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Analytical derivatives of flexible multibody dynamics with the floating frame of reference formulation

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Abstract

In this paper, the analytical derivatives of flexible multibody dynamics with the floating frame of reference formulation are derived in a new way using the invariants and their sensitivities. This enables the decoupling of the sensitivity analysis of flexible multibody dynamics from the finite-element solver and guarantees high accuracy and efficiency of the sensitivity computations. The invariants are shown with both consistent and lumped mass approaches. The latter allows generality towards the formulation of a finite-element type, including beams, shells, and solids. The expressions are fully derived with lumped masses, showing for the first time the compensation term of inertia due to the non-consideration of the mass distribution with this approach. It is then shown that the expressions of the system parameters in the lumped case with the newly introduced inertia compensation term correspond to the general case, and, therefore, the derived approach and equations are of general nature. Crucial for the decoupling of the sensitivity analysis are the analytical derivatives of the system parameters that contain the derivatives of the invariants and whose analytical expressions are derived and provided here for the first time. The partial derivatives arise in the sensitivity analysis with both the direct differentiation method and the adjoint variable method, and the former is shown here. In addition, the partial derivatives arise in the Jacobian matrix of the nonlinear solver for the transient solution of flexible multibody systems.

Keywords Flexible multibody dynamics · Analytical sensitivity analysis · Design sensitivity analysis · Floating frame of reference formulation · Inertia shape integrals

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Nomenclature

$\bigcirc$ zeroth-order tensor or scalar
$\underline{\bigcirc}$ first-order tensor or vector
$\underline{\underline{\bigcirc}}$ second-order tensor or matrix
$\underline{\underline{\underline{\bigcirc}}}$ third-order tensor or 3D matrix
$\underline{\underline{\underline{\underline{\bigcirc}}}}$ fourth-order tensor or 4D matrix
$\dot{\bigcirc}$ first time derivative
$\ddot{\bigcirc}$ second time derivative
$\overline{\bigcirc}$ expressed in floating coordinates
$\underline{\underline{\tilde{\bigcirc}}}$ skew symmetric matrix
$\overline{\nabla}\bigcirc$ total derivative of $\bigcirc$ w.r.t. $\underline{\mathsf{x}}$
$\partial\bigcirc$ partial derivative of $\bigcirc$ w.r.t. $\underline{\mathsf{x}}$
${}^{\bullet}\partial\bigcirc$ partial derivative of $\bigcirc$ w.r.t. $\bullet$
${}^{\bullet}\mathrm{J}\bigcirc$ Jacobian of $\bigcirc$ w.r.t. $\bullet$
$\underline{\underline{d}}$ damping matrix
$\underline{e}$ unit vector
$\underline{\underline{e}}$ unit matrix (identity matrix)
$\underline{\underline{k}}$ stiffness matrix
m mass
$\underline{\underline{m}}$ mass matrix
$\underline{q}$ generalized position vector
$\underline{\dot{q}}$ generalized velocity vector
$\underline{\ddot{q}}$ generalized acceleration vector
$\underline{r}$ position vector to inertial frame expressed in intertial coordinates
t time
$\underline{\overline{u}}$ position vector to floating frame expressed in floating coordinates
$\underline{u}$ position vector to floating frame expressed in inertial coordinates
$\underline{\mathsf{x}}$ vector of design variables
$\underline{\underline{A}}$ rotation matrix
$\underline{\underline{B}}$ Boolean matrix
$\mathcal{F}(\bigcirc)$ function of $\bigcirc$
$\underline{\underline{G}}$ angular velocity matrix that relates $\underline{\omega}$ and $\underline{\dot{\theta}}$
$\overline{\mathcal{I}}$ invariant (inertia shape integral)
K kinetic energy
$\underline{Q}_{\mathrm{e}}$ generalized external force vector
$\underline{Q}_{\mathrm{v}}$ quadratic velocity force vector
$\underline{\underline{\overline{S}}}$ matrix of shape functions expressed in floating coordinates
$\overline{V}$ volume
$\underline{\zeta}$ vector of modal coordinates
$\underline{\theta}$ vector of orientation coordinates
$\underline{\lambda}$ vector of Lagrange multipliers
ρ density
$\underline{\tau}$ position vector of floating frame
$\underline{\chi}_{\mathrm{o}}$ center of mass of undeformed body
$\underline{\omega}$ angular velocity vector
$\underline{\underline{\Theta}}_{\mathrm{o}}$ inertia tensor of undeformed body
$\underline{\Phi}$ vector of kinematic constraints
$\underline{\underline{{}^{q}\mathrm{J}\Phi}}$ Jacobian matrix of kinematic constraints

$\underline{\underline{\overline{\Psi}}}$ modal matrix expressed in floating coordinates

1 Introduction

In this paper, the analytical sensitivities for flexible multibody dynamics with the floating frame of reference formulation are derived in a new way using invariants and their sensitivities. Sensitivity analysis is especially relevant for design optimization. Engineers aim for the best possible design for given requirements, and to enable this, design optimization is an algorithm-based design tool to achieve superior engineering designs. Structural optimization, defined by design optimization in concert with structural analysis via finite-element analysis, is well established, e.g., [1, 13, 20]. Design optimization of multibody systems goes back at least to [2, 3, 14, 15], and recent developments of design optimization with flexible multibody systems are reviewed in [12]. Gradient-based algorithms enable efficient design optimization despite the high computational effort related to flexible multibody dynamics [12, 31]. The driving motor is the efficient evaluation of gradients or design sensitivities. Available methods for design sensitivities include numerical and analytical methods and are reviewed in [19]. Because of their high accuracy and efficiency, analytical methods are most suitable for flexible multibody systems, and these include direct differentiation [6, 9, 25] and the adjoint variable method [4, 16, 22, 23].

The floating frame of reference formulation (FFRF) describes flexible bodies by the position and orientation of a reference frame and the flexible deformations [27]. The system vectors and matrices of FFRF are based on constant "ingredients". Specifically, these ingredients are the inertia shape integrals, which are commonly referred to as invariants in literature [7, 33, 35] and commercial software [8, 21]. It should be noted that these terms are not tensor invariants, but instead invariant with respect to time, i.e., remain constant during the dynamic simulation. The invariants, in turn, depend on the linear-elastic structural finite-element model and allow for the decoupling of the finite-element solver and the multibody solver. In this context, decoupling means the initial assembly of the finite-element model and the invariants prior to the multibody simulation, which do not have to be calculated again for a single design. The FFRF system parameters that require input from the finite-element model are the linear elastic stiffness matrix (for linear-elastic material models), the highly nonlinear mass matrix, and the highly nonlinear quadratic velocity force vector. Material damping can be modeled with proportional damping and thus also depends on the finite-element model. In the following, the finite-element-dependent system parameters containing the mass, damping, and stiffness matrices as well as the quadratic velocity force vector are referred to as structural system parameters. The generalized external force vector instead is loading specific and does not depend on the finite-element model. The kinematic constraints depend on the joint description and are also independent of the finite-element model.

To avoid the evaluation of the inertia shape integrals in the conventional FFRF, commercial software packages use lumped mass approaches to compute the invariants [8, 21, 27]. The evaluation of the invariants with the lumped mass approach is agnostic to the element type, while the conventional FFRF using a consistent mass matrix requires a separate derivation of the inertia shape integrals for each element type. Another option is the so-called nodal-based FFRF, where the evaluation of the inertia shape integrals is avoided altogether [34, 36]. Because of the widespread use and the mentioned agnosticity to the finite-element type, the lumped mass FFRF is used here. The derivation of the system parameters with the lumped mass approach is only partially shown in literature [27] and in the documentation of commercial software [8, 21] and therefore is provided here for the first time. The generality of the lumped mass approach is given by the compensation terms of inertia due to the non-consideration of the mass distribution with lumped masses, introduced here for the first time. It is shown that the lumped mass approach is the general case and therefore the derived method and equations are of general nature. The sensitivity analysis and the partial derivatives of the system parameters are based on the derived lumped mass approach and are shown with the singularity-free and the widespread used Euler parameters but can be extended to any orientation parametrization for generality.

In this work, the sensitivity analysis with respect to general design variables is of interest. This is carried out with the direct differentiation method [12, 29, 31]. A semi-analytical method for flexible multibody systems was introduced in [9], where the equations of motions are differentiated analytically, and the partial derivatives of the system parameters are computed numerically with forward differencing. To increase the accuracy and the efficiency of the method, the present work expands the analytical differentiation by a further level. Therefore, the partial derivatives of the system parameters depending on the finite-element model, i.e. the mass matrix, the damping matrix, the stiffness matrix and the quadratic velocity force vector are derived analytically and shown here for the first time. The analytical derivatives of the system parameters guarantee both high efficiency and high accuracy. The partial derivatives appear in the sensitivity analysis with the shown direct differentiation method and are partially required for the adjoint variable method as well. Applications that require the evaluation of the partial derivatives of the system parameters in addition to the design sensitivity analysis for design optimization include the uncertainty analysis and the Jacobian matrix of the nonlinear solver for the transient solution of flexible multibody systems, regardless of solving for positions, velocities or accelerations.

In the following, the equations of motion of flexible multibody systems are introduced, and the system parameters and invariants of the floating frame of reference formulation are fully derived with the lumped mass approach in § 2. In § 3, the equations of motion are differentiated with respect to the design variables for the sensitivity analysis and with respect to positions, velocities and accelerations for the Jacobians of the nonlinear solver. In § 4, the analytical expressions of the partial derivatives of the system parameters, derived in § 2 and required in § 3, are provided.

2 Equations of motion

The equations of motion of flexible multibody dynamics are given by the index-3 differential–algebraic equations,

$$\underline{\underline{m}}\,\underline{\ddot{q}} + \underline{\underline{d}}\,\underline{\dot{q}} + \underline{\underline{k}}\,\underline{q} + {}^{q}\underline{\underline{\mho\Phi}}^{\mathsf{T}}\,\underline{\lambda} = \underline{Q}_{\mathrm{e}} + \underline{Q}_{\mathrm{v}}, \tag{1}$$

$$\underline{\Phi} = \underline{0}, \tag{2}$$

where $\underline{q}$ is the vector of generalized positions, $\underline{\dot{q}}$ is the vector of generalized velocities, $\underline{\ddot{q}}$ is the vector of generalized accelerations, $\underline{\lambda}$ is the vector of Lagrange multipliers, $\underline{\underline{m}}$ is the mass matrix, $\underline{\underline{d}}$ is the damping matrix, $\underline{\underline{k}}$ is the stiffness matrix, $\underline{\Phi}$ is the vector of kinematic constraints, ${}^{q}\underline{\underline{\mho\Phi}}$ is the Jacobian matrix of the kinematic constraint given by the partial derivative of $\underline{\Phi}$ w.r.t. $\underline{q}$, $\underline{Q}_{\mathrm{e}}$ is the vector of generalized external forces, and $\underline{Q}_{\mathrm{v}}$ is the quadratic velocity force vector. Symbols with one underline $\underline{\bigcirc}$ denote a vector, symbols with two underlines $\underline{\underline{\bigcirc}}$ denote a matrix, and overdots denote the first $\dot{\bigcirc}$ and the second $\ddot{\bigcirc}$ derivative w.r.t. time.

In the following, the system parameters for the FFRF are derived. With FFRF, the most complicated terms are the inertia forces, including the mass matrix that is nonlinear in positions, and the quadratic velocity force vector that is nonlinear in positions and velocities. The complete derivation of these terms is shown with the widely used lumped mass approach as used in commercial software, see, e.g., [8, 21]. This allows calculating the invariants by sums instead of integrals that depend on the shape functions and therefore makes the calculation of the invariants independent of the finite element type and formulation.

2.1 Definitions and preparations

The generalized coordinates with FFRF are given by the position coordinates $\underline{\tau}$ and the rotation coordinates $\underline{\theta}$ of the reference frame and the flexible deformations of the body $\overline{\underline{q}}_{\mathrm{f}}$, leading to the generalized position vector given by

$$\underline{q} = \begin{bmatrix} \underline{\tau}^{\mathsf{T}} & \underline{\theta}^{\mathsf{T}} & \overline{\underline{q}}_{\mathrm{f}}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}. \tag{3}$$

In many applications, FFRF is used with the component mode synthesis (CMS) [5, 17, 18, 26] to limit the computational effort by reducing the degrees of freedom. The generalized position vector of FFRF with CMS is given by

$$\underline{q} \approx \begin{bmatrix} \underline{\tau}^{\mathsf{T}} & \underline{\theta}^{\mathsf{T}} & \underline{\zeta}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}, \tag{4}$$

where $\underline{\zeta}$ are the modal coordinates associated with CMS, and the flexible deformation is approximated by

$$\overline{\underline{q}}_{\mathrm{f}} \approx \overline{\underline{\underline{\Psi}}}\,\underline{\zeta}, \tag{5}$$

with the modal matrix $\overline{\underline{\underline{\Psi}}}$ containing column-wise the component modes that usually include eigenmodes and static modes of the flexible body. The overline on the symbols $\overline{\bigcirc}$ denotes that the coordinates of the quantity are expressed in the floating frame. The reduced coordinates shown in (4) are considered as the general form of the generalized position vector and will be used in the following. If no model reduction is used, the modal coordinates $\underline{\zeta}$ are replaced by the flexible coordinates $\overline{\underline{q}}_{\mathrm{f}}$ and the modal matrix $\overline{\underline{\underline{\Psi}}}$ is replaced by the identity matrix $\underline{\underline{e}}$. The flexible coordinates are given by

$$\overline{\underline{q}}_{\mathrm{f}} = \begin{bmatrix} \cdots & \overline{\underline{u}}_{\mathrm{f}}^{(j)\mathsf{T}} & \overline{\underline{\theta}}_{\mathrm{f}}^{(j)\mathsf{T}} & \cdots \end{bmatrix}^{\mathsf{T}}, \tag{6}$$

with the flexible nodal translations $\overline{\underline{u}}_{\mathrm{f}}^{(j)}$ and the flexible nodal rotations $\overline{\underline{\theta}}_{\mathrm{f}}^{(j)}$ of node j. The flexible nodal translation of one node is obtained by

$$\overline{\underline{u}}_{\mathrm{f}}^{(j)} = \underline{\underline{B}}_{\mathrm{t}}^{(j)}\overline{\underline{q}}_{\mathrm{f}}, \tag{7}$$

