \geq 0. \end{aligned}$$

A one-parameter semigroup is called strongly continuous if for every $x \in \mathcal{X}$, the map

$$\mathbb{R}_{\geq 0} \ni t \mapsto T_t x \in \mathcal{X}$$

is continuous w.r.t. the norm topology on $\mathcal{X}$.
A one-parameter semigroup is called uniformly-continuous or norm-continuous, if the map

$$\mathbb{R}_{\geq 0} \ni t \mapsto T_t \in \mathcal{B}(\mathcal{X})$$

is continuous w.r.t. the norm-topology on $\mathcal{B}(\mathcal{X})$.

Dynamical semigroups appear in variety of contexts (see [54, Chapter 6]) due to their intimate relation to first order linear differential equations – and those appear naturally. This relation appears morally (but not technically), because every strongly continuous semigroup is if the form $T_t = e^{tL}$ for some operator L, called generator. We now go into the details of how a technically correct version of this meta-theorem looks like. The main issue is that L is generally unbounded. We define the generator of a strongly continuous one-parameter semigroup as follows:

Definition 5.2.2. (Generator of a strongly continuous one-parameter semigroup) For a strongly continuous one-parameter semigroup $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{X})$, define the "domain-set" $D(\mathcal{L})$ as

$$D(L) := \{x \in \mathcal{X} \mid \lim_{h\downarrow 0} \frac{1}{h}\left(T_h(x) - x\right) \text{ exists in } \mathcal{X}\}.$$

The operator $L : D(L) \to \mathcal{X}$, defined by

$$Lx = \lim_{h\downarrow 0} \frac{1}{h}\left(T_h(x) - x\right)$$

for all $x \in D(L)$ is called the generator of $\{T_t\}_{t\geq 0}$.

The following central theorem, whose ideas go back to Hille and Yoshida [55, 56], gives a precise relation between strongly continuous one-parameter semigroups and their generators (for a proof see e.g. [54, Theorem II.3.8, Corollary III.5.5])

Theorem 5.2.3. ***(Hille-Yosida)*** *For a Banach space $\mathcal{X}$ and a linear subspace $D(L) \subseteq \mathcal{X}$, let $L : D(L) \subseteq \mathcal{X}$ be a linear operator and let $\omega \in \mathbb{R}$, $M \geq 1$ be constants. The following are equivalent.*

1. *L is the generator of a strongly continuous one-parameter semigroup $\{T_t\}_{t\geq 0}$ satisfying $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$.*
2. *L is closed, densely defined, and every $\lambda > \omega$ belongs to the resolvent set[1] of L and for all $n \in \mathbb{N}$, we have*

$$\left\|(\lambda \mathbb{1} - L)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}.$$

In that case, the semigroup can be recovered uniquely from the generator by the exponential formula

$$T_t(x) = \lim_{n\to\infty} \left(\mathbb{1} - \frac{tA}{n}\right)^{-n} x,$$

which converges for all $x \in \mathcal{X}$ and even uniformly for t in compact intervals.

Remark 5.2.4. To understand the meaning of the previous theorem, it should be noted that for every strongly continuous one-parameter, there are $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$ (see e.g. [54, Proposition 1.5.5]). So every strongly continuous one-parameter semigroup satisfied the hypothesis of the Hille-Yosida theorem. However, not every closed, densely defined operator is necessarily a generator of some semigroup.

Unbounded generators, as in Part 2 of Theorem 5.2.3 appear frequently in applications, e.g. in the form of partial differential operators. However, the unboundedness is a technical obstacle that is very hard to deal with. It should therefore not come as a surprise that only for very special cases (such as for strongly continuous one-parameter unitary groups), a constructive characterization of the generators is known. These cases, unfortunately, do not include the case that is most interesting to us – the case of quantum dynamical semigroups.
The situation is much better, from an analytical point of view, if one restricts the attention to norm-continuous one-parameter semigroups. As it turns out, those are exactly those one-parameter semigroups for which the generator is bounded.

Lemma 5.2.5. *[54, Corollary II.1.5] The generator of a strongly continuous one-parameter semigroup $\{T_t\}_{t\geq 0}$ is a bounded linear operator if and only if $\{T_t\}_{t\geq 0}$ is norm-continuous.*

As mentioned before, for norm-continuous dynamical semigroups, full characterizations for several important cases are known. This will be the content of the next section.

[1]The resolvent set of L consists of those $\lambda \in \mathbb{C}$ for which $(\lambda\mathbb{1} - L)$ is invertible (with bounded inverse, but this is already implied if L is closed.

5.3 Special Cases

In the previous section, we have seen that there is a one-to-one correspondence between dynamical semigroups and their generators. It is often the case that certain properties of the operators comprising a dynamical semigroup are known. For example, the dynamical semigroup might consist of stochastic matrices or completely positive maps. In that case it is a central task to figure out how a certain property of the semigroup translates to a corresponding property of its generator. That this is a non-trivial endeavor is hinted by the names attached to the respective equations: “Lindblad equation", “Schrödinger equation", “Fokker-Plank equation".

5.3.1 Quantum Dynamical Semigroups

We have already motivated the use of quantum dynamical semigroups at the beginning of this chapter. So, let us proceed here with a formal definition

Definition 5.3.1. A strongly/uniformly continuous one parameter semigroup $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{S}_1(\mathcal{H}))$ is called strongly/uniformly continuous Schrödinger quantum dynamical semigroup if T_t is a CP-map for all $t \geq 0$.
A strongly/uniformly continuous one parameter semigroup $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{B}(\mathcal{H}))$ is called strongly/uniformly continuous Heisenberg quantum dynamical semigroup if T_t is a normal CP-map for all $t \geq 0$.

Remark 5.3.2. Clearly, Heisenberg and Schrödinger quantum dynamical semigroups are their respective duals and preduals.

The central theorem in study of quantum dynamical semigroups is a characterization of the generators of uniformly continuous quantum dynamical semigroups. The result was found independently by Gorini, Kossakowski, Sudarshan [57] and Linbdlad [58] in 1976. The corresponding normal form bears their name: the Gorini-Kossakowski-Sudarshan-Linbdlad-form or GKSL-form.

Theorem 5.3.3. ***(GLSK-form)*** *A linear map $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ generates a uniformly continuous Heisenberg quantum dynamical semigroup if and only if there is a normal CP-map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and an operator $K \in \mathcal{B}(\mathcal{H})$ such that*

$$L(X) = \Phi(X) - K^\dagger X - XK,$$

for all $X \in \mathcal{B}(\mathcal{H})$.

Remark 5.3.4. The GKSL-form can be written in several different (and more common) equivalent ways: First, by Stinespring's dilation theorem (Theorem 2.2.3), we have

$$L(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$$

for all $X \in \mathcal{B}(\mathcal{X})$, where $V \in \mathcal{B}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$.
Second, if we write Φ in Kraus-form, then

$$L(X) = \sum_n L_n^\dagger X L_n - K^\dagger X - XK$$

for all $X \in \mathcal{B}(\mathcal{X})$ and some collection of operators $\{L_n\}_n \subseteq \mathcal{B}(\mathcal{H})$ such that the series SOT-converges. Moreover, by differentiation, we can see that T_t is a quantum channel (i.e., $T(\mathbb{1}) = \mathbb{1}$) for all $t \geq 0$ if and only if $L(\mathbb{1}) = 0$. This implies

$$\Phi(\mathbb{1}) = V^\dagger V = \sum_n L_n^\dagger L_n = K^\dagger + K.$$

So, we have a restriction for the real part of K. Let us thus define the self-adjoint operator $H \in \mathcal{B}(\mathcal{H})$ as $H = \frac{1}{2i}\left(K - K^\dagger\right)$. After writing $K = \frac{1}{2}\left(K + K^\dagger\right) + iH$, we obtain

$$\begin{aligned} L(X) &= i[H, X] + \Phi(X) - \frac{1}{2}\{\Phi(\mathbb{1}), X\} \\ &= i[H, X] + V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V - \frac{1}{2}\left\{V^\dagger V, X\right\} \\ &= i[H, X] + \sum_n L_n^\dagger X L_n - \frac{1}{2}\left\{L_n^\dagger L_n, X\right\}, \end{aligned}$$

where the last form is the most widely used one.
All these results can easily be translated to the Schrödinger picture, because if $\{T_t\}_{t\geq 0}$ is a uniformly continuous Heisenberg quantum dynamical semigroup with generator L (which is a normal map), then $\{(T_t)_*\}_{t\geq 0}$ is a uniformly continuous Schrödinger quantum dynamical semigroup with generator $L_* : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$. For example, for Schrödinger channels, the GKLS-form reads

$$L_*(\rho) = -i[H, \rho] + \sum_n L_n \rho L_n^\dagger - \frac{1}{2}\left\{L_n^\dagger L_n, \rho\right\}.$$

5.3.2 Quantum Dynamical Semigroups of Semicausal Maps

Suppose we have a uniformly continuous quantum dynamical semigroup $\{T_t\}_{t\geq 0}$. Further, suppose that for all $t \geq 0$ the map T_t is semicausal. Thus, for every $t \geq 0$ we can understand T_t in terms of semilocalizability. Can we understand the generators of these semigroups in a similarly explicit way? Is there a normal form for generators of uniformly continuous quantum dynamical semigroups of semicausal maps? The answer to this question was the main technical contribution in Article [2]. The answer reads as follows:

Theorem 5.3.5. ***(Normal form generators of semicausal CP-maps, [2, Theorem V.6])*** *Let $L : \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ be defined by $L(X) = \Phi(X) - K^\dagger X - XK$, with a normal CP-map $\Phi \in \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ and $K \in \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$. Then L is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal if and only if there exists a (separable) Hilbert space $\mathcal{H}_{\mathtt{E}}$, a unitary $U \in \mathcal{B}(\mathcal{H}_{\mathtt{E}} \otimes \mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$, a self-adjoint operator $H_{\mathtt{B}} \in \mathcal{B}(\mathcal{H}_{\mathtt{B}})$, and arbitrary operators $A \in \mathcal{B}(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{E}})$, $B \in \mathcal{B}(\mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$ and $K_{\mathtt{A}} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}})$, such that*

$$\begin{aligned} \Phi(X) &= V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V, \ \text{with } V = (\mathbb{1}_{\mathtt{A}} \otimes U)(A \otimes \mathbb{1}_{\mathtt{B}}) + (\mathbb{1}_{\mathtt{A}} \otimes B), \\ K &= (\mathbb{1}_{\mathtt{A}} \otimes B^\dagger U)(A \otimes \mathbb{1}_{\mathtt{B}}) + \frac{1}{2}\mathbb{1}_{\mathtt{A}} \otimes B^\dagger B + K_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}} + \mathbb{1}_{\mathtt{A}} \otimes iH_{\mathtt{B}}. \end{aligned}$$

If $\mathcal{H}_{\mathtt{A}}$ and $\mathcal{H}_{\mathtt{B}}$ are finite-dimensional, with dimensions $d_{\mathtt{A}}$ and $d_{\mathtt{B}}$, then $\mathcal{H}_{\mathtt{E}}$ can be chosen such that $\dim(\mathcal{H}_{\mathtt{E}}) \leq (d_{\mathtt{A}} d_{\mathtt{B}})^2$.

A very interesting property of the normal form in Theorem 5.3.5 is that the CP-part Φ does not need to be semicausal. One way to think about the result is in terms of building blocks: It is immediate that if the CP-part and the K-part of the GKSL-form are both semicausal, then the generator will be semicausal. Since semicausal CP-maps are semilocalizable (Theorem 3.1.5), for this building block the operator V_{sc} has the form $(\mathbb{1}_{\mathtt{A}} \otimes U)(A \otimes \mathbb{1}_{\mathtt{B}})$. Moreover, it turns out that K is then necessalily of the form $K = K_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}} + \mathbb{1}_{\mathtt{A}} \otimes iH_{\mathtt{B}}$. So, maps of that form are valid semicausal GKSL-generators and are our first building block.
For our second building block, notice that if Φ is of the form $\mathrm{id}_{\mathtt{A}} \otimes \Phi_{\mathtt{B}}$ so that $(\mathrm{id}_{\mathtt{A}} \otimes \Phi_{\mathtt{B}})(X_{\mathtt{AB}}) = (\mathbb{1}_{\mathtt{A}} \otimes B)^{\dagger}(X_{\mathtt{AB}} \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}} \otimes B)$, then the non-semicausality of the CP-part can be compensated by choosing $K = \mathbb{1}_{\mathtt{A}} \otimes B^{\dagger}B$. Thus a GKSL-generator of this form is another building block. Of course, a (convex) combination of these two building blocks yields also a GKSL-generator. The crucial insight that lead to Theorem 5.3.5 is that convex combinations is not all there is, namely, we can also let the Stinespring operator V be a linear combination of the Stinespring operators of the building blocks. One might say that we put them into superposition. It is very interesting to note that if one studies dynamical semigroups of semicausal stochastic maps (i.e. the classical setting - see section IV in [2]) then all there is is the 'obvious' convex combination. This renders this 'superposition' a purely quantum feature.
It is straight-forward to check that the normal form in Theorem 5.3.5 yields a semicausal generator. That the proposed way of construction generators from the two building blocks is the only way one can construct semicausal generators is the main insight of Theorem 5.3.5. We have already seen that a mathematical generalization of semicausal maps are maps that leave a weakly closed *-algebra invariant. The next section deals with GKSL-generators with that property.

5.3.3 Quantum Dynamical Semigroups with Invariant Algebra

We have already seen in Section 3.2 that Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausality is equivalent to the invariance of the type-I factor von Neumann algebra $\mathcal{B}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$. Moreover, in Section 3.2.2 we provided a normal form for CP-maps with an invariant atomic weakly closed *-algebra. In this section, we treat the following problem: given a uniformly continuous quantum dynamical semigroup $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{B}(\mathcal{H})$ such that for all $t \geq 0$, $T_t(\mathcal{A}) \subseteq \mathcal{A}$ for a given weakly closed *-algebra, how do the corresponding generators look like?. The answer to this question is the main result of Article [3]. By differentiation, we obtain that $T_t(\mathcal{A}) \subseteq \mathcal{A}$ holds for all $t \geq 0$ if and only if the corresponding GKSL-generator $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies $L(\mathcal{A}) \subseteq \mathcal{A}$.
In Article [3], the main result is approached in two steps: first one shows that the problem of finding a normal forms for invariant GKSL-generators is equivalent to finding normal forms for invariant CP-maps. Then, as a second step, the result of Section 3.2.2 is readily applied. The first result reads:

Theorem 5.3.6. ***(Reduction Theorem, [3, Theorem 4])***
*Let $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined by $L(X) = V^{\dagger}(X \otimes \mathbb{1}_{\mathtt{E}})V - K^{\dagger}X - XK$, for some $V \in \mathcal{B}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $K \in \mathcal{B}(\mathcal{H})$, and let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an approximately finite-dimensional[2] weakly closed *-algebra. The following are equivalent*

[2] A weakly closed *-algebra $\mathcal{A}$ is approximately finite-dimensional, if there exist an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}$ of finite-dimensional *-subalgebras such that $\cup_{n\in\mathbb{N}}\mathcal{A}_n$ is WOT-dense in $\mathcal{A}$.

1. $L(\mathcal{A}) \subseteq \mathcal{A}$.

2. *There exist operators* $V_0 \in \mathcal{B}(\mathcal{H};\mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$, $A, B \in \mathcal{B}(\mathcal{H};\mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ *and* $K_0 \in \mathcal{B}(\mathcal{H};\mathcal{H}_0)$*; an operator* $K_{\mathcal{A}} \in \mathcal{A}$*; and a self-adjoint operator* $H_{\mathcal{A}'} \in \mathcal{A}'$ *such that*

   (a) $A^\dagger(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})A \in \mathcal{A}$ *and* $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})B = BX_{\mathcal{A}}$*, for all* $X_{\mathcal{A}} \in \mathcal{A}$.

   (b) V *and* K *have the following form:*

$$\begin{aligned} V &= (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_0 + A + B, \\ K &= B^\dagger A + \frac{1}{2}B^\dagger B + K_{\mathcal{A}} + iH_{\mathcal{A}'} + P_0^\dagger K_0. \end{aligned}$$

Since for any vector $|e\rangle \in \mathcal{H}_{\mathtt{E}}$, Part 2a implies $(\mathbb{1} \otimes \langle e|)B \in \mathcal{A}'$, it is easy to decompose B w.r.t. a direct integral decomposition of $\mathcal{A}$. Thus the only thing that remains is to characterize the normal CP-map $\Phi_A : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\Phi_A(X) = A^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})A$. If $\mathcal{A}$ is atomic, such a characterization is given in Section 3.2.2. The final result is:

Theorem 5.3.7. ***(GKSL-generator with invariant algebra, [3, Theorem 6])***
Let $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *be given by* $L(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$ *with* $V \in \mathcal{B}(\mathcal{H};\mathcal{H}\otimes\mathcal{H}_{\mathtt{E}})$ *and* $K \in \mathcal{B}(\mathcal{H})$ *and let* $\mathcal{A}$ *be an atomic *-subalgebra of* $\mathcal{B}(\mathcal{H})$*, with decomposition given by Definition 3.2.9. Then the following are equivalent*

1. $L(\mathcal{A}) \subseteq \mathcal{A}$.

2. *There exist operators* $V_0 \in \mathcal{B}(\mathcal{H};\mathcal{H}_0\otimes\mathcal{H}_{\mathtt{E}})$ *and* $K_0 \in \mathcal{B}(\mathcal{H};\mathcal{H}_0)$*; for all* $i, j \in I$ *a Hilbert space* $\mathcal{H}_{\mathtt{F}_{ij}}$*, operators* $A_{ij} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}_j};\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$*, and isometries* $U_{ij} \in \mathcal{B}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{B}_j};\mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$*; and for every* $i \in I$ *operators* $B_i \in \mathcal{B}(\mathcal{H}_{\mathtt{B}_i};\mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$*,* $K_{\mathtt{A}_i} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}_i})$*, and self-adjoint operators* $H_{\mathtt{B}_i} \in \mathcal{B}(\mathcal{H}_{\mathtt{B}_i})$*, such that*

   - V *and* K *can be decomposed as*

$$\begin{aligned} V &= (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_0 + \sum_{i,j\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_{ij}^{sc}P_j + \sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes B_i)P_i, \\ K &= \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger)V_{ii}^{sc}P_i + \frac{1}{2}\sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger B_i)P_i \\ &\quad + K_{\mathcal{A}} + iH_{\mathcal{A}'} + P_0^\dagger K_0, \end{aligned}$$

     with $V_{ij}^{sc} = (\mathbb{1}_{\mathtt{A}_i}\otimes U_{ij})(A_{ij}\otimes\mathbb{1}_{\mathtt{B}_j})$*,* $K_{\mathcal{A}} = \sum_{i\in I} P_i^\dagger(K_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{B}_i})P_i$*, and* $H_{\mathcal{A}'} = \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i}\otimes H_{\mathtt{B}_i})P_i$*, s.t. all series SOT-converge.*

   - *The relation* $U_{ik}^\dagger U_{il} = \delta_{kl}\mathbb{1}$ *holds for all* $i, k, l \in I$.

This theorem proves useful in deriving and unifying several results in the literature. Among them are the Koashi-Imoto theorem [59]; the refined form of the GKSL-generator due to the invariance of the decoherence-free subalgebra [60–63] and the maximally abelian subalgebra [64–67]; as well as the study of Markovian subsystems [68].

5.3.4 Semigroups of Superchannels

In Section 3.3 we introduced superchannels as the description for "do something" to a channel. Clearly, doing something to a channel and then doing something to the resulting channel also does something to a channel. Thus, superchannels form a semigroup and hence we can study continuous one-parameter semigroups of superchannels and their generators. Besides this being an interesting mathematical endeavor, we propose to use dynamical semigroups of superchannels as a description of quantum devices that are subject to due natural-decay due to "ageing".
Let us begin with the formal mathematical result. By Theorem 3.3.4, superchannels are related to semicausal CP-maps via a simmilarity transformation with transformation matrix given by the Choi–Jamiołkowski isomorphism. Moreover we have already characterized the generators of semigroups of semicausal CP-maps in Theorem 5.3.5. Thus what remains to do is to explicitly calculate the simmilarity transformation. Here, Lemma 2.2.9 comes in handy. The details of the calculations involved can be found in [2]. The (slightly simplified) result reads [2, Theorem V.18]:

Theorem 5.3.8. *A linear map $\hat{L} : \mathcal{B}(\mathcal{B}(\mathcal{H}_{\mathtt{A}}); \mathcal{B}(\mathcal{H}_{\mathtt{B}})) \to \mathcal{B}(\mathcal{B}(\mathcal{H}_{\mathtt{A}}); \mathcal{B}(\mathcal{H}_{\mathtt{B}}))$ generates a semigroup of superchannels if and only if it admits a decomposition $\hat{L}(T) = \hat{D}(T) + \hat{H}(T)$ as follows: The "Hamiltonian part" is of the form*

$$\hat{H}(T)(\rho) = -i[H_{\mathtt{B}}, T(\rho)] - iT([H_{\mathtt{A}}, \rho]),$$

with local Hamiltonians $H_{\mathtt{B}} \in \mathcal{B}(\mathcal{H}_{\mathtt{B}})$ and $H_{\mathtt{A}} \in \mathcal{B}(\mathcal{H}_{\mathtt{A}})$.
The "dissipative part" is of the form $\hat{D}(T)(\rho) = \mathrm{tr}_E \left[\hat{D}'(T)(\rho)\right]$, where

$$\begin{aligned}
\hat{D}'(T)(\rho) = U(T \otimes \mathrm{id}_{\mathtt{E}})(A(\rho \otimes \sigma)A^\dagger)U^\dagger \qquad & - \frac{1}{2}(T \otimes \mathrm{id}_{\mathtt{E}})(\left\{A^\dagger A\, ,\, \rho \otimes \sigma\right\}) && (5.3.3a)\\
+ B(T \otimes \mathrm{id}_{\mathtt{E}})(\rho \otimes \sigma)B^\dagger \qquad & - \frac{1}{2}\left\{B^\dagger B\, ,\, (T \otimes \mathrm{id}_{\mathtt{E}})(\rho \otimes \sigma)\right\} && (5.3.3b)\\
+ \left[U(T \otimes \mathrm{id}_{\mathtt{E}})(A(\rho \otimes \sigma))\, ,\, B^\dagger\right] \qquad & + \left[B\, ,\, (T \otimes \mathrm{id}_{\mathtt{E}})((\rho \otimes \sigma)A^\dagger)U^\dagger\right], && (5.3.3c)
\end{aligned}$$

with unitary $U \in \mathcal{B}(\mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$, density matrix $\sigma \in \mathcal{B}(\mathcal{H}_{\mathtt{E}})$ and arbitrary $A \in \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{E}})$ and $B \in \mathcal{B}(\mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$.

The "Hamiltonian part" generates a semigroup of invertible superchannels with the property that the inverse is a superchannel as well. We discuss the three terms of the "dissipative part": Term (5.3.3a) generates a semigroup of superchannels such that the transformed channel, $\hat{S}_t(T)$, arises from a stochastic application of $T \mapsto \mathrm{tr}_E \left[U(T \otimes \mathrm{id}_E)(A(\rho \otimes \sigma)A^\dagger)U^\dagger\right]$ at different points in time (Dyson series expansion). Term (5.3.3b) generates a semigroup of "post-processing" superchannels of the form $\hat{S}_t(T) = e^{L_B t} \circ T$, where L_B generates a (trace-preserving) quantum dynamical semigroup. Term (5.3.3c) stems from the fact that we superpose the building blocks for the decomposition of the generators of semigroups of semicausal CP-maps, rather than simply taking a convex combination. Although this term is hard to interpret directly, the interseting point here is that such a term does not appear if one looks at semigroups of classical superchannels (see [2, Section IV]). Its presence is thus a pure quantum feature and an important part of our findings.

Now back to the physical significance of this result: The main hypothesis here is that dynamical semigroups of superchannels are a reasonable (first step) towards characterizing the decay-processes that quantum devices are subject to due to “ageing".

Quantum channels have quantum states as inputs and outputs. They can thus be seen as the mathematical description of a quantum device with a single input and a single output. Our interpretation of superchannels was that they are the most general mathematical description of a “do something" to a quantum channel action. Since decay processes certainly “do something" to a quantum channel, there must be a superchannel $\hat{S}_t$ for each $t \geq 0$ such that if the initial device was described by a channel T, then the device will be described by $\hat{S}_t(T)$ at time t. It is of course not necessary that the set $\{\hat{S}_t\}_{t\geq 0}$ forms a semigroup. However, we believe that (similar to quantum dynamical semigroups), this is a valid approximation in many cases. We refer the reader who is interested in a concrete example to the introduction of Article [2].

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Appendix A

Core Articles

A.1 “Interaction-Free” Channel Discrimination

In this work we propose and analyze a generalized model for interaction-free measuremnets in the spirit of the Elitzur-Vaidman bomb-tester experiment [4].
In the introduction we thoroughly review the bomb-tester experiment in its original form due to Elitzur and Vaidman [4] and in its iterated form due to Kwiat et al. [28]. Moreover, we hint at a possible reformulation as a quantum channel discrimination problem.
In Section 2 we outline the path to our main result: A characterization for when totally “interaction-free" measurements are possible.
In Section 3, we axiomatically derive two models that capture what we believe that the term “interaction-free" measurement should mean if we generalize the original bomb-tester experiment as a quantum channel discrimination problem. We conclude this section by giving a formal mathematical definition of “interaction-free" measurements in both cases and by comparing the two models.
Our characterization of when totally “interaction-free" measurements are possible consists of two parts: A protocol, which allows us (under the hypothesis of the main theorem) to discriminate quantum channels in an “interaction-free" manner. And a converse which tells us that totally “interaction-free" measurements are impossible in the remaining cases. The protocol and its analysis is the main content of Section 4, while the no-go theorem is proven in Section 5.
Throughout, we aim to make our protocol as resource-efficient as possible. Our protocol uses (in the worst case) one ancillary qubit (as does the original bomb-tester experiment) and we show that one cannot get rid of such an ancillary qubit for a large class of protocols. Moreover, the “interaction" probability of our protocol decays as $\frac{1}{N}$ where N is the number of channel invocations, and we show that this asymptotic behavior is the best achievable one.
In Section 6, we conduct a thorough analysis of how our model and results relate to other models and results in the literature, with a particular emphasis on the Conterfactual Computation scheme by Mitchison and Josza [69].

This article is a continuation and thorough extension of the authors master thesis [70]. Although the objective to generalize “interaction-free" measurements is the same for the present article and the author’s Master-thesis, most ideas and methods involved are different. In particular,

this article axiomatically derives and uses the "interaction" model instead of the more ad-hoc model used in [70]. This allowed us to fully characterize totally 'interaction-free' measurements as opposed to some results for several special cases, as in [70].
The characteristics of the protocol we propose in Section 4 are vastly better then the one in [70]. In particular, our new protocol **a)** has a wider scope - enabling a full characterization **b)** uses at most a single ancillary qubit system - making it potentially feasible for experimental implementation (this is achieved through a symmetrization step). **c)** has provably optimal decay-rate for the 'interaction' probability **d)** can be implemented with almost every Hamiltonian - not just a precisely tuned one **e)** works for infinite-dimensional systems (which requires a much more sophisticated analysis) Moreover, we also improved the no-go results in order to match the constructive case and proved that the $\frac{1}{N}$ decay-rate of the 'interaction' probability is asymptotically optimal.
I was involved in all parts of the paper and wrote the entire manuscript.

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Annales Henri Poincaré

"Interaction-Free" Channel Discrimination

Markus Hasenöhrl and Michael M. Wolf

Abstract. In this work, we investigate the question of which objects can be discriminated by totally "interaction-free" measurements. To this end, we interpret the Elitzur–Vaidman bomb-tester experiment as a quantum channel discrimination problem and generalize the notion of "interaction-free" measurement to arbitrary quantum channels. Our main result is a necessary and sufficient criterion for when it is possible or impossible to discriminate quantum channels in an "interaction-free" manner (i.e., such that the discrimination error probability and the "interaction" probability can be made arbitrarily small). For the case where our condition holds, we devise an explicit protocol with the property that both probabilities approach zero with an increasing number of channel uses, N. More specifically, the "interaction" probability in our protocol decays as $\frac{1}{N}$ and we show that this rate is the optimal achievable one. Furthermore, our protocol only needs at most one ancillary qubit and might thus be implementable in near-term experiments. For the case where our condition does not hold, we prove an inequality that quantifies the trade-off between the error probability and the "interaction" probability.

Contents

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1. Introduction

In 1993, Elitzur and Vaidman proposed their famous bomb-tester experiment [1] to demonstrate that the arguably most intriguing property of quantum theory—superposition—can be exploited to detect an ultra-sensitive bomb in a black-box, in such a way that there is a non-vanishing probability that the bomb will not explode. Only two years later, Kwiat et al. [2] showed how to employ another fundamental phenomenon—the quantum Zeno effect [3]—to boost the probability that the bomb will not explode as close to 1 as one pleases. These powerful ideas found applications in "interaction-free" imaging [4,5], counterfactual quantum computation [6,7], counterfactual communication [8] and cryptography [9], and even complexity theory [10]. Despite the great success, it became apparent that the aforementioned techniques, which we will generically call "interaction-free" measurements, are subject to some fundamental limitations. Notably, it is impossible to learn the outcome of a decision problem solved by a quantum computer [7] without "running" the computer in at least one of the two cases, and two optically semi-transparent objects cannot be discriminated in such a way that no photon gets absorbed [11,12].

Despite the results mentioned above, there seems to be no framework and analysis sufficiently general to pinpoint which objects can or cannot be discriminated perfectly by "interaction-free" measurements. Encouraged by recent results that generalize the quantum Zeno effect [13–16], we aim to remedy these shortcomings. To this end, we interpret the Elitzur–Vaidman bomb-tester experiment as a quantum channel discrimination problem and generalize the notion of "interaction-free" measurement to quantum channels via two slightly different, but in the end largely equivalent models. The theory of quantum supermaps [17] then provides the right framework to consider all possible (causally ordered) discrimination strategies, allowing us to decide when it is possible or impossible to discriminate two channels in an "interaction-free" manner.

Organization of the Paper This article is structured as follows: In the remainder of this section, we review the bomb-tester experiment in its versions by Elitzur and Vaidman and by Kwait et al. We also try to convey the idea of how the general model should look. Armed with this rough understanding, we will be able to state and discuss the major results of this work in Sect. 2. In

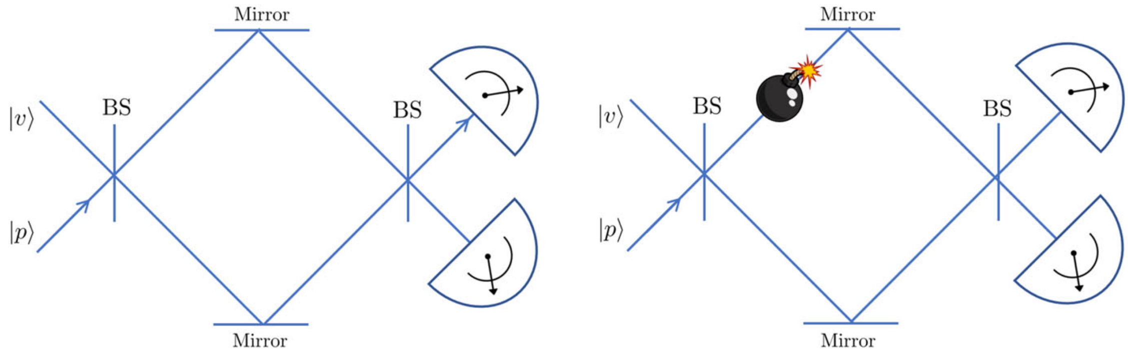

FIGURE 1. Elitzur–Vaidman bomb-tester experiment

Sect. 3, we give a detailed derivation of our model. Our main result, a characterization of what is possible and impossible to do with "interaction-free" measurements, is the combination of two pillars: a no-go theorem, in the form of an inequality, that tells us when it is impossible to discriminate two channels in an "interaction-free" manner; and a protocol that discriminates two channels in those cases that are not touched by the no-go theorem. A quantitative treatment of this protocol is given in Sect. 4, while the main content of Sect. 5 is the no-go theorem. Also in Sect. 5, we prove fundamental limits for the achievable decay rate of the "interaction" probability.

The Bomb-Tester Experiment In the following, we briefly review the bomb-tester experiment in its original version by Elitzur and Vaidman and its iterative version by Kwiat et al. Suppose you have a box and you have been told that inside of this box there is an ultra-sensitive bomb. By ultra-sensitive, we mean that the bomb will explode even if only one photon hits it. As you do not trust the deliverer, you want to check if there is a bomb inside the box. For some reason, the only way to obtain information about the content of the box is by shining light through it. Doing so, however, might trigger the bomb, which is what we want to avoid. If photons were classical particles our task seems to be an impossible one.[1] To circumvent this problem, Elizur and Vaidman proposed to put the box into the upper arm of a Mach–Zehnder interferometer, as depicted in Fig. 1. If we work only with a single photon, then this proposal can be stated abstractly as follows: The Hilbert space of the problem is $\mathcal{H} = \mathcal{H}_U \otimes \mathcal{H}_L$, where $\mathcal{H}_U = \mathcal{H}_L = \operatorname{span}\{v, p\}$ are the Hilbert spaces associated with the upper and lower arm, and the orthogonal unit vectors v and p denote the vacuum and one-photon states, respectively. The 50/50 beamsplitter (BS) can be modeled as a unitary transformation U, defined by

$$\begin{aligned} Uv \otimes v &= v \otimes v, \\ Up \otimes v &= \cos(\theta)\, v \otimes p + \sin(\theta)\, p \otimes v, \\ Uv \otimes p &= -\sin(\theta)\, v \otimes p + \cos(\theta)\, p \otimes v, \end{aligned} \tag{1.1}$$

[1]Note that one needs to be careful about the notion of classicality, since the bomb-tester experiment allows for a formulation in terms of Spekkens toy models [18,19].

where $\theta = 45°$. Suppose we start with a photon in the lower input, then the initial state is $s_0 := |v \otimes p\rangle\langle v \otimes p|$. There are two cases to analyze. On the one hand, if there is no bomb in the box, then the two beamsplitters rotate the state by 90°. Hence, the photon ends up in the upper output. On the other hand, if there is a bomb in the box, then the bomb acts as a measurement device in the upper path. There are three possible outcomes of the experiment. The first possibility is that the photon takes the upper path and thus causes the bomb to explode. This happens with a probability of 50%. If the bomb does not explode, then, by the measurement postulate, the state of the system is still s_0. Since the second beamsplitter has a 50/50 splitting ratio, the probability that we measure the photon in the upper output equals the probability that we measure the photon in the lower output, i.e., the probability for each of them is 25%. The important point here is that in 25% of the cases the photon ends up in the lower path. In that case, we can conclude that there is a bomb in the box, but the bomb has not been triggered. However, we only get this result in 25% of the cases.

Kwiat et al.'s Iterative Version To increase the efficiency of this protocol, the crucial idea is to feed the output back to the input, (thus, to let the photon go through the box many times) and to adjust the splitting ratio of the beamsplitters sensibly (see [2] for the experimental realization). The easiest way to analyze this proposal is to think of the feedback loop in a "rolled out" way. That is, we look at this proposal as if we had N copies of the Mach–Zehnder interferometer (where N is the number of times we let the photon go through the box), in each of which the box is in the upper arm (see Fig. 2).

We further choose the angle $\theta := \frac{90°}{N}$ in (1.1), which defines the action of the beamsplitters. Let us analyze this protocol: If there is no bomb in the box and the photon starts in the lower path, then the photon travels through N beamsplitters, each of which rotates the state by an angle of $\frac{90°}{N}$. So overall the state is rotated by 90°, which means that the photon will be in the upper output. For the case where there is a bomb in the box, let us calculate the probability that the photon always takes the lower path and therefore does not hit the bomb. For each of the beamsplitters, if the photon is in the lower path before the beamsplitter, then the probability that the photon will be in the lower path after the beamsplitter is given by $\cos^2(\theta)$. Since the bomb can be viewed as a measurement device, the probability that the photon always takes the lower path is simply the product of the probabilities at each beamsplitter. Hence, $P(\text{always lower path}) = \cos^{2N}(\theta)$. For $N \to \infty$, we have

$$\cos^{2N}(\theta) = \left(1 - \frac{\pi^2}{8N^2} + \mathcal{O}(N^{-4})\right)^{2N} = 1 - \frac{\pi^2}{4N} + \mathcal{O}(N^{-2}) \xrightarrow{N\to\infty} 1.$$

This simple calculation has the remarkable consequence that (when N is large enough) the photon will always end up in the lower path and the bomb will not explode. Since the photon will always end up in the upper path if there is no bomb in the box, this protocol enables us to tell (with probability approaching 1) whether there is a bomb in the box, while simultaneously ensuring that the bomb will not be triggered.

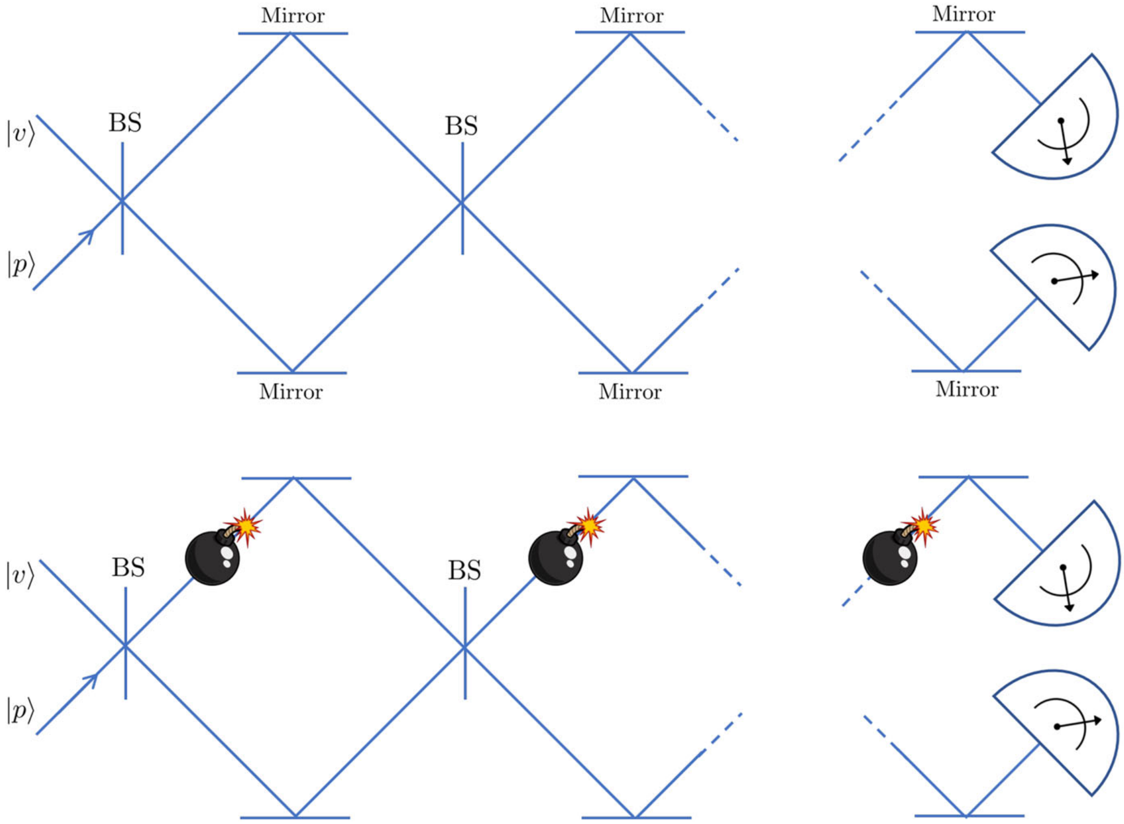

FIGURE 2. Kwiat et al.'s version of the bomb-tester experiment

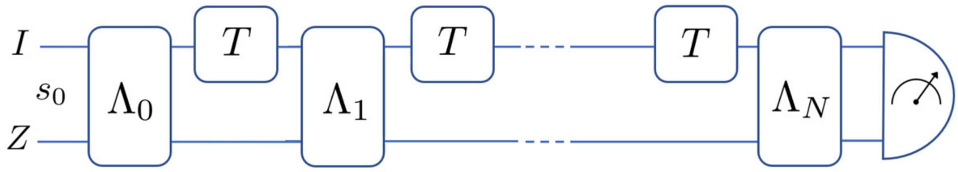

FIGURE 3. N-step discrimination strategy

Interpretation as a Channel Discrimination Problem We have seen in the previous paragraph how to discriminate between a completely transparent object (empty box) and an opaque object (bomb) such that the probability that a photon gets absorbed by the opaque object can be made as small as one pleases. This problem can be reinterpreted as a channel discrimination problem as follows: The channel corresponding to the transparent object is simply the identity channel ($T_{\text{empty}} := \text{id}$), while the action of the opaque object can be identified with the channel[2] $T_{\text{bomb}} : \mathcal{B}_1(\mathcal{H}_U) \to \mathcal{B}_1(\mathcal{H}_U)$, defined by

$$T_{\text{bomb}}(\cdot) = \text{tr}\,[\cdot]\,|v\rangle\langle v|.$$

[2] $\mathcal{B}_1(\mathcal{H})$ denotes the set of traceclass operators on the Hilbert space $\mathcal{H}$ and $\mathcal{S}(\mathcal{H})$ denotes the set of density operators.

According to the theory of quantum combs,[3] the most general (causally ordered) strategy to discriminate channels is given by the sequential scheme, depicted in Fig. 3. That is, if the channels to be discriminated act on the system I (I for interaction), then the most general discrimination strategy[4] allowed by quantum theory can be described as follows: First, we choose an ancillary system Z (which might be arbitrarily large) and an initial state $s_0 \in \mathcal{S}(\mathcal{H}_I \otimes \mathcal{H}_Z)$. Then, we can apply a channel[5] $\Lambda_0 : \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z) \to \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z)$ to s_0. Afterwards, the unknown channel is applied to the system (i.e., if $T : \mathcal{B}_1(\mathcal{H}_I) \to \mathcal{B}_1(\mathcal{H}_I)$ is the unknown channel, then its application transforms the state $\Lambda_0(s_0)$ to $(T \otimes \mathrm{id})(\Lambda_0(s_0))$). Then, we can transform the system by applying a channel $\Lambda_1 : \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z) \to \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z)$. Afterwards, we apply the unknown channel again, followed by an application of a channel $\Lambda_2 : \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z) \to \mathcal{B}_1(\mathcal{H}_I \otimes \mathcal{H}_Z)$. We repeat this process N times overall. In the end, our system is in a state $\rho_N^T \in \mathcal{S}(\mathcal{H}_I \otimes \mathcal{H}_Z)$, which depends on T. Hence, by measuring we can obtain information about the identity of T. Kwiat et al.'s protocol can be integrated in this formalism as follows: We identify the upper path with the system I and the lower path with the system Z and choose $s_0 := |v \otimes p\rangle\langle v \otimes p|$. For $0 \le i \le N-1$, the channels Λ_i are defined by $\Lambda_i(\cdot) := U \cdot U^\dagger =: \hat{U}(\cdot)$, with $\theta = \frac{90^\circ}{N}$ and we set $\Lambda_N := \mathrm{id}$. It is then easy to calculate that

$$\begin{aligned}
\rho_N^{T_{\text{empty}}} &= \hat{U}^N(|v \otimes p\rangle\langle v \otimes p|) = |p \otimes v\rangle\langle p \otimes v|, \\
\rho_N^{T_{\text{bomb}}} &= \Big((T_{\text{bomb}} \otimes \mathrm{id}) \circ \hat{U}\Big)^N (|v \otimes p\rangle\langle v \otimes p|) \\
&= \cos^{2N}(\theta)|v \otimes p\rangle\langle v \otimes p| + (1 - \cos^{2N}(\theta))|v \otimes v\rangle\langle v \otimes v|,
\end{aligned} \tag{1.2}$$

where $\rho_N^{T_{\text{empty}}}$ and $\rho_N^{T_{\text{bomb}}}$ denote the output states of the protocol when the unknown channel is T_{empty} or T_{bomb}. An interesting aspect of the expressions (1.2) is that one can read off the results of the last paragraph, since the states are orthogonal and since the probability that the bomb explodes is simply given by the coefficient of $|v \otimes v\rangle\langle v \otimes v|$. To abstract from the bomb-tester experiment, we want to allow for arbitrary quantum channels and for arbitrary discrimination strategies (Fig. 3). In this more general setting, the concept of the output state does not change. What is not a priori clear is what it means that something was "interaction-free". Since we want to allow for arbitrary strategies (for example, involving many photons in arbitrary superpositions), the output state does not, in general, contain the information if an interaction occurred. Therefore, we need to model separately what "interaction-free" means for general discrimination strategies. A derivation of such a model based on some axioms takes some effort. We will, therefore, postpone this discussion until Sect. 3. For now, let us just describe the essential constituents. First, for the notion of "interaction-free" to have any meaning, there needs to be some

[3] Quantum combs: also known as quantum supermaps, quantum strategies, ...

[4] This includes in particular coherent evolution, the use of entanglement, measurements, adaptive strategies, channels used in parallel, ...

[5] Of course, the application of Λ_0 is redundant, since one could choose s_0 differently. Allowing to apply Λ_0, however, will simplify the notation.

way not to interact with the object in the box. We will thus assume, in analogy to the bomb-tester experiment, the existence of a *vacuum state*. That is, we assume that for the channels under consideration, there exists a pure state $|v\rangle\langle v| \in \mathcal{S}(\mathcal{H}_I)$ such that $|v\rangle\langle v|$ gets mapped to a pure state by the channel and that if the channel is applied to $|v\rangle\langle v|$, then there is no "interaction" with the object in the box. This concept is formalized by the notion of a channel with vacuum.

Definition 1.1 (*Channel with vacuum*). A channel with vacuum $v \in \mathcal{H}$ is a channel $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ together with a unit vector $v \in \mathcal{H}$ such that $T(|v\rangle\langle v|)$ is pure. The unit vector v is called the *vacuum*, and the state $|v\rangle\langle v| \in \mathcal{S}(\mathcal{H})$ is called the vacuum state.

The notion of an object in the box already suggests that we should look at the given channel in the open system picture. To this end, we imagine a Demon sitting in the box and trying to figure out if something else than the vacuum was sent through the box. To do so, we allow the Demon to access the object in the box. In more mathematical terms, the Demon has full access to the output of the conjugate channel [20]. An important implicit assumption underlying the discussion above is that the channels we look at can be applied several times (which means that the channel does not change)—a Markovianity assumption. Given just this Markovianity assumption, it is possible to determine the probability that, for a certain discrimination strategy, the Demon will find out if at any point during the execution of the strategy, the channel was applied to something else than the vacuum state. We will call this probability the *"interaction" probability* (see Definition 3.3), denoted by $P_I^T(D)$, where T denotes the channel and D the discrimination strategy. The central notion of *discrimination in an "interaction-free" manner*, as formalized in Definition 3.4, is then defined by demanding that the discrimination error probability as well as the "interaction" probability can be made arbitrarily small simultaneously. We finish this section by formalizing the notion of a discrimination strategy[6] and by fixing the notation.

Definition 1.2 (*Discrimination strategy*). An N-step discrimination strategy is a tuple $(\mathcal{H}, \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda)$, where $\mathcal{H}$, $\mathcal{H}_Z$, $\mathcal{H}_i$, and $\mathcal{H}_o$ are Hilbert spaces, $s_0 \in \mathcal{S}(\mathcal{H}_i)$ is the initial state and $\Lambda := \{\Lambda_0, \Lambda_1, \ldots, \Lambda_N\}$ is a set of channels, with $\Lambda_0 : \mathcal{B}_1(\mathcal{H}_i) \to \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_Z)$, $\Lambda_n : \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_Z) \to \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_Z)$ for $1 \le n \le N-1$, and $\Lambda_N : \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_Z) \to \mathcal{B}_1(H_o)$.
An N-step discrimination strategy induces the *intermediate state map* $\rho : \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \times \{0, 1, 2, \ldots, N\} \to \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_Z) \cup \mathcal{B}_1(\mathcal{H}_o)$, defined by

$$\begin{aligned} \rho(T, 0) &= \Lambda_0(s_0), \\ \rho(T, n) &= \Lambda_n \circ (T \otimes \mathrm{id}) \circ \rho(T, n-1), \text{ for } 1 \le n \le N. \end{aligned} \tag{1.3}$$

We will always write[7] ρ_n^T for $\rho(T, n)$ and omit $\mathcal{H}_i$ and $\mathcal{H}_o$ if $\mathcal{H}_i = \mathcal{H}_o = \mathcal{H} \otimes \mathcal{H}_Z$.

[6] Note that in this definition, we allow the input and output spaces to be different from $\mathcal{H} \otimes \mathcal{H}_Z$. This is solely for notational flexibility and has no physical significance.
[7] The superscript should not be confused with the transpose.

Notation Throughout, $\mathcal{H}$ (with some subscript) denotes a separable complex Hilbert space and in this paragraph, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. The range of a map $f:\mathcal{X}\to\mathcal{Y}$ is denoted by $\mathrm{ran}(f) := \{f(x)\,|\,x\in\mathcal{X}\}$. The kernel of f is $\ker(f) := \{x\in\mathcal{X}\,|\,f(x)=0\}$. The dual space $\mathcal{X}^*$ of $\mathcal{X}$ is the set of bounded linear functionals on $\mathcal{X}$. The orthogonal complement of a linear subspace $\mathcal{V}\subseteq\mathcal{H}$ is denoted by $\mathcal{V}^\perp$. The open ϵ-ball around $x_0\in\mathcal{X}$ is defined by $B_\epsilon(x_0) := \{x\in\mathcal{X}\,|\,\|x-x_0\|<\epsilon\}$ and the closed δ-disc around $z_0\in\mathbb{C}$ is denoted by $\mathbb{D}_\delta(z_0) := \{z\in\mathbb{C}\,|\,|z-z_0|\leq\delta\}$.

The Banach space of bounded linear operators $\mathcal{X}\to\mathcal{X}$ is denoted by $\mathcal{B}(\mathcal{X})$. The space of trace-class operators $\mathcal{B}_1(\mathcal{H})$ becomes a Banach space with trace-norm $\|\cdot\|_1 := \mathrm{tr}\,[|\cdot|]$. For $A\in\mathcal{B}(\mathcal{H})$, the adjoint is denoted by $A^\dagger$ and the support of A is defined by $\mathrm{supp}(A) := \ker(A)^\perp$. If $A^\dagger = A$, then A is called self-adjoint. A is called positive semi-definite, sometimes denoted by $A\geq 0$, if A is self-adjoint and $\langle\psi|A\psi\rangle\geq 0$ for all $\psi\in\mathcal{H}$. For a closed subspace $\mathcal{V}\subseteq\mathcal{H}$, we denote (in a slight abuse of notation) by $\mathcal{B}(\mathcal{V})\subseteq\mathcal{B}(\mathcal{H})$ the bounded linear operators with range and support in $\mathcal{V}$ and by $\mathcal{B}_1(\mathcal{V})$ the trace-class operators with range and support in $\mathcal{V}$.

A linear operator $T\in\mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ is called a quantum operation if it is completely positive and trace non-increasing. If T is in addition trace-preserving, then T is called a (quantum) channel. If a quantum channel T is written in the form $T(\cdot) = \mathrm{tr}_E\left[V\cdot V^\dagger\right]$, where $V:\mathcal{H}\to\mathcal{H}_E\otimes\mathcal{H}$ is an isometry and where tr_E is the partial trace, then V is called a Stinespring isometry. The set of (quantum) states on $\mathcal{H}$ is given by $\mathcal{S}(\mathcal{H}) := \{\rho\in\mathcal{B}_1(\mathcal{H})\,|\,\rho\geq 0, \mathrm{tr}\,[\rho]=1\}$. The identity channel is denoted by id and the unit matrix by $\mathbb{1}$. For positive semi-definite trace-class operators ρ and σ, the fidelity is defined by $\sqrt{F}(\rho,\sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1$.

For $B\in\mathcal{B}(\mathcal{X})$, the resolvent set is $\rho(B) := \{z\in\mathbb{C}\,|\,z-B \text{ is invertible}\}$ and the spectrum is $\sigma(B) := \mathbb{C}\setminus\rho(B)$. The discrete spectrum of B is the subset of isolated points of $\sigma(B)$ such that the corresponding Riesz projection has finite rank.

2. Results

To state and discuss our main results, we need one more concept, which is similar to that of a decoherence-free subspace.[8]

Definition 2.1 (*Isometric subspace*). Let $\mathcal{V}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$. A channel $T:\mathcal{B}_1(\mathcal{H})\to\mathcal{B}_1(\mathcal{H})$ is said to be *isometric on* $\mathcal{V}$ if there exists an isometry $V:\mathcal{V}\to\mathcal{H}$, such that[9]

$$T|_{\mathcal{B}_1(\mathcal{V})}(\cdot) = V\cdot V^\dagger. \tag{2.1}$$

If T is isometric on $\mathcal{V}$, we call $\mathcal{V}$ an *isometric subspace* w.r.t. T.

[8] An isometric subspace is a decoherence-free subspace if the range of the isometry is $\mathcal{V}$.

[9] $T|_{\mathcal{B}_1(\mathcal{V})}$ denotes the restriction of T to the bounded linear operators with range and support in $\mathcal{V}$.

The significance of channels that are isometric on $\mathcal{V}$ is that they are the analogue to the identity channel in the bomb-tester case. To see why, note that $T|_{\mathcal{B}_1(\mathcal{V})}$ satisfies the Knill–Laflamme error-correcting conditions [21]. Hence, by composing $T|_{\mathcal{B}_1(\mathcal{V})}$ with an appropriate channel, we obtain the identity channel on $\mathcal{B}_1(\mathcal{V})$. Furthermore, as Lemma 3.10 proves in a language adapted to our model, the output of the conjugate channel of T will be the same for all $\rho \in \mathcal{B}_1(\mathcal{V})$. In particular, if we have $v \in \mathcal{V}$, where v is the vacuum, then even though $\rho \in \mathcal{B}_1(\mathcal{V})$ might be different from $|v\rangle\langle v|$, the Demon (having access to the conjugate channel only) has no chance of telling that something other than the vacuum has been sent through the box.

We are now ready to state our main result, which is an easy to check necessary and sufficient criterion that tells us when it is possible (or impossible) to discriminate two quantum channels in an "interaction-free" manner.

Theorem 2.2 (Main result). *Let* $\dim(\mathcal{H}) < \infty$. *Two channels* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *with vacuum* $v \in \mathcal{H}$ *can be discriminated in an "interaction-free" manner if and only if there exists a subspace* $\mathcal{V} \subseteq \mathcal{H}$ *with the following three properties:*

1. $v \in \mathcal{V}$.
2. *At least one of the two channels is isometric on* $\mathcal{V}$.
3. $T_A|_{\mathcal{B}(\mathcal{V})} \neq T_B|_{\mathcal{B}(\mathcal{V})}$.

Note that the central notion "discrimination in an 'interaction-free' manner" has only been defined informally in the paragraph following Definition 1.1. The formal definition, as well as the one for the "interaction" probability, can be found in Sect. 3.3, after a derivation of the mathematical form of these quantities from first principles in Sect. 3.1.

Remark 2.3. At first glance it may seem to be hard to check whether such a subspace exists. This is not so, as one only needs to consider two candidates for $\mathcal{V}$, the so-called maximal vacuum subspaces $\mathcal{V}_{T_A}$ and $\mathcal{V}_{T_B}$, which we define and study in 3.9 and 3.10.

Theorem 2.2 is a direct consequence of two results: a protocol to discriminate two channels and a no-go theorem. We discuss these cases separately in the following two subsections.

2.1. The Constructive Case

We consider the case where our main theorem says that we can discriminate the two channels in an "interaction-free" manner. That is, where there is a subspace $\mathcal{V}$, such that $\mathcal{V}$ contains the vacuum and one of the two channels is isometric on $\mathcal{V}$ and $T_A|_{\mathcal{B}(\mathcal{V})} \neq T_B|_{\mathcal{B}(\mathcal{V})}$. For this case, we propose a protocol (see Sect. 4) that can discriminate two channels in an "interaction-free" manner. We will discuss the properties of this protocol in the following. It turns out that one does not need complete information about the two channels to perform the discrimination task. To account for this, we consider the more general task, where we want to know to which one of two known, disjoint, sets of channels the unknown channel belongs. Of course, Theorem 2.2 puts some restrictions on how these sets may look like. Specifically, we consider the following: Given a channel T with vacuum $v \in \mathcal{V}$ that is isometric on $\mathcal{V}$, we take as our first set (a subset of) the set of channels that equal T if we restrict their domains to $\mathcal{B}_1(\mathcal{V})$. The second set is less restricted in that we only assume that all channels must be channels with (the same) vacuum v and that the restrictions to $\mathcal{B}_1(\mathcal{V})$ must not equal $T|_{\mathcal{B}_1(\mathcal{V})}$. It will then turn out that under these conditions, these two sets can be discriminated in an "interaction-free" manner. Roughly speaking, this tells us that we can test whether the unknown channel is T or some other channel, whose identity is unknown. Putting it yet another way, if the identity channel is interpreted as an empty box and every other channel as a non-empty box, then our result says that one can always find out (in an "interaction-free" manner) if there is something or nothing in the box. Before we state this in mathematical terms, we need to define the *discrimination error probability* for two sets.

Definition 2.4 (*Error probability*). Let $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ be two sets of channels. For an N-step discrimination strategy D and a two-valued POVM $\Pi = \{\pi_A, \pi_B\}$, the *discrimination error probability* is defined by

$$P_e(D, \Pi) := \frac{1}{2}\left[\sup_{T\in\mathcal{C}_A} \operatorname{tr}\left[\pi_B \rho_N^T\right] + \sup_{T\in\mathcal{C}_B} \operatorname{tr}\left[\pi_A \rho_N^T\right]\right]. \tag{2.2}$$

Theorem 2.5 (Discrimination strategy). *For* $\dim(\mathcal{H}) < \infty$*, let* $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *be two closed sets of channels and* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$*, such that*

1. *For all* $T \in \mathcal{C}_A \cup \mathcal{C}_B$*,* T *is a channel with vacuum* $v \in \mathcal{V}$*.*
2. *For all* $T \in \mathcal{C}_A$*,* T *is isometric on* $\mathcal{V}$*.*
3. *The set* $\mathcal{C}_A|_{\mathcal{B}_1(\mathcal{V})} := \left\{T|_{\mathcal{B}_1(\mathcal{V})} \,\middle|\, T \in \mathcal{C}_A\right\}$ *contains exactly one element.*
4. $\mathcal{C}_A|_{\mathcal{B}_1(\mathcal{V})}$ *and* $\mathcal{C}_B|_{\mathcal{B}_1(\mathcal{V})} := \left\{T|_{\mathcal{B}_1(\mathcal{V})} \,\middle|\, T \in \mathcal{C}_B\right\}$ *are disjoint.*

Then there exist a constant C*, and for every* $N \in \mathbb{N}$*, an* N*-step discrimination strategy* D *and a two-valued POVM* Π*, such that*

$$P_e(D, \Pi) \leq \frac{C}{N^2}, \tag{2.3}$$

$$P_I^{T_A}(D) = 0 \quad \text{and} \quad P_I^{T_B}(D) \leq \frac{C}{N}, \tag{2.4}$$

for all $T_A \in \mathcal{C}_A$ *and all* $T_B \in \mathcal{C}_B$*, where* P_I *denotes the "interaction" probability. Thus, the sets* $\mathcal{C}_A$ *and* $\mathcal{C}_B$ *can be discriminated in an "interaction-free" manner.*

Remark 2.6. The strategy we propose that has the properties stated in Theorem 2.5 only requires one ancillary qubit system in the worst-case scenario (as does the Kwiat et al. protocol) and might thus be implementable in the near future. We also show that one cannot get rid of the ancillary qubit in a naive way.

Remark 2.7. Although Theorem 2.5 is formulated for finite-dimensional spaces, a key part of the proof works also in infinite-dimensional spaces (Theorems 4.5 and 4.6).

Remark 2.8. For two channels T_A and T_B with vacuum $v \in \mathcal{H}$, we can define the sets $\mathcal{C}_A := \{T_A\}$ and $\mathcal{C}_B := \{T_B\}$. If there is a subspace $\mathcal{V}$ such that the Conditions 1-3 in the main theorem are fulfilled (and, w.l.o.g, T_A is isometric on $\mathcal{V}$), then clearly $\mathcal{C}_A$ and $\mathcal{C}_B$ satisfy the hypothesis of Theorem 2.5 and thus T_A and T_B can be discriminated in an "interaction-free" manner. This proves the direct part of Theorem 2.2.

Given the result of Theorem 2.5, it is natural to ask whether the bounds on the error probability and the "interaction" probability have the optimal dependence on N. This is clearly not the case for the error probability, as is already evident from the bomb-tester experiment. For the "interaction" probability, we were able to show (under a mild condition on $\mathcal{C}_A$ and $\mathcal{C}_B$) that N^{-1} is indeed the best possible rate. We state this as a meta theorem (see Theorem 5.9).

Theorem. *Subject to a condition stated in Theorem 5.9, there exists a constant $C > 0$ such that*

$$\max(P_I^{T_A}(D), P_I^{T_B}(D)) \geq C\, \frac{(1 - 2P_e(D, \Pi))^4}{N}, \tag{2.5}$$

for all N-step discrimination strategies D and all two-valued POVM's Π.

The result above cannot hold unconditionally. If there is a subspace $\mathcal{V}$ such that $v \in \mathcal{V}$, and *both* channels are isometric on $\mathcal{V}$ and $T_A|_{\mathcal{B}(\mathcal{V})} \neq T_B|_{\mathcal{B}(\mathcal{V})}$, then we can restrict ourselves to probing the channel only with states in $\rho \in \mathcal{B}_1(\mathcal{V})$. Since the Demon cannot tell the difference between these states, the "interaction" probability is zero and the remaining problem is to discriminate two isometric channels. This problem can be solved with discrimination error probability equal to zero, in a finite number of steps [22]. We were unable to show that the case described above is the only one where the N^{-1}-rule can be violated, but this seems plausible.

2.2. The No-Go Case

In this section, we consider the case for which our main theorem tells us that "interaction-free" channel discrimination is impossible; that is, if there exists no subspace satisfying all three properties of Theorem 2.2. In this case the channels T_A and T_B must be such that whenever there is a subspace $\mathcal{V}$ that contains the vacuum and on which at least one of the two channels is isometric, then the two channels must necessarily be the same on that subspace.[10] In this case, we were able to establish the following theorem that shows that there is a trade-off between the error probability and the "interaction" probability, in the sense that not both of them can go to zero simultaneously.

[10] Unfortunately, this case seems to be the generic case. Indeed, on physical grounds (think of two semi-transparent objects) it is reasonable to assume that for both channels, the only isometric subspace that contains the vacuum is simply span$\{v\}$ and that $|v\rangle\langle v|$ is a fixed point.

Theorem 2.9 (No-go theorem). *For* $\dim(\mathcal{H}) < \infty$*, let* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two channels with vacuum* $v \in \mathcal{H}$*. Suppose that no subspace satisfies the properties 1, 2, and 3 of Theorem 2.2 simultaneously.*

Then, there exists a constant $C > 0$*, such that*

$$(1 - 2P_e(D, \Pi))^2 \leq C \max(P_I^{T_A}(D), P_I^{T_B}(D)), \tag{2.6}$$

for all finite-dimensional N*-step discrimination strategies* D *and all two-valued POVMs,* Π*. Hence,* T_A *and* T_B *cannot be discriminated in an "interaction-free" manner.*

Clearly, this implies the converse in Theorem 2.2.

As a by-product, we obtained an inequality for the fidelity, which might be of independent interest.

Proposition 2.10. *For* $\dim(\mathcal{H}) < \infty$*, let* $T_A^{\downarrow}, T_B^{\downarrow} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be quantum operations and let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *such that* $T_A^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})} = T_B^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$ *and* $T_A^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$ *is trace-preserving. Then,*

$$\sqrt{F}(T_A^{\downarrow}(\rho), T_B^{\downarrow}(\sigma)) \geq \sqrt{F}(\rho, \sigma) - 2\sqrt{F}(P^{\perp}\rho P^{\perp}, P^{\perp}\sigma P^{\perp}), \tag{2.7}$$

for all $\rho, \sigma \geq 0$*, where* $P^{\perp}$ *is the orthogonal projection onto* $\mathcal{V}^{\perp}$*.*

3. The Models

In this section, we propose two different, but in the end largely equivalent models that generalize the notion of "interaction-free" measurement to quantum channels. Since the sequential scheme, given in Fig. 3, is the most general causally ordered strategy allowed in quantum theory [17], it suffices to define our notions for this kind of strategy. In both models, we assume the validity of Fig. 3. That is, we assume that the unknown channel T does not change during the execution of the discrimination strategy—the Markovianity assumption. This is a relatively weak assumption, since we are in control of the duration between the individual channel invocations. This section consists of four subsections. In the first two subsections, we derive our two models. The third subsection summarizes the former two by properly defining the quantities of merit and thereby setting the stage for a rigorous analysis in the later sections. In the fourth subsection, we compare the two models by deriving some elementary properties, which will be used later on.

3.1. The "Interaction" Model

In our first model, we interpret the term "interaction-free" in an information-theoretic way. That is, we imagine a Demon sitting in the box trying to figure out, if we interacted with the interior of the box. In more technical terms, this means that the Demon has full access to the output of the conjugate channel. Since our task would be trivially infeasible otherwise, there must be a way not to interact with the box. Therefore, we only consider channels with vacuum. That is, we assume that for all channels under consideration there exists a distinguished pure state, the *vacuum state*, $|v\rangle\langle v|$. This state is assumed

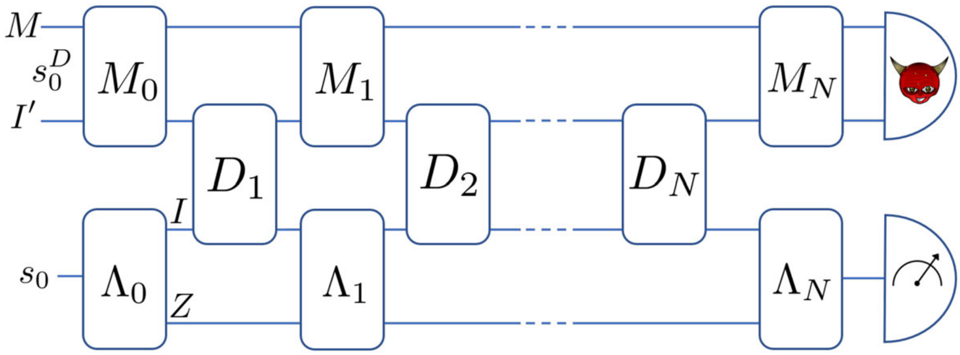

FIGURE 4. General scenario

to have the following two important properties: First, if the vacuum state is sent through the channel, then the Demon concludes that no interaction has occurred. Second, we assume that the channels under consideration map the vacuum state to a pure state. This assumption is physically reasonable as it means that the state of the probe system does not become entangled with the Demon's system. If in contrast, the probe system becomes entangled with the Demon's system, then there must have been an interaction and the term "interaction-free" measurement would be inappropriate. We should mention, however, that the transmission model, which we are going to describe in the next section does not use the ""vacuum maps to pure state"" assumption. This comes at the cost that the transmission functional is no longer a property of a channel (as the "interaction" functional will turn out to be) but rather an object that has to be modeled separately. Together, these two assumptions yield the definition of a channel with vacuum (Definition 1.1). For a given channel T with vacuum $v \in \mathcal{H}_I$ and an N-step discrimination strategy $D = (\mathcal{H}_I, \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda)$, we want to define the *"interaction" probability* $P_I^T(D)$ as the probability that the Demon in the box encounters that, during the execution of D, something other than the vacuum state was sent through the channel. To define this probability, we need to specify how the Demon can obtain information about what was sent through the channel.

A natural way to model this is by assuming that for each of the N channel-uses (indexed by n) in the discrimination strategy, the Demon is allowed to implement the channel T via a channel $D_n : \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_I) \to \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_I)$, where $\mathcal{H}_{I'}$ is the Hilbert space associated with a system I', which the Demon controls. We further allow the Demon to keep an arbitrarily large memory system M (with Hilbert space $\mathcal{H}_M$) which he can manipulate freely (i.e., he can choose the channels M_n, defined below). The most general (causally ordered) scheme that can be obtained from the above description is depicted in Fig. 4. Mathematically, the Demon's strategy is completely determined by an initial state $s_0^D \in \mathcal{S}(\mathcal{H}_M \otimes \mathcal{H}_{I'})$ and channels $M_0, M_1, \ldots, M_N : \mathcal{B}_1(\mathcal{H}_M \otimes \mathcal{H}_{I'}) \to \mathcal{B}_1(\mathcal{H}_M \otimes \mathcal{H}_{I'})$ and $D_1, D_2, \ldots, D_N : \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_I) \to \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_I)$. Given this data, the scheme in Fig. 4 produces the output (or final) state $\rho_F \in$

$\mathcal{S}(\mathcal{H}_M \otimes \mathcal{H}_{I'} \otimes \mathcal{H}_o)$, defined by

$$\rho_F := (M_N \otimes \Lambda_N)(\mathrm{id} \otimes D_N \otimes \mathrm{id}) \ldots (M_1 \otimes \Lambda_1)(\mathrm{id} \otimes D_1 \otimes \mathrm{id})(M_0 \otimes \Lambda_0)(\chi_0),$$

where $\chi_0 := s_0^D \otimes s_0$. In the end, the Demon will measure his system $(M + I')$ and decide, based on the measurement outcome, if an interaction has occurred. The "interaction" probability is then the probability that he detects such an interaction if he chooses his strategy optimally within the given constraints.

