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Error Floors and Efficient Decoding of LDPC Codes

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Contents

Zusammenfassung

Diese Dissertation befasst sich mit binären, nicht-binären und verallgemeinerten Low-Density-Parity-Check (LDPC)-Codes. Die beiden Hauptthemen sind der Entwurf und die Analyse von quantisierten Nachrichtenübermittlungs Dekodern und die Untersuchung von sogenannten Error Floors. Für das letztere Thema analysieren wir die Anzahl der Trapping Sets (TSs), Absorbing Sets (ASs) und Fully Absorbing Sets (FASs), die Dekodierungsfehler verursachen können.

Es werden quantisierte Dekodieralgorithmen für binäre, nicht-binäre und verallgemeinerte LDPC-Codes vorgeschlagen und mit Hilfe der Verteilungsdichteevolution (density evolution) analysiert. Die Algorithmen modellieren die eingehende Nachrichten der variablen Knoten als Beobachtungen eines extrinsischen Kanals. Die Übergangswahrscheinlichkeiten des extrinsischen Kanals sind im Allgemeinen unbekannt, aber genaue Schätzungen werden durch Verteilungsdichteevolution erhalten. Die Verteilungsdichteevolution wird ferner zur Ableitung der asymptotischen iterativen Dekodierungsschwelle verwendet. Code-Ensembles werden entworfen, um diese Schwelle zu optimieren, und numerische Simulationen bestätigen die durch die asymptotische Analyse vorhergesagte Leistung. Eine Stabilitätsanalyse unterstreicht die Rolle, die variable Knoten des Grades 3 spielen.

Die endlichen und asymptotischen Enumeratoren von (elementary) TSs und (fully) ASs für unstrukturierte und Protograph-basierte binäre LDPC-Code-Ensembles werden abgeleitet. Die Enumeratoren werden zur Schätzung der Fehlergrenzen verwendet. In ähnlicher Weise werden die endlichen und asymptotischen Enumeratoren von TSs und (elementary) ASs für unstrukturierte und (eingeschränkte und nicht eingeschränkte) protograph-basierte nicht-binäre LDPC-Code-Ensembles abgeleitet. Die normalisierten logarithmischen asymptotischen Verteilungen werden durch Lösen eines Gleichungssystems erhalten. Die Definitionen von (elementary) TSs und (fully) ASs werden auf verallgemeinerte LDPC-Codes erweitert. Numerische Simulationen zeigen, dass die vorgeschlagenen Definitionen Graphenstrukturen ergeben, die für Bit-Flipping-Dekoder schädlich sind. Die (elementary) TS- und (fully) AS-Enumeratoren für unstrukturierte und protogrammbasierte verallgemeinerte LDPC-Code-Ensembles werden mit Hilfe von Generierungsfunktionen abgeleitet.

Abstract

This dissertation deals with binary, non-binary, and generalized low-density parity-check (LDPC) codes. The two main topics are the design and analysis of quantized message-passing decoders and the study of so-called error floors. For the latter topic, we analyze the number of trapping sets (TSs), absorbing sets (ASs), and fully absorbing sets (FASs) that may cause decoding errors.

Quantized decoding algorithms for binary, non-binary, and generalized LDPC codes are proposed and analyzed using density evolution. The algorithms model the variable node inbound messages as outputs of an extrinsic channel. The transition probabilities of the extrinsic channel are unknown in general but accurate estimates are obtained by density evolution. Density evolution is further used to derive the asymptotic iterative decoding threshold. Code ensembles are designed to optimize this threshold and numerical simulations confirm the performance predicted by the asymptotic analysis. A stability analysis highlights the role played by degree-3 variable nodes.

The finite-length and asymptotic enumerators of (elementary) TSs and (fully) ASs for unstructured and protograph-based binary LDPC code ensembles are derived. The enumerators are used to estimate the error floors. Similarly, the finite-length and asymptotic enumerators of TSs and (elementary) ASs for unstructured and (constrained and unconstrained) protograph-based non-binary LDPC code ensembles are derived. The normalized logarithmic asymptotic distributions are obtained by solving a system of equations. The definitions of (elementary) TSs and (fully) ASs are extended to generalized LDPC codes. Numerical simulations show that the proposed definitions yield graph structures that are harmful for bit flipping decoders. The (elementary) TS and (fully) AS enumerators for unstructured and protograph-based generalized LDPC code ensembles are derived using generating functions.

1

Introduction

An end-to-end communication system model has three key components: transmitter, channel, and receiver. The transmitter maps information to a signal, the channel corrupts the signal by noise, and the receiver estimates the information from its channel output. In his groundbreaking paper [3], Shannon proved that error-correcting codes can enable reliable communication at rates below a channel capacity. Since then, error-correcting codes have received a great deal of attention and many classes of codes have been discovered. For example, Elias showed in 1955 that linear error-correcting codes can achieve the capacity of a discrete memoryless channel (DMC). Channel codes designed in the 1950s and 1960s include Hamming codes [4], Reed-Muller codes [5], Bose-Chaudhuri-Hocquengham (BCH) codes [6], and Reed-Solomon (RS) codes [7].