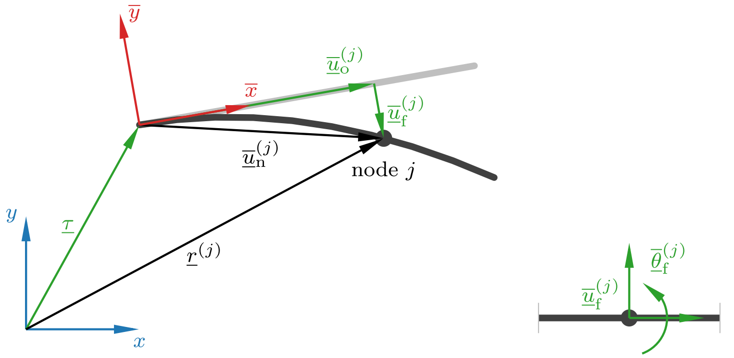
$$\approx \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)}\underline{\zeta}, \tag{8}$$

where Eq. (5) is used, and the translational modal matrix of the node j is given by

$$\overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)} = \underline{\underline{B}}_{\mathrm{t}}^{(j)}\overline{\underline{\underline{\Psi}}}, \tag{9}$$

and the translational Boolean matrix of the node j is defined by

$$\underline{\underline{B}}_{\mathrm{t}}^{(j)} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{e}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}. \tag{10}$$



(a) Global position of a node (b) Nodal coordinates

Fig. 1 Position coordinates on a flexible body with FFRF (colors online)

The flexible nodal rotations are obtained by

$$\underline{\overline{\theta}}_{\mathrm{f}}^{(j)} = \underline{\underline{B}}_{\mathrm{r}}^{(j)} \underline{\overline{q}}_{\mathrm{f}}, \tag{11}$$

$$\approx \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)} \underline{\zeta}, \tag{12}$$

where Eq. (5) has been used, and the rotational modal matrix of the node j is given by

$$\underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)} = \underline{\underline{B}}_{\mathrm{r}}^{(j)} \underline{\underline{\overline{\Psi}}}, \tag{13}$$

and the rotational Boolean matrix of the node j is defined by

$$\underline{\underline{B}}_{\mathrm{r}}^{(j)} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \underline{\underline{e}} & \cdots & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}. \tag{14}$$

As shown in Fig. 1, the position vector $\underline{r}$ of the node j is given by

$$\underline{r}^{(j)} = \underline{\tau} + \underline{\underline{A}}\,\underline{\overline{u}}_{\mathrm{n}}^{(j)}, \tag{15}$$

$$= \underline{\tau} + \underline{\underline{A}} \left(\underline{\overline{u}}_{\mathrm{o}}^{(j)} + \underline{\overline{u}}_{\mathrm{f}}^{(j)} \right), \tag{16}$$

where $\underline{\underline{A}}$ is the rotation matrix, and $\underline{\overline{u}}_{\mathrm{n}}^{(j)}$ is the nodal position of node j relative to the floating frame expressed in floating coordinates consisting of the undeformed position $\underline{\overline{u}}_{\mathrm{o}}^{(j)}$ and the flexible deformation $\underline{\overline{u}}_{\mathrm{f}}^{(j)}$.

The translational velocity of the node j on a flexible body is given by the time derivative of the position, leading to

$$\dot{\underline{r}}^{(j)} = \dot{\underline{\tau}} + \dot{\underline{\underline{A}}}\,\underline{\overline{u}}_{\mathrm{n}}^{(j)} + \underline{\underline{A}}\,\dot{\underline{\overline{u}}}_{\mathrm{f}}^{(j)}, \tag{17}$$

$$= \dot{\underline{\tau}} - \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\underline{\overline{G}}}\,\dot{\underline{\theta}} + \underline{\underline{A}}\,\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)} \dot{\underline{\zeta}}, \tag{18}$$

$$= \begin{bmatrix} \underline{\underline{e}} & -\underline{\underline{A}}\,\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\underline{\overline{G}}} & \underline{\underline{A}}\,\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)} \end{bmatrix} \begin{bmatrix} \dot{\underline{\tau}} \\ \dot{\underline{\theta}} \\ \dot{\underline{\zeta}} \end{bmatrix}, \tag{19}$$

$$= \underline{\underline{L}}_{\mathrm{t}}^{(j)} \dot{\underline{q}}, \tag{20}$$

where $\dot{\underline{\underline{A}}}\,\underline{\overline{u}}_{\mathrm{n}}^{(j)} = -\underline{\underline{A}}\,\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\overline{G}}}\,\underline{\dot{\theta}}$ is used as shown, e.g., in [27], $\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}$ is the skew-symmetric matrix of the vector $\underline{\overline{u}}_{\mathrm{n}}^{(j)}$, $\underline{\underline{\overline{G}}}$ is the angular velocity matrix expressed in floating coordinates that relates the angular velocity of the floating frame to the time derivative of the orientation parametrization $\underline{\overline{\omega}}_{\mathrm{o}} = \underline{\underline{\overline{G}}}\,\underline{\dot{\theta}}$ and in analogy to [33],

$$\underline{\underline{L}}_{\mathrm{t}}^{(j)} = \underline{\underline{{}^{q}\mathrm{J}r^{(j)}}} = \frac{\partial \underline{r}^{(j)}}{\partial \underline{q}}. \tag{21}$$

A further differentiation w.r.t. time leads to the acceleration of node j on the flexible body

$$\underline{\ddot{r}}^{(j)} = \dot{\underline{\underline{L}}}_{\mathrm{t}}^{(j)}\underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{t}}^{(j)}\underline{\ddot{q}}, \tag{22}$$

where in analogy to [33],

$$\dot{\underline{\underline{L}}}_{\mathrm{t}} = \left[\underline{\underline{0}} \quad -\underline{\underline{A}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\overline{G}}} + \dot{\underline{\underline{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)}\underline{\underline{\overline{G}}} + \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\dot{\underline{\underline{\overline{G}}}}\right) \quad \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}\right]. \tag{23}$$

The angular velocity vector of the node j on a flexible body $\underline{\overline{\omega}}_{\mathrm{n}}^{(j)}$ is given by

$$\underline{\overline{\omega}}_{\mathrm{n}}^{(j)} = \underline{\overline{\omega}}_{\mathrm{o}} + \underline{\overline{\omega}}_{\mathrm{f}}^{(j)}, \tag{24}$$

$$= \underline{\underline{\overline{G}}}\,\underline{\dot{\theta}} + \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)}\underline{\dot{\zeta}}, \tag{25}$$

$$= \left[\underline{\underline{0}} \quad \underline{\underline{\overline{G}}} \quad \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)}\right]\begin{bmatrix}\underline{\dot{\tau}}\\ \underline{\dot{\theta}}\\ \underline{\dot{\zeta}}\end{bmatrix} \tag{26}$$

$$= \underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}}, \tag{27}$$

where $\underline{\overline{\omega}}_{\mathrm{f}}^{(j)} = \dot{\underline{\overline{\theta}}}_{\mathrm{f}}^{(j)}$ is used, $\underline{\overline{\omega}}_{\mathrm{o}}$ is the angular velocity vector of the floating frame relative to the inertial frame expressed in floating coordinates, $\underline{\overline{\omega}}_{\mathrm{f}}^{(j)}$ is the flexible angular velocity of node j relative to the floating frame expressed in floating coordinates, and

$$\underline{\underline{L}}_{\mathrm{r}}^{(j)} = \left[\underline{\underline{0}} \quad \underline{\underline{\overline{G}}} \quad \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)}\right]. \tag{28}$$

A further differentiation w.r.t. time leads to the angular acceleration of the node j on the flexible body,

$$\dot{\underline{\overline{\omega}}}_{\mathrm{n}}^{(j)} = \dot{\underline{\underline{L}}}_{\mathrm{r}}^{(j)}\underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\ddot{q}}, \tag{29}$$

where

$$\dot{\underline{\underline{L}}}_{\mathrm{r}}^{(j)} = \left[\underline{\underline{0}} \quad \dot{\underline{\underline{\overline{G}}}} \quad \underline{\underline{0}}\right]. \tag{30}$$

2.2 Inertia forces

The kinetic energy K is given by a translational and a rotational contribution,

$$K = K_{\mathrm{t}} + K_{\mathrm{r}}, \tag{31}$$

and is approximated for the lumped mass approach by

$$K_{\mathrm{t}} \approx \frac{1}{2}\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\dot{r}}^{(j)\mathsf{T}} \underline{\dot{r}}^{(j)}, \tag{32}$$

$$K_{\mathrm{r}} \approx \frac{1}{2}\sum_{j=1}^{n_{\mathrm{n}}} \overline{\underline{\omega}}_{\mathrm{n}}^{(j)\mathsf{T}} \overline{\underline{\underline{\Theta}}}_{\mathrm{n}}^{(j)} \overline{\underline{\omega}}_{\mathrm{n}}^{(j)}, \tag{33}$$

with the nodal mass $m^{(j)}$, the velocity vector of the node $\underline{\dot{r}}^{(j)}$, the angular velocity of the node $\overline{\underline{\omega}}_{\mathrm{n}}^{(j)}$ and the nodal inertia tensor $\overline{\underline{\underline{\Theta}}}_{\mathrm{n}}^{(j)}$. The translational contribution of the kinetic energy K_{t} includes in $\underline{\dot{r}}^{(j)}$ the translational and rotational contribution of the floating frame and the flexible displacements of the node, see Fig. 1. The rotational contribution of the kinetic energy K_{r} includes the contribution of the angular velocity of the node itself as shown in Eq. (24) and Fig. 1.

The abstract derivation suggested by [33, 35] gives the following expressions for the translational part

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K_{\mathrm{t}}}{\partial \underline{\dot{q}}^{\mathsf{T}}}\right) - \frac{\partial K_{\mathrm{t}}}{\partial \underline{q}^{\mathsf{T}}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n_{\mathrm{n}}} \frac{\partial K_{\mathrm{t}}}{\partial \underline{\dot{r}}^{(j)}} \frac{\partial \underline{\dot{r}}^{(j)}}{\partial \underline{\dot{q}}}\right)^{\mathsf{T}} - \left(\sum_{j=1}^{n_{\mathrm{n}}} \frac{\partial K_{\mathrm{t}}}{\partial \underline{\dot{r}}^{(j)}} \frac{\partial \underline{\dot{r}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}}, \tag{34}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\dot{r}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)}\right)^{\mathsf{T}} - \left(\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\dot{r}}^{(j)\mathsf{T}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \underline{r}^{(j)}}{\partial \underline{q}}\right)\right)^{\mathsf{T}}, \tag{35}$$

$$= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\dot{q}}\right) - \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \underline{r}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\dot{q}}, \tag{36}$$

$$= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \left(\underline{\underline{\dot{L}}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\dot{L}}}_{\mathrm{t}}^{(j)} \underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\ddot{q}}\right) +$$

$$- \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\dot{L}}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\dot{q}}, \tag{37}$$

$$= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\dot{L}}}_{\mathrm{t}}^{(j)} \underline{\dot{q}} + \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)} \underline{\ddot{q}}, \tag{38}$$

$$= -\underline{Q}_{\mathrm{v}}^{\mathrm{tran}} + \underline{\underline{m}}^{\mathrm{tran}} \underline{\ddot{q}}. \tag{39}$$

This leads to the mass matrix of the translational contribution

$$\underline{\underline{m}}^{\mathrm{tran}} = \begin{bmatrix} \underline{\underline{m}}_{\mathrm{tt}}^{\mathrm{tran}} & \underline{\underline{m}}_{\mathrm{tr}}^{\mathrm{tran}} & \underline{\underline{m}}_{\mathrm{tf}}^{\mathrm{tran}} \\ & \underline{\underline{m}}_{\mathrm{rr}}^{\mathrm{tran}} & \underline{\underline{m}}_{\mathrm{rf}}^{\mathrm{tran}} \\ \mathrm{sym.} & & \underline{\underline{m}}_{\mathrm{ff}}^{\mathrm{tran}} \end{bmatrix}, \tag{40}$$

$$= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{L}}_{\mathrm{t}}^{(j)}, \tag{41}$$

$$
= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \begin{bmatrix} \underline{\underline{e}} & -\underline{\underline{A}}\,\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\overline{G}}} & \underline{\underline{A}}\,\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)} \\ & \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\overline{G}}} & -\underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)} \\ \text{sym.} & & \underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)} \end{bmatrix}, \tag{42}
$$

and the quadratic velocity force vector of the translational contribution

$$
\underline{Q}_{\mathrm{v}}^{\mathrm{tran}} = \begin{bmatrix} \underline{Q}_{\mathrm{v,t}}^{\mathrm{tran}} \\ \underline{Q}_{\mathrm{v,r}}^{\mathrm{tran}} \\ \underline{Q}_{\mathrm{v,f}}^{\mathrm{tran}} \end{bmatrix}, \tag{43}
$$

$$
= -\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{L}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\dot{L}}}_{\mathrm{t}}^{(j)} \underline{\dot{q}}, \tag{44}
$$

$$
= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \begin{bmatrix} -\underline{\underline{A}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\overline{u}}_{\mathrm{n}}^{(j)} + 2\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\dot{\overline{u}}}_{\mathrm{f}}^{(j)} - \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}\right) \\ \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\overline{u}}_{\mathrm{n}}^{(j)} + 2\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\dot{\overline{u}}}_{\mathrm{f}}^{(j)} - \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}\right) \\ -\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\overline{u}}_{\mathrm{n}}^{(j)} + 2\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\dot{\overline{u}}}_{\mathrm{f}}^{(j)} - \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}\right) \end{bmatrix}, \tag{45}
$$

$$
= \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \begin{bmatrix} \underline{\underline{A}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}\right) \\ -\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}\right) \\ \left(\underline{\underline{e}}_{\zeta} \otimes \underline{\overline{\omega}}_{\mathrm{o}}\right)^{\mathsf{T}}\underline{\underline{\tilde{\overline{\Psi}}}}_{\mathrm{t}}^{(j)\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}}\underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)}\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}}\underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}} \end{bmatrix}. \tag{46}
$$