Before we can analyze what the Demon's optimal strategy is, we need to formulate mathematically the assumption that D_n implements T, and that T must be independent of the Demon's strategy (Markovianity). Precisely, we assume that D_n must be such that if the Demon's system (I') and I are uncorrelated, then the action on the system I must be independent of the state of the system I'. In formulas, we assume that

$$\mathrm{tr}_{I'}\left[D_n(\rho_{I'} \otimes \rho_I)\right] = T(\rho_I) \tag{3.1}$$

for all $\rho_{I'} \in \mathcal{S}(\mathcal{H}_{I'})$, $\rho_I \in \mathcal{S}(\mathcal{H}_I)$, and $n \in \{1, 2, \ldots, N\}$. We note that (3.1) is exactly the definition of a semicausal channel, as introduced in [23]. A structure theorem by Eggeling et al. [24] tells us that semi-causal channels are semi-localizable. That is, D_n can be written in the form:

$$D_n(\rho_{I'I}) = \mathrm{tr}_{E_n}\left[(X_n \otimes \mathrm{id}_I)(\mathrm{id}_{I'} \otimes \hat{V}_n)(\rho_{I'I})\right],$$

where $\hat{V}_n : \mathcal{B}_1(\mathcal{H}_I) \to \mathcal{B}_1(\mathcal{H}_{E_n} \otimes \mathcal{H}_I)$, defined by $\hat{V}_n(\cdot) = V_n \cdot V_n^\dagger$ is the quantum channel associated with a Stinespring isometry $V_n : \mathcal{H}_I \to \mathcal{H}_{E_n} \otimes \mathcal{H}_I$ of T and $X_n : \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_{E_n}) \to \mathcal{B}_1(\mathcal{H}_{I'} \otimes \mathcal{H}_{E_n})$ is some channel. To proceed further in our search for the Demon's optimal strategy, we make a few simplifying observations and definitions. First, the unitary freedom in the Stinespring dilation $\hat{V}_n$ can be absorbed into the channel X_i. We can therefore assume, without loss of generality, that $\mathcal{H}_{E_1} = \mathcal{H}_{E_2} = \cdots = \mathcal{H}_{E_N} =: \mathcal{H}_E$ and $\hat{V}_1 = \hat{V}_2 = \cdots = \hat{V}_N =: \hat{V}$. Second, for $\rho \in \mathcal{S}(\mathcal{H}_M \otimes \mathcal{H}_{I'} \otimes \mathcal{H}_I)$, we have

$$(M_n \otimes \mathrm{id}_I) D_n(\rho) = \mathrm{tr}_{E_n}\left[([(M_n \otimes \mathrm{id}_{E_n})(\mathrm{id}_M \otimes X_n)] \otimes \mathrm{id}_I)(\mathrm{id}_{MI'} \otimes \hat{V}_n)(\rho)\right],$$

which motivates the definition $\underline{X}_n := (M_n \otimes \mathrm{id}_{E_n})(\mathrm{id}_M \otimes X_n)$. In the following, we adopt the convention that if some channel acts trivially on a tensor factor (i.e., as the identity), then we omit these tensor factors in the notation (e.g., $\underline{X}_i \otimes \mathrm{id}_I$ becomes just $\underline{X}_i$). With the newly introduced notation, it follows from the definition of σ_F that the state the Demon obtains is

$$\mathrm{tr}_{IZ}\left[\rho_F\right] = \mathrm{tr}_{IZ}\, \Lambda_N \mathrm{tr}_{E_N} \underline{X}_N \hat{V}_N \Lambda_{N-1} \mathrm{tr}_{E_{N-1}} \underline{X}_{N-1} \ldots \Lambda_1 \mathrm{tr}_{E_1} \underline{X}_1 \hat{V}_1 M_0 \Lambda_0(\chi_0).$$

We can commute the $\underline{X}_i$s and tr_{E_i}s to the left. Thus, upon defining the channel $\Gamma : \mathcal{B}_1(\mathcal{H}_{E_N} \otimes \mathcal{H}_{E_{N-1}} \otimes \cdots \otimes \mathcal{H}_{E_1}) \to \mathcal{B}_1(\mathcal{H}_M \otimes \mathcal{H}_{I'})$ by

$$\Gamma(\rho) = \mathrm{tr}_{E_N} \underline{X}_N \mathrm{tr}_{E_{N-1}} \underline{X}_{N-1} \ldots \mathrm{tr}_{E_1} \underline{X}_1 M_0(s_0^D \otimes \rho),$$

we have

$$\mathrm{tr}_{IZ}\left[\rho_F\right] = \Gamma(\mathrm{tr}_{IZ}\, \Lambda_N \hat{V}_N \Lambda_{N-1} \hat{V}_{N-1} \ldots \Lambda_1 \hat{V}_1 \Lambda_0(s_0)).$$

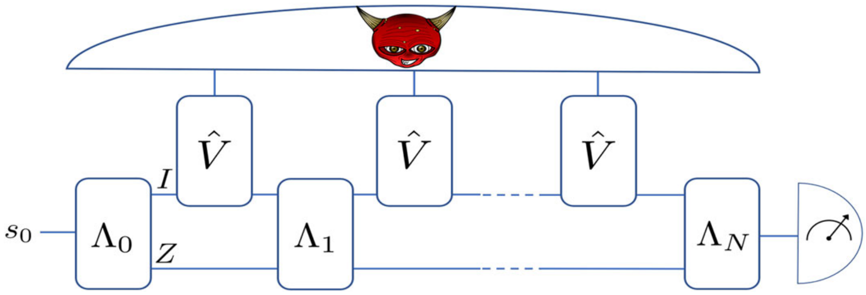

FIGURE 5. Scenario when the Demon's strategy is optimal

To decide if the channel was ever applied to a state different from the vacuum state, the Demon measures his state with a two-valued POVM, $\{Q_1, Q_2\}$. By convention, he will conclude that an interaction occurred (something other than the vacuum was sent through) if the event corresponding to Q_2 occurs. If the state sent through the channel is always the vacuum state, then the Demon's final state is

$$\Gamma(\left[\mathrm{tr}_I \left[V|v\rangle\langle v|V^\dagger\right]\right]^{\otimes N}),$$

where the tensor power is in the space $\mathcal{H}_{E_N} \otimes \mathcal{H}_{E_{N-1}} \otimes \cdots \otimes \mathcal{H}_{E_1}$. Since the Demon must not report an interaction, if the state was always the vacuum state, we demand

$$0 = \mathrm{tr}\left[Q_2\Gamma(\left[\mathrm{tr}_I \left[V|v\rangle\langle v|V^\dagger\right]\right]^{\otimes N})\right] = \mathrm{tr}\left[\Gamma^*(Q_2)\left[\mathrm{tr}_I \left[V|v\rangle\langle v|V^\dagger\right]\right]^{\otimes N}\right],$$

where Γ^* denotes the channel Γ in the Heisenberg picture. Clearly, if $\Gamma^*(Q_2) = \mathbb{1}^{\otimes N} - P_v^{\otimes N}$, where P_v is the orthogonal projection onto the support of $\mathrm{tr}_I \left[V|v\rangle\langle v|V^\dagger\right]$, then this requirement is fulfilled. Since we want to choose the optimal strategy the Demon can pursue, we want to set $\Gamma^*(Q_2) := \mathbb{1}^{\otimes N} - P_v^{\otimes N}$. We can always choose Γ and Q_2 to satisfy the last equation, because this corresponds to the strategy where the Demon simply stores all the states he obtains from the Stinespring dilation in each round. This justifies the graphical representation in Fig. 5. Since we defined the "interaction" probability to be the probability that the Demon concludes that an interaction occurred (if he acts optimally), we have

$$P_I^T(D) := \mathrm{tr}\left[(\mathbb{1}^{\otimes N} - P_v^{\otimes N})\,\mathrm{tr}_{IZ}\hat{V}_N\Lambda_{N-1}\hat{V}_{N-1}\ldots\Lambda_1\hat{V}_1(\rho_0^T)\right], \tag{3.2}$$

where $\rho_0^T := \Lambda_0(s_0)$ is the first intermediate state. We remark that the definition of $P_I^T(D)$ does not depend on the particular choice of the Stinespring dilation, since the unitary freedom in the Stinespring isometries is compensated by the equal and opposite freedom in P_v.

We can simplify this expression a bit. We define $P_v^\perp := \mathbb{1} - P_v$ and note that

$$\mathbb{1}^{\otimes N} - P_v^{\otimes N} = \sum_{n=0}^{N-1} \mathbb{1}^{\otimes N-n-1} \otimes P_v^\perp \otimes P_v^{\otimes n},$$

$$P_v^\perp \otimes P_v^{\otimes K} = \prod_{j=0}^{K-1} P_v^\perp \otimes \mathbb{1}^{\otimes j} \otimes P_v \otimes \mathbb{1}^{\otimes K-j-1}.$$

Using these two expressions and (several times) that Λ_n is trace-preserving, we obtain our final version for $P_I^T(D)$,

$$\begin{aligned}
P_I^T(D) &= \sum_{n=0}^{N-1} \operatorname{tr}\left[\mathbb{1}^{\otimes N-n-1} \otimes P_v^\perp \otimes P_v^{\otimes n} \operatorname{tr}_{IZ} \hat{V}_N \Lambda_{N-1} \hat{V}_{N-1} \ldots \Lambda_1 \hat{V}_1(\rho_0^T)\right] \\
&= \sum_{n=0}^{N-1} \operatorname{tr}\left[P_v^\perp \otimes P_v^{\otimes n} \operatorname{tr}_{IZ} \hat{V}_{n+1} \Lambda_n \hat{V}_n \ldots \Lambda_1 \hat{V}_1(\rho_0^T)\right] \\
&= \sum_{n=0}^{N-1} \operatorname{tr}\left[P_v^\perp \operatorname{tr}_{IZ}(\hat{V}_{n+1}(\Lambda_n(\operatorname{tr}_{E_i}((P_v \otimes \mathbb{1})\hat{V}_n(..\operatorname{tr}_{E_1}((P_v \otimes \mathbb{1})\hat{V}_1(\rho_0^T)..)\right] \\
&= \sum_{n=0}^{N-1} \operatorname{tr}\left[P_v^\perp \operatorname{tr}_I \hat{V}(\operatorname{tr}_Z(\Lambda_i T^\downarrow \Lambda_{n-1} T^\downarrow \ldots \Lambda_1 T^\downarrow(\rho_0^T)))\right] \\
&= \sum_{n=0}^{N-1} \operatorname{tr}\left[P_v^\perp \operatorname{tr}_I \hat{V}(\operatorname{tr}_Z\left[\rho_n^{T\downarrow}\right])\right].
\end{aligned}$$

In the second to last line, we defined $T^\downarrow(\cdot) = \operatorname{tr}_E\left[(P_v \otimes \mathbb{1})V \cdot V^\dagger\right]$ and $\rho_n^{T\downarrow}$ is determined by the intermediate state map. We have thus succeeded in our goal to define the "interaction" probability.

Remark 3.1. It is immediate from (3.2) that an alternative expression for $P_I^T(D)$ is given by

$$P_I^T(D) = 1 - \operatorname{tr}\left[\rho_N^{T\downarrow}\right]. \tag{3.3}$$

There are two reasons to prefer the lengthy version derived above. First, it makes the connection between the "interaction" model and the transmission model (defined below) explicit and thus allows us to treat these points of view on an equal footing. Second, it suggests to approach the problem by looking at the inputs of the individual channel uses, which turns out to be fruitful.

3.2. The Transmission Model

In our second model, we think of an interaction as something that does damage to the system in the box. As a guiding example, we think of a biological system—say a body cell. For the sake of argument, assume that we want to use high-energetic radiation (e.g., X-ray) to resolve the inner structure of the cell. Of course, radiation might damage the cell, which is usually undesirable. A reasonable measure for how much damage has been done to a cell seems to be the number of X-ray photons that were absorbed by the cell. In other words, the damage is quantified by the amount of energy that got *transmitted* from the probe system (X-ray) to the interior of the box (biological cell). Furthermore, if the cell is exposed to radiation several times, then the damage measure should be the sum of the number of photons that were absorbed each time. Let us now abstract away from this example. Assume that the system in the box is modeled quantum mechanically on a Hilbert space $\mathcal{H}_E$ and that the probe system is modeled on $\mathcal{H}_I$. Assume that initially the system E is in the state $\rho_E \in \mathcal{S}(\mathcal{H}_E)$. If we probe the system with a state $\rho_I \in \mathcal{S}(\mathcal{H}_I)$, then the combined evolution is described by a (not necessarily unitary) channel $U : \mathcal{B}_1(\mathcal{H}_E \otimes \mathcal{H}_I) \to \mathcal{B}_1(\mathcal{H}_E \otimes \mathcal{H}_I)$. Thus, the state of the combined system after the evolution is given by

$$\rho'_{EI} = U(\rho_E \otimes \rho_I).$$

Now assume that, in analogy to the number of absorbed photons in the example above, there is some physical quantity (an observable) that got transmitted from the probe system to the interior of the box by the above process, and that this quantity is related to the damage done to the object in the box. We further assume that the process above can only cause damage and cannot repair the system in the box. Thus, the observable must be a positive semi-definite operator Θ on the Hilbert space $\mathcal{H}_E$. Hence, for a single shot experiment, the important object is the positive linear functional $\mathfrak{t} : \mathcal{B}_1(\mathcal{H}_I) \to \mathbb{C}$, defined by

$$\mathfrak{t}(\rho_I) = \mathrm{tr}\left[\Theta\, \mathrm{tr}_I\left[U(\rho_E \otimes \rho_I)\right]\right].$$

For a general N-step discrimination strategy D (with intermediate state map ρ), we assume that the transmitted quantity is extensive. Since the state of the part of the probe system that interacts with the interior of the box in the nth step is given by $\mathrm{tr}_Z\left[\rho_n^T\right]$ (T is the channel defined by $T(\rho_I) = \mathrm{tr}_E\left[U(\rho_E \otimes \rho_I)\right]$), a good definition for the *total transmission* $\mathfrak{T}_T(D)$ is

$$\mathfrak{T}_T(D) := \sum_{n=0}^{N-1} \mathfrak{t}_T\left(\mathrm{tr}_Z\left[\rho_n^T\right]\right).$$

We raise this to a principle by assuming that for every channel T we have a positive linear functional $\mathfrak{t}_T$, which we call the *transmission functional*, that models the damage done to the object. The total transmission then plays the same role for the transmission model as the "interaction" probability does for the "interaction" model.

3.3. Formal Definition

We cast the principles developed in the last sections into formal definitions.

Definition 3.2 (*"Interaction" functional*). Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with vacuum $v \in \mathcal{H}$ and let $V : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$ be any Stinespring isometry of T. The positive linear functional $\mathfrak{i}_T : \mathcal{B}_1(\mathcal{H}) \to \mathbb{C}$, defined by

$$\mathfrak{i}_T(\cdot) := \mathrm{tr}\left[P_v^{\perp} \mathrm{tr}_{\mathcal{H}}\left[V \cdot V^{\dagger}\right]\right], \tag{3.4}$$

is called the *"interaction" functional* of T, where $P_v^{\perp}$ is the orthogonal projection onto the kernel of $\mathrm{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^{\dagger}\right]$.

Definition 3.3 (*"Interaction" probability*). Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with vacuum $v \in \mathcal{H}$ and let $D = (\mathcal{H}, \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda)$ be an N-step discrimination strategy. The *"interaction" probability* is defined by

$$P_I^T(D) := \sum_{n=0}^{N-1} \mathfrak{i}_T \left(\mathrm{tr}_Z \left[\rho_n^{T^\downarrow} \right] \right), \tag{3.5}$$

where the quantum operation $T^\downarrow : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is defined by

$$T^\downarrow(\cdot) = \mathrm{tr}_E \left[(P_v \otimes \mathbb{1}) V \cdot V^\dagger \right], \tag{3.6}$$

and where $V : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$ is any Stinespring isometry of T and P_v is the orthogonal projection onto the support of $\mathrm{tr}_{\mathcal{H}} \left[V|v\rangle\langle v|V^\dagger \right]$.

Definition 3.4 *("Interaction-free" discrimination).* Let $v \in \mathcal{H}$ and $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ be two sets of channels such that for all $T \in \mathcal{C}_A \cup \mathcal{C}_B$, T is a channel with vacuum v. We say that $\mathcal{C}_A$ and $\mathcal{C}_B$ can be *discriminated in an "interaction-free" manner* if for every $\epsilon, \delta > 0$ there exists an N-step discrimination strategy D and a two-valued POVM Π such that

$$P_e(D, \Pi) < \epsilon \quad \text{and} \quad P_I^T(D) < \delta, \tag{3.7}$$

for all $T \in \mathcal{C}_A \cup \mathcal{C}_B$.

Definition 3.5 (*Channel with transmission functional*). A *channel with transmission functional* $\mathfrak{t}_T$ is a channel $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ together with a positive linear functional $\mathfrak{t}_T \in (\mathcal{B}_1(\mathcal{H}))^*$. We call $\mathfrak{t}_T$ the *transmission functional.*

Definition 3.6 *(Total transmission)* Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with transmission functional $\mathfrak{t}_T$. For an N-step discrimination strategy $D = (\mathcal{H}, \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda)$, the *total transmission* is defined by

$$\mathfrak{T}_T(D) := \sum_{n=0}^{N-1} \mathfrak{t}_T \left(\mathrm{tr}_Z \left[\rho_n^T \right] \right). \tag{3.8}$$

Definition 3.7 *(Transmission-free discrimination).* Let $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ be two sets of channels such that for all $T \in \mathcal{C}_A \cup \mathcal{C}_B$, T is a channel with transmission functional $\mathfrak{t}_T$. We say that $\mathcal{C}_A$ and $\mathcal{C}_B$ can be *discriminated in a transmission-free manner* if for every $\epsilon, \delta > 0$ there exists an N-step discrimination strategy D and a two-valued POVM Π such that

$$P_e(D, \Pi) < \epsilon \quad \text{and} \quad \mathfrak{T}_T(D) < \delta, \tag{3.9}$$

for all $T \in \mathcal{C}_A \cup \mathcal{C}_B$.

3.4. Comparison of the Models and Elementary Properties

In this section, we clarify the relation between the transmission model and the "interaction" model. As a rule of thumb, the transmission model can be thought of as a generalization of the "interaction" model. Since we admit arbitrary positive linear functionals as transmission functionals, we have a much greater flexibility when modeling. For example, one could decide that out of the two objects to be discriminated, it does not matter (or is even desirable) if the second one gets destroyed. We should therefore set the transmission functional of the second channel to zero. This is something that is not possible in the "interaction" model. On the other hand, the advantage of the "interaction" model is that the "interaction" probability has a very clear interpretation and that the "interaction" functional is an intrinsic property of the channel. For the relation between these models, we note the following lemma.

Lemma 3.8. *Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with vacuum $v \in \mathcal{H}$ and let $\mathfrak{i}_T$ be its "interaction" functional. If we interpret T as a channel with transmission functional $\mathfrak{i}_T$, then*

$$P_I^T(D) \leq \mathfrak{T}_T(D), \tag{3.10}$$

for all N-step discrimination strategies D.

Proof. Immediate from the definition, since (by induction) $\rho_i^{T\downarrow} \leq \rho_i^T$. □

The insight that should be gained from this lemma is that if we want to prove that a certain discrimination task can be done in an "interaction-free" or in a transmission-free manner, then it suffices to tackle the problem in the transmission model. Thus, the results in Sect. 4 will be formulated in terms of the transmission model. On the other hand, if we want to prove a no-go theorem, then it is sufficient to work in the "interaction" model. At this point, there is a little detail that we do not want to hide, which is that it is possible that certain discrimination tasks can be performed with less resources, if one works in the "interaction" model and not in the transmission model. We will not investigate this possibility any further. We close this section by introducing the concept of a *maximal vacuum subspace*.

Definition 3.9 (*Maximal vacuum subspace*). Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with vacuum $v \in \mathcal{H}$ and let $V : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$ be any Stinespring isometry of T. The subspace $\mathcal{V}_T$ of $\mathcal{H}$, defined by[11]

$$\mathcal{V}_T := V^{-1}\left[\operatorname{supp}(\operatorname{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^\dagger\right]) \otimes \mathcal{H}\right], \tag{3.11}$$

is called the *maximal vacuum subspace* of T.

Lemma 3.10 (Properties of maximal vacuum subspaces). *For $\dim(\mathcal{H}) < \infty$, let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel with vacuum $v \in \mathcal{H}$. The maximal vacuum subspace $\mathcal{V}_T$ has the following properties:*

1. *$v \in \mathcal{V}_T$.*
2. *T is isometric on $\mathcal{V}_T$.*
3. *If T is isometric on a subspace $\mathcal{V}' \subseteq \mathcal{H}$, then either $\mathcal{V}_T \cap \mathcal{V}' = \{0\}$ or $\mathcal{V}' \subseteq \mathcal{V}_T$.*
4. *$\mathcal{V}_T$ is the union of all subspaces that contain v and on which T is isometric.*
5. *There exists a constant $C_T > 0$ such that $\mathfrak{i}_T(\rho) \geq C_T \operatorname{tr}\left[P^\perp \rho\right]$ for all $\rho \geq 0$, where $P^\perp$ is the projection onto $\mathcal{V}_T^\perp$.*

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[11] $V^{-1}[\cdot]$ denotes the preimage operation.

6. *For all* $\rho \geq 0$*, we have* $\mathfrak{i}_T(\rho) \leq \mathrm{tr}\left[P^\perp \rho\right]$*, where* $P^\perp$ *is the projection onto* $\mathcal{V}_T^\perp$*.*

Remark 3.11. The Claims 1–4 and 6 remain true if one lifts the assumption that $\mathcal{H}$ is finite-dimensional. Claim 5, however, would then be wrong.

Proof. We start with the following observation: Let $V : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$ be any Stinespring isometry of T. Since $T(|v\rangle\langle v|)$ is pure, Vv must be a tensor product. Thus, there are two unit vectors $v' \in \mathcal{H}$ and $e \in \mathcal{H}_E$ such that

$$Vv = e \otimes v'.$$

Hence, $\mathrm{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^\dagger\right] = |e\rangle\langle e|$ and

$$\mathrm{supp}(\mathrm{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^\dagger\right]) = \mathrm{span}\{e\}. \tag{3.12}$$

(1) Clearly, $Vv \in \mathrm{supp}(\mathrm{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^\dagger\right]) \otimes \mathcal{H}$. Thus, $v \in V^{-1}\left[Vv\right] \subseteq \mathcal{V}_T$.
(2) For $\phi \in \mathcal{V}_T$, we have $V\phi = e \otimes \psi_\phi$ for a uniquely defined $\psi_\phi \in \mathcal{H}$. We define $U : \mathcal{V}_T \to \mathcal{H}$ by $U\phi := \psi_\phi$. It is easy to check that U is an isometry and that $T(|\phi\rangle\langle\phi|) = U|\phi\rangle\langle\phi|U^\dagger$. Since this holds for all $\phi \in \mathcal{V}_T$, T is isometric on $\mathcal{V}_T$.
(3) Suppose that T is isometric on $\mathcal{V}'$, with isometry $U' : \mathcal{V}' \to \mathcal{H}$. If $\dim(\mathcal{V}') \leq 1$ then the claim is trivially true. So we can assume that $\dim(\mathcal{V}') \geq 2$. Let v_1 and v_2 be two orthogonal unit vectors in $\mathcal{V}'$. By assumption,

$$T(|v_i\rangle\langle v_i|) = \mathrm{tr}_E\left[V|v_i\rangle\langle v_i|V^\dagger\right] = U'|v_i\rangle\langle v_i|U'^\dagger,$$

for $i \in \{1,2\}$. As $U'|v_i\rangle\langle v_i|U'^\dagger$ is pure, there exists a pair of unit vectors $e_1, e_2 \in \mathcal{H}_E$ such that $Vv_i = e_i \otimes U'v_i$. By linearity, we have

$$\begin{aligned} 0 &= T(|v_1+v_2\rangle\langle v_1+v_2|) - \mathrm{tr}_E\left[V|v_1+v_2\rangle\langle v_1+v_2|V^\dagger\right] \\ &= U'|v_1+v_2\rangle\langle v_1+v_2|U'^\dagger - U'|v_1\rangle\langle v_1|U'^\dagger \\ &\quad - \langle e_2|e_1\rangle U'|v_1\rangle\langle v_2|U'^\dagger - \langle e_1|e_2\rangle U'|v_2\rangle\langle v_1|U'^\dagger - U'|v_2\rangle\langle v_2|U'^\dagger \\ &= (1-\langle e_2|e_1\rangle)U'|v_1\rangle\langle v_2|U'^\dagger + (1-\langle e_1|e_2\rangle)U'|v_2\rangle\langle v_1|U'^\dagger. \end{aligned}$$

This can only be true if $\langle e_1|e_2\rangle = 1$, which is true only if $e_1 = e_2$. Thus, by transitivity, there is a unit vector $e' \in \mathcal{H}_E$ such that $Vv' = e' \otimes U'v'$ for all $v' \in \mathcal{V}'$. With the definition of U in the proof of 2, we also have $V\phi = e \otimes U\phi$ for all $\phi \in \mathcal{V}_T$. Assume that $\mathcal{V}_T \cap \mathcal{V}' \neq \{0\}$. For a unit vector $\hat{v} \in \mathcal{V}_T \cap \mathcal{V}'$, the Cauchy–Schwarz inequality yields

$$\begin{aligned} 1 &= |\langle\hat{v}|\hat{v}\rangle| = |\langle V\hat{v}|V\hat{v}\rangle| = |\langle e'|e\rangle|\,|\langle U'\hat{v}|U\hat{v}\rangle| \\ &\leq |\langle e'|e\rangle|\,\|U'\hat{v}\|\,\|U\hat{v}\| = |\langle e'|e\rangle| \leq \|e'\|\,\|e\| = 1. \end{aligned}$$

Hence, the Cauchy–Schwarz inequality is satisfied with equality, which implies that the vectors e and e' differ only by a phase factor. In particular, $\mathrm{span}\{e\} = \mathrm{span}\{e'\}$. Using (3.12), we have for any $v' \in \mathcal{V}'$ that $Vv' = e' \otimes U'v' \in \mathrm{supp}(\mathrm{tr}_{\mathcal{H}}\left[V|v\rangle\langle v|V^\dagger\right]) \otimes \mathcal{H}$. Consequently, $v' \in \mathcal{V}_T$. As v' was arbitrary, this proves $\mathcal{V}' \subseteq \mathcal{V}_T$ as claimed.
4) If an isometric subspace $\mathcal{V}'$ contains v, then (by 1) the intersection with $\mathcal{V}_T$ is non-trivial. Thus, (by 3) $\mathcal{V}'$ is a subspace of $\mathcal{V}_T$. Hence, $\mathcal{V}_T$ contains all isometric subspaces and the claim follows as (by 2) $\mathcal{V}_T$ is isometric itself.

The following consideration is needed in the proof of 5 as well as in the proof of 6. We define the projections $\hat{P} := P_v \otimes \mathbb{1}$ and $\hat{P}^\perp := \mathbb{1} - \hat{P}$, where $P_v := |v\rangle\langle v|$. We further denote by P, the orthogonal projection onto $\mathcal{V}_T$ and define $P^\perp := \mathbb{1} - P$. In the following, let $\rho \geq 0$. By definition, we have

$$\begin{aligned}
\mathfrak{i}_T(\rho) &= \operatorname{tr}\left[P_v^\perp \operatorname{tr}_\mathcal{H}\left[V\rho V^\dagger\right]\right] \\
&= \operatorname{tr}\left[\hat{P}^\perp V\rho V^\dagger\right] \\
&= \operatorname{tr}\left[\hat{P}^\perp VP\rho PV^\dagger\right] + \operatorname{tr}\left[\hat{P}^\perp VP\rho P^\perp V^\dagger\right] \\
&\quad + \operatorname{tr}\left[\hat{P}^\perp VP^\perp\rho PV^\dagger\right] + \operatorname{tr}\left[\hat{P}^\perp VP^\perp\rho P^\perp V^\dagger\right].
\end{aligned}$$

By definition, if $\psi \in \mathcal{V}_T$, then $\hat{P}^\perp V\psi = 0$. Thus, $\hat{P}^\perp VP = 0$ as an operator. Hence, all summands except the last one vanish. Thus, we have

$$\mathfrak{i}_T(\rho) = \operatorname{tr}\left[\hat{P}^\perp VP^\perp\rho P^\perp V^\dagger\right] = \operatorname{tr}\left[V^\dagger\hat{P}^\perp VP^\perp\rho P^\perp\right]. \tag{3.13}$$

We can now prove 5. To this end, note that if $\operatorname{tr}\left[P^\perp\rho P^\perp\right] = 0$, then the claim follows trivially. Otherwise, $\frac{P^\perp\rho P^\perp}{\operatorname{tr}[P^\perp\rho P^\perp]}$ is a density matrix and the spectral theorem implies that

$$\frac{P^\perp\rho P^\perp}{\operatorname{tr}\left[P^\perp\rho P^\perp\right]} = \sum_i p_i|\psi_i^\perp\rangle\langle\psi_i^\perp|,$$

with $p_i \geq 0$, $\sum_i p_i = 1$ and $\psi_i^\perp \in \mathcal{V}_T^\perp$. By convexity, we have

$$\begin{aligned}
\operatorname{tr}\left[\hat{P}^\perp VP^\perp\rho P^\perp V^\dagger\right] &= \operatorname{tr}\left[P^\perp\rho\right]\operatorname{tr}\left[\hat{P}^\perp V\frac{P^\perp\rho P^\perp}{\operatorname{tr}\left[P^\perp\rho P^\perp\right]}V^\dagger\right] \\
&\geq \operatorname{tr}\left[P^\perp\rho\right]\inf_{\substack{\psi^\perp\in\mathcal{V}_T^\perp \\ \|\psi^\perp\|=1}}\operatorname{tr}\left[\hat{P}^\perp V|\psi^\perp\rangle\langle\psi^\perp|V^\dagger\right].
\end{aligned}$$

If the infimum is strictly positive, then this is the C_T we are looking for. To see that this is indeed the case, note that the set $\{\psi^\perp \in \mathcal{V}_T^\perp \mid | \left\|\psi^\perp\right\| = 1\}$ is compact. Thus, the infimum is actually a minimum. Assume for the sake of contradiction that $\operatorname{tr}\left[\hat{P}^\perp V|\psi^\perp\rangle\langle\psi^\perp|V^\dagger\right] = 0$, for some unit vector $\psi^\perp \in \mathcal{V}_T^\perp$. Then $\langle\hat{P}^\perp V\psi^\perp|\hat{P}^\perp V\psi^\perp\rangle = 0$ and consequently $\hat{P}^\perp V\psi^\perp = 0$. Hence, $V\psi^\perp \in \operatorname{supp}(\operatorname{tr}_\mathcal{H}\left[V|v\rangle\langle v|V^\dagger\right]) \otimes \mathcal{H}$ and $\psi^\perp \in \mathcal{V}_T$. As this is a contradiction, the claim follows.

To prove 6, we use Hölder's inequality for Schatten norms. Applying this inequality to the RHS of (3.13) yields

$$\mathfrak{i}_T(\rho) \leq \left\|V^\dagger\hat{P}^\perp V\right\|_\infty \left\|P^\perp\rho P^\perp\right\|_1 = \operatorname{tr}\left[P^\perp\rho\right].$$

The last equality follows, since $V^\dagger\hat{P}^\perp V$ is an orthogonal projection (and thus has norm 1) and since $P^\perp\rho P^\perp \geq 0$. This proves the claim. □

Remark 3.12. Since by the previous theorem, every subspace that is isometric w.r.t. T and contains the vacuum is contained in $\mathcal{V}_T$, checking the conditions in Theorem 2.2 reduces to checking whether

$$T_A|_{\mathcal{B}(\mathcal{V}_{T_A})} \neq T_B|_{\mathcal{B}(\mathcal{V}_{T_A})} \quad \text{or} \quad T_A|_{\mathcal{B}(\mathcal{V}_{T_B})} \neq T_B|_{\mathcal{B}(\mathcal{V}_{T_B})}. \tag{3.14}$$

This can be done efficiently, since $\mathcal{V}_{T_A}$ and $\mathcal{V}_{T_B}$ can be computed by simple linear algebraic methods.

4. The Discrimination Protocol

The main goal of this section is to prove Theorem 2.5. This is done in two steps. At first, we show how to discriminate between the identity channel and a compact set of channels, where some additional conditions are imposed on the channels under consideration. In particular, we obtain the following theorem.

Theorem 4.1. *For* $\dim(\mathcal{H}) < \infty$, *let* $\mathcal{C} \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *be a closed set of channels and let* $v \in \mathcal{H}$ *be a unit vector such that for all* $T \in \mathcal{C}$, *the state* $|v\rangle\langle v|$ *is the only state that is a fixed point of* T. *Then, there exists a constant* C *and for every* $N \in \mathbb{N}$ *an* N*-step discrimination strategy* D *and a two-valued POVM* Π *such that*

$$P_e(D, \Pi) \leq \frac{C}{N^2}, \tag{4.1}$$

where the discrimination error probability is w.r.t the sets $\{\mathrm{id}\}$ *and* $\mathcal{C}$.
Furthermore, if $T \in \mathcal{C}$ *is a channel with transmission functional* $\mathfrak{t}_T$ *and* $\mathfrak{t}_T(|v\rangle\langle v|) = 0$, *then the total transmission* $\mathfrak{T}_T(D)$ *is bounded by*

$$\mathfrak{T}_T(D) \leq \frac{C\,\|\mathfrak{t}_T\|}{N}. \tag{4.2}$$

In particular, if $\mathfrak{t}_{\mathrm{id}} = 0$ *and for all* $T \in \mathcal{C}$, T *is a channel with transmission functional* $\mathfrak{t}_T$, *with* $\mathfrak{t}_T(|v\rangle\langle v|) = 0$; *and if* $\sup_{T\in\mathcal{C}} \|\mathfrak{t}_T\| < \infty$, *then the sets* $\{\mathrm{id}\}$ *and* $\mathcal{C}$ *can be discriminated in a transmission-free manner.*

Proof. This statement is a direct consequence of Theorem 4.10 and the discussion in the paragraph "Description of the discrimination strategy." □

The second step then is to show how to reduce the general case to Theorem 4.1. This is the main content of Sect. 4.2, in which we also prove Theorem 2.5.

4.1. Empty or Not?

In this section we study a special case of the general discrimination task. That is, we study the case where we want to discriminate between the identity channel (empty box) and a compact set of channels $\mathcal{C} \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$, which does not contain the identity channel. We show that under some conditions on the spectrum of the channels in $\mathcal{C}$ and on the transmission functionals, a Kwiat et al.-like strategy suffices to perform the task in a transmission-free manner, even if the underlying Hilbert space is infinite-dimensional. In the finite-dimensional case, our considerations reduce to Theorem 4.1. Before we

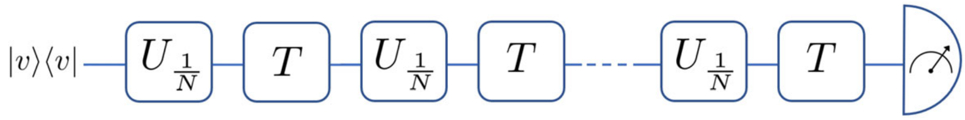

FIGURE 6. General form of a Kwiat et al.-like strategy

go into detail on what we mean by a Kwiat et al.-like strategy, we give an overview of the additional conditions we impose on the channels in $\mathcal{C}$.

Outline of the Assumptions Our first assumption is that there is a pure state $|v\rangle\langle v| \in \mathcal{S}(\mathcal{H})$ (vacuum) that is a fixed point of all channels in $\mathcal{C}$ and that the transmission functionals satisfy $\mathfrak{t}_T(|v\rangle\langle v|) = 0$ for all $T \in \mathcal{C}$. As a remark, note that if there were no state $\rho \in \mathcal{S}(\mathcal{H})$, with $\mathfrak{t}_T(\rho) = 0$ for all $T \in \mathcal{C}$, then, of course, the discrimination task is impossible. On the other hand, if there exists such a state ρ, then, by the spectral theorem and the linearity and positivity of $\mathfrak{t}_T$, there exists a pure state $\rho_v \in \mathcal{S}(\mathcal{H})$, with $\mathfrak{t}_T(\rho_v) = 0$ for all $T \in \mathcal{C}$. But then, if ρ_v is not a fixed point of T, the discrimination task becomes trivial. Thus, assuming a pure fixed point for the current setting is not a strong assumption.

Our second assumption is that all channels in $\mathcal{C}$ have a spectral gap. That is, if we exclude 1 from the spectrum of T, then the remaining part must be contained in a disk of radius less than 1 (remember that since T is a channel, its spectral radius is 1 and 1 is part of the spectrum). In Remark 4.11, we show that the spectral gap assumption cannot be waived completely if a Kwiat et al.-like protocol (defined below) should succeed.

Our third assumption is that the spectral gap assumption is compatible in a certain sense with the discrimination strategy. Expression (4.9) in the statement of Theorem 4.5 makes this statement precise. A sufficient condition for the compatibility assumption to be fulfilled (given our second assumption) is that 1 is a simple eigenvalue of every channel in $\mathcal{C}$. This is the content of Theorem 4.6. Furthermore, in the finite-dimensional case our second assumption is automatically fulfilled (given our first assumption) if 1 is a simple eigenvalue of every channel in $\mathcal{C}$. This is the content of Theorem 4.10.

Our fourth assumption concerns the relation between the channels in $\mathcal{C}$ and their associated transmission functionals. Note that the definition of a transmission functional (Definition 3.5) does not impose such a relation. For our current purpose, however, this is problematic since $\sup_{T\in\mathcal{C}} \|\mathfrak{t}_T\|$ may be infinite. We will thus assume that $\sup_{T\in\mathcal{C}} \|\mathfrak{t}_T\|$ is finite. This is a very mild assumption, since it is implied if $\mathfrak{t}_T$ depends continuously on T (which is very reasonable on physical grounds). Furthermore, note that if $\mathfrak{t}_T$ is an "interaction" functional, then, as a consequence of Claim 6 in Lemma 3.10, we have $\sup_{T\in\mathcal{C}} \|\mathfrak{t}_T\| \leq 1$.

Description of the Discrimination Strategy The next step is to design a strategy that allows us to discriminate between the identity channel and $\mathcal{C}$. An important factor in designing a strategy is the amount of resources that are needed to implement it. To this end, we show that only a bare minimum is required. Let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that v is not an eigenvector of e^{-iH}. In other words, we assume that $C_H := \left|\langle v|e^{-iH}v\rangle\right|$ is strictly less than 1. Then, our strategy is to repeat the N-step discrimination strategy, depicted in Fig. 6, a total of K times. More precisely, upon defining the 1-parameter family of channels $U_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ by $U_t(\cdot) = e^{-iHt} \cdot e^{iHt}$, the discrimination strategy is given by the initial state $s_0 := |v\rangle\langle v|$ and the set of channels Λ, with $\Lambda_i := U_{\frac{1}{N}}$ for $0 \le i \le N-1$ and $\Lambda_N := \mathrm{id}$. After each execution of the discrimination strategy, we perform a measurement described by the two-valued POVM $\{P^\perp, |v\rangle\langle v|\}$, where $P^\perp := \mathbb{1} - |v\rangle\langle v|$. If all K outcomes correspond to the second event, then we decide that the unknown channel is in $\mathcal{C}$ and if otherwise we decide that the unknown channel is the identity. Of course, this protocol can be cast into the form of an NK-step discrimination strategy by using an ancillary system and the principle of deferred measurement (see [25], p. 186). We call this strategy $D_{H,N,K}$. By Definition 2.4, the error probability is then given by

$$P_e(D_{H,N,K}, \Pi) = \frac{1}{2}\left(\mathrm{tr}\left[|v\rangle\langle v|\rho_N^{\mathrm{id}}\right]^K + \sup_{T\in\mathcal{C}}\left\{\mathrm{tr}\left[P^\perp \rho_N^T\right]\sum_{k=0}^{K-1}\mathrm{tr}\left[|v\rangle\langle v|\rho_N^T\right]^k\right\}\right),$$

where ρ is the intermediate state map and where Π denotes the measurement scheme described above. Explicitly, we have

$$\rho_N^{\mathrm{id}} = U_{\frac{1}{N}}^N(|v\rangle\langle v|) = e^{-iH}|v\rangle\langle v|e^{iH} \quad \text{and} \quad \rho_N^T = (T\circ U_{\frac{1}{N}})^N(|v\rangle\langle v|).$$

In general, this leads to the estimate

$$P_e(D_{H,N,K}, \Pi) \le \frac{1}{2}\left(C_H^{2K} + K\sup_{T\in\mathcal{C}}\mathrm{tr}\left[P^\perp\rho_N^T\right]\right). \tag{4.3}$$

Now suppose that $P_M := \sup_{T\in\mathcal{C}}\mathrm{tr}\left[P^\perp\rho_N^T\right]$ approaches zero as $N\to\infty$ (we show this below). Then, for given $\epsilon > 0$, we can choose $K := \left\lceil\frac{\ln(\epsilon)}{\ln(C_H)}\right\rceil$ and N such that $KP_M < \epsilon$. It follows from (4.3) that $P_e(D_{H,N,K}, \Pi) < \epsilon$. In other words, $P_e(D_{H,N,K}, \Pi)$ approaches zero if and only if P_M does. Furthermore, for a channel $T \in \mathcal{C}$, the total transmission is given by

$$\mathfrak{T}_T(D_{H,N,K}) = K\sum_{n=0}^{N-1}\mathfrak{t}_T\left(\rho_n^T\right) = K\mathfrak{T}_T(D_{H,N,1}).$$

Thus, also $\mathfrak{T}_T(D_{H,N,K})$ approaches zero if and only if $\mathfrak{T}_T(D_{H,N,1})$ does. In addition to that, we could always choose H such that $\langle v|e^{-iH}v\rangle = 0$. In that case, it suffices to set $K = 1$, which yields the simple expression

$$P_e(D_{H,N,1}, \Pi) = \frac{1}{2}\mathrm{tr}\left[P^\perp(T\circ U_{\frac{1}{N}})^N(|v\rangle\langle v|)\right],$$

for the error probability. Hence, in order to find a strategy that discriminates between the identity channel and the set $\mathcal{C}$, we only need to show that the quantities P_M and $\sup_{T\in\mathcal{C}}\mathfrak{T}_T(D_{H,N,1})$ approach zero for $N\to\infty$. Moreover, since $\mathfrak{t}_T$ can be written in the form $\mathfrak{t}_T(\cdot) = \mathrm{tr}\left[\Theta_T\cdot\right]$ for some positive semi-definite operator $\Theta_T \in \mathcal{B}(\mathcal{H})$ and since, by assumption $\mathfrak{t}_T(|v\rangle\langle v|) = 0$, we can conclude that for $\rho \geq 0$,

$$\mathfrak{t}_T(\rho) \leq \|\mathfrak{t}_T\| \operatorname{tr}\left[P^\perp \rho\right].$$

The important conclusion that we draw from the discussion above is that in order to prove Theorem 4.1, it suffices to show (under the hypotheses of Theorem 4.1) that for any self-adjoint $H \in \mathcal{B}(\mathcal{H})$, there is a constant C such that the inequalities

$$\operatorname{tr}\left[P^\perp (T \circ U_{\frac{1}{N}})^N(|v\rangle\langle v|)\right] \leq \frac{C}{N^2}, \tag{4.4}$$

$$\operatorname{tr}\left[P^\perp \sum_{n=0}^{N-1} (U_{\frac{1}{N}} \circ T)^n(|v\rangle\langle v|)\right] \leq \frac{C}{N}, \tag{4.5}$$

hold for all $N \in \mathbb{N}$. This is precisely the statement of Theorem 4.10. Taking the validity of Theorem 4.10 for granted, we conclude that Theorem 4.1 holds.

Technical Theorems The remainder of this section is devoted to the proof of Theorem 4.10 and its infinite-dimensional versions. The following lemmas serve this purpose.

Lemma 4.2 ([26], p. 202). *Let $T \in \mathcal{B}(\mathcal{H})$, let $z \in \mathbb{C}$ be in the unbounded component of the resolvent $\rho(T)$, and let X be a closed invariant subspace of T. Then, X is an invariant subspace of $(z - T)^{-1}$.*

Lemma 4.3. *Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel such that 1 is in the discrete spectrum of T. Then, for any $n \in \mathbb{N}$ and any (rectifiable) path inside the resolvent set of T that encloses 1, and separates 1 from $\sigma(T) \setminus \{1\}$, we have*

$$\frac{1}{2\pi i} \oint\limits_{\Gamma_1} \frac{z^n}{z - T} \, \mathrm{d}z = \frac{1}{2\pi i} \oint\limits_{\Gamma_1} \frac{1}{z - T} \, \mathrm{d}z. \tag{4.6}$$

Proof. See "Appendix A". □

Lemma 4.4 (Invariant subspace lemma). *Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel, where $\mathcal{H}$ can be finite or infinite dimensional. Let $v \in \mathcal{H}$ be such that $|v\rangle\langle v|$ is a fixed point of T and set $V_v := \operatorname{span}\{v\}$. Then, the subspaces*

$$\mathcal{B}_{v\perp} := \left\{|v\rangle\langle\phi| \,\middle|\, \phi \in V_v^\perp\right\}, \tag{4.7}$$

$$\mathcal{B}_{\perp v} := \left\{|\phi\rangle\langle v| \,\middle|\, \phi \in V_v^\perp\right\}, \tag{4.8}$$

are invariant under T.

Proof. We prove that $\mathcal{B}_{v\perp}$ is invariant. The invariance of $\mathcal{B}_{\perp v}$ follows as T is Hermiticity-preserving. Let $\{K_i\}$ be a set of (non-zero) Kraus-operators of T. By assumption we have

$$|v\rangle\langle v| = T(|v\rangle\langle v|) = \sum_i \operatorname{tr}\left[K_i^\dagger K_i\right] \frac{K_i|v\rangle\langle v|K_i^\dagger}{\operatorname{tr}\left[K_i^\dagger K_i\right]},$$

where the series converges in trace norm. As the pure state $|v\rangle\langle v|$ is an extreme point of the closed and convex set of quantum states and the RHS is a convex combination of states, we must have that $K_i|v\rangle\langle v|K_i^\dagger$ is proportional to $|v\rangle\langle v|$. Henceforth, v is an eigenvector of K_i for all i. We denote the corresponding eigenvalue by λ_i. So for $\psi \in V_v^\perp$, we get

$$T(|v\rangle\langle\psi|) = \sum_i K_i|v\rangle\langle\psi|K_i^\dagger = |v\rangle\langle\phi|,$$

where $\phi = \sum_i \overline{\lambda_i} K_i\,\psi$. As T is trace-preserving, we have

$$0 = \mathrm{tr}\left[|v\rangle\langle\phi|\right] = \mathrm{tr}\left[T(|v\rangle\langle\psi|)\right] = \mathrm{tr}\left[|v\rangle\langle\phi|\right] = \langle\phi|v\rangle.$$

Hence, $\phi \in V_v^\perp$. This proves the claim. □

The following theorem is the main technical result. In fact, everything else in this section can (to some extent) be regarded as a corollary to this theorem.

Theorem 4.5. *Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel such that 1 is in the discrete spectrum of T, and let $v \in \mathcal{H}$ be a unit vector such that $|v\rangle\langle v|$ is a fixed point of T. Furthermore, let $H \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $\tau > 0$ and $0 < \delta < 1$ such that*

$$\sigma(U_t \circ T) \subseteq \mathbb{D}_{1-\delta}(0) \cup \{1\}, \tag{4.9}$$

for $0 \le t \le \tau$, where $U_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is defined by $U_t(\cdot) := e^{-iHt} \cdot e^{iHt}$. Then, the inequalities

$$\mathrm{tr}\left[P^\perp (T \circ U_{\frac{1}{N}})^N(|v\rangle\langle v|)\right] \le \frac{C}{N^2}, \tag{4.10}$$

$$\mathrm{tr}\left[P^\perp \sum_{n=0}^{N-1} (U_{\frac{1}{N}} \circ T)^n(|v\rangle\langle v|)\right] \le \frac{C}{N}, \tag{4.11}$$

hold for all $N \in \mathbb{N}$. Here, $P^\perp := \mathbb{1} - |v\rangle\langle v|$ and

$$C := \max\left\{\tau^{-2},\ 18\delta^{-1}\,\|H\|^2_{\mathcal{B}(\mathcal{H})} \max_{\substack{0\le t\le\tau\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\| \left\|(z-U_tT)^{-1}\right\|\right\} < \infty,$$

where $\Gamma := \left\{z \in \mathbb{C} \,\middle|\, |z| = 1 - \frac{\delta}{2}\right\} \cup \left\{z \in \mathbb{C} \,\middle|\, |z-1| = \frac{\delta}{2}\right\}$.

Proof. We need to calculate the quantities (4.10) and (4.11). To do so, we employ the holomorphic functional calculus. For $0 \le t \le \tau$ and $n \in \mathbb{N}$, we have

$$(U_tT)^n = \frac{1}{2\pi i}\oint_{|z-1|=\frac{\delta}{2}} \frac{z^n}{z-U_tT}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{|z|=1-\frac{\delta}{2}} \frac{z^n}{z-U_tT}\,\mathrm{d}z \tag{4.12}$$

$$= \frac{1}{2\pi i}\oint_{|z-1|=\frac{\delta}{2}} \frac{1}{z-U_tT}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{|z|=1-\frac{\delta}{2}} \frac{z^n}{z-U_tT}\,\mathrm{d}z, \tag{4.13}$$

where we used Lemma 4.3 to obtain the second line. Under the trace, we can (crudely) estimate this term as follows:

$$\begin{aligned}\left|\mathrm{tr}\left[P^{\perp}(U_tT)^n(|v\rangle\langle v|)\right]\right| &\leq \frac{\delta}{2}\max_{|z-1|=\frac{\delta}{2}}\left|\mathrm{tr}\left[P^{\perp}\frac{1}{z-U_tT}(|v\rangle\langle v|)\right]\right| \\ &\quad + \left(1-\frac{\delta}{2}\right)^{n+1}\max_{|z|=1-\frac{\delta}{2}}\left|\mathrm{tr}\left[P^{\perp}\frac{1}{z-U_tT}(|v\rangle\langle v|)\right]\right| \\ &\leq \max_{z\in\Gamma}\left|\mathrm{tr}\left[P^{\perp}\frac{1}{z-U_tT}(|v\rangle\langle v|)\right]\right|.\end{aligned} \tag{4.14}$$

In everything that follows, we assume that $z \in \Gamma$. To proceed, we need two auxiliary calculations. First, we use the second resolvent identity ([27], p. 84) twice to obtain

$$\begin{aligned}\frac{1}{z-U_tT} &= \frac{1}{z-T} + \frac{1}{z-T}(U_t-\mathrm{id})\frac{T}{z-T} \\ &\quad + \frac{1}{z-U_tT}(U_t-\mathrm{id})\frac{T}{z-T}(U_t-\mathrm{id})\frac{T}{z-T}.\end{aligned} \tag{4.15}$$

Second, an elementary application of Taylor's formula yields

$$\|U_t-\mathrm{id}\| \leq 2\,\|H\|_{\mathcal{B}(\mathcal{H})}\,t, \tag{4.16}$$

$$(U_t-\mathrm{id})(\rho) = i[\rho,H]t + \mathfrak{U}t^2, \tag{4.17}$$

with $||\mathfrak{U}|| \leq 2\,\|H\|^2_{\mathcal{B}(\mathcal{H})}$. When looking at (4.15), it is clear that the summands are of zeroth, first and second order in t, as $t \to 0$. The crucial step is to show that under the trace, the second term is $\mathcal{O}(t^2)$. Using (4.17), we get

$$\begin{aligned}\frac{1}{z-T}(U_t-\mathrm{id})\frac{T}{z-T}(|v\rangle\langle v|) &= \frac{1}{z-1}\frac{1}{z-T}(U_t-\mathrm{id})(|v\rangle\langle v|) \\ &= \frac{it}{z-1}\frac{1}{z-T}(|v\rangle\langle Hv| - |Hv\rangle\langle v|) \\ &\quad + \frac{t^2}{z-1}\frac{1}{z-T}(\mathfrak{U}(|v\rangle\langle v|)).\end{aligned} \tag{4.18}$$

It is easily verified, using the self-adjointness of H, that $|v\rangle\langle Hv| - |Hv\rangle\langle v| = |v\rangle\langle\phi| - |\phi\rangle\langle v|$, with $\phi := (H - \langle v|Hv\rangle)v$. Clearly, $\langle\phi|v\rangle = 0$. Thus, $|\phi\rangle\langle v| \in \mathcal{B}_{\perp v}$ and $|v\rangle\langle\phi| \in \mathcal{B}_{v\perp}$, where $\mathcal{B}_{\perp v}$ and $\mathcal{B}_{v\perp}$ are both invariant subspaces of T (by Lemma 4.4). As z is in the unbounded component of the resolvent set of T, Lemma 4.2 implies that also $(z-T)^{-1}(|\phi\rangle\langle v|) \in \mathcal{B}_{\perp v}$ and $(z-T)^{-1}(|v\rangle\langle\phi|) \in \mathcal{B}_{v\perp}$. Thus, the first term in (4.18) vanishes under the trace, and we get

$$\left|\mathrm{tr}\left[P^{\perp}(4.18)\right]\right| \leq t^2\frac{2\,\|H\|^2_{\mathcal{B}(\mathcal{H})}}{|z-1|}\left\|(z-T)^{-1}\right\|. \tag{4.19}$$

So under the trace, this term is indeed quadratic in t. For the other two terms in (4.15), we have

$$\left|\mathrm{tr}\left[P^{\perp}\frac{1}{z-T}(|v\rangle\langle v|)\right]\right| = \frac{1}{|z-1|}\mathrm{tr}\left[P^{\perp}|v\rangle\langle v|)\right] = 0 \tag{4.20}$$

and

$$\begin{aligned}
&\left|\operatorname{tr}\left[P^\perp \frac{1}{z-U_tT}(U_t-\mathrm{id})\frac{T}{z-T}(U_t-\mathrm{id})\frac{T}{z-T}(|v\rangle\langle v|)\right]\right| \\
&\quad\le \frac{1}{|z-1|}\left\|(z-U_tT)^{-1}\right\|\left\|U_t-\mathrm{id}\right\|^2\left\|\frac{T}{z-T}\right\| \\
&\quad\le t^2\frac{4\left\|H\right\|^2_{\mathcal{B}(\mathcal{H})}}{|z-1|}\left\|(z-U_tT)^{-1}\right\|\left\|(z-T)^{-1}\right\|, \qquad (4.21)
\end{aligned}$$

where we used the estimate (4.16) and $\|T\|=1$ to obtain the last line. We can now use the results obtained in (4.19), (4.20), and (4.21) to estimate the quantity of interest, (4.14). We have

$$\begin{aligned}
(4.14) &\le 2t^2\left\|H\right\|^2_{\mathcal{B}(\mathcal{H})}\max_{z\in\Gamma}\frac{\left\|(z-T)^{-1}\right\|\left(1+2\left\|(z-U_tT)^{-1}\right\|\right)}{|z-1|} \\
&\le t^2\left(18\delta^{-1}\left\|H\right\|^2_{\mathcal{B}(\mathcal{H})}\max_{\substack{0\le t'\le\tau\\ z\in\Gamma}}\left\|(z-T)^{-1}\right\|\left\|(z-U_{t'}T)^{-1}\right\|\right) \\
&=: t^2C_0. \qquad (4.22)
\end{aligned}$$

To obtain the second estimate, we used $\max_{z\in\Gamma}|z-1|^{-1}=2\delta^{-1}$ and $\left\|(z-U_tT)^{-1}\right\|\ge\left\|(z-U_tT)\right\|^{-1}\ge(|z|+1)^{-1}\ge\frac{2}{5}$. Equation (4.22) is a bound for $t\le\tau$. To prove the theorem, we need a bound for all $t\ge 0$. To this end, we note that $\operatorname{tr}\left[P^\perp(U_tT)^n(|v\rangle\langle v|)\right]\le 1$, since the expression represents a probability. We further define $C:=\max(\tau^{-2},C_0)$. If $t\le\tau$, then by Eq. (4.22),

$$\operatorname{tr}\left[P^\perp(U_tT)^n(|v\rangle\langle v|)\right]\le t^2C_0\le Ct^2.$$

And if $t>\tau$, then

$$\operatorname{tr}\left[P^\perp(U_tT)^n(|v\rangle\langle v|)\right]\le 1\le\frac{t^2}{\tau^2}\le Ct^2.$$

Hence,

$$\operatorname{tr}\left[P^\perp(U_tT)^n(|v\rangle\langle v|)\right]\le Ct^2,$$

for all $t\ge 0$. This is a bound independent of n. Inequality (4.11) is then easily obtained by setting $t:=\frac{1}{N}$ and summing over all n, which yields an additional factor N. It remains to show inequality (4.10), in which U_t and T have switched order. Since $|v\rangle\langle v|$ is a fixed point of T, we have $\operatorname{tr}\left[P^\perp(TU_t)^N(|v\rangle\langle v|)\right]=\operatorname{tr}\left[P^\perp T(U_tT)^N(|v\rangle\langle v|)\right]$. We set $\rho:=(U_tT)^N(|v\rangle\langle v|)$ and $\phi:=P^\perp\rho v$ and write

$$\rho=\langle v|\rho v\rangle|v\rangle\langle v|+|v\rangle\langle\phi|+|\phi\rangle\langle v|+P^\perp\rho P^\perp.$$

Clearly, $|v\rangle\langle\phi|\in\mathcal{B}_{v\perp}$ and $|\phi\rangle\langle v|\in\mathcal{B}_{\perp v}$. Hence, by Lemma 4.4, we have $T(|v\rangle\langle\phi|)\in\mathcal{B}_{v\perp}$ and $T(|\phi\rangle\langle v|)\in\mathcal{B}_{\perp v}$. Thus,

$$\operatorname{tr}\left[P^\perp T(\rho)\right]=\operatorname{tr}\left[P^\perp T(P^\perp\rho P^\perp)\right]\le\operatorname{tr}\left[T(P^\perp\rho P^\perp)\right]=\operatorname{tr}\left[P^\perp\rho\right].$$

Hence,

$$\mathrm{tr}\left[P^{\perp}(TU_t)^N(|v\rangle\langle v|)\right] \leq \frac{C}{N^2}.$$

This finishes the proof. □

Theorem 4.6. *Let $\mathcal{C} \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ be a compact set of channels, and let $v \in \mathcal{H}$ be a unit vector. Assume that*

1. *For all $T \in \mathcal{C}$, the quantity*
$$r_T := \sup_{z\in\sigma(T)\setminus\{1\}} |z|$$
is strictly less than 1. In other words, the spectral gap is nonzero.
2. *For each $T \in \mathcal{C}$, the state $|v\rangle\langle v|$ is a fixed point of T.*
3. *For all $T \in \mathcal{C}$, the algebraic multiplicity[12] of the isolated point $1 \in \sigma(T)$ is 1. In other words, 1 is a simple eigenvalue.*

Furthermore, let $H \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $U_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be defined by $U_t(\cdot) = e^{-iHt} \cdot e^{iHt}$. Then, there exists a constant $C_{\mathcal{C}} < \infty$, such that

$$\mathrm{tr}\left[P^{\perp}\left(T \circ U_{\frac{1}{N}}\right)^N(|v\rangle\langle v|)\right] \leq \frac{C_{\mathcal{C}}\,\|H\|^2_{\mathcal{B}(\mathcal{H})}}{N^2}, \tag{4.23}$$

$$\mathrm{tr}\left[P^{\perp}\sum_{n=0}^{N-1}\left(U_{\frac{1}{N}} \circ T\right)^n(|v\rangle\langle v|)\right] \leq \frac{C_{\mathcal{C}}\,\|H\|^2_{\mathcal{B}(\mathcal{H})}}{N}, \tag{4.24}$$

for all $N \in \mathbb{N}$, where $P^{\perp} := \mathbb{1} - |v\rangle\langle v|$.

Remark 4.7. The distinctive feature of the preceding theorems is the N^{-2} bound in (4.23). It seems that such a bound cannot be obtained directly from the results in [13–16], because those results are of the form $\left(U_{\frac{1}{N}} \circ T\right)^N \approx \mathcal{P} + \mathcal{O}(\frac{1}{N})$ for $N \to \infty$, where $\mathcal{P}$ denotes the spectral projection on the eigenspace with eigenvalue 1.

Proof. The basic strategy is to reduce the claim to an application of Theorem 4.5. To this end, we basically need to show that Conditions 1–3 imply that condition (4.9) can be satisfied uniformly, i.e., that there exist $\tau > 0$ and $0 < \delta < 1$ such that (4.9) is satisfied for all $T \in \mathcal{C}$. The main tool to show this is the upper semi-continuity of the spectrum. To use that property, we import the following two theorems.

Theorem 4.8 ([28], p. 208). *For a Banach space $\mathcal{X}$, let $T, S \in \mathcal{B}(\mathcal{X})$, and let Γ be a compact subset of the resolvent set $\rho(T)$.*

If $\|T - S\| < \min_{z\in\Gamma}\left\|(z - T)^{-1}\right\|^{-1}$, then $\Gamma \subseteq \rho(S)$. Furthermore, for any open set $V \subseteq \mathbb{C}$, with $\sigma(T) \subset V$, there exists $\gamma > 0$, such that $\sigma(S) \subseteq V$ whenever $\|S - T\| < \gamma$.

[12] For an isolated point $\lambda \in \sigma(T)$, the algebraic multiplicity is the dimension of the range of the spectral projection.

Theorem 4.9 ([29], p. 67). *For a Banach space $\mathcal{X}$, let $P, Q \in \mathcal{B}(\mathcal{X})$ be bounded projections with $\|P - Q\| < 1$. Then, there exists an invertible operator $A \in \mathcal{B}(\mathcal{X})$, such that $Q = APA^{-1}$. In particular* $\operatorname{ran}(P)$ *and* $\operatorname{ran}(Q)$ *are isomorphic.*

To start, we show that not only $r_T < 1$ for all $T \in \mathcal{C}$, but that $\sup_{T\in\mathcal{C}} r_T < 1$. To this end, we show that the function $r : \mathcal{C} \to \mathbb{R}$, $T \mapsto r_T$ is upper semi-continuous. That is, we need to show that for every $T \in \mathcal{C}$ and every $\epsilon > 0$, there is a set $U \subseteq \mathcal{C}$, which is open in the relative topology on $\mathcal{C}$, such that $r_S \le r_T + \epsilon$ for all $S \in U$. For fixed T and $\epsilon > 0$, define $\epsilon' := \min(\epsilon, \frac{1-r_T}{3})$ and the open set $V_{\epsilon'} := B_{r_T+\epsilon'}(0) \cup B_{\epsilon'}(1) \subseteq \mathbb{C}$. By construction, $\sigma(T) \subseteq V_{\epsilon'}$. Thus, Theorem 4.8 implies that there exists $\gamma > 0$ such that $\sigma(S) \subseteq V_{\epsilon'}$ for all $S \in B_\gamma(T)$. Thus, for $S \in B_\gamma(T)$ the projection P_S onto the spectral subspace associated with the spectral subset $\sigma(S) \cap B_{\epsilon'}(1)$ is given by

$$P_S := \frac{1}{2\pi i} \oint_{|z-1|=\frac{1-r_T}{2}} \frac{1}{z - T_n}\, \mathrm{d}z = P_T + \frac{1}{2\pi i} \oint_{|z-1|=\frac{1-r_T}{2}} \frac{1}{z-S}(S-T)\frac{1}{z-T}\, \mathrm{d}z,$$

where we used the second resolvent identity to obtain the last equation. A standard estimate yields

$$\|P_S - P_T\| \le \frac{1 - r_T}{2} \|S - T\| \max_{|z-1|=\frac{1-r_T}{2}} \left\{\left\|(z-S)^{-1}\right\| \left\|(z-T)^{-1}\right\|\right\}.$$

Since the set $S_0 := \overline{B_{\frac{\gamma}{2}}(T)} \cap \mathcal{C}$ is compact, the constant

$$C_0 := \max_{\substack{|z-1|=\frac{1-r_T}{2} \\ S\in S_0}} \left\{\left\|(z-S)^{-1}\right\| \left\|(z-T)^{-1}\right\|\right\}$$

is finite. We set $\gamma' := \min\{\frac{\gamma}{2}, \frac{1}{(1-r_T)C_0}\}$ and $U := B_{\gamma'}(T) \cap \mathcal{C}$. By construction, U is open in the relative topology on $\mathcal{C}$ and we have $\sigma(S) \subseteq V_{\epsilon'}$ and $\|P_S - P_T\| \le \frac{1}{2} < 1$ for all $S \in U$. By Assumption 3, $\operatorname{ran}(P_T)$ is 1-dimensional. Thus, by Theorem 4.9, also $\operatorname{ran}(P_S)$ is one-dimensional, for $S \in U$. Thus, there can be only one point in $\sigma(S) \cap B_{\epsilon'}(1)$, and this point must be 1, as 1 is in the spectrum of every channel. Hence, for $S \in U$, we have $\sigma(S) \setminus \{1\} \subseteq B_{r_T+\epsilon'}(0)$. So $r(S) = r_S \le r_T + \epsilon' \le r_T + \epsilon = r(T) + \epsilon$. In other words, r is upper semi-continuous. The upper semi-continuous function r assumes its maximum on the compact set $\mathcal{C}$. This maximum cannot be equal to 1, as this would contradict Assumption 1. Thus $\max_{T\in\mathcal{C}} r_T < 1$, as claimed.
In preparation for the application of Theorem 4.5, we define the joint spectral gap

$$\delta_J := 1 - \max_{T\in\mathcal{C}} r(T). \tag{4.25}$$

We have $0 < \delta_J < 1$ and

$$\sigma(T) \subseteq \mathbb{D}_{1-\delta_J}(0) \cup \{1\},$$

for all $T \in \mathcal{C}$. We define $\Gamma := \mathbb{D}_{1+\frac{\delta_J}{3}}(0) \setminus (B_{\frac{\delta_J}{3}}(1) \cup B_{1-\frac{2\delta_J}{3}}(0))$, which is a compact subset of $\rho(T)$ for all $T \in \mathcal{C}$, and we set

$$\tau := \frac{1}{7\,\|H\|_{\mathcal{B}(\mathcal{H})}} \min_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^{-2},$$

which is nonzero, as the minimization is over a strictly positive function on a compact set. For this particular choice of τ, we show that

$$\sigma(U_tT) \subseteq D_{1-\frac{2\delta_J}{3}}(0) \cup \{1\}$$

for $0 \leq t \leq \tau$ and then use Theorem 4.5. From now on, let $0 \leq t \leq \tau$ and $T \in \mathcal{C}$. Using the Taylor estimate (4.16) and the definition of τ yields

$$\begin{aligned} \|T - U_tT\| &\leq \|U_t - \mathrm{id}\|\,\|T\| \leq 2\,\|H\|_{\mathcal{B}(\mathcal{H})}\, t \\ &\leq \frac{2}{7} \min_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^{-2}. \end{aligned} \tag{4.26}$$

This inequality has two important implications. First, for $z \in \Gamma$ we have $\left\|(z-T)^{-1}\right\|^{-1} \leq \|z-T\| \leq |z|+1 \leq \frac{7}{3}$. Hence, (4.26) $< \min_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^{-1}$ and we can apply Theorem 4.8, which tells us that $\Gamma \subseteq \rho(U_tT)$ for all $T \in \mathcal{C}$ and $0 \leq t \leq \tau$. Equivalently,

$$\sigma(U_tT) \subseteq \mathbb{D}_{1-\frac{2\delta_J}{3}}(0) \cup \mathbb{D}_{\frac{\delta_J}{3}}(1).$$

Thus, we only have to show that $\sigma(U_tT) \cap \mathbb{D}_{\frac{\delta_J}{3}}(1) = \{1\}$.