A class of capacity-approaching codes called turbo codes appeared in 1993 [8]. Shortly afterward, low-density parity-check (LDPC) codes introduced by Gallager [9] were rediscovered. LDPC codes are a class of linear block codes characterized by a sparse parity-check matrix. The decoding algorithms of LDPC codes operate by iteratively exchanging messages between the nodes of the code graph. LDPC codes with an associated iterative decoding algorithm were shown to be capacity-approaching by MacKay [10]. Non-binary LDPC codes have an outstanding error correction capability, outperforming their binary counterparts substantially at short block lengths [11]. A more general class of LDPC codes called generalized low-density parity-check (GLDPC) codes was introduced by Tanner in [12]. GLDPC codes offer a trade-off between error floor and waterfall performance due to their good distance properties and the improved block codes used at the check nodes (CNs) (compared to single parity-check (SPC) codes employed by the CNs of LDPC codes). This comes at the cost of increasing decoding complexity.

After their rediscovery, LDPC codes found widespread use in many modern communication standards due to their outstanding performance. Much research has been devoted to developing encoding and decoding algorithms for these codes. The growing requirement for high data rates makes designing low-complexity and high-throughput decoding algorithms crucial. The study of low-complexity message passing algorithms for LDPC codes originates from the seminal work by Gallager [9]. In [9], Gallager introduced two decoding algorithms, known as Gallager A and Gallager B, where the variable and check nodes exchange binary messages. Coarse message quantization reduces the amount of information exchanged in the decoder, but decoding complexity can also be reduced by using simplified update rules at the CNs. Several works studied reducing the decoding complexity of LDPC and GLDPC codes [13–32].

The performance of an LDPC code in terms of frame error rate (FER)/bit error rate (BER) versus the signal-to-noise ratio (SNR) is characterized by two regions: the waterfall region characterized by a fast decline of the error probability with the SNR and the error floor region characterized by a flattening of the error probability. Density evolution (DE) analysis evaluates the iterative decoding thresholds of LDPC code ensembles, i.e., the worst channel parameter for which reliable transmission is possible for infinite block length. Thus, the DE can be used to design code ensembles with good waterfall performance. Almost all codes in the ensemble will have nearly the same waterfall performance [33]. In the error floor region, the performance under iterative decoding of LDPC codes is frequently dominated by the presence of specific graphical configurations in the code Tanner graphs [34–36]. Such structures, called trapping sets or near-codewords, were studied in [35, 36] for binary LDPC codes. A subclass of trapping sets, called (fully) absorbing sets, was introduced in [37]. The definitions of trapping sets (TSs) and absorbing sets (ASs) were extended to non-binary codes in [38–40]. As pointed out in [37], not all trapping sets cause decoding failures. Nevertheless, characterizing (e.g., enumerating) trapping and (fully) absorbing sets for LDPC code ensembles is useful to gain a deeper understanding of error floors. This is especially interesting for applications requiring very low error floors [41, 42], where Monte Carlo simulation is impractical. While notable exceptions exist (e.g. [43–45]), the impact of trapping and (fully) absorbing sets on code performance has generally been studied using code ensembles, following the approach of Gallager [37, 46–49].

1.1 Outline

The two main topics of this dissertation are reduced-complexity decoding algorithms of binary, non-binary, and generalized LDPC codes and the finite-length and asymptotic enumerators of trapping and (fully) absorbing sets of binary, non-binary, and generalized LDPC code ensembles. The thesis is structured as follows.

- ▷ **Chapter 2** provides basic notation and definitions necessary for the following chapters. We also introduce the channel models that we used.
- ▷ **Chapter 3** reviews linear error-correcting codes and their properties. The second section is dedicated to the LDPC codes. We begin by defining binary, non-binary, and generalized LDPC codes and code ensembles (unstructured and protograph-based). We briefly explain the enumeration methods for analyzing LDPC codes, which will be used to enumerate the trapping and (fully) absorbing sets in Chapters 6-8. Further, we review iterative decoding algorithms for LDPC codes.
- ▷ **Chapter 4** is devoted to the design and analysis of reduced complexity message passing decoding algorithms for binary and non-binary LDPC codes. For binary LDPC codes, we introduce the *matched* quantized min-sum (MQMS) decoder, where the exchanged check and variable node messages are represented by b bits. We consider two cases for the binary-input additive white Gaussian noise (biAWGN) channel output: unquantized symbols and quantized symbols. For the latter case, the biAWGN channel output is quantized using a b_0-bit uniform quantizer. For the non-binary case, we consider transmission over q-ary symmetric channels (QSCs), q-ary erasure channels (QECs), as well as additive white Gaussian noise (AWGN) and Poisson channels with pulse position modulation (PPM). We introduce and analyze the following decoders: symbol message passing (SMP), symbol and erasure message passing (SEMP), scaled reliability list message passing (SRLMP), and 1 and 2- bit reliability-based symbol message passing (RSMP).
- ▷ **Chapter 5** studies the performance of GLDPC codes under binary message passing (BMP) and ternary message passing (TMP) decoding. At the CNs, the binary and ternary messages are obtained either by using bounded distance decoding (BDD) or a posteriori probability (APP) soft-input soft-output (SISO) decoding.
- ▷ **Chapter 6** deals with the (elementary) trapping and (fully) absorbing set enumerators for binary LDPC code ensembles. First, we review the random matrix enumeration

approach, which was previously applied to obtain the asymptotic enumerators for (elementary) TSs for binary irregular LDPC code ensembles and the (elementary) (fully) ASs of regular LDPC code ensembles. We extend the analysis to obtain the (elementary) AS and fully absorbing set (FAS) enumerators of irregular LDPC code ensembles. Next, we provide alternative derivations of the (elementary) trapping and (fully) absorbing sets enumerators for binary unstructured LDPC codes based on generating functions. We also derive the finite-length and asymptotic (elementary) trapping and (fully) absorbing set enumerators for binary protograph-based LDPC code ensembles. Numerical results illustrate how the proposed enumeration technique can be used to estimate the error floor of LDPC codes.