Analogously to the translational part, following the abstract derivation suggested by [33, 35] gives the expressions for the rotational part,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K_{\mathrm{r}}}{\partial \underline{\dot{q}}^{\mathsf{T}}}\right) - \frac{\partial K_{\mathrm{r}}}{\partial \underline{q}^{\mathsf{T}}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n_{\mathrm{n}}} \frac{\partial K_{\mathrm{r}}}{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}} \frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{\dot{q}}}\right)^{\mathsf{T}} - \left(\sum_{j=1}^{n_{\mathrm{n}}} \frac{\partial K_{\mathrm{r}}}{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}} \frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}}, \tag{47}
$$

$$
= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n_{\mathrm{n}}} \left(\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\overline{\omega}}_{\mathrm{n}}^{(j)}\right)^{\mathsf{T}} \underline{\underline{L}}_{\mathrm{r}}^{(j)}\right)^{\mathsf{T}} - \left(\sum_{j=1}^{n_{\mathrm{n}}} \left(\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\overline{\omega}}_{\mathrm{n}}^{(j)}\right)^{\mathsf{T}} \frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}}, \tag{48}
$$

$$
= \sum_{j=1}^{n_{\mathrm{n}}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\underline{L}}_{\mathrm{r}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}}\right) - \sum_{j=1}^{n_{\mathrm{n}}} \left(\frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}}, \tag{49}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n_{\mathrm{n}}} \left(\underline{\underline{\dot{L}}}_{\mathrm{r}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{r}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{L}}}_{\mathrm{r}}^{(j)}\underline{\dot{q}} + \underline{\underline{L}}_{\mathrm{r}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\ddot{q}}\right) + \\
&\quad - \sum_{j=1}^{n_{\mathrm{n}}} \left(\frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}},
\end{aligned} \tag{50}
$$

$$
= \sum_{j=1}^{n_{\mathrm{n}}} \left(\underline{\underline{\dot{L}}}_{\mathrm{r}}^{(j)} - \frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}}\right)^{\mathsf{T}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{L}}_{\mathrm{r}}^{(j)}\underline{\dot{q}} + \sum_{j=1}^{n_{\mathrm{n}}} \underline{\underline{L}}_{\mathrm{r}}^{(j)\mathsf{T}}\underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)}\underline{\underline{\dot{L}}}_{\mathrm{r}}^{(j)}\underline{\dot{q}} +
$$

$$+\sum_{j=1}^{n_\mathrm{n}} \underline{\underline{L}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{L}}_\mathrm{r}^{(j)} \ddot{\underline{q}}, \tag{51}$$

$$= -\underline{Q}_\mathrm{v}^\mathrm{rot} + \underline{\underline{m}}^\mathrm{rot} \ddot{\underline{q}}, \tag{52}$$

where the following partial derivatives have been used

$$\frac{\partial \underline{\overline{\omega}}_\mathrm{n}^{(j)}}{\partial \underline{q}} = \begin{bmatrix} \underline{\underline{0}} & \frac{\partial \underline{\overline{\omega}}_\mathrm{o}^{(j)}}{\partial \underline{\theta}} & \underline{\underline{0}} \end{bmatrix}. \tag{53}$$

This leads to the mass matrix of the rotational contribution

$$\underline{\underline{m}}^\mathrm{rot} = \begin{bmatrix} \underline{\underline{m}}_\mathrm{tt}^\mathrm{rot} & \underline{\underline{m}}_\mathrm{tr}^\mathrm{rot} & \underline{\underline{m}}_\mathrm{tf}^\mathrm{rot} \\ & \underline{\underline{m}}_\mathrm{rr}^\mathrm{rot} & \underline{\underline{m}}_\mathrm{rf}^\mathrm{rot} \\ \text{sym.} & & \underline{\underline{m}}_\mathrm{ff}^\mathrm{rot} \end{bmatrix}, \tag{54}$$

$$= \sum_{j=1}^{n_\mathrm{n}} \underline{\underline{L}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{L}}_\mathrm{r}^{(j)}, \tag{55}$$

$$= \sum_{j=1}^{n_\mathrm{n}} \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ & \underline{\underline{\overline{G}}}^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{G}}} & \underline{\underline{\overline{G}}}^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)} \\ \text{sym.} & & \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)} \end{bmatrix}, \tag{56}$$

and the quadratic velocity force vector of the rotational contribution

$$\underline{Q}_\mathrm{v}^\mathrm{rot} = \begin{bmatrix} \underline{Q}_\mathrm{v,t}^\mathrm{rot} \\ \underline{Q}_\mathrm{v,r}^\mathrm{rot} \\ \underline{Q}_\mathrm{v,f}^\mathrm{rot} \end{bmatrix}, \tag{57}$$

$$= -\sum_{j=1}^{n_\mathrm{n}} \left(\dot{\underline{\underline{L}}}_\mathrm{r}^{(j)} - \frac{\partial \underline{\overline{\omega}}_\mathrm{n}^{(j)}}{\partial \underline{q}} \right)^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{L}}_\mathrm{r}^{(j)} \dot{\underline{q}} - \sum_{j=1}^{n_\mathrm{n}} \underline{\underline{L}}_\mathrm{r}{}^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \dot{\underline{\underline{L}}}_\mathrm{r}^{(j)} \dot{\underline{q}}, \tag{58}$$

$$= -\sum_{j=1}^{n_\mathrm{n}} \begin{bmatrix} \underline{0} \\ \left(\dot{\underline{\underline{\overline{G}}}} - \frac{\partial \underline{\overline{\omega}}_\mathrm{o}^{(j)}}{\partial \underline{\theta}} \right)^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \left(\underline{\overline{\omega}}_\mathrm{o} + \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)} \dot{\underline{\zeta}} \right) + \underline{\underline{\overline{G}}}^\mathsf{T} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \dot{\underline{\underline{\overline{G}}}} \dot{\underline{\theta}} \\ \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \dot{\underline{\underline{\overline{G}}}} \dot{\underline{\theta}} \end{bmatrix}. \tag{59}$$

Assembling the translational and rotational contribution leads to the full FFRF mass matrix

$$\underline{\underline{m}} = \begin{bmatrix} \underline{\underline{m}}_\mathrm{tt} & \underline{\underline{m}}_\mathrm{tr} & \underline{\underline{m}}_\mathrm{tf} \\ & \underline{\underline{m}}_\mathrm{rr} & \underline{\underline{m}}_\mathrm{rf} \\ \text{sym.} & & \underline{\underline{m}}_\mathrm{ff} \end{bmatrix}, \tag{60}$$

$$= \underline{\underline{m}}^\mathrm{tran} + \underline{\underline{m}}^\mathrm{rot}, \tag{61}$$

$$= \sum_{j=1}^{n_\mathrm{n}} \begin{bmatrix} m^{(j)} \underline{\underline{e}} & -\underline{\underline{A}}\, m^{(j)} \tilde{\underline{\underline{\overline{u}}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{G}}} & \underline{\underline{A}}\, m^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{t}^{(j)} \\ & \underline{\underline{\overline{G}}}^\mathsf{T} \left(m^{(j)} \tilde{\underline{\underline{\overline{u}}}}_\mathrm{n}^{(j)\mathsf{T}} \tilde{\underline{\underline{\overline{u}}}}_\mathrm{n}^{(j)} + \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \right) \underline{\underline{\overline{G}}} & -\underline{\underline{\overline{G}}}^\mathsf{T} \left(m^{(j)} \tilde{\underline{\underline{\overline{u}}}}_\mathrm{n}^{(j)\mathsf{T}} \underline{\underline{\overline{\Psi}}}_\mathrm{t}^{(j)} - \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)} \right) \\ \text{sym.} & & m^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{t}^{(j)\mathsf{T}} \underline{\underline{\overline{\Psi}}}_\mathrm{t}^{(j)} + \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\overline{\Psi}}}_\mathrm{r}^{(j)} \end{bmatrix}, \tag{62}$$

where the node-dependent terms are isolated with the expression shown in [35] for

$$\underline{\underline{\tilde{u}}}_{\mathrm{n}}^{(j)} = \underline{\underline{\tilde{u}}}_{\mathrm{o}}^{(j)} + \underline{\underline{\tilde{u}}}_{\mathrm{f}}^{(j)}, \tag{63}$$

$$= \underline{\underline{\tilde{u}}}_{\mathrm{o}}^{(j)} + \underline{\underline{\tilde{\Psi}}}_{\mathrm{t}}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right), \tag{64}$$

where $\underline{\underline{\tilde{\Psi}}}_{\mathrm{t}}^{(j)}$ contains the skew-symmetric matrices of the columns of $\underline{\underline{\Psi}}_{\mathrm{t}}^{(j)}$ [35], $\otimes$ denotes Kronecker's product, and $\underline{\underline{e}}$ is the identity matrix.

The term $\underline{\underline{m}}_{\mathrm{tt}}$ is given by

$$\underline{\underline{m}}_{\mathrm{tt}} = \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{e}}, \tag{65}$$

$$= m\, \underline{\underline{e}}, \tag{66}$$

where the total mass of the body m is obtained by

$$m\, \underline{\underline{e}} = \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{e}}, \quad \in \mathbb{R}^{3\times 3}. \tag{67}$$

The term $\underline{\underline{m}}_{\mathrm{tr}}$ is given by

$$\underline{\underline{m}}_{\mathrm{tr}} = -\underline{\underline{A}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{u}}}_{\mathrm{n}}^{(j)} \underline{\underline{\overline{G}}}, \tag{68}$$

$$= -\underline{\underline{A}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \left(\underline{\underline{\tilde{u}}}_{\mathrm{o}}^{(j)} + \underline{\underline{\tilde{\Psi}}}_{\mathrm{t}}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\overline{G}}}, \tag{69}$$

$$= -\underline{\underline{A}} \sum_{j=1}^{n_{\mathrm{n}}} \left(m^{(j)} \underline{\underline{\tilde{u}}}_{\mathrm{o}}^{(j)} + m^{(j)} \underline{\underline{\tilde{\Psi}}}_{\mathrm{t}}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\overline{G}}}, \tag{70}$$

$$= -\underline{\underline{A}} \left(m\, \underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\overline{G}}}, \tag{71}$$

where the center of mass of the undeformed body w.r.t. the floating frame $\underline{\chi}_{\mathrm{o}}$, and its skew-symmetric matrix $\underline{\underline{\tilde{\chi}}}_{\mathrm{o}}$ are given by

$$\underline{\chi}_{\mathrm{o}} = \frac{1}{m} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\overline{u}}_{\mathrm{o}}^{(j)}, \quad \in \mathbb{R}^{3\times 1}, \tag{72}$$

$$\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} = \frac{1}{m} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{u}}}_{\mathrm{o}}^{(j)}, \quad \in \mathbb{R}^{3\times 3}, \tag{73}$$

and the invariants $\underline{\underline{\mathcal{I}}}_{\psi}$ and $\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}$ are defined by

$$\underline{\underline{\mathcal{I}}}_{\psi} = \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\Psi}}_{\mathrm{t}}^{(j)}, \quad \in \mathbb{R}^{3\times n_{\mathrm{m}}}, \tag{74}$$

$$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}} = \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)}, \quad \in \mathbb{R}^{3\times 3n_\mathrm{m}}. \tag{75}$$

The term $\underline{\underline{m}}_\mathrm{tf}$ is given by

$$\underline{\underline{m}}_\mathrm{tf} = \underline{\underline{A}} \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \overline{\underline{\underline{\Psi}}}_\mathrm{t}^{(j)}, \tag{76}$$

$$= \underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\psi}, \tag{77}$$

where the invariant $\underline{\underline{\mathcal{I}}}_{\psi}$ has already been defined.

The term $\underline{\underline{m}}_\mathrm{rr}$ is given by

$$\underline{\underline{m}}_\mathrm{rr} = \overline{\underline{\underline{G}}}^\mathsf{T} \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{n}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{n}^{(j)} + \overline{\underline{\underline{\Theta}}}_\mathrm{n}^{(j)} \right) \overline{\underline{\underline{G}}}, \tag{78}$$

$$= \overline{\underline{\underline{G}}}^\mathsf{T} \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \left(\tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)} + \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right)^\mathsf{T} \left(\tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)} + \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) + \overline{\underline{\underline{\Theta}}}_\mathrm{n}^{(j)} \right) \overline{\underline{\underline{G}}}, \tag{79}$$

$$\begin{aligned} = \overline{\underline{\underline{G}}}^\mathsf{T} \sum_{j=1}^{n_\mathrm{n}} \Bigg(& m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)} + \overline{\underline{\underline{\Theta}}}_\mathrm{n}^{(j)} + \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) + \\ & + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^\mathsf{T} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^\mathsf{T} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \Bigg) \overline{\underline{\underline{G}}}, \end{aligned} \tag{80}$$

$$= \overline{\underline{\underline{G}}}^\mathsf{T} \left(\underline{\underline{\Theta}}_\mathrm{o} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^\mathsf{T} \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}}^\mathsf{T} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^\mathsf{T} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \overline{\underline{\underline{G}}}, \tag{81}$$

where the inertia tensor of the undeformed body expressed in the floating frame is given by

$$\underline{\underline{\Theta}}_\mathrm{o} = \underline{\underline{\Theta}}_\mathrm{o,t} + \underline{\underline{\Theta}}_\mathrm{o,r}, \tag{82}$$

$$= \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)} + \sum_{j=1}^{n_\mathrm{n}} \overline{\underline{\underline{\Theta}}}_\mathrm{n}^{(j)}, \quad \in \mathbb{R}^{3\times 3}, \tag{83}$$

and the invariants $\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}}$ and $\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}$ are defined by

$$\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}} = \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{o}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)}, \quad \in \mathbb{R}^{n_\mathrm{m}\times 3n_\mathrm{m}}, \tag{84}$$

$$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} = \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)\,\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_\mathrm{t}^{(j)}, \quad \in \mathbb{R}^{3n_\mathrm{m}\times 3n_\mathrm{m}}. \tag{85}$$