Second, $\left\|(U_tT-T)(z-T)^{-1}\right\| \leq \frac{2}{7} \min_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^{-1} \leq \frac{2}{3}$. Thus, the series

$$\frac{1}{z-T} \sum_{k=0}^{\infty} \left[(U_tT-T)(z-T)^{-1}\right]^k = (z-U_tT)^{-1}$$

converges. A term-by-term estimate yields

$$\left\|(z-U_tT)^{-1}\right\| \leq 3\left\|(z-T)^{-1}\right\|, \tag{4.27}$$

Let $P_t := \frac{1}{2\pi i} \oint_{|z-1|=\frac{\delta_J}{3}} \frac{1}{z-U_tT}\,\mathrm{d}z$ be the spectral projection, then

$$\begin{aligned} \|P_t - P_0\| &= \left\| \frac{1}{2\pi i} \oint_{|z-1|=\frac{\delta_J}{3}} \frac{1}{z-U_tT} - \frac{1}{z-T}\,\mathrm{d}z \right\| \\ &\leq \frac{\delta_J}{3} \max_{|z-1|=\frac{\delta_J}{3}} \left\|(z-U_tT)^{-1} - (z-T)^{-1}\right\| \\ &= \frac{\delta_J}{3} \max_{|z-1|=\frac{\delta_J}{3}} \left\|(z-U_tT)^{-1}(U_tT-T)(z-T)^{-1}\right\| \\ &\leq \delta_J \,\|U_tT - T\| \max_{z\in\Gamma} \left\|(z-T)^{-1}\right\|^2 \end{aligned}$$

$$\leq \frac{2\delta_J}{7} < 1,$$

where we used the second resolvent identity to obtain the third line, (4.27) for the fourth line and (4.26) for the fifth line. Hence, by Theorem 4.9, the dimension of $\mathrm{ran}(P_t)$ equals the dimension of $\mathrm{ran}(P_0)$ for all $0 \leq t \leq \tau$, and the latter dimension is 1. Thus, $\sigma(U_t T) \cap \mathbb{D}_{\frac{\delta_J}{3}}(1)$ contains exactly one point, which must be 1, as $U_t T$ is a channel. In conclusion, we have

$$\sigma(U_t T) \subseteq \mathbb{D}_{1-\delta}(0) \cup \{1\},$$

for all $T \in \mathcal{C}$ and $0 \leq t \leq \tau$, with $\delta := \frac{2\delta_J}{3}$. Finally, a direct application of Theorem 4.5 proves the claim. We can also get an explicit bound for $C_{\mathcal{C}}$. To this end, we need to bound the constant that appears in Theorem 4.5. We have

$$\tau^{-2} = 49 \, \|H\|^2_{\mathcal{B}(\mathcal{H})} \max_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^4$$

and, by (4.27), the second term can be bounded by

$$36\delta_J^{-1} \, \|H\|^2_{\mathcal{B}(\mathcal{H})} \max_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^2. \tag{4.28}$$

Furthermore, by the spectral mapping theorem, the spectral radius of $(z-T)^{-1}$ is given by $(\inf_{s\in\sigma(T)} \|z-s\|)^{-1} = (\mathrm{dist}(z,\sigma(T)))^{-1}$. Since the norm of any operator is an upper bound for the spectral radius, we have

$$\max_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\| \geq \max_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\{\mathrm{dist}(z,\sigma(T))^{-1}\right\} \geq 3\delta_J^{-1} \geq 3.$$

By applying this bound to (4.28), we see that $\tau^{-2} \geq$ (4.28). Thus, we can choose

$$C_{\mathcal{C}} := 49 \max_{\substack{T\in\mathcal{C}\\ z\in\Gamma}} \left\|(z-T)^{-1}\right\|^4 < \infty. \tag{4.29}$$

□

Theorem 4.10. *For* $\dim(\mathcal{H}) < \infty$*, let* $\mathcal{C}$ *be a closed set of channels* $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *and let* $v \in \mathcal{H}$ *be a unit vector such that for every* $T \in \mathcal{C}$*, the state* $|v\rangle\langle v|$ *is the only state that is a fixed point of* T*.*

Furthermore, let $H \in \mathcal{B}(\mathcal{H})$ *be self-adjoint and* $U_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be defined by* $U_t(\cdot) = e^{-iHt} \cdot e^{iHt}$*. Then, there exists a constant* $C_{\mathcal{C}} < \infty$*, such that for all* $N \in \mathbb{N}$*,*

$$\mathrm{tr}\left[P^{\perp}\,(T \circ U_{\frac{1}{N}})^N(|v\rangle\langle v|)\right] \leq \frac{C_{\mathcal{C}}\,\|H\|^2_{\mathcal{B}(\mathcal{H})}}{N^2} \tag{4.30}$$

$$\mathrm{tr}\left[P^{\perp}\sum_{n=0}^{N-1}(U_{\frac{1}{N}} \circ T)^n(|v\rangle\langle v|)\right] \leq \frac{C_{\mathcal{C}}\,\|H\|^2_{\mathcal{B}(\mathcal{H})}}{N}, \tag{4.31}$$

where $P^{\perp} := \mathbb{1} - |v\rangle\langle v|$*.*

Proof. The claim follows from Theorem 4.6 and from results by Burgarth and Giovannetti [30]. In particular, in their terminology, a channel T is called *ergodic* if there is a unique state that is a fixed point of T. And (according to Theorem 7 in [30]), T is called *mixing* if 1 is the only eigenvalue with modulus 1 and the eigenvalue 1 is simple. Thus, in particular, if T is mixing, then the spectral gap is nonzero. Theorem 8 in [30] says that ergodic channels are mixing if the unique state that is a fixed point is pure. By assumption, every $T \in \mathcal{C}$ is ergodic and the only state that is a fixed point is the pure state $|v\rangle\langle v|$. Thus, all $T \in \mathcal{C}$ are mixing and the conditions in Theorem 4.6 are automatically satisfied. This proves the claim. □

Remark 4.11. In the previous theorem, it is important that $|v\rangle\langle v|$ is the only state that is a fixed point. To demonstrate this, we define the Hamiltonian on a qubit system, $\mathcal{H}_Q := \operatorname{span}\{v, q_1\}$, as $H := \frac{\pi}{2}\sigma_y$, where σ_y is the Pauli matrix.[13] So, $U_t(\cdot) := e^{-iHt} \cdot e^{iHt}$. The channel $T : \mathcal{B}_1(\mathcal{H}_Q) \to \mathcal{B}_1(\mathcal{H}_Q)$ is then defined by

$$T(\cdot) := \operatorname{tr}\left[|v\rangle\langle v| \cdot\right] |v\rangle\langle v| + \operatorname{tr}\left[|q_1\rangle\langle q_1| \cdot\right] |q_1\rangle\langle q_1|.$$

It is not hard to verify by induction that

$$(U_{\frac{1}{N}} \circ T)^n = U_{\frac{1}{N}} \left(\frac{1}{2}(1 + \cos^n(2\theta))|v\rangle\langle v| + \frac{1}{2}(1 - \cos^n(2\theta))|q_1\rangle\langle q_1| \right),$$

where $\theta := \frac{\pi}{2N}$. The formula for the sum of the geometric progression yields

$$\sum_{n=0}^{N-1} (U_{\frac{1}{N}} \circ T)^n(|v\rangle\langle v|) = U_{\frac{1}{N}} \left(\frac{1}{2}\left(N + \lambda\right)|v\rangle\langle v| + \frac{1}{2}\left(N - \lambda\right)|q_1\rangle\langle q_1| \right),$$

with $\lambda := \frac{1 - \cos^N(2\theta)}{2\sin^2(\theta)}$. It is an exercise in elementary calculus (or a query in your favorite computer algebra system) to show that

$$\lim_{N\to\infty} (N - \lambda) = \frac{\pi^2}{4}. \tag{4.32}$$

Since $U_{\frac{1}{N}} \to \mathrm{id}$, when $N \to \infty$, it follows that the quantity on the RHS of (4.31) does not vanish as $N \to \infty$. In particular, our example shows that the Kwiat et al.-like protocol cannot be applied naively. Thus, the reduction process described in the next section is needed in some cases.

Remark 4.12. If the channel in Theorem 4.10 is a qubit channel ($\mathcal{H} = \operatorname{span}\{v, p\}$), then one can determine the precise asymptotics in a rather tedious calculation.

[13] In coordinates, $\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $e^{-iHt} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, with $\theta := \frac{\pi t}{2}$

We only state the result, which is that if $H := \frac{\pi}{2}\sigma_y$, then

$$\lim_{N\to\infty} N^2 \mathrm{tr}\left[P^\perp \left(T \circ U_{\frac{1}{N}}\right)^N (|v\rangle\langle v|)\right] = \lim_{N\to\infty} N \mathrm{tr}\left[P^\perp \sum_{n=0}^{N-1} (U_{\frac{1}{N}} \circ T)^n (|v\rangle\langle v|)\right]$$
$$= \frac{\pi^2}{4} \frac{1 - |\tau_0|^2}{(1-\tau)\,|1-\tau_0|^2}, \tag{4.33}$$

where $\tau := \mathrm{tr}\left[P^\perp T(P^\perp)\right]$ and $\tau_0 := \mathrm{tr}\left[|p\rangle\langle v|\, T(|v\rangle\langle p|)\right]$.
This result contains as a special case the result for semi-transparent objects [31,32].

Remark 4.13. It is a direct consequence of the results in the next section that the N^{-1} form of the bound is optimal.

4.2. The Reduction Protocol

In this section, in which we assume that all Hilbert spaces are finite-dimensional, we want to transform our given channel in such a way that the Kwiat et al.-like strategy, which was described in the previous section, can be applied. The general idea is that instead of inserting the unknown channel directly into the circuit of Fig. 6, we preprocess and postprocess the states that go in and out of the channel. In other words, we replace the channel T in Fig. 6 by the construction that is depicted on the RHS of Fig. 7. In Fig. 7, $\mathcal{H}_Q$ and $\mathcal{H}_A$ are Hilbert spaces and $R_0 : \mathcal{B}_1(\mathcal{H}_Q) \to \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_A)$ and $R_0' : \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_A) \to \mathcal{B}_1(\mathcal{H}_Q)$ are channels. The resulting transformation can be viewed as a map $R : \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \to \mathcal{B}(\mathcal{B}_1(\mathcal{H}_Q))$, defined by $R(T) := R_0'(T \otimes \mathrm{id})R_0$. Maps of this kind are usually called superchannels [33]. Clearly, if T is a channel with transmission functional $\mathfrak{t}_T$, then $R(T)$ is a channel with transmission functional $\mathfrak{t}_{R(T)} := \mathfrak{t}_T \circ \mathrm{tr}_A \circ R_0$. We say that the superchannel R transforms the transmission functional $\mathfrak{t}_T$ to $\mathfrak{t}_{R(T)}$. For consistency reasons, we also remark the following: As is shown in [33], for any superchannel $S : \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \to \mathcal{B}(\mathcal{B}_1(\mathcal{H}_Q))$, there exists a Hilbert space $\mathcal{H}_{A'}$ and channels $S_0 : \mathcal{B}_1(\mathcal{H}_Q) \to \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_{A'})$ and $S_0' : \mathcal{B}_1(\mathcal{H} \otimes \mathcal{H}_{A'}) \to \mathcal{B}_1(\mathcal{H}_Q)$ such that $S(T) = S_0'(T \otimes \mathrm{id})S_0$ for all $T \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$. Of course, the choice of $\mathcal{H}_{A'}$, S_0, and S_0' is not unique. The transformation of the transmission functional, however, is unique. To see this, assume that we apply S to the map T_B, defined by $T_B(\cdot) = \mathrm{tr}\,[B\,\cdot\,]\,\rho_0$, where $\rho_0 \in \mathcal{S}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ are arbitrary. Since S_0' is trace-preserving, we have for $\sigma \in \mathcal{B}(\mathcal{H}_Q)$, that $\mathrm{tr}\,[S(T)(\sigma)] = \mathrm{tr}\,[(T \otimes \mathrm{id})S_0(\sigma)] = \mathrm{tr}\,[B \mathrm{tr}_{A'}[S_0(\sigma)]]$. Since B and σ were arbitrary, it follows that $\mathrm{tr}_{A'} \circ S_0$ is independent of the choice of $\mathcal{H}_{A'}$, S_0, and S_0'. Hence, the transformation of the transmission functional is independent of the particular implementation of a superchannel. Formally, the replacement described above yields a transformation of the discrimination strategy. That is, given a discrimination strategy $D = (\mathcal{H}_Q, \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda)$, with $\Lambda = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_N\}$, then we obtain the transformed discrimination strategy $D^R := (\mathcal{H}, \mathcal{H}_A \otimes \mathcal{H}_Z, \mathcal{H}_i, \mathcal{H}_o, s_0, \Lambda_R)$, with $\Lambda_0^R := (R_0 \otimes \mathrm{id}_Z)\Lambda_0$, $\Lambda_N^R := \Lambda_N(R_0' \otimes \mathrm{id}_Z)$, and $\Lambda_n^R := (R_0 \otimes \mathrm{id}_Z)\Lambda_n(R_0' \otimes \mathrm{id}_Z)$, for $1 \le n \le N-1$.

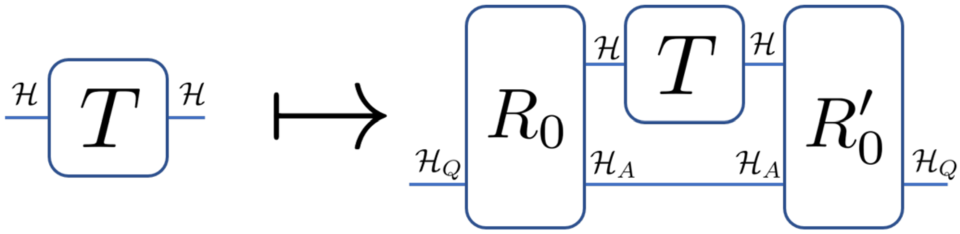

FIGURE 7. General transformation scheme: a superchannel

The task of this section is to show the existence of a superchannel such that the general discrimination task reduces to the one described in the last section. It will be evident from the proof of the following theorem that such a superchannel can be implemented by using only one ancillary qubit and classical resources. Furthermore, we show in Remark 4.18 that in general the implementation of such a superchannel is impossible without using an ancillary qubit.

Theorem 4.14 (Reduction superchannel). *For* $\dim(\mathcal{H}) < \infty$, *let* $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be a channel and let* $\mathcal{V} \subseteq \mathcal{H}$ *be a subspace such that* T *is isometric on* $\mathcal{V}$. *Furthermore, let* $v \in \mathcal{V}$ *be a unit vector. Then, there exists a two-dimensional Hilbert space* $\mathcal{H}_Q$, *with orthonormal basis* $\{q_0, q_1\}$ *and a superchannel* $R : \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \to \mathcal{B}(\mathcal{B}_1(\mathcal{H}_Q))$ *with the following properties:*

1. *If* $T' \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *satisfies* $T|_{\mathcal{B}_1(\mathcal{V})} = T'|_{\mathcal{B}_1(\mathcal{V})}$, *then* $R(T') = \mathrm{id}$.
2. *If* $T' \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *is a channel such that* $T|_{\mathcal{B}_1(\mathcal{V})} \neq T'|_{\mathcal{B}_1(\mathcal{V})}$, *then the only state that is a fixed point of* $R(T')$, *is* $|q_0\rangle\langle q_0|$.
3. *If* $T' \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *is a channel with transmission functional* $\mathfrak{t}_{T'}$ *and* $\mathfrak{t}_T(|v\rangle\langle v|) = 0$, *then the transformed transmission functional* $\mathfrak{t}_{R(T')}$ *is given by*

$$\mathfrak{t}_{R(T')}(\cdot) = \begin{cases} \frac{1}{2}\mathfrak{t}_{T'}(\frac{P^\perp}{d-1})\mathrm{tr}\left[|q_1\rangle\langle q_1| \cdot\right] & \text{if } d > 1 \\ 0 & \text{if } d = 1 \end{cases}, \tag{4.34}$$

where $d := \dim(\mathcal{V})$ *and where* $P^\perp$ *denotes the orthogonal projection onto* $\{\psi \in \mathcal{V} \,|\, \langle \psi | v \rangle = 0\}$.

Before we prove the theorem, let us explore its consequences. First, we establish the analog of Theorem 2.5 for the transmission functional model.

Corollary 4.15. *For* $\dim(\mathcal{H}) < \infty$, *let* $\mathcal{C}_A, \mathcal{C}_B \subseteq \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ *be two closed sets of channels. Furthermore, let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *and let* $v \in \mathcal{V}$ *be a unit vector such that*

1. *For all* $T \in \mathcal{C}_A \cup \mathcal{C}_B$, T *is a channel with transmission functional* $\mathfrak{t}_T$.
2. *For all* $T \in \mathcal{C}_A$, T *is isometric on* $\mathcal{V}$.
3. *For all* $T \in \mathcal{C}_A$, $\mathfrak{t}_T|_{\mathcal{B}_1(\mathcal{V})} = 0$.
4. *For all* $T \in \mathcal{C}_B$, $\mathfrak{t}_T(|v\rangle\langle v|) = 0$.
5. $\sup_{T \in \mathcal{C}_B} \left\| \mathfrak{t}_T|_{\mathcal{B}_1(\mathcal{V})} \right\| < \infty$
6. *The set* $\mathcal{C}_A|_{\mathcal{B}_1(\mathcal{V})} := \left\{ T|_{\mathcal{B}_1(\mathcal{V})} \,\middle|\, T \in \mathcal{C}_A \right\}$ *contains exactly one element.*
7. $\mathcal{C}_A|_{\mathcal{B}_1(\mathcal{V})}$ *and* $\mathcal{C}_B|_{\mathcal{B}_1(\mathcal{V})} := \left\{ T|_{\mathcal{B}_1(\mathcal{V})} \,\middle|\, T \in \mathcal{C}_B \right\}$ *are disjoint.*

Then, there exist a constant C and for every $N \in \mathbb{N}$, an N-step discrimination strategy D and a two-valued POVM Π such that

$$P_e(D,\Pi) \leq \frac{C}{N^2}, \tag{4.35}$$

$$\mathfrak{T}_{T_A}(D) = 0 \quad \text{and} \quad \mathfrak{T}_{T_B}(D) \leq \frac{C}{N}, \tag{4.36}$$

for all $T_A \in \mathcal{C}_A$ and all $T_B \in \mathcal{C}_B$, where the discrimination error probability is w.r.t. the sets $\mathcal{C}_A$ and $\mathcal{C}_B$. Hence, the sets $\mathcal{C}_A$ and $\mathcal{C}_B$ can be discriminated in a transmission-free manner.

Proof. We combine Theorems 4.1 and 4.14. Let us fix some $T_A \in \mathcal{C}_A$. From Theorem 4.14 (with $T = T_A$), we obtain the map R, with the properties (1), (2), and (3). We want to apply Theorem 4.1 with $\mathcal{C} := R(\mathcal{C}_B)$. Since $\mathcal{C}_B$ is (as a closed subset of the compact set of channels) compact and R is continuous, $\mathcal{C}$ is compact and hence closed. Furthermore, since by Assumption 7, the sets $\mathcal{C}_A|_{\mathcal{B}_1(\mathcal{V})}$ and $\mathcal{C}_B|_{\mathcal{B}_1(\mathcal{V})}$ are disjoint, we have $T'|_{\mathcal{B}_1(\mathcal{V})} \neq T_A|_{\mathcal{B}_1(\mathcal{V})}$ for all $T' \in \mathcal{C}_B$. Hence, property (2) implies that for all $T \in \mathcal{C}$, the state $|q_0\rangle\langle q_0|$ is the only state that is a fixed point of T. In particular, id $\notin \mathcal{C}$. Furthermore, Assumption 6 implies that $T'|_{\mathcal{B}_1(\mathcal{V})} = T_A|_{\mathcal{B}_1(\mathcal{V})}$, for all $T' \in \mathcal{C}_A$. Hence, by property (1), $R(\mathcal{C_A}) = \{\mathrm{id}\}$. Thus, Theorem 4.1 yields a discrimination strategy $\tilde{D}$ and a two-valued POVM such that $P_e(\tilde{D},\Pi) \leq \tilde{C}N^{-2}$, for some constant $\tilde{C}$. By construction, $P_e(\tilde{D},\Pi)$ is the discrimination probability w.r.t. the sets $\mathcal{C}$ and $\{\mathrm{id}\}$, but since we have for $T' \in \mathcal{C}_A \cup \mathcal{C}_B$ that $R(T') \in \{\mathrm{id}\}$ iff $T' \in \mathcal{C}_A$ and $R(T') \in \mathcal{C}$ iff $R(T') \in \mathcal{C}_B$, it follows that $P_e(\tilde{D}^R,\Pi) = P_e(\tilde{D},\Pi)$, where $\tilde{D}^R$ is the transformed discrimination strategy, as defined in the main text. For $T' \in \mathcal{C}_A$, condition 3 and property (3) imply that the transformed transmission functional $\mathfrak{t}_{R(T')} = 0$. Thus, $\mathfrak{T}_{T'}(\tilde{D}^R) = 0$. Furthermore, for $T' \in \mathcal{C}_B$ with transmission functional $\mathfrak{t}_{T'}$, property (3) implies that the norm of the transformed transmission functional satisfies $\left\|\mathfrak{t}_{R(T')}\right\| = \frac{1}{2}\mathfrak{t}_{T'}\left(\frac{P^\perp}{d-1}\right) \leq \frac{1}{2}\left\|\mathfrak{t}_{T'}|_{\mathcal{B}_1(\mathcal{V})}\right\|$. Since we have $\mathfrak{T}_{T'}(\tilde{D}^R) = \mathfrak{T}_{R(T')}(\tilde{D})$, Theorem 4.1 implies that $\mathfrak{T}_{T'}(D^R) \leq \frac{\tilde{C}\left\|\mathfrak{t}_{T'}|_{\mathcal{B}_1(\mathcal{V})}\right\|}{2N}$. We finish the proof by identifying D with $\tilde{D}^R$ and defining

$$C := \max\left[\tilde{C}, \frac{\tilde{C}}{2}\sup_{T' \in \mathcal{C}_B}\left\|\mathfrak{t}_{T'}|_{\mathcal{B}_1(\mathcal{V})}\right\|\right] < \infty. \tag{4.37}$$

□

As a direct consequence of the previous result, we get the validity of Theorem 2.5.

Proof. (Theorem 2.5) We interpret every channel T with "interaction" functional $\mathfrak{i}_T$ as channel with transmission functional $\mathfrak{i}_T$. By Lemma 3.8, it suffices to check Conditions 1-7 of Corollary 4.15. 1, 2, 6, and 7 follow by assumption and 3, 4, and 5 follow directly from Lemma 3.10 (6). □

The remainder of this section is devoted to the proof of Theorem 4.14. We show that the transformation depicted in Fig. 8 has the desired properties. We

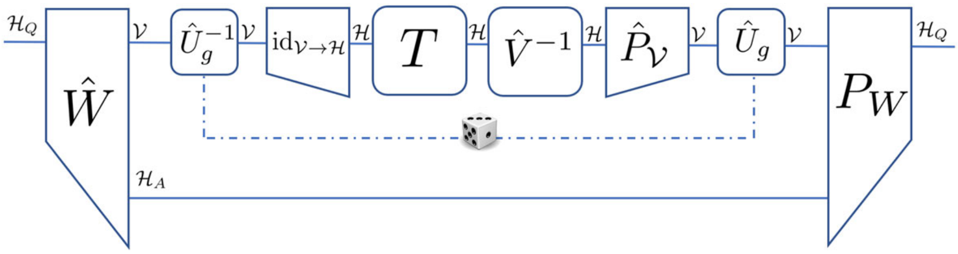

FIGURE 8. The reduction superchannel for $\dim(\mathcal{V}) > 1$

define this superchannel precisely in the proof of Theorem 4.14. An important part is the so-called twirling operation, which we study here for a special group.

Lemma 4.16 (Twirling). *For $2 \le d := \dim(\mathcal{H}) < \infty$, let $v \in \mathcal{H}$ be a unit vector and set $V_v := \operatorname{span}\{v\}$. We define the group*

$$G := \left\{ g = \mathbb{1}_{V_v} \oplus U_g \in \mathcal{B}(V_v \oplus V_v^{\perp}) \,\middle|\, U_g \in \mathcal{B}(V_v^{\perp}) \text{ is unitary} \right\} \tag{4.38}$$

and the twirling superchannel $S : \mathcal{B}(\mathcal{B}_1(\mathcal{H})) \to \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ by

$$S(T) = \int \hat{U}_g \circ T \circ \hat{U}_g^{-1} \,\mathrm{d}\mu_G(g), \tag{4.39}$$

where μ_G is the Haar measure on G and $\hat{U}_g : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is the quantum channel obtained by conjugation with the group element $g \in G$, i.e., $\hat{U}_g(\cdot) = g \cdot g^{-1}$. Then, the following statements hold.

- *Let $\psi \in V_v^{\perp}$ be any unit vector and $\phi := \frac{1}{\sqrt{2}}(v + \psi)$. If $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is a channel and $|\phi\rangle\langle\phi|$ is a fixed point of $S(T)$, then $T = \mathrm{id}$. Conversely, $S(\mathrm{id}) = \mathrm{id}$ and thus $|\phi\rangle\langle\phi|$ is a fixed point of $S(\mathrm{id})$.*
- *For a functional $\mathfrak{t} : \mathcal{B}_1(\mathcal{H}) \to \mathbb{C}$, we have*

$$\int \mathfrak{t} \circ \hat{U}_g^{-1} \,\mathrm{d}\mu_G(g) = \mathfrak{t}\left(\frac{P^{\perp}}{d-1}\right) \operatorname{tr}\left[P^{\perp} \cdot\right] + \mathfrak{t}(|v\rangle\langle v|) \operatorname{tr}\left[|v\rangle\langle v| \cdot\right]. \tag{4.40}$$

Remark 4.17. The integration over the Haar measure can be replaced by a unitary t-design [34]. We can thus implement the superchannel S without using an ancillary quantum system.

Proof. We start by showing that the range of S is spanned by the following seven operators:

$$\begin{aligned}
&\operatorname{tr}\left[|v\rangle\langle v| \cdot\right] |v\rangle\langle v|, && \operatorname{tr}\left[P^{\perp} \cdot\right] |v\rangle\langle v|, && \operatorname{tr}\left[|v\rangle\langle v| \cdot\right] \frac{P^{\perp}}{d-1}, \\
&\operatorname{tr}\left[P^{\perp} \cdot\right] \frac{P^{\perp}}{d-1}, && P^{\perp} \cdot |v\rangle\langle v|, && |v\rangle\langle v| \cdot P^{\perp}, \\
& && && P^{\perp} \cdot P^{\perp} - \operatorname{tr}\left[P^{\perp} \cdot\right] \frac{P^{\perp}}{d-1}.
\end{aligned} \tag{4.41}$$

Using the definition of the Haar measure, we obtain that the range of S consists of precisely those operators $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ that commute with $\hat{U}_g$ for all $g \in G$. We calculate the commutant on the level of Choi matrices. To this end, we identify $\mathcal{B}_1(\mathcal{H})$ with $\mathcal{H}\otimes\mathcal{H}$ via the Choi isomorphism ($|h_i\rangle\langle h_j| \leftrightarrow h_i \otimes h_j$), where $h_0, h_1, \dots h_{d-1}$ is an orthonormal basis of $\mathcal{H}$ such that $h_0 = v$. The operator corresponding to $\hat{U}_g$ is $g\otimes\overline{g}$, where the complex conjugation is w.r.t the aforementioned basis. We can rewrite this operator as:

$$\begin{aligned} g\otimes\overline{g} &= (\mathbb{1}_{V_v} \oplus U_g)\otimes(\mathbb{1}_{V_v}\oplus\overline{U}_g) \\ &= (\mathbb{1}_{V_v}\otimes\mathbb{1}_{V_v})\oplus(\mathbb{1}_{V_v}\otimes\overline{U}_g)\oplus(U_g\otimes\mathbb{1}_{V_v})\oplus(U_g\otimes\overline{U}_g). \end{aligned}$$

The maps $g\mapsto \mathbb{1}_{V_v}\otimes\mathbb{1}_{V_v}$, $g\mapsto\mathbb{1}_{V_v}\oplus\overline{U}_g$, and $g\mapsto U_g\otimes\mathbb{1}_{V_v}$ are inequivalent irreducible representations of G. If $d=2$, the representation $g\mapsto(U_g\otimes\overline{U}_g)$ is the trivial 1-dimensional representation. A simple consequence of Schur's lemma is that the commutant then is $2^2+1^2+1^2=6$ dimensional (see [35], p. 60 for the dimension formula). For $d=2$, the span of the operators in (4.41) is also 6-dimensional ($P^\perp\cdot P^\perp - \mathrm{tr}\left[P^\perp\cdot\right]\frac{P^\perp}{d-1}=0$). So in this case, we have proven the claim. If $d\geq 3$, then the representation $g\mapsto(U_g\otimes\overline{U}_g)$ is the direct sum of the trivial 1-dimensional representation and an irreducible $((d-1)^2-1)$-dimensional representation (see [36]). Hence, the dimension of the commutant is $2^2+1^2+1^2+1^2=7$. Also, the dimension of the span of the operators in (4.41) is 7-dimensional. This proves that the range of S is indeed given by the span of the operators in (4.41).

For our first claim, we clearly have $S(\mathrm{id})=\mathrm{id}$. Conversely, let T be a channel such that $|\phi\rangle\langle\phi|$ is a fixed point of $S(T)$. Let $\alpha_1,\alpha_2,\dots,\alpha_7$ be the coefficients of an expansion of $S(T)$ in terms of the operators in (4.41). Note that for $d=2$, this expansion is not unique but can be made that way by demanding $\alpha_7 := 1$. As $|\phi\rangle\langle\phi|$ is a fixed point of $S(T)$, we have

$$\begin{aligned} |\phi\rangle\langle\phi| &= \frac{1}{2}\left(|v\rangle\langle v| + |v\rangle\langle\psi| + |\psi\rangle\langle v| + |\psi\rangle\langle\psi|\right) \\ &= S(T)(|\phi\rangle\langle\phi|) \\ &= \frac{1}{2}\Bigg((\alpha_1+\alpha_2)|v\rangle\langle v| + (\alpha_3+\alpha_4-\alpha_7)\frac{P^\perp}{d-1} + \alpha_5|\psi\rangle\langle v| + \alpha_6|v\rangle\langle\psi| \\ &\qquad + \alpha_7|\psi\rangle\langle\psi|\Bigg). \end{aligned}$$

By comparing the second and the last expression, it follows that $\alpha_1+\alpha_2=1$ and $\alpha_5=\alpha_6=1$. If $d=2$, then $P^\perp=|\psi\rangle\langle\psi|$ and $\alpha_3+\alpha_4=1$. Otherwise, we have $\alpha_7=1$ and $\alpha_3+\alpha_4-\alpha_7=0$, hence also $\alpha_3+\alpha_4=1$. Furthermore,

$$\begin{aligned} S(T)(|v\rangle\langle v|) &= \alpha_1|v\rangle\langle v| + \alpha_3\frac{P^\perp}{d-1}, \\ S(T)(|\psi\rangle\langle\psi|) &= \alpha_2|v\rangle\langle v| + (\alpha_4-\alpha_7)\frac{P^\perp}{d-1} + \alpha_7|\psi\rangle\langle\psi|. \end{aligned} \tag{4.42}$$

As $S(T)$ is trace-preserving, we obtain $\alpha_1+\alpha_3=1$ and $\alpha_3+\alpha_4=1$. Our equations imply that $\alpha_2=1-\alpha_1, \alpha_3=1-\alpha_1$, and $\alpha_4=\alpha_1$. Positivity of $S(T)$ in (4.42) implies that $\alpha_1\geq 0$ and $\alpha_3\geq 0$. Thus, $0\leq\alpha_1\leq 1$. We want to show that complete positivity of $S(T)$ even implies $\alpha_1=1$. To this end, we define $\mathcal{H}_A := \mathrm{span}\{v, \psi\}$ and $\Omega^+, \Omega^- \in \mathcal{H}_A \otimes \mathcal{H}$ by

$$\Omega^+ := v \otimes v + \psi \otimes \psi, \qquad \Omega^- := v \otimes v - \psi \otimes \psi.$$

As $S(T)$ is completely positive, we have

$$\begin{aligned}
0 &\leq \langle \Omega^- | (\mathrm{id}_A \otimes S(T))(|\Omega^+\rangle\langle\Omega^+|)\, \Omega^-\rangle \\
&= \langle \Omega^- | \left(|v\rangle\langle v| \otimes \left(\alpha_1 |v\rangle\langle v| + (1-\alpha_1)\frac{P^\perp}{d-1} \right)\right)\, \Omega^-\rangle \\
&\quad + \langle \Omega^- | \left(|\psi\rangle\langle v| \otimes |\psi\rangle\langle v| \right)\, \Omega^-\rangle + \langle \Omega^- | \left(|v\rangle\langle \psi| \otimes |v\rangle\langle \psi| \right)\, \Omega^-\rangle \\
&\quad + \langle \Omega^- | \left(|\psi\rangle\langle \psi| \otimes \left((1-\alpha_1)|v\rangle\langle v| + \alpha_1 \frac{P^\perp}{d-1} + |\psi\rangle\langle\psi| - \frac{P^\perp}{d-1} \right)\right)\, \Omega^-\rangle \\
&= \alpha_1 - 2 + \frac{\alpha_1 - 1}{d-1} + 1 \\
&= d\frac{\alpha_1 - 1}{d-1}.
\end{aligned}$$

Thus, $\alpha_1 \geq 1$. This further implies that $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 0$, and $\alpha_4 = 1$. Together with the earlier result that $\alpha_5 = \alpha_6 = \alpha_7 = 1$, we obtain

$$\begin{aligned}
S(T) &= \mathrm{tr}\left[|v\rangle\langle v| \cdot\right] |v\rangle\langle v| + \mathrm{tr}\left[P^\perp \cdot\right] \frac{P^\perp}{d-1} + P^\perp \cdot |v\rangle\langle v| + |v\rangle\langle v| \cdot P^\perp \\
&\quad + P^\perp \cdot P^\perp - \mathrm{tr}\left[P^\perp \cdot\right] \frac{P^\perp}{d-1} \\
&= \mathrm{id}.
\end{aligned}$$

Thus, we have shown that if $|\phi\rangle\langle\phi|$ is a fixed point of $S(T)$, then $S(T) = \mathrm{id}$. To see that this also implies that $T = \mathrm{id}$, we note that $S(T)$ is a convex combination of the channels $\hat{U}_g \circ T \circ \hat{U}_g^{-1}$. But as the identity is an extremal element of the convex set of quantum channels, $\hat{U}_g \circ T \circ \hat{U}_g^{-1}$ must be proportional to the identity μ_G-almost everywhere. In particular, $\hat{U}_g \circ T \circ \hat{U}_g^{-1} = \mathrm{id}$, for some $g \in G$. Thus, $T = \mathrm{id}$. This proves the first claim.

It remains to prove the second claim. For $\mathfrak{t}(\cdot) = \mathrm{tr}\left[L \cdot\right]$ and $\rho \in \mathcal{B}_1(\mathcal{H})$, we have

$$S'(\mathfrak{t})(\rho) := \int \mathfrak{t} \circ \hat{U}_g^{-1}(\rho)\, \mathrm{d}\mu_G(g) = \mathrm{tr}\left[\int gLg^{-1}\, \mathrm{d}\mu_G(g)\, \rho\right].$$

By the definition of the Haar measure, the integral must commute with all $g \in G$. The representation $g \mapsto \mathbb{1}_{V_v} \oplus U_g$ is the sum of two inequivalent irreducible representations of G. Thus, the commutant is 2-dimensional. It is easy to check that $P^\perp$ and $|v\rangle\langle v|$ are in the commutant. Thus,

$$\int gLg^{-1}\, \mathrm{d}\mu_G(g) = \lambda_1 P^\perp + \lambda_2 |v\rangle\langle v|,$$

for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Therefore, we can write

$$S'(\mathfrak{t})(\rho) = \lambda_1 \mathrm{tr}\left[P^\perp \rho\right] + \lambda_2 \mathrm{tr}\left[|v\rangle\langle v| \rho\right].$$

Substituting $P^\perp$ and $|v\rangle\langle v|$ for ρ yields $\lambda_1 = (d-1)^{-1} S'(\mathfrak{t})(P^\perp)$ and $\lambda_2 = S'(\mathfrak{t})(|v\rangle\langle v|)$. As $P^\perp$ and $|v\rangle\langle v|$ commute with all $g \in G$, we have

$$S'(\mathfrak{t})(P^\perp) = \mathfrak{t}\left(\int g^{-1} P^\perp g \, \mathrm{d}\mu_G(g)\right) = \mathfrak{t}(P^\perp),$$
$$S'(\mathfrak{t})(|v\rangle\langle v|) = \mathfrak{t}\left(\int g^{-1} |v\rangle\langle v| g \, \mathrm{d}\mu_G(g)\right) = \mathfrak{t}(|v\rangle\langle v|).$$

We plug this into (4.43) and obtain the desired result, Eq. (4.40). Thus, we have proven our last claim. □

We are now ready to prove Theorem 4.14.

Proof. As already mentioned, the proof consists of an explicit construction of the superchannel R. The construction is depicted in Fig. 8. We start by defining the components of this circuit from left to right. For the definition of the first component, we define $\mathcal{H}_A$ to be a two-dimensional Hilbert space with orthonormal basis $\{a_0, a_1\}$. The channel $\hat{W} : \mathcal{B}_1(\mathcal{H}_Q) \to \mathcal{B}_1(\mathcal{V} \otimes \mathcal{H}_A)$ is defined by $\hat{W}(\cdot) = W \cdot W^\dagger$, with isometry $W : \mathcal{H}_Q \to \mathcal{V} \otimes \mathcal{H}_A$ defined by

$$W q_0 = v \otimes a_0,$$
$$W q_1 = \begin{cases} \frac{1}{\sqrt{2}}(v + \psi) \otimes a_1, & \text{if } \dim(\mathcal{V}) > 1 \\ v \otimes a_1, & \text{if } \dim(\mathcal{V}) = 1 \end{cases},$$

where $\psi \in \mathcal{V}$ is any unit vector that is orthogonal to v. This channel is designed in order to exhibit the second conclusion of Lemma 4.16.

The second component is the twirling operation $S : \mathcal{B}(\mathcal{B}_1(\mathcal{V})) \to \mathcal{B}(\mathcal{B}_1(\mathcal{V}))$, which is a superchannel on its own and which we only define for $\dim(\mathcal{V}) > 1$. This operation is depicted by the two unitary channels $\hat{U}_g$ and $\hat{U}_g^{-1}$ connected by a dashed line and acts as

$$S(\cdot) := \int \hat{U}_g \circ (\cdot) \circ \hat{U}_g^{-1} \, \mathrm{d}\mu_G(g), \tag{4.43}$$

where μ_G is the Haar measure on the compact group G, defined by (cf. Lemma 4.16)

$$G := \left\{ g = \mathbb{1}_{V_v} \oplus U_g \in \mathcal{B}(V_v \oplus V_v^\perp) \,\middle|\, U_g \in \mathcal{B}(V_v^\perp) \text{ is unitary} \right\},$$

with $V_v := \operatorname{span}\{v\}$. The channels $\hat{U}_g, \hat{U}_g^{-1} : \mathcal{B}_1(\mathcal{V}) \to \mathcal{B}_1(\mathcal{V})$ are defined by

$$\hat{U}_g(\cdot) := (\mathbb{1}_{V_v} \oplus U_g)(\cdot)(\mathbb{1}_{V_v} \oplus U_g^\dagger) \quad \text{and} \quad \hat{U}_g^{-1}(\cdot) := (\mathbb{1}_{V_v} \oplus U_g^\dagger)(\cdot)(\mathbb{1}_{V_v} \oplus U_g).$$

The channel $\mathrm{id}_{\mathcal{V}\to\mathcal{H}} : \mathcal{B}_1(\mathcal{V}) \to \mathcal{B}_1(\mathcal{H}), \rho \mapsto \rho$ embeds $\mathcal{B}_1(\mathcal{V})$ into $\mathcal{B}_1(\mathcal{H})$.

To define the channel $\hat{V}^{-1} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, we use that by assumption, T is isometric on $\mathcal{V}$. This means that there exists an isometry $\tilde{V} : \mathcal{V} \to \mathcal{H}$ such that $T|_{\mathcal{B}_1(\mathcal{V})}(\cdot) = \tilde{V} \cdot \tilde{V}^\dagger$. This isometry can be extended (in a non-unique way) to a unitary and therefore invertible operation $V : \mathcal{H} \to \mathcal{H}$. We then define

$$\hat{V}^{-1}(\cdot) := V^\dagger \cdot V.$$

We define the channel $\hat{P}_{\mathcal{V}} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{V})$ by

$$\hat{P}_{\mathcal{V}}(\cdot) := P_{\mathcal{V}} \cdot P_{\mathcal{V}}^{\dagger} + \operatorname{tr}\left[(\mathbb{1} - P_{\mathcal{V}}^{\dagger} P_{\mathcal{V}})(\cdot)\right] |v\rangle\langle v|,$$

where $P_{\mathcal{V}} : \mathcal{H} \to \mathcal{V}$ is the orthogonal projection onto $\mathcal{V}$. To finish the channel definitions, we define the channel $P_W : \mathcal{B}_1(\mathcal{V} \otimes \mathcal{H}_A) \to \mathcal{B}_1(\mathcal{H}_Q)$ by

$$P_W(\cdot) := W^{\dagger} \cdot W + \operatorname{tr}\left[(\mathbb{1} - WW^{\dagger})(\cdot)\right] |q_0\rangle\langle q_0|.$$

We can now define the superchannel R. If $\dim(\mathcal{V}) > 1$, we define

$$R(\cdot) := P_W \circ \left(\left[\int \hat{U}_g \circ \hat{P}_{\mathcal{V}} \circ \hat{V}^{-1} \circ (\cdot) \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} \circ \hat{U}_g^{-1} \, \mathrm{d}\mu_G(g) \right] \otimes \mathrm{id}_A \right) \circ \hat{W}, \tag{4.44}$$

and if $\dim(\mathcal{V}) = 1$, we define

$$R(\cdot) := P_W \circ \hat{V}^{-1} \circ (\cdot) \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} \circ \hat{W}. \tag{4.45}$$

With the definition in place, it only remains to show that the superchannel R has the claimed properties. To prove the first claim, let $T' \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ such that $T|_{\mathcal{B}_1(\mathcal{V})} = T'|_{\mathcal{B}_1(\mathcal{V})}$. For $\dim(\mathcal{V}) > 1$, we use that by construction $\hat{V}^{-1} \circ T'|_{\mathcal{B}_1(\mathcal{V})} = \mathrm{id}_{\mathcal{V}}$ and that operators in $\mathcal{B}_1(\mathcal{V})$ are fixed points of $P_{\mathcal{V}}$. We get

$$\begin{aligned} R(T') &= P_W \circ \left(\left[\int \hat{U}_g \circ \mathrm{id}_{\mathcal{V}} \circ \hat{U}_g^{-1} \, \mathrm{d}\mu_G(g) \right] \otimes \mathrm{id}_A \right) \circ \hat{W} \\ &= P_W \circ \hat{W} \\ &= \mathrm{id}_Q. \end{aligned}$$

By means of a similar argument, it follows that the claim also holds for $\dim(\mathcal{V}) = 1$. To prove the second claim, we start by showing that $|q_0\rangle\langle q_0|$ is a fixed point of $R(T')$, for every channel T'. For $\dim(\mathcal{V}) > 1$, we have

$$\begin{aligned} R(T')(|q_0\rangle\langle q_0|) &= P_W \circ \left(S(\hat{P}_{\mathcal{V}} \circ \hat{V}^{-1} \circ T' \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}}) \otimes \mathrm{id}_A \right) \circ W(|q_0\rangle\langle q_0|) \\ &= P_W \left(S(\hat{P}_{\mathcal{V}} \circ \hat{V}^{-1} \circ T' \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}})(|v\rangle\langle v|) \otimes |a_0\rangle\langle a_0| \right) \\ &= |q_0\rangle\langle q_0|, \end{aligned}$$

where the last line follows as P_W maps every state of the form $\sigma \otimes |a_0\rangle\langle a_0|$ to $|q_0\rangle\langle q_0|$. An analogous argument yields that $|q_0\rangle\langle q_0|$ is also a fixed point of $R(T')$ if $\dim(\mathcal{V}) = 1$. Conversely, assume that $T' \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ is a channel such that $T|_{\mathcal{B}_1(\mathcal{V})} \neq T'|_{\mathcal{B}_1(\mathcal{V})}$ and $\rho \in \mathcal{S}(\mathcal{H}_Q)$ is a fixed point of $R(T')$. We prove that $\rho = |q_0\rangle\langle q_0|$. We do so by first showing that if $\rho \neq |q_0\rangle\langle q_0|$, then $|q_1\rangle\langle q_1|$ is also a fixed point of $R(T')$, which will lead to a contradiction. By part 1 of the theorem, $|q_0\rangle\langle q_0|$ is a fixed point of $R(T')$. Hence, Lemma 4.4 implies that $\mathrm{span}\{|q_0\rangle\langle q_1|\}$ and $\mathrm{span}\{|q_1\rangle\langle q_0|\}$ are invariant subspaces of $R(T')$. Thus,

$$\langle q_0 | R(T')(|q_0\rangle\langle q_1|)\, q_0\rangle = \langle q_0 | R(T')(|q_1\rangle\langle q_0|)\, q_0\rangle = 0.$$

We then have

$$\begin{aligned}\langle q_0|\rho\, q_0\rangle &= \langle q_0|R(T')(\rho)\, q_0\rangle \\ &= \sum_{i,j=0}^{1}\langle q_i|\rho\, q_j\rangle\langle q_0|R(T')(|q_i\rangle\langle q_j|)\, q_0\rangle \\ &= \sum_{i=0}^{1}\langle q_i|\rho\, q_i\rangle\langle q_0|R(T')(|q_i\rangle\langle q_i|)\, q_0\rangle \\ &= \langle q_0|\rho\, q_0\rangle + \langle q_1|\rho\, q_1\rangle\langle q_0|R(T')(|q_1\rangle\langle q_1|)\, q_0\rangle.\end{aligned}$$

Hence,

$$\langle q_1|\rho\, q_1\rangle\langle q_0|R(T')(|q_1\rangle\langle q_1|)\, q_0\rangle = 0.$$

If $\langle q_1|\rho\, q_1\rangle = 0$, then positivity of ρ implies that $\rho = |q_0\rangle\langle q_0|$, which contradicts the assumption that $\rho \neq |q_0\rangle\langle q_0|$. It follows that

$$\langle q_0|R(T')(|q_1\rangle\langle q_1|)\, q_0\rangle = 0.$$

Positivity of $R(T')(\rho)$ yields $R(T')(|q_1\rangle\langle q_1|) = |q_1\rangle\langle q_1|$, which shows that $|q_1\rangle\langle q_1|$ is a fixed point of $R(T')$. We now show that this leads to a contradiction. With the abbreviations $\tilde{S} := S(\hat{P}_{\mathcal{V}}\circ\hat{V}^{-1}\circ T'\circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}})$ and $\phi := \frac{1}{\sqrt{2}}(v+\psi)$, we get

$$\begin{aligned}|q_1\rangle\langle q_1| &= R(T')(|q_1\rangle\langle q_1|) \\ &= P_W\left(\tilde{S}(|\phi\rangle\langle\phi|)\otimes|a_1\rangle\langle a_1|\right) \\ &= \mathrm{tr}\left[|\phi\rangle\langle\phi|\,\tilde{S}(|\phi\rangle\langle\phi|)\right]|q_1\rangle\langle q_1| + \mathrm{tr}\left[(\mathbb{1}-WW^\dagger)\,\tilde{S}(|\phi\rangle\langle\phi|)\right]|q_0\rangle\langle q_0|.\end{aligned}$$

Comparing the last with the first line implies that $\mathrm{tr}\left[|\phi\rangle\langle\phi|\,\tilde{S}(|\phi\rangle\langle\phi|)\right] = 1$. We observe the latter equation says that the Cauchy–Schwarz inequality (w.r.t. the Hilbert–Schmidt inner product) is satisfied with equality. Thus, $\tilde{S}(|\phi\rangle\langle\phi|) = |\phi\rangle\langle\phi|$. Lemma 4.16 then implies

$$\hat{P}_{\mathcal{V}}\circ\hat{V}^{-1}\circ T'\circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} = \mathrm{id}_{\mathcal{V}}.$$

Note that $P_{\mathcal{V}}$ is the sum of the two completely positive trace non-increasing maps, $P_1(\cdot) := P_{\mathcal{V}}\cdot P_{\mathcal{V}}$ and $P_2(\cdot) := \mathrm{tr}\left[(\mathbb{1}-P_{\mathcal{V}}^\dagger P_{\mathcal{V}})(\cdot)\right]|v\rangle\langle v|$. Thus, with the appropriate normalization, the extremal point of the convex set of completely positive maps, $\mathrm{id}_{\mathcal{V}}$, can be written as a convex combination of $P_i\circ\hat{V}^{-1}\circ T'\circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}}$. Thus,

$$\hat{V}^{-1}\circ T'\circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} = \mathrm{id}_{\mathcal{V}\to\mathcal{H}}. \tag{4.46}$$

As $\hat{V}^{-1}$ is invertible and $T'\circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}}$, identity (4.46) is equivalent to

$$T'|_{\mathcal{B}_1(\mathcal{V})} = \hat{V}|_{\mathcal{B}_1(\mathcal{V})}.$$

By construction of $\hat{V}$, the RHS equals $T|_{\mathcal{B}_1(\mathcal{V})}$. But this contradicts the assumption that $T|_{\mathcal{B}_1(\mathcal{V})} \neq T'|_{\mathcal{B}_1(\mathcal{V})}$. Thus, $|q_1\rangle\langle q_1|$ cannot be a fixed point of $R(T')$. Consequently, $\rho = |q_0\rangle\langle q_0|$, which proves that $|q_0\rangle\langle q_0|$ is the only state that is a fixed point of $R(T')$. This proves the second claim. To prove the third claim, we must calculate how our protocol transforms the transmission functional. For $\dim(\mathcal{V}) = 1$, we get directly from the definition (4.45) that $\mathfrak{t}_{R(T)}(\cdot) = \operatorname{tr}[\cdot]\, \mathfrak{t}_T(|v\rangle\langle v|) = 0$. For $\dim(\mathcal{V}) > 1$, the transmission functional $\mathfrak{t}_T$ transforms to $\mathfrak{t}_{R(T)}$, given by

$$\mathfrak{t}_{R(T)} := \int \mathfrak{t}_T \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} \circ \hat{U}_g^{-1} \circ \operatorname{tr}_A \circ \hat{W}\, \mathrm{d}\mu_G(g). \tag{4.47}$$

To evaluate (4.47), we use (4.40) and get

$$\mathfrak{t}_{R(T)}(\cdot) = \mathfrak{t}_T \circ \mathrm{id}_{\mathcal{V}\to\mathcal{H}} \left(\frac{P^\perp}{d-1}\right) \operatorname{tr}\left[P^\perp \operatorname{tr}_A \left[\hat{W}(\cdot)\right]\right].$$

A direct calculation then yields the claim. □

Remark 4.18. With our protocol, we achieved a transformation from channels on $\mathcal{H}$ to qubit channels with certain properties. This was achieved by using classical communication and one ancillary qubit. To demonstrate that our implementation of this transformation uses the quantum resources in the most economic way possible, we show that in general one cannot use only classical communication to implement a transformation which has the desired properties. To this end, we consider the following procedure. First, we use an instrument to transform the state and to obtain classical information. Then, we apply the channel, which should be transformed. Afterwards, we apply some quantum channel, where the choice of the channel may depend on the classical information that we obtained in the first step. Our instrument described by a collection of nonzero quantum operations $I_1, I_2, \ldots, I_N$, such that $\sum_i I_i$ is trace-preserving. We denote the associated channels that are applied in the last step by $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$. Our protocol then implements the following transformation:

$$T \mapsto \sum_i \Lambda_i \circ T \circ I_i. \tag{4.48}$$

Assume that the channel T of the Theorem 4.14 is the identity and $\dim(\mathcal{H}) = \dim(\mathcal{V}) = 2$. Our first requirement is that $\mathrm{id} \mapsto \mathrm{id}$. Thus,

$$\mathrm{id} = \sum_i \Lambda_i \circ I_i. \tag{4.49}$$

Since id is an extreme point of the convex set of quantum operations, there must be non-negative coefficients $p_1, p_2, \ldots, p_N$, such that

$$\Lambda_i \circ I_i = p_i \cdot \mathrm{id}, \text{ for } i = 1, 2, \ldots, N. \tag{4.50}$$

This implies that Λ_i and I_i must be proportional to a unitary conjugation, i.e., $\Lambda_i(\cdot) = U_i^\dagger \cdot U_i$ and $I_i(\cdot) = p_i U_i \cdot U_i^\dagger$, for some unitary operator U_i. Our second requirement is that (since $\mathcal{V} = \mathcal{H}$) every channel except id must be transformed to a state whose only fixed point is $|q_0\rangle\langle q_0| =: P_0$. In particular, for the pinching channel, defined by $T_P(\cdot) = P_0 \cdot P_0 + P_1 \cdot P_1$, with $P_1 := \mathbb{1} - P_0$, we have

$$P_0 = \sum_i \sum_{j=0}^{1} p_i (U_i^\dagger P_j U_i) P_0 (U_i^\dagger P_j U_i). \tag{4.51}$$

Since P_0 is an extremal point of the convex set $\{\rho \geq 0 \,|\, \mathrm{tr}\,[\rho] \leq 1\}$, we get that

$$(U_i^\dagger P_j U_i) P_0 (U_i^\dagger P_j U_i) = \lambda_{ij} P_0, \tag{4.52}$$

for some $\lambda_{ij} \geq 0$. From this, we conclude that either $U_i^\dagger P_j U_i = P_0$ or $U_i^\dagger P_j U_i = P_1$. But then the application of the transformed channel to P_1 yields

$$\sum_i \sum_{j=0}^{1} p_i (U_i^\dagger P_j U_i) P_1 (U_i^\dagger P_j U_i) = P_1. \tag{4.53}$$

Thus, P_0 is not the only state that is a fixed point of the transformed channel. Hence, to achieve our transformation, an ancillary system is needed.

5. No-Go Results

In this section, we consider the case for which we claimed in our main theorem that it is impossible to discriminate two channels in an "interaction-free" manner. There are two major results in this section: Theorem 5.7 which claims an inequality between the error probability and the "interaction" probability; and Theorem 5.9, which claims that, under a certain condition, the best achievable rate (in terms of the number of channel uses, N) for the "interaction" probability is proportional to N^{-1}. Both theorems are consequences of our main technical results: Propositions 5.2 and 5.3. The proof techniques for these results are inspired by the techniques used in two papers by Mitchison, Massar, and Pironio [11,12], who proved an analogous no-go result for the special case of a semitransparent object. Before we state the first proposition, we define a quantity that will appear as proportionality constant in the results of this section. As this may seem complicated, we want to stress that in all relevant cases, $C_{\mathcal{V},\mathcal{W}}^{(T_A^\downarrow, T_B^\downarrow)}$ can be bounded by 2.

Definition 5.1. For $\dim(\mathcal{H}) < \infty$, let $T_A^\downarrow, T_B^\downarrow : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be two linear maps, let $\mathcal{V}$ be a linear subspace of $\mathcal{H}$, and let $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_K\}$ be a collection of mutually orthogonal subspaces of $\mathcal{V}^\perp$ with the property that $\mathcal{V}^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_K$. Furthermore, let P and $P_1, P_2, \ldots, P_K$ be the orthogonal projections onto $\mathcal{V}$ and $\mathcal{W}_1, \mathcal{W}_2, \ldots \mathcal{W}_K$.

We define the quantity $C_{\mathcal{V},\mathcal{W}}^{(T_A^\downarrow, T_B^\downarrow)}$ to be the infimum of the (possibly empty) set of real numbers r with the property that there exists a finite-dimensional Hilbert space $\mathcal{H}_E$, isometries $V_A, V_B : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$, and orthogonal projections $P_A, P_B : \mathcal{H}_E \to \mathcal{H}_E$ such that[14]

$$r = \max_{1 \leq k \leq K} \left\| P_k ({V_A}^\dagger (P_A P_B \otimes \mathbb{1}) V_B - \mathbb{1}) P_k \right\|, \tag{5.1a}$$

[14] $\|\cdot\|$ is the operator norm on $\mathcal{B}(\mathcal{H})$.

$$V_A P = V_B P, \tag{5.1b}$$

$$T^{\downarrow}_X(\cdot) = \mathrm{tr}_E\left[(P_X \otimes \mathbb{1})V_X \cdot V_X^{\dagger}\right], \tag{5.1c}$$

for $X \in \{A, B\}$.

We are now ready to state the first important proposition, which establishes, for a single channel use, an uncertainty relation between the "information-gain" (RHS of (5.2)) about the identity of the channel (is it T_A or T_B?) and a quantity that depends on the probability that if we would measure the input states, we would find that they are supported in the orthogonal complement of a subspace $\mathcal{V}$. Later on, this subspace will be chosen to be a maximum vacuum subspace.

Proposition 5.2 (Information-interaction tradeoff). *For* $\dim(\mathcal{H}) < \infty$*, let* $T^{\downarrow}_A$, $T^{\downarrow}_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be quantum operations and let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *such that* $T^{\downarrow}_A|_{\mathcal{B}_1(\mathcal{V})}$ *is trace-preserving and* $T^{\downarrow}_A|_{\mathcal{B}_1(\mathcal{V})} = T^{\downarrow}_B|_{\mathcal{B}_1(\mathcal{V})}$*. Let* $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_K\}$ *be a collection of mutually orthogonal subspaces of* $\mathcal{V}^{\perp}$*, such that* $\mathcal{V}^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_K$*. Denote the orthogonal projections onto these subspaces by* $P_1, P_2, \ldots, P_K$*. Then,* $C^{(T^{\downarrow}_A, T^{\downarrow}_B)}_{\mathcal{V},\mathcal{W}} \leq 2$ *and*

$$\sqrt{F}(\rho,\sigma) - \sqrt{F}(T^{\downarrow}_A(\rho), T^{\downarrow}_B(\sigma)) \leq C^{(T^{\downarrow}_A, T^{\downarrow}_B)}_{\mathcal{V},\mathcal{W}} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\,[P_k\rho]\,\mathrm{tr}\,[P_k\sigma]}, \tag{5.2}$$

for all $\rho, \sigma \geq 0$*.*

Before proving the proposition, let us remark that Proposition 2.10 is a direct consequence thereof.

Proof. (Proposition 2.10) This follows directly from the fact that the fidelity can be characterized in terms of the minimum over measurements of expressions of the form given on the RHS of (5.2) (see [25], p. 412). □

Proof. (Proposition 5.2) We first establish that $C^{(T^{\downarrow}_A, T^{\downarrow}_B)}_{\mathcal{V},\mathcal{W}} \leq 2$. Let P, $P^{\perp}$ be the orthogonal projections onto $\mathcal{V}$ and $\mathcal{V}^{\perp}$. By applying the triangular inequality and the sub-multiplicativity of the operator norm to the definition of $C^{(T^{\downarrow}_A, T^{\downarrow}_B)}_{\mathcal{V},\mathcal{W}}$, it follows that if there exist $\mathcal{H}_E, V_A, V_B, P_A$, and P_B with the properties of Definition 5.1, then $C^{(T^{\downarrow}_A, T^{\downarrow}_B)}_{\mathcal{V},\mathcal{W}} \leq 2$. Therefore, we start our proof by showing the existence of the aforementioned quantities. It is a basic property of completely positive trace non-increasing maps (see [25], p. 365) that there exist finite-dimensional Hilbert spaces $\mathcal{H}_{E_A}$ and $\mathcal{H}_{E_B}$, isometries $\tilde{V}_A : \mathcal{H} \to \mathcal{H}_{E_A} \otimes \mathcal{H}$ and $\tilde{V}_B : \mathcal{H} \to \mathcal{H}_{E_B} \otimes \mathcal{H}$, and orthogonal projections $\tilde{P}_A : \mathcal{H}_{E_A} \to \mathcal{H}_{E_A}$ and $\tilde{P}_B : \mathcal{H}_{E_B} \to \mathcal{H}_{E_B}$, such that $T^{\downarrow}_A(\cdot) = \mathrm{tr}_{E_A}\left[(\tilde{P}_A \otimes \mathbb{1})\tilde{V}_A \cdot \tilde{V}_A^{\dagger}\right]$ and $T^{\downarrow}_B(\cdot) = \mathrm{tr}_{E_B}\left[(\tilde{P}_B \otimes \mathbb{1})\tilde{V}_B \cdot \tilde{V}_B^{\dagger}\right]$. By enlarging the smaller of the two ancillary Hilbert spaces and identifying two orthonormal basis, we can achieve that $\mathcal{H}_{E_A}$ and $\mathcal{H}_{E_B}$ are the same space, $\mathcal{H}_E$. By assumption, $T_A|_{\mathcal{B}_1(\mathcal{V})}$ and $T_B|_{\mathcal{B}_1(\mathcal{V})}$ are trace-preserving. It follows that $(\tilde{P}_A \otimes \mathbb{1})\tilde{V}_A|_{\mathcal{V}}$ and $(\tilde{P}_A \otimes \mathbb{1})\tilde{V}_B|_{\mathcal{V}}$ are isometries and thus $(\tilde{P}_A \otimes \mathbb{1})\tilde{V}_A|_{\mathcal{V}} = \tilde{V}_A|_{\mathcal{V}}$ and $(\tilde{P}_B \otimes \mathbb{1})\tilde{V}_B|_{\mathcal{V}} = \tilde{V}_B|_{\mathcal{V}}$. Hence, $\tilde{V}_A|_{\mathcal{V}}$ and $\tilde{V}_B|_{\mathcal{V}}$ are Stinespring isometries of the same channel and thus are related by a unitary operator on $\mathcal{H}_E$. Precisely, there exists a unitary operator $W : \mathcal{H}_E \to \mathcal{H}_E$ such that $\tilde{V}_B|_{\mathcal{V}} = (W \otimes \mathbb{1})\tilde{V}_A|_{\mathcal{V}}$. Equivalently, $\tilde{V}_B P = (W \otimes \mathbb{1})\tilde{V}_A P$. It is then easy to verify that the operators $V_A := (W \otimes \mathbb{1})\tilde{V}_A$, $V_B := \tilde{V}_B$ and $P_A := W\tilde{P}_A W^{-1}, P_B := \tilde{P}_B$ satisfy the requirements (5.1c) and (5.1b). In particular, we have

$$(P_A \otimes \mathbb{1})V_A P = V_A P = V_B P = (P_B \otimes \mathbb{1})V_B P. \tag{5.3}$$

This finishes the proof of the first part of the proposition. For the second part, we fix V_A, V_B, P_A, and P_B such that the Conditions (5.1c) and (5.1b) are satisfied. In particular, this implies that (5.3) holds. To prove the inequality, we proceed as follows: for two positive operators $\rho, \sigma \geq 0$, Uhlmann's theorem implies that there exists a finite-dimensional Hilbert space $\mathcal{H}_Q$ and two vectors $\psi, \phi \in \mathcal{H}_Q \otimes \mathcal{H}$ (purifications) such that $\mathrm{tr}_Q\left[|\psi\rangle\langle\psi|\right] = \rho$ and $\mathrm{tr}_Q\left[|\phi\rangle\langle\phi|\right] = \sigma$ and $\sqrt{F}(\rho,\sigma) = |\langle\psi|\phi\rangle|$. We further note that $(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A)|\psi\rangle$ and $(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B)|\phi\rangle$ are purifications of $T_A^{\downarrow}(\rho)$ and $T_B^{\downarrow}(\sigma)$. Hence, Uhlmann's theorem implies that

$$\sqrt{F}(T_A^{\downarrow}(\rho), T_B^{\downarrow}(\sigma)) \geq |\langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A)\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B)\phi\rangle|\,. \tag{5.4}$$

By inserting $\mathbb{1}_Q \otimes P + \mathbb{1}_Q \otimes P^{\perp}$ (which is equal to the identity) and expanding the scalar product, we obtain

$$\begin{aligned}\text{RHS of (5.4)} = |&\langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P)\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P)\phi\rangle & \text{(5.5a)}\\ &+ \langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P)\phi\rangle & \text{(5.5b)}\\ &+ \langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P)\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P^{\perp})\phi\rangle & \text{(5.5c)}\\ &+ \langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P^{\perp})\phi\rangle|. & \text{(5.5d)}\end{aligned}$$

It is not hard to see from (5.3) that the terms (5.5b) and (5.5c) vanish. Explicitly, we have

$$\begin{aligned}\text{(5.5b)} &= \langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P)\phi\rangle\\ &= \langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P)\phi\rangle\\ &= \langle(\mathbb{1}_Q \otimes V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P)\phi\rangle\\ &= \langle(\mathbb{1}_Q \otimes V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes V_A P)\phi\rangle\\ &= \langle\psi|(\mathbb{1}_Q \otimes P^{\perp}P)\phi\rangle\\ &= 0,\end{aligned}$$

and similarly for (5.5c). Adding and subtracting $\langle(\mathbb{1}_Q \otimes P^{\perp})\psi|(\mathbb{1}_Q \otimes P^{\perp})\phi\rangle$ and using the inverse triangular inequality yields

$$\begin{aligned}\text{(5.5)} \geq\ &|\langle(\mathbb{1}_Q \otimes P)\psi|(\mathbb{1}_Q \otimes P)\phi\rangle + \langle(\mathbb{1}_Q \otimes P^{\perp})\psi|(\mathbb{1}_Q \otimes P^{\perp})\phi\rangle|\\ &- |\langle(\mathbb{1}_Q \otimes (P_A \otimes \mathbb{1})V_A P^{\perp})\psi|(\mathbb{1}_Q \otimes (P_B \otimes \mathbb{1})V_B P^{\perp})\phi\rangle\\ &- \langle(\mathbb{1}_Q \otimes P^{\perp})\psi|(\mathbb{1}_Q \otimes P^{\perp})\phi\rangle|.\end{aligned} \tag{5.6}$$

We further use $P^\perp P = 0$ (thus $\sqrt{F}(\rho,\sigma) = |\langle(\mathbb{1}_Q \otimes P)\psi|(\mathbb{1}_Q \otimes P)\phi\rangle + \langle(\mathbb{1}_Q \otimes P^\perp)\psi|(\mathbb{1}_Q \otimes P^\perp)\phi\rangle|$) and some rearrangement to arrive at

$$(5.6) = \sqrt{F}(\rho,\sigma) - |\langle(\mathbb{1}_Q \otimes P^\perp)\psi|(\mathbb{1}_Q \otimes P^\perp(V_A^\dagger(P_A P_B \otimes \mathbb{1})V_B - \mathbb{1})P^\perp)\phi\rangle|. \tag{5.7}$$

As by assumption, $P^\perp = \sum_k P_k$ and $P_k P_l = 0$ for $k \neq l$, we get

$$\begin{aligned}(5.7) &\geq \sqrt{F}(\rho,\sigma) - \sum_{k=1}^{K} |\langle(\mathbb{1}_Q \otimes P_k)\psi|(\mathbb{1}_Q \otimes P_k(V_A^\dagger(P_A P_B \otimes \mathbb{1})V_B - \mathbb{1})P_k)\phi\rangle| \\ &\geq \sqrt{F}(\rho,\sigma) - \sum_{k=1}^{K} \left\{ \left\| P_k(V_A^\dagger(P_A P_B \otimes \mathbb{1})V_B - \mathbb{1})P_k \right\| \right. \\ &\qquad \left. \|(\mathbb{1}_Q \otimes P_k)\psi\| \, \|(\mathbb{1}_Q \otimes P_k)\phi\| \right\} \\ &= \sqrt{F}(\rho,\sigma) - \sum_{k=1}^{K} \left\| P_k(V_A^\dagger(P_A P_B \otimes \mathbb{1})V_B - \mathbb{1})P_k \right\| \sqrt{\mathrm{tr}\,[P_k\rho]\,\mathrm{tr}\,[P_k\sigma]},\end{aligned} \tag{5.8}$$

where we used the Cauchy–Schwarz inequality and the sub-multiplicativity of the matrix norm to get from the first to the second line. For the last line, we used

$$\begin{aligned}\|\mathbb{1}_Q \otimes P_k\psi\|^2 = \langle\psi|(\mathbb{1}_Q \otimes P_k)\psi\rangle = \mathrm{tr}\,[(\mathbb{1}_Q \otimes P_k)|\psi\rangle\langle\psi|] = \mathrm{tr}\,[P_k \mathrm{tr}_Q\,[|\psi\rangle\langle\psi|]] \\ = \mathrm{tr}\,[P_k\rho]\,.\end{aligned}$$

As the only constraints that V_A, V_B, P_A, P_B, and $\mathcal{H}_E$ have to satisfy are the ones of Definition 5.1, we conclude that

$$(5.8) \geq \sqrt{F}(\rho,\sigma) - C_{\mathcal{V},\mathcal{W}}^{(T_A^\downarrow, T_B^\downarrow)} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\,[P_k\rho]\,\mathrm{tr}\,[P_k\sigma]}.$$

This proves the claim. □

Proposition 5.2 does not allow for ancillary systems. In the following proposition, which is an iterated refinement of the preceding one, we show that this problem can be solved by applying Proposition 5.2 to $T^\downarrow \otimes \mathrm{id}$.