- ▷ **Chapter 7** addresses trapping and (elementary) absorbing set enumerators for non-binary LDPC code ensembles. We consider unstructured and constrained, and unconstrained protograph-based code ensembles. We provide numerical evidence that these sets contribute to the error probability under certain hard-decision message passing decoding algorithms.

- ▷ **Chapter 8** focuses on (elementary) trapping and (fully) absorbing set enumerators for irregular and protograph-based GLDPC code ensembles. We propose new definitions of (elementary) trapping and (fully) absorbing sets for GLDPC codes. We derive the finite-length and asymptotic distributions of (elementary) TSs, ASs, and FASs for GLDPC code ensembles. The derivation is based on generating functions. The impact of these sets on the performance of a GLDPC code is confirmed through simulations. The enumerators are used to estimate the error floor of GLDPC codes.

- ▷ **Chapter 9** concludes the thesis and discusses future research directions.

1.2 Contributions of the Thesis

Most results in this thesis appeared in the following conference proceedings and journal publications:

- ▷ **E. Ben Yacoub**, F. Steiner, B. Matuz, G. Liva, "Protograph-Based LDPC Code Design for Ternary Message Passing Decoding," *Proc. ITG. Int. Conf. Syst., Commun. and Coding (SCC)*, Rostock, Germany, pp. 17-22, Feb. 2019. [50]

- ▷ **E. Ben Yacoub**, F. Lazaro, A. Graell i Amat, G. Liva, "Symbol Message Passing Decoding of Nonbinary Spatially-Coupled Low-Density Parity-Check Codes," *Proc.*

Int. Annual Conf. (AEIT), Florence, Italy, pp. 1-6, Sep. 2019. [51]

- ▷ **E. Ben Yacoub**, G. Liva, "Asymptotic Absorbing Set Enumerators for Irregular LDPC Code Ensembles," *Proc. Int. Zurich Seminar (IZS)*, Zurich, Switzerland, pp. 36-40, Feb. 2020. [52]
- ▷ **E. Ben Yacoub**, G. Liva, G. Kramer, "Efficient Evaluation of Asymptotic Trapping Set Enumerators for Irregular LDPC Code Ensembles," *Proc. Int. Zurich Seminar (IZS)*, Zurich, Switzerland, pp. 49-52, Feb. 2020. [53]
- ▷ **E. Ben Yacoub**, G. Liva, "Asymptotic Absorbing Set Enumerators for Non-Binary Protograph-Based LDPC Code Ensembles," *Proc. Int. Symp. Inf. Theory (ISIT)*, Los Angeles, California, United States, pp. 355-360, Jun. 2020. [54]
- ▷ **E. Ben Yacoub**, "Matched Quantized Min-Sum Decoding of Low-Density Parity-Check Codes," *Proc. Inf. Theory (ITW)*, Riva del Gada, Italy, pp. 1-5, Apr. 2021. [55]
- ▷ **E. Ben Yacoub**, "List Message Passing Decoding of Non-binary Low-Density Parity-Check Codes," *Proc. Int. Symp. Inf. Theory (ISIT)*, Melbourne, Australia, pp. 84-89, Jul. 2021. [56]
- ▷ **E. Ben Yacoub**, B. Matuz, A. Graell i Amat, G. Liva, "Quaternary Message Passing Decoding of LDPC Codes: Density Evolution Analysis and Error Floor," *Proc. Int. Symp. on Topics in Coding (ISTC)*, Montreal, Canada, pp. 1-5, Aug. 2021. [57]
- ▷ **E. Ben Yacoub**, B. Matuz, "Symbol Message Passing Decoding of LDPC Codes for Orthogonal Modulations," *Proc. Int. Symp. on Topics in Coding (ISTC)*, Montreal, Canada, pp. 1-5, Aug. 2021. [58]
- ▷ **E. Ben Yacoub**, G. Liva, "Analysis of Binary and Ternary Message Passing Decoding for Generalized LDPC Codes," *Proc. Int. Symp. Problems of Redundancy in Inf. and Control Syst. (REDUNDANCY)*, Moscow, Russian Federation, pp. 137-142, Oct. 2021. [1]
- ▷ **E. Ben Yacoub**, "Reliability-Based Message Passing Decoding of Non-binary Low-Density Parity-Check Codes," *Proc. Int. Zurich Seminar (IZS)*, Zurich, Switzerland, pp. 39-43, Mar. 2022. [59]
- ▷ **E. Ben Yacoub**, B. Matuz, "Analysis of Symbol Message Passing LDPC Decoder for the Poisson PPM Channel," *Proc. Int. Symp. Inf. Theory (ISIT)*, Espoo, Finland, pp. 1193-1198, Jun. 2022. [60]