The term $\underline{\underline{m}}_\mathrm{rf}$ is given by

$$\underline{\underline{m}}_\mathrm{rf} = -\overline{\underline{\underline{G}}}^\mathsf{T} \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_\mathrm{n}^{(j)\,\mathsf{T}} \overline{\underline{\underline{\Psi}}}_\mathrm{t}^{(j)} - \overline{\underline{\underline{\Theta}}}_\mathrm{n}^{(j)} \overline{\underline{\underline{\Psi}}}_\mathrm{r}^{(j)} \right), \tag{86}$$

$$= -\underline{\underline{\overline{G}}}^{\mathsf{T}} \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \left(\underline{\underline{\tilde{u}}}_\mathrm{o}^{(j)} + \underline{\underline{\tilde{\Psi}}}_\mathrm{t}^{(j)} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right)^{\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} - \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\Psi}}_\mathrm{r}^{(j)} \right), \tag{87}$$

$$= -\underline{\underline{\overline{G}}}^{\mathsf{T}} \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \underline{\underline{\tilde{u}}}_\mathrm{o}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} m^{(j)} \underline{\underline{\tilde{\Psi}}}_\mathrm{t}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} - \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\Psi}}_\mathrm{r}^{(j)} \right), \tag{88}$$

$$= -\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi} \right), \tag{89}$$

where the invariants $\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi}$ and $\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi}$ are defined by

$$\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi} = \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi,\mathrm{t}} - \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi,\mathrm{r}}, \tag{90}$$

$$= \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \underline{\underline{\tilde{u}}}_\mathrm{o}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} - \sum_{j=1}^{n_\mathrm{n}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\Psi}}_\mathrm{r}^{(j)}, \quad \in \mathbb{R}^{3\times n_\mathrm{m}}, \tag{91}$$

$$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi} = \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \underline{\underline{\tilde{\Psi}}}_\mathrm{t}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)}, \quad \in \mathbb{R}^{3n_\mathrm{m}\times n_\mathrm{m}}. \tag{92}$$

The term $\underline{\underline{m}}_\mathrm{ff}$ is given by

$$\underline{\underline{m}}_\mathrm{ff} = \sum_{j=1}^{n_\mathrm{n}} \left(m^{(j)} \underline{\underline{\Psi}}_\mathrm{t}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} + \underline{\underline{\Psi}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\Psi}}_\mathrm{r}^{(j)} \right), \tag{93}$$

$$= \underline{\underline{\mathcal{I}}}_{\psi\psi}, \tag{94}$$

where the invariant $\underline{\underline{\mathcal{I}}}_{\psi\psi}$ is defined by

$$\underline{\underline{\mathcal{I}}}_{\psi\psi} = \sum_{j=1}^{n_\mathrm{n}} m^{(j)} \underline{\underline{\Psi}}_\mathrm{t}^{(j)\mathsf{T}} \underline{\underline{\Psi}}_\mathrm{t}^{(j)} + \sum_{j=1}^{n_\mathrm{n}} \underline{\underline{\Psi}}_\mathrm{r}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_\mathrm{n}^{(j)} \underline{\underline{\Psi}}_\mathrm{r}^{(j)}, \quad \in \mathbb{R}^{n_\mathrm{m}\times n_\mathrm{m}}. \tag{95}$$

with the components expressed in terms of the invariants by

$$\underline{\underline{m}}_\mathrm{tt} = m\, \underline{\underline{e}}, \tag{96}$$

$$\underline{\underline{m}}_\mathrm{tr} = -\underline{\underline{A}} \left(m\, \underline{\underline{\tilde{\chi}}}_\mathrm{o} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\overline{G}}}, \tag{97}$$

$$\underline{\underline{m}}_\mathrm{tf} = \underline{\underline{A}}\, \underline{\underline{\mathcal{I}}}_{\psi}, \tag{98}$$

$$\underline{\underline{m}}_\mathrm{rr} = \underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\overline{\Theta}}}_\mathrm{o} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\overline{G}}}, \tag{99}$$

$$\underline{\underline{m}}_\mathrm{rf} = -\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_\mathrm{o}\psi} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi} \right), \tag{100}$$

$$\underline{\underline{m}}_\mathrm{ff} = \underline{\underline{\mathcal{I}}}_{\psi\psi}, \tag{101}$$

Assembling the translational and rotational contribution of the quadratic velocity force vector leads to the full expressions,

$$\underline{Q}_{\mathrm{v}} = \begin{bmatrix} \underline{Q}_{\mathrm{v,t}} \\ \underline{Q}_{\mathrm{v,r}} \\ \underline{Q}_{\mathrm{v,f}} \end{bmatrix}, \tag{102}$$

$$= \underline{Q}_{\mathrm{v}}^{\mathrm{tran}} + \underline{Q}_{\mathrm{v}}^{\mathrm{rot}}, \tag{103}$$

where the components are given by

$$\underline{Q}_{\mathrm{v,t}} = \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\tilde{\overline{u}}}_{\mathrm{n}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{A}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{A}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}, \tag{104}$$

$$\begin{aligned} \underline{Q}_{\mathrm{v,r}} = & -\underline{\underline{\overline{G}}}^{\mathsf{T}} \underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} - 2\underline{\underline{\overline{G}}}^{\mathsf{T}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}} \underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} + \\ & - \underline{\underline{\overline{G}}}^{\mathsf{T}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)\mathsf{T}} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}} - \underline{\underline{\overline{G}}}^{\mathsf{T}} \sum_{j=1}^{n_{\mathrm{n}}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)} \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}} + \\ & - \sum_{j=1}^{n_{\mathrm{n}}} \left(\underline{\underline{\dot{\overline{G}}}} - \frac{\partial \underline{\overline{\omega}}_{\mathrm{o}}^{(j)}}{\partial \underline{\theta}} \right)^{\mathsf{T}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)} \left(\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)} \underline{\dot{\zeta}} \right), \end{aligned} \tag{105}$$

$$\begin{aligned} \underline{Q}_{\mathrm{v,f}} = & \left(\underline{e}_{\zeta} \otimes \underline{\overline{\omega}}_{\mathrm{o}} \right)^{\mathsf{T}} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\tilde{\overline{\Psi}}}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} + 2 \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)} \underline{\overline{\omega}}_{\mathrm{o}} + \\ & + \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{\overline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}} \underline{\underline{\tilde{\overline{u}}}}_{\mathrm{n}}^{(j)} \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}} - \sum_{j=1}^{n_{\mathrm{n}}} \underline{\underline{\overline{\Psi}}}_{\mathrm{r}}^{(j)\mathsf{T}} \underline{\underline{\overline{\Theta}}}_{\mathrm{n}}^{(j)} \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}. \end{aligned} \tag{106}$$

Isolating the node-dependent expressions as shown in [35] for positions by Eq. (64) and for velocities by

$$\underline{\underline{\dot{\tilde{\overline{u}}}}}_{\mathrm{f}}^{(j)} = \underline{\underline{\tilde{\overline{\Psi}}}}_{\mathrm{t}}^{(j)} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}} \right) \tag{107}$$

and using the introduced definitions of the invariants lead to the full expression of the quadratic velocity force vector in terms of the invariants given by

$$\begin{aligned} \underline{Q}_{\mathrm{v,t}} = & \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}} \left(m\, \underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}} \right) \underline{\overline{\omega}}_{\mathrm{o}} + \\ & + \underline{\underline{A}} \left(m\, \underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}}, \end{aligned} \tag{108}$$

$$\begin{aligned} \underline{Q}_{\mathrm{v,r}} = & -\underline{\underline{\overline{G}}}^{\mathsf{T}} \underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}} \left(\underline{\underline{\Theta}}_{\mathrm{o,t}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + \\ & - 2\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + \end{aligned}$$

$$-\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\Theta}}_{\mathrm{o}}+\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)+\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}+\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\dot{\underline{\underline{\overline{G}}}}\,\underline{\dot{\theta}}+$$

$$-\left(\dot{\underline{\underline{\overline{G}}}}-\frac{\partial\underline{\overline{\omega}}_{\mathrm{o}}^{(j)}}{\partial\underline{\theta}}\right)^{\mathsf{T}}\left(\underline{\underline{\Theta}}_{\mathrm{o,r}}\underline{\overline{\omega}}_{\mathrm{o}}+\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi,\mathrm{r}}\underline{\dot{\zeta}}\right), \tag{109}$$

$$\underline{Q}_{\mathrm{v,f}}=\left(\underline{\underline{e}}_{\zeta}\otimes\underline{\overline{\omega}}_{\mathrm{o}}\right)^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}+\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}}+2\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\underline{\overline{\omega}}_{\mathrm{o}}+$$

$$+\left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi}+\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\dot{\underline{\underline{\overline{G}}}}\,\underline{\dot{\theta}}, \tag{110}$$

where blue highlighted terms are only present for lumped mass approach, and red highlighted terms are not needed when using Euler parameters (colors online). In fact, when comparing the derived Eqs. (96)–(101) and Eqs. (108)–(110) using the lumped mass approach with the derivations using the integral formulation in [24, 27, 28] and the nodal-based FFRF in [32, 35], it becomes clear that the obtained equations are identical with the exception of one additional term in Eq. (109) highlighted in blue (colors online), see § 2.2.2 and Table 1 for more details. This term appears only with the lumped mass approach and is interpreted as the compensation term of inertia due to the non-consideration of the mass distribution when using the lumped mass approach. This additional inertia compensation term originates in the model for the kinetic energy with a translational and a rotational contribution when using lumped masses, see Eq. (31). This is not necessary with the consistent approach because the terms off the diagonal of the consistent mass matrix are occupied, and thus the mass distribution is already considered. In the rotational contribution of the kinetic energy with lumped masses, the contribution of the nodal inertia tensor and the angular velocity of the node itself is considered, including the angular velocity of the floating frame and the flexible angular velocity of the node, see Eq. (24) and Fig. 1. These lead to the additional inertia compensation term with the lumped mass approach.

The shown derivation is general for any orientation parametrization. In the following, Euler parameters will be used since they are free from singularities and for their widespread use. In addition, the Euler parameters simplifies the equations of the quadratic velocity vector, and the terms highlighted in red in Eqs. (108)–(110) can be removed (colors online), see § 2.2.1 for more details.

2.2.1 Simplifications with Euler parameters

Euler parameters describe the orientation of the reference frame with four parameters related by one algebraic constraint. These include one value related to the angle of rotation θ_0 and the direction cosines of a unique orientational axis of rotation $\underline{\theta}_{\mathrm{s}}=\begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^{\mathsf{T}}$,

$$\underline{\theta}=\begin{bmatrix}\theta_0 & \underline{\theta}_{\mathrm{s}}^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}}=\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3\end{bmatrix}^{\mathsf{T}}. \tag{111}$$

The rotation matrix with Euler parameters is given by

$$\underline{\underline{A}}=\underline{\underline{e}}+2\underline{\underline{\tilde{\theta}}}_{\mathrm{s}}\left(\theta_0\underline{\underline{e}}+\underline{\underline{\tilde{\theta}}}_{\mathrm{s}}\right), \tag{112}$$

$$=\begin{bmatrix}1-2\theta_2^2-2\theta_3^2 & 2\left(\theta_1\theta_2-\theta_0\theta_3\right) & 2\left(\theta_1\theta_3+\theta_0\theta_2\right)\\ 2\left(\theta_1\theta_2+\theta_0\theta_3\right) & 1-2\theta_1^2-2\theta_3^2 & 2\left(\theta_2\theta_3-\theta_0\theta_1\right)\\ 2\left(\theta_1\theta_3-\theta_0\theta_2\right) & 2\left(\theta_2\theta_3+\theta_0\theta_1\right) & 1-2\theta_1^2-2\theta_2^2\end{bmatrix}. \tag{113}$$

The angular velocity matrix expressed in floating coordinates is given in terms of Euler parameters by

$$\underline{\underline{\overline{G}}} = 2\left[-\underline{\theta}_{\mathrm{s}} \qquad -\underline{\underline{\tilde{\theta}}}_{\mathrm{s}} + \theta_0 \underline{\underline{e}}\right], \tag{114}$$

$$= 2\begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}. \tag{115}$$

The angular velocity vector of the floating frame relative to the inertial frame expressed in floating coordinates is given by

$$\underline{\overline{\omega}}_{\mathrm{o}} = \underline{\underline{\overline{G}}}\,\underline{\dot{\theta}}, \tag{116}$$

$$= -\underline{\underline{\dot{\overline{G}}}}\,\underline{\theta}. \tag{117}$$

Therefore, the differentiation of the angular velocity vector with respect to Euler parameters that is present among others in Eqs. (53) and (109) is given by

$$\frac{\partial \underline{\overline{\omega}}_{\mathrm{o}}^{(j)}}{\partial \underline{\theta}} = -\underline{\underline{\dot{\overline{G}}}}, \tag{118}$$

leading to

$$\frac{\partial \underline{\overline{\omega}}_{\mathrm{n}}^{(j)}}{\partial \underline{q}} = -\underline{\underline{\dot{L}}}_{\mathrm{r}}. \tag{119}$$

Inserting these expressions in the shown derivation and using the following Euler parameter identities as shown in [24],

$$\underline{\underline{\dot{\overline{G}}}}\,\underline{\dot{\theta}} = \underline{0}, \tag{120}$$

$$2\underline{\underline{\dot{\overline{G}}}}^{\mathsf{T}} = \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}, \tag{121}$$

does not affect the terms of the mass matrix shown in Eqs. (96)–(101) but leads to simplified expressions of the quadratic velocity force vector with the use of Euler parameters given by

$$\underline{Q}_{\mathrm{v,t}} = \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\underline{\overline{\omega}}_{\mathrm{o}}, \tag{122}$$

$$\begin{aligned}\underline{Q}_{\mathrm{v,r}} = {} & -\underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + \\ & -2\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} \quad -\underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi,\mathrm{r}}\underline{\dot{\zeta}},\end{aligned} \tag{123}$$

$$\underline{Q}_{\mathrm{v,f}} = \left(\underline{\underline{e}}_{\zeta}\otimes\underline{\overline{\omega}}_{\mathrm{o}}\right)^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\underline{\overline{\omega}}_{\mathrm{o}}. \tag{124}$$

These expressions with the use of Euler parameters will be used for the analytical derivatives of the system parameters in § 4. The inertia compensation term in Eq. (123) that is highlighted in blue appears only with the lumped mass approach, and the related derivation terms will also be highlighted in blue in the following (colors online).