Proposition 5.3 (Technical no-go theorem). *For* $\dim(\mathcal{H}) < \infty$*, let* $T_A^\downarrow, T_B^\downarrow : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two completely positive trace non-increasing maps. Let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *such that* $T_A^\downarrow|_{\mathcal{B}_1(\mathcal{V})}$ *is trace-preserving and* $T_A^\downarrow|_{\mathcal{B}_1(\mathcal{V})} = T_B^\downarrow|_{\mathcal{B}_1(\mathcal{V})}$*. Let* $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_K\}$ *be a collection of mutually orthogonal subspaces of* $\mathcal{V}^\perp$*, such that* $\mathcal{V}^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_K$*. We denote the orthogonal projections onto these subspaces by* $P_1, P_2, \ldots, P_K$*. Furthermore, let* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be completely positive maps such that* $T_A - T_A^\downarrow$ *and* $T_B - T_B^\downarrow$ *are also completely positive. Then, for every finite-dimensional* N*-step discrimination* *strategy* $D = (\mathcal{H}, \mathcal{H}_Z, s_0, \Lambda)$*, we have*

$$1 - \sqrt{F}(\rho_N^{T_A}, \rho_N^{T_B}) \leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\rho_i^{T_A^{\downarrow}}\right]\right] \cdot \mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\rho_i^{T_B^{\downarrow}}\right]\right]}, \tag{5.9}$$

where ρ *is the intermediate state map of* D*. Furthermore,* $C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \leq 2$*.*

Corollary 5.4. *For* $\dim(\mathcal{H}) < \infty$*, let* $T_A^{\downarrow}, T_B^{\downarrow} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two completely positive trace non-increasing maps. Let* $\mathcal{V}$ *be a subspace of* $\mathcal{H}$ *such that* $T_A^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$ *is trace-preserving and* $T_A^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})} = T_B^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$*. Then,*

$$1 - \sqrt{F}(\rho_N^{T_A}, \rho_N^{T_B}) \leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\rho_i^{T_A^{\downarrow}}\right]\right] \cdot \mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\rho_i^{T_B^{\downarrow}}\right]\right]}. \tag{5.10}$$

Proof. To reduce the overhead in notation, we define $\rho_i := \rho_i^{T_A}$, $\rho_i^{\downarrow} := \rho_i^{T_A^{\downarrow}}$ and $\sigma_i := \rho_i^{T_B}$, $\sigma_i^{\downarrow} := \rho_i^{T_B^{\downarrow}}$. We start to prove the proposition by showing that

$$1 - \sqrt{F}(\rho_N, \sigma_N) \leq 1 - \sqrt{F}(\rho_N^{\downarrow}, \sigma_N^{\downarrow}). \tag{5.11}$$

This inequality follows from the strong concavity of the fidelity and the observation that $\rho_N - \rho_N^{\downarrow} \geq 0$ and $\sigma_N - \sigma_N^{\downarrow} \geq 0$. The latter statement follows inductively, as $\rho_0 - \rho_0^{\downarrow} = 0 \geq 0$ and

$$\begin{aligned} \rho_{i+1} - \rho_{i+1}^{\downarrow} &= \Lambda_i((T_A \otimes \mathrm{id})(\rho_i) - (T_A^{\downarrow} \otimes \mathrm{id})(\rho_i^{\downarrow})) \\ &= \Lambda_i((T_A \otimes \mathrm{id})(\rho_i - \rho_i^{\downarrow}) + ((T_A - T_A^{\downarrow}) \otimes \mathrm{id})(\rho_i^{\downarrow})) \\ &\geq 0. \end{aligned}$$

The last line follows, as by induction $\rho_i - \rho_i^{\downarrow} \geq 0$ and $T_A - T_A^{\downarrow}$ is, by assumption, completely positive. Replacing ρ by σ and A by B in the argument above shows that also $\sigma_N - \sigma_N^{\downarrow} \geq 0$. We write $\Delta\rho := \rho_N - \rho_N^{\downarrow}$ and $\Delta\sigma := \sigma_N - \sigma_N^{\downarrow}$ and use the strong concavity (see [25], p. 414) and the non-negativity of the fidelity, to obtain the following inequality:

$$\begin{aligned} \sqrt{F}(\rho_N, \sigma_N) &= \sqrt{F}(\rho_N^{\downarrow} + \Delta\rho, \sigma_N^{\downarrow} + \Delta\sigma) \\ &\geq \sqrt{F}(\rho_N^{\downarrow}, \sigma_N^{\downarrow}) + \sqrt{F}(\Delta\rho, \Delta\sigma) \\ &\geq \sqrt{F}(\rho_N^{\downarrow}, \sigma_N^{\downarrow}), \end{aligned}$$

which is equivalent to (5.11). To prove (5.9), it remains to show that

$$1 - \sqrt{F}(\rho_N^{\downarrow}, \sigma_N^{\downarrow}) \leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sum_{k=0}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\rho_i^{\downarrow}\right]\right] \cdot \mathrm{tr}\left[P_k \mathrm{tr}_Z \left[\sigma_i^{\downarrow}\right]\right]}. \tag{5.12}$$

To this end, notice that if $T_A^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})} = T_B^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$, then $(T_A^{\downarrow} \otimes \mathrm{id})|_{\mathcal{B}_1(\mathcal{V} \otimes \mathcal{H}_Z)} = (T_B^{\downarrow} \otimes \mathrm{id})|_{\mathcal{B}_1(\mathcal{V} \otimes \mathcal{H}_Z)}$. Hence, $T_A' := (T_A^{\downarrow} \otimes \mathrm{id})$, $T_B' := (T_B^{\downarrow} \otimes \mathrm{id})$, $\mathcal{V}' := \mathcal{V} \otimes \mathcal{H}_Z$ and

$\mathcal{W}' := \{\mathcal{W}_1 \otimes \mathcal{H}_Z, \ldots, \mathcal{W}_K \otimes \mathcal{H}_Z\}$ satisfy the assumptions of Proposition 5.2. Furthermore, as the fidelity is non-decreasing under the action of the channel Λ_i (see [25], p. 414), we have

$$\begin{aligned}\sqrt{F}(\rho_i^{\downarrow}, \sigma_i^{\downarrow}) - \sqrt{F}(\rho_{i+1}^{\downarrow}, \sigma_{i+1}^{\downarrow}) &= \sqrt{F}(\rho_i^{\downarrow}, \sigma_i^{\downarrow}) - \sqrt{F}(\Lambda_i \circ T'_A(\rho_i^{\downarrow}), \Lambda_i \circ T'_B(\sigma_i^{\downarrow})) \\ &\leq \sqrt{F}(\rho_i^{\downarrow}, \sigma_i^{\downarrow}) - \sqrt{F}(T'_A(\rho_i^{\downarrow}), T'_B(\sigma_i^{\downarrow})).\end{aligned}$$

We want to apply Proposition 5.2 to the RHS of this expression. To do this correctly, we should notice that the projections, appearing in (5.2), project onto $\mathcal{W}_k \otimes \mathcal{H}_Z$, and hence are equal to $P_k \otimes \mathbb{1}$. Also, if V_A, V_B, P_A, and P_B satisfy Conditions (5.1b) and (5.1c), then $V_A \otimes \mathbb{1}, V_B \otimes \mathbb{1}, P_A \otimes \mathbb{1}$, and $P_B \otimes \mathbb{1}$ satisfy the Conditions (5.1b) and (5.1c) for T'_A and T'_B. If we plug this into (5.1a) and use that in general $\|X \otimes \mathbb{1}\| = \|X\|$, we obtain

$$C_{\mathcal{V}',\mathcal{W}'}^{(T_A^{\downarrow}\otimes \mathrm{id}, T_B^{\downarrow}\otimes \mathrm{id})} \leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})}.$$

Using these observations, we get

$$\begin{aligned}\sqrt{F}(\rho_i^{\downarrow}, \sigma_i^{\downarrow}) - \sqrt{F}(\rho_{i+1}^{\downarrow}, \sigma_{i+1}^{\downarrow}) &\leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[(P_k \otimes \mathbb{1})\rho_i^{\downarrow}\right] \mathrm{tr}\left[(P_k \otimes \mathbb{1})\sigma_i^{\downarrow}\right]} \\ &= C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\rho_i^{\downarrow}\right]\right] \mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\sigma_i^{\downarrow}\right]\right]}.\end{aligned}$$

Equivalently,

$$\sqrt{F}(\rho_{i+1}^{\downarrow}, \sigma_{i+1}^{\downarrow}) \geq \sqrt{F}(\rho_i^{\downarrow}, \sigma_i^{\downarrow}) - C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\rho_i^{\downarrow}\right]\right] \mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\sigma_i^{\downarrow}\right]\right]}.$$

If we iterate this inequality, we obtain

$$\sqrt{F}(\rho_N^{\downarrow}, \sigma_N^{\downarrow}) \geq \sqrt{F}(\rho_0^{\downarrow}, \sigma_0^{\downarrow}) - C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sum_{k=1}^{K} \sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\rho_i^{\downarrow}\right]\right] \mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\sigma_i^{\downarrow}\right]\right]}.$$

Using $\sqrt{F}(\rho_0^{\downarrow}, \sigma_0^{\downarrow}) = \sqrt{F}(s_0, s_0) = 1$ and some rearrangement establishes (5.12) and completes the proof of the theorem. $\square$

To connect this technical result with the main results of this section, we need two auxiliary lemmas.

Lemma 5.5. *For* $\dim(\mathcal{H}) < \infty$*, let* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two channels and let* D *be a finite-dimensional* N*-step discrimination strategy and* Π *be a two-valued POVM. Then,*

$$\frac{(1 - 2P_e(D, \Pi))^2}{2} \leq 1 - \sqrt{F}(\rho_N^{T_A}, \rho_N^{T_B}), \tag{5.13}$$

where ρ *is the intermediate state map of* D*.*

Proof. By definition,

$$P_e(D, \Pi) = \frac{1}{2}\left[\mathrm{tr}\left[\pi_B \rho_N^{T_A}\right] + \mathrm{tr}\left[\pi_A \rho_N^{T_B}\right]\right]. \tag{5.14}$$

If we minimize over the possible two-valued POVMs Π', the famous Holevo-Helstrom formula reads

$$P_e^m(D) := \min_{\Pi'} P_e(D,\Pi') = \frac{1}{2}\left[1 - \frac{1}{2}\left\|\rho_N^{T_A} - \rho_N^{T_B}\right\|_1\right].$$

Since $0 \leq P_e(D,\Pi) \leq \frac{1}{2}$, we have $1 - 2P_e(D,\Pi) \geq 0$. Thus,

$$\frac{(1-2P_e(D,\Pi))^2}{2} \leq \frac{(1-2P_e^m(D))^2}{2}. \tag{5.15}$$

By the Fuchs–van de Graaf inequality (see [25], p. 416),

$$\frac{1}{2}\|\rho-\sigma\|_1 \leq \sqrt{1-\sqrt{F}(\rho,\sigma)^2}.$$

Thus,

$$\begin{aligned}
\frac{(1-2P_e^m(D))^2}{2} &= \frac{\left(\frac{1}{2}\left\|\rho_N^{T_A}-\rho_N^{T_B}\right\|_1\right)^2}{2}\\
&\leq \frac{1-\sqrt{F}(\rho_N^{T_A},\rho_N^{T_B})^2}{2}\\
&= (1-\sqrt{F}(\rho_N^{T_A},\rho_N^{T_B}))\frac{1+\sqrt{F}(\rho_N^{T_A},\rho_N^{T_B})}{2}\\
&\leq 1-\sqrt{F}(\rho_N^{T_A},\rho_N^{T_B}).
\end{aligned}$$

Together with (5.15), this proves the claim. □

Lemma 5.6. *For* $\dim(\mathcal{H}) < \infty$, *let* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two channels with vacuum* $v \in \mathcal{H}$. *Let* $\mathcal{V}_{T_A}$ *and* $\mathcal{V}_{T_B}$ *be the respective maximal vacuum subspaces and let* $T_A^{\downarrow}$ *and* $T_B^{\downarrow}$ *be as in Definition 3.3 (Eq. 3.6). Furthermore, let* $\mathcal{V}$ *be a subspace such that* $v \in \mathcal{V}$ *and* $\mathcal{V} \subseteq \mathcal{V}_{T_A} \cap \mathcal{V}_{T_B}$. *Let* $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_K\}$ *be a collection of mutually orthogonal subspaces of* $\mathcal{V}^{\perp}$, *such that* $\mathcal{V}^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_K$. *Denote the orthogonal projections onto these subspaces by* $P_1, P_2, \ldots, P_K$. *Then,*

$$\frac{(1-2P_e(D,\Pi))^2}{2} \leq C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})} \sum_{i=0}^{N-1}\sum_{k=1}^{K}\sqrt{\mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right]\cdot \mathrm{tr}\left[P_k \mathrm{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right]}, \tag{5.16}$$

for all finite-dimensional N*-step discrimination strategies* $D = (\mathcal{H}, \mathcal{H}_Z, s_0, \Lambda)$ *and all two-valued POVMs,* Π.

Proof. By Lemma 5.5, we have for any finite-dimensional N-step discrimination strategy D and any two-valued POVM, Π, that

$$\frac{(1-2P_e(D,\Pi))^2}{2} \leq 1-\sqrt{F}(\rho_N^{T_A},\rho_N^{T_B}). \tag{5.17}$$

We want to apply Proposition 5.3 to the RHS of this inequality. To this end, we have to define the quantities appearing in that proposition. We identify $T_A, T_B, \mathcal{V}$, and $\mathcal{W}$ with the objects that bear the same name. In the following let $X \in \{A, B\}$. We define $T_X^{\downarrow}$ as in Definition 3.3 and need to check that

$T_X - T_X^{\downarrow}$ is completely positive and that $T_X^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$ is trace-preserving. To this end, we fix a Stinespring isometry $V_X : \mathcal{H} \to \mathcal{H}_E \otimes \mathcal{H}$ of T_X. Then, $T_X^{\downarrow}$ is defined by

$$T_X^{\downarrow}(\cdot) = \mathrm{tr}_E \left[(P_v^{(X)} \otimes \mathbb{1}) V_X \cdot V_X^{\dagger} \right],$$

where $P_v^{(X)}$ is the projection onto the support of $\mathrm{tr}_{\mathcal{H}} \left[V_X |v\rangle\langle v| V_X^{\dagger} \right]$. It follows immediately from this expression that $T_X - T_X^{\downarrow}$ is completely positive. To see that $T_X^{\downarrow}|_{\mathcal{B}_1(\mathcal{V}_{T_X})}$ is trace-preserving, note that by Definition 3.9

$$\mathcal{V}_{T_X} = V_X^{-1} \left[\mathrm{supp}(\mathrm{tr}_{\mathcal{H}} \left[V_X |v\rangle\langle v| V_X^{\dagger} \right]) \otimes \mathcal{H} \right].$$

Thus, for any[15] $\rho \in \mathcal{B}_1(\mathcal{V}_{T_X})$,

$$V_X \rho V_X^{\dagger} \in \mathcal{B}_1(\mathrm{supp}(\mathrm{tr}_{\mathcal{H}} \left[V_X |v\rangle\langle v| V_X^{\dagger} \right]) \otimes \mathcal{H}).$$

As $P_v^{(X)} \otimes \mathbb{1}$ is the projection onto $\mathrm{supp}(\mathrm{tr}_{\mathcal{H}} \left[V_X |v\rangle\langle v| V_X^{\dagger} \right]) \otimes \mathcal{H}$, we have

$$T_X^{\downarrow}|_{\mathcal{B}_1(\mathcal{V}_{T_X})}(\cdot) = \mathrm{tr}_E \left[(P_v^{(X)} \otimes \mathbb{1}) V_X \cdot V_X^{\dagger} \right] = \mathrm{tr}_E \left[V_X \cdot V_X^{\dagger} \right] = T_X|_{\mathcal{B}_1(\mathcal{V}_{T_X})}(\cdot).$$

Thus, $T_X^{\downarrow}|_{\mathcal{B}_1(\mathcal{V}_{T_X})}$ is trace-preserving, as $T_X|_{\mathcal{B}_1(\mathcal{V}_{T_X})}$ is. As $\mathcal{V}$ is a subspace of $\mathcal{V}_{T_X}$, also $T_X^{\downarrow}|_{\mathcal{B}_1(\mathcal{V})}$ is trace-preserving. This is what we have claimed. As all assumptions are satisfied, we can invoke Proposition 5.3, which directly yields the desired inequality. □

The next result has already been stated in the results section.

Theorem 5.7 (No-go theorem). *For $\dim(\mathcal{H}) < \infty$, let $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be two channels with vacuum $v \in \mathcal{H}$. If there exists no subspace $\mathcal{V} \subseteq \mathcal{H}$ such that $v \in \mathcal{V}$, at least one of the channels T_A or T_B is isometric on $\mathcal{V}$ and $T_A|_{\mathcal{B}_1(\mathcal{V})} \neq T_B|_{\mathcal{B}_1(\mathcal{V})}$, then there exists a constant $C < \infty$, such that*

$$(1 - 2P_e(D, \Pi))^2 \leq C \sqrt{P_I^{T_A}(D) \cdot P_I^{T_B}(D)} \leq C \max(P_I^{T_A}(D), P_I^{T_B}(D)), \tag{5.18}$$

for all finite-dimensional N-step discrimination strategies D and all two-valued POVMs, Π. Hence, T_A and T_B cannot be discriminated in an "interaction-free" manner.

Remark 5.8. The assumption "The statement that T_A or T_B is isometric on a subspace $\mathcal{V}$, with $v \in \mathcal{V}$, already implies that $T_A|_{\mathcal{B}_1(\mathcal{V})} = T_B|_{\mathcal{B}_1(\mathcal{V})}$" can be rephrased in two equivalent ways. The first one is that the Conditions 1, 2, and 3 in the Main Theorem (Sect. 2) cannot be fulfilled simultaneously. The second reformulation is that for the maximum vacuum subspaces $\mathcal{V}_{T_A}$ and $\mathcal{V}_{T_B}$, we have $\mathcal{V}_{T_A} = \mathcal{V}_{T_B}$ and $T_A|_{\mathcal{B}_1(\mathcal{V}_{T_A})} = T_B|_{\mathcal{B}_1(\mathcal{V}_{T_B})}$. The equivalence follows directly from the characterization of maximal vacuum subspaces in

[15] Remember that for a subspace $\mathcal{V}_0 \subseteq \mathcal{H}$, the operators in $\mathcal{B}_1(\mathcal{V}_0)$ are those that can be written in the form $\sum_{i,j} \alpha_{ij} |\psi_i\rangle\langle\psi_i|$, with $\alpha_{ij} \in \mathbb{C}$ and $\psi_i \in \mathcal{V}_0$.

Lemma 3.10 4. This second reformulation is not only important in the proof, but also if one wants to check this criterion, as $\mathcal{V}_{T_A}$ and $\mathcal{V}_{T_B}$ are efficiently computable directly from Definition 3.9.

Proof. We use the second characterization in Remark 5.8. That is, $\mathcal{V}_{T_A} = \mathcal{V}_{T_B}$ and $T_A|_{\mathcal{B}_1(\mathcal{V}_{T_A})} = T_B|_{\mathcal{B}_1(\mathcal{V}_{T_B})}$. We set $\mathcal{V} := \mathcal{V}_{T_A}$ and let $T_A^{\downarrow}$ and $T_B^{\downarrow}$ be as in Definition 3.3. Furthermore, we define $\mathcal{W} := \{\mathcal{W}_1\}$, with $\mathcal{W}_1 := \mathcal{V}^{\perp}$. Then, by Lemma 5.6, we have

$$(1 - 2P_e(D,\Pi))^2 \leq 2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})} \sum_{i=0}^{N-1} \sqrt{\mathrm{tr}\left[P^{\perp}\mathrm{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right] \cdot \mathrm{tr}\left[P^{\perp}\mathrm{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right]}, \tag{5.19}$$

where $P^{\perp}$ is the orthogonal projection onto $\mathcal{W}_1 = \mathcal{V}^{\perp}$. As $\mathcal{V}$ is the maximum vacuum subspace of T_A and T_B, Lemma 3.10 5 implies that for $X \in \{A, B\}$, there is a constant $C_{T_X} > 0$ such that $\mathfrak{i}_{T_X}(\rho) \geq C_{T_X}\mathrm{tr}\left[P^{\perp}\rho\right]$ for all $\rho \geq 0$. As $\mathrm{tr}_Z\left[\rho_i^{T_X^{\downarrow}}\right] \geq 0$, we get

$$\begin{aligned}
(5.19) &\leq \frac{2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})}}{\sqrt{C_{T_A}C_{T_B}}} \sum_{i=0}^{N-1} \sqrt{\mathfrak{i}_{T_A}\left(\mathrm{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right)\mathfrak{i}_{T_B}\left(\mathrm{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right)} \\
&\leq \frac{2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})}}{\sqrt{C_{T_A}C_{T_B}}} \sqrt{\left(\sum_{i=0}^{N-1}\mathfrak{i}_{T_A}\left(\mathrm{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right)\right)\left(\sum_{i=0}^{N-1}\mathfrak{i}_{T_B}\left(\mathrm{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right)\right)} \\
&= \frac{2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})}}{\sqrt{C_{T_A}C_{T_B}}} \sqrt{P_I^{T_A}(D)\cdot P_I^{T_B}(D)},
\end{aligned}$$

where we used the Cauchy–Schwarz inequality (on $\mathbb{C}^N$) to obtain the second line and the definition of the "interaction" probability in the last line. We note that the last inequality in the statement of the theorem is trivial. Thus, by setting $C := \frac{2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow},T_B^{\downarrow})}}{\sqrt{C_{T_A}C_{T_B}}}$, we have proven the claim. □

The following theorem is the technical version of the result stated in the results section.

Theorem 5.9 (Rate limit theorem). *For* $\dim(\mathcal{H}) < \infty$, *let* $T_A, T_B : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ *be two channels with vacuum* $v \in \mathcal{H}$. *Let* $\mathcal{V}_{T_A}$ *and* $\mathcal{V}_{T_B}$ *be the respective maximal vacuum subspace of* T_A *and* T_B. *Set* $\mathcal{V} := \mathcal{V}_{T_A} \cap \mathcal{V}_{T_B}$. *Suppose that* $T_A|_{\mathcal{B}_1(\mathcal{V})} = T_B|_{\mathcal{B}_1(\mathcal{V})}$ *and that* $\mathcal{V}^{\perp} \cap \mathcal{V}_{T_A}$ *and* $\mathcal{V}^{\perp} \cap \mathcal{V}_{T_B}$ *are orthogonal.*
Then there exists a constant $C > 0$ *such that*

$$\max(P_I^{T_A}(D), P_I^{T_B}(D)) \geq C\,\frac{(1 - 2P_e(D,\Pi))^4}{N}, \tag{5.20}$$

for all finite-dimensional N*-step discrimination strategies* D, *and any two-valued POVM* Π.

Proof. The proof is similar to the one of the no-go theorem. Let $T_A^{\downarrow}$ and $T_B^{\downarrow}$ be as in Definition 3.3 and set $\mathcal{V} := \mathcal{V}_{T_A} \cap \mathcal{V}_{T_B}$. Furthermore, define $\mathcal{W} := \{\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3\}$ with $\mathcal{W}_1 := \mathcal{V}^{\perp} \cap \mathcal{V}_{T_A}$, $\mathcal{W}_2 := \mathcal{V}^{\perp} \cap \mathcal{V}_{T_B}$ and $\mathcal{W}_3 := (\mathcal{W}_1 \oplus \mathcal{W}_2)^{\perp} \cap \mathcal{V}^{\perp}$. Clearly, $\mathcal{W}_1, \mathcal{W}_2$, and $\mathcal{W}_3$ are mutually orthogonal and their direct sum is $\mathcal{V}^{\perp}$. Furthermore, $\mathcal{W}_2 \oplus \mathcal{W}_3 = \mathcal{V}_{T_A}^{\perp}$ and $\mathcal{W}_1 \oplus \mathcal{W}_3 = \mathcal{V}_{T_B}^{\perp}$. Thus, by Lemma 5.6, we have

$$(1 - 2P_e(D, \Pi))^2 \leq 2C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sum_{k=1}^{3} \sqrt{\operatorname{tr}\left[P_k \operatorname{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right] \cdot \operatorname{tr}\left[P_k \operatorname{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right]}, \tag{5.21}$$

where for $k \in \{1, 2, 3\}$, P_k is the orthogonal projection onto $\mathcal{W}_k$. Using the Cauchy–Schwarz inequality (on $\mathbb{C}^3$), and the fact that probabilities are less than one, and afterwards the Cauchy–Schwarz inequality on $\mathbb{C}^N$, we get

$$\begin{aligned}(5.21) &\leq \sqrt{12} C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sum_{i=0}^{N-1} \sqrt{\operatorname{tr}\left[(P_2 + P_3)\operatorname{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right] + \operatorname{tr}\left[(P_1 + P_3)\operatorname{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right]} \\ &\leq \sqrt{12N} C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sqrt{\sum_{i=0}^{N-1} \operatorname{tr}\left[P_{\mathcal{V}_{T_A}}^{\perp} \operatorname{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right] + \operatorname{tr}\left[P_{\mathcal{V}_{T_B}}^{\perp} \operatorname{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right]},\end{aligned} \tag{5.22}$$

where $P_{\mathcal{V}_{T_A}}^{\perp}$ and $P_{\mathcal{V}_{T_B}}^{\perp}$ are the projections onto $\mathcal{V}_{T_A}^{\perp}$ and $\mathcal{V}_{T_B}^{\perp}$. Lemma 3.10, 5 implies that for $X \in \{A, B\}$, there is a constant $C_{T_X} > 0$ such that $\mathfrak{i}_{T_X}(\rho) \geq C_{T_X} \operatorname{tr}\left[P_{\mathcal{V}_{T_X}}^{\perp} \rho\right]$ for all $\rho \geq 0$. As $\operatorname{tr}_Z\left[\rho_i^{T_X^{\downarrow}}\right] \geq 0$, we get

$$\begin{aligned}(5.22) &\leq \sqrt{12N} C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sqrt{C_{T_A}^{-1} \sum_{i=0}^{N-1} \mathfrak{i}_{T_A}\left(\operatorname{tr}_Z\left[\rho_i^{T_A^{\downarrow}}\right]\right) + C_{T_B}^{-1} \sum_{i=0}^{N-1} \mathfrak{i}_{T_B}\left(\operatorname{tr}_Z\left[\rho_i^{T_B^{\downarrow}}\right]\right)} \\ &\leq C_{\mathcal{V}}^{(T_A^{\downarrow}, T_B^{\downarrow})} \sqrt{\frac{24}{\min(C_{T_A}, C_{T_B})}} \sqrt{N \max(P_I^{T_A}(D), P_I^{T_B}(D))}.\end{aligned}$$

Taking the square and defining $C := \frac{\min(C_{T_A}, C_{T_B})}{24 C_{\mathcal{V},\mathcal{W}}^{(T_A^{\downarrow}, T_B^{\downarrow})\,2}}$ proves the claim. □

6. Related Work

In this section we compare our setup and results to selected other works in the literature.[16] We start with a detailed comparison with the work on counterfactual computation (CFC) by Mitchison and Josza [7]. CFC aims to determine the outcome of a quantum computation without switching on the computer. Expressed in a language closer to ours, CFC aims to discriminate (counterfactually, the term analogous to "interaction-free") between two unitaries U_0 and

[16] Although we tried to make the discussion as self-contained as possible, this section is intended for the reader who is at last partially familiar with the referenced works.

U_1 defined on a bipartite system $\mathcal{H}_O \otimes \mathcal{H}_S$ (where "O" stands for output and "S" for switch).

Before defining what counterfactual means, we need to discuss the allowed discrimination strategies. Here, it is allowed to use the unknown unitary many times while performing unitary operations and measurements in between. It is also allowed to add an ancillary system $\mathcal{H}_Z$ of arbitrary size and to let the unitary operations act on the space $\mathcal{H}_O \otimes \mathcal{H}_S \otimes \mathcal{H}_Z$. This implies that measurements can be deferred until the unknown unitary was applied for the last time. Thus, if the unknown unitary U_r, $r \in \{0,1\}$ is used N times, the initial state is $\psi_I \in \mathcal{H}_O \otimes \mathcal{H}_S \otimes \mathcal{H}_Z$ and the intermediary unitaries are $V_1, V_2, \dots, V_{N-1} \in \mathcal{B}(\mathcal{H}_O \otimes \mathcal{H}_S \otimes \mathcal{H}_Z)$, then the ($r$-dependent) state before the final measurement is

$$\psi_F^r = (U_r \otimes \mathbb{1}_Z) V_{N-1} (U_r \otimes \mathbb{1}_Z) V_{N-2} \cdots (U_r \otimes \mathbb{1}_Z) V_1 (U_r \otimes \mathbb{1}_Z) \psi_I. \tag{6.1}$$

In preparation for defining the term counterfactual, one assumes that for each $r \in \{0,1\}$ we can split the switch space into two orthogonal spaces $\mathcal{H}_S = \mathcal{H}_S^{r,\text{off}} \oplus \mathcal{H}_S^{r,\text{on}}$, called the off and on subspaces, respectively. The interpretation here is that if we apply U_r to a state in $\mathcal{H}_O \otimes \mathcal{H}_S^{r,\text{off}}$, then the computer does not run. Consistent with this interpretation, it is also assumed that

$$U_r \psi = \psi, \ \text{ for all } \psi \in \mathcal{H}_O \otimes \mathcal{H}_S^{r,\text{off}}. \tag{6.2}$$

One then introduces a decomposition into so-called histories. To this end, one imagines that after each application of U_r a measurement was performed, projecting either onto $\mathcal{H}_O \otimes \mathcal{H}_S^{r,\text{off}} \otimes \mathcal{H}_Z$ or onto $\mathcal{H}_O \otimes \mathcal{H}_S^{r,\text{on}} \otimes \mathcal{H}_Z$. We denote the corresponding projections by P^r_{off} and P^r_{on}. On can then decompose ψ_F^r as:

$$\begin{aligned} \psi_F^r &= \sum_{h \in \{\text{on},\text{off}\}^N} v_h^r, \ \text{ with} \\ v_h^r &= P^r_{h_N} (U_r \otimes \mathbb{1}_Z) V_{N-1} \cdots P^r_{h_2} (U_r \otimes \mathbb{1}_Z) V_1 P^r_{h_1} (U_r \otimes \mathbb{1}_Z) \psi_I. \end{aligned} \tag{6.3}$$

Each of the on/off sequences h in (6.3) is called a history.

Suppose we perform a projective measurement on the final state with possible outcomes $m \in \{1, 2, \dots, M\}$ and associated projections $\{Q_1, Q_2, \dots, Q_M\}$. Mitchison and Josza (Definition 5.1 in [7]) then define an outcome m to be a *counterfactual outcome of type* $r \in \{0,1\}$, if

1. $Q_m v_h^r = 0$, if h is not the all-off history,
2. $Q_m \psi_F^{1-r} = 0$.

The first condition says that the only history consistent with the outcome m must be the all-off history and the second condition demands that the outcome m can only occur if the unknown unitary is U_r (and not U_{1-r}).

Now, how does CFC relate to "interaction-free" channel discrimination? First, one can interpret "interaction-free" channel discrimination in terms of CFC after some modifications, as follows. Consider a channel T with vacuum $v \in \mathcal{H}_I$, given by $T(\cdot) = \operatorname{tr}_E \left[V \cdot V^\dagger \right]$. In Sect. 3.1, we determined that the Demon's optimal strategy is to perform a two-outcome measurement on E (with corresponding projections P_v and $P_v^\perp$). After extending V to a unitary

U, we can interpret the whole space $\mathcal{H}_E \otimes \mathcal{H}_I$ as the switch space $\mathcal{H}_S$ and set $\mathcal{H}_O := \mathbb{C}$. A natural way to introduce the splitting of $\mathcal{H}_S$ into on and off subspace is then to define $\mathcal{H}_S^{\text{off}} = \text{range}(P_v) \otimes \mathcal{H}_I$ and $\mathcal{H}_S^{on} = \text{range}(P_v^\perp) \otimes \mathcal{H}_I$. Note, however, that this definition does not satisfy (6.2).[17] A violation of assumption (6.2) does not prevent us from defining histories, nor does it interfere with the definition of a counterfactual outcome as above. So, one might consider broadening the definition of CFC by dropping it. However, upon close investigation one finds that (the proofs of) all theorems in [7] rely crucially on that assumption. In any case, even after dropping that assumption, the definition of a counterfactual outcome above is still too restrictive to fully cover "interaction-free" channel discrimination, since we do not require that the "interaction" probability or the error probability are exactly zero (as demanded by CFC) but rather that they can be made arbitrarily small. This requires a probabilistic modification of the definition of a counterfactual outcome, such as the one suggested in the discussion section in [7]. We therefore conclude that "interaction-free" channel discrimination is consistent with a sufficiently broadened definition of CFC. Unfortunately, however, we do not think that this point of view has any important direct implications for the feasibility of the "interaction-free" channel discrimination task. The main reasons for this belief are that even after reformulation into the language of CFC, the allowed discrimination strategies differ considerably and that the only result in [7] that goes beyond the qubit case is that the number of insertions of U_r must tend to infinity for an optimal success probability.[18]

What about implications of our results for CFC? We believe that a conceptual weakness of CFC is that there are (in general) no observable consequences—in the sense that (the surroundings of) the apparatus changes—regardless of whether a computation was performed counterfactually or not. This is so because the imagined measurements after each application of the unknown unitary are not actually performed. We think that the question about a change of (the surroundings of) the apparatus is the relevant one for technical applications, which is our main focus. If one demands that the imaginary measurements are actually performed, then CFC becomes a special case of "interaction-free" channel discrimination by assigning to the unitary $U_r \in \mathcal{B}(\mathcal{H}_O \otimes \mathcal{H}_S)$ the channel $T_r : \mathcal{B}(\mathcal{H}_O \otimes \mathcal{H}_S)$ given by

$$T_r(\rho) = (\mathbb{1}_O \otimes P_S^{r,\text{off}})U\rho U^\dagger(\mathbb{1}_O \otimes P_S^{r,\text{off}}) + (\mathbb{1}_O \otimes P_S^{r,\text{on}})U\rho U^\dagger(\mathbb{1}_O \otimes P_S^{r,\text{on}}), \tag{6.4}$$

for all $\rho \in \mathcal{B}_1(\mathcal{H}_O \otimes \mathcal{H}_S)$, where $P_S^{r,\text{off}}$ and $P_S^{r,\text{on}}$ are the projections according to the splitting of $\mathcal{H}_S$ into on and off subspace. It follows from (6.2) that T_r

[17] For example, if U is defined on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by $U|00\rangle = |00\rangle, U|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + \langle 10|), U|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle), U|11\rangle = |11\rangle$ and $|0\rangle$ is the vacuum, then $P_v = |0\rangle\langle 0|$. However, the corresponding off subspace $\mathcal{H}_S^{\text{off}} = \text{span}\{|00\rangle, |01\rangle\}$ is not even left invariant by U.

[18] Note that this conclusion does not apply to all cases of our setting, since the number of applications for a perfect discrimination of two unitaries is finite.

is a channel with vacuum, where the vacuum can be taken to be any vector in $\mathcal{H}_O \otimes \mathcal{H}_S^{r,\mathrm{off}}$. Hence, our results apply to this setting.

From our technological point of view, some interpretational discussions in the literature can be avoided. For example, in [37], Hosten et. al claimed that they could discriminate counterfactually between four unitaries associated with the result of a Grover search. We agree with [38,39] that the proposal in [37] does not constitute a CFC for all possible outcomes in the sense of [7]. However, from the point of view of our model, this is a rather artificial debate. Since a unitarily evolving system does not interact with its surroundings (the Demon), there is no way to tell whether a computation has been performed or not by looking at the surroundings. In that sense, the task in [37] was (as every other discrimination task involving only unitary operations) performed in an "interaction-free" manner.

A work with a title similar to ours is "Interaction-free measurement as quantum channel discrimination" by Zhou and Yung [32]. The objective of their work was to determine if the Kwiat et. al protocol for detecting a semi-transparent object can be enhanced by using an entangled initial state. The study was conducted by employing tools from quantum channel theory, but no attempts were made to generalize the notion of "interaction-free" measurements. Generalizing this notion, however, is the main focus of the present work.

7. Conclusion and Open Problems

In our work, we have characterized when it is possible and impossible to discriminate quantum channels in an "interaction-free" manner. This answers the question, what can be done perfectly with "interaction-free" measurements. However, there are still some open questions. One question that is in direct succession of our work is, under which conditions two channels can be discriminated such that the "interaction" probability decays faster than N^{-1}. Another question would ask for a more quantitative treatment, i.e., even though one might not be able to discriminate two channels in an "interaction-free" manner, there still might be a significant quantum advantage over classical strategies. A related question suggested to us by an anonymous reviewer is what kind of information about the discriminator's strategy the Demon can obtain. In this context, we showed that the Demon cannot distinguish (under our conditions) between a strategy that always sends the vacuum through the channels and our proposed one. However, the more general question remains open. A big question concerns the influence of noise and decoherence. We note that noise may influence what can or cannot be done in both directions, since the noise can also be on the Demon's side and hence make his detection skills weaker. Before the no-go results for semitransparent objects were established [11,12], one anticipated application of "interaction-free" measurement was to eliminate the exposure of humans to radiation in medical applications such as X-ray scans. This is not possible. However, our no-go theorem does not touch the case of asymmetric "interaction-free" discrimination. That is, we may allow that one of the two objects to be discriminated gets destroyed (for example, by simply setting its transmission functional to zero). This might even be a desirable effect. For example, in a medical context, we would love to design a procedure such that a tumor gets destroyed, while the healthy tissue stays intact.

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Appendix A.

Lemma A.1 (Semi-simplicity of the peripheral spectrum). *Let $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ be a channel such that 1 is in the discrete spectrum of T. Then, for any $n \in \mathbb{N}$ and any (rectifiable) path inside the resolvent set of T that encloses 1, and separates 1 from $\sigma(T) \setminus \{1\}$, we have*

$$\frac{1}{2\pi i} \oint_{\Gamma_1} \frac{z^n}{z - T} \, \mathrm{d}z = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{1}{z - T} \, \mathrm{d}z. \tag{A.1}$$

Proof. For brevity, we denote the Riesz-Projection on the RHS of (A.1) by P. As 1 is in the discrete spectrum of T, Corollary 2.3.6 in [29] says that $TP = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{z}{z-T} \, \mathrm{d}z = P + N$, where N is a nilpotent operator that commutes with P. Hence $T = P + N + T_0$, where $T_0 := (\mathrm{id} - P)T(\mathrm{id} - P)$. By the analytic functional calculus, we have

$$(P+N)^n = \left(\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{z}{z-T}\,\mathrm{d}z\right)^n = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{z^n}{z-T}\,\mathrm{d}z.$$

If $N = 0$, then the claim follows, since P is a projection ($P^n = P$). To this end, assume that $N \neq 0$. Since N is nilpotent, there exists an integer D such that $N^D \neq 0$ and $N^{D+1} = 0$. As $N \neq 0$, we have $D \geq 1$. Choose $\rho \in \mathcal{B}_1(\mathcal{H})$ such that $N^D(\rho) \neq 0$ and $P(\rho) = \rho$. Note that $PT_0 = T_0 P = 0$. Thus, $T^n(\rho) = (P+N)^n(\rho) + T_0^n(\rho) = (P+N)^n(\rho)$. In particular, since T is a channel, $\|T^n\| = 1$ and thus

$$\|(P+N)^n(\rho)\| \leq \|\rho\|\,. \tag{A.2}$$

For $n \geq D$, we have

$$(P+N)^n(\rho) = \sum_{i=0}^{D} \binom{n}{i} N^i(\rho).$$

Furthermore, the vectors $\rho, N(\rho), N^2(\rho), \ldots N^D(\rho)$ are linearly independent. The coordinate function of $N(\rho)$ is $\binom{n}{1}$, which is unbounded for $n \to \infty$. Since the coordinate function can be extended to a continuous linear functional on $\mathcal{B}_1(\mathcal{H})$ (Hahn–Banach), the unboundedness contradicts (A.2). Hence, $N = 0$. □

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A.2 Quantum and classical dynamical semigroups of superchannels and semicausal channels

In this work, we investigate dynamical semigroups of superchannels and the related semicausal CP-maps in the quantum as well as in the classical setting. Superchannels are the most general transformation, transforming quantum channels into quantum channels, while preserving the probabilistic structure, in the same vein as quantum channels are the most general transformation of quantum states.
In the introduction, we motivate the use of dynamical semigroups as a model for the decay-processes that physical devices are subject to while they “age".
The main part is then concerned with the mathematical description of dynamical semigroups of superchannels in terms of their generators. That part of the paper is divided into two parts: One covering dynamical semigroups of classical superchannels and classical semicausal maps, and a part covering dynamical semigroups of quantum superchannels. Although the main focus is on the quantum part, the classical part is important, because its treatment allows us to tell which features we found in the quantum case have a classical counterpart and which features are purely quantum. We found indeed that in the quantum case the most general form for generators of normal CP-maps admit a superposition structure that does not have a classical counterpart.
Both the classical and the quantum sections are subdivided into three parts, where part one introduces the relevant notions of superchannels and semicausal maps (and in the classical case establishes the fundamental representation theorems - analogous to those of the quantum case [14,16]); part two studies dynamical semigroups of semicausal maps. We provide a full characterization of the generators of dynamical semigroups of semicausal maps. The characterization has two pillars: first a computationally efficient method that allows one to tell if a given generator generates a semigroup of semicausal CP-maps; and second a normal form - which in the quantum case is a refinement of the famous GKLS-form - that allows us to list all possible generators of semicausal CP-maps. This is accomplished by using a certain symmetrisation procedure that allows us to transfer parts of the known characterization of semicausal CP-maps (known under the name semilocalizability [15, 16]) to the corresponding generator. Moreover, we provide an algorithm that allows us to compute the various components of our normal form. In part three of the classical and the quantum part, we use that semicausal CP-maps and superchannels are related via a similarity transformation, where the transformation matrix is the Choi–Jamiołkowski isomorphism. This allows us to obtain a full characterization of the generators of dynamical semigroups of superchannels by translating the corresponding results from the generators of dynamical semigroups of semicausal CP-maps.

The study of dynamical semigroups of superchannels was suggested independently to both authors by Prof. Michael M. Wolf. It was subsequently decided to join forces to work on the problem together. I am the principal author of this paper. In particular, I came up with the idea for how to reduce the problem of characterizing semicausal maps in GKSL-form to characterizing semicausal CP-map – an insight ultimately leading to our main technical contribution, Theorem V.6. The formulation of the resulting proof, as structured by Lemmas V.9, V.10, V.11, V.12

and V.13 was also my responsibility. In the quantum part section, I also contributed Lemma V.17 and Theorem V.18 and the translation Lemmas V.20 and V.21. In the classical part of the paper, I developed the details of the prove strategy for Theorems IV.4 and IV.10 and I was heavily involved in Theorem IV.3. I was involved in all parts of the paper and did the majority of the writing.

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ABSTRACT

Quantum devices are subject to natural decay. We propose to study these decay processes as the Markovian evolution of quantum channels, which leads us to dynamical semigroups of superchannels. A superchannel is a linear map that maps quantum channels to quantum channels while satisfying suitable consistency relations. If the input and output quantum channels act on the same space, then we can consider dynamical semigroups of superchannels. No useful constructive characterization of the generators of such semigroups is known. We characterize these generators in two ways: First, we give an efficiently checkable criterion for whether a given map generates a dynamical semigroup of superchannels. Second, we identify a normal form for the generators of semigroups of quantum superchannels, analogous to the Gorini-Kossakowski-Lindblad-Sudarshan form in the case of quantum channels. To derive the normal form, we exploit the relation between superchannels and semicausal completely positive maps, reducing the problem to finding a normal form for the generators of semigroups of semicausal completely positive maps. We derive a normal for these generators using a novel technique, which applies also to infinite-dimensional systems. Our work paves the way for a thorough investigation of semigroups of superchannels: Numerical studies become feasible because admissible generators can now be explicitly generated and checked. Analytic properties of the corresponding evolution equations are now accessible via our normal form.

I. INTRODUCTION AND MOTIVATION

Anybody who has ever owned an electronic device knows that these devices have a finite lifespan after which they stop working properly. At least from a consumer perspective, a long lifespan is a desirable property for such devices. Thus, it is important for an engineer to know which kind of decay processes can affect a device in order to suppress them by an appropriate design. Certainly, these considerations will also become important for the design of quantum devices. We, therefore, propose to systematically study the decay processes that quantum devices can be subject to.

In this work, we take a first step in this direction by deriving the general form of linear time-homogeneous master equations that govern how quantum channels behave when inserted into a circuit board at different points in time. This leads to the study of dynamical semigroups of superchannels. Here, superchannels are linear transformations between quantum channels.[1]

Let us consider a concrete example (see Fig. 1). Suppose we are trying to estimate the optical transmissivity of some material (M), which we assume to depend on the polarization of the incident light. A simple approach is to send photons from a light source (S) through the material and to count how many photons arrive at the detector (D). We model the material by a quantum channel T_M, acting on the states of photons described as three-level systems, with the levels corresponding to vacuum, horizontal, and vertical polarization. In an idealized world, with a perfect vacuum in the regions between the source, the material, and the detector, we can infer the transmissivity from the measurement statistics of the state $T_M(\sigma)$, where σ is the state of the photon emitted from the source. However, in a more realistic scenario, even though we might have created an (almost) perfect vacuum between the devices at construction time, some particles are leaked into that region over time.

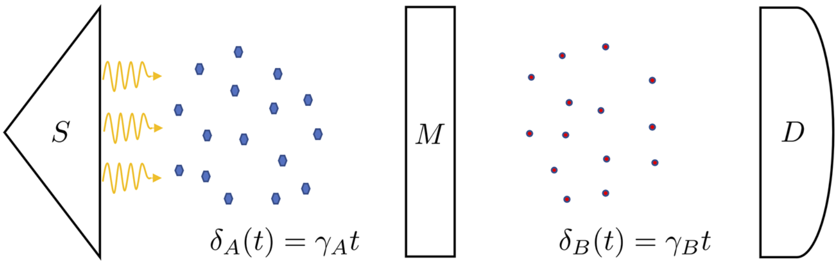

FIG. 1. Estimating the transmissivity of a material under the influence of an influx of particles into the regions between the components.

FIG. 2. If the particle density is low, then the incident photon interacts with the particles in the region sequentially and independently. The effect of a single interaction can be described by a channel ΔQ. Hence, the state after the first interaction is $\Delta Q(\sigma)$, the state after the second interaction is $\Delta Q(\Delta Q(\sigma))$, and so forth. The number of interactions is given by the product of the particle density δ and the volume V. Hence, the effect of an region with fixed volume is described by the channel $Q_\delta = (\Delta Q)^{\delta V}$. It follows that if $\delta = \delta_1 + \delta_2$, then $Q_{\delta_1+\delta_2} = (\Delta Q)^{\delta_1 V}(\Delta Q)^{\delta_2 V} = Q_{\delta_1} \circ Q_{\delta_2}$. The semigroup property for real δ can then be obtained in the continuum limit.

Then, interactions between the photons and these particles might occur, causing absorption or a change in polarization. Hence, the situation is no longer described accurately by T_M alone but also requires a description of the particle-filled regions.

To find such a description, we argue that the effect of particles in some region (here, either between S and M, or M and D) can be modeled by a quantum dynamical semigroup, parameterized by the particle density δ. If the particle density is reasonably low and Q_δ is the quantum channel describing the effect of the particles on the incident light at a given δ, then, as explained in Fig. 2, Q_δ satisfies the semigroup property $Q_{\delta_1+\delta_2} = Q_{\delta_1} \circ Q_{\delta_2}$. Furthermore, if there are no particles, then there should be no effect. Hence, $Q_0 = \mathrm{id}$. After adding continuity in the parameter δ as a further natural assumption, the family $\{Q_\delta\}_{\delta \geq 0}$ forms a quantum dynamical semigroup. That is, we can write $Q_\delta = e^{L\delta}$ for some generator L in Gorini-Kossakowski-Lindblad-Sudarshan (GKLS)-form.

If we assume in our example that particles of type A are leaked into the region between S and M at a rate γ_A and that particles of type B are leaked into the region between M and D at a rate γ_B, then the overall channel describing the transformation that emitted photons undergo at time t is given by

$$\hat{S}_t(T_M) = e^{\gamma_B L_B t} \circ T_M \circ e^{\gamma_A L_A t},$$

where L_A and L_B are the generators of the dynamical semigroups describing the effect of the particles in the respective regions.

We note that at any fixed time, $\hat{S}_t$ interpreted as a map on quantum channels is a superchannel written in "circuit"-form. This means that $\hat{S}_t$ describes a transformation of quantum channels implemented via pre- and post-processing. Furthermore, $\hat{S}_t(T_M)$ can be determined by solving the time-homogenous master equation

$$\frac{d}{dt}T(t) = \hat{L}(T(t)),$$

where $\hat{L}(T) = \gamma_A L_A \circ T + \gamma_B T \circ L_B$, with the initial condition $T(0) = T_M$. In other words, we have

$$\hat{S}_t = e^{\hat{L}t},$$

and thus, the family $\{\hat{S}_t\}_{t\geq 0}$ forms a dynamical semigroup of superchannels.

By inductive reasoning, we, thus, arrive at our central physical hypothesis: Decay-processes of quantum devices with some sort of influx are well described by dynamical semigroups of superchannels. It follows that such decay-processes can be understood by characterizing dynamical semigroups of superchannels. Such a characterization is the main goal of our work.

In particular, we aim to understand dynamical semigroups of superchannels in terms of their generators. We characterize these generators fully by providing two results: First, we give an efficiently checkable criterion for whether a given map generates a dynamical semigroup of superchannels. Second, we identify a normal form for the generators of semigroups of quantum superchannels, analogous to the GKLS form in the case of quantum channels. Interestingly, we find that the most general form of dynamical semigroups of superchannels goes beyond the simple introductory example above.

We arrive at these results through a path (see Fig. 3) that also illuminates the connection to the classical case. We start by studying dynamical semigroups of classical superchannels, which (analogously to quantum superchannels being transformations between quantum channels)

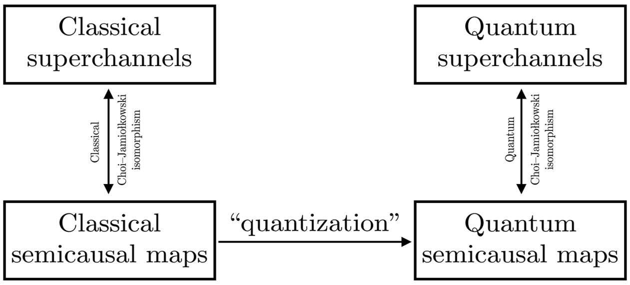

FIG. 3. Schematic of the concepts studied in this work.

are transformations between stochastic matrices. We do so by establishing a one-to-one correspondence between classical superchannels and certain classical semicausal channels, that is, stochastic matrices on a bipartite system (AB) that do not allow for communication from B to A (see Definition IV.2). We can then obtain a full characterization of the generators of semigroups of classical superchannels by characterizing generators of semigroups of classical semicausal maps first and then translating the results back to the level of superchannels. The study of (dynamical semigroups of) classical superchannels and classical semicausal channels is the content of Sec. IV.

Armed with the intuition obtained from the classical case, we then go on to study the quantum case. We start by characterizing the generators of semigroups of semicausal[2] completely positive maps (CP-maps)—our main technical result and one of independent interest. This characterization can be obtained from the classical case by a "quantization"-procedure that allows us to see exactly which features of semigroups of semicausal CP-maps are "fully quantum." Dynamical semigroups of semicausal CP-maps are discussed Sec. V B. Finally, in Sec. V C, we use the one-to-one correspondence (via the quantum Choi–Jamiołkowski isomorphism) between certain semicausal CP-maps and quantum superchannels to obtain a full characterization of the generators of semigroups of quantum superchannels. While the classical section (Sec. IV) and the quantum section (Sec. V) are heuristically related, they are logically independent and can be read independently.

This work is structured as follows: In the remainder of this section, we discuss results related to ours. Section II contains an overview over our main results. In Sec. III, we recall relevant notions from functional analysis and quantum information, as well as some notation. The (logically) independent sections (Secs. IV and V) comprise the main body of our paper, containing complete statements and proofs of our results on dynamical semigroups of superchannels and semicausal channels. We study the classical case in Sec. IV and the quantum case in Sec. V. Finally, we conclude with a summary and an outlook to future research in Sec. VI.

A. Related work

The study of quantum superchannels goes back to Ref. 1 and has since evolved to the study of higher-order quantum maps.[3–5] A peculiar feature of higher-order quantum theory is that it allows for indefinite causal order.[6,7] However, it was recently discovered that the causal order is preserved under (certain) continuous evolutions.[8,9] It, therefore, seems interesting to study continuous evolutions of higher-order quantum maps systematically. Our work can be seen as an initial step into his direction.

The study of (semi-)causal and (semi-)localizable quantum channels goes back to Ref. 2. By proving the equivalence of semicausality and semilocalizability for quantum channels, the authors of Ref. 10 resolved a conjecture raised in Ref. 2 (and attributed to DiVincenzo). Later, the authors of Ref. 11 provided an alternative proof for this equivalence and further investigated causal and local quantum operations.

II. RESULTS

We give an overview over our answers to the questions identified in Sec. I. In our first result, we identify a set of constraints that a linear map satisfies if and only if it generates a semigroup of quantum superchannels.

Result 1.1 (Lemma V.17—informal). *Checking whether a linear map* $\hat{L} : \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B)) \to \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B))$ *generates a semigroup of quantum superchannels can be phrased as a semidefinite constraint satisfaction problem.*

Therefore, we can efficiently check whether a given linear map is a valid generator of a semigroup of quantum superchannels. We can even solve optimization problems over such generators in terms of semidefinite programs. Thereby, this first characterization of generators of semigroups of quantum superchannels facilitates working with them computationally.

As our second result, we determine a normal form for generators of semigroups of quantum superchannels. Similar to the GKLS-form, we decompose the generator into a "dissipative part" and a "Hamiltonian part," where the latter generates a semigroup of invertible superchannels such that the inverse is a superchannel as well.

Result 1.2 (Theorem V.18—informal). *A linear map* $\hat{L}:\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))\to\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))$ *generates a semigroup of quantum superchannels if and only if it can be written as* $\hat{L}(T)=\hat{D}(T)+\hat{H}(T)$*, where the "Hamiltonian part" is of the form*

$$\hat{H}(T)(\rho)=-i[H_B,T(\rho)]-iT([H_A,\rho]),$$

with local Hamiltonians H_B *and* H_A*, and where the "dissipative part" is of the form* $\hat{D}(T)(\rho)=\operatorname{tr}_E\big[\hat{D}'(T)(\rho)\big]$*, where*

$$\hat{D}'(T)(\rho)=U(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma)A^\dagger)U^\dagger-\frac{1}{2}(T\otimes\mathrm{id}_E)(\{A^\dagger A,\rho\otimes\sigma\}) \tag{1a}$$

$$+B(T\otimes\mathrm{id}_E)(\rho\otimes\sigma)B^\dagger-\frac{1}{2}\{B^\dagger B,(T\otimes\mathrm{id}_E)(\rho\otimes\sigma)\} \tag{1b}$$

$$+\Big[U(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma)),B^\dagger\Big]+\Big[B,(T\otimes\mathrm{id}_E)((\rho\otimes\sigma)A^\dagger)U^\dagger\Big], \tag{1c}$$

with unitary U *and arbitrary* A *and* B*.*

The "dissipative part" consists of three terms: Term (1a) itself generates a semigroup of superchannels (for $B=0$), with the interpretation that the transformed channel $[\hat{S}_t(T)]$ arises due to the stochastic application of $T\mapsto\operatorname{tr}_E\big[U(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma)A^\dagger)U^\dagger\big]$ at different points in time (Dyson series expansion). Term (1b) itself generates a semigroup of superchannels (for $A=0$) of the form $\hat{S}_t(T)=e^{L_Bt}\circ T$, where L_B is a generator of a quantum dynamical semigroup (and hence in GKLS-form). Term (1c) is a "superposition" term, which is harder to interpret. It will become apparent from the path taken via the "quantization" of semicausal semigroups that this term is a pure quantum feature with no classical analog. Therefore, the presence of (1c) can be regarded as one of our main findings. It is also worth noting that the normal form in Result 1.2 is more general than the form of the generator we found in our introductory example. Hence, nature allows for more general decay-processes than the simple ones with an independent influx of particles before and after the target object. We also complement this structural result by an algorithm that determines the operators U, A, B, H_A, and H_B if the conditions in Result 1.1 are met.

The proof of these results relies on the relation (via the Choi–Jamiołkowski isomorphism) between superchannels and semicausal CP-maps. Our next findings—and from a technical standpoint our main contributions—are the corresponding results for semigroups of semicausal CP-maps.

Result 2.1 (Lemma V.5—informal). *Checking whether a linear map* $L:\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)\to\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$ *generates a semigroup of* $B\not\to A$ *semicausal CP-maps can be phrased as a semidefinite constraint satisfaction problem for its Choi-matrix.*

Based on this insight, we can efficiently check whether a given linear map is a valid generator of a semigroup of semicausal CP-maps.

Since semigroups of semicausal CP-maps are, in particular, semigroups of CP-maps, our normal form for generators giving rise to semigroups of semicausal CP-maps is a refining of the GKLS-form.

Result 2.2 (Theorem V.6—informal). *A linear map* $L:\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)\to\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$ *generates a semigroup of* $B\not\to A$ *semicausal CP-maps (in the Heisenberg picture) if and only if it can be written as* $L(X)=\Phi(X)-K^\dagger X-XK$*, where the CP part* Φ *is of the form*

$$\Phi(X)=V^\dagger(X\otimes\mathbb{1}_E)V,\ \textit{with}\ V=(\mathbb{1}_A\otimes U)(A\otimes\mathbb{1}_B)+(\mathbb{1}_A\otimes B),$$

with a unitary $U\in\mathcal{B}(\mathcal{H}_E\otimes\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_E)$ *and arbitrary* $A\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_A\otimes\mathcal{H}_E)$ *and* $B\in\mathcal{B}(\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_E)$*, and the* K *in the non-CP part is of the form*

$$K=(\mathbb{1}_A\otimes B^\dagger U)(A\otimes\mathbb{1}_B)+\frac{1}{2}\mathbb{1}_A\otimes B^\dagger B+K_A\otimes\mathbb{1}_B+\mathbb{1}_A\otimes iH_B,$$

with a self-adjoint H_B *and an arbitrary* K_A*.*

This characterization has both computational and analytical implications: On the one hand, it provides a recipe for describing semicausal GKLS generators in numerical implementations. On the other hand, the constructive characterization of semicausal GKLS generators makes a more detailed analysis of their (e.g., spectral) properties tractable. It is also worth noting that in Result 2.2, we can allow for (separable) infinite-dimensional spaces. In the finite-dimensional case, we also provide an algorithm to compute the operators U, A, B, K_A, and H_B, if the conditions of Result 2.1 are met.

Let us now turn to the corresponding results in the classical case. Here, instead of looking at (semigroups of) CP-maps and quantum channels, we look at (entry-wise) non-negative matrices and row-stochastic matrices (see Secs. III and IV for details) that we assume to act on $\mathbb{R}^{\mathbb{X}}$ for (finite) alphabets $\mathbb{X}\in\{\mathbb{A},\mathbb{B},\mathbb{E}\}$.

The following result is the classical analog of Result 2.2:

Result 3 (Corollary IV.8—informal). *A linear map* $Q:\mathbb{R}^{\mathbb{A}}\otimes\mathbb{R}^{\mathbb{B}}\to\mathbb{R}^{\mathbb{A}}\otimes\mathbb{R}^{\mathbb{B}}$ *generates a semigroup of (Heisenberg)* $B\not\to A$ *semicausal non-negative matrices if and only if it can be written as*

$$Q=(A\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes U)-K_A\otimes\mathbb{1}_B+\sum_{i=1}^{|\mathbb{A}|}|a_i\rangle\langle a_i|\otimes B^{(i)},$$

with a row-stochastic matrix $U\in\mathcal{B}(\mathbb{R}^{\mathbb{B}};\mathbb{R}^{\mathbb{E}}\otimes\mathbb{R}^{\mathbb{B}})$, *a non-negative matrix* $A\in\mathcal{B}(\mathbb{R}^{\mathbb{A}}\otimes\mathbb{R}^{\mathbb{E}};\mathbb{R}^{\mathbb{A}})$, *a diagonal matrix* K_A, *and maps* $B^{(i)}\in\mathcal{B}(\mathbb{R}^{\mathbb{B}})$ *that generate semigroups of row-stochastic matrices.*

We will discuss in detail how Result 2.2 arises as the "quantization" of Result 3 in the paragraph following the Proof of Lemma V.5. Here, we highlight that in both the quantum and the classical case, the generators of semicausal semigroups are constructed from two basic building blocks. In the quantum case, these are a $B\not\to A$ semicausal CP-map Φ_{sc}, with $\Phi_{sc}(X)=V_{sc}^{\dagger}(X\otimes\mathbb{1}_E)V_{sc}$ and $V_{sc}=(\mathbb{1}_A\otimes U)(A\otimes\mathbb{1}_B)$ and a GKLS generator of the form $\mathrm{id}_A\otimes\hat{B}$. In the classical case, they are a $B\not\to A$ semicausal non-negative map $\Phi_{sc}=(\mathbb{1}_A\otimes U)(A\otimes\mathbb{1}_B)$ and operators of the form $|a_i\rangle\langle a_i|\otimes B^{(i)}$, where $B^{(i)}$ generates a semigroup of row-stochastic maps. The difference between the quantum case and the classical case then lies in the way the general form is constructed from the building blocks. While we simply take convex combinations of the building-blocks in the classical case, we have to take superpositions of the building-blocks, by which we mean that we need to combine the corresponding Strinespring operators, in the quantum case.

As our last result, we present the normal form for generators of semigroups of classical superchannels.

Result 4. *A linear map* $\hat{Q}:\mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})\to\mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ *generates a semigroup of classical superchannels if and only if it can be written as*

$$\hat{Q}(M)=U(M\otimes\mathbb{1}_E)A-\sum_{i=1}^{|\mathbb{A}|}\langle\mathbf{1}_{AE}|Aa_i\rangle M|a_i\rangle\langle a_i|+\sum_{i=1}^{|\mathbb{A}|}B^{(i)}M|a_i\rangle\langle a_i|,$$

with a column-stochastic matrix $U\in\mathcal{B}(\mathbb{R}^{\mathbb{E}}\otimes\mathbb{R}^{\mathbb{B}};\mathbb{R}^{\mathbb{B}})$, *a non-negative matrix* $A\in\mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{A}}\otimes\mathbb{R}^{\mathbb{E}})$, *a diagonal matrix* K_A, *and a collection of generators of semigroups of column-stochastic matrices* $B^{(i)}\in\mathcal{B}(\mathbb{R}^{\mathbb{B}})$.

As in the quantum case, we have two kinds of evolutions: a stochastic application of $M\mapsto U(M\otimes\mathbb{1}_E)A$ at different points in time and a conditioned post-processing evolution of the form $\sum_i e^{B^{(i)}t}M|a_i\rangle\langle a_i|$. Note that there are no "superposition" terms, such as (1c).

III. NOTATION AND PRELIMINARIES

In this section, we review basic notions from functional analysis, quantum information theory, and the theory of dynamical semigroups. We also fix our notation for these settings as well as for a classical counterpart of the quantum setting.

A. Functional analysis

Throughout this paper, $\mathcal{H}$ (with some subscript) denotes a (in general, infinite-dimensional) separable complex Hilbert space. Whenever $\mathcal{H}$ is assumed to be finite-dimensional, we explicitly state this assumption. We denote the Banach space of bounded linear operators with domain $\mathcal{H}_A$ and codomain $\mathcal{H}_B$, equipped with the operator norm, by $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ and write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H};\mathcal{H})$. For $X\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$, the adjoint $X^{\dagger}\in\mathcal{B}(\mathcal{H}_B;\mathcal{H}_A)$ of X is the unique linear operator such that $\langle\psi_B|X\psi_A\rangle=\langle X^{\dagger}\psi_B|\psi_A\rangle$ for all $|\psi_A\rangle\in\mathcal{H}_A$ and all $|\psi_B\rangle\in\mathcal{H}_B$. Here, and throughout this paper, we use the standard Dirac notation.

An operator $Y\in\mathcal{B}(\mathcal{H})$ is called self-adjoint if $Y^{\dagger}=Y$. A self-adjoint $Y\in\mathcal{B}(\mathcal{H})$ is called positive semidefinite, denoted by $Y\geq 0$, if there exists an operator $Z\in\mathcal{B}(\mathcal{H})$ such that $Y=Z^{\dagger}Z$. If Y is positive semidefinite, then there exists a unique positive semidefinite operator $\sqrt{Y}$ such that $Y=\sqrt{Y}\sqrt{Y}$ (Ref. 12, p. 196). The operator $\sqrt{Y}$ is called the square-root of Y. The absolute value $|Y|\in\mathcal{B}(\mathcal{H})$ of Y is defined by $|Y|=\sqrt{Y^{\dagger}Y}$.

We define the set of trace-class operators $\mathcal{S}_1(\mathcal{H}_A;\mathcal{H}_B)=\{\rho\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)\,|\,\mathrm{tr}[|\rho|]<\infty\}$, which becomes a Banach space when endowed with the norm $\|\rho\|_1:=\mathrm{tr}[|\rho|]$. We write $\mathcal{S}_1(\mathcal{H})$ for $\mathcal{S}_1(\mathcal{H};\mathcal{H})$. The set $\mathcal{S}_1(\mathcal{H}_A;\mathcal{H}_B)$ satisfies the two-sided$*$-ideal property: If $\rho\in\mathcal{S}_1(\mathcal{H}_A;\mathcal{H}_B)$ and $Y\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$, then $\rho^{\dagger}\in\mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_A)$, $\rho^{\dagger}Y\in\mathcal{S}_1(\mathcal{H}_A)$, and $Y\rho^{\dagger}\in\mathcal{S}_1(\mathcal{H}_B)$.

Besides the norm topology, we will use the strong operator topology and the ultraweak topology. The strong operator topology is the smallest topology on $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ such that for all $|\psi_A\rangle\in\mathcal{H}_A$, the map $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)\ni Y\mapsto Y|\psi_A\rangle\in\mathcal{H}_B$ is continuous, where $\mathcal{H}_B$ is equipped with the norm topology. The ultraweak topology on $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ is the smallest topology such that the map $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)\ni Y\mapsto\mathrm{tr}[\rho^{\dagger}Y]\in\mathbb{C}$ is continuous for all $\rho\in\mathcal{S}_1(\mathcal{H}_A;\mathcal{H}_B)$. Since $\mathcal{H}_A$ and $\mathcal{H}_B$ are separable, so is $\mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_A)$. Hence, the sequential Banach Alaoglu theorem implies that every bounded sequence in $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ has an ultraweakly convergent subsequence. Here, we view $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ as the continuous dual of $\mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_A)$. The aforementioned results can be found in many books, e.g., Ref. 12 (ch. VI.6), however, usually only for the case $\mathcal{H}_A=\mathcal{H}_B$.

The general results stated above can be obtained from this case by considering $\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ and $\mathcal{S}_1(\mathcal{H}_A;\mathcal{H}_B)$ as subspaces of $\mathcal{B}(\mathcal{H}_A\oplus\mathcal{H}_B)$ and $\mathcal{S}_1(\mathcal{H}_A\oplus\mathcal{H}_B)$, respectively.

An operator $V\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ is called an isometry if $\|V|\psi_A\rangle\| = \||\psi_A\rangle\|$ for all $|\psi_A\rangle\in\mathcal{H}_A$. The (possibly empty) set of unitaries, the surjective isometries, is denoted by $\mathcal{U}(\mathcal{H}_A;\mathcal{H}_B)$, and we write $\mathcal{U}(\mathcal{H})$ for $\mathcal{U}(\mathcal{H};\mathcal{H})$. As a special notation, if $\mathcal{H}'_A$ and $\mathcal{H}'_B$ are closed linear subspaces of $\mathcal{H}_A$ and $\mathcal{H}_B$, with (canonical) isometric embeddings $\mathbb{1}_{A'\to A}\in\mathcal{B}(\mathcal{H}'_A;\mathcal{H}_A)$ and $\mathbb{1}_{B'\to B}\in\mathcal{B}(\mathcal{H}'_B;\mathcal{H}_B)$, respectively, then we will write $\mathcal{U}_P(\mathcal{H}'_A;\mathcal{H}'_B) = \{\mathbb{1}_{B'\to B}U\mathbb{1}^\dagger_{A'\to A}\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)\,|\,U\in\mathcal{U}(\mathcal{H}'_A;\mathcal{H}'_B)\}$ and $\mathcal{U}_P(\mathcal{H})$ for $\mathcal{U}_P(\mathcal{H};\mathcal{H})$. That is, this is the set of partial isometries.