- **E. Ben Yacoub**, G. Liva, “Fully Absorbing Set Enumerators for Protograph-Based LDPC Code Ensembles,” *IEEE Commun. Letters*, vol. 26, no. 12, pp. 2831-2835, Dec. 2022. [61]

- **E. Ben Yacoub**, G. Liva, “Trapping and Absorbing Set Enumerators for Nonbinary Protograph-Based Low-Density Parity-Check Code Ensembles,” *IEEE Trans. Commun.*, vol. 71, no. 4, pp. 1847-1862, Apr. 2023. [62]

- **E. Ben Yacoub**, “Trapping and Absorbing Set Enumerators for Irregular Generalized Low-Density Parity-Check Code Ensembles,” *IEEE Trans. Inf. Theory*, vol. 69 , no. 6, pp. 3637-3662, Jun. 2023. [63]

- **E. Ben Yacoub**, “Weight and Trapping Set Distributions for Non-Binary Regular Low-Density Parity-Check Code Ensembles,” *Proc. Int. Symp. Inf. Theory (ISIT)*, Taipei, Taiwan, pp. 2141-2146, Jun. 2023. [64]

- **E. Ben Yacoub**, G. Liva, “Analysis of Binary and Ternary Message Passing Decoding for Generalized LDPC Codes,” *IEEE Trans. Commun.*, vol. 71 , no. 9 , pp. 5078-5092, Sep. 2023. [65]

- **E. Ben Yacoub**, “Trapping and Absorbing Set Enumerators for Protograph-Based Generalized Low-Density Parity-Check Code Ensembles,” *Proc. Int. Symp. Problems of Redundancy in Inf. and Control Syst. (REDUNDANCY)*, Moscow, Russian Federation, pp. 142-147, Oct. 2023. [66]

1.3 Contributions Outside the Scope of the Thesis

The publications of the author during the thesis period, which are not included in the main results of the thesis, are as follows:

- F. Steiner, **E. Ben Yacoub**, B. Matuz, G. Liva, A. Graell i Amat, “One and Two Bit Message Passing for SC-LDPC Codes with Higher-Order Modulation,” *IEEE/OSA J. Lightw. Technol.*, vol. 37, no. 23, pp. 5914-5925, Dec. 2019. [67]

- H. Bartz, **E. Ben Yacoub**, L. Bertarelli, G. Liva, “Protograph-Based Decoding of LDPC Codes with Hamming Weight Amplifiers,” *Proc. Int. Workshop on Code-Based Cryptography*, Zagreb, Croatia, pp. 80-93, May 2020. [68]

- ▷ **E. Ben Yacoub**, G. Liva, E. Paolini, "Bounds on the Error Probability of Nonbinary Linear Codes over the Q-ary Symmetric Channel," *Proc. Conf. on Inf. Sciences and Syst. (CISS)*, Baltimore, USA, pp. 1-6, Mar. 2021. [69]
- ▷ D. Lentner, **E. Ben Yacoub**, S. Calabro, G.Böcherer, N. Stojanovic, G. Kramer, "Concatenated Forward Error Correction with KP4 and Single Parity Check Codes," *IEEE/OSA J. Lightw. Technol.*, vol. 41, no. 17, pp. 5641-5652, Sep. 2023. [70]
- ▷ **E. Ben Yacoub**, "Trapping and Absorbing Set Enumerators for Multi-Edge Type LDPC Code Ensembles," *Proc. 59th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, USA, pp. 1-8, Sep. 2023. [71]
- ▷ A. Zahr, **E. Ben Yacoub**, B. Matuz, G. Liva, "Rate-Adaptive Protograph-Based MacKay-Neal Codes," *submitted*, 2024. [72]
- ▷ **E. Ben Yacoub**, G. Liva, E. Paolini, M. Chiani, "Combinatorial Analysis of Non-Adaptive Group Testing with Sparse Pooling Graphs," *submitted*, 2024. [73]

2

Preliminaries

2.1 Notation and Definitions

This chapter introduces notation and definitions. The indicator function $\mathbb{I}(\mathcal{A})$ takes on the value 1 if the proposition $\mathcal{A}$ is true and 0 otherwise. For $\boldsymbol{z} = (z_1, z_2, \ldots, z_d)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d)$, we use the shorthand

$$\boldsymbol{z}^{\boldsymbol{\beta}} = \prod_{t=1}^{d} z_t^{\beta_t}.$$

We denote random variables (RVs) with capital letters and the corresponding realizations with lowercase letters. The probability mass function (PMF) of the RV X is referred to as P_X, while a probability density function (PDF) is written as p_X.