Table 1 FFRF invariants with different approaches

Invariant symbols	Inertia shape integrals	Consistent mass	Lumped mass
$m\,\underline{\underline{e}}$	$\int \rho \mathrm{d}V \underline{\underline{e}}$	$\left(\underline{1} \otimes \underline{\underline{e}}\right)^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \left(\underline{1} \otimes \underline{\underline{e}}\right)$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \underline{\underline{e}}$
$\underline{\chi}_{\mathrm{o}}$	$\frac{1}{m} \int \rho \overline{\underline{u}}_{\mathrm{o}} \mathrm{d}V$	$\left(\underline{1} \otimes \underline{\underline{e}}\right)^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \overline{\underline{u}}_{\mathrm{o}}$	$\frac{1}{m} \sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \overline{\underline{u}}_{\mathrm{o}}^{(j)}$
$\underline{\underline{\Theta}}_{\mathrm{o}}$	$\int \rho \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}} \mathrm{d}V$	$\tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{(j)\mathsf{T}} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{(j)} + \sum_{j=1}^{n_{\mathrm{n}}} \overline{\underline{\underline{\Theta}}}_{\mathrm{n}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\psi}$	$\int \rho \overline{\underline{\underline{S}}} \mathrm{d}V$	$\left(\underline{1} \otimes \underline{\underline{e}}\right)^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \overline{\underline{\underline{\Psi}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}$	$\int \rho \tilde{\overline{\underline{\underline{S}}}} \mathrm{d}V$	$\left(\underline{1} \otimes \underline{\underline{e}}\right)^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \tilde{\overline{\underline{\underline{\Psi}}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{\Psi}}}}_{\mathrm{t}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi}$	$\int \rho \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \tilde{\overline{\underline{\underline{S}}}} \mathrm{d}V$	$\tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \overline{\underline{\underline{\Psi}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{(j)\mathsf{T}} \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)} + \sum_{j=1}^{n_{\mathrm{n}}} \overline{\underline{\underline{\Theta}}}_{\mathrm{n}}^{(j)} \overline{\underline{\underline{\Psi}}}_{\mathrm{r}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{o}\tilde{\psi}}$	$\int \rho \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \tilde{\overline{\underline{\underline{S}}}} \mathrm{d}V$	$\tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \tilde{\overline{\underline{\underline{\Psi}}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{u}}}}_{\mathrm{o}}^{(j)\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_{\mathrm{t}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\psi\psi}$	$\int \rho \overline{\underline{\underline{S}}}^{\mathsf{T}} \overline{\underline{\underline{S}}} \mathrm{d}V$	$\overline{\underline{\underline{\Psi}}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \overline{\underline{\underline{\Psi}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)\mathsf{T}} \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)} + \sum_{j=1}^{n_{\mathrm{n}}} \overline{\underline{\underline{\Psi}}}_{\mathrm{r}}^{(j)\mathsf{T}} \overline{\underline{\underline{\Theta}}}_{\mathrm{n}}^{(j)} \overline{\underline{\underline{\Psi}}}_{\mathrm{r}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi}$	$\int \rho \tilde{\overline{\underline{\underline{S}}}}^{\mathsf{T}} \overline{\underline{\underline{S}}} \mathrm{d}V$	$\tilde{\overline{\underline{\underline{\Psi}}}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \overline{\underline{\underline{\Psi}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{\Psi}}}}_{\mathrm{t}}^{(j)\mathsf{T}} \overline{\underline{\underline{\Psi}}}_{\mathrm{t}}^{(j)}$
$\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}$	$\int \rho \tilde{\overline{\underline{\underline{S}}}}^{\mathsf{T}} \tilde{\overline{\underline{\underline{S}}}} \mathrm{d}V$	$\tilde{\overline{\underline{\underline{\Psi}}}}^{\mathsf{T}} \underline{\underline{m}}_{\mathrm{FE}} \tilde{\overline{\underline{\underline{\Psi}}}}$	$\sum_{j=1}^{n_{\mathrm{n}}} m^{(j)} \tilde{\overline{\underline{\underline{\Psi}}}}_{\mathrm{t}}^{(j)\mathsf{T}} \tilde{\overline{\underline{\underline{\Psi}}}}_{\mathrm{t}}^{(j)}$

2.2.2 Expressions of inertia terms with different approaches

Table 1 shows a comparison of the invariants when using the integral formulation [24, 27], the consistent mass formulation with nodal-based FFRF [32, 35] and the lumped mass approach used in [8, 21] and fully derived here. In Table 1, $\overline{\underline{\underline{S}}}$ is the matrix of shape functions expressed in the floating frame, $\tilde{\overline{\underline{\underline{S}}}}$ is generated from $\overline{\underline{\underline{S}}}$ in analogy to Eq. (64) and [35], and $\underline{\underline{m}}_{\mathrm{FE}}$ is the consistent mass matrix of the FE model.

The generality of the lumped mass approach is given by the compensation term of inertia. A first discretization is given by the geometric discretization with the FE meshing of the flexible bodies and is applied equally to all approaches. With the lumped mass approach, a second discretization is given by the diagonalization of the FE-mass matrix leading to the additional compensation term of inertia due to the non-consideration of the mass distribution. When using another approach for the calculation of the invariants, the expressions of the inertia terms remain exactly the same, see Eqs. (96)–(101) for the mass matrix and Eqs. (122)–(124) for the quadratic velocity force vector. The only exception is the compensation term of inertia that only appears when using the lumped mass approach and is blue highlighted (colors online) in Eq. (123).

2.3 Stiffness matrix

When using a linear-elastic material model, the elastic forces can be expressed in terms of a linear stiffness matrix,

$$\underline{\underline{k}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{k}}_{\mathrm{ff}} \end{bmatrix}. \tag{125}$$

The sub-matrix related to the flexible coordinates of the stiffness matrix corresponds to that from the structural analysis, and the other terms are equal to zero.

2.4 Damping matrix

For material damping, the damping matrix is linear when using a linear-elastic material model,

$$\underline{\underline{d}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{d}}_{\mathrm{ff}} \end{bmatrix}. \tag{126}$$

The sub-matrix $\underline{\underline{d}}_{\mathrm{ff}}$ corresponds to the flexible coordinates and is the damping matrix of structural dynamics. All other terms of the FFRF damping matrix are equal to zero.

2.5 Dependencies

The equations of motion shown in Eq. (1) and (2) are solved for the system responses that generally depend on the vector of design variables $\underline{\mathsf{x}}$ and on time t,

$$\underline{q} = \mathcal{F}\left(\underline{\mathsf{x}}, t\right), \qquad \underline{\dot{q}} = \mathcal{F}\left(\underline{\mathsf{x}}, t\right), \qquad \underline{\ddot{q}} = \mathcal{F}\left(\underline{\mathsf{x}}, t\right), \qquad \underline{\lambda} = \mathcal{F}\left(\underline{\mathsf{x}}, t\right), \tag{127}$$

where $\bigcirc = \mathcal{F}\left(\bullet\right)$ denotes that $\bigcirc$ is a function of $\bullet$. The dependencies of the system parameters are given by

$$\begin{aligned} \underline{\underline{m}} &= \mathcal{F}\left(\underline{\mathsf{x}}, \underline{q}\left(\underline{\mathsf{x}}, t\right)\right), & \underline{Q}_{\mathrm{v}} &= \mathcal{F}\left(\underline{\mathsf{x}}, \underline{\dot{q}}\left(\underline{\mathsf{x}}, t\right), \underline{q}\left(\underline{\mathsf{x}}, t\right)\right), \\ \underline{\underline{d}} &= \mathcal{F}\left(\underline{\mathsf{x}}\right), & \underline{Q}_{\mathrm{e}} &= \mathcal{F}\left(\underline{\mathsf{x}}, \underline{\dot{q}}\left(\underline{\mathsf{x}}, t\right), \underline{q}\left(\underline{\mathsf{x}}, t\right)\right), \\ \underline{\underline{k}} &= \mathcal{F}\left(\underline{\mathsf{x}}\right), & \underline{\Phi} &= \mathcal{F}\left(\underline{\mathsf{x}}, \underline{\dot{q}}\left(\underline{\mathsf{x}}, t\right), \underline{q}\left(\underline{\mathsf{x}}, t\right)\right), \end{aligned} \tag{128}$$

and a summary is given in Table 2. A filled circle shows a dependency, a half-filled circle shows that there can be a dependency, and an empty circle shows that there is no dependency. Explicit and implicit dependencies are shown in black and gray, respectively. A clear understanding of the dependencies is fundamental for the differentiation of the system parameters in the following sections.

The mass matrix and the quadratic velocity force vector explicitly depend on generalized positions, and the quadratic velocity force vector explicitly also depends on generalized velocities. The generalized external forces may depend on generalized positions and velocities; if so, loading is specific and depends on the application. Similarly, the type of the kinematic constraints determines its dependencies. Holonomic constraints explicitly depend on generalized positions, and nonholonomic constraints explicitly depend on generalized velocities. Rheonomic constraints explicitly depend on time, and scleronomic constraints are time-independent.

The dependency of the system parameters w.r.t. the design variables changes for different types of design variables. Here, the design variables are categorized in structural, control, and kinematic. The structural system parameters, including the mass, damping, and stiffness matrices, as well as the quadratic velocity force vector explicitly, depend on structural design variables such as cross-sectional dimensions of beam elements or the thickness of shell elements. The generalized external forces explicitly depend on control design variables. The kinematic constraints explicitly depend on the design variables when kinematic design variables are used, e.g., joint positions. Certain design variables are of composite nature, i.e., affect, for example, the structure and the kinematics, e.g., when the length of a flexible body is used as a design variable, and a joint is located at the end of the body.

Table 2 Dependencies of system parameters: ● explicitly dependent, ◉ implicitly dependent, ○ independent, ◐ denotes possible dependency; structural system parameters requiring input from finite-element model are highlighted in green (colors online)

	$\underline{\underline{m}}$	$\underline{\underline{d}}$	$\underline{\underline{k}}$	$\underline{Q}_{\mathrm{v}}$	$\underline{Q}_{\mathrm{e}}$	$\underline{\Phi}$
positions	●	○	○	●	◐	◐
velocities	○	○	○	●	◐	◐
structural design variables	●	●	●	●	◑	◑
control design variables	◉	○	○	◉	●	◑
kinematic design variables	◉	○	○	◉	◑	●

3 Derivatives of the equations of motion

Design sensitivities or gradients are the derivatives of a function or a parameter w.r.t. the vector of design variables. A generic parameter $\bigcirc$ that depends on the vector of design variables $\underline{\mathrm{x}}$, the position vector $\underline{q}$, and the velocity vector $\underline{\dot{q}}$ is given by

$$\bigcirc = \mathcal{F}\left(\underline{\mathrm{x}}, \underline{q}, \underline{\dot{q}}\right), \tag{129}$$

and its gradient is obtained by the application of the chain rule leading to

$$\frac{\mathrm{d}\bigcirc}{\mathrm{d}\underline{\mathrm{x}}} = \frac{\partial\bigcirc}{\partial\underline{\mathrm{x}}} + \frac{\partial\bigcirc}{\partial\underline{q}}\frac{\mathrm{d}\underline{q}}{\mathrm{d}\underline{\mathrm{x}}} + \frac{\partial\bigcirc}{\partial\underline{\dot{q}}}\frac{\mathrm{d}\underline{\dot{q}}}{\mathrm{d}\underline{\mathrm{x}}}, \tag{130}$$

$$\underline{\nabla\bigcirc} = \underline{\partial\bigcirc} + \underline{{}^{q}\mathcal{J}\bigcirc}\,\underline{\underline{\nabla q}} + \underline{{}^{\dot{q}}\mathcal{J}\bigcirc}\,\underline{\underline{\nabla\dot{q}}}, \tag{131}$$

where $\underline{\nabla\bigcirc}$ is the total derivative of $\bigcirc$ w.r.t. $\underline{\mathrm{x}}$, $\underline{\partial\bigcirc}$ is the partial derivative of $\bigcirc$ w.r.t. $\underline{\mathrm{x}}$, and $\underline{{}^{\bullet}\mathcal{J}\bigcirc}$ is the Jacobian or partial derivative of $\bigcirc$ w.r.t. the coordinates $\underline{\bullet}$. The derivative w.r.t. the vector of design variables leads to an additional dimension of the gradient. As shown in Fig. 2, the gradient of a scalar is one-dimensional, the gradient of a vector is two-dimensional, and the gradient of a matrix is three-dimensional. Analogously to the differentiation with respect to the design variables, the total position Jacobian, which is the total derivative of the generic parameter $\bigcirc$ shown in Eq. (129) w.r.t. the position vector, is given by

$$\frac{\mathrm{d}\bigcirc}{\mathrm{d}\underline{q}} = \frac{\partial\bigcirc}{\partial\underline{q}} + \frac{\partial\bigcirc}{\partial\underline{\dot{q}}}\frac{\mathrm{d}\underline{\dot{q}}}{\mathrm{d}\underline{q}}, \tag{132}$$

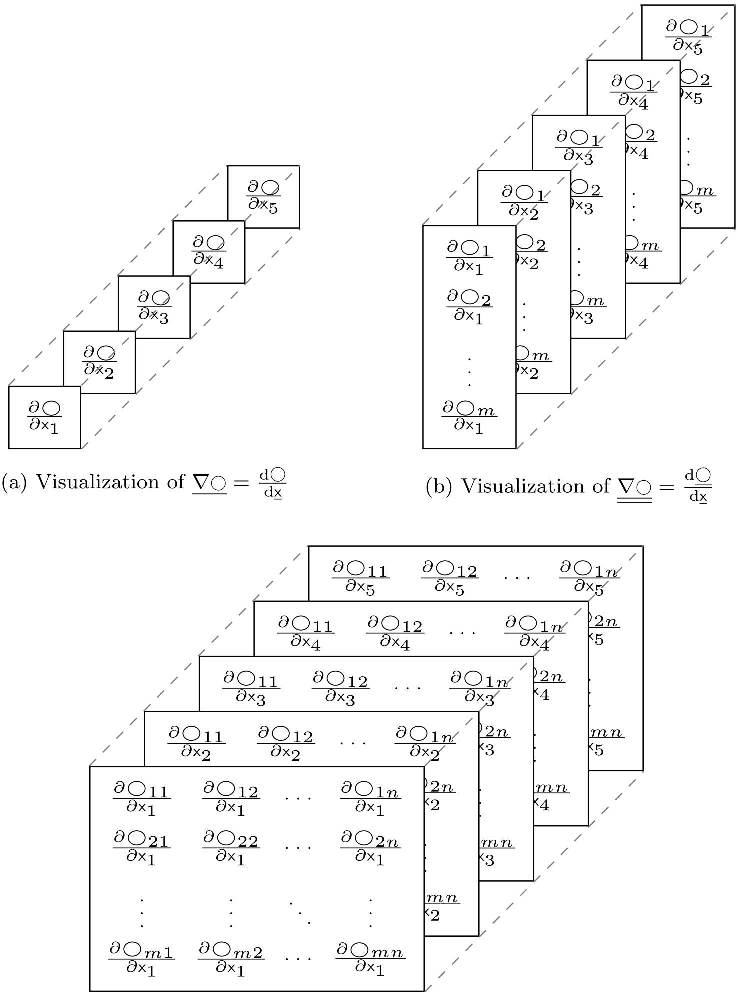
$$\underline{{}^{q}\mathrm{J}\bigcirc} = \underline{{}^{q}\mathcal{J}\bigcirc} + \underline{{}^{\dot{q}}\mathcal{J}\bigcirc}\,\underline{\underline{{}^{q}\mathrm{J}\dot{q}}}, \tag{133}$$

where $\underline{{}^{\bullet}\mathrm{J}\bigcirc}$ is the total Jacobian of $\bigcirc$ w.r.t. $\bullet$, and $\underline{{}^{\bullet}\mathcal{J}\bigcirc}$ is the partial Jacobian of $\bigcirc$ w.r.t. $\bullet$. Similarly, the total velocity Jacobian and the total acceleration Jacobian of the generic parameter $\bigcirc$ are given by

$$\underline{{}^{\dot{q}}\mathrm{J}\bigcirc} = \underline{{}^{\dot{q}}\mathcal{J}\bigcirc} + \underline{{}^{q}\mathcal{J}\bigcirc}\,\underline{\underline{{}^{\dot{q}}\mathrm{J}q}}, \tag{134}$$

$$\underline{{}^{\ddot{q}}\mathrm{J}\bigcirc} = \underline{{}^{\dot{q}}\mathcal{J}\bigcirc}\,\underline{\underline{{}^{\ddot{q}}\mathrm{J}\dot{q}}} + \underline{{}^{q}\mathcal{J}\bigcirc}\,\underline{\underline{{}^{\ddot{q}}\mathrm{J}q}}. \tag{135}$$

