LETTER

Comment on "Free surface tension in incompressible smoothed particle hydrodynamics (ISPH)" [Comput. Mech. 2020, 65, 487–502]

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Abstract

We comment on a recent article [Comput. Mech. 2020, 65, 487–502] about surface-tension modeling for free-surface flows with Smoothed Particle Hydrodynamics. The authors motivate part of their work related to a novel principal curvature approximation by the wrong claim that the classical curvature formulation in SPH overestimates the curvature in 3D by a factor of 2. In this note we confirm the correctness of the classical formulation and point out the misconception of the commented article.

Keywords Curvature · Surface-tension · SPH · Free-surface

1 Introduction

The authors of the paper "Free surface tension in incompressible smoothed particle hydrodynamics (ISPH)" [2] present "a Dirichlet pressure boundary condition for ISPH [...]" and "[...] a new approach to compute the curvature more exactly for three-dimensional cases [...]". This development is motivated by the claim that the established SPH curvature estimates give wrong results in three dimensions. As we will show below, this claim is based on a straightforward misconception in using the curvature term.

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2 Curvature definition

The singular surface-tension force $\mathbf{F_s}$ at a phase interface considering capillary forces only is given by

$$\mathbf{F_s} = \sigma \kappa_f \mathbf{n}, \tag{1}$$

where σ, κ_f and $\mathbf{n}$ denote the surface-tension coefficient, the curvature and the surface normal direction, respectively. Assuming constant material properties, the classical Young-Laplace formula for a quiescent spherical drop is simply $\Delta p = \sigma \kappa_f$.

Fluid mechanical curvature The *fluid mechanical curvature* is defined as

$$\kappa_f = -\nabla \cdot \mathbf{n} = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = (\kappa_1 + \kappa_2) \ , \tag{2}$$

where R_1 and R_1 are the principal radii of the surface, and κ_1 and κ_2 its respective principal curvature (see, e.g., [1]). Note, for a sphere in 3D with $R_1 = R_2 = R$, the curvature is given by $\kappa_f = \frac{2}{R}$. In 2D, this curvature is simply $\kappa_f = \frac{1}{R}$ (considering a cylindrical surface with $R_1 = R$ and $R_2 \to \infty$).

Mean curvature The *mean curvature* or *geometrical curvature* [5] is mathematically defined as

$$\kappa_g = -\frac{1}{2}\nabla \cdot \mathbf{n} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{1}{2}(\kappa_1 + \kappa_2) \ . \tag{3}$$

Here, for a sphere in 3D with $R_1 = R_2 = R$, the curvature reduces to $\kappa_g = \frac{1}{R}$. In 2D, there is only a single principal radius yielding $\kappa_g = \frac{1}{R}$.

3 Discussion

Obviously, Fürstenau et al have confused the two definitions and compared the numerical approximation for the fluid mechanical curvature (their eq. 34) with the mean curvature. This can be implied from a comparison of the two Figs. 2 and 3 in the article, where the analytical curvature for two bubbles in 2D and 3D is compared with the numerical approximations. Both analytical values are obtained from $\kappa_g = \frac{1}{R}$ for the given radii. From Fig. 3 showing the numerical curvature κ_f and analytical mean curvature $\kappa_g = \frac{1}{2}\kappa_f$ the authors conclude:

> "The approach was tested by comparing the curvatures of spherical bubbles in 2D and 3D test cases (see Fig. 1). When plotting the curvatures over the width (see Figs. 2, 3) it is obvious that in 2D the difference between ours and the standard approach is small, but in 3D the standard approach overestimates the curvature by a factor of 2 while ours is close to the analytical value." (p. 493, [2])

This statement or conclusion is wrong and needs to be rectified in order to avoid proliferation. We point out that identical results and erroneous claims also have been published in [3].

The geometrical curvature κ_g coincidences with the fluid mechanical curvature κ_f in 2D. In 3D, however, they differ by definition by a factor of 2. Amongst other references, the following quote from Taylor nicely clarifies this issue:

> "1.1 The *mean curvature* is $H = \kappa_1 + \kappa_2$. The most elementary approach of classical differential geometry [5] is to define principal curvatures κ_1 and κ_2, and then to define the mean curvature to be $(\kappa_1 + \kappa_2)/2$. The "mean" in "mean curvature" refers to this idea of the average of the curvatures. But in many ways, as will become clear below, it is much more natural not to divide by that 2, and it has become common to leave it out. Thus we will use $H = \kappa_1 + \kappa_2$". [6]

4 Conclusion

We emphasize that the well-known methods to compute the curvature via the divergence of the surface normals (e.g. [1,4]) give the correct results. Nonetheless, the proposed approach of the authors to extract the mean curvature still is valid and applicable. Yet, this methodology does not improve on the prediction accuracy of existing formulations and does not justify the additional computational effort for the local coordinate transformation with principal curvature extraction.

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