For positive values $\alpha_1, \ldots, \alpha_d$ that sum to one, we define the natural entropy function as

$$H(\alpha_1, \ldots, \alpha_d) = -\sum_{i=1}^{d} \alpha_i \ln(\alpha_i). \tag{2.1}$$

For $d = 2$, we use the shorthand

$$H_{\mathsf{b}}(\alpha) = -\alpha \ln(\alpha) - (1-\alpha) \ln(1-\alpha). \tag{2.2}$$

Definition 2.1. Let $x(n)$ and $y(n)$ be two real-valued sequences, where $y(n) \neq 0$ $\forall n$, $x(n)$

is exponentially equivalent to $y(n)$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{x(n)}{y(n)}\right) = 0.$$

We will use the notation $x(n) \doteq y(n)$ to specify that $x(n)$ is exponentially equivalent to $y(n)$.

2.2 Probability Theory

2.2.1 Random Variables

Let Ω be the set of all possible outcomes of a probabilistic experiment called a sample space. A RV X is a mapping from Ω into another set $\mathcal{X}$. The distribution of a discrete RV X is characterized by the PMF P_X, which gives the probability that X is equal to some number, i.e., for all $x \in \mathcal{X}$

$$P_X(x) = \Pr\{X = x\}. \tag{2.3}$$

We have

$$P_X(x) \geq 0 \quad \forall x \in \mathcal{X} \text{ and } \sum_{x\in\mathcal{X}} P_X(x) = 1. \tag{2.4}$$

The support of X is defined as

$$\text{supp}(P_X) = \{x \in \mathcal{X} : P_X(x) > 0\}. \tag{2.5}$$

If $\mathcal{X} = \mathbb{R}$, the continuous RV is specified by its cumulative distribution function (CDF),

$$F_X(x) = \Pr\{X \leq x\} \quad \forall x \in \mathbb{R}. \tag{2.6}$$

If the CDF is continuous and differentiable, the PDF of X is defined as

$$p_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}. \tag{2.7}$$

The PDF fulfills

$$p_X(x) \geq 0 \quad \text{and} \int_{-\infty}^{+\infty} p_X(x)\,\mathrm{d}x = 1. \tag{2.8}$$

We will define the moments and some information-theoretic quantities of discrete RVs in the following. The extensions to continuous RVs are straightforward; one needs to replace PMFs with PDFs and summation by integration.

2.2.2 Moments of Random Variables

Consider a discrete RV X and the function $f : \mathcal{X} \to \mathbb{R}$. The expectation of $f(X)$ is

$$\mathbb{E}[f(X)] = \sum_{x \in \text{supp}(P_X)} f(x) P_X(x). \tag{2.9}$$

Let Y be a discrete RV. The conditional expectation of X given Y is defined for $y \in \text{supp}(P_Y)$ as follows:

$$\mathbb{E}[X|Y = y] = \sum_{x \in \text{supp}(P_{X|Y}(\cdot|y))} x P_{X|Y}(x|y). \tag{2.10}$$

The variance of a X is

$$\mathbb{V}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{2.11}$$

2.2.3 Information Measures

The self-information $i(x) = -\log_2(P_X(x))$ measures the amount of information associated with the realization x of X. The entropy of X is defined as

$$H(X) = \mathbb{E}[i(X)] = -\sum_{x \in \text{supp}(P_X)} P_X(x) \log_2 (P_X(x)). \tag{2.12}$$

Let P_X and P_Y be two PMFs with $\text{supp}(P_X) \subseteq \text{supp}(P_Y)$. The information divergence measures a difference of these distributions and is defined as

$$D(P_X||P_Y) = \sum_{x \in \text{supp}(P_X)} P_X(x) \log_2 \left(\frac{P_X(x)}{P_Y(x)} \right). \tag{2.13}$$

The mutual information of two discrete RVs X and Y is defined as

$$I(X;Y) = D(P_{XY}||P_X P_Y). \tag{2.14}$$

The capacity of a DMC with input X and output Y is the maximum rate for which reliable communication can be achieved and is given by

$$C = \max_{P_X} I(X;Y). \tag{2.15}$$

2.3 Channel Models

2.3.1 Q-ary Erasure Channel

A q-ary erasure channel (QEC) has input alphabet $\mathcal{X} = \mathbb{F}_q$ and output alphabet $\mathcal{Y} = \{\mathsf{E}\} \cup \mathbb{F}_q$, where E is an erasure denoting complete uncertainty about the transmitted symbol. The transition probabilities of this channel are

$$P_{Y|X}(y|x) = \begin{cases} 1-\epsilon & y = x \\ \epsilon & y = \mathsf{E} \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

The channel capacity (in bits per channel use) of the QEC is achieved by a uniform input distribution and is computed as

$$C_{\text{QEC}} = (1-\epsilon)\log_2(q). \tag{2.17}$$

2.3.2 Q-ary Symmetric Channel

Consider a q-ary symmetric channel (QSC) with error probability ϵ and input and output alphabet $\mathcal{X} = \mathcal{Y} = \mathbb{F}_q$. The QSC is illustrated in Fig. 2.1. The transition probabilities of this QSC are

$$P(y|x) = \begin{cases} 1-\epsilon & \text{if } y = x \\ \frac{\epsilon}{q-1} & \text{otherwise.} \end{cases} \tag{2.18}$$

As the channel is strongly symmetric, the capacity in bits per channel use, is given by

$$C_{\text{QSC}} = \log_2(q) + \epsilon \log_2\left(\frac{\epsilon}{q-1}\right) + (1-\epsilon)\log_2(1-\epsilon). \tag{2.19}$$