This section shows the derivatives of the equations of motion. This is, first of all, the sensitivity analysis as the driving motor for efficient design optimization and uncertainty



(a) Visualization of $\underline{\nabla \bigcirc} = \frac{\mathrm{d}\underline{\bigcirc}}{\mathrm{d}\underline{\mathsf{x}}}$

(b) Visualization of $\underline{\underline{\nabla \bigcirc}} = \frac{\mathrm{d}\underline{\underline{\bigcirc}}}{\mathrm{d}\underline{\mathsf{x}}}$

(c) Visualization of $\underline{\underline{\underline{\nabla \bigcirc}}} = \frac{\mathrm{d}\underline{\underline{\underline{\bigcirc}}}}{\mathrm{d}\underline{\mathsf{x}}}$

Fig. 2 Visualization of design sensitivities with $n_{\mathsf{x}} = 5$

analysis with the differentiation of the equations of motion w.r.t. the design variables. Using the analytical direct differentiation method, it is shown where the partial derivatives of the system parameters appear. Numerical methods, in addition to the design sensitivity analysis, that require the evaluation of the partial derivatives of the system parameters include the nonlinear solver in the time integration of a flexible multibody system. These usually require the Jacobian matrix of the nonlinear solver, which is the derivative of the equations of motion w.r.t. the system responses for which the system is solved. There are different time integration methods, which solve either for positions, velocities or accelerations. To cover all methods, the derivatives of the equations of motion w.r.t. positions velocities and accelerations are shown.

3.1 Derivatives of the equations of motion w.r.t. the design variables

The direct differentiation of the equations of motion w.r.t. the design variables leads to the governing equations of the sensitivity analysis,

$$\underline{\underline{\underline{\nabla m}}}\,\underline{\ddot{q}} + \underline{\underline{m}}\,\underline{\underline{\nabla \ddot{q}}} + \underline{\underline{\underline{\nabla d}}}\,\underline{\dot{q}} + \underline{\underline{d}}\,\underline{\underline{\nabla \dot{q}}} + \underline{\underline{\underline{\nabla k}}}\,\underline{q} + \underline{\underline{k}}\,\underline{\underline{\nabla q}} + \underline{\underline{\underline{\nabla {}^{q}\mathfrak{J}\Phi}}}^{\mathsf{T}}\underline{\lambda} + \underline{\underline{{}^{q}\mathfrak{J}\Phi}}^{\mathsf{T}}\underline{\underline{\nabla\lambda}} = \underline{\underline{\nabla Q_{\mathrm{e}}}} + \underline{\underline{\nabla Q_{\mathrm{v}}}}, \tag{136}$$

$$\underline{\underline{\nabla\Phi}} = \underline{\underline{0}}. \tag{137}$$

These equations are solved for the design sensitivities of the system responses $\underline{\underline{\nabla \ddot{q}}}$, $\underline{\underline{\nabla \dot{q}}}$, $\underline{\underline{\nabla q}}$ and $\underline{\underline{\nabla\lambda}}$. The dependencies shown in Eq. (128) and Table 2, as well as the application of the chain rule, lead to the derivatives of the system parameters,

$$\underline{\underline{\underline{\nabla m}}} = \underline{\underline{\underline{\partial m}}} + \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}\,\underline{\underline{\nabla q}}, \tag{138}$$

$$\underline{\underline{\underline{\nabla d}}} = \underline{\underline{\underline{\partial d}}}, \tag{139}$$

$$\underline{\underline{\underline{\nabla k}}} = \underline{\underline{\underline{\partial k}}}, \tag{140}$$

$$\underline{\underline{\underline{\nabla {}^{q}\mathfrak{J}\Phi}}} = \underline{\underline{\underline{\partial {}^{q}\mathfrak{J}\Phi}}} + \underline{\underline{\underline{{}^{q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}\,\underline{\underline{\nabla q}} + \underline{\underline{\underline{{}^{\dot{q}}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}\,\underline{\underline{\nabla \dot{q}}}, \tag{141}$$

$$\underline{\underline{\nabla Q_{\mathrm{e}}}} = \underline{\underline{\partial Q_{\mathrm{e}}}} + \underline{\underline{{}^{q}\mathfrak{J}Q_{\mathrm{e}}}}\,\underline{\underline{\nabla q}} + \underline{\underline{{}^{\dot{q}}\mathfrak{J}Q_{\mathrm{e}}}}\,\underline{\underline{\nabla \dot{q}}}, \tag{142}$$

$$\underline{\underline{\nabla Q_{\mathrm{v}}}} = \underline{\underline{\partial Q_{\mathrm{v}}}} + \underline{\underline{{}^{q}\mathfrak{J}Q_{\mathrm{v}}}}\,\underline{\underline{\nabla q}} + \underline{\underline{{}^{\dot{q}}\mathfrak{J}Q_{\mathrm{v}}}}\,\underline{\underline{\nabla \dot{q}}}, \tag{143}$$

$$\underline{\underline{\nabla \Phi}} = \underline{\underline{\partial \Phi}} + \underline{\underline{{}^{q}\mathfrak{J}\Phi}}\,\underline{\underline{\nabla q}} + \underline{\underline{{}^{\dot{q}}\mathfrak{J}\Phi}}\,\underline{\underline{\nabla \dot{q}}}. \tag{144}$$

The same solution strategy used in the primal analysis can be applied to the solution of the sensitivity analysis. In addition to the sensitivity analysis of the equations of motion, the sensitivity analysis must be performed for the time integration method and the nonlinear solver [29]. A solution routine for flexible multibody dynamics, including the sensitivity analysis with generalized-α time integration and Baumgarte stabilization with a numerical computation of the partial derivatives, is shown in [9, 11, 30].

3.2 Derivatives of the equations of motion w.r.t. position

The nonlinear solver of time integration methods that solve for positions requires the position Jacobian that is the differentiation of the equations of motion w.r.t. positions. The total derivative of the equations of motion w.r.t. the generalized positions is given by

$$\underline{\underline{\underline{{}^{q}\mathcal{J}m}}}\,\underline{\ddot{q}} + \underline{\underline{m}}\,\underline{\underline{{}^{q}\mathcal{J}\ddot{q}}} + \underline{\underline{d}}\,\underline{\underline{{}^{q}\mathcal{J}\dot{q}}} + \underline{\underline{k}} + \underline{\underline{\underline{{}^{q}\mathcal{J}{}^{q}\mathfrak{J}\Phi}}}^{\mathsf{T}}\underline{\lambda} = \underline{\underline{{}^{q}\mathcal{J}Q_{\mathrm{e}}}} + \underline{\underline{{}^{q}\mathcal{J}Q_{\mathrm{v}}}}, \tag{145}$$

$$\underline{\underline{{}^{q}\mathcal{J}\Phi}} = \underline{\underline{0}}, \tag{146}$$

where the application of the chain rule leads to

$$\underline{\underline{\underline{{}^{q}\mathcal{J}m}}} = \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}, \tag{147}$$

$$\underline{\underline{\underline{{}^{q}\mathcal{J}{}^{q}\mathfrak{J}\Phi}}} = \underline{\underline{\underline{{}^{q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}} + \underline{\underline{\underline{{}^{\dot{q}}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}\,\underline{\underline{{}^{q}\mathcal{J}\dot{q}}}, \tag{148}$$

$$\underline{\underline{{}^{q}\mathsf{J}Q_\mathrm{e}}} = \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{e}}} + \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{e}}}\,\underline{\underline{{}^{q}\mathsf{J}\dot q}}, \tag{149}$$

$$\underline{\underline{{}^{q}\mathsf{J}Q_\mathrm{v}}} = \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{v}}} + \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{v}}}\,\underline{\underline{{}^{q}\mathsf{J}\dot q}}, \tag{150}$$

$$\underline{\underline{{}^{q}\mathsf{J}\Phi}} = \underline{\underline{{}^{q}\mathfrak{J}\Phi}} + \underline{\underline{{}^{\dot q}\mathfrak{J}\Phi}}\,\underline{\underline{{}^{q}\mathsf{J}\dot q}}. \tag{151}$$

With implicit time integration, the system responses may have dependencies among themselves and therefore $\underline{\underline{{}^{q}\mathsf{J}\ddot q}} \neq \underline{0}$ and $\underline{\underline{{}^{q}\mathsf{J}\dot q}} \neq \underline{0}$ in the general case. However, if $\underline{\underline{{}^{q}\mathsf{J}\ddot q}} = \underline{0}$ and $\underline{\underline{{}^{q}\mathsf{J}\dot q}} = \underline{0}$, then the differentiation of the equations of motion w.r.t. the generalized positions reduces to

$$\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}\,\underline{\ddot q} + \underline{k} + \underline{\underline{\underline{{}^{q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}^{\mathsf{T}}\underline{\lambda} = \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{e}}} + \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{v}}}, \tag{152}$$

$$\underline{\underline{{}^{q}\mathfrak{J}\Phi}} = \underline{0}. \tag{153}$$

3.3 Derivatives of the equations of motion w.r.t. velocity

When using time integration methods that solve for velocities, the nonlinear solver requires the velocity Jacobian that is given by the differentiation of the equations of motion w.r.t. the generalized velocities. This is given by

$$\underline{\underline{\underline{{}^{\dot q}\mathsf{J}m}}}\,\underline{\ddot q} + \underline{\underline{m}}\,\underline{\underline{{}^{\dot q}\mathsf{J}\ddot q}} + \underline{d} + \underline{\underline{k}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}} + \underline{\underline{\underline{{}^{\dot q}\mathsf{J}{}^{q}\mathfrak{J}\Phi}}}^{\mathsf{T}}\underline{\lambda} = \underline{\underline{{}^{\dot q}\mathsf{J}Q_\mathrm{e}}} + \underline{\underline{{}^{\dot q}\mathsf{J}Q_\mathrm{v}}}, \tag{154}$$

$$\underline{\underline{{}^{\dot q}\mathsf{J}\Phi}} = \underline{0}, \tag{155}$$

where the chain rule leads to

$$\underline{\underline{\underline{{}^{\dot q}\mathsf{J}m}}} = \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}}, \tag{156}$$

$$\underline{\underline{\underline{{}^{\dot q}\mathsf{J}{}^{q}\mathfrak{J}\Phi}}} = \underline{\underline{\underline{{}^{\dot q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}} + \underline{\underline{\underline{{}^{q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}}, \tag{157}$$

$$\underline{\underline{{}^{\dot q}\mathsf{J}Q_\mathrm{e}}} = \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{e}}} + \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{e}}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}}, \tag{158}$$

$$\underline{\underline{{}^{\dot q}\mathsf{J}Q_\mathrm{v}}} = \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{v}}} + \underline{\underline{{}^{q}\mathfrak{J}Q_\mathrm{v}}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}}, \tag{159}$$

$$\underline{\underline{{}^{\dot q}\mathsf{J}\Phi}} = \underline{\underline{{}^{\dot q}\mathfrak{J}\Phi}} + \underline{\underline{{}^{q}\mathfrak{J}\Phi}}\,\underline{\underline{{}^{\dot q}\mathsf{J}q}}. \tag{160}$$

When using implicit time integration, the derivatives of the system responses are generally not zero, $\underline{\underline{{}^{\dot q}\mathsf{J}\ddot q}} \neq \underline{0}$ and $\underline{\underline{{}^{\dot q}\mathsf{J}q}} \neq \underline{0}$. In cases where the derivatives of the system responses are zero $\underline{\underline{{}^{\dot q}\mathsf{J}\ddot q}} = \underline{0}$ and $\underline{\underline{{}^{\dot q}\mathsf{J}q}} = \underline{0}$, the differentiation of the equations of motion w.r.t. the generalized velocities reduces to

$$\underline{d} + \underline{\underline{\underline{{}^{\dot q}\mathfrak{J}{}^{q}\mathfrak{J}\Phi}}}^{\mathsf{T}}\underline{\lambda} = \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{e}}} + \underline{\underline{{}^{\dot q}\mathfrak{J}Q_\mathrm{v}}}, \tag{161}$$

$$\underline{\underline{{}^{\dot q}\mathfrak{J}\Phi}} = \underline{0}. \tag{162}$$