B. Flip operator, partial trace, complete positivity, and duality

The flip operator $\mathbb{F}_{A;B}\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_A)$ is the unique operator satisfying $\mathbb{F}_{A;B}(|\psi_A\rangle\otimes|\psi_B\rangle) = |\psi_B\rangle\otimes|\psi_A\rangle$ for all $|\psi_A\rangle\in\mathcal{H}_A$ and all $|\psi_B\rangle\in\mathcal{H}_B$.

The partial trace with respect to the space $\mathcal{H}_A$ is the unique linear map $\mathrm{tr}_A:\mathcal{S}_1(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)\to\mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_C)$ that satisfies $\mathrm{tr}[X\mathrm{tr}_A[\rho]] = \mathrm{tr}[(\mathbb{1}_A\otimes X)\rho]$ for all $\rho\in\mathcal{S}_1(\mathcal{H}_A\otimes\mathcal{H}_B)$ and all $X\in\mathcal{B}(\mathcal{H}_C;\mathcal{H}_B)$. If the spaces involved have subscripts, the partial trace will always be denoted with the corresponding subscript. The partial trace with respect to $\rho\in\mathcal{S}_1(\mathcal{H}_A)$ is the unique linear map $\mathrm{tr}_\rho:\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)\to\mathcal{B}(\mathcal{H}_B;\mathcal{H}_C)$ that satisfies $\mathrm{tr}[\sigma\mathrm{tr}_\rho[X]] = \mathrm{tr}[(\rho\otimes\sigma)X]$ for all $\sigma\in\mathcal{S}_1(\mathcal{H}_C;\mathcal{H}_B)$ and all $X\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)$. Proofs of existence and uniqueness can be found in Ref. 13 (Theorem 2.28 and Theorem 2.30), where we used again the observation that the results above follow from the usual ones for $\mathcal{H}_B=\mathcal{H}_C$, by looking at the operators on $\mathcal{H}_A\otimes(\mathcal{H}_B\oplus\mathcal{H}_C)$.

Let $T\in\mathcal{B}(\mathcal{B}(\mathcal{H}_B);\mathcal{B}(\mathcal{H}_A))$. The map T is called positive if $T(X_B)$ is positive semidefinite whenever $X_B\in\mathcal{B}(\mathcal{H}_B)$ is positive semidefinite. For $n\in\mathbb{N}_0$, the map $T_n:\mathcal{B}(\mathbb{C}^n\otimes\mathcal{H}_B)\to\mathcal{B}(\mathbb{C}^n\otimes\mathcal{H}_A)$ is uniquely defined by the requirement that $T_n(X_n\otimes X_B) = X_n\otimes T(X_B)$ for all $X_n\in\mathcal{B}(\mathbb{C}^n)$ and all $X_B\in\mathcal{B}(\mathcal{H}_B)$. The map T is completely positive (CP) if the map T_n is positive for all $n\in\mathbb{N}_0$. A CP-map T is called normal if T is continuous when $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ are both equipped with the ultraweak topology. We denote the set of normal CP-maps by $\mathrm{CP}_\sigma(\mathcal{H}_B;\mathcal{H}_A)$ and write $\mathrm{CP}_\sigma(\mathcal{H})$ for $\mathrm{CP}_\sigma(\mathcal{H};\mathcal{H})$. By the Stinespring dilation theorem (in its form for normal CP-maps), T is a normal CP-map if and only if there exist a (separable) Hilbert space $\mathcal{H}_E$ and an operator $V\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B\otimes\mathcal{H}_E)$ such that for all $X_B\in\mathcal{B}(\mathcal{H}_B)$, we have $T(X_B) = V^\dagger(X_B\otimes\mathbb{1}_E)V$. Furthermore, the Stinespring dilation can be chosen to be minimal, that is, the pair $(V,\mathcal{H}_E)$ can be chosen such that $\mathrm{span}\{(X_B\otimes\mathbb{1}_E)V|\psi_A\rangle\,|\,X_B\in\mathcal{B}(\mathcal{H}_B),|\psi_A\rangle\in\mathcal{H}_A\}$ is norm-dense in $\mathcal{H}_B\otimes\mathcal{H}_E$. Furthermore, if $(V',\mathcal{H}'_E)$ is another Stinespring dilation, then there exists an isometry $U\in\mathcal{B}(\mathcal{H}_E;\mathcal{H}'_E)$ such that $V' = (\mathbb{1}_B\otimes U)V$. Another equivalent characterization is the so-called Kraus form: T is a normal CP-map if and only if there exists a countable set of operators $\{L_i\}_i\subset\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$, the Kraus operators, such that for all $X_B\in\mathcal{B}(\mathcal{H}_B)$, we have $T(X_B) = \sum_i L_i^\dagger X_B L_i$, where the series converges in the strong operator topology. One can obtain Kraus operators from a Stinespring dilation $(V,\mathcal{H}_E)$ by choosing an orthonormal basis $\{|e_i\rangle\}_i$ of $\mathcal{H}_E$ and defining $L_i = (\mathbb{1}_B\otimes\langle e_i|)V$. A map T is unital if $T(\mathbb{1}_B) = \mathbb{1}_A$, and a unital normal CP-map is called a Heisenberg (quantum) channel.

Let $S\in\mathcal{B}(\mathcal{S}_1(\mathcal{H}_A);\mathcal{S}_1(\mathcal{H}_B))$. The dual map $S^*\in\mathcal{B}(\mathcal{B}(\mathcal{H}_B);\mathcal{B}(\mathcal{H}_A))$ is the unique linear map that satisfies $\mathrm{tr}\big[X_B^\dagger S(\rho)\big] = \mathrm{tr}\big[\big(S^*(X_B)\big)^\dagger\rho\big]$ for all $X_B\in\mathcal{B}(\mathcal{H}_B)$ and all $\rho\in\mathcal{S}_1(\mathcal{H}_A)$. We call S the Schrödinger picture map and S^* the Heisenberg picture map. The map S is called completely positive if S^* is completely positive in the sense defined above. In that case, S^* is automatically normal. In fact, T is a normal CP-map if and only if there exists $S\in\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))$ such that $S^* = T$. It follows that S is completely positive if and only if there exist a separable Hilbert space $\mathcal{H}_E$ and an operator $V\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B\otimes\mathcal{H}_E)$ such that $S(\rho) = \mathrm{tr}_E\big[V\rho V^\dagger\big]$ for all $\rho\in\mathcal{S}_1(\mathcal{H}_A)$. Furthermore, S is completely positive if and only if there exist a countable set of operators $\{L_i\}_i\subset\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ such that $S(\rho) = \sum_i L_i\rho L_i^\dagger$ and the series converges in trace-norm. A map S is trace-preserving if $\mathrm{tr}[S(\rho_A)] = \mathrm{tr}[\rho_A]$ for all $\rho_A\in\mathcal{S}_1(\mathcal{H}_A)$. A trace-preserving CP-map is called a (quantum) channel. The facts in this section are contained or follow directly from the results in Refs. 14 and 15.

C. Choi–Jamiołkowski isomorphism, partial transposition

In this section, let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be finite-dimensional Hilbert spaces with fixed orthonormal bases $\{|a_i\rangle\}_i$, $\{|b_j\rangle\}_j$, and $\{|c_k\rangle\}_k$, respectively. The transpose (with respect to $\{|a_i\rangle\}_i$ and $\{|b_j\rangle\}_j$) of an operator $X\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_B)$ is the unique linear operator $X^T\in\mathcal{B}(\mathcal{H}_B;\mathcal{H}_A)$ such that $\langle b_j|Xa_i\rangle = \langle a_i|X^Tb_j\rangle$ for all elements of the orthonormal bases. The partial transposition (with respect to $\{|a_i\rangle\}_i$) of an operator $X\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)$ is the unique linear operator $X^{T_A}\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)$ such that $(\langle a_i|\otimes\mathbb{1}_C)X(|a_j\rangle\otimes\mathbb{1}_B) = (\langle a_j|\otimes\mathbb{1}_C)X^{T_A}(|a_i\rangle\otimes\mathbb{1}_B)$ for all elements of the orthonormal basis.

The (quantum) Choi–Jamiołkowski isomorphism,[16,17] defined with respect to an orthonormal basis $\{|a_i\rangle\}_i$ of $\mathcal{H}_A$, is the bijective linear map $\mathfrak{C}_{A;B}:\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))\to\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$, $\mathfrak{C}_{A;B}(T) = (\mathrm{id}_A\otimes T)(|\Omega\rangle\langle\Omega|)$, and its inverse is given by $\mathfrak{C}^{-1}_{A;B}(\tau)(\rho) = \mathrm{tr}_A\big[(\rho^T\otimes\mathbb{1})\tau\big]$, where $|\Omega\rangle := \sum_i|a_i\rangle\otimes|a_i\rangle$. A map $S\in\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))$ is completely positive if and only if $\mathfrak{C}_{A;B}(S)\geq 0$; S is trace-preserving if and only if $\mathrm{tr}_B[\mathfrak{C}_{A;B}(S)] = \mathbb{1}_A$, and we have the identity $\mathrm{tr}_A[\mathfrak{C}_{A;B}(S)] = S(\mathbb{1}_A)$. We will occasionally call elements of the image of $\mathfrak{C}_{A;B}$ Choi matrices.

D. Non-negative matrices and duality

As we provide characterizations for both the quantum and the classical case, we now also introduce the notation and definitions required for the latter. With a classical system A, we associate a finite alphabet $\mathbb{A} = \{a_1,a_2,\ldots,a_{|\mathbb{A}|}\}$ and a "state-space" $\mathbb{R}^{\mathbb{A}}$, with the orthonormal basis

$\{|a_i\rangle\}_{i=1}^{|\mathbb{A}|}$. We define by $|\mathbf{1}_A\rangle := \sum_i |a_i\rangle$ the all-one-vector. A vector $|x\rangle \in \mathbb{R}^{\mathbb{A}}$ is called non-negative if $\langle a|x\rangle \geq 0$ for all $a \in \mathbb{A}$. A linear operator $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is called non-negative if $M|x\rangle$ is non-negative whenever $|x\rangle$ is non-negative (equivalently, all matrix elements are non-negative). A non-negative M is called column-stochastic if $\langle \mathbf{1}_B|M = \langle \mathbf{1}_A|$, column-sub-stochastic if there exists a non-negative P such that $M + P$ is column-stochastic, row-stochastic if $M|\mathbf{1}_A\rangle = |\mathbf{1}_B\rangle$, and row-sub-stochastic if there exists a non-negative P such that $M + P$ is row-stochastic. Given $|x\rangle$ or $\langle x|$, we denote by $\operatorname{diag}(|x\rangle) = \operatorname{diag}(\langle x|)$ the diagonal matrix with the components of x on the diagonal. Finally, we will use the "classical Choi–Jamiołkowski isomorphism" (also known as vectorization), which is a convenient notation to make the connection to the quantum case more transparent. The classical Choi–Jamiołkowski isomorphism, defined with respect to $\{|a_i\rangle\}_i$, is the linear map $\mathfrak{C}^C_{A;B} : \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}}) \to \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ defined by $\mathfrak{C}^C_{A;B}(M) = (\mathbb{1}_A \otimes M)|\Omega\rangle$, where $|\Omega\rangle := \sum_i |a_i\rangle \otimes |a_i\rangle$. The inverse $(\mathfrak{C}^C_{A;B})^{-1}$ is then given by $(\mathfrak{C}^C_{A;B})^{-1}(|x\rangle) = (\langle\Omega| \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes |x\rangle)$ We will sometimes refer to elements of the range of $\mathfrak{C}^C_{A;B}$ as Choi vectors.

E. Dynamical semigroups

Let $\mathcal{X}$ be a Banach space. A family of operators $\{T_t\}_{t\geq 0}$, with $T_t \in \mathcal{B}(\mathcal{X})$ for all $t \geq 0$, is called a norm-continuous one-parameter semigroup on $\mathcal{X}$ or, short, dynamical semigroup if $T_0 = \mathbb{1}$, $T_{s+t} = T_s T_t$ for all $t, s \geq 0$ and the map $\mathbb{R}_{\geq 0} \ni t \mapsto T_t$ is norm-continuous. Norm-continuous dynamical semigroups are automatically differentiable and have bounded generators, that is, there exists $L \in \mathcal{B}(\mathcal{X})$ such that $T_t = e^{tL}$ for all $t \geq 0$ and $L = \frac{d}{dt}\big|_{t=0_+} T_t$ (Ref. 18, Theorem I.3.7).

Lindblad[19] proved that $T_t \in \mathrm{CP}_\sigma(\mathcal{H})$ for all $t \geq 0$ if and only if there exist $\Phi \in \mathrm{CP}_\sigma(\mathcal{H})$ and $K \in \mathcal{B}(\mathcal{H})$ such that $T_t = e^{tL}$, with $L(X) = \Phi(X) - K^\dagger X - XK$. In this case, we refer to $\{T_t\}_{t\geq 0}$ as a CP semigroup. We call the corresponding form of the generator L the GKLS form[19,20] and Φ its CP part. If $\mathcal{H}$ is finite-dimensional, then $T_t = e^{tL} \in \mathrm{CP}_\sigma(\mathcal{H})$ for all $t \geq 0$ if and only if the operator $\mathfrak{L} := \mathfrak{C}_{A;B} = (\mathrm{id} \otimes L)(|\Omega\rangle\langle\Omega|)$ is self-adjoint and $P^\perp \mathfrak{L} P^\perp \geq 0$, where $|\Omega\rangle = \sum_i |a_i\rangle \otimes |a_i\rangle$ for some orthonormal basis $\{a_i\}$ of $\mathcal{H}$ and $P^\perp \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is the orthogonal projection onto the orthogonal complement of $\{|\Omega\rangle\}$.[21,22] The corresponding classical result is as follows: $\{T_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ is a dynamical semigroup of non-negative linear maps if and only if there exist a non-negative linear map $\Phi \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ and a diagonal map $K \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ (with respect to the basis orthogonal basis $\{|a_i\rangle\}_i$) such that the generator L has the form $\Phi - K$.[23]

IV. THE CLASSICAL CASE

Before studying the quantum scenario, we consider the classical version of our main question. That is, we study continuous semigroups of classical superchannels and their generators. On the one hand, this allows us to develop an intuition that we can build upon for the quantum case. On the other hand, a comparison between the classical and the quantum case elucidates which features of the latter are actually quantum. For the purpose of this section, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{E}$ denote finite alphabets as in Subsection III D.

A classical superchannel is a map that maps classical channels, i.e., stochastic matrices, to classical channels while preserving the probabilistic structure of the classical theory. To achieve the latter requirement, we require that a classical superchannel is a linear map and that probabilistic transformations, i.e., sub-stochastic matrices, are mapped to probabilistic transformations. Expressed more formally, we have the following definition:

Definition IV.1 (classical superchannels). A linear map $\hat{S} : \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}}) \to \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is called a classical superchannel if $\hat{S}(M) \in \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is column sub-stochastic whenever $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is column sub-stochastic and $\hat{S}(M) \in \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is column stochastic whenever $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}};\mathbb{R}^{\mathbb{B}})$ is column stochastic.

A related concept is that of a classical semicausal channel, which is a stochastic matrix on a bipartite space $\mathbb{A} \times \mathbb{B}$ such that no communication from B to A is allowed. We formalize this as follows:

Definition IV.2 (classical semicausality). An operator $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is called column $B \not\to A$ semicausal if there exists $M^A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ such that $(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)M = M^A(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)$.

Similarly, $N \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is called row $B \not\to A$ semicausal if there exists $N^A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ such that $N(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = N^A \otimes |\mathbf{1}_B\rangle$.

Clearly, M is column $B \not\to A$ semicausal if and only if M^T is row $B \not\to A$ semicausal. To emphasize the analogy to the quantum case, we will often refer to a column $B \not\to A$ semicausal map as a Schrödinger $B \not\to A$ semicausal map and to a row $B \not\to A$ semicausal map as a Heisenberg $B \not\to A$ semicausal map. In both cases, the maps M^A and N^A will be called the reduced maps.

The structure of this section is as follows: We start by establishing the connection between classical superchannels and classical non-negative semicausal maps, followed by a characterization of classical non-negative semicausal maps as a composition of known objects; such a characterization is known in the quantum case as the equivalence between semicausality and semilocalizability. We then turn to the study of the generators of semigroups of semicausal and non-negative maps and finally use the correspondence between superchannels and semicausal channels to obtain the corresponding results for the generators of semigroups of superchannels.

A. Correspondence between classical superchannels and semicausal non-negative linear maps

We first show, with a proof inspired by the one given in Ref. 1 for the analogous correspondence in the quantum case, that we can understand classical superchannels in terms of classical semicausal channels. To concisely state this correspondence, we use the classical version of the Choi–Jamiołkowski isomorphism. Let us mention here once again that we assume all alphabets $(\mathbb{A}, \mathbb{B}, \ldots)$ to be finite for our treatment of the classical case.

Theorem IV.3. *Let $\hat{S} : \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}}) \to \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}})$ be a linear map and define $S \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ via $S = \mathfrak{C}^C_{A;B} \circ \hat{S} \circ (\mathfrak{C}^C_{A;B})^{-1}$. Then, $\hat{S}$ is a classical superchannel if and only if S is non-negative and (Schrödinger $B \not\to A$) semicausal such that the reduced map S^A satisfies $S^A|\mathbf{1}_A\rangle = |\mathbf{1}_A\rangle$. In this case, S^A is automatically non-negative.*

Proof. We first show the "if"-direction, i.e., that if S is non-negative and (Schrödinger $B \not\to A$) semicausal, then $\hat{S} = (\mathfrak{C}^C_{A;B})^{-1} \circ S \circ \mathfrak{C}^C_{A;B}$ is a superchannel. Suppose M is a non-negative matrix. Then, $\hat{S}(M)$ is non-negative, since $\mathfrak{C}^C_{A;B}$ maps non-negative matrices to non-negative vectors, S maps non-negative vectors to non-negative vectors, and $(\mathfrak{C}^C_{A;B})^{-1}$ maps non-negative vectors to non-negative matrices.

Furthermore, if M is column stochastic, then

$$\begin{aligned}
\langle \mathbf{1}_B | \hat{S}(M) &= \langle \mathbf{1}_B | \big(\mathfrak{C}^C_{A;B}\big)^{-1} \circ S \circ \mathfrak{C}^C_{A;B}(M) \\
&= \big(\langle \Omega| \otimes \langle \mathbf{1}_B|\big)\big(\mathbb{1}_A \otimes S\big(\mathfrak{C}^C_{A;B}(M)\big)\big) \\
&= \langle \Omega| \big(\mathbb{1}_A \otimes S^A\big((\mathbb{1}_A \otimes \langle \mathbf{1}_B|)\mathfrak{C}^C_{A;B}(M)\big)\big) \\
&= \langle \Omega| \big(\mathbb{1}_A \otimes S^A\big((\mathbb{1}_A \otimes (\langle \mathbf{1}_B| M))|\Omega\rangle\big)\big) \\
&= \langle \Omega| \big(\mathbb{1}_A \otimes S^A |\mathbf{1}_A\rangle\big) \\
&= \langle \Omega| \big(\mathbb{1}_A \otimes |\mathbf{1}_A\rangle\big) \\
&= \langle \mathbf{1}_A|,
\end{aligned}$$

so $\hat{S}(M)$ is stochastic. In the preceding calculation, we used that S is semicausal in the third line, that M is stochastic in the fifth line, and that $S^A|\mathbf{1}_A\rangle = |\mathbf{1}_A\rangle$ in the sixth line.

Now suppose that M is sub-stochastic such that $M + Q$ is stochastic, with Q being non-negative. Then, $\hat{S}(M + Q) = \hat{S}(M) + \hat{S}(Q)$ is stochastic, and since $\hat{S}(Q)$ is non-negative, $\hat{S}(M)$ is sub-stochastic. This proves that $\hat{S}$ is a superchannel. The claim about the non-negativity of S^A now follows directly from the semicausality condition.

For the converse, suppose $\hat{S}$ is a superchannel. Since for all $a \in \mathbb{A}$ and all $b \in \mathbb{B}$, the matrix $|b\rangle\langle a|$ is sub-stochastic, it follows by linearity of $\hat{S}$ that $\hat{S}(M)$ is non-negative whenever M is non-negative. Thus, since $(\mathfrak{C}^C_{A;B})^{-1}$ maps non-negative vectors to non-negative matrices, $\hat{S}$ maps non-negative matrices to non-negative matrices, and $\mathfrak{C}^C_{A;B}$ maps non-negative matrices to non-negative vectors, it follows that S is non-negative.

Next, we want to show that S is Schrödinger $B \not\to A$ semicausal. Since $\hat{S}$ is a superchannel, S maps Choi vectors of stochastic matrices to Choi vectors of stochastic matrices, that is, $(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S|x\rangle = |\mathbf{1}_A\rangle$ for all non-negative vectors $|x\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}}$ that satisfy $(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle = |\mathbf{1}_A\rangle$. As a tool, we define the set of scaled differences of Choi vectors of stochastic matrices by

$$C_0 := \big\{\lambda(|p\rangle - |n\rangle) \,|\, \lambda \in \mathbb{R};\ |p\rangle, |n\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}} \text{ non}-\text{negative, with } (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|p\rangle = (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|n\rangle = |\mathbf{1}_A\rangle\big\}. \tag{2}$$

We claim that

$$C_0 = C_0' := \big\{|x'\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}} \,|\, (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x'\rangle = 0\big\}.$$

To see this, first note that $C_0 \subseteq C_0'$ follows directly from the definition. For the other inclusion, $C_0 \supseteq C_0'$, we decompose $|x'\rangle \in C_0'$ as $|x'\rangle = |p'\rangle - |n'\rangle$ for two non-negative vectors $|p'\rangle, |n'\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}}$. It follows that $(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|p'\rangle = (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|n'\rangle$. Furthermore, for $\varepsilon > 0$ small enough, we have that $|y'\rangle := |\mathbf{1}_A\rangle - \varepsilon(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|p'\rangle$ is non-negative. However, for any non-negative unit $|v\rangle \in \mathbb{R}^{\mathbb{B}}$, with $\langle \mathbf{1}_B|v\rangle = 1$, the vectors $|p\rangle := \varepsilon|p'\rangle + |y'\rangle \otimes |v\rangle$ and $|n\rangle := \varepsilon|n'\rangle + |y'\rangle \otimes |v\rangle$ are Choi vectors of stochastic matrices. Hence, $|x'\rangle = \frac{1}{\varepsilon}(|p\rangle - |n\rangle) \in C_0$.

We define $P^{\perp} \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ by $P^{\perp}|x\rangle = \frac{1}{|\mathbb{B}|}[(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle] \otimes |\mathbf{1}_B\rangle$ and $P := \mathbb{1}_{AB} - P^{\perp}$. Then, since $(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)P|x\rangle = (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle - (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle = 0$, we have that $P|x\rangle \in C_0$ for all $|x\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}}$. We define $S^A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ by $S^A|x_A\rangle = \frac{1}{|\mathbb{B}|}(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)P^{\perp}S(|x_A\rangle \otimes |\mathbf{1}_B\rangle) = \frac{1}{|\mathbb{B}|}(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S(|x_A\rangle \otimes |\mathbf{1}_B\rangle)$ and calculate

$$\begin{aligned}
(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S|x\rangle &= (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S(P|x\rangle) + (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S(P^{\perp}|x\rangle) \\
&= (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S(P^{\perp}|x\rangle) \\
&= (\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S\left(\frac{1}{|\mathbb{B}|}[(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle] \otimes |\mathbf{1}_B\rangle\right) \\
&= S^A((\mathbb{1}_A \otimes \langle \mathbf{1}_B|)|x\rangle),
\end{aligned}$$

where we used in the second line that C_0 is invariant under S, a fact that follows directly from (2). This calculation exactly shows that S is Schödinger $A \not\to B$ semicausal.

It remains to show that $S^A|\mathbf{1}_A\rangle = |\mathbf{1}_A\rangle$. This follows easily, since

$$\begin{aligned} S^A|\mathbf{1}_A\rangle &= \frac{1}{|\mathbb{B}|}(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)S(|\mathbf{1}_A\rangle \otimes |\mathbf{1}_B\rangle) \\ &= \frac{1}{|\mathbb{B}|}(\mathbb{1}_A \otimes \langle \mathbf{1}_B|)\mathfrak{C}^C_{A;B} \circ \hat{S} \circ (\mathfrak{C}^C_{A;B})^{-1}(|\mathbf{1}_A\rangle \otimes |\mathbf{1}_B\rangle) \\ &= \mathbb{1}_A \otimes \left[\langle \mathbf{1}_B|\hat{S}\left(\frac{1}{|\mathbb{B}|}|\mathbf{1}_B\rangle\langle \mathbf{1}_A|\right)\right]|\Omega\rangle \\ &= (\mathbb{1}_A \otimes \langle \mathbf{1}_A|)|\Omega\rangle \\ &= |\mathbf{1}_A\rangle, \end{aligned}$$

where we used that $\frac{1}{|\mathbb{B}|}|\mathbf{1}_B\rangle\langle \mathbf{1}_A|$ is stochastic and that thus $\hat{S}(\frac{1}{|\mathbb{B}|}|\mathbf{1}_B\rangle\langle \mathbf{1}_A|)$ is stochastic. □

In summary, Theorem IV.3 tells us that, via the classical Choi–Jamiołkowski isomorphism, we can view classical superchannels equivalently also as suitably normalized semicausal non-negative maps.

B. Relation between classical semicausality and semilocalizability

The goal of this section is to get a better understanding of the structure of semicausal maps. For non-negative semicausal maps, we have the following structure theorem:

Theorem IV.4. *A non-negative map $N \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is row $B \not\to A$ semicausal if and only if there exist a (finite) alphabet $\mathbb{E}$, a (non-negative) row-stochastic matrix $U \in \mathcal{B}(\mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}})$, and a non-negative matrix $A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}}; \mathbb{R}^{\mathbb{A}})$ such that*

$$N = (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U). \tag{3}$$

In that case, we can choose $|\mathbb{E}| = |\mathbb{A}|^2$.

Borrowing the terminology from the quantum case,[2,10] the preceding theorem tells us that non-negative semicausal maps are semilocalizable. We formally define the latter notion for the classical case as follows:

Definition IV.5. A non-negative map $N \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is called Heisenberg $B \not\to A$ semilocalizable if it can be written in the form of Eq. (3).

Similarly, a non-negative map $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is called Schrödinger $B \not\to A$ semilocalizable if it can be written as $M = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B)$ for a (non-negative) column-stochastic matrix $U \in \mathcal{B}(\mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{B}})$ and a non-negative matrix $A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}})$.

The requirement that U is stochastic and A is non-negative in the decomposition above is essential. In fact, if one drops these requirements, then a decomposition $M = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B)$ can be found for any matrix $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$.

Due to Theorem IV.4, a non-negative Schrödinger $B \not\to A$ semicausal and column-stochastic map M admits an operational interpretation. First, note that if M is not only semicausal but also stochastic, then also the matrix A in Eq. (3) is stochastic. Thus, the interpretation of the decomposition is as follows: First, Alice applies some probabilistic operation (A) to the composite system $\mathbb{A} \times \mathbb{E}$. Then, she transmits the E-part to Bob, who now applies a stochastic operation (U) to his part of the system.

Given this interpretation, the idea behind the construction in the Proof of Theorem IV.4 is that Alice first looks the input of system A and generates the output of system A according to the distribution given by the matrix N^A. Then, she copies the input as well as her generated output and sends this information to Bob, who is then able to complete the operation by generating an output conditional on his input and the information he got from Alice. Given that this construction requires copying, it might be considered surprising that a quantum analog is true nevertheless.[10]

Proof (Theorem IV.4). If N is Schrödinger $B \not\to A$ semilocalizable, then

$$N(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U|\mathbf{1}_B\rangle) = (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes |\mathbf{1}_{EB}\rangle) = (A(\mathbb{1}_A \otimes |\mathbf{1}_E\rangle)) \otimes |\mathbf{1}_B\rangle.$$

Hence, N is row $B \not\to A$ semicausal.

Conversely, if N is row $B \not\to A$ semicausal, we choose $\mathbb{E} := \mathbb{A} \times \mathbb{A}$ and define

$$
\begin{aligned}
A &:= \sum_{i,j,k} \langle a_j | N^A a_k \rangle \, |a_j\rangle\langle a_k| \otimes \langle a_k \otimes a_j|, \\
U &:= \sum_{\substack{m,n,r,s \\ \langle a_n|N^A a_m\rangle \neq 0}} \frac{\langle a_n \otimes b_r | N\, a_m \otimes b_s\rangle}{\langle a_n | N^A a_m\rangle} |a_m \otimes a_n \otimes b_r\rangle\langle b_s| + \left[\sum_{\substack{m,n \\ \langle a_n|N^A a_m\rangle = 0}} |a_m \otimes a_n\rangle \right] \otimes \mathbb{1}_B.
\end{aligned}
\tag{4}
$$

To show that $N = (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U)$, we calculate

$$
\begin{aligned}
(A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U) = &\sum_{\substack{i,j,k \\ m,n,r,s \\ \langle a_n|N^A a_m\rangle \neq 0}} \frac{\langle a_j|N^A a_k\rangle \langle a_n \otimes b_r | N\, a_m \otimes b_s\rangle}{\langle a_n|N^A a_m\rangle} \left[(|a_j\rangle\langle a_k| \otimes \langle a_k \otimes a_j| \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes |a_m \otimes a_n \otimes b_r\rangle\langle b_s|) \right] \\
&+ \sum_{\substack{i,j,k \\ m,n \\ \langle a_n|N^A a_m\rangle = 0}} \langle a_j|N^A a_k\rangle (|a_j\rangle\langle a_k| \otimes \langle a_k \otimes a_j| \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes |a_m \otimes a_n\rangle \otimes \mathbb{1}_B) \\
&= \sum_{\substack{i,j,k,r,s \\ \langle a_j|N^A a_k\rangle \neq 0}} \frac{\langle a_j|N^A a_k\rangle \langle a_j \otimes b_r | N\, a_k \otimes b_s\rangle}{\langle a_j|N^A a_k\rangle} \, |a_j\rangle\langle a_k| \otimes |b_r\rangle\langle b_s| \\
&+ \sum_{\substack{i,j,k \\ \langle a_j|N^A a_k\rangle = 0}} \langle a_j|N^A a_k\rangle |a_j\rangle\langle a_k| \otimes \mathbb{1}_B \\
&= N.
\end{aligned}
$$

For the last step, observe that the second sum vanishes and that one can drop the constraint that $\langle a_j|N^A a_k\rangle \neq 0$ in the first sum (after cancellation) because $\langle a_j \otimes b_r | N\, a_k \otimes b_s\rangle = 0$ if $\langle a_j|N^A a_k\rangle = 0$. To see this last claim, note that, since N is non-negative and semicausal, we have

$$
0 \leq \langle a_j \otimes b_r | N\, a_k \otimes b_s\rangle \leq \langle a_j \otimes b_r | N\, a_k \otimes \mathbf{1}_B\rangle = \langle a_j|N^A a_k\rangle \langle b_r | \mathbf{1}_B\rangle = 0.
$$

It is clear that A and U are non-negative since N and, thus, also N^A are non-negative by assumption. It remains to show that U is row-stochastic. We have

$$
\begin{aligned}
U|\mathbf{1}_B\rangle &= \sum_{\substack{m,n,r,s \\ \langle a_n|N^A a_m\rangle \neq 0}} \frac{\langle a_n \otimes b_r | N\, a_m \otimes b_s\rangle}{\langle a_n|N^A a_m\rangle} |a_m \otimes a_n \otimes b_r\rangle + \sum_{\substack{m,n,s \\ \langle a_n|N^A a_m\rangle = 0}} |a_m \otimes a_n \otimes b_s\rangle \\
&= \sum_{\substack{m,n,r \\ \langle a_n|N^A a_m\rangle \neq 0}} \frac{\langle a_n \otimes b_r | N\, a_m \otimes \mathbf{1}_B\rangle}{\langle a_n|N^A a_m\rangle} |a_m \otimes a_n \otimes b_r\rangle + \sum_{\substack{m,n,s \\ \langle a_n|N^A a_m\rangle = 0}} |a_m \otimes a_n \otimes b_s\rangle \\
&= \sum_{\substack{m,n,r \\ \langle a_n|N^A a_m\rangle \neq 0}} |a_m \otimes a_n \otimes b_r\rangle + \sum_{\substack{m,n,s \\ \langle a_n|N^A a_m\rangle = 0}} |a_m \otimes a_n \otimes b_s\rangle \\
&= |\mathbf{1}_{EB}\rangle,
\end{aligned}
$$

where we used the condition that N is semicausal to obtain the third line. This finishes the proof. □

Remark IV.6. Theorem IV.4 can be extended to weak $-$ * continuous non-negative maps on the Banach space of bounded real sequences, but this requires extra care and does not yield additional insight beyond the previous proof.

C. Generators of semigroups of classical semicausal non-negative maps

The main goal of this section is to establish a structure theorem for the generators of semigroups of non-negative semicausal maps. First, recall that a (norm)-continuous semigroup $\{N_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ has a generator $Q \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ such that $N_t = e^{tQ}$. A classical result states that N_t is non-negative for all $t \geq 0$ if and only if the generator Q can be written in the form $Q = \Phi - K$, where Φ is non-negative and K is a diagonal matrix with respect to the canonical basis.[24] A second, crucial observation is that N_t is Heisenberg $B \not\to A$ semicausal for all $t \geq 0$

if and only if Q is Heisenberg $B \not\to A$ semicausal. To see this, let us first show that the reduced maps $\{N_t^A\}_{t\geq 0}$ also form a norm-continuous semigroup of non-negative maps. Since non-negativity is clear, we derive the semigroup properties ($N_0^A = \mathbb{1}_A$, $N_{t+s}^A = N_t^A N_s^A$, and continuity) from the corresponding ones of $\{N_t\}_{t\geq 0}$,

$$
\begin{aligned}
N_0^A &= (\mathbb{1}_A \otimes \langle b_1|)(N_0^A \otimes |\mathbf{1}_B\rangle) = (\mathbb{1}_A \otimes \langle b_1|)N_0(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = (\mathbb{1}_A \otimes \langle b_1|)(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = \mathbb{1}_A,\\
N_{t+s}^A &= (\mathbb{1}_A \otimes \langle b_1|)(N_{t+s}^A \otimes |\mathbf{1}_B\rangle) = (\mathbb{1}_A \otimes \langle b_1|)N_{t+s}(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = (\mathbb{1}_A \otimes \langle b_1|)N_t N_s(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)\\
&= (\mathbb{1}_A \otimes \langle b_1|)N_t(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)N_s^A = (\mathbb{1}_A \otimes \langle b_1|)(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)N_t^A N_s^A = N_t^A N_s^A,\\
\|N_t^A - N_s^A\| &= \sup_{\|x\|_\infty = 1} \|(N_t^A - N_s^A)|x\rangle\|_\infty = \sup_{\|x\|_\infty = 1} \|((N_t^A - N_s^A)|x\rangle) \otimes |\mathbf{1}_B\rangle\|_\infty = \sup_{\|x\|_\infty = 1} \|(N_t - N_s)(|x\rangle \otimes |\mathbf{1}_B\rangle)\|_\infty\\
&\leq \sup_{\|y\|_\infty = 1} \|(N_t - N_s)|y\rangle\| = \|N_t - N_s\|.
\end{aligned}
$$

Thus, we conclude that $N_t^A = e^{tQ^A}$ for some generator $Q^A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$. We further have

$$
\begin{aligned}
Q(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) &= \frac{d}{dt}\Big|_{t=0} N_t(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)\\
&= \frac{d}{dt}\Big|_{t=0} (\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)N_t^A\\
&= (\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)Q^A.
\end{aligned}
$$

Thus, Q is semicausal if N_t is semicausal for all $t \geq 0$. Conversely, if Q is semicausal, then N_t is semicausal, since

$$
\begin{aligned}
N_t(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) &= e^{tQ}(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)\\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)\\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)\big(Q^A\big)^k\\
&= (\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)e^{tQ^A}.
\end{aligned}
$$

Therefore, our task reduces to characterizing semicausal maps of the form $Q = \Phi - K$. Let us first remark that it is straight-forward to check (numerically) whether a given map satisfies these two conditions: We just need to check for non-negativity of the off-diagonal elements and whether $(\mathbb{1}_A \otimes \langle b|)Q|a_i \otimes \mathbf{1}_B\rangle = 0$ for all $a_i \in \mathbb{A}$ and all $b \in \{|\mathbf{1}_B\rangle\}^\perp$. That is, semicausality can be checked in terms of $|\mathbb{A}|(|\mathbb{B}| - 1)$ linear equations and $|\mathbb{A}||\mathbb{B}|(|\mathbb{A}||\mathbb{B}| - 1)$ linear inequalities. Thus, a desirable result would be a normal form for all Heisenberg $B \not\to A$ semicausal generators Q, which allows for generating such maps rather than checking whether a given maps is of the desired form. The main result of this section is exactly such a normal form.

To understand our normal form below, note that there are two natural ways of constructing a generator (remember that the matrix elements are interpreted as transition rates) that does not transmit information from system B to system A. First, we can leave system A unchanged and have transitions only on system B. The most basic form of such a map is $|a_i\rangle\langle a_i| \otimes B^{(i)}$ for some $1 \leq i \leq |\mathbb{A}|$ and for some $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ that is itself a valid generator of a semigroup of row-stochastic maps. That means that $B^{(i)} = \Phi^{(i)} - \mathrm{diag}(\Phi^{(i)}|\mathbf{1}_B\rangle)$ for some non-negative matrix $\Phi^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$. Second, if we want to act non-trivially on system A, we can make both the two parts of a generator $Q = \Phi - K$, the non-negative part $\Phi \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ and the diagonal part $K \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$, semicausal separately. Such a map has the form $\Phi_{sc} - K_A \otimes \mathbb{1}_B$, where Φ_{sc} is semicausal non-negative and $K_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$ is diagonal. The fact that (convex) combinations of these basic building blocks already give rise to the most general form of semicausal generators for semigroups of non-negative bounded linear maps is the content of our next theorem, which establishes the desired normal form.

Theorem IV.7 (generators of classical semigroups of semicausal non-negative maps). *A map $Q \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is the generator of a (norm-continuous) semigroup of Heisenberg $B \not\to A$ semicausal non-negative linear maps if and only if there exist a non-negative Heisenberg $B \not\to A$ semicausal map $\Phi_{sc} \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$, a diagonal map $K_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$, and linear maps $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ that generate (norm-continuous) semigroups of row-stochastic maps, for $1 \leq i \leq |\mathbb{A}|$, such that*

$$
Q = \Phi_{sc} - K_A \otimes \mathbb{1}_B + \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes B^{(i)}.
$$

In that case, Φ_{sc} can be chosen "block-off-diagonal," i.e., $\Phi_{sc} = \sum_{i\neq j} |a_i\rangle\langle a_j| \otimes \Phi_{sc}^{(ij)}$ for some collection of (non-negative) maps $\Phi_{sc}^{(ij)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$.

Proof. It is straight-forward to check that a generator Q of the given form has non-negative off-diagonal entries with respect to the standard basis and is Heisenberg $B \not\to A$ semicausal. By the above discussion, this means that such a generator indeed gives rise to a semigroup of semicausal non-negative maps.

We prove the converse. Suppose Q is the generator of a semigroup of non-negative linear maps. Then, we can expand it as $Q = \sum_{i,\ j=1}^{|\mathbb{A}|} |a_i\rangle\langle a_j| \otimes Q^{(ij)}$, where the operators $Q^{(ij)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ are non-negative for $i \neq j$ and of the form of a generator of a non-negative semigroup (i.e., non-negative minus diagonal) for $i = j$. This decomposition, together with semicausality, implies that for all $1 \leq i, j \leq |\mathbb{A}|$,

$$Q^{(ij)}|\mathbf{1}_B\rangle = (\langle a_i| \otimes \mathbf{1}_B)Q(|a_j\rangle \otimes |\mathbf{1}_B\rangle) = \langle a_i|Q^A|a_j\rangle \cdot |\mathbf{1}_B\rangle.$$

In other words, $|\mathbf{1}_B\rangle$ is an eigenvector of every $Q^{(ij)}$, with the corresponding eigenvalue $\lambda^{(ij)} = \langle a_i|Q^A|a_j\rangle$. Hence, if we define $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ as $B^{(i)} := Q^{(ii)} - \lambda^{(ii)}\mathbb{1}_B$, then B^i generates a semigroup of non-negative maps (since $Q^{(ij)}$ does and $\lambda^{(ii)}\mathbb{1}_B$ is diagonal) and satisfies (by construction) $B^{(i)}|\mathbf{1}_B\rangle = 0$. Hence, $B^{(i)}$ generates a semigroup of row-stochastic maps.

With this notation, we can rewrite Q as

$$Q = \underbrace{\sum_{i\neq j} |a_i\rangle\langle a_j| \otimes Q^{(ij)}}_{=:\Phi_{sc}} - \underbrace{\sum_{i=1}^{|\mathbb{A}|} -\lambda^{(ii)}|a_i\rangle\langle a_i|}_{=:K_A} \otimes \mathbb{1}_B + \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes B^{(i)}.$$

Note that Φ_{sc} is semicausal, since it can be written as the linear combination of the three semicausal maps Q, $K_A \otimes \mathbb{1}_B$, and $\sum_i |a_i\rangle\langle a_i| \otimes B^{(i)}$. Thus, we have reached the claimed form. □

By applying Theorem IV.4, we can further expand the Φ part.

Corollary IV.8. A map $Q \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is the generator of a (norm-continuous) semigroup of Heisenberg $B \not\to A$ semicausal non-negative linear maps if and only if there exist a (finite) alphabet $\mathbb{E}$, a (non-negative) row-stochastic matrix $U \in \mathcal{B}(\mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}})$, a non-negative matrix $A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}}; \mathbb{R}^{\mathbb{A}})$, a diagonal matrix $K_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$, and maps $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ that generate (norm-continuous) semigroups of (row-)stochastic maps, for $1 \leq i \leq |\mathbb{A}|$, such that

$$Q = (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U) - K_A \otimes \mathbb{1}_B + \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes B^{(i)}.$$

In that case, we can choose $|\mathbb{E}| = |\mathbb{A}|^2$.

One should also note that with the notation of Corollary IV.8, the reduced map is given by $Q^A = (A(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle)) - K_A$. Hence, the reduced dynamics only depends on the operators A and K_A. Further note that if we require the semigroup to consist of non-negative semicausal maps that are also row-stochastic, then we obtain the additional requirement that $K_A|\mathbf{1}_A\rangle = A|\mathbf{1}_{AE}\rangle$, which completely determines K_A. For completeness and later use, we write down the form of the generators non-negative semigroups that are Schrödinger $B \not\to A$ semicausal.

Corollary IV.9. A map $Q \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$ is the generator of a (norm-continuous) semigroup of Schrödinger $B \not\to A$ semicausal non-negative linear maps if and only if there exist a (finite) alphabet $\mathbb{E}$, a (non-negative) column-stochastic matrix $U \in \mathcal{B}(\mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{B}})$, a non-negative matrix $A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}})$, a diagonal matrix $K_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{B}})$, and maps $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ that generate (norm-continuous) semigroups of column-stochastic maps, for $1 \leq i \leq |\mathbb{A}|$, such that

$$Q = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B) - K_A \otimes \mathbb{1}_B + \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes B^{(i)}.$$

In that case, we can choose $|\mathbb{E}| = |\mathbb{A}|^2$.

Similar to the row-stochastic case, $B^{(i)}$ generates a semigroup of column-stochastic maps if and only if $B^{(i)} = \Phi^{(i)} - \mathrm{diag}(\langle \mathbf{1}_B|\Phi^{(i)})$ for some non-negative matrix $\Phi^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$.

D. Generators of semigroups of classical superchannels

We finally turn to semigroups of classical superchannels, that is, a collection of classical superchannels $\{\hat{S}_t\}_{t\geq 0}$, such that $\hat{S}_0 = \mathrm{id}$, $\hat{S}_{t+s} = \hat{S}_t\hat{S}_s$, and the map $t \mapsto \hat{S}_t$ is continuous (with respect to any and, thus, all of the equivalent norms in finite dimensions). To formulate a technically slightly stronger result, we call a linear map $\hat{S}$ a preselecting supermap if $\mathfrak{C}^C_{A;B} \circ \hat{S} \circ (\mathfrak{C}^C_{A;B})^{-1}$ is a non-negative Schrödinger

$B \not\to A$ semicausal map. Theorem IV.3 then tells us that a superchannel is a special preselecting supermap. The result of this section is the following theorem:

Theorem IV.10. *A linear map $\hat{Q} : \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}}) \to \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}})$ generates a semigroup of classical preselecting supermaps if and only if there exist a (finite) alphabet $\mathbb{E}$, a column-stochastic matrix $U \in \mathcal{B}(\mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{B}})$, a non-negative matrix $A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}})$, a diagonal matrix $K_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$, and a collection of generators of semigroups of column-stochastic matrices $B^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ such that*

$$\hat{Q}(M) = U(M \otimes \mathbb{1}_E)A - MK_A + \sum_{i=1}^{|\mathbb{A}|} B^{(i)} M|a_i\rangle\langle a_i|. \tag{5}$$

Furthermore, $\hat{Q}$ generates a semigroup of classical superchannels if and only if $\hat{Q}$ generates a semigroup of preselecting supermaps and $\langle a_i|K_A a_i\rangle = \langle \mathbf{1}_{AE}|A a_i\rangle$ for all $1 \le i \le |\mathbb{A}|$. In this case, $\hat{Q}$ is given by

$$\hat{Q}(M) = U(M \otimes \mathbb{1}_E)A - \sum_{i=1}^{|\mathbb{A}|} \langle \mathbf{1}_{AE}|A a_i\rangle M|a_i\rangle\langle a_i| + \sum_{i=1}^{|\mathbb{A}|} B^{(i)} M|a_i\rangle\langle a_i|. \tag{6}$$

Proof. The main idea is to relate the generators of superchannels to those of semicausal maps. This relation is given by definition for preselecting supermaps and by Theorem IV.3 for superchannels. For a generator $\hat{Q}$ of a semigroup of preselecting supermaps $\{\hat{S}_t\}_{t\ge 0}$, we have

$$\hat{Q} = \frac{d}{dt}\bigg|_{t=0} \hat{S}_t = (\mathfrak{C}^C_{A;B})^{-1} \frac{d}{dt}\bigg|_{t=0} \left[\mathfrak{C}^C_{A;B} \circ \hat{S}_t \circ (\mathfrak{C}^C_{A;B})^{-1}\right] \mathfrak{C}^C_{A;B}.$$

Thus, $\hat{Q}$ generates a semigroup of preselecting supermaps if and only if $\hat{Q}$ can be written as $\hat{Q} = (\mathfrak{C}^C_{A;B})^{-1} \circ Q \circ \mathfrak{C}^C_{A;B}$ for some generator Q of a semigroup of non-negative Schrödinger $B \not\to A$ semicausal maps. Thus, to prove the first part of our theorem, we simply take the normal form in Corollary IV.9 and compute the similarity transformation above.

For $|\Omega\rangle = \sum_i |a_i\rangle \otimes |a_i\rangle \in \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{A}}$ and an operator $X_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$, the well-known identity $(X_A \otimes \mathbb{1}_A)|\Omega\rangle = (\mathbb{1}_A \otimes X_A^T)|\Omega\rangle$ can be proven by a direct calculation. Similarly, it is easy to show that for $X_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}})$, the slightly more general identity $(X_A \otimes \mathbb{1}_A)|\Omega\rangle = (\mathbb{1}_A \otimes \mathbb{F}_{A;E} X_A^{T_A})|\Omega\rangle$ holds, where $\mathbb{F}_{A;E}$ is the flip operator that exchanges systems A and E. We use these two identities in the following calculations.

For $\tilde{A} \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{A}} \otimes \mathbb{R}^{\mathbb{E}})$ and $\tilde{U} \in \mathcal{B}(\mathbb{R}^{\mathbb{E}} \otimes \mathbb{R}^{\mathbb{B}}; \mathbb{R}^{\mathbb{B}})$, we have, for any $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}})$,

$$\begin{aligned}
(\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes \tilde{U})(\tilde{A} \otimes \mathbb{1}_B)\mathfrak{C}^C_{A;B}(M) &= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes \tilde{U})(\tilde{A} \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes M)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes (\tilde{U}(\mathbb{1}_E \otimes M)))(\tilde{A} \otimes \mathbb{1}_A)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes (\tilde{U}(\mathbb{1}_E \otimes M)))(\mathbb{1}_A \otimes \mathbb{F}_{A;E}\tilde{A}^{T_A})|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes (\tilde{U}(\mathbb{1}_E \otimes M)\mathbb{F}_{A;E}\tilde{A}^{T_A}))|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}\mathfrak{C}^C_{A;B}(\tilde{U}(\mathbb{1}_E \otimes M)\mathbb{F}_{A;E}\tilde{A}^{T_A}) \\
&= \tilde{U}(\mathbb{1}_E \otimes M)\mathbb{F}_{A;E}\tilde{A}^{T_A} \\
&= (\tilde{U}\mathbb{F}_{B;E})(M \otimes \mathbb{1}_E)\tilde{A}^{T_A}.
\end{aligned}$$

For $\tilde{K}_A \in \mathcal{B}(\mathbb{R}^{\mathbb{A}})$, we get, for any $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}})$,

$$\begin{aligned}
(\mathfrak{C}^C_{A;B})^{-1}(K_A \otimes \mathbb{1}_B)\mathfrak{C}^C_{A;B}(M) &= (\mathfrak{C}^C_{A;B})^{-1}(\tilde{K}_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes M)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes M)(\tilde{K}_A \otimes \mathbb{1}_A)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes M)(\mathbb{1}_A \otimes \tilde{K}_A^T)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A \otimes M\tilde{K}_A^T)|\Omega\rangle \\
&= (\mathfrak{C}^C_{A;B})^{-1}\mathfrak{C}^C_{A;B}(M\tilde{K}_A^T) \\
&= M\tilde{K}_A^T.
\end{aligned}$$

Finally, for an operator $\tilde{B}^{(i)} \in \mathcal{B}(\mathbb{R}^{\mathbb{B}})$ and for any $1 \le i \le |\mathbb{A}|$, we have, for any $M \in \mathcal{B}(\mathbb{R}^{\mathbb{A}}; \mathbb{R}^{\mathbb{B}})$,

$$\begin{aligned}(\mathfrak{C}^C_{A;B})^{-1}(|a_i\rangle\langle a_i|\otimes B^{(i)})\mathfrak{C}^C_{A;B}(M) &= (\mathfrak{C}^C_{A;B})^{-1}(|a_i\rangle\langle a_i|\otimes B^{(i)})(\mathbb{1}_A\otimes M)|\Omega\rangle\\ &= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A\otimes B^{(i)}M)(|a_i\rangle\langle a_i|\otimes \mathbb{1}_A)|\Omega\rangle\\ &= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A\otimes B^{(i)}M)(\mathbb{1}_A\otimes |a_i\rangle\langle a_i|)|\Omega\rangle\\ &= (\mathfrak{C}^C_{A;B})^{-1}(\mathbb{1}_A\otimes B^{(i)}M|a_i\rangle\langle a_i|)|\Omega\rangle\\ &= (\mathfrak{C}^C_{A;B})^{-1}\mathfrak{C}^C_{A;B}(B^{(i)}M|a_i\rangle\langle a_i|)\\ &= B^{(i)}M|a_i\rangle\langle a_i|.\end{aligned}$$

Applying the results of these calculations term by term to the normal form in Corollary IV.9 yields the first claim, where we defined $A = \tilde{A}^{T_A}$, $U = \tilde{U}\mathbb{F}_{B;E}$, $K_A = \tilde{K}_A^T$, and $B^{(i)} = \tilde{B}^{(i)}$.

If the semigroup $\{\hat{S}_t\}_{t\geq 0}$ consists of superchannels, that is, preselecting maps such that (by Theorem IV.3) the reduced maps S_t^A of the semigroup of semicausal maps $S_t := \mathfrak{C}^C_{A;B}\circ\hat{S}_t\circ(\mathfrak{C}^C_{A;B})^{-1}$ (which are defined by the requirement that $(\mathbb{1}_A\otimes|\mathbf{1}_B\rangle)S_t = S_t^A(\mathbb{1}_A\otimes|\mathbf{1}_B\rangle))$ satisfy $S_t^A|\mathbf{1}_A\rangle = |\mathbf{1}_A\rangle$, then differentiating this relation yields

$$Q^A|\mathbf{1}_A\rangle = \frac{d}{dt}\bigg|_{t=0} S_t^A|\mathbf{1}_A\rangle = \frac{d}{dt}\bigg|_{t=0}|\mathbf{1}_A\rangle = 0.$$

We conclude that $\hat{Q}$ generates a semigroup of superchannels if and only if Q generates a semigroup of semicausal maps and $Q^A|\mathbf{1}_A\rangle = 0$. We obtain directly from Corollary IV.9 that $Q^A = (\mathbb{1}_A\otimes|\mathbf{1}_E\rangle)\tilde{A} - \tilde{K}_A$. It follows that

$$\langle a_i\mathbf{1}_E|\tilde{A}\mathbf{1}_A\rangle = \langle a_i\mathbf{1}_E|A^{T_A}\mathbf{1}_A\rangle = \langle\mathbf{1}_{AE}|Aa_i\rangle = \langle a_i|\tilde{K}_A\mathbf{1}_A\rangle = \langle a_i|K_Aa_i\rangle, \tag{7}$$

where we used that $\tilde{K}_A = K_A$ is diagonal in the last step. This is the condition claimed in the theorem. Finally, (6) is obtained by combining this condition with (5). □

V. THE QUANTUM CASE

We now turn to the quantum case. As introduced and described in more detail in Ref. 1, a quantum superchannel is a map that maps quantum channels to quantum channels while preserving the probabilistic structure of the theory. To achieve the latter, it is usually required that a quantum superchannel is a linear map and that probabilistic transformations, i.e., trace non-increasing CP-maps, should be mapped to probabilistic transformations even if we add an innocent bystander. When dealing with superchannels, we will restrict ourselves to the finite-dimensional case and leave the infinite-dimensional case[25] for future work. We follow[1] and define superchannels as follows:

Definition V.1 (superchannels). A linear map $\hat{S}:\mathcal{B}(\mathcal{S}_1(\mathcal{H}_A);\mathcal{S}_1(\mathcal{H}_B))\to\mathcal{B}(\mathcal{S}_1(\mathcal{H}_A);\mathcal{S}_1(\mathcal{H}_B))$ is called a superchannel if for all $n\in\mathbb{N}$ the map $\hat{S}_n = \mathrm{id}_{\mathcal{B}(\mathcal{S}_1(\mathbb{C}^n))}\otimes\hat{S}$ satisfies that $\hat{S}_n(T)$ is a probabilistic transformation whenever $T\in\mathcal{B}(\mathcal{S}_1(\mathbb{C}^n\otimes\mathcal{H}_A);\mathcal{S}_1(\mathbb{C}^n\otimes\mathcal{H}_B))$ is a probabilistic transformation and that $\hat{S}_n(T)$ is a quantum channel whenever $T\in\mathcal{B}(\mathcal{S}_1(\mathbb{C}^n\otimes\mathcal{H}_A);\mathcal{S}_1(\mathbb{C}^n\otimes\mathcal{H}_B))$ is a quantum channel.

A related concept is that of a semicausal quantum channel, which is a quantum channel on a bipartite space $\mathcal{H}_A\otimes\mathcal{H}_B$ such that no communication from B to A is allowed. Following Refs 2 and 10, we formalize this as follows:

Definition V.2 (semicausality). A bounded linear map $L_*:\mathcal{S}_1(\mathcal{H}_A\otimes\mathcal{H}_B)\to\mathcal{S}_1(\mathcal{H}_A\otimes\mathcal{H}_B)$ is called Schrödinger $B\not\to A$ semicausal if there exists $L_*^A:\mathcal{S}_1(\mathcal{H}_A)\to\mathcal{S}_1(\mathcal{H}_A)$ such that $\mathrm{tr}_B[L_*(\rho)] = L_*^A(\mathrm{tr}_B[\rho])$, for all $\rho\in\mathcal{S}_1(\mathcal{H}_A\otimes\mathcal{H}_B)$. Similarly, $L:\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)\to\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$ is called Heisenberg $B\not\to A$ semicausal if there exists $L^A:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_A)$ such that $L(X_A\otimes\mathbb{1}_B) = L^A(X_A)\otimes\mathbb{1}_B$ for all $X_A\in\mathcal{B}(\mathcal{H}_A)$.

The map L_* is Schrödinger $B\not\to A$ semicausal if and only if the dual map $L := L_*^*$ is normal and Heisenberg $B\not\to A$ semicausal. We will often omit the Schrödinger or Heisenberg attribute if it is clear from the context. This section is structured analogously to the section about the classical case. In particular, we will start by reminding the reader of the connection between semicausal maps and superchannels as well as the characterization of semicausal CP-maps in terms of semilocalizable maps, as schematically shown in Fig. 4. We then turn to the study of the generators of semigroups of semicausal CP-maps and finally use the correspondence between superchannels and semicausal channels to obtain the corresponding results of the generators of semigroups of superchannels.

A. Superchannels, semicausal channels, and semilocalizable channels

We first state the characterization of superchannels in terms of semicausal maps, obtained in Ref. 1.

Theorem V.3. *For finite-dimensional spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, let $\hat{S}:\mathcal{B}(\mathcal{S}_1(\mathcal{H}_A);\mathcal{S}_1(\mathcal{H}_B))\to\mathcal{B}(\mathcal{S}_1(\mathcal{H}_A);\mathcal{S}_1(\mathcal{H}_B))$ be a linear map and define $S = \mathfrak{C}_{A;B}\circ\hat{S}\circ\mathfrak{C}_{A;B}^{-1}$. Then, $\hat{S}$ is a superchannel if and only if S is CP and Schrödinger $B\not\to A$ semicausal such that the reduced map S^A satisfies $S^A(\mathbb{1}_A) = \mathbb{1}_A$.*

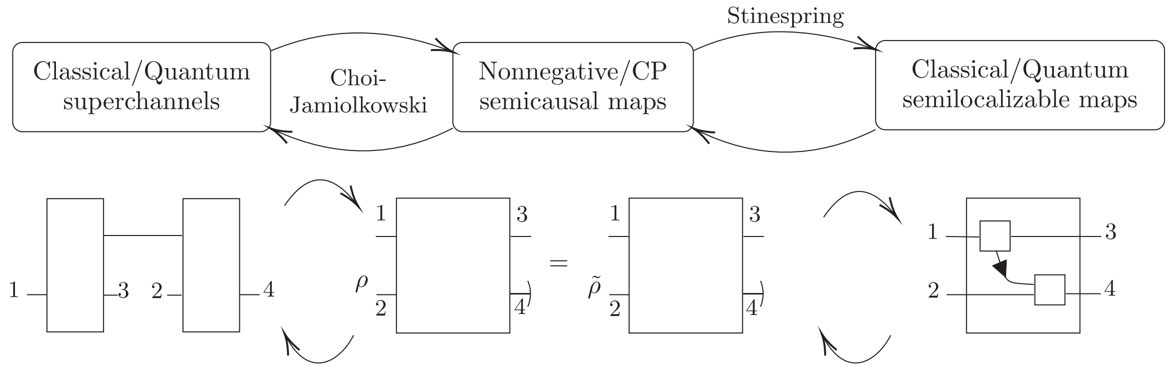

FIG. 4. Visualization of the relation between the notions of superchannels, semicausal maps, and semilocalizable maps. Superchannels and semicausal maps are related via a similarity transform with the Choi–Jamiołkowski isomorphism. Schrödinger $B \not\to A$ semicausal maps are those maps whose output, after tracing out system 4, does not depend on input 2 (ρ or $\tilde{\rho}$). Semicausal maps are precisely those maps that allow for one-way communication only. This is called semilocalizability.

The next result is due to Eggeling, Schlingemann, and Werner,[10] who proved it in the finite-dimensional setting. The following form, which is a generalization of Ref. 10 to the infinite-dimensional case and which has previously been shown in (Ref. 26, Theorem 4), can be obtained from our main result (Theorem V.6) by setting $K = 0$.

Theorem V.4. *A map $\Phi \in CP_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ is Heisenberg $B \not\to A$ semicausal if and only if there exist a (separable) Hilbert space $\mathcal{H}_E$, a unitary operator $U \in \mathcal{U}(\mathcal{H}_E \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, and an arbitrary operator $A \in \mathcal{B}(\mathcal{H}_A; \mathcal{H}_A \otimes \mathcal{H}_E)$ such that*

$$\Phi(X) = V^\dagger (X \otimes \mathbb{1}_E) V, \text{ with } V = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B). \tag{8}$$

If $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional, with dimensions d_A and d_B, then $\mathcal{H}_E$ can be chosen such that $\dim(\mathcal{H}_E) \leq (d_A d_B)^2$.

We call a normal CP-map $\Phi \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ *semilocalizable* if its Stinespring dilation can be written in the form of Eq. (8). With that nomenclature, the above theorem is exactly the quantum analog of Theorem IV.4.

B. Generators of semigroups of semicausal CP maps

The main goal of this section is to establish a structure theorem for the generators of semigroups of semicausal CP-maps, the proof-structure of which is highlighted in Fig. 5. This is our main technical contribution. To get started, recall that a generator $L \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$ generates a norm-continuous semigroup $\{T_t\}_{t \geq 0} \subseteq \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ of CP-maps (i.e., $T_t = e^{tL}$) if and only if L can be written in GKLS-form, i.e., if and only if there exist $\Phi \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that

$$L(X) = \Phi(X) - K^\dagger X - XK, \quad X \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B). \tag{9}$$

As in the classical case, we continue by showing that T_t is Heisenberg $B \not\to A$ semicausal for all $t \geq 0$ if and only if L is Heisenberg $B \not\to A$ semicausal. We start by showing that the family of reduced maps $\{T_t^A\}_{t\geq 0}$ also forms a norm-continuous semigroup of normal CP-maps. That T_t^A is normal and CP follows, since for any density operator $\rho_B \in \mathcal{S}_1(\mathcal{H}_B)$, we have

$$T_t^A = \mathrm{tr}_{\rho_B} \circ T_t \circ D,$$

where $D \in \mathrm{CP}_\sigma(\mathcal{H}_A; \mathcal{H}_A \otimes \mathcal{H}_B)$ is defined by $D(X_A) = X_A \otimes \mathbb{1}_B$. Hence, T_t^A is a normal CP-map as composition of normal CP-maps. It remains to check the semigroup properties ($T_0^A = \mathrm{id}_A$, $T_{t+s}^A = T_t^A T_s^A$, and norm-continuity). We have

$$\begin{aligned}
T_0^A(X_A) &= \mathrm{tr}_{\rho_B}[T_0(X_A \otimes \mathbb{1}_B)] = \mathrm{tr}_{\rho_B}[X_A \otimes \mathbb{1}_B] = X_A, \\
T_{t+s}^A(X_A) &= \mathrm{tr}_{\rho_B}[T_{t+s}(X_A \otimes \mathbb{1}_B)] = \mathrm{tr}_{\rho_B}[T_t(T_s(X_A \otimes \mathbb{1}_B))] = \mathrm{tr}_{\rho_B}\big[T_t(T_s^A(X_A) \otimes \mathbb{1}_B)\big] = \mathrm{tr}_{\rho_B}\big[(T_t^A T_s^A(X_A)) \otimes \mathbb{1}_B\big] = T_t^A T_s^A(X_A), \\
\big\|T_t^A - T_s^A\big\| &= \sup_{\|X_A\|_{\mathcal{B}(\mathcal{H}_A)}=1} \big\|T_t^A(X_A) - T_s^A(X_A)\big\|_{\mathcal{B}(\mathcal{H}_A)} = \sup_{\|X_A\|_{\mathcal{B}(\mathcal{H}_A)}=1} \big\|(T_t^A(X_A) - T_s^A(X_A)) \otimes \mathbb{1}_B\big\|_{\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)} \\
&= \sup_{\|X_A\|_{\mathcal{B}(\mathcal{H}_A)}=1} \|T_t(X_A \otimes \mathbb{1}_B) - T_s(X_A \otimes \mathbb{1}_B)\|_{\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)} \leq \sup_{\|X\|_{\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)}=1} \|T_t(X) - T_s(X)\|_{\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)} = \|T_t - T_s\|.
\end{aligned}$$

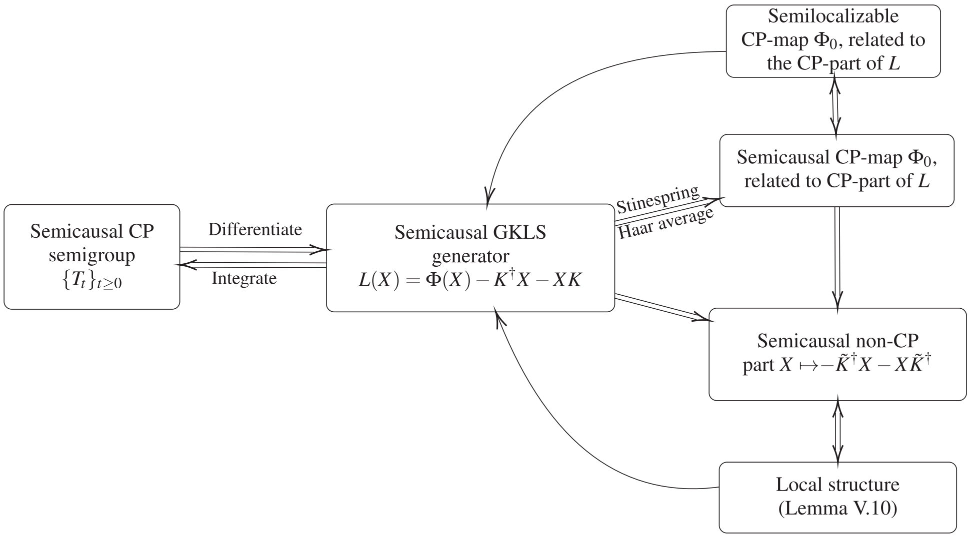

FIG. 5. Overview of the proof structure leading to the normal form for semicausal Lindblad generators (Theorem V.6). We first observe that semicausality of the CP semigroup is equivalent to semicausality of the corresponding GKLS generator L. The insight is then that we can construct a CP-map Φ_0 that is closely related to the CP-part of L and that is semicausal (Lemma V.13). From the semilocalizable form of Φ_0, we then obtain an explicit form for the CP-part of L. This, together with the observation that a semicausal non-CP part has to have a local form, yields the desired normal form.

Thus, we conclude that $T_t^A = e^{tL^A}$ for some generator $L^A \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A))$ of normal CP-maps. We further have

$$L(X_A \otimes \mathbb{1}_B) = \frac{d}{dt}\Big|_{t=0} T_t(X_A \otimes \mathbb{1}_B) = \frac{d}{dt}\Big|_{t=0} T_t^A(X_A) \otimes \mathbb{1}_B = L^A(X_A) \otimes \mathbb{1}_B.$$

Thus, L is semicausal if T_t is semicausal for all $t \geq 0$. Conversely, if L is semicausal, then T_t is semicausal for all $t \geq 0$, since

$$\begin{aligned} T_t(X_A \otimes \mathbb{1}_B) &= e^{tL}(X_A \otimes \mathbb{1}_B) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k(X_A \otimes \mathbb{1}_B) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L^A\right)^k (X_A) \otimes \mathbb{1}_B \\ &= e^{tL^A}(X_A) \otimes \mathbb{1}_B. \end{aligned}$$

Therefore, our task reduces to characterizing semicausal maps in the GKLS-form, i.e., we want to determine the corresponding Φ and K. Our main result (Theorem V.6) is a normal form, which allows us to list all semicausal generators L.

Before we delve into this, we treat the inverse question: Given some $L \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$, is it a semicausal generator? A computationally efficiently chackable criterion can be constructed via the Choi–Jamiołkowski isomorphism. If $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional and $L \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$ is given, then we define $\mathfrak{L} = \mathfrak{C}_{AB;AB}(L) \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, where the Choi–Jamiołkowski isomorphism is defined with respect to the orthogonal bases $\{|a_i\rangle\}_{i=1}^{\dim(\mathcal{H}_A)}$ and $\{|b_j\rangle\}_{j=1}^{\dim(\mathcal{H}_B)}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and where the spaces $\mathcal{H}_{A_1} = \mathcal{H}_{A_2} = \mathcal{H}_A$ and $\mathcal{H}_{B_1} = \mathcal{H}_{B_2} = \mathcal{H}_B$ are introduced for notational convenience. Furthermore, define $P^\perp \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$ to be the orthogonal projection onto the orthogonal complement of $\{|\Omega\rangle\}$, where $|\Omega\rangle = \sum_{i,j} |a_i\rangle \otimes |b_j\rangle \otimes |a_i\rangle \otimes |b_j\rangle$.