2.3.3 Q-ary Error and Erasure Channel

Consider a q-ary error and erasure channel (QEEC) with error probability ϵ, erasure probability θ, input alphabet $\mathcal{X} = \mathbb{F}_q$ and output alphabet $\mathcal{Y} = \{\mathsf{E}\} \cup \mathbb{F}_q$, where E

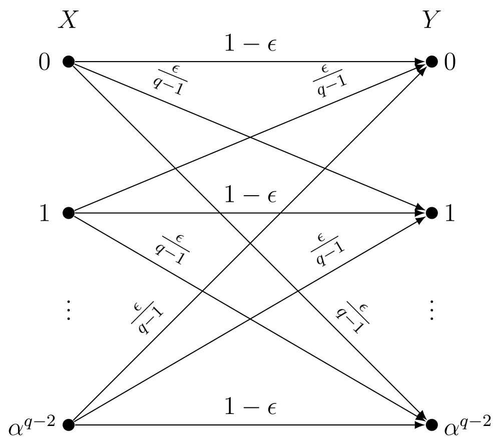


Figure 2.1: The QSC with error probability ϵ.

represents an erasure. The transition probabilities of the QEEC are given by

$$P(y|x) = \begin{cases} 1-\epsilon-\theta & \text{if } y = x \\ \theta & \text{if } y = \mathsf{E} \\ \frac{\epsilon}{q-1} & \text{otherwise.} \end{cases} \tag{2.20}$$

The capacity in bits per channel use is

$$C_{\text{QEEC}} = \epsilon \log_2\left(\frac{\epsilon}{q-1}\right) + (1-\epsilon-\theta)\log_2(1-\epsilon-\theta) - (1-\theta)\log_2\left(\frac{1-\theta}{q}\right). \tag{2.21}$$

2.3.4 Poisson Pulse Position Modulation Channel

Consider the finite field $\mathbb{F}_q = \{0, 1, \alpha, \ldots, \alpha^{q-2}$, where q is a power of two and α is a primitive element of $\mathbb{F}_q$. A pulse position modulation (PPM) symbol $\boldsymbol{x} = \left(x_0, x_1, \ldots, x_{\alpha^{q-2}}\right)$ spans q time slots of which one slot has a pulse and the remaining $q-1$ slots are blank. With slight abuse of notation we may write the slot index u as an element of a finite field. We denote by P_u a PPM symbol for which the u-th time slot contains a pulse. We denote the channel input alphabet by $\mathcal{X} = \{\mathsf{P}_0, \mathsf{P}_1, \ldots, \mathsf{P}_{\alpha^{q-2}}\}$. For orthogonal modulations, for all $a, a' \in \mathbb{F}_q$ we have

$$\langle \mathsf{P}_a, \mathsf{P}_{a'} \rangle = \begin{cases} 1 & a = a' \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

We consider transmission over an optical channel with direct detection at the receiver. Let $\boldsymbol{y} = \left(y_0, y_1, \ldots, y_{\alpha^{q-2}}\right)$ be the received sequence, where y_u is the number of received photons in the u-th slot. Let n_{s} be the average number of received signal photons per pulsed slot and let n_{b} be the average number of received background noise photons per slot. Considering the u-th slot, the channel transition probabilities follow a Poisson distribution, i.e., for all $y \in \mathbb{N}_0$ we have

$$P_{Y_u|\boldsymbol{X}}(y|\mathsf{P}_{u'}) = \begin{cases} \frac{\exp(-(n_{\mathsf{b}}+n_{\mathsf{s}}))(n_{\mathsf{b}}+n_{\mathsf{s}})^y}{y!} & u' = u \\ \frac{\exp(-n_{\mathsf{b}})n_{\mathsf{b}}^y}{y!} & \text{else.} \end{cases} \tag{2.23}$$

For $a \in \mathbb{F}_q$, we have the likelihood

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\mathsf{P}_a) = \prod_{u\in\mathbb{F}_q} P_{Y_u|\boldsymbol{X}}(y_u|\mathsf{P}_a) = \left(1 + \frac{n_{\mathsf{s}}}{n_{\mathsf{b}}}\right)^{y_a} \exp(-(n_{\mathsf{s}} + qn_{\mathsf{b}})) \prod_{u\in\mathbb{F}_q} \frac{n_{\mathsf{b}}^{y_u}}{y_u!}. \tag{2.24}$$

Let $\gamma = n_{\mathsf{s}}/q$ be the average number of received signal photons per slot. For the Poisson channel with q-ary PPM, the channel capacity in bits per channel use is given by

$$C_{\text{PPM Poisson}} = \log_2(q) - \mathbb{E}\left[\log_2\left(\sum_{u\in\mathbb{F}_q} \frac{\Gamma(Y_u)}{\Gamma(Y_0)}\right) \middle| \boldsymbol{X} = \mathsf{P}_0\right] \tag{2.25}$$

where for $u \in \mathbb{F}_q$

$$\Gamma(Y_u) = \left(\frac{n_{\mathsf{s}}}{n_{\mathsf{b}}} + 1\right)^{Y_u} \exp(-n_{\mathsf{s}}). \tag{2.26}$$