3.4 Derivatives of the equations of motion w.r.t. acceleration

The nonlinear solver for time integration methods that solve for accelerations as shown in [11, 29, 30] requires the acceleration Jacobian given by the total derivative of the equations of motion w.r.t. the generalized accelerations. This is given by

$$\underline{\underline{\underline{{}^{\ddot{q}}\mathsf{J}m}}}\,\underline{\ddot{q}}+\underline{\underline{m}}+\underline{\underline{d}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}}+\underline{\underline{k}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}+\underline{\underline{\underline{{}^{\ddot{q}}\mathsf{J}{}^{q}\mathcal{J}\Phi}}}^{\mathsf{T}}\underline{\lambda}=\underline{\underline{{}^{\ddot{q}}\mathsf{J}Q_{\mathrm{e}}}}+\underline{\underline{{}^{\ddot{q}}\mathsf{J}Q_{\mathrm{v}}}},\tag{163}$$

$$\underline{\underline{{}^{\ddot{q}}\mathsf{J}\Phi}}=\underline{\underline{0}},\tag{164}$$

where the chain rule leads to

$$\underline{\underline{\underline{{}^{\ddot{q}}\mathsf{J}m}}}=\underline{\underline{\underline{{}^{q}\mathcal{J}m}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}},\tag{165}$$

$$\underline{\underline{\underline{{}^{\ddot{q}}\mathsf{J}{}^{q}\mathcal{J}\Phi}}}=\underline{\underline{\underline{{}^{q}\mathcal{J}{}^{q}\mathcal{J}\Phi}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}+\underline{\underline{\underline{{}^{\dot{q}}\mathcal{J}{}^{q}\mathcal{J}\Phi}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}},\tag{166}$$

$$\underline{\underline{{}^{\ddot{q}}\mathsf{J}Q_{\mathrm{e}}}}=\underline{\underline{{}^{q}\mathcal{J}Q_{\mathrm{e}}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}+\underline{\underline{{}^{\dot{q}}\mathcal{J}Q_{\mathrm{e}}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}},\tag{167}$$

$$\underline{\underline{{}^{\ddot{q}}\mathsf{J}Q_{\mathrm{v}}}}=\underline{\underline{{}^{q}\mathcal{J}Q_{\mathrm{v}}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}+\underline{\underline{{}^{\dot{q}}\mathcal{J}Q_{\mathrm{v}}}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}},\tag{168}$$

$$\underline{\underline{{}^{\ddot{q}}\mathsf{J}\Phi}}=\underline{\underline{{}^{q}\mathcal{J}\Phi}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}+\underline{\underline{{}^{\dot{q}}\mathcal{J}\Phi}}\,\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}}.\tag{169}$$

With implicit time integration methods, the positions and velocities may depend on the accelerations and therefore $\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}}\neq\underline{\underline{0}}$ and $\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}\neq\underline{\underline{0}}$ in general. In cases where positions and velocities are independent of accelerations, the derivatives of the system responses are zero $\underline{\underline{{}^{\ddot{q}}\mathsf{J}\dot{q}}}=\underline{\underline{0}}$ and $\underline{\underline{{}^{\ddot{q}}\mathsf{J}q}}=\underline{\underline{0}}$, and the mass matrix $\underline{\underline{m}}$ is the only term that remains.

4 Partial derivatives of the system parameters

The derivatives of the equations of motion have been performed in § 3 where the derivatives of the system parameters appear. In previous works [9, 10, 29, 30], the derivatives of the system parameters have been computed numerically for a semi-analytical approach. To further increase the efficiency of the method and the precision of the results, the analytical derivatives of the system parameters will be introduced in the following. The derivation is limited to the structural system parameters, which can be formulated in general terms, including the mass matrix shown in Eqs. (96)–(101), the quadratic velocity force vector with the use of Euler parameters shown in Eqs. (122)–(124), the stiffness matrix shown in Eq. (125), and the damping matrix shown in Eq. (126). The derivatives of the loading specific generalized external forces and the joint specific kinematic constraints need to be differentiated separately and are not shown here.

4.1 Partial derivatives w.r.t. design variables

The structural system parameters, including the mass matrix, the stiffness matrix, the damping matrix, and the quadratic velocity force vector explicitly depend on the design variables. In particular, the finite-element mass and stiffness matrix and the inertia shape integrals or invariants are the depending terms.

The partial derivative of the mass matrix w.r.t. the design variables as required in Eq. (138) is given by

$$\underline{\underline{\underline{\partial m}}}=\begin{bmatrix}\underline{\underline{\underline{\partial m_{\mathrm{tt}}}}} & \underline{\underline{\underline{\partial m_{\mathrm{tr}}}}} & \underline{\underline{\underline{\partial m_{\mathrm{tf}}}}}\\ & \underline{\underline{\underline{\partial m_{\mathrm{rr}}}}} & \underline{\underline{\underline{\partial m_{\mathrm{rf}}}}}\\ \text{sym.} & & \underline{\underline{\underline{\partial m_{\mathrm{ff}}}}}\end{bmatrix},\tag{170}$$

where

$$\underline{\underline{\underline{\partial m}}}_{\text{tt}} = \underline{\partial m}\,\underline{\underline{e}}, \tag{171}$$

$$\underline{\underline{\underline{\partial m}}}_{\text{tr}} = -\underline{\underline{A}}\left(\underline{\partial m}\,\underline{\underline{\tilde{\chi}}}_{\text{o}} + m\,\underline{\underline{\underline{\partial \tilde{\chi}}}}_{\text{o}} + \underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right)\right)\underline{\underline{\overline{G}}}, \tag{172}$$

$$\underline{\underline{\underline{\partial m}}}_{\text{tf}} = \underline{\underline{A}}\,\underline{\underline{\underline{\partial \mathcal{I}}}}_{\psi}, \tag{173}$$

$$\begin{aligned}\underline{\underline{\underline{\partial m}}}_{\text{rr}} = \underline{\underline{\overline{G}}}^{\mathsf{T}}\Big(&\underline{\underline{\underline{\partial \Theta}}}_{\text{o}} + \underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right) + \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}^{\mathsf{T}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}} + \\ &+ \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right)\Big)\underline{\underline{\overline{G}}},\end{aligned} \tag{174}$$

$$\underline{\underline{\underline{\partial m}}}_{\text{rf}} = -\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\text{u}}_{\text{o}}\psi} + \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}\psi}\right), \tag{175}$$

$$\underline{\underline{\underline{\partial m}}}_{\text{ff}} = \underline{\underline{\underline{\partial \mathcal{I}}}}_{\psi\psi}. \tag{176}$$

This shows that the partial derivatives of the invariants are required for the partial derivative of the mass matrix.

The partial derivative of the quadratic velocity force vector w.r.t. the design variables as required in Eq. (143) is given by

$$\underline{\underline{\partial Q}}_{\text{v}} = \begin{bmatrix} \underline{\underline{\partial Q}}_{\text{v,t}} \\ \underline{\underline{\partial Q}}_{\text{v,r}} \\ \underline{\underline{\partial Q}}_{\text{v,f}} \end{bmatrix}, \tag{177}$$

where

$$\underline{\underline{\partial Q}}_{\text{v,t}} = \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\text{o}}\left(\underline{\partial m}\,\underline{\underline{\tilde{\chi}}}_{\text{o}} + m\,\underline{\underline{\underline{\partial \tilde{\chi}}}}_{\text{o}} + \underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\text{o}} + 2\underline{\underline{A}}\,\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}}\left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right)\underline{\overline{\omega}}_{\text{o}}, \tag{178}$$

$$\begin{aligned}\underline{\underline{\partial Q}}_{\text{v,r}} = -\underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\text{o}}\Big(&\underline{\underline{\underline{\partial \Theta}}}_{\text{o}} + \underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right) + \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}^{\mathsf{T}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}} + \\ &+ \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right)\Big)\underline{\overline{\omega}}_{\text{o}} + \\ &- 2\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}}\left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right) + \left(\underline{\zeta} \otimes \underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\text{o}} \quad -\underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\text{o}}\underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\text{u}}_{\text{o}}\psi,\text{r}}\underline{\dot{\zeta}},\end{aligned} \tag{179}$$

$$\underline{\underline{\partial Q}}_{\text{v,f}} = \left(\underline{\underline{e}}_{\zeta} \otimes \underline{\overline{\omega}}_{\text{o}}\right)^{\mathsf{T}}\left(\underline{\underline{\underline{\partial \mathcal{I}}}}^{\mathsf{T}}_{\tilde{\text{u}}_{\text{o}}\tilde{\psi}} + \underline{\underline{\underline{\partial \mathcal{I}}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\text{o}} + 2\underline{\underline{\underline{\partial \mathcal{I}}}}^{\mathsf{T}}_{\tilde{\psi}\psi}\left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right)\underline{\overline{\omega}}_{\text{o}}, \tag{180}$$

where the blue highlighted term is for the lumped mass approach only (colors online), see § 2.2. As with the mass matrix, the derivatives of the invariants are required.

The partial derivative of the stiffness matrix w.r.t. the design variables as required in Eq. (140) is given by

$$
\underline{\underline{\underline{\partial k}}} = \begin{bmatrix} \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} \\ \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} \\ \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{\partial k}}}_{\mathrm{ff}} \end{bmatrix}, \tag{181}
$$

where $\underline{\underline{\underline{\partial k}}}_{\mathrm{ff}}$ is the partial derivative of the finite-element stiffness matrix.

The partial derivative of the damping matrix w.r.t. the design variables as required in Eq. (139) is given by

$$
\underline{\underline{\underline{\partial d}}} = \begin{bmatrix} \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} \\ \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} \\ \underline{\underline{\underline{0}}} & \underline{\underline{\underline{0}}} & \underline{\underline{\underline{\partial d}}}_{\mathrm{ff}} \end{bmatrix}, \tag{182}
$$

where $\underline{\underline{\underline{\partial d}}}_{\mathrm{ff}}$ is the partial derivative of the damping matrix of structural dynamics.

It can be summarized that the calculation of the partial derivatives of the system parameters w.r.t. the design variables require the partial derivatives of the finite element mass and stiffness matrices, the damping matrix and the invariants. Since these are constant in time, a single evaluation for each design evaluation at the beginning of the sensitivity analysis is sufficient. This enables the decoupling of the sensitivity analysis of the multibody system from the finite-element analysis and guarantees the high efficiency of the method. In addition, the decoupling allows computing the partial derivatives of the finite-element mass and stiffness matrix, the damping matrix and the invariants with numerical differentiation since a single evaluation per design is sufficient. However, the authors recommend the implementation of analytical differentiation for typical design variables that are used in many optimization formulations and numerical differentiation for other design variables to keep the method general.

4.2 Partial derivatives w.r.t. generalized positions

The structural system parameters that directly depend on the generalized positions are the mass matrix and the quadratic velocity force vector. In particular, the rotation matrix, the angular velocity vector, the angular velocity matrix, and the flexible coordinates directly depend on the generalized positions.

The Jacobian of the mass matrix w.r.t. generalized positions as required in Eqs. (138), (147), (156), (165) is given by

$$
\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}} = \begin{bmatrix} \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tt}} & \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tr}} & \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tf}} \\ & \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{rr}} & \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{rf}} \\ \text{sym.} & & \underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{ff}} \end{bmatrix}, \tag{183}
$$

where

$$
\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tt}} = \underline{\underline{\underline{0}}}, \tag{184}
$$

$$
\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tr}} = -\underline{\underline{\underline{{}^{q}\mathfrak{J}A}}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta} \otimes \underline{e}\right)\right)\underline{\underline{\overline{G}}} - \underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}} \otimes \underline{e}\right)\underline{\underline{\overline{G}}}+
$$

$$- \underline{\underline{A}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{\underline{{}^{q}\mathfrak{J}\overline{G}}}}, \tag{185}$$

$$\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{tf}} = \underline{\underline{\underline{{}^{q}\mathfrak{J}A}}}\,\underline{\underline{\mathcal{I}}}_{\psi}, \tag{186}$$

$$\begin{aligned}
\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{rr}} = {} & \underline{\underline{\underline{{}^{q}\mathfrak{J}\overline{G}}}}^{\mathsf{T}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{\overline{G}}}+\\
& + \underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right) + \left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}}+\right.\\
& \qquad \left. + \left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{\overline{G}}}+\\
& + \underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{\underline{{}^{q}\mathfrak{J}\overline{G}}}},
\end{aligned} \tag{187}$$

$$\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{rf}} = -\underline{\underline{\underline{{}^{q}\mathfrak{J}\overline{G}}}}^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi}\right) - \underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\psi}, \tag{188}$$

$$\underline{\underline{\underline{{}^{q}\mathfrak{J}m}}}_{\mathrm{ff}} = \underline{\underline{\underline{0}}}. \tag{189}$$

The Jacobian of the quadratic velocity force vector w.r.t. generalized positions as required in Eqs. (143), (150), (159), (168) is given by

$$\underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v}} = \begin{bmatrix} \underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v,t}} \\ \underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v,r}} \\ \underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v,f}} \end{bmatrix}, \tag{190}$$

where

$$\begin{aligned}
\underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v,t}} = {} & \underline{\underline{\underline{{}^{q}\mathfrak{J}A}}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{A}}\,\underline{\underline{\underline{{}^{q}\mathfrak{J}\tilde{\overline{\omega}}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}}+\\
& + \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{{}^{q}\mathfrak{J}\overline{\omega}}}_{\mathrm{o}}+\\
& + 2\underline{\underline{\underline{{}^{q}\mathfrak{J}A}}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\underline{\underline{{}^{q}\mathfrak{J}\overline{\omega}}}_{\mathrm{o}},
\end{aligned} \tag{191}$$

$$\begin{aligned}
\underline{\underline{{}^{q}\mathfrak{J}Q}}_{\mathrm{v,r}} = {} & -\underline{\underline{\underline{{}^{q}\mathfrak{J}\overline{G}}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}}+\\
& - \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\underline{{}^{q}\mathfrak{J}\tilde{\overline{\omega}}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}}+\\
& - \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right) + \left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}}+\right.\\
& \qquad \left. + \left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\underline{{}^{q}\mathfrak{J}\zeta}}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}}+\\
& - \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}^{\mathsf{T}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\underline{{}^{q}\mathfrak{J}\overline{\omega}}}_{\mathrm{o}}+
\end{aligned}$$

$$
\begin{aligned}
&- 2\,\underline{\underline{\underline{{}^{q}\mathrm{J}\overline{G}}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + \\
&- 2\,\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\left(\underline{\underline{{}^{q}\mathrm{J}\zeta}} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + \\
&- 2\,\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) + \left(\underline{\zeta} \otimes \underline{\underline{e}} \right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{{}^{q}\mathrm{J}\overline{\omega}}}_{\mathrm{o}} + \\
&- \underline{\underline{\underline{{}^{q}\mathrm{J}\overline{G}}}}^{\mathsf{T}} \underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi,\mathrm{r}} \dot{\underline{\zeta}} \right) - \underline{\underline{\overline{G}}}^{\mathsf{T}} \underline{\underline{\underline{{}^{q}\mathrm{J}\tilde{\overline{\omega}}}}}_{\mathrm{o}} \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\psi,\mathrm{r}} \dot{\underline{\zeta}} \,,
\end{aligned}
\tag{192}
$$

$$
\begin{aligned}
\underline{\underline{{}^{q}\mathrm{J}Q}}_{\mathrm{v,f}} = {} & \left(\underline{\underline{e}}_{\zeta} \otimes \underline{\overline{\omega}}_{\mathrm{o}} \right)^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\underline{{}^{q}\mathrm{J}\zeta}} \otimes \underline{\underline{e}} \right) \right) \underline{\overline{\omega}}_{\mathrm{o}} + \left(\underline{\underline{e}}_{\zeta} \otimes \underline{\overline{\omega}}_{\mathrm{o}} \right)^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta} \otimes \underline{\underline{e}} \right) \right) \underline{\underline{{}^{q}\mathrm{J}\overline{\omega}}}_{\mathrm{o}} + \\
& + 2\,\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi} \left(\dot{\underline{\zeta}} \otimes \underline{\underline{e}} \right) \underline{\underline{{}^{q}\mathrm{J}\overline{\omega}}}_{\mathrm{o}},
\end{aligned}
\tag{193}
$$

where the blue highlighted term is for the lumped mass approach only (colors online), see § 2.2.