Lemma V.5. A linear map $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the generator of a semigroup of Heisenberg $B \not\to A$ semicausal CP-maps if and only if

- *$\mathfrak{L}$ is self-adjoint and $P^\perp \mathfrak{L} P^\perp \geq 0$, and*
- *$\mathrm{tr}_{B_1}[\mathfrak{L}] = \mathfrak{L}^A \otimes \mathbb{1}_{B_2}$ for some (then necessarily self-adjoint) $\mathfrak{L}^A \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2})$.*

The generated semigroup is unital (i.e., $T_t(\mathbb{1}_{AB}) = \mathbb{1}_{AB}$ for $t \geq 0$) if and only if $\mathrm{tr}_{A_1}[\mathfrak{L}^A] = 0$.

Furthermore, a linear map $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the generator of a semigroup of Schrödinger $B \not\to A$ semicausal CP-maps if and only if

- *$\mathfrak{L}$ is self-adjoint and $P^\perp \mathfrak{L} P^\perp \geq 0$ and*
- $(\mathbb{F}_{A_1;B_1} \otimes \mathbb{1}_{A_2})\mathrm{tr}_{B_2}[\mathfrak{L}](\mathbb{F}_{A_1;B_1} \otimes \mathbb{1}_{A_2}) = \mathbb{1}_{B_1} \otimes \mathfrak{L}^A$ *for some (then necessarily self-adjoint)* $\mathfrak{L}^A \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2})$.

The generated semigroup is trace-preserving (i.e., $\mathrm{tr}[T_t(\rho)] = \mathrm{tr}[\rho]$ *for* $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *and* $t \geq 0$*) if and only if* $\mathrm{tr}_{A_2}[\mathfrak{L}^A] = 0$.

Thus, checking whether a map L is the generator of a semigroup of semicausal CP-maps reduces to checking several semidefinite constraints. In particular, the problem to optimize over all semicausal generators is a semidefinite program.

Proof. It is known (see, e.g., the appendix in Ref. 21) that L generates a semigroup of CP-maps if and only if $\mathfrak{L}$ is self-adjoint and $P^\perp \mathfrak{L} P^\perp \geq 0$. This criterion goes by the name of conditional complete positivity.[22] Thus, it remains to translate the other criteria to the level of Choi–Jamiołkowski operators. If L is Heisenberg $B \not\to A$ semicausal, then

$$\begin{aligned}\mathrm{tr}_{B_1}[\mathfrak{L}] &= \mathrm{tr}_{B_1}[(\mathrm{id}_{A_1B_1} \otimes L)(|\Omega\rangle\langle\Omega|)] \\ &= (\mathrm{id}_{A_1} \otimes L)(|\Omega_A\rangle\langle\Omega_A| \otimes \mathbb{1}_{B_2}) \\ &= (\mathrm{id}_{A_1} \otimes L^A)(|\Omega_A\rangle\langle\Omega_A|) \otimes \mathbb{1}_{B_2} \\ &= \mathfrak{L}^A \otimes \mathbb{1}_{B_2},\end{aligned}$$

where we defined $|\Omega_A\rangle = \sum_i |a_i\rangle \otimes |a_i\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathfrak{L}^A = (\mathrm{id}_{A_1} \otimes L^A)(|\Omega_A\rangle\langle\Omega_A|)$. Conversely, if $\mathrm{tr}_{B_1}[\mathfrak{L}] = \mathfrak{L}^A \otimes \mathbb{1}_{B_2}$, define $L^A = \mathfrak{C}^{-1}_{A;A}(\mathfrak{L}^A)$. Then,

$$\begin{aligned}L(X_A \otimes \mathbb{1}_{B_1}) &= \mathrm{tr}_{A_1B_1}\Big[\Big((X_A^T \otimes \mathbb{1}_{B_1}) \otimes \mathbb{1}_{A_2B_2}\Big)\mathfrak{L}\Big] \\ &= \mathrm{tr}_{A_1}\Big[\Big(X_A^T \otimes \mathbb{1}_{A_2B_2}\Big)\mathrm{tr}_{B_1}[\mathfrak{L}]\Big] \\ &= \mathrm{tr}_{A_1}\Big[(X_A^T \otimes \mathbb{1}_{A_2B_2})(\mathfrak{L}^A \otimes \mathbb{1}_{B_2})\Big] \\ &= \mathrm{tr}_{A_1}\Big[(X_A^T \otimes \mathbb{1}_{A_2})\mathfrak{L}^A\Big] \otimes \mathbb{1}_{B_2} \\ &= \mathfrak{C}^{-1}_{A;A}(\mathfrak{L}^A)(X_A) \otimes \mathbb{1}_{B_2} \\ &= L^A(X_A) \otimes \mathbb{1}_B.\end{aligned}$$

Finally, it is known that a semigroup of CP-maps is unital if and only if $L(\mathbb{1}_{A_2B_2}) = 0$. However, this is equivalent to our criterion, since a simple calculation shows that

$$\mathrm{tr}_{A_1B_1}[\mathfrak{L}] = L(\mathbb{1}_{A_2B_2}).$$

This finishes the proof for the Heisenberg picture case. The Schrödinger case can be proven along similar lines or be obtained directly from the Heisenberg case via the identity $\mathfrak{C}_{AB;AB}(L^*) = \mathbb{F}_{A_1B_1;A_2B_2}[\mathfrak{C}_{AB;AB}(L)]^T \mathbb{F}_{A_1B_1;A_2B_2}$. □

Let us now return to the main goal of this section: finding a normal form for semicausal generators in GKLS-form. We motivate (and interpret) our normal form as the "quantization" of the normal form for generators of classical semicausal semigroups (Theorem IV.7). In the classical case, the normal form had two building blocks: an operator of the form $Q_1 = \Phi_{sc} - K_A \otimes \mathbb{1}_B$, where Φ_{sc} is non-negative and semicausal, and an operator of the form $Q_2 = \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes B^{(i)}$, where the $B^{(i)}$'s are generators of row-stochastic maps, (i.e., $B^{(i)}$ generates a non-negative semigroup and $B^{(i)}|\mathbf{1}_B\rangle = 0$). It is straightforward to guess a quantum analog for the first building block: a generator $L_1 \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$ defined by

$$L_1(X) = \Phi_{sc}(X) - (K_A \otimes \mathbb{1}_B)^\dagger X - X(K_A \otimes \mathbb{1}_B), \tag{10}$$

where $\Phi_{sc} \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$, given in the Stinespring form by $\Phi_{sc}(X) = V_{sc}^\dagger(X \otimes \mathbb{1}_E)V_{sc}$, is semicausal. One readily verifies that L_1 defines a semicausal generator. To "quantize" the second building block, note that Q_2 does not induce any change on system A. Indeed, since

$$e^{tQ_2}(\mathbb{1}_A \otimes |\mathbf{1}_B\rangle) = \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes (e^{tB^{(i)}}|\mathbf{1}_B\rangle) = \sum_{i=1}^{|\mathbb{A}|} |a_i\rangle\langle a_i| \otimes |\mathbf{1}_B\rangle = \mathbb{1}_A \otimes |\mathbf{1}_B\rangle, \tag{11}$$

the generated semigroup looks like the identity on system A. In the quantum case, semigroups that do not induce any change on system A are more restricted, since any information-gain about system A inevitably disturbs system A—so there can be no conditioning as in the classical case. Indeed, if one requires that $T_t \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ satisfies the quantum analog of Eq. (11), namely,

$$T_t(X_A \otimes \mathbb{1}_B) = X_A \otimes \mathbb{1}_B \tag{12}$$

for all $X_A \in \mathcal{B}(\mathcal{H}_A)$, then $T_t = \mathrm{id}_A \otimes \Theta_t$ for some unital map $\Theta_t \in \mathrm{CP}_\sigma(\mathcal{H}_B)$ (see Appendix B for a proof). Differentiation of $T_t = \mathrm{id}_A \otimes \Theta_t$ at $t = 0$ now implies that the generator of a semigroup of CP-maps that satisfy (12) are of the form $\mathrm{id}_A \otimes \hat{B}$, where $\hat{B}$ generates a semigroup of unital CP-maps [i.e., $\hat{B}(\mathbb{1}_B) = 0$]. To conclude, the two building blocks are operators of the form of L_1 in Eq. (10) and maps L_2 of the form

$$L_2(X) = (\mathbb{1}_A \otimes B)^\dagger (X \otimes \mathbb{1}_E)(\mathbb{1}_A \otimes B) - \frac{1}{2}\left\{\mathbb{1}_A \otimes B^\dagger B,\, X\right\} + i[\mathbb{1}_A \otimes H_B,\, X],$$

with $B \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$ and a self-adjoint $H_B \in \mathcal{B}(\mathcal{H}_B)$.

In the classical case, we obtained the normal form (Theorem IV.7) by taking a convex combination of the basic building blocks. This corresponds to probabilistically choosing one or the other. In quantum theory, there is a more general concept: superposition. To account for this, we construct our normal form not as a convex combination of the maps L_1 and L_2 but by taking a linear combination (superposition) of the Stinespring operators V_{sc} and $\mathbb{1}_A \otimes B$ as the Stinespring operator of the CP-part of the GKLS-form (note here that the coefficients can be absorbed into V_{sc} and $\mathbb{1}_A \otimes B$, respectively). This means that if L is given by Eq. (9) with $\Phi(X) = V^\dagger (X \otimes \mathbb{1}_E) V$, then we take $V = V_{sc} + \mathbb{1}_A \otimes B$. It turns out that K can then be chosen such that L becomes semicausal. Also note that we can further decompose $V_{sc} = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B)$, as in Theorem V.4.

Our main technical result is that the heuristics employed in the "quantization" procedure above is sound, i.e., that the generators constructed in that way are the only semicausal generators in the GKLS-form.

Theorem V.6. *Let $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be defined by $L(X) = \Phi(X) - K^\dagger X - XK$, with $\Phi \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then, L is Heisenberg $B \not\to A$ semicausal if and only if there exist a (separable) Hilbert space $\mathcal{H}_E$, a unitary $U \in \mathcal{U}(\mathcal{H}_E \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, a self-adjoint operator $H_B \in \mathcal{B}(\mathcal{H}_B)$, and arbitrary operators $A \in \mathcal{B}(\mathcal{H}_A; \mathcal{H}_A \otimes \mathcal{H}_E)$, $B \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, and $K_A \in \mathcal{B}(\mathcal{H}_A)$ such that*

$$\Phi(X) = V^\dagger (X \otimes \mathbb{1}_E) V, \text{ with } V = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B) + (\mathbb{1}_A \otimes B), \tag{13a}$$

$$K = (\mathbb{1}_A \otimes B^\dagger U)(A \otimes \mathbb{1}_B) + \frac{1}{2}\mathbb{1}_A \otimes B^\dagger B + K_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes iH_B. \tag{13b}$$

If $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional, with dimensions d_A and d_B, then $\mathcal{H}_E$ can be chosen such that $\dim(\mathcal{H}_E) \le (d_A d_B)^2$.

Remark V.7. Note that the characterization in Theorem V.6 is for generators of Heisenberg $B \not\to A$ semicausal dynamical semigroups. There are two special cases of interest: First, if we want the dynamical semigroup to be unital, then we need to further impose $L(\mathbb{1}_A \otimes \mathbb{1}_B) = 0$ in the normal form above, which is equivalent to $A^\dagger A = K_A + K_A^\dagger$—a constraint that also appears in the usual Linblad form. Second, if the dynamical semigroup corresponds (in the sense of Theorem V.3) to a semigroup of superchannels, then we additionally require that the reduced generator satisfies $L_*^A(\mathbb{1}_A) = 0$. We will use this in the "translation step" in Theorem V.18.

Remark V.8. In the finite-dimensional case, the Proof of Theorem V.6 is constructive. In Appendix C, we discuss in detail how to obtain the operators A, U, K_A, B, and H_B starting from the conditions in Lemma V.5.

The remainder of this section is devoted to the Proof of Theorem V.6, whose structure is highlighted in Fig. 5.

We begin with a technical observation about certain Haar integrals.

Lemma V.9. Let $\mathcal{H}_n$ be an n-dimensional subspace of $\mathcal{H}_A$ with orthogonal projection $P_n \in \mathcal{B}(\mathcal{H}_A)$, and let $V \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_C)$. Then,

$$\int_{\mathcal{U}_P(\mathcal{H}_n)} (U \otimes \mathbb{1}_C)\, V\, (U^\dagger \otimes \mathbb{1}_B)\, dU = P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}[V], \tag{14}$$

where the integration is with respect to the Haar measure on $\mathcal{U}_P(\mathcal{H}_n)$. It follows that $\left\|P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}[V]\right\| \le \|V\|$. Furthermore, if $\mathcal{H}$ is separable infinite-dimensional, with orthonormal basis $\{|e_i\rangle\}_{i\in\mathbb{N}}$ and $\mathcal{H}_n = \mathrm{span}\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$, then there exist $B \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_C)$ and an ultraweakly convergent subsequence of $\left(P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}[V]\right)_{n\in\mathbb{N}}$ with the limit $\mathbb{1}_A \otimes B$.

Proof. To calculate the integral, we employ the Weingarten formula,[27–29] which for the relevant case reads

$$\int_{\mathcal{U}_P(\mathcal{H}_n)} U_{ij} U^\dagger_{j'\,i'}\, dU = \frac{1}{n}\delta_{i\,i'}\delta_{j\,j'},$$

where $U_{ij} = \langle f_i|Uf_j\rangle$ and $U^\dagger_{j',i'} = \langle f_{j'}|U^\dagger f_{i'}\rangle$ for some orthonormal basis $\{|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle\}$ of $\mathcal{H}_n$. A basis expansion then yields

$$\int_{\mathcal{U}_P(\mathcal{H}_n)} (U \otimes \mathbb{1}_C)\, V\, (U^\dagger \otimes \mathbb{1}_B)\, dU = \sum_{i,j,i',j'=1}^{n} \Big[|f_i\rangle\langle f_{i'}| \otimes \big((\langle f_j| \otimes \mathbb{1}_C)\, V\, (|f_{j'}\rangle \otimes \mathbb{1}_B)\big) \int_{\mathcal{U}_P(\mathcal{H}_n)} U_{ij} U^\dagger_{j',i'}\, dU\Big] = P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}[V].$$

For the second claim, we note that a standard estimate of the integral yields $\|\frac{1}{n}\mathrm{tr}_{P_n}[V]\| = \|P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}[V]\| \le \|V\|$. Thus, the sequence $\big(\frac{1}{n}\mathrm{tr}_{P_n}[V]\big)_{n\in\mathbb{N}}$ is bounded and hence, by Banach–Alaoglu, has an ultraweakly convergent subsequence, whose limit we call B. The claim then follows by observing that under the separability assumption, $(P_n)_{n\in\mathbb{N}}$ converges ultraweakly to $\mathbb{1}_A$ and that the tensor product of two ultraweakly convergent sequences converges ultraweakly. □

As a first step toward our main result, we provide a characterization of those semicausal Lindblad generators that can be written with the vanishing CP part.

Lemma V.10. Let $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $L(X) := -K^\dagger X - XK$, *with* $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. *Then, L is Heisenberg* $B \not\to A$ *semicausal if and only if there exist* $K_A \in \mathcal{B}(\mathcal{H}_A)$ *and a self-adjoint* $H_B \in \mathcal{B}(\mathcal{H}_B)$, *with* $K = K_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes iH_B$.

Proof. If $K = K_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes iH_B$, then $L(X_A \otimes \mathbb{1}_B) = (-K_A^\dagger X_A - K_A X_A) \otimes \mathbb{1}_B + X_A \otimes (iH_B - iH_B) = (-K_A^\dagger X_A - X_A K_A) \otimes \mathbb{1}_B$. Hence, L is semicausal. Conversely, suppose L is semicausal with $L(X_A \otimes \mathbb{1}_B) = L^A(X_A) \otimes \mathbb{1}_B$. Let $\mathcal{H}_n$ be an n-dimensional subspace of $\mathcal{H}_A$ and $U \in \mathcal{U}_P(\mathcal{H}_n)$. Then,

$$(L(U \otimes \mathbb{1}_B))(U^\dagger \otimes \mathbb{1}_B) = -K^\dagger (P_n \otimes \mathbb{1}_B) - (U \otimes \mathbb{1}_B)K(U^\dagger \otimes \mathbb{1}_B) = (L^A(U)\, U^\dagger) \otimes \mathbb{1}_B,$$

where $P_n \in \mathcal{B}(\mathcal{H}_A)$ is the orthogonal projection onto $\mathcal{H}_n$. We integrate both sides with respect to the Haar measure on $\mathcal{U}_P(\mathcal{H}_n)$. Lemma V.9 and some rearrangement and taking the conjugate yields

$$(P_n \otimes \mathbb{1}_B)K = -P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}\big[K^\dagger\big] - L_n^A \otimes \mathbb{1}_B \tag{15}$$

for some operator $L_n^A \in \mathcal{B}(\mathcal{H}_A)$. If $\mathcal{H}_A$ is finite-dimensional, we can take $\mathcal{H}_n = \mathcal{H}_A$ so that $P_n = \mathbb{1}_A$. Hence, $K = -\tilde{K}_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes B$, with $B = \frac{1}{n}\mathrm{tr}_A\big[K^\dagger\big]$ and $\tilde{K}_A = L_n^A$. If $\mathcal{H}_A$ is separable infinite-dimensional, we obtain the same result via a limiting procedure $n \to \infty$ as follows: Let $\{|e_i\rangle\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_A$ and set $\mathcal{H}_n = \mathrm{span}\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$. Then, the second part of Lemma V.9 allows us to pass to a subsequence of $\big(P_n \otimes \frac{1}{n}\mathrm{tr}_{P_n}\big[K^\dagger\big]\big)_{n\in\mathbb{N}}$ that converges ultraweakly to a limit $\mathbb{1}_A \otimes B$. The corresponding subsequence of $((P_n \otimes \mathbb{1}_B)K)_{n\in N}$ converges ultraweakly to K, and hence, that subsequence of $\big(L_n^A \otimes \mathbb{1}_B\big)_{n\in\mathbb{N}}$ converges ultraweakly to a limit $\tilde{K}_A \otimes \mathbb{1}_B$. That is, we get $K = -\tilde{K}_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes B$. Therefore,

$$0 = L(X_A \otimes \mathbb{1}_B) - L(X_A \otimes \mathbb{1}_B) = (L^A(X_A) - \tilde{K}_A^\dagger X_A - X_A \tilde{K}_A) \otimes \mathbb{1}_B - X_A \otimes (B + B^\dagger),$$

which can only be true for all X_A if $B + B^\dagger$ is proportional to $\mathbb{1}_B$. Since $B + B^\dagger$ is self-adjoint, we have $B + B^\dagger = 2r\mathbb{1}_B$ for some $r \in \mathbb{R}$. We can then set $iH_B := r\mathbb{1}_B - B$ and $K_A := -\tilde{K}_A - r\mathbb{1}$ so that H_B is self-adjoint and $K = K_A \otimes \mathbb{1} + \mathbb{1} \otimes iH_B$. □

If we had restricted our attention to Hamiltonian generators and unitary groups in finite dimensions, an analog of this lemma would have already followed from the fact that semicausal unitaries are tensor products, which was proved in Ref. 2 (and reproved in Ref. 11).

As another technical ingredient, the following lemma establishes a closedness property of the set of semicausal maps:

Lemma V.11. Let $(V_m)_{m\in\mathbb{N}}$ *and* $(W_n)_{n\in\mathbb{N}}$ *be ultraweakly convergent sequences in* $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$, *with limits V and W. Suppose that for all* $m, n \in \mathbb{N}$, *the map* $\Phi_{m,n} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, *defined by* $\Phi_{m,n}(X) = V_m^\dagger (X \otimes \mathbb{1}_E) W_n$, *is Heisenberg* $B \not\to A$ *semicausal. Then, the map* $\Phi : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, *defined by* $\Phi(V) = V^\dagger (X \otimes \mathbb{1}_E) W$, *is also Heisenberg* $B \not\to A$ *semicausal.*

Proof. For $X_A \in \mathcal{B}(\mathcal{H}_A)$ and $\rho \in \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have that $\rho\, V_m^\dagger (X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E) \in \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E; \mathcal{H}_A \otimes \mathcal{H}_B)$, since the trace-class operators are an ideal in the bounded operators. Hence, by definition of the ultraweak topology,

$$\mathrm{tr}\Big[\rho\, V_m^\dagger (X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E) W\Big] = \lim_{n\to\infty} \mathrm{tr}\Big[\rho\, V_m^\dagger (X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E) W_n\Big] = \lim_{n\to\infty} \mathrm{tr}\big[\rho\, (\Phi_{m,n}^A(X_A) \otimes \mathbb{1}_B)\big].$$

Since $\mathrm{tr}\big[\rho\, \Phi_{m,n}^A(X_A) \otimes \mathbb{1}_B\big]$ converges as $n \to \infty$ for every $\rho \in \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B)$, the sequence $\big(\Phi_{m,n}^A(X_A) \otimes \mathbb{1}_B\big)_{n\in\mathbb{N}}$ converges ultraweakly.[30] We call the limit $\Phi_m^A(X_A) \otimes \mathbb{1}_B$. It is then easy to see that $\Phi_m^A(X_A)$, viewed as a map on $\mathcal{B}(\mathcal{H}_A)$, is linear and continuous. This tells us that the map $\Phi_m : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, defined by $\Phi_m(X) = V_m^\dagger (X \otimes \mathbb{1}_E) W$, is semicausal for all $m \in \mathbb{N}$. Furthermore, we have that $\rho^\dagger\, W^\dagger (X_A^\dagger \otimes \mathbb{1}_B \otimes \mathbb{1}_E) \in \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E; \mathcal{H}_A \otimes \mathcal{H}_B)$ for all $X_A \in \mathcal{B}(\mathcal{H}_A)$ and $\rho \in \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B)$, and thus,

$$\operatorname{tr}\Big[\rho\, V^{\dagger}(X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E)\, W\Big] = \overline{\operatorname{tr}\Big[\rho^{\dagger}\, W^{\dagger}(X_A^{\dagger} \otimes \mathbb{1}_B \otimes \mathbb{1}_E)\, V\Big]} = \lim_{m\to\infty} \overline{\operatorname{tr}\Big[\rho^{\dagger}\, W^{\dagger}(X_A^{\dagger} \otimes \mathbb{1}_B \otimes \mathbb{1}_E)\, V_m\Big]} = \lim_{m\to\infty} \operatorname{tr}\Big[\rho\, V_m^{\dagger}(X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E)\, W\Big]$$
$$= \lim_{m\to\infty} \operatorname{tr}\Big[\rho\,(\Phi_m^A(X_A) \otimes \mathbb{1}_E)\Big].$$

Repeating the argument above then shows that Φ is semicausal. □

As a final preparatory step, we observe that, given a semicausal Lindblad generator, we can use its CP part to define a family of semicausal CP-maps.

Lemma V.12. Let $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be defined by $L(X) := V^{\dagger}(X \otimes \mathbb{1}_E)V - K^{\dagger}X - XK$, with $V \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ and $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. If L is Heisenberg $B \not\to A$ semicausal, then the map $S_{Y,Z} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, defined by

$$S_{Y,Z}(X) = [V(Z \otimes \mathbb{1}_B) - (Z \otimes \mathbb{1}_B \otimes \mathbb{1}_E)V]^{\dagger}\ (X \otimes \mathbb{1}_E)\ [V(Y \otimes \mathbb{1}_B) - (Y \otimes \mathbb{1}_B \otimes \mathbb{1}_E)V],$$

is Heisenberg $B \not\to A$ semicausal for every $Y, Z \in \mathcal{B}(\mathcal{H}_A)$.

Proof. For every $M \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we define the map $\Psi_M : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ by

$$\Psi_M(X) = L(M^{\dagger}XM) - M^{\dagger}L(XM) - L(M^{\dagger}X)M + M^{\dagger}L(X)M$$
$$= [(M \otimes \mathbb{1}_E)V - VM]^{\dagger}(X \otimes \mathbb{1}_E)[(M \otimes \mathbb{1}_E)V - VM].$$

This map has already been used, for a different purpose, in Lindblad's original work [Ref. 19, Eq. (5.1)]. It follows from the semicausality of L that if we choose $M = M_A \otimes \mathbb{1}_B$ for some $M_A \in \mathcal{B}(\mathcal{H}_A)$, then Ψ_M is semicausal. Furthermore, a calculation shows that

$$\frac{1}{4}\sum_{k=0}^{3} i^k \Psi_{M+i^k N}(X) = [VN - (N \otimes \mathbb{1}_E)V]^{\dagger}\ (X \otimes \mathbb{1}_E)\ [VM - (M \otimes \mathbb{1}_E)V].$$

By choosing $N = Z \otimes \mathbb{1}_B$ and $M = Y \otimes \mathbb{1}_B$, it follows that $S_{Y,Z}$ is the linear combination of four semicausal maps and, hence, is itself semicausal. □

We now combine this lemma with an integration over the Haar measure to obtain the key lemma in our proof.

Lemma V.13. Let $L : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be defined by $L(X) := V^{\dagger}(X \otimes \mathbb{1}_E)V - K^{\dagger}X - XK$, with $V \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ and $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. If L is Heisenberg $B \not\to A$ semicausal, then there exists $B \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$ such that the map $S : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, defined by

$$S(X) = [V - \mathbb{1}_A \otimes B]^{\dagger}(X \otimes \mathbb{1}_E)[V - \mathbb{1}_A \otimes B],$$

is also Heisenberg $B \not\to A$ semicausal.

Furthermore, if $\mathcal{H}_A$ is finite-dimensional, then we can choose $B = \operatorname{tr}_A[V]/\dim(\mathcal{H}_A)$.

Proof. Let $\mathcal{H}_n$ and $\mathcal{H}_m$ be n and m dimensional subspaces of $\mathcal{H}_A$ with respective orthogonal projections $P_n \in \mathcal{B}(\mathcal{H}_A)$ and $P_m \in \mathcal{B}(\mathcal{H}_A)$. Since for every $U \in \mathcal{U}_P(\mathcal{H}_n)$ and $W \in \mathcal{U}_P(\mathcal{H}_m)$, the map $S_{U,W}$, defined in Lemma V.12, is semicausal and also the map $\overline{S} : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, defined by

$$\overline{S}(X) := \int_{\mathcal{U}_P(\mathcal{H}_n)} \int_{\mathcal{U}_P(\mathcal{H}_m)} (U \otimes \mathbb{1}_B) S_{U,W}(X)(W^{\dagger} \otimes \mathbb{1}_B)\ dW dU,$$

is semicausal. Writing out the definition of $S_{U,W}$ yields

$$\overline{S}(X) = \left[V(P_n \otimes \mathbb{1}_B) - \int_{\mathcal{U}_P(\mathcal{H}_n)} (U \otimes \mathbb{1}_B \otimes \mathbb{1}_E)V(U^{\dagger} \otimes \mathbb{1}_E)dU\right]^{\dagger}(X \otimes \mathbb{1}_E)\left[V(P_m \otimes \mathbb{1}_B) - \int_{\mathcal{U}_P(\mathcal{H}_m)} (W \otimes \mathbb{1}_B \otimes \mathbb{1}_E)V(W^{\dagger} \otimes \mathbb{1}_B)dW\right]$$
$$= \left[V(P_n \otimes \mathbb{1}_B) - P_n \otimes \frac{1}{n}\operatorname{tr}_{P_n}[V]\right]^{\dagger}(X \otimes \mathbb{1}_E)\left[V(P_m \otimes \mathbb{1}_B) - P_m \otimes \frac{1}{m}\operatorname{tr}_{P_m}[V]\right],$$

where the last line was obtained by using Lemma V.9. If $\mathcal{H}_A$ is finite-dimensional, we can choose $\mathcal{H}_n = \mathcal{H}_m = \mathcal{H}_A$ so that $P_n = P_m = \mathbb{1}_A$ and obtain the desired result immediately. If $\mathcal{H}_A$ is separable infinite-dimensional and $\{|e_i\rangle\}_{i\in\mathbb{N}}$ is an orthonormal basis and $\mathcal{H}_k := \operatorname{span}\{|e_1\rangle, |e_2\rangle, \ldots, |e_k\rangle\}$, then by Lemma V.9, the sequence $\left(P_k \otimes \frac{1}{k}\operatorname{tr}_{P_k}[V]\right)_{k\in\mathbb{N}}$ has an ultraweakly convergent subsequence with a limit $\mathbb{1}_A \otimes B$, where $B \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$. Furthermore, since $(P_k)_{k\in\mathbb{N}}$ converges ultraweakly to $\mathbb{1}_A$, we have that the sequence $\left(V(P_k \otimes \mathbb{1}_B) - P_k \otimes \frac{1}{k}\operatorname{tr}_{P_k}[V]\right)_{k\in\mathbb{N}}$ has a subsequence that converges ultraweakly to $V - \mathbb{1}_A \otimes B$. Hence, by passing to subsequences, we can apply Lemma V.11, which yields that S is semicausal. □

Remark V.14. The previous two lemmas are at the heart of our result. They illustrate a (to the best of our knowledge) novel technique that allows to characterize GKLS generators with a certain constraint if this constraint is well understood for completely positive maps. It seems useful to develop this method more generally, but this is beyond the scope of the present work.

With these tools at hand, we can now prove our main result.

Proof (Theorem V.6). A straightforward calculation shows that L, defined via (22a) and (22b), is semicausal. To prove the converse, note that by the Stinespring dilation theorem, there exist a separable Hilbert space $\tilde{\mathcal{H}}_E$ and $\tilde{V} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \tilde{\mathcal{H}}_E)$ such that $\Phi(X) = \tilde{V}^\dagger (X \otimes \mathbb{1}_E)\tilde{V}$. It is well known [see, e.g., Ref. 31 (Theorems 2.1 and 2.2)] that if $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional with dimensions d_A and d_B, then $\tilde{\mathcal{H}}_E$ can be chosen such that $\dim(\tilde{\mathcal{H}}_E) \le (d_A d_B)^2$. By Lemma V.13, there exists $\tilde{B} \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \tilde{\mathcal{H}}_E)$ such that the map $\Phi_0 \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$, defined by $\Phi_0(X) = \left[\tilde{V} - \mathbb{1}_A \otimes \tilde{B}\right]^\dagger (X \otimes \mathbb{1}_E)\left[\tilde{V} - \mathbb{1}_A \otimes \tilde{B}\right]$, is semicausal. We define $V_{sc} = \tilde{V} - \mathbb{1} \otimes \tilde{B}$ and obtain

$$\Phi(X_A \otimes \mathbb{1}_B) = \Phi_0(X_A \otimes \mathbb{1}_B) + \kappa^\dagger (X_A \otimes \mathbb{1}_B) + (X_A \otimes \mathbb{1}_B)\kappa,$$

where $\kappa = (\mathbb{1}_A \otimes \tilde{B}^\dagger)V_{sc} + \frac{1}{2}(\mathbb{1}_A \otimes \tilde{B}^\dagger \tilde{B})$. Since L and Φ_0 are semicausal, we can write $L(X_A \otimes \mathbb{1}) = L^A(X_A) \otimes \mathbb{1}_B$ and $\Phi_0(X_A \otimes \mathbb{1}_B) = \Phi_0^A(X_A) \otimes \mathbb{1}_B$ for all $X_A \in \mathcal{B}(\mathcal{H}_A)$. Hence,

$$L(X_A \otimes \mathbb{1}_B) - \Phi_0(X_A \otimes \mathbb{1}_B) = \left(L^A(X_A) - \Phi_0^A(X_A)\right) \otimes \mathbb{1}_B = -(K-\kappa)^\dagger (X_A \otimes \mathbb{1}_B) - (X_A \otimes \mathbb{1}_B)(K-\kappa). \tag{16}$$

It follows that the map defined by $X \mapsto -(K-\kappa)^\dagger X - X(K-\kappa)$ is semicausal. Thus, Lemma V.10 implies that there exist $K_A \in \mathcal{B}(\mathcal{H}_A)$ and a self-adjoint $H_B \in \mathcal{B}(\mathcal{H}_B)$ such that $K - \kappa = K_A \otimes \mathbb{1} + \mathbb{1} \otimes iH_B$.

What we have achieved so far is that $\tilde{V} = V_{sc} + \mathbb{1} \otimes \tilde{B}$ and $K = (\mathbb{1}_A \otimes \tilde{B}^\dagger)V_{sc} + \frac{1}{2}\mathbb{1} \otimes \tilde{B}^\dagger \tilde{B} + K_A \otimes \mathbb{1} + \mathbb{1} \otimes iH_B$. Hence, if we can decompose $V_{sc} = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B)$, then we are basically done. However, this decomposition is given (up to details) by the equivalence between semicausal and semilocalizable channels.[10] Since the conclusion in Ref. 10 was in the finite-dimensional setting, we will repeat the argument here, showing that it goes through also for infinite-dimensional spaces while paying special attention to the dimensions of the spaces involved. Since $\Phi_0 \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\Phi_0(X_A \otimes \mathbb{1}_B) = \Phi_0^A(X_A) \otimes \mathbb{1}_B$, we also have $\Phi_0^A \in \mathrm{CP}_\sigma(\mathcal{H}_A)$. By the Stinespring dilation theorem (for normal CP-maps), there exist a separable Hilbert space $\mathcal{H}_F$ and $W \in \mathcal{B}(\mathcal{H}_A; \mathcal{H}_A \otimes \mathcal{H}_F)$ such that $\Phi_0^A(X_A) = W^\dagger (X_A \otimes \mathbb{1}_F)W$ and such that $\mathrm{span}\{(X_A \otimes \mathbb{1}_F)W|\psi\rangle | X_A \in \mathcal{B}(\mathcal{H}_A), |\psi\rangle \in \mathcal{H}_A\}$ is dense in $\mathcal{H}_A \otimes \mathcal{H}_F$. The last condition is called the minimality condition. We then get

$$V_{sc}^\dagger (X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_{\tilde{E}})V_{sc} = (W \otimes \mathbb{1}_B)^\dagger (X_A \otimes \mathbb{1}_F \otimes \mathbb{1}_B)(W \otimes \mathbb{1}_B).$$

Clearly, $\mathrm{span}\{(X_A \otimes \mathbb{1}_F \otimes \mathbb{1}_B)(W \otimes \mathbb{1}_B)|\psi\rangle | X_A \in \mathcal{B}(\mathcal{H}_A), |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B\}$ is dense in $\mathcal{H}_A \otimes \mathcal{H}_F \otimes \mathcal{H}_B$. Thus, by minimality, there exists an isometry $\tilde{U} \in \mathcal{B}(\mathcal{H}_F \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \tilde{\mathcal{H}}_E)$ such that $V_{sc} = (\mathbb{1}_A \otimes \tilde{U})(W \otimes \mathbb{1}_B)$. In the finite-dimensional case, the fact that $\tilde{U}$ is an isometry then implies that $\dim(\mathcal{H}_F) \le \dim(\tilde{\mathcal{H}}_E)$ such that we can think of $\mathcal{H}_F$ as a subspace of $\tilde{\mathcal{H}}_E$. Thus, $\tilde{U}$ can be extended to a unitary operator $\hat{\tilde{U}} \in \mathcal{U}(\tilde{\mathcal{H}}_E \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \tilde{\mathcal{H}}_E)$. Then, defining $\mathcal{H}_E = \tilde{\mathcal{H}}_E$, $U = \hat{\tilde{U}}$, $B = \tilde{B}$, and $A = W$ proves the claim in this case. In the infinite-dimensional case, we can take $\mathcal{H}_E = \mathcal{H}_F \oplus \tilde{\mathcal{H}}_E$. We can now view both $\tilde{\mathcal{H}}_E \otimes \mathcal{H}_B$ and $\mathcal{H}_F \otimes \mathcal{H}_B$ as closed subspaces of $\mathcal{H}_E \otimes \mathcal{H}_B$. Then, $\left(\tilde{U}(\mathcal{H}_F \otimes \mathcal{H}_B)\right)^\perp$ and $(\mathcal{H}_F \otimes \mathcal{H}_B)^\perp$ are isomorphic. Hence, $\tilde{U}$ can be extended to a unitary operator $\hat{\tilde{U}} \in \mathcal{U}(\mathcal{H}_E \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$. We finish the proof by defining $U = \hat{\tilde{U}}$, $B = (\mathbb{1}_B \otimes \mathbb{1}_{\tilde{E}\to E})\tilde{B}$, and $A = (\mathbb{1}_A \otimes \mathbb{1}_{F\to E})W$, where $\mathbb{1}_{\tilde{E}\to E}$ and $\mathbb{1}_{F\to E}$ denote the isometric embeddings of $\tilde{\mathcal{H}}_E$ and $\mathcal{H}_F$ into $\mathcal{H}_E$, respectively. □

As a first consequence, we obtain the analogous theorem for semigroups of Schrödinger $B \not\to A$ semicausal CP-maps.

Corollary V.15. Let $L : \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B)$ be defined by $L(\rho) = \Phi_S(\rho) - K\rho - \rho K^\dagger$, where $\Phi_S \in \mathrm{CP}_S(\mathcal{H}_A \otimes \mathcal{H}_B)$, with $\Phi_S(\rho) = \mathrm{tr}_E\left[V\rho V^\dagger\right]$ and $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then, L is Schrödinger $B \not\to A$ semicausal if and only if K, V, and $\mathcal{H}_E$ can be chosen as in (22a) and (22b).

As a further corollary, we translate the results above to the familiar representation in terms of jump-operators (by going from Stinespring to Kraus).

Corollary V.16. A map $L : \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{S}_1(\mathcal{H}_A \otimes \mathcal{H}_B)$ generates a (trace-)norm-continuous semigroup of trace-preserving Schrödinger $B \not\to A$ semicausal CP-maps if and only if there exist $\{\phi_j\}_j \subset \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\{B_j\}_j \subset \mathcal{B}(\mathcal{H}_B)$, $H_A \in \mathcal{B}(\mathcal{H}_A)$, and $H_B \in \mathcal{B}(\mathcal{H}_B)$ such that $\{\phi_j\}_j$ is a set of Kraus operators of a Schrödinger $B \not\to A$ semicausal CP-map and $\{B_j\}_j$ is a set of Kraus operators of some CP-map such that

$$L(\rho) = -i[H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B, \rho] + \sum_j (\phi_j + \mathbb{1}_A \otimes B_j)\rho(\phi_j + \mathbb{1}_A \otimes B_j)^\dagger - \frac{1}{2}\left\{\mathbb{1}_A \otimes B_j^\dagger B_j + \phi_j^\dagger \phi_j, \rho\right\} - (\mathbb{1}_A \otimes B_j^\dagger)\phi_j\rho - \rho\phi_j^\dagger(\mathbb{1}_A \otimes B_j).$$

Proof. A simple calculation by defining the Kraus operators as $(\mathbb{1}_{AB} \otimes |e_i\rangle)V$, with $\{|e_j\rangle\}_j$ being an orthonormal basis of $\mathcal{H}_E$ and V given by Theorem V.6. □

We conclude this section about semicausal semigroups with an example that uses our normal form in full generality.

Example. We consider the scenario of two 2-level atoms that can interact according to the processes specified in Fig. 6. We can describe this process either via a dilation (as in Theorem V.6) or via the Kraus operators (as in Corollary V.16). In the dilation picture, we introduce an auxiliary Hilbert space $\mathcal{H}_E := \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i$ is for the ith photon. Then, the process is described by $V = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B) + (\mathbb{1}_A \otimes B)$, with

$$A \in \mathcal{B}(\mathcal{H}_A;\mathcal{H}_A \otimes \mathcal{H}_E),\ \ A = |0\rangle\langle 1|_A \otimes |11\rangle_E,$$
$$B \in \mathcal{B}(\mathcal{H}_B;\mathcal{H}_B \otimes \mathcal{H}_E),\ \ B = |10\rangle_E \otimes |0\rangle\langle 1|_B,$$
$$U \in \mathcal{U}(\mathcal{H}_E \otimes \mathcal{H}_B;\mathcal{H}_B \otimes \mathcal{H}_E),\ \ U = \mathbb{F}_{E;B}(\mathbb{1}_{\mathcal{H}_1} \otimes \tilde{U}),$$

where $\tilde{U} \in \mathcal{U}(\mathcal{H}_2 \otimes \mathcal{H}_B)$ is determined by

$$\tilde{U}|00\rangle_{\mathcal{H}_2 B} = |00\rangle_{\mathcal{H}_2 B},\ \ \tilde{U}|10\rangle_{\mathcal{H}_2 B} = |01\rangle_{\mathcal{H}_2 B},\ \ \tilde{U}|11\rangle_{\mathcal{H}_2 B} = |11\rangle_{\mathcal{H}_2 B}.$$

The crucial feature of this example is that the CP-part of the generator $(\mathrm{tr}_E[V \cdot V^\dagger])$ cannot be written as a convex combination of the two building blocks (Φ_{sc} and $\mathrm{id}_A \otimes \hat{B}$). As mentioned also in the quantization procedure before, this is a pure quantum feature and stems from the fact that it cannot be determined if a photon arriving at the detector D_1 came from B or A. Hence, the system remains in a superposition state.

We can also look at the usual representation via jump operators. This can be achieved by switching from dilations to Kraus operators. We obtain the two jump-operators

$$L_1 := \underbrace{L_e \otimes L_a}_{=:\phi_1} + \mathbb{1}_A \otimes \underbrace{L_e}_{B_1},\ \ L_2 := \underbrace{L_e \otimes |1\rangle\langle 1|}_{=:\phi_2},$$

where $L_e = |0\rangle\langle 1|$ and $L_a = L_e^\dagger$ describe emission and absorption of a photon, respectively. Thus, the usual Lindblad equation reads

$$\frac{d\rho}{dt} = (L_e \otimes L_a + \mathbb{1}_A \otimes L_e)\rho(L_e \otimes L_a + \mathbb{1}_A \otimes L_e) + (\mathbb{1}_A \otimes L_e)\rho(\mathbb{1}_A \otimes L_e) - \frac{1}{2}\Big\{\mathbb{1}_A \otimes L_e^\dagger L_e + L_e^\dagger L_e \otimes \mathbb{1}_B\, ,\, \rho\Big\}.$$

It is also possible and instructive to consider the reduced dynamics on system A, which can also be described by a Lindblad equation, since B does not communicate to A (this is not true otherwise),

$$\frac{d\rho_A}{dt} = L_e \rho_A L_e^\dagger - \frac{1}{2}\Big\{L_e^\dagger L_e\, ,\, \rho_A\Big\},$$

where $\rho_A(t) = \mathrm{tr}_B[\rho(t)]$. Not surprisingly (given our model), this describes an atom emitting photons.

C. Generators of semigroups of quantum superchannels

We finally turn to semigroups of quantum superchannels (on finite-dimensional spaces), that is, a collection of quantum superchannels $\{\hat{S}_t\}_{t\geq 0} \subseteq \mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B)))$, such that $\hat{S}_0 = \mathrm{id}$, $\hat{S}_{t+s} = \hat{S}_t\hat{S}_s$, and the map $t \mapsto \hat{S}_t$ is continuous [with respect to any and, thus, all of the equivalent norms on the finite-dimensional space $\mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B)))$]. To formulate a technically slightly stronger result, we call a map $\hat{S} \in \mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B)))$ a preselecting supermap if $\mathfrak{C}_{A;B} \circ \hat{S} \circ \mathfrak{C}_{A;B}^{-1}$ is a Schrödinger $B \not\to A$ semicausal CP-map. Theorem V.3 then tells us that a superchannel is a special preselecting supermap. Again, as for semicausal CP-maps, we characterize the generators of semigroups of preselecting supermaps and superchannels in two ways: First, we answer how to determine if a given map $\hat{L} \in \mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B)))$ is such a generator. Second, we provide a normal form for all generators.

The answer to the first question is really a corollary of Lemma V.5 together with Theorem V.3. To this end, define $\hat{\mathfrak{L}} := \mathfrak{C}_{AB;AB}\big(\mathfrak{C}_{A;B} \circ \hat{L} \circ \mathfrak{C}_{A;B}^{-1}\big) \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, where we fix some orthonormal bases $\{|a_i\rangle\}_{i=1}^{\dim(\mathcal{H}_A)}$ and $\{|b_j\rangle\}_{j=1}^{\dim(\mathcal{H}_B)}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$ such that $\mathfrak{C}_{A;B}$ is defined with respect to $\{|a_i\rangle\}_{i=1}^{\dim(\mathcal{H}_A)}$ and $\mathfrak{C}_{AB;AB}$ is defined with respect to the product of the two bases. Furthermore, we introduced the spaces $\mathcal{H}_{A_1} = \mathcal{H}_{A_2} = \mathcal{H}_A$ and $\mathcal{H}_{B_1} = \mathcal{H}_{B_2} = \mathcal{H}_A$ for notational convenience. Finally, we define $P^\perp \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$ to be the orthogonal projection onto the orthogonal complement of $\{|\Omega\rangle\}$, where $|\Omega\rangle = \sum_{i,j}|a_i\rangle \otimes |b_j\rangle \otimes |a_i\rangle \otimes |b_j\rangle$. We then have the following lemma:

Lemma V.17. A linear map $\hat{L} \in \mathcal{B}(\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B)))$ generates a semigroup of quantum superchannels if and only if

- *$\hat{\mathfrak{L}}$ is self-adjoint and $P^\perp \hat{\mathfrak{L}} P^\perp \geq 0$,*

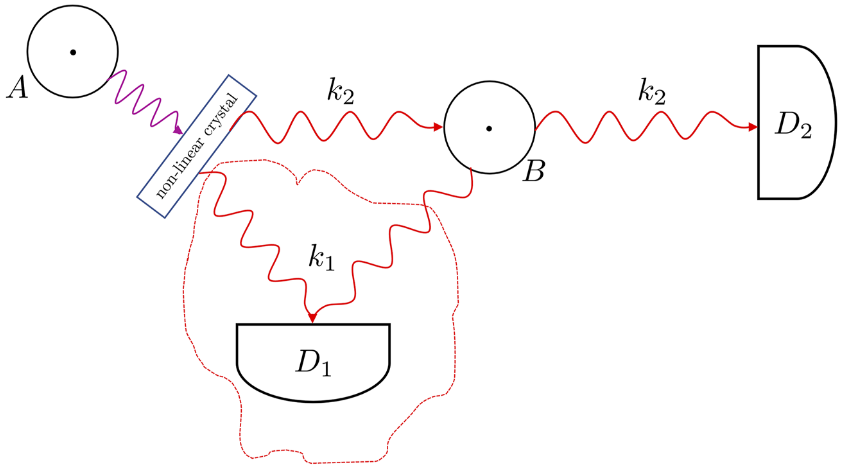

FIG. 6. Systems A and B describe 2-level systems, respectively. The allowed interactions are infinitesimally described as follows: If A is in its excited state, it can emit a photon. Through parametric down-conversion, the photon is converted into two photons (of lower energy). One of those two photons, k_1, is sent to a detector D_1. The other, k_2, is sent to B. If B is in its ground state, it absorbs k_2. If B is in its excited state, it cannot absorb k_2, so k_2 passes through B and travels to a detector D_2. Additionally, in this case, B can emit a photon, indistinguishable from k_1, to D_1.

- $(\mathbb{F}_{A_1;B_1} \otimes \mathbb{1}_{A_2})\mathrm{tr}_{B_2}\big[\hat{\mathfrak{L}}\big](\mathbb{F}_{A_1;B_1} \otimes \mathbb{1}_{A_2}) = \mathbb{1}_{B_1} \otimes \hat{\mathfrak{L}}^A$ *for some (then necessarily self-adjoint)* $\hat{\mathfrak{L}}^A \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2})$, *and*
- $\mathrm{tr}_{A_1}\big[\hat{\mathfrak{L}}^A\big] = 0$.*$\hat{L}$ is preselecting if and only if the first two conditions hold.*

Proof. Theorem V.3 tells us that $\{\hat{S}_t\}_{t\geq 0}$ forming a semigroup of superchannels is eqivalent to $S_t = \mathfrak{C}_{A;B} \circ \hat{S}_t \circ \mathfrak{C}_{A;B}^{-1}$ forming a semigroup of Schrödinger $B \not\to A$ semicausal CP-maps and that the reduced map S_t^A satisfies $S_t^A(\mathbb{1}_A) = \mathbb{1}_A$. By Lemma V.5, the semicausal semigroup property is equivalent to the first two conditions in the statement. This proves the claim about preselecting $\hat{L}$.

By differentiation, it follows that $S_t^A(\mathbb{1}_A) = \mathbb{1}_A$ is satisfied if and only if L^A, the generator of $\{S_t^A\}_{t\geq 0}$, satisfies $L^A(\mathbb{1}_A) = 0$. However, since $\mathrm{tr}_{A_1}\big[\hat{\mathfrak{L}}^A\big] = L^A(\mathbb{1}_A)$, the claim follows. □

We finally turn to a normal form for generators of semigroups of preselecting supermaps and superchannels.

Theorem V.18. *A linear map $\hat{L} : \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B)) \to \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B))$ generates a semigroup of hyper-preselecting supermaps if and only if there exist a Hilbert space $\mathcal{H}_E$, a state $\sigma \in \mathcal{B}(\mathcal{H}_E)$, a unitary $U \in \mathcal{U}(\mathcal{H}_B \otimes \mathcal{H}_E)$, a self-adjoint operator $H_B \in \mathcal{B}(\mathcal{H}_B)$, and arbitrary operators $A \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_E)$, $B \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_E)$, and $K_A \in \mathcal{B}(\mathcal{H}_A)$ such that $\hat{L}$ acts on $T \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B))$ as $\hat{L}(T) = \hat{\Phi}(T) - \hat{\kappa}_L(T) - \hat{\kappa}_R(T)$ with*

$$\begin{aligned}\hat{\Phi}(T)(\rho) = &\mathrm{tr}_E\Big[U\,(T \otimes \mathrm{id}_E)(A(\rho \otimes \sigma)A^\dagger)\,U^\dagger\Big] + \mathrm{tr}_E\Big[B\,(T \otimes \mathrm{id}_E)((\rho \otimes \sigma)A^\dagger)\,U^\dagger\Big] \\ &+ \mathrm{tr}_E\Big[U\,(T \otimes \mathrm{id}_E)(A(\rho \otimes \sigma))\,B^\dagger\Big] + \mathrm{tr}_E\Big[B\,(T \otimes \mathrm{id}_E)((\rho \otimes \sigma))\,B^\dagger\Big],\end{aligned} \tag{17}$$

$$\hat{\kappa}_L(T)(\rho) = \mathrm{tr}_E\Big[B^\dagger U\,(T \otimes \mathrm{id}_E)(A(\rho \otimes \sigma))\Big] + \frac{1}{2}\mathrm{tr}_E\Big[B^\dagger B(T \otimes \mathrm{id}_E)(\rho \otimes \sigma)\Big] + T(K_A\,\rho) + iH_B\,T(\rho), \tag{18a}$$

$$\hat{\kappa}_R(T)(\rho) = \mathrm{tr}_E\Big[(T \otimes \mathrm{id}_E)((\rho \otimes \sigma)A^\dagger)\,U^\dagger B\Big] + \frac{1}{2}\mathrm{tr}_E\Big[(T \otimes \mathrm{id}_E)(\rho \otimes \sigma)B^\dagger B\Big] + T(\rho\,K_A^\dagger) - T(\rho)\,iH_B. \tag{18b}$$

We can choose σ to be pure and $\mathcal{H}_E$ with $\dim(\mathcal{H}_E) \leq (d_A d_B)^2$, where d_A and d_B are the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Furthermore, $\hat{L}$ generates a semigroup of superchannels if and only if $\hat{L}$ generates a semigroup of preselecting supermaps and $\mathrm{tr}_\sigma\big[A^\dagger A\big] = K_A + K_A^\dagger$. In that case, we can split $\hat{L}$ into a dissipative part $\hat{D}$ and a "Hamiltonian" part $\hat{H}$, i.e., a part that generates a (semi-)group of invertible superchannels whose inverses are superchannels as well. We have $\hat{L}(T) = \hat{D}(T) + \hat{H}(T)$, with

$$\hat{D}(T)(\rho) = \mathrm{tr}_E\big[\hat{D}'(T)(\rho)\big] \quad \text{and} \quad \hat{H}(T)(\rho) = -i[H_B\,,\,T(\rho)] - iT([H_A\,,\,\rho]),$$

where H_A is the imaginary part of K_A, where

$$\hat{D}'(T)(\rho) = U(T\otimes \mathrm{id}_E)(A(\rho\otimes\sigma)A^\dagger)U^\dagger - \frac{1}{2}(T\otimes\mathrm{id}_E)(\{A^\dagger A, \rho\otimes\sigma\}) \tag{19a}$$

$$+ B(T\otimes\mathrm{id}_E)(\rho\otimes\sigma)B^\dagger - \frac{1}{2}\{B^\dagger B, (T\otimes\mathrm{id}_E)(\rho\otimes\sigma)\} \tag{19b}$$

$$+ \left[U(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma)), B^\dagger\right] + \left[B, (T\otimes\mathrm{id}_E)((\rho\otimes\sigma)A^\dagger)U^\dagger\right] \tag{19c}$$

and where $[\cdot,\cdot]$ *and* $\{\cdot,\cdot\}$ *denote the commutator and anticommutator, respectively.*

Remark V.19. Similar to Theorem V.6, the Proof of Theorem V.18 is constructive. In Appendix D, we discuss in detail how to obtain the operators A, U, K_A, B, H_A, and H_B starting from the conditions in Theorem V.17.

As in the classical case, the proof strategy is to use the relation between superchannels and semicausal channels and Theorem V.6. As this translation process is more involved than in the classical case, we need two auxiliary lemmas.

Lemma V.20. Let $S: \mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)\to\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$ *be given by*

$$S(X) = \mathrm{tr}_E\left[(\mathbb{1}_A\otimes L_B)(L_A\otimes\mathbb{1}_B)X(R_A^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes R_B^\dagger)\right], \tag{20}$$

with Hilbert spaces $\mathcal{H}_C$ *and* $\mathcal{H}_E$*, operators* $L_A, R_A \in \mathcal{B}(\mathcal{H}_A;\mathcal{H}_A\otimes\mathcal{H}_C)$*, and* $L_B, R_B\in\mathcal{B}(\mathcal{H}_C\otimes\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_E)$*. Then, for* $T\in\mathcal{B}(\mathcal{B}(\mathcal{H}_A);\mathcal{B}(\mathcal{H}_B))$ *and* $\rho\in\mathcal{B}(\mathcal{H}_A)$*,*

$$\left[\mathfrak{C}_{A;B}^{-1}\circ S\circ\mathfrak{C}_{A;B}\right](T)(\rho) = \mathrm{tr}_E\left[V_L(T\otimes\mathrm{id}_C)\left(W_L\rho W_R^\dagger\right)V_R^\dagger\right], \tag{21}$$

with $V_L = L_B\mathbb{F}_{B;C}$*,* $V_R = R_B\mathbb{F}_{B;C}$*, and* $W_L = L_A^{T_A}$*,* $W_R = R_A^{T_A}$*. Here, the partial transpose on* $\mathcal{H}_A$ *is taken with respect to the basis used to define the Choi–Jamiołkowski isomorphism.*

Proof. The proof is a direct calculation. We present it in detail in Appendix A. □

Lemma V.21. Let $X\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_C;\mathcal{H}_A\otimes\mathcal{H}_B)$*,* $Y\in\mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B;\mathcal{H}_A\otimes\mathcal{H}_C)$*,* $\rho\in\mathcal{S}_1(\mathcal{H}_B)$*. Then,* $\mathrm{tr}_\rho[XY]^T = \mathrm{tr}_C\left[Y^{T_A}(\mathbb{1}_A\otimes\rho)X^{T_A}\right]$*.*

Proof. The proof is a direct calculation. We present it in detail in Appendix A. □

We are finally ready to prove Theorem V.18

Proof (Theorem V.18). The idea is to relate the generators of superchannels to semicausal maps. This relation is given by definition for preselecting supermaps and by Theorem V.3 for superchannels. For a generator $\hat{L}$ of a semigroup of preselecting supermaps $\{\hat{S}_t\}_{t\ge 0}$, we have

$$\hat{L} = \mathfrak{C}_{A;B}^{-1}\circ\frac{d}{dt}\bigg|_{t=0}[\mathfrak{C}_{A;B}\circ\hat{S}_t\circ\mathfrak{C}_{A;B}^{-1}]\circ\mathfrak{C}_{A;B}.$$

Thus, $\hat{L}$ generates a semigroup of preselecting supermaps if and only if $\hat{L}$ can be written as $\hat{L} = \mathfrak{C}_{A;B}^{-1}\circ L\circ\mathfrak{C}_{A;B}$ for some generator L of a semigroup of Schrödinger $B\not\to A$ semicausal CP-maps. Thus, to prove the first part of our theorem, we can take the normal form in Corollary V.15 and compute the similarity transformation above. We now execute this in detail. To start with, Corollary V.15 tells us that $L(\rho) = \Phi_S(\rho) - K\rho - \rho K^\dagger$, where

$$\Phi_S(\rho) = \mathrm{tr}_E\left[V\rho V^\dagger\right], \text{ with } V = (\mathbb{1}_A\otimes\tilde{U})(\tilde{A}\otimes\mathbb{1}_B) + (\mathbb{1}_A\otimes\tilde{B}), \tag{22a}$$

$$K = (\mathbb{1}_A\otimes\tilde{B}^\dagger\tilde{U})(\tilde{A}\otimes\mathbb{1}_B) + \frac{1}{2}\mathbb{1}_A\otimes\tilde{B}^\dagger\tilde{B} + \tilde{K}_A\otimes\mathbb{1}_B + \mathbb{1}_A\otimes i\tilde{H}_B, \tag{22b}$$

for some unitary $\tilde{U}\in\mathcal{U}(\mathcal{H}_E\otimes\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_E)$, some self-adjoint $\tilde{H}_B\in\mathcal{B}(\mathcal{H}_B)$, and some operators $\tilde{A}\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_A\otimes\mathcal{H}_E)$, $\tilde{B}\in\mathcal{B}(\mathcal{H}_B;\mathcal{H}_B\otimes\mathcal{H}_E)$, and $\tilde{K}_A\in\mathcal{B}(\mathcal{H}_A)$. In order to apply Lemma V.20, we fix a unit vector $|\xi\rangle\in\mathcal{H}_E$ and define $\Xi_A := \mathbb{1}_A\otimes|\xi\rangle\in\mathcal{B}(\mathcal{H}_A;\mathcal{H}_A\otimes\mathcal{H}_E)$ and $\Xi_B := |\xi\rangle\otimes\mathbb{1}_B\in\mathcal{B}(\mathcal{H}_B;\mathcal{H}_E\otimes\mathcal{H}_B)$ so that $\mathbb{1}_A\otimes\tilde{B} = (\mathbb{1}_A\otimes\tilde{B}\Xi_B^\dagger)(\Xi_A\otimes\mathbb{1}_B)$. We can then write

$$\Phi_S(\rho) = \mathrm{tr}_E\left[(\mathbb{1}_A\otimes\tilde{U})(\tilde{A}\otimes\mathbb{1}_B)\rho(\tilde{A}^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes U^\dagger)\right] + \mathrm{tr}_E\left[(\mathbb{1}_A\otimes\tilde{B}\Xi_B^\dagger)(\Xi_A\otimes\mathbb{1}_B)\rho(\Xi_A^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\Xi_B\tilde{B}^\dagger)\right]$$
$$+ \mathrm{tr}_E\left[(\mathbb{1}_A\otimes\tilde{U})(\tilde{A}\otimes\mathbb{1}_B)\rho(\Xi_A^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\Xi_B\tilde{B}^\dagger)\right] + \mathrm{tr}_E\left[(\mathbb{1}_A\otimes\tilde{B}\Xi_B^\dagger)(\Xi_A\otimes\mathbb{1}_B)\rho(\tilde{A}^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes U^\dagger)\right],$$

which is an expression suitable for a term by term application of Lemma V.20. Doing so yields

$$\begin{aligned}\hat{\Phi}(T)(\rho) &:= (\mathfrak{C}_{A;B}^{-1} \circ \Phi_S \circ \mathfrak{C}_{A;B})(T)(\rho)\\ &= \mathrm{tr}_E\Big[U\,(T\otimes \mathrm{id}_E)(A(\rho\otimes\sigma)A^\dagger)\;U^\dagger\Big] + \mathrm{tr}_E\Big[B\,(T\otimes\mathrm{id}_E)((\rho\otimes\sigma)A^\dagger)\;U^\dagger\Big]\\ &\quad+ \mathrm{tr}_E\Big[U\,(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma))\;B^\dagger\Big] + \mathrm{tr}_E\Big[B\,(T\otimes\mathrm{id}_E)((\rho\otimes\sigma))\;B^\dagger\Big],\end{aligned}$$

where we defined $U := \tilde{U}\mathbb{F}_{B;E}$, $B := \tilde{B}\Xi_B^\dagger\mathbb{F}_{B;E}$, $A := \tilde{A}^{\,T_A}\Xi_A^\dagger$, and $\sigma := |\xi\rangle\langle\xi|$. This proves Eq. (17). Similarly, upon defining $\kappa_L(\rho) := K\rho$, we can write[32]

$$\begin{aligned}\kappa_L(\rho) &= \mathrm{tr}_E\Big[(\mathbb{1}_A\otimes\mathbb{F}_{E;B}\Xi_B\tilde{B}^\dagger\tilde{U})(\tilde{A}\otimes\mathbb{1}_B)\rho(\Xi_A^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\mathbb{F}_{B;E})\Big] + \mathrm{tr}_E\Big[(\mathbb{1}_A\otimes\mathbb{F}_{E;B}\Xi_B\tilde{B}^\dagger\tilde{B}\Xi_B^\dagger)(\Xi_A\otimes\mathbb{1}_B)\rho(\Xi_A^\dagger\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\mathbb{F}_{B;E})\Big]\\ &\quad+ \mathrm{tr}_{\mathbb{C}}\big[(\mathbb{1}_A\otimes\mathbb{1}_B)(\tilde{K}_A\otimes\mathbb{1}_B)\rho(\mathbb{1}_A\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\mathbb{1}_B)\big] + \mathrm{tr}_{\mathbb{C}}[(\mathbb{1}_A\otimes iH_B)(\mathbb{1}_A\otimes\mathbb{1}_B)\rho(\mathbb{1}_A\otimes\mathbb{1}_B)(\mathbb{1}_A\otimes\mathbb{1}_B)]\end{aligned}$$

and apply Lemma V.20 term by term, which yields

$$\begin{aligned}\hat{\kappa}_L(T)(\rho) &:= (\mathfrak{C}_{A;B}^{-1}\circ\kappa_L\circ\mathfrak{C}_{A;B})(T)(\rho)\\ &= \mathrm{tr}_E\Big[B^\dagger U\,(T\otimes\mathrm{id}_E)(A(\rho\otimes\sigma))\Big] + \frac{1}{2}\mathrm{tr}_E\Big[B^\dagger B(T\otimes\mathrm{id}_E)(\rho\otimes\sigma)\Big] + T(K_A\,\rho) + iH_B\,T(\rho),\end{aligned}$$

where U, A, and B are defined as above and $K_A := (\tilde{K}_A)^T$ and $H_B := \tilde{H}_B$. An analogous calculation with $\kappa_R(\rho) := \rho K^\dagger$ and $\hat{\kappa}_R(T) := (\mathfrak{C}_{A;B}^{-1}\circ\kappa_R\circ\mathfrak{C}_{A;B})(T)$ finishes the proof of the first part, since the claim about the dimension of $\mathcal{H}_E$ follows form the corresponding statements in Theorem V.6.

To prove the second part, first remember that we have observed above that Theorem V.3 implies that L is Schrödinger $B \not\to A$ semicausal, with $\mathrm{tr}_B[L(\rho)] = L^A(\mathrm{tr}_B[\rho])$. Furthermore, if we write $S_t = \mathfrak{C}_{A;B}\circ\hat{S}_t\circ\mathfrak{C}_{A;B}^{-1}$, then Theorem V.3 implies that S_t is Schrödinger $B\not\to A$ semicausal for all $t\ge 0$, with $\mathrm{tr}_B[S_t(\rho)] = S_t^A(\mathrm{tr}_B[\rho])$, and also $S_t^A(\mathbb{1}_A) = \mathbb{1}_A$ holds. Differentiating that expression at $t = 0$ yields the equivalent condition $L^A(\mathbb{1}_A) = 0$. Hence, our goal is to incorporate the last condition into the form of (22). To do so, we determine L^A by calculating $\mathrm{tr}_B[L(\rho)]$, where L is in the form of (22). We obtain $\mathrm{tr}_B[L(\rho)] = \mathrm{tr}_E[\tilde{A}\,\mathrm{tr}_B[\rho]\,\tilde{A}^\dagger] - \tilde{K}_A\mathrm{tr}_B[\rho] - \mathrm{tr}_B[\rho]\tilde{K}_A^\dagger$. Thus, the condition $L^A(\mathbb{1}) = 0$ holds if and only if $\mathrm{tr}_E[\tilde{A}\tilde{A}^\dagger] = \tilde{K}_A + \tilde{K}_A^\dagger$. Transposing both sides of this equation and using that the definition of A implies that $\tilde{A} = A^{T_A}\Xi_A$ yield $\big(\mathrm{tr}_E\big[A^{T_A}(\mathbb{1}_A\otimes\sigma)(A^\dagger)^{T_A}\big]\big)^T = K_A + K_A^\dagger$. However, the left-hand side is, by Lemma V.21, equal to $\mathrm{tr}_\sigma\big[A^\dagger A\big]$. This proves the claim that $\hat{L}$ generates a semigroup of superchannels if and only if $\hat{L}$ is hyper-preselecting and $\mathrm{tr}_\sigma\big[A^\dagger A\big] = K_A + K_A^\dagger$. Finally, defining $H_A := \frac{1}{2i}(K_A - K_A^\dagger)$ and a few rearrangements lead to (19). □

VI. CONCLUSION

A. Summary

The underlying question of this work is as follows: How can we mathematically characterize the processes that describe the aging of quantum devices? We have argued that, under a Markovianity assumption, such processes can be modeled by continuous semigroups of quantum superchannels. Therefore, the goal of this work was to provide a full characterization of such semigroups of superchannels.

We have derived such a general characterization in terms of the generators of these semigroups. Crucially, we have exploited that superchannels correspond to certain semicausal maps and that, therefore, it suffices to characterize generators of semigroups of semicausal maps. We have demonstrated both an efficient procedure for checking whether a given generator is indeed a valid semicausal GKLS generator and a complete characterization of such valid semicausal GKLS generators. The latter is constructive in the sense that it can be used to describe parametrizations of these generators. Aside from the theoretical relevance of these results, they will be valuable in studying properties of these generators numerically. Finally, we have translated these results back to the level of superchannels, thus answering our initial question.

We have also posed and answered the classical counterpart of the above question. That is, we have characterized the generators semigroups of classical superchannels and of semicausal non-negative maps. These results for the classical case might be of independent interest. From the perspective of quantum information theory, they provide a comparison helpful to understand and interpret the characterizations in the quantum case.

B. Outlook and open questions

We conclude by presenting some open questions raised by our work. First, in our proof of the characterization of semicausal GKLS generators, we have described a procedure for constructing a semicausal CP-map associated with such a generator. We believe that this method can be applied to a wide range of problems. Determining the exact scope of this method is currently work in progress.

Second, there is a wealth of results on the spectral properties of quantum channels and, in particular, semigroups of quantum channels. With the explicit form of generators of semigroups of superchannels now known, we can conduct analogous studies for semigroups of quantum superchannels. Understanding such spectral properties, and potentially how they differ from the properties in the scenario of quantum channels, would, in particular, lead to a better understanding of the asymptotic behavior of semigroups of superchannels, e.g., with respect to entropy production,[33,34] the thermodynamics of quantum channels,[35] or entanglement-breaking properties.[36]

A further natural question would be a quantum superchannel analog of the Markovianity problem: When can a quantum superchannel $\hat{S}$ be written as $e^{\hat{L}}$ for some $\hat{L}$ that generates a semigroup of superchannels? Several works have investigated the Markovianity problem for quantum channels[21,37–39] and a divisibility variant of this question, both for quantum channels and for stochastic matrices.[40–42] It would be interesting to see how these results translate to quantum or classical superchannels. Similarly, we can now ask questions of reachability along Markovian paths. Yet another question aiming at understanding Markovianity is as follows: If we consider master equations arising from a Markovianity assumption on the underlying process formalized not via semigroups of channels but instead via semigroups of superchannels, what are the associated classes of (time-dependent) generators and corresponding CPTP evolutions?

Two related directions, both of which will lead to a better understanding of Markovian structures in higher order quantum operations, are as follows: support our mathematical characterization of the generators of semigroups of superchannels by a physical interpretation, similar to the Monte Carlo wave function interpretation of Lindblad generators of quantum channels, and extend our characterization from superchannels to general higher order maps.