2.3.5 Additive White Noise Channel with Pulse Position Modulation

Consider transmission over an additive white Gaussian noise (AWGN) channel with q-ary PPM, where q is a power of two. The channel output is

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n} \tag{2.27}$$

where $\boldsymbol{n} = (n_0, n_1, \ldots, n_{\alpha^{q-2}})$ is the length-q noise vector sampled from q independent and identically distributed Gaussian RVs with zero-mean and variance σ^2. The likelihood

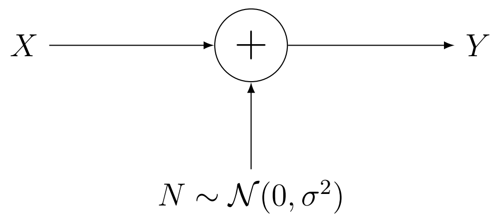


Figure 2.2: The biAWGN channel.

$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$ can be written as

$$p(\boldsymbol{y}|\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^q \exp\left(-\frac{||\boldsymbol{y}-\boldsymbol{x}||^2}{2\sigma^2}\right) \propto \exp\left(-\frac{1}{\sigma^2}\langle \boldsymbol{y}, \boldsymbol{x}\rangle\right). \tag{2.28}$$

Let E_{b} denote the energy per information bit, E_{s} the energy per modulation symbol, and N_0 is the one-sided noise power spectral density. Then, we have

$$\frac{E_{\mathsf{b}}}{N_0} = \frac{1}{R\log_2(q)}\frac{E_{\mathsf{s}}}{N_0} = \frac{1}{R\log_2(q)}\frac{1}{2\sigma^2}. \tag{2.29}$$

The capacity of the AWGN with PPM is given by

$$C_{\text{PPM AWGN}} = \log_2(q) - \mathbb{E}\left[\log_2\left(\sum_{u\in\mathbb{F}_q}\exp\left(\frac{Y_u - Y_0}{\sigma^2}\right)\right)\middle|\boldsymbol{X} = \mathsf{P}_0\right]. \tag{2.30}$$

2.3.6 Binary Input Additive White Noise Channel

The binary-input additive white Gaussian noise (biAWGN) channel is depicted in Fig. 2.2 and takes as input a RV $X \in \{-1, +1\}$ and outputs $Y = X + N$, where N is a zero-mean Gaussian RV with variance σ^2. Thus, the channel transition density is given by

$$p_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-x)^2}{2\sigma^2}\right). \tag{2.31}$$

The SNR is defined as

$$\text{SNR} = \frac{E_{\mathsf{s}}}{N_0} \tag{2.32}$$

where $N_0 = 2\sigma^2$ and E_{s} is the average signal energy, which is, in this case, equal to 1. Another important measure is the normalized SNR

$$\frac{E_{\mathsf{b}}}{N_0} = \frac{E_{\mathsf{s}}}{N_0 R} = \frac{1}{2\sigma^2 R} \tag{2.33}$$

where E_b is the energy per information bit.

The capacity of the biAWGN is achieved by a uniform input distribution and is given by

$$C_{\text{biAWGN}} = 1 - \int_{-\infty}^{+\infty} p_{Y|X}(y|+1) \log_2 \left(1 + \exp\left(-\frac{2y}{\sigma^2}\right)\right) \mathrm{d}y. \tag{2.34}$$

3

LDPC Codes

3.1 Linear Codes

Definition 3.1 (Linear codes)**.** A q-ary (n, k) linear block code $\mathscr{C}$ of length n and dimension k is a k-dimensional linear subspace of $\mathbb{F}_q^n$.

An (n, k) linear code $\mathscr{C}$ can be characterized by a generator matrix $\boldsymbol{G} \in \mathbb{F}_q^{k\times n}$, whose rows span $\mathscr{C}$. The generator matrix $\boldsymbol{G}$ encodes the information sequence $\boldsymbol{u}$ into the codeword $\boldsymbol{c} \in \mathscr{C}$ as $\boldsymbol{c} = \boldsymbol{u} \cdot \boldsymbol{G}$. The code $\mathscr{C}$ can be equivalently defined by its parity-check matrix $\boldsymbol{H} \in \mathbb{F}_q^{(n-k)\times n}$, whose null space is $\mathscr{C}$, i.e.,

$$\mathscr{C} = \left\{\boldsymbol{c} \in \mathbb{F}_q^n | \boldsymbol{c} \cdot \boldsymbol{H}^\mathsf{T} = \boldsymbol{0}\right\}. \tag{3.1}$$

Definition 3.2 (Hamming weight)**.** The Hamming weight of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$ is defined as the number of its non-zero entries. Formally,

$$\mathsf{w}_\mathrm{H}(\boldsymbol{x}) = \sum_{i=1}^{n} \mathbb{I}(x_i \neq 0). \tag{3.2}$$

The weight distribution of a code $\mathscr{C}$ is typically described using its weight enumerator function (WEF)

$$W(x) = \sum_{i=0}^{n} W_i x^i \tag{3.3}$$

where x is a dummy variable and $\mathrm{coeff}(W(x), x^i)$ gives the cardinality of codewords with weight i.