The Jacobian of the rotation matrix w.r.t. generalized positions is independent of the position of the reference frame and the flexible deformations, leading to

$$
\underline{\underline{\underline{{}^{q}\mathrm{J}A}}} = 2\,\underline{\underline{\underline{{}^{q}\mathrm{J}\tilde{\theta}}}}_{\mathrm{s}} \left(\theta_0 \underline{\underline{e}} + \underline{\underline{\tilde{\theta}}}_{\mathrm{s}} \right) + 2\,\underline{\underline{\tilde{\theta}}}_{\mathrm{s}} \left(\underline{\underline{{}^{q}\mathrm{J}\theta_0 \underline{e}}} + \underline{\underline{\underline{{}^{q}\mathrm{J}\tilde{\theta}}}}_{\mathrm{s}} \right), \tag{194}
$$

$$
= \left[\underline{\underline{\underline{{}^{\tau}\mathrm{J}A}}} \quad \underline{\underline{\underline{{}^{\theta}\mathrm{J}A}}} \quad \underline{\underline{\underline{{}^{\zeta}\mathrm{J}A}}} \right], \tag{195}
$$

$$
= \left[\underline{\underline{\underline{0}}} \quad \underline{\underline{\underline{{}^{\theta}\mathrm{J}A}}} \quad \underline{\underline{\underline{0}}} \right], \tag{196}
$$

with

$$
\begin{aligned}
\underline{\underline{\underline{{}^{\theta}\mathrm{J}A}}} = \Bigg[& \begin{bmatrix} 0 & -2\theta_3 & 2\theta_2 \\ 2\theta_3 & 0 & -2\theta_1 \\ -2\theta_2 & 2\theta_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2\theta_2 & 2\theta_3 \\ 2\theta_2 & -4\theta_1 & -2\theta_0 \\ 2\theta_3 & 2\theta_0 & -4\theta_1 \end{bmatrix} \cdots \\
& \cdots \begin{bmatrix} -4\theta_2 & 2\theta_1 & 2\theta_0 \\ 2\theta_1 & 0 & 2\theta_3 \\ -2\theta_0 & 2\theta_3 & -4\theta_2 \end{bmatrix} \begin{bmatrix} -4\theta_3 & -2\theta_0 & 2\theta_1 \\ 2\theta_0 & -4\theta_3 & 2\theta_2 \\ 2\theta_1 & 2\theta_2 & 0 \end{bmatrix} \Bigg].
\end{aligned}
\tag{197}
$$

The Jacobian of the angular velocity matrix expressed in floating coordinates w.r.t. generalized positions is given by

$$
\underline{\underline{\underline{{}^{q}\mathrm{J}\overline{G}}}} = \left[-\underline{\underline{{}^{q}\mathrm{J}\theta}}_{\mathrm{s}} \qquad -\underline{\underline{\underline{{}^{q}\mathrm{J}\tilde{\theta}}}}_{\mathrm{s}} + \underline{\underline{{}^{q}\mathrm{J}\theta_0 \underline{e}}} \right] \tag{198}
$$

$$
= \left[\underline{\underline{\underline{{}^{\tau}\mathrm{J}\overline{G}}}} \quad \underline{\underline{\underline{{}^{\theta}\mathrm{J}\overline{G}}}} \quad \underline{\underline{\underline{{}^{\zeta}\mathrm{J}\overline{G}}}} \right] \tag{199}
$$

$$
= \left[\underline{\underline{\underline{0}}} \quad \underline{\underline{\underline{{}^{\theta}\mathrm{J}\overline{G}}}} \quad \underline{\underline{\underline{0}}} \right] \tag{200}
$$

with

$$
\underline{\underline{\underline{{}^{\theta}\mathrm{J}\overline{G}}}} = 2 \left[\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right]. \tag{201}
$$

The Jacobian of the angular velocity vector of the floating frame relative to the inertial frame expressed in floating coordinates w.r.t. generalized positions is given by

$$
\begin{aligned}
{}^{q}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} &= {}^{q}\underline{\underline{\underline{\mathrm{J}\overline{G}}}}\,\underline{\dot{\theta}}, && (202)\\
&= \left[\,{}^{\tau}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} \quad {}^{\theta}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} \quad {}^{\zeta}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}}\,\right], && (203)\\
&= \left[\,\underline{\underline{0}} \quad {}^{\theta}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} \quad \underline{\underline{0}}\,\right], && (204)
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{\theta}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} &= {}^{\theta}\underline{\underline{\underline{\mathrm{J}\overline{G}}}}\,\underline{\dot{\theta}}, && (205)\\
&= -\underline{\underline{\dot{\overline{G}}}}, && (206)\\
&= 2\begin{bmatrix} \dot{\theta}_1 & -\dot{\theta}_0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_2 & \dot{\theta}_3 & -\dot{\theta}_0 & -\dot{\theta}_1 \\ \dot{\theta}_3 & -\dot{\theta}_2 & \dot{\theta}_1 & -\dot{\theta}_0 \end{bmatrix}. && (207)
\end{aligned}
$$

The Jacobian of the flexible coordinates w.r.t. the generalized positions is given by

$$
\begin{aligned}
{}^{q}\underline{\underline{\mathrm{J}\zeta}} &= \left[\,{}^{\tau}\underline{\underline{\mathrm{J}\zeta}} \quad {}^{\theta}\underline{\underline{\mathrm{J}\zeta}} \quad {}^{\zeta}\underline{\underline{\mathrm{J}\zeta}}\,\right], && (208)\\
&= \left[\,\underline{\underline{0}} \quad \underline{\underline{0}} \quad \underline{\underline{e}}\,\right]. && (209)
\end{aligned}
$$

4.3 Partial derivatives w.r.t. generalized velocities

The quadratic velocity vector depends on the generalized velocities. In particular, the angular velocity vector and velocity of the elastic coordinates directly depend on the generalized velocities.

The Jacobian of the quadratic velocity force vector w.r.t. the generalized velocities as required in Eqs. (143), (150), (159), (168) is given by

$$
{}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v}} = \begin{bmatrix} {}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,t}} \\ {}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,r}} \\ {}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,f}} \end{bmatrix}, \qquad (210)
$$

where

$$
\begin{aligned}
{}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,t}} = {}& \underline{\underline{A}}\,{}^{\dot{q}}\underline{\underline{\underline{\mathrm{J}\tilde{\overline{\omega}}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} + \underline{\underline{A}}\,\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(m\,\underline{\underline{\tilde{\chi}}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right){}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} +\\
&+ 2\underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left({}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}}\otimes\underline{\underline{e}}\right)\underline{\overline{\omega}}_{\mathrm{o}} + 2\underline{\underline{A}}\,\underline{\underline{\mathcal{I}}}_{\tilde{\psi}}\left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right){}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}}, \qquad (211)\\
{}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,r}} = {}& -\underline{\underline{\overline{G}}}^{\mathsf{T}}\,{}^{\dot{q}}\underline{\underline{\underline{\mathrm{J}\tilde{\overline{\omega}}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} +\\
&- \underline{\underline{\overline{G}}}^{\mathsf{T}}\underline{\underline{\tilde{\overline{\omega}}}}_{\mathrm{o}}\left(\underline{\underline{\Theta}}_{\mathrm{o}} + \underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}} + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right){}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{\mathrm{o}} +\\
&- 2\underline{\underline{\overline{G}}}^{\mathsf{T}}\left(\underline{\underline{\mathcal{I}}}_{\tilde{\mathrm{u}}_{\mathrm{o}}\tilde{\psi}}\left({}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}}\otimes\underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}}\underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}}\left({}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}}\otimes\underline{\underline{e}}\right)\right)\underline{\overline{\omega}}_{\mathrm{o}} +
\end{aligned}
$$

$$
\begin{aligned}
&- 2\underline{\underline{\overline{G}}}^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}_{\tilde{u}_o\tilde{\psi}} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right) + \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)^{\mathsf{T}} \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\dot{\zeta}}\otimes\underline{\underline{e}}\right)\right) {}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} + \\
&- \underline{\underline{\overline{G}}}^{\mathsf{T}} \, {}^{\dot{q}}\underline{\underline{\underline{\mathrm{J}\tilde{\overline{\omega}}}}}_{o} \underline{\underline{\mathcal{I}}}_{\tilde{u}_o\psi,\mathrm{r}} \underline{\dot{\zeta}} - \underline{\underline{\overline{G}}}^{\mathsf{T}} \underline{\underline{\tilde{\overline{\omega}}}}_{o} \underline{\underline{\mathcal{I}}}_{\tilde{u}_o\psi,\mathrm{r}} {}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}} \, ,
\end{aligned} \tag{212}
$$

$$
\begin{aligned}
{}^{\dot{q}}\underline{\underline{\mathrm{J}Q}}_{\mathrm{v,f}} = {} & \left(\underline{\underline{e}}_{\zeta} \otimes {}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o}\right)^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{u}_o\tilde{\psi}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right) \underline{\overline{\omega}}_{o} + \\
& + \left(\underline{\underline{e}}_{\zeta} \otimes \underline{\overline{\omega}}_{o}\right)^{\mathsf{T}} \left(\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{u}_o\tilde{\psi}} + \underline{\underline{\mathcal{I}}}_{\tilde{\psi}\tilde{\psi}} \left(\underline{\zeta}\otimes\underline{\underline{e}}\right)\right) {}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} + \\
& + 2\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi} \left({}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}} \otimes \underline{\underline{e}}\right) \underline{\overline{\omega}}_{o} + 2\underline{\underline{\mathcal{I}}}^{\mathsf{T}}_{\tilde{\psi}\psi} \left(\underline{\dot{\zeta}} \otimes \underline{\underline{e}}\right) {}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o},
\end{aligned} \tag{213}
$$

where the blue highlighted term is for the lumped mass approach only (colors online), see § 2.2.

The Jacobian of the angular velocity vector of the floating frame relative to the inertial frame expressed in floating coordinates is given by

$$
{}^{\dot{q}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} = \left[{}^{\dot{r}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} \quad {}^{\dot{\theta}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} \quad {}^{\dot{\zeta}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} \right], \tag{214}
$$

$$
= \left[\underline{\underline{0}} \quad {}^{\dot{\theta}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} \quad \underline{\underline{0}} \right], \tag{215}
$$

with

$$
{}^{\dot{\theta}}\underline{\underline{\mathrm{J}\overline{\omega}}}_{o} = \underline{\underline{\overline{G}}}, \tag{216}
$$

and the Jacobian of the flexible velocity coordinates w.r.t. the generalized velocities is given by

$$
{}^{\dot{q}}\underline{\underline{\mathrm{J}\dot{\zeta}}} = \left[{}^{\dot{r}}\underline{\underline{\mathrm{J}\dot{\zeta}}} \quad {}^{\dot{\theta}}\underline{\underline{\mathrm{J}\dot{\zeta}}} \quad {}^{\dot{\zeta}}\underline{\underline{\mathrm{J}\dot{\zeta}}} \right], \tag{217}
$$

$$
= \left[\underline{\underline{0}} \quad \underline{\underline{0}} \quad \underline{\underline{e}} \right]. \tag{218}
$$

5 Conclusion

In this paper, the equations of motion of flexible multibody dynamics with FFRF are shown, and the system parameters are fully derived with the widely used lumped mass approach. It is shown that the lumped mass approach is general in relation to the finite-element type, and thus, the derived equations are of general nature. The sensitivity analysis and the partial derivatives of the system parameters are derived for the general lumped mass approach and are shown with Euler parameters but can be extended to any orientation parametrization for generality.

The use of the lumped mass approach enables computing the invariants by numerical integration that is agnostic to the type of finite elements. The system parameters are derived for any orientation parametrization, and the associated simplifications for the singularity-free and the widespread used Euler parameters are shown. The expressions of the system parameters derived with the lumped mass approach correspond to the general case with the additional compensation terms of inertia due to the non-consideration of the mass distribution with lumped masses.

Analytical sensitivity analysis via direct differentiation is applied to the equations of motion for flexible multibody systems. This is the driving motor for efficient design optimization and uncertainty analysis, and this is where the partial derivatives of the system parameters appear. In addition to the sensitivity analysis, the equations of motion are differentiated w.r.t. the generalized coordinates, including positions, velocities and accelerations. These are required in the Jacobian matrix of the nonlinear solver when solving flexible multibody systems. Here the partial derivatives of the system parameters also appear.

In previous studies based on a semi-analytical approach, the partial derivatives of the system parameters have been computed numerically. To further increase the efficiency and the accuracy, the analytical derivatives of the structural system parameters of FFRF are provided here. The analytical derivatives are based on FFRF with the lumped mass approach, including the derivatives of the invariants enabling the decoupling of the sensitivity analysis of flexible multibody dynamics from the FE solver. The partial derivatives are shown with Euler parameters but can be extended to any orientation parametrization for generality. The derived partial derivatives are useful for the shown direct differentiation method and for the adjoint variable method as well. In addition, the partial Jacobians w.r.t. positions and velocities can be used in the nonlinear solver when solving the governing equations of flexible multibody dynamics for higher efficiency and accuracy.

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Author contributions V.G.: conceptualization, data curation, formal analysis, investigation, methodology, resources, software, validation, visualization, writing – original draft, writing – review & editing. A.Z.: conceptualization, formal analysis, investigation, methodology, supervision, validation, writing – original draft, writing – review & editing. E.W.: conceptualization, funding acquisition, methodology, project administration, supervision, writing – original draft, writing – review & editing.

Declarations

Competing interests The authors declare no competing interests.

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