This work has focused on generators of general semigroups of superchannels, without further restrictions. For quantum channels and their Lindblad generators, there exists a well-developed theory of locality, at the center of which are Lieb–Robinson bounds.[43] If we put locality restrictions on generators of superchannels, how do these translate to the generated superchannels?

Finally, an important conceptual direction for future work is to identify further applications of our theory of dynamical semigroups of superchannels. In the Introduction, we gave a physical meaning to semigroups of superchannels by relating them to the decay process of quantum devices. This, however, is only one possible interpretation. For example, semigroups of superchannels might also describe a manufacturing process, where a quantum device is created layer-by-layer. We hope that other use-cases will be found in the future.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: PROOF OF LEMMAS V.20 AND V.21

In this appendix, we provide a complete proof of Lemmas V.20 and V.21.

Lemma A.1 (restatement of Lemma V.20). Let $S : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be given by

$$S(X) = \mathrm{tr}_E\Big[(\mathbb{1}_A \otimes L_B)(L_A \otimes \mathbb{1}_B)X(R_A^\dagger \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes R_B^\dagger)\Big],$$

with Hilbert spaces $\mathcal{H}_C$ and $\mathcal{H}_E$, operators $L_A, R_A \in \mathcal{B}(\mathcal{H}_A; \mathcal{H}_A \otimes \mathcal{H}_C)$, and $L_B, R_B \in \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$. Then, for $T \in \mathcal{B}(\mathcal{B}(\mathcal{H}_A); \mathcal{B}(\mathcal{H}_B))$ and $\rho \in \mathcal{B}(\mathcal{H}_A)$,

$$\big[\mathfrak{C}_{A;B}^{-1} \circ S \circ \mathfrak{C}_{A;B}\big](T)(\rho) = \mathrm{tr}_E\Big[V_L(T \otimes \mathrm{id}_C)\Big(W_L \rho W_R^\dagger\Big)V_R^\dagger\Big],$$

with $V_L = L_B \mathbb{F}_{B;C}$, $V_R = R_B \mathbb{F}_{B;C}$ and $W_L = L_A^{T_A}$, $W_R = R_A^{T_A}$. Here, the partial transpose on $\mathcal{H}_A$ is taken with respect to the basis used to define the Choi–Jamiołkowski isomorphism.

Proof. Let $\{|e_i\rangle\}_i$ be the orthonormal basis of $\mathcal{H}_A$ with respet to which the Choi–Jamiołkowski isomorphism is defined. Let $\{|c_n\rangle\}_n$ be an orthonormal basis of $\mathcal{H}_C$. Then, the formal calculation, which is an algebraic version of drawing the corresponding tensor-network pictures, can be executed as follows:

$$\begin{aligned}
\big[\mathfrak{C}_{A;B}^{-1} \circ S \circ \mathfrak{C}_{A;B}\big](T)(\rho) &= \mathrm{tr}_A\Big[(\rho^T \otimes \mathbb{1}_B)\,\mathrm{tr}_E\Big[(\mathbb{1}_A \otimes L_B)(L_A \otimes \mathbb{1}_B)\mathfrak{C}_{A;B}(T)(R_A^\dagger \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes R_B^\dagger)\Big]\Big] \\
&= \mathrm{tr}_E\Big[L_B\,\mathrm{tr}_A\Big[(\rho^T \otimes \mathbb{1}_C \otimes \mathbb{1}_B)(L_A \otimes \mathbb{1}_B)\mathfrak{C}_{A;B}(T)(R_A^\dagger \otimes \mathbb{1}_B)\Big]R_B^\dagger\Big] \\
&= \sum_{i,j}\mathrm{tr}_E\Big[L_B\Big(\mathrm{tr}_A\Big[(\rho^T \otimes \mathbb{1}_C)L_A|e_i\rangle\langle e_j|R_A^\dagger\Big] \otimes T(|e_i\rangle\langle e_j|)\Big)R_B^\dagger\Big] \\
&= \sum_{i,j,k,m,n}\langle e_k\, c_n|\Big((\rho^T \otimes \mathbb{1}_C)L_A|e_i\rangle\langle e_j|R_A^\dagger\Big)\, e_k\, c_m\rangle\ \mathrm{tr}_E\Big[L_B\,(|c_n\rangle\langle c_m| \otimes T(|e_i\rangle\langle e_j|))R_B^\dagger\Big] \\
&= \sum_{i,j,m,n}\langle e_i|\Big(L_A^T(\rho \otimes |c_n\rangle\langle c_m|)\overline{R}_A\Big)\, e_j\rangle\ \mathrm{tr}_E\Big[L_B\,(|c_n\rangle\langle c_m| \otimes T(|e_i\rangle\langle e_j|))R_B^\dagger\Big] \\
&= \sum_{m,n}\mathrm{tr}_E\Big[L_B\,\Big(|c_n\rangle\langle c_m| \otimes T\Big(L_A^T(\rho \otimes |c_n\rangle\langle c_m|)\overline{R}_A\Big)\Big)R_B^\dagger\Big] \\
&= \mathrm{tr}_E\Bigg[L_B\mathbb{F}_{B;C}(T \otimes \mathrm{id}_C)\Bigg(\Bigg[\sum_n(\mathbb{1}_A \otimes |c_n\rangle)L_A^T(\mathbb{1}_A \otimes |c_n\rangle)\Bigg]\rho\Bigg[\sum_m(\mathbb{1}_A \otimes |c_m\rangle)R_A^T(\mathbb{1}_A \otimes |c_m\rangle)\Bigg]^\dagger\Bigg)\mathbb{F}_{B;C}R_B^\dagger\Bigg] \\
&= \mathrm{tr}_E\Big[V_L(T \otimes \mathrm{id}_C)\Big(W_L\rho W_R^\dagger\Big)V_R^\dagger\Big].
\end{aligned}$$

□

Lemma A.2. Let $X \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_C; \mathcal{H}_A \otimes \mathcal{H}_B)$, $Y \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_C)$, $\rho \in \mathcal{S}_1(\mathcal{H}_B)$. *Then,* $\mathrm{tr}_\rho[XY]^T = \mathrm{tr}_C\big[Y^{T_A}(\mathbb{1}_A \otimes \rho)X^{T_A}\big]$.

Proof. Let $\{|a_i\rangle\}_i$ be the orthonormal basis with respect to which the transposition is taken. Using the general identity $\mathrm{tr}\big[M^T\big] = \mathrm{tr}[M]$, the definition of the trace with respect to a trace-class operator, and the cyclicity of the trace, we obtain, for every $\sigma \in \mathcal{S}_1(\mathcal{H}_A)$,

$$\begin{aligned}
\mathrm{tr}\Big[\sigma\mathrm{tr}_\rho[XY]^T\Big] &= \mathrm{tr}\Big[\sigma^T\mathrm{tr}_\rho[XY]\Big] \\
&= \mathrm{tr}\Big[(\sigma^T \otimes \rho)XY\Big] \\
&= \sum_{i,j,k}\mathrm{tr}\Big[(\langle a_i| \otimes \mathbb{1}_B)(\sigma^T \otimes \rho)(|a_j\rangle\langle a_j| \otimes \mathbb{1}_B)X(|a_k\rangle\langle a_k| \otimes \mathbb{1}_C)Y(|a_i\rangle \otimes \mathbb{1}_B)\Big] \\
&= \sum_{i,j,k}\mathrm{tr}\Big[(\langle a_j| \otimes \mathbb{1}_B)(\sigma \otimes \rho)(|a_i\rangle\langle a_k| \otimes \mathbb{1}_B)X^{T_A}(|a_j\rangle\langle a_i| \otimes \mathbb{1}_C)Y^{T_A}(|a_k\rangle \otimes \mathbb{1}_B)\Big] \\
&= \sum_k\mathrm{tr}\Bigg[\rho(\langle a_k| \otimes \mathbb{1}_B)X^{T_A}\Bigg(\Bigg(\sum_{i,j}\langle a_j|\sigma\, a_i\rangle|a_i\rangle\langle a_j|\Bigg) \otimes \mathbb{1}_C\Bigg)Y^{T_A}(|a_k\rangle \otimes \mathbb{1}_B)\Bigg] \\
&= \mathrm{tr}\Big[(\mathbb{1}_A \otimes \rho)X^{T_A}(\sigma \otimes \mathbb{1}_C)Y^{T_A}\Big] \\
&= \mathrm{tr}\Big[\sigma\mathrm{tr}_C\Big[Y^{T_A}(\mathbb{1}_A \otimes \rho)X^{T_A}\Big]\Big].
\end{aligned}$$

This proves the claim. □

APPENDIX B: NO INFORMATION WITHOUT DISTURBANCE

Here, we prove a "no information without disturbance"-like lemma that yielded a useful interpretation in the main text.

Lemma B.1. Let $T \in \mathrm{CP}_\sigma(\mathcal{H}_A \otimes \mathcal{H}_B)$ *be such that*

$$T(X_A \otimes \mathbb{1}_B) = X_A \otimes \mathbb{1}_B \tag{B1}$$

for all $X_A \in \mathcal{B}(\mathcal{H}_A)$. *Then,* $T(X) = (\mathbb{1}_A \otimes W^\dagger)(X \otimes \mathbb{1}_E)(\mathbb{1}_A \otimes W)$ *for all* $X \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *and some isometry* $W \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, *where* $\mathcal{H}_E$ *is some Hilbert space.*

Proof. This claim follows from the uniqueness of the minimal Stinespring dilation in the same way as the "semicausal = semilocalizable" theorem. Write Eq. (B1) in the Stinespring form as

$$V^\dagger(X_A \otimes \mathbb{1}_B \otimes \mathbb{1}_E)V = X_A \otimes \mathbb{1}_B$$

for some $V \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Then, V and $\mathbb{1}_{AB}$ are the Stinespring operators of the same CP-map $(X_A \mapsto X_A \otimes \mathbb{1}_B)$ and the latter clearly belongs to a minimal dilation. Thus, there exists an isometry $W \in \mathcal{B}(\mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$ such that $V = (\mathbb{1}_A \otimes W)\mathbb{1}_{AB}$. This is the claim. □

Note that the lemma above is just a formulation of the "obvious" fact that if system A undergoes a closed system evolution (id_A), then there is no interaction with an external system B.

APPENDIX C: CONSTRUCTIVE APPROACH TO THEOREM V.6

In this appendix, we are going to describe in detail how one can computationally construct the operators A, U, B, K_A, and H_B in Theorem V.6 if the conditions of Lemma V.5 are met.

Since it is important for an actual implementation on a computer, let us be very precise about notation. We introduce indexed copies of $\mathcal{H}_A$ and $\mathcal{H}_B$, i.e., $\mathcal{H}_{A_0} = \mathcal{H}_{A_1} = \mathcal{H}_{A_2} = \mathcal{H}_A$ and $\mathcal{H}_{B_0} = \mathcal{H}_{B_1} = \mathcal{H}_{B_2} = \mathcal{H}_B$. Furthermore, we fix orthonormal bases $\{|a_i\rangle\}_{i=1}^{d_A}$ and $\{|b_i\rangle\}_{i=1}^{d_B}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We use the symbol Ω with some subscript to denote the maximally entangled state on various systems. For example, $|\Omega_{A_1;A_2}\rangle := \sum_i |a_i\rangle \otimes |a_i\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $|\Omega_{A_1B_1;A_2B_2}\rangle = \sum_{i,j} |a_i\rangle \otimes |b_j\rangle \otimes |a_i\rangle \otimes |b_j\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$. We further reserve $P \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$ for the orthogonal projection onto $\mathrm{span}\{|\Omega_{A_1B_1;A_2B_2}\rangle\}$ (i.e., $P = (d_A d_B)^{-1} |\Omega_{A_1B_1;A_2B_2}\rangle\langle\Omega_{A_1B_1;A_2B_2}|$) and take $P^\perp = \mathbb{1}_{A_1B_1A_2B_2} - P$.

Now, let $\mathfrak{L} \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$ be given as in Lemma V.5, then we can compute the operators A, U, B, K_A, and H_B via the following 15 steps:

1. Compute $\tau = P^\perp \mathfrak{L} P^\perp$.
2. Compute $V = (\mathbb{1}_{A_0B_0} \otimes \sqrt{\tau})(|\Omega_{A_0B_0;A_1B_1}\rangle \otimes \mathbb{1}_{A_2B_2})$.
3. Define $\mathcal{H}_E := \mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$ so that $V \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ (identification).
4. Compute $B = \frac{1}{d_A} \mathrm{tr}_A[V]$.
5. Compute $V_{sc} = V - \mathbb{1}_A \otimes B$.
6. Compute $\tau_{sc} = (\mathbb{1}_{A_1B_1} \otimes V_{sc})^\dagger (|\Omega_{A_1B_1;AB}\rangle\langle\Omega_{A_1B_1;AB}| \otimes \mathbb{1}_E)(\mathbb{1}_{A_1B_1} \otimes V_{sc}) \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_A \otimes \mathcal{H}_B)$.
7. Choose any unit vector $|\beta\rangle \in \mathcal{H}_B$.
8. Compute $\tau_{sc}^A = (\mathbb{1}_{A_1A_2} \otimes \langle\beta|)\mathrm{tr}_{B_1}[\tau_{sc}](\mathbb{1}_{A_1A_2} \otimes |\beta\rangle)$.
9. Compute $\mathcal{H}_F = \mathrm{range}(\sqrt{\tau_{sc}^A})$ so that $\sqrt{\tau_{sc}^A} \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}; \mathcal{H}_F)$ is surjective.
10. Compute $A = (\mathbb{1}_{A_0} \otimes \sqrt{\tau_{sc}^A})(|\Omega_{A_0;A_1}\rangle \otimes \mathbb{1}_{A_2})$.
11. Compute U as *the* solution of the system of linear equations $\mathcal{M}(U) = V_{sc}$, where the $d_A^2 d_B^2 d_E \times d_F d_B^2 d_E$-matrix $\mathcal{M} : \mathcal{B}(\mathcal{H}_F \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B; \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ is defined by $\mathcal{M}(U) = (\mathbb{1}_A \otimes U)(A \otimes \mathbb{1}_B)$. Clearly, we must first represent $\mathcal{M}$ with respect to some basis.
12. Compute $K = -\mathrm{tr}_{A_1B_1}\left[P\mathfrak{L}P^\perp + \frac{1}{2}\mathrm{tr}[P\mathfrak{L}]P\right]$, where we identify $\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2} = \mathcal{H}_A \otimes \mathcal{H}_B$ so that $K \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.
13. Compute $K_{sc} = K - (\mathbb{1}_A \otimes B^\dagger)V_{sc} - \frac{1}{2}\mathbb{1}_A \otimes B^\dagger B$.
14. Compute $K_A = \frac{1}{d_B}\mathrm{tr}_B[K_{sc}]$.
15. Compute $H_B = \frac{-i}{d_A}\mathrm{tr}_A[K_{sc} - K_A \otimes \mathbb{1}_B]$.

Note that the procedure above computes an isometry $U \in \mathcal{B}(\mathcal{H}_F \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, which can then be extended to a unitary, if necessary. In that case, we also have to embed $\mathcal{H}_F$ into $\mathcal{H}_E$ and redefine A accordingly. More precisely, we need to execute the following additional steps:

16. Compute $\mathbb{1}_{F\to E} = \mathbb{1}_{A_1} \otimes |\beta\rangle_{B_1} \otimes \mathbb{1}_{A_2} \otimes |\beta\rangle_{B_2}$.
17. Redefine $A \leftarrow (\mathbb{1}_{A_0} \otimes \mathbb{1}_{F\to E})A$.
18. Extend U via the following steps:
    (a) Compute $\hat{U} = U(\mathbb{1}_{F\to E}^\dagger \otimes \mathbb{1}_B)$.
    (b) Compute an orthonormal basis $\{|f_i^\perp\rangle\}_{i=1}^N$ of $\mathrm{range}(\mathbb{1}_{EB} - \hat{U}^\dagger\hat{U})$.
    (c) Compute an orthonormal basis $\{|r_i^\perp\rangle\}_{i=1}^N$ of $\mathrm{range}(\mathbb{1}_{BE} - \hat{U}\hat{U}^\dagger)$.
    (d) Redefine $U \leftarrow \hat{U} + \sum_{i=1}^N |r_i^\perp\rangle\langle f_i^\perp|$.

Let us comment on why the steps above give the right result. In general, we have

$$\mathfrak{L} = P^\perp\mathfrak{L}P^\perp + P\mathfrak{L}P^\perp + P^\perp\mathfrak{L}P + P\mathfrak{L}P = \tau + \left(P\mathfrak{L}P^\perp + \frac{1}{2}\mathrm{tr}[P\mathfrak{L}]P\right) + \left(P^\perp\mathfrak{L}P + \frac{1}{2}\mathrm{tr}[P\mathfrak{L}]P\right).$$

Thus, the maps Φ and K appearing in the GKLS-form in Theorem V.6 can be extracted from the previous equation by applying the inverse of the Choi–Jamiołkowski isomorphism. One readily obtains $\Phi = \mathfrak{C}_{AB;AB}^{-1} \circ \tau$ and $K = -\mathrm{tr}_{A_1B_1}\left[P\mathfrak{L}P^\perp + \frac{1}{2}\mathrm{tr}[P\mathfrak{L}]P\right]$.

- Step 2 computes the Stinespring dilation of a CP-map whose Choi–Jamiołkowski operator is τ. A direct computation shows that $\tau = (\mathbb{1}_{A_1B_1} \otimes V)^\dagger(|\Omega_{A_1B_1;A_2B_2}\rangle\langle\Omega_{A_1B_1;A_2B_2}| \otimes \mathbb{1}_E)(\mathbb{1}_{A_1B_1} \otimes V)$.
- Step 4 computes the operator B in the representation. In the Proof of Theorem V.6, B was obtained from $\tilde{B}$, which, in turn, was obtained from V and Lemma V.13. In the finite-dimensional setting, Lemma V.13 constructs B exactly as is written down above.
- Steps 6, 7, and 8 define τ_{sc} as the Choi–Jamiołkowski operator of a CP-map with the Stinespring operator V_{sc}. Thus, according to the Proof of Theorem V.6, τ is the Choi–Jamiołkowski of a Heisenberg $B \not\to A$ semicausal map. Semicausality is expressed on the level of Choi–Jamiołkowski operators by the existence of an operator τ_{sc}^A such that $\mathrm{tr}_{B_1}[\tau_{sc}] = \tau_{sc}^A \otimes \mathbb{1}_{B_2}$ (compare with the Proof of Lemma V.5). Using this relation makes clear that step 8 extracts τ_{sc}^A from τ_{sc} and that the result is independent of the choice of $|\beta\rangle$.
- Step 10 defines A as the Stinespring dilation of the (reduced) map whose Choi–Jamiołkowski operator is τ_{sc}^A. The dilation constructed in this way is minimal. This is exactly the way in which the operator $W = A$ was constructed in the Proof of Theorem V.6.
- Step 11 obtains U by solving the defining relation (for $\tilde{U}$) in the Proof of Theorem V.6. One might wonder why the solution to this system of equations is unique (even though $\mathcal{M}$ is not a square matrix). Uniqueness follows from the minimality of $A \otimes \mathbb{1}_B$, that is, vectors of the form $(X_A \otimes \mathbb{1}_{FB})(A \otimes \mathbb{1}_B)|\psi\rangle$ span $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$. In detail, if U and U' satisfy $\mathcal{M}(U) = \mathcal{M}(U')$, then $0 = (\mathbb{1}_A \otimes (U - U'))(A \otimes \mathbb{1}_B)$ and hence $0 = (\mathbb{1}_A \otimes (U - U'))(X_A \otimes \mathbb{1}_{FB})(A \otimes \mathbb{1}_B)|\psi\rangle$. By linearity, this implies $U - U' = 0$.
- Step 12 computes the operator K in the GKLS-form according to the discussion above.
- Step 13 defines an operator K_{sc}, which according the statement of Theorem V.6 and also due to the discussion below Eq. (16) is of the form $K_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes iH_B$.
- Steps 14 and 15 extract K_A and H_B from K_{sc}. Note that such a decomposition is not unique, since for any $\lambda \in \mathbb{R}$, the transformation $K_A \to K_A + i\lambda\mathbb{1}_A$, $H_B \to H_B - \lambda\mathbb{1}_B$ leaves K_{sc} invariant. This transformation, however, allows us to choose H_B traceless. In that case, steps 14 and 15 determine K_A and H_B.

APPENDIX D: CONSTRUCTIVE APPROACH TO THEOREM V.18

In this appendix, we are going to describe in detail how one can computationally construct the operators A, U, B, H_A, and H_B in Theorem V.18 if the conditions of Lemma V.17 are met. We use the notation from Appendix C.

Given the operator $\hat{\mathfrak{L}} \in \mathcal{B}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$ as in Lemma V.17, then we can compute the operators A, U, B, H_A, and H_B via the following eight steps:

1. Apply steps 1–18 in the protocol in Appendix C to $\hat{\mathfrak{L}}$. This yields $\mathcal{H}_E = \mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$, $\tilde{A} \in \mathcal{B}(\mathcal{H}_{A_2}; \mathcal{H}_{A_0} \otimes \mathcal{H}_E)$, $\tilde{U} \in \mathcal{B}(\mathcal{H}_E \otimes \mathcal{H}_B; \mathcal{H}_B \otimes \mathcal{H}_E)$, $\tilde{K}_A \in \mathcal{B}(\mathcal{H}_A)$, and $\tilde{H}_B \in \mathcal{B}(\mathcal{H}_B)$.
2. Choose any unit vector $|\xi\rangle \in \mathcal{H}_E$.
3. Compute $\sigma = |\xi\rangle\langle\xi|$.
4. Compute $A = (\mathbb{1}_{A_{-1}} \otimes \mathbb{1}_E \otimes \langle\Omega_{A_0;A_3}|)(\mathbb{1}_{A_{-1}} \otimes \mathbb{F}_{A_0;E}\tilde{A} \otimes \mathbb{1}_{A_3})(|\Omega_{A_{-1};A_2}\rangle \otimes \mathbb{1}_{A_3} \otimes \langle\xi|)$.
5. Compute $B = \tilde{B}(\mathbb{1}_B \otimes \langle\xi|)$.
6. Compute $U = \tilde{U}\mathbb{F}_{B;E}$.
7. Set $H_B = \tilde{H}_B$.
8. Calculate $H_A = \frac{1}{2i}(\tilde{K}_A^T - \tilde{K}_A^{\dagger T})$, where the transposition is with respect to the $\{|a_i\rangle\}$ basis defined in Appendix C.

Let us comment on why the steps above yield the right result:

- Step 1 can be executed, since the assumptions of Lemma V.5 are the first two assumptions in Lemma V.17.
- Steps 2 and 3 define σ as in the Proof of Theorem V.18.
- Step 4 is a more explicit expression for $\tilde{A}^{T_A}\Xi_A^\dagger$ in the Proof of Theorem V.18.
- Steps 5, 6, and 7 are exactly the definitions of B, U, and $\mathcal{H}_B$, respectively, in the Proof of Theorem V.18.
- For step 8, we note that the condition $\mathrm{tr}_{A_1}[\hat{\mathfrak{L}}^A] = 0$ implies $L^A(\mathbb{1}) = 0$ so that we can follow the last few sentences in the Proof of Theorem V.18.

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Appendix B

Further articles as principal author under review

B.1 On the generators of quantum dynamical semigroups with invariant subalgebras

In this article, we consider quantum dynamical semigroups with an invariant algebra. More precisely, we look at uniformly continuous one-parameter semigroups $\{T_t\}_{t\geq 0} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ that have the property that for every $t \geq 0$, the map T_t is a normal CP-map that leaves an (atomic) weakly closed *-algebra $\mathcal{A}$ invariant. The generators of quantum dynamical semigroups are known to be representable in GKSL-form. That is, every generator $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be written as $L(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$, for some $V \in \mathcal{B}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $K \in \mathcal{B}(\mathcal{H})$. Moreover, the invariance assertion $T_t(\mathcal{A}) \subseteq \mathcal{A}$ for all $t \geq 0$ is equivalent to $L(\mathcal{A}) \subseteq \mathcal{A}$. So, the real question is, what can we say about V and K if we know that $L(\mathcal{A}) \subseteq \mathcal{A}$?
This research question has its origin in the days I was finalizing Article [2] and is the answer to the question “How can the techniques developed in Article [2] (where they were specifically targeted at semicausal and superchannels) be generalized". To see that the current question is a generalization, note that a linear map $T : \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{B}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ is Heisenberg $\mathtt{B} \not\to \mathtt{A}$ semicausal if and only if $T(\mathcal{B}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}) \subseteq \mathcal{B}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$. We further remark that $\mathcal{B}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$ has the canonical form of a type-I factor von Neumann algebra.
Our result consists of two parts. First, we show that for approximately finite-dimensional algebras $\mathcal{A}$, the question: “How do GKSL-generators L with $L(\mathcal{A}) \subseteq \mathcal{A}$ look like?" can be reduced to “How do CP-maps T with $T(\mathcal{A}) \subseteq \mathcal{A}$ look like?". This is the content of Theorem 4 in [3]. In the second part, we show how to characterize CP-maps that leave an *atomic* weakly closed *-algebra $\mathcal{A}$ invariant (Theorem 5). This result is a generalization of the equivalence of semicausal and semilocalizable CP-maps [16].
Combining the two partial results leads to a normal form for GKSL-generators with invariant atomic weakly closed *-algebra (Theorem 6). As a further result, we characterize the gauge-freedom in our normal form (Theorem 7).
Our result can be seen as a unifying approach to derive several important results in the literature. Among those are the normal form for quantum dynamical semigroups with an invariant

decoherence-free subalgebra [60, Theorem 3.2(1)], quantum dynamical semigroups with an invariant maximally abelian subalgebra [66, 67] and the Koashi-Imoto theorem [59, Eq. (85) and Theorem 3].

Let me reiterate that the idea for the article originates from the request to generalize the techniques of Article [2]. I am the principal author of this article. My main contributions include (but are not limited to) the ideas and the technical write-ups of Theorems 4, 5, 6 and 7.

On the generators of quantum dynamical semigroups with invariant subalgebras

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Abstract. The problem of characterizing GKLS-generators and CP-maps with an invariant von Neumann subalgebra $\mathcal{A}$ appeared in different guises in the literature. We prove two unifying results: First, we show how to construct a normal form for $\mathcal{A}$-invariant GKLS-generators, if a normal form for $\mathcal{A}$-invariant CP-maps is known — rendering the two problems essentially equivalent. Second, we provide a normal form for $\mathcal{A}$-invariant CP-maps if $\mathcal{A}$ is atomic (which includes the finite-dimensional case). As an application we reproduce several results from the literature as direct consequences of our characterizations and thereby point out connections between different fields.

1. Introduction

Quantum dynamical semigroups play in important role in many areas of physics. A (norm-continuous) quantum dynamical semigroup is a collection of normal completely positive maps $(T_t)_{t\geq 0}$ on $\mathcal{L}(\mathcal{H})$ such that $T_0 = \mathrm{id}$, $T_{s+t} = T_s \circ T_t$ for all $s, t \geq 0$ and the map $t \mapsto T_t$ is norm-continuous. By the general theory of continuous one-parameter semigroups [9, Theorem 3.7], there exists a bounded operator L, called generator, such that $T_t = e^{tL}$ for all $t \geq 0$. The fundamental result due to Gorini, Kossakowski, Sudarshan [12] and Lindblad [17] is that L generates a norm-continuous quantum dynamical semigroup if and only if L is of the form

$$L(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V - K^\dagger X - XK, \quad X \in \mathcal{L}(\mathcal{H}), \tag{1}$$

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for some $V \in \mathcal{L}(\mathcal{H};\mathcal{H}\otimes\mathcal{H}_{\mathtt{E}})$ and $K \in \mathcal{L}(\mathcal{H})$.

In the past, special cases of the following question on restricted GKLS-generators arose in the literature: Suppose $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is a von Neumann algebra such that $T_t(\mathcal{A}) \subseteq \mathcal{A}$ for all $t \geq 0$ or, equivalently, such that $L(\mathcal{A}) \subseteq \mathcal{A}$. How does this condition constrain the operators V and K? In this work, we provide an answer to this question, if $\mathcal{A}$ is atomic — thus covering many interesting cases, in particular the finite-dimensional case. It should be noted here that $T_t(\mathcal{A}) \subseteq \mathcal{A}$ for all $t \geq 0$ is equivalent to $L(\mathcal{A}) \subseteq \mathcal{A}$.

Among the results for which the answer to the question posed above is useful are: The Koashi-Imoto theorem [16], an important result in the theory of quantum communication, giving the general form of a quantum channel leaving a certain set of density matrices invariant; the form of the GKLS-generator imposed by the invariance of the decoherence-free subalgebra [1,7,10,24]; general questions about decoherence, where the form of the GKLS-generator imposed by the invariance of a maximally abelian subalgebra is important [5,11,21–23]; the study of Markovian subsystems [26] and the study of the aging process of quantum devices via dynamical semigroups of superchannels [13].

This paper is structured as follows: In Section 2., we introduce the notation and remind the reader of several facts related to completely positive (CP) maps, GKLS-generators, and weakly closed *-algebras. In Section 3.1., we show how to reduce the general problem of classifying GKLS-generators with an invariant approximately finite-dimensional algebra to the one of classifying normal CP-maps with the same invariant algebra. In Section 3.2. we classify normal completely positive maps with an invariant *atomic* algebra. Section 3.3. combines the results from Sections 3.1. and 3.2. to obtain a classification of GKLS-generators with invariant atomic algebras. In Section 4., we use our results to reproduce several results from the literature discussed above. Finally, in Section 5., we conclude our work and outline possible further lines of research.

2. Preliminaries and Notation

Functional analysis: Throughout, $\mathcal{H}$ (with some subscript) denotes a *separable* complex Hilbert space. For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{L}(\mathcal{X};\mathcal{Y})$ the set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, which becomes a Banach space when equipped with the operator norm. We abbreviate $\mathcal{L}(\mathcal{X};\mathcal{X})$ by $\mathcal{L}(\mathcal{X})$. The *strong operator topology (SOT)* on $\mathcal{L}(\mathcal{H}_{\mathtt{A}};\mathcal{H}_{\mathtt{B}})$ is the smallest topology such that for all $|\psi_{\mathtt{A}}\rangle \in \mathcal{H}_{\mathtt{A}}$, the map $X \mapsto X|\psi_{\mathtt{A}}\rangle$ is continuous. The *weak operator topology (WOT)* on $\mathcal{L}(\mathcal{H}_{\mathtt{A}};\mathcal{H}_{\mathtt{B}})$ is the smallest topology such for all $|\psi_{\mathtt{A}}\rangle \in \mathcal{H}_{\mathtt{A}}$ and $|\psi_{\mathtt{B}}\rangle \in \mathcal{H}_{\mathtt{B}}$, the map $X \mapsto \langle\psi_{\mathtt{B}}|X\psi_{\mathtt{A}}\rangle$ is continuous. The *ultraweak (or weak-*) topology* on $\mathcal{L}(\mathcal{H}_A;\mathcal{H}_B)$ is the smallest topology such that for all $\rho \in \mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_A)$, the map $X \mapsto \mathrm{tr}\,[X\rho]$ is continuous. Here, tr denotes the trace and $\mathcal{S}_1(\mathcal{H}_B;\mathcal{H}_A)$ the set of *trace-class* *operators*, that is those $\rho \in \mathcal{L}(\mathcal{H}_B; \mathcal{H}_A)$ for which $\mathrm{tr}\left[\sqrt{\rho^\dagger \rho}\right] < \infty$. A subset $T \subseteq \mathcal{H}$ is *total* in $\mathcal{H}$ if its linear span is dense in $\mathcal{H}$. An operator $V \in \mathcal{L}(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{B}})$ is called an *isometry* if $\|V|\psi\rangle\| = \||\psi\rangle\|$ for all $|\psi\rangle \in \mathcal{H}_{\mathtt{A}}$. A surjective isometry is called *unitary*.

CP-maps and GKLS-generators: A linear map $\Phi : \mathcal{L}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{L}(\mathcal{H}_{\mathtt{B}})$ is a *normal CP-map* if there exists a Hilbert space $\mathcal{H}_{\mathtt{E}}$ and an operator $V \in \mathcal{L}(\mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{E}})$ such that $\Phi(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V$. We denote the set of such normal CP-maps by $\mathrm{CP}_\sigma(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{B}})$ and abbreviate $\mathrm{CP}_\sigma(\mathcal{H}; \mathcal{H})$ by $\mathrm{CP}_\sigma(\mathcal{H})$. The pair $(V, \mathcal{H}_{\mathtt{E}})$ is called a *Stinespring representation* of Φ. An equivalent characterization of normal CP-map is that they admit a *Kraus representation*. That is, there exist operators $\{v_i\}_i \subset \mathcal{L}(\mathcal{H})$ such that $\Phi(X) = \sum_i v_i^\dagger X v_i$ for all $X \in \mathcal{L}(\mathcal{H})$, where the series is SOT-convergent. The choice of $(V, \mathcal{H}_{\mathtt{E}})$ representing Φ is not unique. However, the following well-known theorem (see e.g. [19, Theorem 29.6]) quantifies the freedom.

THEOREM 1. *Let $\tilde{V} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\tilde{\mathtt{E}}})$ and define $\Phi(X) = \tilde{V}^\dagger (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \tilde{V}$ for all $X \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$. Then there exist $\mathcal{H}_{\mathtt{E}}$ and $V \in \mathcal{L}(\mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{E}})$ such that a) $\Phi(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V$ for all $X \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$ and such that b) $\{(X \otimes \mathbb{1}_{\mathtt{E}}) V |\psi\rangle \mid X \in \mathcal{L}(\mathcal{H}_{\mathtt{A}}), |\psi\rangle \in \mathcal{H}_{\mathtt{B}}\}$ is total in $\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{E}}$.*
If $(\mathcal{H}_{\mathtt{E}}, V)$ is any pair such that a) and b) are satisfied, and if $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V})$ is another pair such that a) is satisfied, then there exists an isometry $W \in \mathcal{L}(\mathcal{H}_{\mathtt{E}}; \mathcal{H}_{\tilde{\mathtt{E}}})$ such that $\tilde{V} = (\mathbb{1}_{\mathtt{A}} \otimes W) V$. If b) is also satisfied for $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V})$, then W is unitary.

If V satisfies condition b) above, then it is called *minimal*.

A linear map $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is called *GKLS-generator* (or generator in GKLS-form) if there exists $\Phi \in \mathrm{CP}_\sigma(\mathcal{H})$ and $K \in \mathcal{L}(\mathcal{H})$, such that $L(X) = \Phi(X) - K^\dagger X - XK$ for all $X \in \mathcal{L}(\mathcal{X})$. As for normal CP-maps, the representation is not unique. The following characterization of the freedom can be extracted from [19, Chapter 30], in particular from the proof of Proposition 30.14. We give a complete proof in Appendix B.

THEOREM 2. *Let $\tilde{V} \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\tilde{\mathtt{E}}})$ and $\tilde{K} \in \mathcal{L}(\mathcal{H})$ and define $L(X) = \tilde{V}^\dagger (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \tilde{V} - \tilde{K}^\dagger X - X\tilde{K}$ for all $X \in \mathcal{L}(\mathcal{H})$. Then there exist $\mathcal{H}_{\mathtt{E}}$, $V \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $K \in \mathcal{L}(\mathcal{H})$ such that a) $L(X) = V^\dagger (X \otimes \mathbb{1}_{\mathtt{E}}) V - K^\dagger X - XK$ for all $X \in \mathcal{L}(\mathcal{H})$ and such that b) $\{((X \otimes \mathbb{1}_{\mathtt{E}}) V - VX) |\psi\rangle \mid X \in \mathcal{L}(\mathcal{H}), |\psi\rangle \in \mathcal{H}\}$ is total in $\mathcal{H} \otimes \mathcal{H}_{\mathtt{E}}$.*
If $(\mathcal{H}_{\mathtt{E}}, V, K)$ is any triplet such that a) and b) are satisfied, and if $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V}, \tilde{K})$ is another triplet such that a) is satisfied, then there exists an isometry $W \in \mathcal{L}(\mathcal{H}_{\mathtt{E}}; \mathcal{H}_{\tilde{\mathtt{E}}})$, a vector $|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{\mathtt{E}}}$, and a number $\mu \in \mathbb{R}$ such that

$$\tilde{V} = (\mathbb{1} \otimes W) V + \mathbb{1} \otimes |\tilde{\psi}\rangle, \quad \tilde{K} = K + (\mathbb{1} \otimes \langle \tilde{\psi} | W) V + \frac{1}{2} \|\tilde{\psi}\|^2 + i\mu. \tag{2}$$

If b) is also satisfied for $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V}, \tilde{K})$, then W is unitary.

Weakly closed *-algebras: We introduce several conventions that will be useful in simplifying the notation throughout. A *weakly closed *-algebra* $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed w.r.t. the WOT[1] and w.r.t. taking adjoints. A weakly closed *-algebra does not necessarily contain the identity — if it does then it is called a *von Neumann-algebra* (abbr. *vN-algebra*). Every weakly closed *-algebra $\mathcal{A}$ is (unitarily equivalent to) the direct sum of a zero-dimensional algebra 0_0 and a vN-Algebra [14, Proposition 5.1.8]. That is, $\mathcal{A} = U_{\mathcal{A}} \left(0_0 \oplus \mathcal{A}_{\bar{0}} \right) U_{\mathcal{A}}^\dagger$, where $U_{\mathcal{A}} : \mathcal{H}_\oplus \to \mathcal{H}$ is a unitary on $\mathcal{H}_\oplus = \mathcal{H}_0 \oplus \mathcal{H}_{\bar{0}}$, $0_0 = \{0\} \subseteq \mathcal{L}(\mathcal{H}_0)$ and $\mathcal{A}_{\bar{0}}$ is a vN-algebra in $\mathcal{L}(\mathcal{H}_{\bar{0}})$. If $P_0^\oplus \in \mathcal{L}(\mathcal{H}_\oplus; \mathcal{H}_0)$ and $P_{\bar{0}}^\oplus \in \mathcal{L}(\mathcal{H}_\oplus; \mathcal{H}_{\bar{0}})$ are the orthogonal projections onto $\mathcal{H}_0$ and $\mathcal{H}_{\bar{0}}$, then we define $P_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0)$ and $P_{\bar{0}} \in \mathcal{L}(\mathcal{H}; \mathcal{H}_{\bar{0}})$ by $P_0 = P_0^\oplus U_{\mathcal{A}}^\dagger$ and $P_{\bar{0}} = P_{\bar{0}}^\oplus U_{\mathcal{A}}^\dagger$.

Two special types of weakly closed *-algebras are of particular importance to us: the approximately finite-dimensional ones and the atomic ones. A weakly closed *-algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is called *approximately finite-dimensional (AFD)* if there exists an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \cdots \subseteq \mathcal{A}$ of finite-dimensional (hence weakly closed) sub-*-algebras of $\mathcal{A}$ such that $\cup_{n \in \mathbb{N}} \mathcal{A}_n$ is WOT-dense in $\mathcal{A}$. *Atomic* weakly closed *-algebras are usually defined by the requirement that every non-zero projection in $\mathcal{A}$ majorizes a non-zero minimal projection [25, Definition 5.9] — a property always fulfilled in finite dimensions. For our purposes, it is more convenient to think of them as those weakly closed *-algebras that are the direct sum of type-I factors. A proof of this equivalence can be found in the appendix of [7].

DEFINITION 3. A weakly closed *-algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is called *atomic* if

$$\mathcal{A} = U_{\mathcal{A}} \left(0_0 \oplus \bigoplus_{i \in I} (\mathcal{L}(\mathcal{H}_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{B}_i}) \right) U_{\mathcal{A}}^\dagger, \tag{3}$$

for a Hilbert space $\mathcal{H}_0$, sequences of Hilbert spaces $\{\mathcal{H}_{A_i}\}_{i \in I}$ and $\{\mathcal{H}_{B_i}\}_{i \in I}$ indexed by a countable index set I, and a unitary $U_{\mathcal{A}} : \mathcal{H}_\oplus \to \mathcal{H}$, where $\mathcal{H}_\oplus = \mathcal{H}_0 \oplus \bigoplus_{i \in I} (\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i})$.

We further define for all $i \in I$ the Hilbert space $\mathcal{H}_i = \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{B}_i}$. For all $k \in I \cup \{0\}$, let $P_k^\oplus \in \mathcal{L}(\mathcal{H}_\oplus; \mathcal{H}_k)$ be the orthogonal projection onto $\mathcal{H}_k$ and let us define $P_k \in \mathcal{L}(\mathcal{H}; \mathcal{H}_k)$ as $P_k = P_k^\oplus U_{\mathcal{A}}^\dagger$.[2] Hence, an arbitrary element $X_{\mathcal{A}} \in \mathcal{A}$ can be written as SOT-convergent series $X_{\mathcal{A}} = \sum_{i \in I} P_i^\dagger (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) P_i$, for some operators $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, with $\sup_{i \in I} \|X_{\mathtt{A}_i}\| < \infty$.

For $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{A}' := \{X \in \mathcal{L}(\mathcal{H}) \,|\, X X_{\mathcal{A}} = X_{\mathcal{A}} X, \forall X_{\mathcal{A}} \in \mathcal{A}\}$ its *commutant*. If $\mathcal{A}$ is an atomic weakly closed *-algebra with decomposition given

[1] Equivalently, one can use the SOT or the ultraweak topology.

[2] Note that this definition is consistent with the one introduced in the first paragraph above.

above, then as a special case of general theory of direct integral decompositions of vN-algebras (see e.g. [15, Proposition 14.1.24, Theorem 11.2.16])), $\mathcal{A}'$ is given by

$$\mathcal{A}' = U_{\mathcal{A}} \left(\mathcal{L}(\mathcal{H}_0) \oplus \bigoplus_{i\in I} (\mathbb{1}_{\mathtt{A}_i} \otimes \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})) \right) U_{\mathcal{A}}^\dagger. \tag{4}$$

Hence, an arbitrary element $X'_{\mathcal{A}} \in \mathcal{A}'$ can be written as SOT-convergent series $X_{\mathcal{A}'} = P_0^\dagger X_0 P_0 + \sum_{i\in I} P_i^\dagger (\mathbb{1}_{\mathtt{A}_i} \otimes X_{\mathtt{B}_i}) P_i$, for some operators $X_0 \in \mathcal{L}(\mathcal{H}_0)$ and $X_{\mathtt{B}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$, with $\sup_{i\in I} \|X_{\mathtt{B}_i}\| < \infty$. And $X_{\mathcal{A}'}$ is self-adjoint if and only if all the operators in the decomposition are self-adjoint [15, Proposition 14.1.8].

3. Results

3.1. GKLS-generators with invariant *-algebra

In this section we state and prove our first main result, namely a theorem that allows us to reduce the problem of characterizing GKLS-generators with invariant weakly closed *-algebras to characterizing CP-maps with invariant weakly closed *-algebras. Since CP-maps are special GKLS-generators (for $K = 0$), this renders these problems essentially equivalent. For technical reasons, we need to restrict ourselves to AFD algebras. The notation in the following theorem and the subsequent proof follows Section 2..

THEOREM 4. *Let $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be defined by $L(X) = \Phi(X) - K^\dagger X - XK$, for some $\Phi \in \mathrm{CP}_\sigma(\mathcal{H})$ and $K \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ be an AFD weakly closed *-algebra. The following are equivalent*

1. *$L(\mathcal{A}) \subseteq \mathcal{A}$.*
2. *(Stinespring) Suppose Φ is given in Stinespring representation $\Phi(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V$, where $V \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$. Then there exist operators $V_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$, $A, B \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $K_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0)$; an operator $K_{\mathcal{A}} \in \mathcal{A}$; and a self-adjoint operator $H_{\mathcal{A}'} \in \mathcal{A}'$ such that*
   (a) *$A^\dagger(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})A \in \mathcal{A}$ and $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})B = BX_{\mathcal{A}}$, for all $X_{\mathcal{A}} \in \mathcal{A}$.*
   (b) *V and K have the following form:*

$$V = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_0 + A + B, \tag{5a}$$

$$K = B^\dagger A + \frac{1}{2}B^\dagger B + K_{\mathcal{A}} + iH_{\mathcal{A}'} + P_0^\dagger K_0. \tag{5b}$$

3. *(Kraus) Suppose Φ is given in Kraus representation $\Phi(X) = \sum_{n\in N} \phi_n^\dagger X \phi_n$. Then there exists a countable index set N; collections of operators $\{v_n\}_{n\in N} \subset$*

$\mathcal{L}(\mathcal{H};\mathcal{H}_0)$ and $\{a_n\}_{n\in N}, \{b_n\}_{n\in N} \subset \mathcal{L}(\mathcal{H})$ such that $\sum_{n\in N} v_n^\dagger v_n$, $\sum_{n\in N} a_n^\dagger a_n$ and $\sum_{n\in N} b_j^\dagger b_j$ SOT-converge; an operator $K_0 \in \mathcal{L}(\mathcal{H};\mathcal{H}_0)$; an operator $K_\mathcal{A} \in \mathcal{A}$; and a self-adjoint operator $H_{\mathcal{A}'} \in \mathcal{A}'$ such that

(a) $\sum_{n\in N} a_n^\dagger X_\mathcal{A} a_n \in \mathcal{A}$ for all $X_\mathcal{A} \in \mathcal{A}$ and $b_n \in \mathcal{A}'$ for all $n \in N$.

(b) $\{\phi_n\}_{n\in N}$ and K have the following form:

$$\phi_n = P_0^\dagger v_n + a_n + b_n, \text{ for all } n \in N, \tag{6a}$$

$$K = \sum_{n\in N} b_n^\dagger a_n + \frac{1}{2}\sum_{n\in N} b_n^\dagger b_n + K_\mathcal{A} + iH_{\mathcal{A}'} + P_0^\dagger K_0. \tag{6b}$$

Proof. We prove 1 $\iff$ 2 and obtain 3 as a corollary. For the implication 2 $\implies$ 1, let $X_\mathcal{A} \in \mathcal{A}$ be arbitrary. We have

$$\begin{aligned}
\Phi(X_\mathcal{A}) = &\quad (V_0^\dagger(P_0 \otimes \mathbb{1}_\mathtt{E}) + A^\dagger + B^\dagger)(X_\mathcal{A} \otimes \mathbb{1}_\mathtt{E})((P_0^\dagger \otimes \mathbb{1}_\mathtt{E})V_0 + A + B) \\
\overset{P_0 X_\mathcal{A} = 0 = X_\mathcal{A} P_0^\dagger}{=} &\quad (A^\dagger + B^\dagger)(X_\mathcal{A} \otimes \mathbb{1}_\mathtt{E})(A + B) \\
\overset{(X_\mathcal{A}\otimes \mathbb{1}_\mathtt{E})B = BX_\mathcal{A}}{=} &\quad A^\dagger(X_\mathcal{A} \otimes \mathbb{1}_\mathtt{E})A + (B^\dagger A + \frac{1}{2}B^\dagger B)^\dagger X_\mathcal{A} + X_\mathcal{A}(B^\dagger A + \frac{1}{2}B^\dagger B)
\end{aligned}$$

and

$$\begin{aligned}
K^\dagger X_\mathcal{A} + X_\mathcal{A} K = &(B^\dagger A + \frac{1}{2}B^\dagger B)^\dagger X_\mathcal{A} + X_\mathcal{A}(B^\dagger A + \frac{1}{2}B^\dagger B) + K_\mathcal{A}^\dagger X_\mathcal{A} + X_\mathcal{A} K_\mathcal{A} \\
&- \underbrace{iH_{\mathcal{A}'}X_\mathcal{A} + iX_\mathcal{A} H_{\mathcal{A}'}}_{=0, \text{ since } H_{\mathcal{A}'} \in \mathcal{A}'} + \underbrace{K_0^\dagger P_0 X_\mathcal{A} + X_\mathcal{A} P_0^\dagger K_0}_{=0, \text{ since } P_0 X_\mathcal{A} = 0 = X_\mathcal{A} P_0^\dagger}.
\end{aligned}$$

Combining the calculations above yields

$$L(X_\mathcal{A}) = A^\dagger(X_\mathcal{A} \otimes \mathbb{1}_\mathtt{E})A - K_\mathcal{A}^\dagger X_\mathcal{A} - X_\mathcal{A} K_\mathcal{A},$$

which belongs to $\mathcal{A}$ since by assumption $A^\dagger(X_\mathcal{A} \otimes \mathbb{1}_E)A \in \mathcal{A}$, $K_\mathcal{A} \in \mathcal{A}$ and $K_\mathcal{A}^\dagger \in \mathcal{A}$. The proof of the converse proceeds in two main steps: First we show that there are operators V_0, A and B such that the conditions in 2a and Eq. (5a) hold. Second, we derive the form of K. As a first step, we construct a family of linear maps on $\mathcal{L}(\mathcal{H})$ each of which is closely related to L and leaves $\mathcal{A}$ invariant. Since $L(\mathcal{A}) \subseteq \mathcal{A}$, and since $\mathcal{A}$ is a *-algebra,

$$\Psi(X, Y, Z) := L(Y^\dagger XZ) - Y^\dagger L(XZ) - L(Y^\dagger X)Z + Y^\dagger L(X)Z \tag{7}$$

is an element of $\mathcal{A}$ whenever $X, Y, Z \in \mathcal{A}$. A direct calculation using the representation $\Phi(X) = V^\dagger(X \otimes \mathbb{1}_\mathtt{E})V$ reveals that

$$\Psi(X, Y, Z) = [VY - (Y \otimes \mathbb{1}_E)V]^\dagger (X \otimes \mathbb{1}_E) [(VZ - (Z \otimes \mathbb{1}_E)V]. \tag{8}$$

With the notation introduced in Section 2.: Since $\mathcal{A}$ is AFD, so is $U_{\mathcal{A}}^{\dagger}\mathcal{A}U_{\mathcal{A}} = 0_0 \oplus \mathcal{A}_{\bar{0}}$ and so is the vN-algebra $\mathcal{A}_{\bar{0}} \subseteq \mathcal{L}(\mathcal{H}_{\bar{0}})$. Let $\tilde{\mathcal{A}}_1 \subseteq \tilde{\mathcal{A}}_2 \subseteq \tilde{\mathcal{A}}_3 \subseteq \cdots$ be an increasing sequence of finite-dimensional *-subalgebras of $\mathcal{A}_{\bar{0}}$, such that $\cup_{n\in\mathbb{N}}\tilde{\mathcal{A}}_n$ is WOT-dense in $\mathcal{A}_{\bar{0}}$. For every $n \in \mathbb{N}$, define $\mathcal{A}_n := \operatorname{span}\{\tilde{\mathcal{A}}_n \cup \mathbb{C}\mathbb{1}_{\bar{0}}\}$. Clearly, also $\cup_{n\in\mathbb{N}}\mathcal{A}_n$ is WOT-dense in $\mathcal{A}_{\bar{0}}$, but now $\mathcal{A}_n$ is a vN-algebra for every $n \in \mathbb{N}$. In the following, we will often need to assign to operators in $\mathcal{A}_{\bar{0}}$ the corresponding ones in $\mathcal{A}$. For notational convenience, we define for each $X \in \mathcal{A}_{\bar{0}}$ the operator $\hat{X} = P_{\bar{0}}^{\dagger} X P_{\bar{0}} \in \mathcal{L}(\mathcal{H})$. We denote by $\mathcal{U}(\mathcal{A}_n)$ the unitary group in $\mathcal{A}_n$. As $\mathcal{A}_n$ is finite-dimensional, $\mathcal{U}(\mathcal{A}_n)$ is a compact group, so there exists a unique Haar probability measure on $\mathcal{U}(\mathcal{A}_n)$. For any $n, m \in \mathbb{N}$ and $X \in \mathcal{L}(\mathcal{H})$, we obtain the following Haar average

$$\begin{aligned}\int_{\mathcal{U}(\mathcal{A}_m)}\int_{\mathcal{U}(\mathcal{A}_n)} &\Psi(\hat{U}_n X \hat{W}_m^{\dagger}, \hat{U}_n^{\dagger}, \hat{W}_m)\,\mathrm{d}U_n \mathrm{d}W_m \\ &= \left((\hat{\mathbb{1}}_{\bar{0}} \otimes \mathbb{1}_{\mathtt{E}})V - \mathbb{E}_n(V)\right)^{\dagger} (X \otimes \mathbb{1}_{\mathtt{E}}) \left((\hat{\mathbb{1}}_{\bar{0}} \otimes \mathbb{1}_{\mathtt{E}})V - \mathbb{E}_m(V)\right),\end{aligned} \tag{9}$$

where

$$\mathbb{E}_k(V) := \int_{\mathcal{U}(\mathcal{A}_k)} (\hat{U}_k^{\dagger} \otimes \mathbb{1}_{\mathtt{E}}) V \hat{U}_k \,\mathrm{d}U_k, \quad k \in \mathbb{N}.$$

Since we integrate over a probability measure, $\|\mathbb{E}_k(V)\| \leq \|V\|$ and hence the (sequential) Banach–Alaoglu theorem implies that the sequence $(\mathbb{E}_k(V))_{k\in\mathbb{N}}$ has an ultraweakly convergent subsequence whose limit we denote by $\mathbb{E}(V)$. For $X_{\mathcal{A}} \in \mathcal{A}$, the RHS of Eq. (9) is an element of $\mathcal{A}$ for all $n, m \in \mathbb{N}$, since the integrand is in $\mathcal{A}$ and the Bochner integral converges in norm. Furthermore, since $\mathcal{A}$ is ultraweakly closed, passing to subsequences and taking the limit $n \to \infty$ and then $m \to \infty$ yields that

$$\Psi(X_{\mathcal{A}}) := \left((\hat{\mathbb{1}}_{\bar{0}} \otimes \mathbb{1}_{\mathtt{E}})V - \mathbb{E}(V)\right)^{\dagger} (X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}}) \left((\hat{\mathbb{1}}_{\bar{0}} \otimes \mathbb{1}_{\mathtt{E}})V - \mathbb{E}(V)\right) \tag{10}$$

is an element of $\mathcal{A}$ for all $X_{\mathcal{A}} \in \mathcal{A}$. In other words, Ψ interpreted as a CP-map satisfies $\Psi(\mathcal{A}) \subseteq \mathcal{A}$. We now define V_0, B and A as follows: $V_0 = (P_0 \otimes \mathbb{1}_{\mathtt{E}})V$, $B = \mathbb{E}(V)$ and $A = V - (P_0^{\dagger} \otimes \mathbb{1}_{\mathtt{E}})V_0 - B$. Thus $V = A + B + (P_0^{\dagger} \otimes \mathbb{1}_{\mathtt{E}})V$, which is precisely Eq. (5a). It follows directly from Eq. (10) that $A^{\dagger}(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})A \in \mathcal{A}$ for all $X_{\mathcal{A}} \in \mathcal{A}$ — verifying the first part of condition 2a. By the definition of the Haar measure and since $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is an increasing sequence, we have $(\hat{U}_k \otimes \mathbb{1}_{\mathtt{E}})\mathbb{E}(V) = \mathbb{E}(V)\hat{U}_k$ for all $U_k \in \mathcal{U}(\mathcal{A}_k)$. But since every $X_k \in \mathcal{A}_k$ can be written as a finite linear combination of elements in $\mathcal{U}(\mathcal{A}_k)$ (see [14, Theorem 4.1.7]), we have $(\hat{X}_k \otimes \mathbb{1}_E)\mathbb{E}(V) = \mathbb{E}(V)\hat{X}_k$, for all $X_k \in \mathcal{A}_k$ and hence $(\hat{X} \otimes \mathbb{1}_{\mathtt{E}})\mathbb{E}(V) = \mathbb{E}(V)\hat{X}$ for all $X \in \cup_{n\in\mathbb{N}}\mathcal{A}_n$. Evidently, this equation is also preserved under ultraweak limits. Thus $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})\mathbb{E}(V) = \mathbb{E}(V)X_{\mathcal{A}}$, for all $X_{\mathcal{A}} \in \mathcal{A}$. Since $B = \mathbb{E}(V)$, this implies the second part of condition 2a.

It remains to show that K has the desired form. To this end, note that for any $X_{\mathcal{A}} \in \mathcal{A}$, we have $L(X_{\mathcal{A}}) \in \mathcal{A}$ by assumption, but since $V = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_0 + A + B$, $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})B = BX_{\mathcal{A}}$ and $P_0 X_{\mathcal{A}} = 0$, we also have

$$\begin{aligned} L(X_{\mathcal{A}}) &= [A+B]^\dagger (X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}}) [A+B] - K^\dagger X_{\mathcal{A}} - X_{\mathcal{A}} K \\ &= A^\dagger (X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}}) A - (K - B^\dagger A - \frac{1}{2} B^\dagger B)^\dagger X_{\mathcal{A}} - X_{\mathcal{A}} (K - B^\dagger A - \frac{1}{2} B^\dagger B). \end{aligned}$$

Since $A^\dagger (X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}}) A \in \mathcal{A}$, this implies that

$$-\kappa^\dagger X_{\mathcal{A}} - X_{\mathcal{A}} \kappa \in \mathcal{A}, \tag{11}$$

for all $X_{\mathcal{A}} \in \mathcal{A}$, where $\kappa = K - B^\dagger A - \frac{1}{2} B^\dagger B$. For $U_n \in \mathcal{U}(\mathcal{A}_n)$, we choose $X_{\mathcal{A}} = \hat{U}_n$, multiply Eq. (11) from the left by $\hat{U}_n^\dagger$ and integrate over the Haar measure. Thus, we see that

$$-\int_{\mathcal{U}(\mathcal{A}_n)} \hat{U}_n^\dagger \kappa \hat{U}_n \, \mathrm{d}U_n - \hat{\mathbb{1}}_{\overline{0}} \kappa \tag{12}$$

belongs to $\mathcal{A}$. By the same arguments as above, we can pass to a subsequence such that for $n \to \infty$, expression Eq. (12) converges to

$$-\kappa_{\mathcal{A}'} - \hat{\mathbb{1}}_{\overline{0}} \kappa, \tag{13}$$

for some $\kappa_{\mathcal{A}'} \in \mathcal{A}'$ and such that the whole expression belongs to $\mathcal{A}$. We now define the self-adjoint operator $H_{\mathcal{A}'} = -\frac{1}{2i}(\kappa_{\mathcal{A}'} - \kappa_{\mathcal{A}'}^\dagger) \in \mathcal{A}'$, the operator $K_{\mathcal{A}} := \hat{\mathbb{1}}_{\overline{0}} \kappa - i H_{\mathcal{A}'} = \kappa - P_0^\dagger P_0 \kappa - i H_{\mathcal{A}'}$ and $K_0 = P_0 \kappa$. By the definition of K_0 and $K_{\mathcal{A}}$ we thus get $\kappa = K_{\mathcal{A}} + i H_{\mathcal{A}'} + P_0^\dagger K_0$, which is the desired form if $K_{\mathcal{A}} \in \mathcal{A}$. This last assertion can be seen as follows:

$$\begin{aligned} K_{\mathcal{A}} &= \frac{1}{2}\left(K_{\mathcal{A}} + K_{\mathcal{A}}^\dagger\right) + \frac{1}{2}\left(K_{\mathcal{A}} - K_{\mathcal{A}}^\dagger\right) \\ &= \underbrace{\frac{1}{2}\left(\kappa^\dagger \hat{\mathbb{1}}_{\overline{0}} + \hat{\mathbb{1}}_{\overline{0}} \kappa\right)}_{\in \mathcal{A}, \text{ by } \hat{\mathbb{1}}_{\overline{0}} \in \mathcal{A} \text{ and Eq. (11)}} + \frac{1}{2}\Bigg(\underbrace{(\hat{\mathbb{1}}_{\overline{0}} \kappa + \kappa_{\mathcal{A}'})}_{\in \mathcal{A}, \text{ by Eq. (13)}} - \underbrace{(\hat{\mathbb{1}}_{\overline{0}} \kappa + \kappa_{\mathcal{A}'})^\dagger}_{\in \mathcal{A}, \text{ by Eq. (13)}} \Bigg). \end{aligned}$$

This finishes the proof of 1 $\iff$ 2.

Part 3 is a matter of going from the Stinespring representation of normal CP-maps to their Kraus representation and back. This is a standard procedure and a very nice account can be found in [3]. We just mention here that after choosing an orthonormal basis $\{|e_n\rangle\}_{n \in N}$ of $\mathcal{H}_{\mathtt{E}}$, the collections $\{v_n\}_{n \in N}$, $\{a_n\}_{n \in N}$ and $\{b_n\}_{n \in N}$ and the operators V_0, A and B are related via $v_n := (\mathbb{1} \otimes \langle e_n|)V_0$, $a_n := (\mathbb{1} \otimes \langle e_n|)A$ and $b_n := (\mathbb{1} \otimes \langle e_n|)B$. The corresponding properties are then routinely verifiable. □

3.2. CP-maps with invariant atomic algebra

In this section, we study the problem of finding a normal form for (normal) CP-maps with an *atomic* invariant subalgebra. Slightly more generally, we aim to find normal a normal form for normal CP-maps Φ with the property that $\Phi(\mathcal{A}) \subseteq \mathcal{C}$, for two atomic weakly closed *-algebras $\mathcal{A}$ and $\mathcal{C}$. Since we are now dealing with two algebras, we need to distinguish them in the notation in Definition 3. For the algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{A})$: the index set is called I; the Hilbert spaces $\{\mathcal{H}_i\}_{i\in I\cup\{0\}}$ are denoted by $\mathcal{H}_{i:\mathcal{A}}$, with $\mathcal{H}_{i:\mathcal{A}} = \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{B}_i}$ ($i \in I$); and the operators P_i are called $P_{i:\mathcal{A}} \in \mathcal{L}(\mathcal{H}_\mathcal{A}; \mathcal{H}_{i:\mathcal{A}})$. For the algebra $\mathcal{C} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{C})$: the index set is called J; the Hilbert spaces $\{\mathcal{H}_j\}_{j\in J\cup\{0\}}$ are denoted by $\mathcal{H}_{j:\mathcal{C}}$, with $\mathcal{H}_{j:\mathcal{C}} = \mathcal{H}_{\mathtt{C}_j} \otimes \mathcal{H}_{\mathtt{D}_j}$ ($j \in J$); and the operators P_j are called $P_{j:\mathcal{C}} \in \mathcal{L}(\mathcal{H}_\mathcal{C}; \mathcal{H}_{j:\mathcal{C}})$. With this notation in place, we can state our second main result:

THEOREM 5. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{A})$ and $\mathcal{C} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{C})$ be two atomic weakly closed *-algebras. For $\Phi \in \mathrm{CP}_\sigma(\mathcal{H}_\mathcal{A}; \mathcal{H}_\mathcal{C})$ defined by $\Phi(X) = V^\dagger(X \otimes \mathbb{1}_\mathtt{E})V$, with $V \in \mathcal{L}(\mathcal{H}_\mathcal{C}; \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathtt{E})$, the following are equivalent*

1. $\Phi(\mathcal{A}) \subseteq \mathcal{C}$.
2. *There exist an operator $V_0 \in \mathcal{L}(\mathcal{H}_\mathcal{C}; \mathcal{H}_{0:\mathcal{A}} \otimes \mathcal{H}_\mathtt{E})$; and for all $i \in I$ and $j \in J$ Hilbert spaces $\mathcal{H}_{\mathtt{F}_{ij}}$, operators $A_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{C}_j}; \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$, and isometries $U_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{D}_j}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_\mathtt{E})$, such that*
   - *V can be decomposed as*
   $$V = (P_{0:\mathcal{A}}^\dagger \otimes \mathbb{1}_\mathtt{E})V_0 + \sum_{i\in I,\, j\in J} (P_{i:\mathcal{A}}^\dagger \otimes \mathbb{1}_\mathtt{E})V_{ij}P_{j:\mathcal{C}}, \tag{14}$$
   with $V_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j})$, s.t. the series SOT-converges.
   - *The relation $U_{ik}^\dagger U_{il} = \delta_{kl}\mathbb{1}$ holds for all $i \in I$ and $k, l \in J$.*

The representation in 2 can be chosen such that $\{(X \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}|\psi\rangle \,|\, X \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi\rangle \in \mathcal{H}_{\mathtt{C}_j}\}$ is total in $\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}}$.

Remark. The theorem above tells us that if V is written as a block matrix (w.r.t. a basis determined by the structure of $\mathcal{A}$ and $\mathcal{C}$), then all the blocks are necessarily of the "semicloalizable" form $(\mathbb{1} \otimes U)(A \otimes \mathbb{1})$ — and that the U's need to satisfy an orthogonality relation.

Proof. Let us start by showing that 2 $\implies$ 1. We know that $\mathcal{X}_\mathcal{A} \in \mathcal{A}$ if and only if it can be decomposed as SOT-convergent series $\mathcal{X}_\mathcal{A} = \sum_{i\in I} P_{i:\mathcal{A}}^\dagger (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_{i:\mathcal{A}}$, for $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$ with $\sum_{i\in I} \|X_{\mathtt{A}_i}\| < \infty$. Since $\mathcal{A}$ and $\mathcal{C}$ are ultraweakly closed, Φ is ultraweakly continuous, and the operators in $\mathcal{A}_F := \{X_\mathcal{A} \,|\, X_{\mathtt{A}_i} \neq$

0 only for finitely many $i \in I$ } are ultraweakly dense in $\mathcal{A}$, it suffices to show the claim for $X_{\mathcal{A}} \in \mathcal{A}_F$ so that convergence issues (w.r.t. the I-summation) play no role in the following calculation (where the J-summation SOT-converges):

$$\begin{aligned}
\Phi(X_{\mathcal{A}}) &= \sum_{i\in I} \left[(P_{i:\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})V\right]^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) \left[(P_{i:\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})V\right] \\
&= \sum_{i\in I} \sum_{k,l\in J} P_{k:\mathcal{C}}^{\dagger} V_{ik}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{il} P_{l:\mathcal{C}} \\
&= \sum_{i\in I} \sum_{k,l\in J} P_{k:\mathcal{C}}^{\dagger} (A_{ik}^{\dagger} \otimes \mathbb{1}_{\mathtt{D}_k}) (X_{\mathtt{A}_i} \otimes U_{ik}^{\dagger} U_{il}) (A_{il} \otimes \mathbb{1}_{\mathtt{D}_l}) P_{l:\mathcal{C}} \\
&= \sum_{i\in I} \sum_{j\in J} P_{j:\mathcal{C}}^{\dagger} (A_{ij}^{\dagger} \otimes \mathbb{1}_{\mathtt{D}_j}) (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}} \otimes \mathbb{1}_{\mathtt{D}_j}) (A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j}) P_{j:\mathcal{C}} \\
&= \sum_{j\in J} P_{j:\mathcal{C}}^{\dagger} \left[\left(\sum_{i\in I} A_{ij}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}}) A_{ij} \right) \otimes \mathbb{1}_{\mathtt{D}_j} \right] P_{j:\mathcal{C}},
\end{aligned}$$

where we used the expansion of $X_{\mathcal{A}}$ in the first line, Eq. (14) in the second line (in particular the orthogonality of the projections), the explicit form $V_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j})$ in the third line, the orthogonality relation $U_{ik}^{\dagger} U_{il} = \delta_{kl} \mathbb{1}$ in the fourth line and algebraic manipulations in the fifth line. But the last line is just the decomposed form of an element of $\mathcal{C}$. Thus we have shown that $\Phi(\mathcal{A}) \subseteq \mathcal{C}$.

For the converse, suppose that Φ, defined by $\Phi(X) = V^{\dagger}(X \otimes \mathbb{1}_{\mathtt{E}})V$ satisfies $\Phi(\mathcal{A}) \subseteq \mathcal{C}$. Let $I_0 = I \cup \{0\}$ and $J_0 = J \cup \{0\}$. Then $\sum_{i_0 \in I_0} P_{i_0:\mathcal{A}}^{\dagger} P_{i_0:\mathcal{A}} = \mathbb{1}_{\mathcal{A}}$ and $\sum_{j_0 \in J_0} P_{j_0:\mathcal{C}}^{\dagger} P_{j_0:\mathcal{C}} = \mathbb{1}_{\mathcal{C}}$, where the series SOT-converge. Hence, we can expand

$$V = (P_{0:\mathcal{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{E}})V_0 + \sum_{i\in I} (P_{i:\mathcal{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{E}}) V_{i0} P_{0:\mathcal{C}} + \sum_{i\in I, j\in J} (P_{i:\mathcal{A}}^{\dagger} \otimes \mathbb{1}_{\mathtt{E}}) V_{ij} P_{j:\mathcal{C}}, \tag{15}$$

where we defined $V_0 = (P_{0:\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})V$ and for all $i \in I$ and $j_0 \in J_0$ the operator $V_{ij_0} = (P_{i:\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}}) V P_{j_0:\mathcal{C}}^{\dagger}$. Thus it remains to show that $V_{i0} = 0$ and that V_{ij} has our specific form.

By definition, every $X_{\mathcal{C}} \in \mathcal{C}$ is of the form $X_{\mathcal{C}} = \sum_{j\in J} P_{j:\mathcal{C}}^{\dagger} (X_{\mathtt{C}_j} \otimes \mathbb{1}_{\mathtt{D}_j}) P_{j:\mathcal{C}}$. Thus, in particular $\Phi(X_{\mathcal{A}})$ assumes that form for all $X_{\mathcal{A}} \in \mathcal{A}$. This has the following three implications: First, for every $i \in I$, $j \in J$ and every $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, there exists $X_{\mathtt{C}_j} \in \mathcal{L}(\mathcal{H}_{\mathtt{C}_j})$ such that

$$P_{j:\mathcal{C}} \Phi\left(P_{i:\mathcal{A}}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) P_{i:\mathcal{A}}\right) P_{j:\mathcal{C}}^{\dagger} = V_{ij}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{ij} = X_{\mathtt{C}_j} \otimes \mathbb{1}_{\mathtt{D}_j}. \tag{16}$$

Second, for every $i \in I$ and $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, we have

$$P_{0:\mathcal{C}}\, \Phi\left(P_{i:\mathcal{A}}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) P_{i:\mathcal{A}}\right)\, P_{0:\mathcal{C}}^{\dagger} = V_{i0}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{i0} = 0. \tag{17}$$

Third, for every $i \in I$ and $k,l \in J$ with $k \neq l$ and all $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, we have

$$P_{k:\mathcal{C}}\, \Phi \left(P_{i:\mathcal{A}}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) P_{i:\mathcal{A}} \right) P_{l:\mathcal{C}}^{\dagger} = V_{ik}^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{il} = 0. \tag{18}$$

It is a direct consequence of Eq. (17), by choosing $X_{\mathtt{A}_i} = \mathbb{1}_{\mathtt{A}_i}$, that $V_{i0} = 0$ for all $i \in I$. Hence the second term in Eq. (15) vanishes as desired.

We will now see that Eq. (16) implies that $V_{ij} = (\mathbb{1} \otimes U_{ij})(A_{ij} \otimes \mathbb{1})$. This is due to the equivalence between semicausal and semilocalizable CP-maps, established for finite-dimensional systems in [8]. We reproduce the argument here for the infinite-dimensional case. Choose some unit vector $|\psi\rangle \in \mathcal{H}_{\mathtt{D}_j}$ and define the normal CP-maps $\Phi_{ij} \in \mathrm{CP}_{\sigma}(\mathcal{H}_{i:\mathcal{A}}; \mathcal{H}_{j:\mathcal{C}})$ and $\Psi_{ij} \in \mathrm{CP}_{\sigma}(\mathcal{H}_{\mathtt{A}_i}; \mathcal{H}_{\mathtt{C}_j})$ by $\Phi_{ij}(X_i) = P_{j:\mathcal{C}} \Phi \left(P_{i:\mathcal{A}}^{\dagger} X_i P_{i:\mathcal{A}} \right) P_{j:\mathcal{C}}^{\dagger}$ and $\Psi_{ij}(X_{\mathtt{A}_i}) = (\mathbb{1}_{\mathtt{C}_j} \otimes \langle\psi|) \Phi_{ij}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})(\mathbb{1}_{\mathtt{C}_j} \otimes |\psi\rangle)$. Eq. (16) then implies that

$$\Phi_{ij}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) = \Psi_{ij}(X_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{D}_j}, \tag{19}$$

for all $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. By Stinespring's dilation theorem (see Theorem 1), there exists a minimal dilation given by $\mathcal{H}_{\mathtt{F}_{ij}}$ and $A_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{C}_j}; \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$ such that $\Psi_{ij}(X_{\mathtt{A}_i}) = A_{ij}^{\dagger}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}}) A_{ij}$. It follows that $A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j}$ is a minimal dilation for $X_{\mathtt{A}_i} \mapsto \Psi_{ij}(X_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{D}_j}$. But Eqs. (19) and (16) then imply that

$$V_{ij}^{\dagger}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{ij} = (A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j})^{\dagger} (X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}} \otimes \mathbb{1}_{\mathtt{D}_j})(A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j}), \tag{20}$$

for all $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. In other words, V_{ij} and $A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j}$ are Stinespring operators of the same map. Since $A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j}$ is minimal, there exists an isometry $U_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{D}_j}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$ such that $V_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{D}_j})$. This is the desired form.

It remains to show that $U_{ik}^{\dagger} U_{il} = \delta_{kl} \mathbb{1}$ for all $i \in I$ and $k,l \in J$. Since the U_{ij}'s are isometries, this condition is fulfilled for $k = l$. For $k \neq l$, we choose arbitrary $|\psi_k\rangle \in \mathcal{H}_{k:\mathcal{C}}$ and $|\psi_l\rangle \in \mathcal{H}_{l:\mathcal{C}}$ and $X_L, X_R \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. Eq. (18), with $X_{\mathtt{A}_i} = X_L^{\dagger} X_R$ implies

$$\begin{aligned} 0 &= \langle (X_L \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{ik} \psi_k | (X_R \otimes \mathbb{1}_{\mathtt{B}_i} \otimes \mathbb{1}_{\mathtt{E}}) V_{il} \psi_l \rangle \\ &= \langle (X_L \otimes \mathbb{1}_{\mathtt{F}_{ik}} \otimes \mathbb{1}_{\mathtt{D}_k})(A_{ik} \otimes \mathbb{1}_{\mathtt{D}_k}) \psi_k | \left[\mathbb{1}_{\mathtt{A}_i} \otimes U_{ik}^{\dagger} U_{il} \right] (X_R \otimes \mathbb{1}_{\mathtt{F}_{il}} \otimes \mathbb{1}_{\mathtt{D}_l})(A_{il} \otimes \mathbb{1}_{\mathtt{D}_l}) \psi_l \rangle. \end{aligned}$$

Since $\{(X_L \otimes \mathbb{1}_{\mathtt{F}_{ik}} \otimes \mathbb{1}_{\mathtt{D}_k})(A_{ik} \otimes \mathbb{1}_{\mathtt{D}_k})|\psi_k\rangle \,|\, X_L \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi_k\rangle \in \mathcal{H}_{k:\mathcal{C}}\}$ being total and $\{(X_R \otimes \mathbb{1}_{\mathtt{F}_{il}} \otimes \mathbb{1}_{\mathtt{D}_l})(A_{il} \otimes \mathbb{1}_{\mathtt{D}_l})|\psi_l\rangle \,|\, X_R \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi_l\rangle \in \mathcal{H}_{l:\mathcal{C}}\}$ being total is the definition of minimality of $A_{ik} \otimes \mathbb{1}_{\mathtt{D}_k}$ and $A_{il} \otimes \mathbb{1}_{\mathtt{D}_l}$, respectively, we can conclude from the equation above (using sesquilinearity of the inner product) that $\mathbb{1}_{\mathtt{A}_i} \otimes U_{ik}^{\dagger} U_{il} = 0$ and hence that $U_{ik}^{\dagger} U_{il} = 0$, as desired. Finally, note that the claim about the totality of $\{(X \otimes \mathbb{1}_{\mathtt{F}_{ij}}) A_{ij} |\psi\rangle \,|\, X \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi\rangle \in \mathcal{H}_{\mathtt{C}_j}\}$ in $\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}}$ follows by construction. □

3.3. GKLS-generators with invariant atomic algebra

The notation in the following theorem and its proof follows Section 2..

THEOREM 6. *Let $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be given by $L(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$ with $V \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ and $K \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{A}$ be an atomic *-subalgebra of $\mathcal{L}(\mathcal{H})$, with decomposition given by Definition 3. Then the following are equivalent*

1. $L(\mathcal{A}) \subseteq \mathcal{A}$.
2. *There exist operators $V_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$ and $K_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0)$; for all $i, j \in I$ a Hilbert space $\mathcal{H}_{\mathtt{F}_{ij}}$, operators $A_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_j}; \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$, and isometries $U_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{B}_j}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$; and for every $i \in I$ operators $B_i \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$, $K_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, and self-adjoint operators $H_{\mathtt{B}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$, such that*
   - *V and K can be decomposed as*
   $$V = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_0 + \sum_{i,j\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_{ij}^{sc}P_j + \sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes B_i)P_i,$$
   $$K = \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger)V_{ii}^{sc}P_i + \frac{1}{2}\sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger B_i)P_i + K_{\mathcal{A}} + iH_{\mathcal{A}'} + P_0^\dagger K_0,$$
   with $V_{ij}^{sc} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{B}_j})$, $K_{\mathcal{A}} = \sum_{i\in I} P_i^\dagger(K_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i$, and $H_{\mathcal{A}'} = \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes H_{\mathtt{B}_i})P_i$, s.t. all series SOT-converge.
   - *The relation $U_{ik}^\dagger U_{il} = \delta_{kl}\mathbb{1}$ holds for all $i, k, l \in I$.*

Proof. The basic strategy is to use Theorem 4 to reduce the problem to CP-maps with invariant algebra $\mathcal{A}$, followed by an application of Theorem 5. In detail: Part 2 of Theorem 4 provides us with operators $\tilde{A}$, $\tilde{B}$, $\tilde{V}_0$, $\tilde{K}_0$, $K_{\mathcal{A}}$ and $\tilde{H}_{\mathcal{A}'}$ such that

$$V = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})\tilde{V}_0 + \tilde{A} + \tilde{B}, \tag{21a}$$

$$K = \tilde{B}^\dagger\tilde{A} + \frac{1}{2}\tilde{B}^\dagger\tilde{B} + K_{\mathcal{A}} + i\tilde{H}_{\mathcal{A}'} + P_0^\dagger\tilde{K}_0. \tag{21b}$$

We observe the following:

- Since $K_{\mathcal{A}} \in \mathcal{A}$, it can be decomposed as $K_{\mathcal{A}} = \sum_{i\in I} P_i^\dagger(K_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i$ for operators $K_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$.
- Since $\Phi(X_{\mathcal{A}}) := \tilde{A}^\dagger(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})\tilde{A} \in \mathcal{A}$ for all $X_{\mathcal{A}} \in \mathcal{A}$, we can apply Theorem 5, which implies that there exist $A_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$; and for all $i, j \in I$, a Hilbert space $\mathcal{H}_{\mathtt{F}_{ij}}$, operators $A_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_j}; \mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}})$, and isometries $U_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}} \otimes \mathcal{H}_{\mathtt{B}_j}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$ such that $\tilde{A} = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})A_0 + \sum_{i,j\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_{ij}^{sc}P_j$, where $V_{ij}^{sc} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})(A_{ij} \otimes \mathbb{1}_{\mathtt{B}_j})$ and $U_{ik}^\dagger U_{il} = \delta_{kl}\mathbb{1}$ for all $i, k, l \in I$.

- Since $\tilde{B}$ satisfies $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})\tilde{B} = \tilde{B}X_{\mathcal{A}}$ for all $X_{\mathcal{A}} \in \mathcal{A}$, a calculation executed in Lemma 11 shows that there exist $B_0 \in \mathcal{L}(\mathcal{H}_0; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$ and operators $B_i \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$ such that $\tilde{B} = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})B_0 P_0 + \sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes B_i)P_i$.
- Since $\tilde{H}_{\mathcal{A}'} \in \mathcal{A}'$, the discussion around Eq. (4) yields that it can be decomposed as $\tilde{H}_{\mathcal{A}'} = P_0^\dagger H_0 P_0 + \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes H_{\mathtt{B}_i})P_i$, for self-adjoint $H_0 \in \mathcal{L}(\mathcal{H}_0)$ and $H_{\mathtt{B}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$.

Each of the points above provides an explicit representation of the operators in Eqs. (21a) and (21b). Plugging them in yields

$$\begin{aligned} V = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})\tilde{V}_0 + (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})A_0 + (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})B_0 P_0 \\ + \sum_{i,j\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})V_{ij}^{sc}P_j + \sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{A_i} \otimes B_i)P_i, \end{aligned}$$

which has the desired form after defining $V_0 = \tilde{V}_0 + A_0 + B_0 P_0$. And

$$\begin{aligned} K = P_0^\dagger \tilde{K}_0 + P_0^\dagger B_0^\dagger A_0 P_0 + \frac{1}{2}P_0^\dagger B_0^\dagger B_0 P_0 + iP_0^\dagger H_0 P_0 \\ + \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger)V_{ii}^{sc}P_i + \frac{1}{2}\sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger B_i)P_i + K_{\mathcal{A}} + iH_{\mathcal{A}'}, \end{aligned}$$

which has the desired form after defining $K_0 = \tilde{K}_0 + B_0^\dagger A_0 P_0 + \frac{1}{2}B_0^\dagger B_0 P_0 + iH_0 P_0$. □

The representation in Part 2 of Theorem 6 is not unique. The following theorem quantifies the freedom in that representation.

THEOREM 7. *The operators and spaces in Part 2 of Theorem 6 can be chosen to satisfy the following minimality conditions: a) For all $i \in I$, the set $\{(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - A_{ii}X_{\mathtt{A}_i}|\psi\rangle \mid X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi\rangle \in \mathcal{H}_{\mathtt{A}_i}\}$ is total in $\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ii}}$, and b) the set $\{(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}|\psi\rangle \mid X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i}), |\psi\rangle \in \mathcal{H}_{\mathtt{A}_j}\}$ is total in $\mathcal{H}_{\mathtt{A}_i} \otimes \mathcal{H}_{\mathtt{F}_{ij}}$ for all $i, j \in I$ with $i \neq j$.*
*Let $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ and $\tilde{L} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be given by $L(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$ and $\tilde{L}(X) = \tilde{V}^\dagger(X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{K}^\dagger X - X\tilde{K}$, with $V \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$, $\tilde{V} \in \mathcal{L}(\mathcal{H}; \mathcal{H} \otimes \mathcal{H}_{\tilde{\mathtt{E}}})$, and $K, \tilde{K} \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{A}$ be an atomic *-subalgebra of $\mathcal{L}(\mathcal{H})$, with decomposition given by Definition 3. Suppose that $L(\mathcal{A}) \subseteq \mathcal{A}$ and $\tilde{L}(\mathcal{A}) \subseteq \mathcal{A}$ and let the corresponding representations (Theorem 6) $(V_0, K_0, \{\mathcal{H}_{\mathtt{F}_{ij}}\}, \{U_{ij}\}, \{A_{ij}\}, \{B_i\}, \{K_{\mathtt{A}_i}\}, \{H_{\mathtt{B}_i}\})$ and $(\tilde{V}_0, \tilde{K}_0, \{\mathcal{H}_{\tilde{\mathtt{F}}_{ij}}\}, \{\tilde{U}_{ij}\}, \{\tilde{A}_{ij}\}, \{\tilde{B}_i\}, \{\tilde{K}_{\mathtt{A}_i}\}, \{\tilde{H}_{\mathtt{B}_i}\})$ both satisfy conditions a) and b) above. Then, the following hold:*

1. *If $L(X_{\mathcal{A}}) = \tilde{L}(X_{\mathcal{A}})$ for all $X_{\mathcal{A}} \in \mathcal{A}$, then, for every $i \in I$, there exist a unitary $W_{ii} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ii}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ii}})$, vectors $|\tilde{\psi}_i\rangle \in \mathcal{H}_{\tilde{\mathtt{F}}_{ii}}$ with $\sup_{i\in I}\|\tilde{\psi}_i\| < \infty$, and numbers $\mu_i \in \mathbb{R}$, with $\sup_{i\in I}|\mu_i| < \infty$, such that $\tilde{A}_{ii} = (\mathbb{1}_{\mathtt{A}_i} \otimes W_{ii})A_{ii} + \mathbb{1}_{\mathtt{A}_i} \otimes |\tilde{\psi}_i\rangle$ and*

$\tilde{K}_{\mathtt{A}_i} = K_{\mathtt{A}_i} + (\mathbb{1}_{\mathtt{A}_i} \otimes \langle\tilde{\psi}_i|W_{ii})A_{ii} + \frac{1}{2}\|\tilde{\psi}_i\|^2 + i\mu_i$. *Moreover, for all* $i, j \in I$ *with* $i \neq j$ *there exists a unitary* $W_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ij}})$ *such that* $\tilde{A}_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes W_{ij})A_{ij}$.

2. *If* $\tilde{V} = V$ *and* $\tilde{K} = K$, *then* $\tilde{V}_0 = V_0$, $\tilde{K}_0 = K_0$, $\tilde{B}_i = B_i - U_{ii}(|W_{ii}^\dagger\tilde{\psi}_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i})$, *and* $\tilde{H}_{\mathtt{B}_i} = H_{\mathtt{B}_i} + \frac{i}{2}(G - G^\dagger) - \mu_i\mathbb{1}_{\mathtt{B}_i}$, *where* $G = B_i^\dagger U_{ii}(|W_{ii}^\dagger\tilde{\psi}_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i})$. *Moreover, for all* $i, j \in I$, *we have* $\tilde{U}_{ij} = U_{ij}(W_{ij}^\dagger \otimes \mathbb{1}_{\mathtt{B}_i})$.

Remark It is the matter of a straightforward calculation to show that the (simultaneous) substitutions in 1 and 2 above leave the operators V and K invariant. Thus Theorem 7 quantifies exactly the freedom in our representation. Moreover, Theorem 2 quantifies the freedom in the choice of $(\mathcal{H}_{\mathtt{E}}, V, K)$.

Proof. We start with proving the possibility of a reduction to minimality, as claimed in the first part of the theorem. Suppose L is given according to Part 2 of Theorem 6, with data $(\tilde{V}_0, \tilde{K}_0, \{\mathcal{H}_{\tilde{\mathtt{F}}_{ij}}\}, \{\tilde{U}_{ij}\}, \{\tilde{A}_{ij}\}, \{\tilde{B}_i\}, \{\tilde{K}_{\mathtt{A}_i}\}, \{\tilde{H}_{\mathtt{B}_i}\})$. For any $i \in I$, $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$, we have $P_iL(P_i^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{B}_i})P_i)P_i^\dagger = \Big[\tilde{A}_{ii}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\tilde{\mathtt{F}}_{ii}})\tilde{A}_{ii} - \tilde{K}_{\mathtt{A}_i}^\dagger X_{\mathtt{A}_i} - X_{\mathtt{A}_i}\tilde{K}_{\mathtt{A}_i}\Big]\otimes \mathbb{1}_{\mathtt{B}_i} =: L_{ii}^\downarrow(X_{\mathtt{A}_i})\otimes\mathbb{1}_{\mathtt{B}_i}$. By Theorem 2, there exists $(\mathcal{H}_{\mathtt{F}_{ii}}, A_{ii}, K_{\mathtt{A}_i})$ such that $L_{ii}^\downarrow(X_{\mathtt{A}_i}) = A_{ii}^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - K_{\mathtt{A}_i}^\dagger X_{\mathtt{A}_i} - X_{\mathtt{A}_i}K_{\mathtt{A}_i}$, A_{ii} satisfies condition a), $\tilde{A}_{ii} = (\mathbb{1}_{\mathtt{A}_i}\otimes W_{ii})A_{ii} + \mathbb{1}_{\mathtt{A}_i} \otimes |\tilde{\psi}_i\rangle$, and $\tilde{K}_{\mathtt{A}_i} = K_{\mathtt{A}_i} + (\mathbb{1}_{\mathtt{A}_i} \otimes \langle\tilde{\psi}_i|W_{ii})A_{ii} + \frac{1}{2}\|\tilde{\psi}_i\|^2 + i\mu_i$, for an isometry $W_{ii} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ii}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ii}})$, a vector $|\tilde{\psi}_i\rangle \in \mathcal{H}_{\tilde{\mathtt{F}}_{ii}}$, and a number $\mu_i \in \mathbb{R}$. We define $U_{ii} = \tilde{U}_{ii}(W_{ii}\otimes\mathbb{1}_{\mathtt{B}_i})$, so that U_{ii} is an isometry. Furthermore, we define $B_i = \tilde{B}_i + |\tilde{\psi}_i\rangle\otimes\mathbb{1}_{\mathtt{B}_i}$ and $H_{\mathtt{B}_i} = \tilde{H}_{\mathtt{B}_i} + \frac{i}{2}(\tilde{G} - \tilde{G}^\dagger) + \mu_i\mathbb{1}_{\mathtt{B}_i}$, where $\tilde{G} = \tilde{B}_i^\dagger\tilde{U}_{ii}(|\tilde{\psi}_i\rangle\otimes\mathbb{1}_{\mathtt{B}_i})$. A direct calculation shows that replacing the operators with 'tilde' by the ones without, does not change V and K, but now A_{ii} satisfies condition a). The claim that $\sup_{i\in I}\|\tilde{\psi}_i\| < \infty$ and $\sup_{i\in I}|\mu_i| < \infty$ follows, since L would be unbounded otherwise. For $i, j \in I$ with $i \neq j$, we have $P_jL(P_i^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i)P_j^\dagger = \Big[\tilde{A}_{ij}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\tilde{\mathtt{F}}_{ij}})\tilde{A}_{ij}\Big] \otimes \mathbb{1}_{\mathtt{B}_j} =: L_{ij}^\downarrow(X_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{B}_j}$. By Theorem 1, there exists $(\mathcal{H}_{\tilde{\mathtt{F}}_{ij}}, A_{ij})$ such that $L_{ij}^\downarrow(X_{\mathtt{A}_i}) = A_{ij}^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{F}_{ij}})A_{ij}$, condition b) holds, and $\tilde{A}_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes W_{ij})A_{ij}$, for some isometry $W_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ij}})$. We define $U_{ij} = \tilde{U}_{ij}(W_{ij} \otimes \mathbb{1}_{\mathtt{B}_j})$. It follows that U_{ij} is an isometry and also that $U_{ik}^\dagger U_{il} = \delta_{kl}\mathbb{1}$. Again, a calculation shows that replacing the operators with 'tilde' by the ones without, does not change K and V.

Next, we want to prove 1. Since $L(X_{\mathcal{A}}) = \tilde{L}(X_{\mathcal{A}})$ for all $X_{\mathcal{A}} \in \mathcal{A}$, we have in particular $P_iL(P_i^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i)P_i^\dagger = P_i\tilde{L}(P_i^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i)P_i^\dagger$ for all $i \in I$. This is equivalent to $A_{ii}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - K_{\mathtt{A}_i}^\dagger X_{\mathtt{A}_i} - X_{\mathtt{A}_i}K_{\mathtt{A}_i} = \tilde{A}_{ii}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\tilde{\mathtt{F}}_{ii}})\tilde{A}_{ii} - \tilde{K}_{\mathtt{A}_i}^\dagger X_{\mathtt{A}_i} - X_{\mathtt{A}_i}\tilde{K}_{\mathtt{A}_i}$. Since a) holds for L and $\tilde{L}$, Theorem 2 implies the existence of unitaries $W_{ii} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ii}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ii}})$, vectors $|\tilde{\psi}_i\rangle \in \mathcal{H}_{\tilde{\mathtt{F}}_{ii}}$, and numbers $\mu_i \in \mathbb{R}$ s.t. $\tilde{A}_{ii} = (\mathbb{1}_{\mathtt{A}_i}\otimes W_{ii})A_{ii} + \mathbb{1}_{\mathtt{A}_i}\otimes|\tilde{\psi}_i\rangle$ and $\tilde{K}_{\mathtt{A}_i} = K_{\mathtt{A}_i} + (\mathbb{1}_{\mathtt{A}_i}\otimes\langle\tilde{\psi}_i|W_{ii})A_{ii} + \frac{1}{2}\|\tilde{\psi}_i\|^2 + i\mu_i$. The claim that $\sup_{i\in I}\|\tilde{\psi}_i\| < \infty$ and $\sup_{i\in I}|\mu_i| < \infty$ follows, since $\tilde{L}$ would be unbounded otherwise. For $i,j \in I$ with $i \neq j$ we have $P_j L(P_i^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i)P_j^\dagger = P_j \tilde{L}(P_i^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i})P_i)P_j^\dagger$, which is equivalent to $A_{ij}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij} = \tilde{A}_{ij}^\dagger(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\tilde{\mathtt{F}}_{ij}})\tilde{A}_{ij}$ for all $X_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. Since b) holds for L and $\tilde{L}$, Theorem 1 implies the existence of a unitary $W_{ij} \in \mathcal{L}(\mathcal{H}_{\mathtt{F}_{ij}}; \mathcal{H}_{\tilde{\mathtt{F}}_{ij}})$ s.t. $\tilde{A}_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes W_{ij})A_{ij}$. This is claim 1.

For part 2, we first notice that $\tilde{V} = V$ and $\tilde{K} = K$ immediately implies (by projecting into the respective subspace) that $\tilde{V}_0 = V_0$ and $\tilde{K}_0 = K_0$. Moreover, for any $i \in I$,

$$V_{ii} := (\mathbb{1}_{\mathtt{A}_{ii}} \otimes U_{ii})(A_{ii} \otimes \mathbb{1}_{\mathtt{B}_{ii}}) + \mathbb{1}_{\mathtt{A}_i} \otimes B_i = (\mathbb{1}_{\mathtt{A}_{ii}} \otimes \tilde{U}_{ii})(\tilde{A}_{ii} \otimes \mathbb{1}_{\mathtt{B}_{ii}}) + \mathbb{1}_{\mathtt{A}_i} \otimes \tilde{B}_i =: \tilde{V}_{ii}\,. \tag{22}$$

Thus,

$$\begin{aligned}
&(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i\mathtt{E}})\tilde{V}_{ii} - \tilde{V}_{ii}(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) = (\mathbb{1}_{\mathtt{A}_i} \otimes \tilde{U}_{ii})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\tilde{\mathtt{F}}_{ii}})\tilde{A}_{ii} - \tilde{A}_{ii}X_{\mathtt{A}_i}] \otimes \mathbb{1}_{\mathtt{B}_i})\\
&= (\mathbb{1}_{\mathtt{A}_i} \otimes (\tilde{U}_{ii}(W_{ii} \otimes \mathbb{1}_{\mathtt{B}_i})))([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - A_{ii}X_{\mathtt{A}_i}] \otimes \mathbb{1}_{\mathtt{B}_i})\\
&= (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ii})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - A_{ii}X_{\mathtt{A}_i}] \otimes \mathbb{1}_{\mathtt{B}_i})\,,
\end{aligned} \tag{23}$$

where the second line was obtained by using the relation between A_{ii} and $\tilde{A}_{ii}$ in Part 1. From the equality of the last two lines and the totality implied by a), we conclude $U_{ii} = \tilde{U}_{ii}(W_{ii} \otimes \mathbb{1}_{\mathtt{B}_i})$. Using this relation, the relation between A_{ii} and $\tilde{A}_{ii}$, and Eq. (22) yields $\tilde{B}_i = B_i - U_{ii}(|W_{ii}^\dagger\tilde{\psi}_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i})$. Moreover, from $P_iKP_i^\dagger = (\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger)(\mathbb{1}_{\mathtt{A}_i} \otimes U_{ii})(A_{ii} \otimes \mathbb{1}_{\mathtt{B}_i}) + \frac{1}{2}(\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger B_i) + (K_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) + (\mathbb{1}_{\mathtt{A}_i} \otimes iH_{\mathtt{B}_i}) = (\mathbb{1}_{\mathtt{A}_i} \otimes \tilde{B}_i^\dagger)(\mathbb{1}_{\mathtt{A}_i} \otimes \tilde{U}_{ii})(\tilde{A}_{ii} \otimes \mathbb{1}_{\mathtt{B}_i}) + \frac{1}{2}(\mathbb{1}_{\mathtt{A}_i} \otimes \tilde{B}_i^\dagger\tilde{B}_i) + (\tilde{K}_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i}) + (\mathbb{1}_{\mathtt{A}_i} \otimes i\tilde{H}_{\mathtt{B}_i}) = P_i\tilde{K}P_i^\dagger$ and the already established relations between the operators with and without 'tilde', we obtain $\tilde{H}_{\mathtt{B}_i} = H_{\mathtt{B}_i} + \frac{i}{2}(G - G^\dagger) - \mu_i\mathbb{1}_{\mathtt{B}_i}$. Finally, for $i,j \in I$ with $i \neq j$ we get $(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i\mathtt{E}})V_{ij} = (\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}] \otimes \mathbb{1}_{\mathtt{B}_j}) = (\mathbb{1}_{\mathtt{A}_i} \otimes \tilde{U}_{ij})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})\tilde{A}_{ij}] \otimes \mathbb{1}_{\mathtt{B}_j}) = (\mathbb{1}_{\mathtt{A}_i} \otimes (\tilde{U}_{ij}(W_{ij} \otimes \mathbb{1}_{\mathtt{B}_j})))([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}] \otimes \mathbb{1}_{\mathtt{B}_j})$. By the totality condition b), we can conclude $U_{ij} = \tilde{U}_{ij}(W_{ij} \otimes \mathbb{1}_{\mathtt{B}_j})$. Since W_{ij} is unitary, this finishes the proof. □

For later convenience, we also note the following.

COROLLARY 8. *Let $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be given by $L(X) = -K^\dagger X - XK$, with $K \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{A}$ be an atomic *-subalgebra of $\mathcal{L}(\mathcal{H})$, with decomposition given by Definition 3. If $L(\mathcal{A}) \subseteq \mathcal{A}$, then we can choose $A_{ij} = 0$ and $B_i = 0$, for all $i,j \in I$ in the corresponding representation of Part 2 of Theorem 6. Thus, $K = K_{\mathcal{A}} + iH_{\mathcal{A}'} + P_0^\dagger K_0$.*

Proof. If $V = 0$ and K are given via $(V_0, K_0, \{\mathcal{H}_{\mathtt{F}_{ij}}\}, \{U_{ij}\}, \{A_{ij}\}, \{B_i\}, \{K_{\mathtt{A}_i}\}, \{H_{\mathtt{B}_i}\})$ such that a) and b) in Theorem 7 hold, $0 = V_{ij} = (P_i \otimes \mathbb{1}_{\mathtt{E}})VP_j$ implies $(\mathbb{1}_{\mathtt{A}_i} \otimes U_{ij})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ij}})A_{ij}] \otimes \mathbb{1}_{\mathtt{B}_j}) = 0$ for $i \neq j$ and $(\mathbb{1}_{\mathtt{A}_i} \otimes U_{ii})([(X_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{F}_{ii}})A_{ii} - A_{ii}X_{\mathtt{A}_i}] \otimes \mathbb{1}_{\mathtt{B}_i}) = 0$ for all $i \in I$ (compare Eqs. (22) and (23)). Thus by the totality conditions a) and b), we conclude $U_{ij} = 0$ for all $i,j \in I$, which implies that $\mathcal{H}_{\mathtt{F}_{ij}}$ is zero-dimensional. Hence, $A_{ij} = 0$ and consequently also $B_i = 0$ for all $i \in I$. □

4. Applications

4.1. Semicausal quantum dynamical semigroups

As a first application of our results, we use them to reprove the main result of [13], namely the characterization of GKLS generators of semicausal quantum dynamical semigroups, a crucial step towards characterizing the generators of continuous one-parameter semigroups of quantum superchannels. Here, we call a CP-map $\Phi : \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ semicausal if there is a CP-map $\Phi_{\mathtt{A}} : \mathcal{L}(\mathcal{H}_{\mathtt{A}}) \to \mathcal{L}(\mathcal{H}_{\mathtt{A}})$ such that $\Phi(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) = \Phi_{\mathtt{A}}(X_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$ holds for all $X_{\mathtt{A}} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$. That is, Φ is semicausal if and only if $\Phi(\mathcal{A}_{sc}) \subseteq \mathcal{A}_{sc}$ holds for the atomic vN-subalgebra $\mathcal{A}_{sc} := \mathcal{L}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}} \subseteq \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$.

Using Theorem 6, we see that a GKLS generator $L : \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}}) \to \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$, $L(X) = V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK$, satisfies $L(\mathcal{A}_{sc}) \subseteq \mathcal{A}_{sc}$ if and only if there exist a Hilbert space $\mathcal{H}_{\mathtt{F}}$, an operator $A \in \mathcal{L}(\mathcal{H}_{\mathtt{A}}; \mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{F}})$, and an isometry $U \in \mathcal{L}(\mathcal{H}_{\mathtt{F}} \otimes \mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$; operators $B \in \mathcal{L}(\mathcal{H}_{\mathtt{B}}; \mathcal{H}_{\mathtt{B}} \otimes \mathcal{H}_{\mathtt{E}})$, $K_{\mathtt{A}} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$, and a self-adjoint operator $H_{\mathtt{B}} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}})$, such that $V = (\mathbb{1}_{\mathtt{A}} \otimes U)(A \otimes \mathbb{1}_{\mathtt{B}}) + \mathbb{1}_{\mathtt{A}} \otimes B$ and $K = (\mathbb{1}_{\mathtt{A}} \otimes B^\dagger U)(A \otimes \mathbb{1}_{\mathtt{B}}) + \frac{1}{2}\mathbb{1}_{\mathtt{A}} \otimes B^\dagger B + K_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}} + \mathbb{1}_{\mathtt{A}} \otimes iH_{\mathtt{B}}$. This is exactly [13, Theorem V.6].

4.2. Quantum dynamical semigroups with an atomic decoherence-free subalgebra

For our second application, we first recall that for a uniformly continuous and unital quantum dynamical semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ acting on $\mathcal{L}(\mathcal{H})$ the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the largest vN-subalgebra of $\mathcal{L}(\mathcal{H})$ on which every T_t acts as a $*$-automorphism [10, Theorem 3] or, equivalently, such that $T_t(X_{\mathcal{N}(\mathcal{T})}) = e^{i\tilde{H}t} X_{\mathcal{N}(\mathcal{T})} e^{-i\tilde{H}t}$ for all $X_{\mathcal{N}(\mathcal{T})} \in \mathcal{N}(\mathcal{T})$, where $\tilde{H} \in \mathcal{L}(\mathcal{H})$ is self-adjoint [10, Proposition 2]. In particular, note that every T_t leaves $\mathcal{N}(\mathcal{T})$ invariant. As shown in [7], a quantum dynamical semigroup inherits structure from its decoherence-free subalgebra. This is the content of the following:

THEOREM 9. *[7, Theorem 3.2(1)] Let $\mathcal{T}$ and $\mathcal{N}(\mathcal{T})$ be as above, and assume that $\mathcal{N}(\mathcal{T})$ is atomic, with normal form as in Definition 3. Then, for any GKLS-generator L, given in Kraus form by $L(X) = \sum_{n \in N} \phi_n^\dagger X \phi_n - K^\dagger X - XK$, where $K = \frac{1}{2}\sum_{n \in N} \phi_n^\dagger \phi_n + i\,\mathrm{Im}(K)$, we have*

$$\phi_n = \sum_{i \in I} P_i^\dagger (\mathbb{1}_{\mathtt{A}_i} \otimes \beta_{n;i}) P_i, \quad \mathrm{Im}(K) = \sum_{i \in I} P_i^\dagger (\kappa_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} + \mathbb{1}_{\mathtt{A}_i} \otimes \kappa_{\mathtt{B}_i}) P_i, \tag{24}$$

for some $\beta_{n;i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$ and some self-adjoint $\kappa_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$ and $\kappa_{\mathtt{B}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$.

We can recover Theorem 9 as a special case of our results as follows. By the characterization of the decoherence-free subalgebra described above, we have $L(X_{\mathcal{N}(\mathcal{T})}) = i[\tilde{H}, X_{\mathcal{N}(\mathcal{T})}] \in \mathcal{N}(\mathcal{T})$ for all $X_{\mathcal{N}(\mathcal{T})} \in \mathcal{N}(\mathcal{T})$. Therefore, if we define $\tilde{L}(X) := -(i\tilde{H})^\dagger X - X(i\tilde{H})$ for all $X \in \mathcal{L}(\mathcal{H})$, then Corollary 8 implies $\tilde{H} = \sum_{i\in I} P_i^\dagger(\tilde{H}_{\mathtt{A}_i} \otimes \mathbb{1}_{\mathtt{B}_i} + \mathbb{1}_{\mathtt{A}_i} \otimes \tilde{H}_{\mathtt{B}_i})P_i$ for self-adjoint $\tilde{H}_{\mathtt{A}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$ and $\tilde{H}_{\mathtt{B}_i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$. This already provides a normal form representation of $\tilde{L}$ as in Theorem 6, and the minimality conditions of Theorem 7 are satisfied, since $\tilde{A}_{ij} = 0$. Now, if L is w.l.o.g. also minimal, Part 1 of Theorem 7 (with the roles of the operators with and without 'tilde' interchanged) implies that $A_{ii} = \mathbb{1}_{\mathtt{A}_i} \otimes |\psi_i\rangle$ for some vectors $|\psi_i\rangle \in \mathcal{H}_{\mathtt{F}_{ii}}$ and that $A_{ij} = 0$ for $i \neq j$. On the level of the Kraus operators, this yields $\phi_n = (\mathbb{1} \otimes \langle e_n|)\big(\sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes U_{ii}(|\psi_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i}) + \mathbb{1}_{\mathtt{A}_i} \otimes B_i)P_i\big)$, which has the desired form with $\beta_{n;i} = (\mathbb{1}_{\mathtt{B}_i} \otimes \langle e_n|)(U_{ii}(|\psi_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i}) + B_i)$. (Note: If L is not already minimal, we can first follow the steps in the proof of Theorem 7 to reduce to a minimal generator and then apply the above reasoning.) Moreover, using the representation of Theorem 6 and again Part 1 of Theorem 7,

$$\begin{aligned}\mathrm{Im}(K) &= \sum_{i\in I} P_i^\dagger\big(\mathrm{Im}((\mathbb{1}_{\mathtt{A}_i} \otimes B_i^\dagger U_{ii})(A_{ii} \otimes \mathbb{1}_{\mathtt{B}_i})) + \mathrm{Im}(K_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{B}_i} + \mathbb{1}_{\mathtt{A}_i} \otimes H_{\mathtt{B}_i}\big)P_i \\ &= \sum_{i\in I} P_i^\dagger\big(\mathbb{1}_{\mathtt{A}_i} \otimes \mathrm{Im}(B_i^\dagger U_{ii}(|\psi_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i})) + \mathrm{Im}(\tilde{K}_{\mathtt{A}_i}) \otimes \mathbb{1}_{\mathtt{B}_i} + \mu_i + \mathbb{1}_{\mathtt{A}_i} \otimes H_{\mathtt{B}_i}\big)P_i,\end{aligned}$$

which has the desired form with $\kappa_{\mathtt{A}_i} := \mathrm{Im}(\tilde{K}_{\mathtt{A}_i})$ and $\kappa_{\mathtt{B}_i} := H_{\mathtt{B}_i} + \mathrm{Im}(B_i^\dagger U_{ii}(|\psi_i\rangle \otimes \mathbb{1}_{\mathtt{B}_i})) + \mu_i$.

As a final remark on our short discussion of the decoherence-free subalgebra, we point out that [24, Corollary 21] showed that $\mathcal{N}(\mathcal{T})$ is atomic whenever the quantum dynamical semigroup $\mathcal{T}$ admits a normal faithful invariant state. In many situations of interest, we can therefore focus on $\mathcal{N}(\mathcal{T})$ being atomic, as in Theorems 6 and 9.

4.3. Quantum dynamical semigroups and CP-maps with an invariant maximal abelian subalgebra

Our third application is concerned with the following question: Given a maximal abelian vN-subalgebra $\mathcal{C}$ of $\mathcal{L}(\mathcal{H})$, that is $\mathcal{C}' = \mathcal{C}$, what is the most general form of a GKLS-generator that leaves $\mathcal{C}$ invariant? According to Theorem 4, we can reduce the above question to characterizing CP-maps with an invariant maximal abelian vN-subalgebra. The latter question was previously investigated in [11, 23]. More precisely, [23, Theorem 1] gave an abstract characterization of such GKLS-generators in terms of a commutation relation and a sufficient condition on the Kraus operators of the CP part of the GKLS-generator. Ref. [11] extended these deliberations and, in particular, gave a necessary and sufficient condition for a normal CP-map to leave a maximal abelian vN-subalgebra invariant:

THEOREM 10. *[11, Corollary 3.4] Let Φ be a normal CP-map on $\mathcal{L}(\mathcal{H})$ with Kraus decomposition $\Phi(X) = \sum_{n\in N} \phi_n^\dagger X \phi_n$. Let $\mathcal{C}$ be a maximal abelian vN-subalgebra of $\mathcal{L}(\mathcal{H})$. Then, Φ leaves $\mathcal{C}$ invariant if and only if for every $c \in \mathcal{C}$ there exist $c_{mn} = c_{mn}(c) \in \mathcal{C}$ for $m, n \in N$ s.t. 1) $c_{mn}(c^\dagger) = c_{nm}(c)^\dagger$ and 2) $[c, \phi_m] = \sum_{n\in N} c_{mn}\phi_n$.*

The "if"-direction in Theorem 10 can be seen using the von Neumann bicommutant theorem. We can recover the "only if"-direction, albeit only for atomic $\mathcal{C}$, as a consequence of Theorem 5 as follows: If $\mathcal{C}$ is a maximal abelian and atomic vN-subalgebra of $\mathcal{L}(\mathcal{H})$, then its decomposition as in Definition 3 becomes particularly simple, with $\dim(\mathcal{H}_{\mathtt{A}_i}) = 1 = \dim(\mathcal{H}_{\mathtt{B}_i})$ for all $i \in I$. Thus, for any $i \in I$, there exists $|p_i\rangle \in \mathcal{H}$ such that $P_i = \langle p_i| \in \mathcal{L}(\mathcal{H}; \mathbb{C})$ such that $\{|p_i\rangle\}_{i\in I}$ forms an orthonormal basis of $\mathcal{H}$. The decomposition of V in Theorem 5 in this case simplifies to $V = \sum_{i,j\in I} |p_i\rangle\langle p_j| \otimes |\psi_{ij}\rangle$ for some vectors $|\psi_{ij}\rangle \in \mathcal{H}_{\mathtt{E}}$ satisfying $\langle \psi_{ij}|\psi_{ik}\rangle = 0$ for all $i, j, k \in I$ with $j \neq k$. Accordingly, the Kraus operators of Φ can be written as $\phi_n = \sum_{i,j\in I} \langle e_n|\psi_{ij}\rangle |p_i\rangle\langle p_j|$ for some orthonormal basis $\{|e_n\rangle\}_{n\in N}$ of $\mathcal{H}_{\mathtt{E}}$.

Since any $c \in \mathcal{C}$ can be decomposed as $c = \sum_{i\in I} c_i |p_i\rangle\langle p_i|$, we obtain for any $m \in N$:

$$[c, \phi_m] = \sum_{i,j\in I} (c_i - c_j) \langle e_m|\psi_{ij}\rangle |p_i\rangle\langle p_j| \,. \tag{25}$$

As $\langle \psi_{ij}|\psi_{ik}\rangle = 0$ for $j \neq k$, we can define $C_i \in \mathcal{L}(\mathcal{H}_{\mathtt{E}})$ by linearly extending

$$C_i |\psi_{ij}\rangle = (c_i - c_j)|\psi_{ij}\rangle, \qquad \text{for all } i, j \in I \,. \tag{26}$$

We consider a basis expansion $C_i := \sum_{m,n\in N} c_{mn;i} |e_m\rangle\langle e_n|$ and define $c_{mn} = \sum_{i\in I} c_{mn;i} |p_i\rangle\langle p_i| \in \mathcal{C}$. Then, we get

$$\begin{aligned}
\sum_{n\in N} c_{mn}\phi_n &= \sum_{n\in N}\sum_{i,j\in I} \langle e_n|\psi_{ij}\rangle c_{mn;i} |p_i\rangle\langle p_j| \\
&= \sum_{i,j\in I} |p_i\rangle\langle p_j| \cdot \underbrace{\sum_{n\in N} \langle e_n|\psi_{ij}\rangle c_{mn;i}}_{=\langle e_m|C_i|\psi_{ij}\rangle \overset{(26)}{=} (c_i - c_j)\langle e_m|\psi_{ij}\rangle} \\
&= \sum_{i,j\in I} (c_i - c_j)\langle e_m|\psi_{ij}\rangle |p_i\rangle\langle p_j| \,,
\end{aligned}$$

which equals $[c, \phi_m]$ by Eq. (25). Thus, 2) is satisfied. Also 1) holds, since the replacement $c \mapsto c^\dagger$ in the above reasoning leads to $c_i \mapsto \bar{c}_i$, which in turn gives $C_i \mapsto C_i^\dagger$, finally implying $c_{mn;i} \mapsto \bar{c}_{nm;i}$ and thus $c_{mn} \mapsto c_{nm}^\dagger$.

We note that our Theorem 5 not only gives rise to Theorem 10, but also provides a concrete characterization of the most general CP-maps satisfying the criterion of Theorem 10 for the atomic case. Similarly, we can reproduce and concretize [21, Theorem 1.2] for atomic $\mathcal{C}$.

4.4. Completely positive and trace-preserving maps with fixed points

In this section, we look at the Koashi-Imoto Theorem [16, Eq. (85) and Theorem 3] and restrict ourselves to finite-dimensional systems. The Koashi-Imoto Theorem characterizes the form of CP-maps $T \in \mathrm{CP}_\sigma(\mathcal{H})$ that are trace-preserving and state-preserving, in the sense that $T(\rho_s) = \rho_s$ for some set of density matrices $\{\rho_s\}_{s\in S}$. If $T(\rho) = \mathrm{tr}_{\mathtt{E}}\left[V\rho V^\dagger\right]$, with $V \in \mathcal{L}(\mathcal{H};\mathcal{H}\otimes\mathcal{H}_{\mathtt{E}})$; $\mathcal{F}_T$ is the set of fixed points of T; $\mathcal{H}_{\mathcal{F}_T} := \cup_{\rho\in\mathcal{F}_T}\mathrm{supp}(\rho) \subseteq \mathcal{H}$ is the support of a maximal rank fixed point; and $Q : \mathcal{H} \to \mathcal{H}_{\mathcal{F}_T}$ is the corresponding projection, then the Koashi-Imoto Theorem states that there is a decomposition of $\mathcal{H}_{\mathcal{F}_T} = U_{\mathcal{F}_{\tilde{T}^*}}(\bigoplus_{i\in I}(\mathcal{H}_{\mathtt{A}_i}\otimes\mathcal{H}_{\mathtt{B}_i}))$ for some unitary $U_{\mathcal{F}_{\tilde{T}^*}}$ (the notation will soon make sense) and an index set I such that

$$\hat{V} = \bigoplus_{i\in I}(\mathbb{1}_{\mathtt{A}_i}\otimes V_i) \quad \text{and} \quad \mathcal{F}_T = Q^\dagger U_{\mathcal{F}_{\tilde{T}^*}}\bigoplus_{i\in I}(\mathcal{L}(\mathcal{H}_{\mathtt{A}_i})\otimes\sigma_i)U^\dagger_{\mathcal{F}_{\tilde{T}^*}}Q, \tag{27}$$

where $\hat{V} = (U_{\mathcal{F}_{\tilde{T}^*}}\otimes\mathbb{1}_{\mathtt{E}})\left[(Q\otimes\mathbb{1}_{\mathtt{E}})VQ^\dagger\right]U^\dagger_{\mathcal{F}_{\tilde{T}^*}}$, all $V_i \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i};\mathcal{H}_{\mathtt{B}_i}\otimes\mathcal{H}_{\mathtt{E}})$ are isometries and all $\sigma_i \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$ are density matrices.

We can reproduce this result as follows: By elementary considerations (exploiting the positivity of T), the map $\tilde{T} : \mathcal{L}(\mathcal{H}_{\mathcal{F}_T}) \to \mathcal{L}(\mathcal{H}_{\mathcal{F}_T})$, $\tilde{T}(X) = QT(Q^\dagger XQ)Q^\dagger$ is a trace-preserving CP-map (see [27, Proof of Lemma 6.4] for details), which, by construction, has a full-rank fixed-point. Since Lindblad [18, Section 3] we know that if $\tilde{T}$ has a full rank fixed-point, then the set of fixed points of the (unital) dual map $\tilde{T}^*$ forms a vN-algebra, $\mathcal{F}_{\tilde{T}^*}$. As $\dim(\mathcal{H}) < \infty$, $\mathcal{F}_{\tilde{T}^*}$ it is atomic and can be decomposed according to Definition 3. Moreover, $\tilde{T}^*(X) = W^\dagger(X\otimes\mathbb{1}_{\mathtt{E}})W$ with $W = (Q\otimes\mathbb{1}_{\mathtt{E}})VQ^\dagger$ and $\tilde{T}^*(\mathcal{F}_{\tilde{T}^*}) \subseteq \mathcal{F}_{\tilde{T}^*}$. Hence, W decomposes according to Theorem 5. This implies $P_i\tilde{T}^*(P_i^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{B}_i})P_i)P_i^\dagger = \left[A_{ii}^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{F}_{ii}})A_{ii}\right]\otimes\mathbb{1}_{\mathtt{B}_i} = X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{B}_i}$ for $i\in I$ and $X_{\mathtt{A}_i}\in\mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. Since we can choose A_{ii} minimal, this implies $\mathcal{H}_{\mathtt{F}_{ii}} = \mathbb{C}$ and $A_{ii} = \mathbb{1}_{\mathtt{A}_i}$. For $i,j\in I$ with $i\neq j$, we have $P_j\tilde{T}^*(P_i^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{B}_i})P_i)P_j^\dagger = \left[A_{ij}^\dagger(X_{\mathtt{A}_i}\otimes\mathbb{1}_{\mathtt{F}_{ij}})A_{ij}\right]\otimes\mathbb{1}_{\mathtt{B}_i} = 0$ for all $X_{\mathtt{A}_i}\in\mathcal{L}(\mathcal{H}_{\mathtt{A}_i})$. Thus $A_{ij} = 0$. In conclusion, we have $W = \sum_{i\in I}P_i^\dagger(\mathbb{1}_{\mathtt{A}_i}\otimes U_{ii})P_i$, with P_i given by Definition 3. From the definition of P_i this is exactly the first part of Eq. (27), if we identify V_i with U_{ii}.

For the second part, first note that by construction, $\mathcal{F}_T = Q^\dagger\mathcal{F}_{\tilde{T}}Q$. By the Brouwer fixed-point theorem, there exists a density matrix σ_i such that $\mathrm{tr}_{\mathtt{E}}\left[V_i\sigma_iV_i^\dagger\right] = \sigma_i$. With that choice, it is easy to see that operators of the form of the second part in Eq. (27) are fixed-points. But since, as a general property of linear maps on finite-dimensional spaces, the dimension of the fixed-point space of $\tilde{T}$ equals the dimension the fixed-point space of $\tilde{T}^*$, the claim follows. Thus we have arrived at the Koashi-Imoto Theorem.

5. Conclusion

In this work, we have fully characterized the generators of quantum dynamical semigroups with an invariant vN-subalgebra. We have provided a constructive normal form for such restricted GKLS-generators and determined the freedom in their representation. In particular, these results encompass corresponding characterizations for CP-maps with invariant vN-subalgebras.

The assumption of an invariant atomic vN-subalgebra implies that the restriction of the quantum dynamical semigroup to that subalgebra is again a valid quantum dynamical semigroup. This means that we can also interpret Theorem 6 as providing, given a GKLS generator on a vN-subalgebra, a complete characterization of the possible extensions to a GKLS generator on $\mathcal{L}(\mathcal{H})$. In particular, Theorem 5 can be regarded as a constructive version of Arveson's extension theorem [2, 20], describing the most general CP extension on $\mathcal{L}(\mathcal{H})$ of a given CP-map defined on a vN-subalgebra, if that subalgebra is atomic.

As demonstrated in Section 4., our characterization of GKLS-generators with an invariant vN-subalgebra provides a unifying perspective on the results of different prior works. We expect that this point of view can be useful for further scenarios, such as the study of dynamical semigroups of higher-order quantum maps [4, 6], generalizing dynamical semigroups of quantum superchannels [13].

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A Auxiliary Lemma

In this appendix, we provide a full statement and a complete proof of a lemma useful in proving Theorem 6. Here, we use the notation from Section 3.

LEMMA 11. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ be an atomic weakly closed *-algebra. An operator $B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H} \otimes \mathcal{H}_{\mathtt{E}})$ satisfies $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})B = B(X_{\mathcal{A}} \otimes \mathbb{1}_{\tilde{\mathtt{E}}})$ for all $X_{\mathcal{A}} \in \mathcal{A}$ if and only if there exist $B_0 \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$ and for every $i \in I$ an operator $B_i \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$ such that B is given by the SOT-convergent series*

$$B = (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})B_0(P_0 \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) + \sum_{i \in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes B_i)(P_i \otimes \mathbb{1}_{\tilde{\mathtt{E}}}).$$

Proof. We fix orthonormal bases $\{|e_n\rangle\}_{n\in N}$ and $\{|\tilde{e}_m\rangle\}_{m\in M}$ of $\mathcal{H}_{\mathtt{E}}$ and $\mathcal{H}_{\tilde{\mathtt{E}}}$, respectively. Since $(X_{\mathcal{A}} \otimes \mathbb{1}_{\mathtt{E}})B = B(X_{\mathcal{A}} \otimes \mathbb{1}_{\tilde{\mathtt{E}}})$ for all $X_{\mathcal{A}} \in \mathcal{A}$, it follows that the operators $\beta_{nm} := (\mathbb{1}_{\mathcal{H}} \otimes \langle e_n|)B(\mathbb{1}_{\mathcal{H}} \otimes |\tilde{e}_m\rangle)$ belong to $\mathcal{A}'$. Thus (following the discussion around Eq. (4)), there are operators $\beta_{nm;0} \in \mathcal{L}(\mathcal{H}_0)$ and $\beta_{nm;i} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i})$ such that $\beta_{nm} = P_0^\dagger \beta_{nm;0} P_0 + \sum_{i\in I} P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes \beta_{nm;i})P_i$. We define $B_0 = \sum_{n\in N, m\in M}(\mathbb{1}_0 \otimes |e_n\rangle)\beta_{nm;0}(\mathbb{1}_0 \otimes \langle\tilde{e}_m|) \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H}_0 \otimes \mathcal{H}_{\mathtt{E}})$ and for all $i \in I$ the operator $B_i = \sum_{n\in N, m\in M}(\mathbb{1}_{\mathtt{B}_i} \otimes |e_n\rangle)\beta_{nm;i}(\mathbb{1}_{\mathtt{B}_i} \otimes \langle\tilde{e}_m|) \in \mathcal{L}(\mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H}_{\mathtt{B}_i} \otimes \mathcal{H}_{\mathtt{E}})$. We then have

$$\begin{aligned}
B &= \sum_{n\in N, m\in M} (\mathbb{1}_{\mathcal{H}} \otimes |e_n\rangle)\beta_{nm}(\mathbb{1}_{\mathcal{H}} \otimes \langle\tilde{e}_m|) \\
&= \sum_{n\in N, m\in M} (\mathbb{1}_{\mathcal{H}} \otimes |e_n\rangle)P_0^\dagger \beta_{nm;0} P_0(\mathbb{1}_{\mathcal{H}} \otimes \langle\tilde{e}_m|) \\
&\qquad + \sum_{\substack{n\in N, m\in M,\\ i\in I}} (\mathbb{1}_{\mathcal{H}} \otimes |e_n\rangle)P_i^\dagger(\mathbb{1}_{\mathtt{A}_i} \otimes \beta_{n;i})P_i(\mathbb{1}_{\mathcal{H}} \otimes \langle\tilde{e}_m|) \\
&= \sum_{n\in N, m\in M} (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_0 \otimes |e_n\rangle)\beta_{nm;0}(\mathbb{1}_0 \otimes \langle\tilde{e}_m|)(P_0 \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \\
&\qquad + \sum_{\substack{n\in N, m\in M,\\ i\in I}} (P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})\left(\mathbb{1}_{\mathtt{A}_i} \otimes [(\mathbb{1}_{\mathtt{B}_i} \otimes |e_n\rangle)\beta_{nm;i}(\mathbb{1}_{\mathtt{B}_i} \otimes \langle\tilde{e}_m|)]\right)(P_i \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \\
&= (P_0^\dagger \otimes \mathbb{1}_{\mathtt{E}})B_0(P_0 \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) + \sum_{i\in I}(P_i^\dagger \otimes \mathbb{1}_{\mathtt{E}})(\mathbb{1}_{\mathtt{A}_i} \otimes B_i)(P_i \otimes \mathbb{1}_{\tilde{\mathtt{E}}}).
\end{aligned}$$

This is the claimed result. □

B Proof of Theorem 2

LEMMA 12. *Two operators $K, \tilde{K} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}} \otimes \mathcal{H}_{\mathtt{B}})$ satisfy a) $(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})K^\dagger + K(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) = (X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})\tilde{K}^\dagger + \tilde{K}(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})$ for all $X_{\mathtt{A}} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$ if and only if there exists a self-adjoint $H_{\mathtt{B}} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}})$ such that b) $\tilde{K} = K + \mathbb{1}_{\mathtt{A}} \otimes iH_{\mathtt{B}}$.*

Proof. If $\tilde{K} = K + \mathbb{1}_{\mathtt{A}} \otimes iH_{\mathtt{B}}$, then $a)$ holds trivially. For the converse, decompose K and $\tilde{K}$ into real and imaginary part as $K = R + iH$ and $\tilde{K} = \tilde{R} + i\tilde{H}$. By choosing $X_{\mathtt{A}} = \mathbb{1}_{\mathtt{A}}$, we obtain $\tilde{R} = R$. The relation $a)$ the simplifies to $iH(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) - (X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})iH = i\tilde{H}(X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}) - (X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}})i\tilde{H}$. In terms of the commutator, this reads $\left[i(H - \tilde{H}), X_{\mathtt{A}} \otimes \mathbb{1}_{\mathtt{B}}\right] = 0$ for all $X_{\mathtt{A}} \in \mathcal{L}(\mathcal{H}_{\mathtt{A}})$. Thus $i(H - \tilde{H})$ is in the commutant of $\mathcal{L}(\mathcal{H}_{\mathtt{A}}) \otimes \mathbb{1}_{\mathtt{B}}$, which is $\mathbb{1}_{\mathtt{A}} \otimes \mathcal{L}(\mathcal{H}_{\mathtt{B}})$. Hence, there exists a self-adjoint $H_{\mathtt{B}} \in \mathcal{L}(\mathcal{H}_{\mathtt{B}})$ such that $\tilde{H} = H + \mathbb{1}_{\mathtt{A}} \otimes H_{\mathtt{B}}$. With $\tilde{K} = R + i\tilde{H}$, the claim follows. □

We provide a proof of Theorem 2. The proof follows Chapter 30 in [19], in particular the proof of Proposition 30.14 therein.

Proof. For any triplets $(\mathcal{H}_{\mathtt{E}}, V, K)$ and $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V}, \tilde{K})$, we introduce the shorthand $\pi(X) := (X \otimes \mathbb{1}_{\mathtt{E}})V - VX$ and $\tilde{\pi}(X) := (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{V}X$. For the triplet $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V}, \tilde{K})$ given in the statement of the theorem, the space $\tilde{S} := \mathrm{span}\{\tilde{\pi}(X)|\psi\rangle \mid X \in \mathcal{L}(\mathcal{H}), |\psi\rangle \in \mathcal{H}\}$ is invariant under the action of $Y \otimes \mathbb{1}_{\tilde{\mathtt{E}}}$ for all $Y \in \mathcal{L}(\mathcal{H})$, since for any $\tilde{S} \ni |\phi\rangle = \sum_i \tilde{\pi}(X_i)|\psi_i\rangle$, we have

$$\begin{aligned}(Y \otimes \mathbb{1}_{\tilde{\mathtt{E}}})|\phi\rangle &= \sum_i (Y \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{\pi}(X_i)|\psi_i\rangle \\ &= \sum_i \underbrace{\tilde{\pi}(YX_i)|\psi_i\rangle}_{\in \tilde{S}} - \underbrace{\tilde{\pi}(Y)X_i|\psi_i\rangle}_{\in \tilde{S}} \in \tilde{S}.\end{aligned}$$

Thus, the closure of $\tilde{S}$ is of the form $\mathcal{H} \otimes \mathcal{H}_{\mathtt{E}}$, for some subspace $\mathcal{H}_{\mathtt{E}} \subseteq \mathcal{H}_{\tilde{\mathtt{E}}}$. Denote by $P \in \mathcal{L}(\mathcal{H}_{\tilde{\mathtt{E}}}; \mathcal{H}_{\mathtt{E}})$ the associated orthogonal projection onto $\mathcal{H}_{\mathtt{E}}$. We define $V := (\mathbb{1} \otimes P)\tilde{V}$. By construction, V satisfies b). So, to prove the first part of the theorem, it remains to construct a suitable K. Since $\mathbb{1} \otimes P^\dagger P$ is the projection onto the closure of $\tilde{S}$, we obtain

$$\begin{aligned}(X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{V}X &= (\mathbb{1} \otimes P^\dagger P)((X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{V}X) \\ &= (\mathbb{1} \otimes P^\dagger)((X \otimes \mathbb{1}_{\mathtt{E}})V - VX)\end{aligned}$$

for all $X \in \mathcal{L}(\mathcal{H})$. A rearrangement yields

$$(X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})(\tilde{V} - (\mathbb{1} \otimes P^\dagger)V) = (\tilde{V} - (\mathbb{1} \otimes P^\dagger)V)X$$

By Lemma 11 (with $\mathcal{A} = \mathcal{L}(\mathcal{H})$), this implies that there exists $|\tilde{\phi}\rangle \in \mathcal{H}_{\tilde{\mathtt{E}}}$ such that $\tilde{V} - (\mathbb{1} \otimes P^\dagger)V = \mathbb{1} \otimes |\tilde{\phi}\rangle$. We define $K := \tilde{K} - (\mathbb{1} \otimes \langle\tilde{\phi}|)\tilde{V} + \frac{1}{2}\|\tilde{\phi}\|^2$, and notice that $PP^\dagger = \mathbb{1}_{\mathtt{E}}$. Thus,

$$\begin{aligned}V^\dagger(X \otimes \mathbb{1}_{\mathtt{E}})V &= \left[(\mathbb{1} \otimes P^\dagger)V\right]^\dagger (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \left[(\mathbb{1} \otimes P^\dagger)V\right] \\ &= \left[\tilde{V} - \mathbb{1} \otimes |\tilde{\phi}\rangle\right]^\dagger (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}}) \left[\tilde{V} - \mathbb{1} \otimes |\tilde{\phi}\rangle\right].\end{aligned}$$

From here, it is easy to see that $V^\dagger(X\otimes \mathbb{1}_{\mathtt{E}})V - K^\dagger X - XK = \tilde{V}^\dagger(X\otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{K}^\dagger X - X\tilde{K} = L(X)$ for all $X\in\mathcal{L}(\mathcal{H})$. This is our first claim.

For the second claim suppose that for $(\mathcal{H}_{\mathtt{E}}, V, K)$, a) and b) are satisfied, and that for $(\mathcal{H}_{\tilde{\mathtt{E}}}, \tilde{V}, \tilde{K})$, a) is satisfied. A direct calculation reveals that

$$\begin{aligned}\Psi(X,Y) &:= L(X^\dagger Y) - X^\dagger L(Y) - L(X^\dagger)Y + X^\dagger L(\mathbb{1})Y\\ &= \pi(X)^\dagger\pi(Y) = \tilde{\pi}(X)^\dagger\tilde{\pi}(Y),\end{aligned} \tag{28}$$

for all $X,Y\in\mathcal{L}(\mathcal{H})$. On $S := \operatorname{span}\{\pi(X)|\psi\rangle \mid X\in\mathcal{L}(\mathcal{H}), |\psi\rangle\in\mathcal{H}\}$, we define a map W_0 by linear extension of the relation $W_0\pi(X)|\psi\rangle := \tilde{\pi}(X)|\psi\rangle$. This is well-defined, since if $\sum_i \pi(X_i)|\psi_i\rangle = \sum_j \pi(Y_j)|\psi_j\rangle$, then

$$\begin{aligned}
&\|\sum_i \tilde{\pi}(X_i)|\psi_i\rangle - \sum_j \tilde{\pi}(Y_j)|\psi_j\rangle\|^2\\
&= \sum_{i,i'}\langle\psi_i|\tilde{\pi}(X_i)^\dagger\tilde{\pi}(X_{i'})\psi_{i'}\rangle + \sum_{i,j'}\langle\psi_i|\tilde{\pi}(X_i)^\dagger\tilde{\pi}(Y_{j'})\psi_{j'}\rangle\\
&\quad + \sum_{j,i'}\langle\psi_j|\tilde{\pi}(Y_j)^\dagger\tilde{\pi}(X_{i'})\psi_{i'}\rangle + \sum_{j,j'}\langle\psi_j|\tilde{\pi}(Y_j)^\dagger\tilde{\pi}(Y_{j'})\psi_{j'}\rangle\\
&\overset{(28)}{=} \sum_{i,i'}\langle\psi_i|\pi(X_i)^\dagger\pi(X_{i'})\psi_{i'}\rangle + \sum_{i,j'}\langle\psi_i|\pi(X_i)^\dagger\pi(Y_{j'})\psi_{j'}\rangle\\
&\quad + \sum_{j,i'}\langle\psi_j|\pi(Y_j)^\dagger\pi(X_{i'})\psi_{i'}\rangle + \sum_{j,j'}\langle\psi_j|\pi(Y_j)^\dagger\pi(Y_{j'})\psi_{j'}\rangle\\
&= \|\sum_i \pi(X_i)|\psi_i\rangle - \sum_j \pi(Y_i)|\psi_j\rangle\|^2 = 0.
\end{aligned}$$

Furthermore, W_0 can be extended to an isometry $\overline{W}_0$ on the closure of S, since for any $|\phi\rangle = \sum_i \pi(X_i)|\psi_i\rangle \in S$, we have

$$\begin{aligned}\|W_0|\phi\rangle\|^2 &= \sum_{i,i'}\langle\psi_i|\tilde{\pi}(X_i)^\dagger\tilde{\pi}(X_{i'})\psi_{i'}\rangle\\ &\overset{(28)}{=} \sum_{i,i'}\langle\psi_i|\pi(X_i)^\dagger\pi(X_{i'})\psi_{i'}\rangle = \|\phi\|^2 .\end{aligned}$$

Moreover, from

$$\begin{aligned}(X\otimes\mathbb{1}_{\tilde{\mathtt{E}}})\overline{W}_0\pi(Y)|\psi\rangle &= (X\otimes\mathbb{1}_{\tilde{\mathtt{E}}})\tilde{\pi}(Y)|\psi\rangle = \tilde{\pi}(XY)|\psi\rangle - \tilde{\pi}(X)Y|\psi\rangle\\ &= \overline{W}_0\pi(XY)|\psi\rangle - \overline{W}_0\pi(X)Y|\psi\rangle = \overline{W}_0(X\otimes\mathbb{1}_{\mathtt{E}})\pi(Y)|\psi\rangle\end{aligned}$$

and totality of S, we conclude that $(X\otimes\mathbb{1}_{\tilde{\mathtt{E}}})\overline{W}_0 = \overline{W}_0(X\otimes\mathbb{1}_{\mathtt{E}})$. Lemma 11 (with $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and the roles of $\mathcal{H}_{\mathtt{E}}$ and $\mathcal{H}_{\tilde{\mathtt{E}}}$ interchanged) yields that there is an isometry

$W \in \mathcal{L}(\mathcal{H}_{\mathtt{E}}; \mathcal{H}_{\tilde{\mathtt{E}}})$ such that $\overline{W}_0 = \mathbb{1} \otimes W$. We note that W_0 maps S surjectively onto $\tilde{S}$. Thus, (since isometries have closed ranges) if $\tilde{S}$ is dense, $\overline{W}_0$ is surjective and hence W is a unitary. This is the claim of the last sentence in the theorem.

It remains to verify Eq. (2). To this end, note that by definition $(\mathbb{1} \otimes W)((X \otimes \mathbb{1}_{\mathtt{E}})V - VX) = (X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})\tilde{V} - \tilde{V}X$, which can be expressed as

$$(X \otimes \mathbb{1}_{\tilde{\mathtt{E}}})((\mathbb{1} \otimes W)V - \tilde{V}) = ((\mathbb{1} \otimes W)V - \tilde{V})X.$$

Since this holds for all $X \in \mathcal{L}(\mathcal{H})$, Lemma 11 (with $\mathcal{A} = \mathcal{L}(\mathcal{H})$) tells us that there exists a vector $|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{\mathtt{E}}}$ such that $(\mathbb{1} \otimes W)V - \tilde{V} = -\mathbb{1} \otimes |\tilde{\psi}\rangle$. This is the first part of Eq. (2). To find the relation between K and $\tilde{K}$ we equate versions two of $L(X)$ in a) (with and without the tilde) and substitute $\tilde{V} = (\mathbb{1} \otimes W)V + \mathbb{1} \otimes |\tilde{\psi}\rangle$ After expanding the quadratic term and some cancellations and rearrangements, we arrive at

$$\hat{K}^\dagger X + X\hat{K} = \tilde{K}^\dagger X + X\tilde{K}, \quad \text{for all } X \in \mathcal{L}(\mathcal{H}),$$

with $\hat{K} = K + (\mathbb{1} \otimes \langle\tilde{\psi}|W)V + \frac{1}{2}\|\tilde{\psi}\|^2$. By Lemma 12 (with $\mathcal{H}_{\mathtt{B}} = \mathbb{C}$), there is $\mu \in \mathbb{R}$ such that $\tilde{K} = \hat{K} + i\mu$. This finishes the proof. □

The end