The Hamming distance between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^n$ is the number of entries in which they differ. We have

$$\mathsf{d}_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \mathbb{I}(x_i \neq y_i) = \mathsf{w}_{\mathrm{H}}(\boldsymbol{x} - \boldsymbol{y}). \tag{3.4}$$

The minimum distance of a code is the minimum number of entries in which any two codewords differ. For a linear code $\mathscr{C}$, the minimum distance is equal to the minimal Hamming weight among all non-zero codewords, i.e.,

$$\mathsf{d}_{\min} = \min_{\substack{\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathscr{C} \\ \boldsymbol{c}_1 \neq \boldsymbol{c}_2}} \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}_1, \boldsymbol{c}_2) = \min_{\boldsymbol{c} \in \mathscr{C} \setminus \boldsymbol{0}} \mathsf{w}_{\mathrm{H}}(\boldsymbol{c}). \tag{3.5}$$

Another code parameter of interest is the decoding radius, i.e., its guaranteed error correction capability under bounded distance decoding (BDD)

$$t = \left\lfloor \frac{\mathsf{d}_{\min} - 1}{2} \right\rfloor. \tag{3.6}$$

Bounded Distance Decoding

Consider a linear code $\mathscr{C}$ with minimum distance $\mathsf{d}_{\min}$. BDD can correct all error patterns of Hamming weight $t = \lfloor (\mathsf{d}_{\min} - 1)/2 \rfloor$ or less. For a received sequence $\boldsymbol{z}$, the decoded vector $\hat{\boldsymbol{c}}$ is given by

$$\hat{\boldsymbol{c}} = \begin{cases} \boldsymbol{c} \in \mathscr{C} & \text{if } \exists \boldsymbol{c} \in \mathscr{C} \text{ with } \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{z}) \leq t \\ \boldsymbol{z} & \text{otherwise} \end{cases}$$

where $\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{z})$ is the Hamming distance between $\boldsymbol{c}$ and $\boldsymbol{z}$.

Bounded Distance Decoding with Erasures

Consider a linear code $\mathscr{C}$ with minimum distance $\mathsf{d}_{\min}$. For a received sequence $\boldsymbol{z}$ with v erasures, the decoded vector $\hat{\boldsymbol{c}}$ is given by

$$\hat{\boldsymbol{c}} = \begin{cases} \boldsymbol{c} \in \mathscr{C} & \text{if } \exists \boldsymbol{c} \in \mathscr{C} \text{ with } 2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{z}) + v \leq \mathsf{d}_{\min} - 1 \\ \mathbf{E} & \text{otherwise} \end{cases}$$

where $\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{z})$ is the Hamming distance between $\boldsymbol{c}$ and $\boldsymbol{z}$ and $\mathbf{E} = (\mathsf{E}, \ldots, \mathsf{E})$.

3.2 Binary LDPC Codes

3.2.1 Binary Unstructured LDPC Codes

Binary LDPC codes are linear block codes characterized by an $m \times n$ sparse binary parity-check matrix $\boldsymbol{H}$. The parity-check matrix can be represented by a Tanner graph, which is a bipartite graph $G = (\mathcal{V} \cup \mathcal{C}, \mathscr{E})$ consisting of n variable nodes (VNs) corresponding to codeword bits and m CNs corresponding to parity checks. The set $\mathscr{E}$ of edges contains the elements e_{ij}, where e_{ij} is an edge between VN $\mathsf{v}_j \in \mathcal{V}$ and CN $\mathsf{c}_i \in \mathcal{C}$. Note that e_{ij} belongs to the set $\mathcal{E}$ if and only if the parity-check matrix element h_{ij} equals 1.

The sets $\mathcal{N}(\mathsf{v}_j)$ and $\mathcal{N}(\mathsf{c}_i)$ denote the neighbors of VN v_j and CN c_i, respectively. The degree of a VN v_j is the cardinality of the set $\mathcal{N}(\mathsf{v}_j)$. Similarly, the degree of a CN c_i is the cardinality of the set $\mathcal{N}(\mathsf{c}_i)$.

The node-oriented degree distribution polynomials of an LDPC code graph are given by

$$\Lambda(x) = \sum_i \Lambda_i x^i, \quad \mathrm{P}\,(x) = \sum_i \mathrm{P}_i x^i \tag{3.7}$$

where Λ_i corresponds to the fraction of VNs with degree i and P_i corresponds to the fraction of CNs with degree i. The VN edge-oriented degree distribution polynomial of an LDPC code graph is given by

$$\lambda(x) = \sum_i \lambda_i x^{i-1} \tag{3.8}$$

where λ_i corresponds to the fraction of edges incident to VNs with degree i. Similarly, the CN edge-oriented degree distribution polynomial is given by

$$\rho\,(x) = \sum_i \rho_i x^{i-1} \tag{3.9}$$

where ρ_i corresponds to the fraction of edges incident to CNs with degree i. One can convert a node perspective degree distribution into an edge perspective degree distribution as follows:

$$\lambda_i = \frac{i\Lambda_i}{\sum_j j\Lambda_j}, \quad \rho_i = \frac{i\mathrm{P}_i}{\sum_j j\mathrm{P}_j}. \tag{3.10}$$

Let $\mathsf{d}_\mathsf{v}^{\max}$ ($\mathsf{d}_\mathsf{c}^{\max}$) be the maximum VN (CN) degree. We denote by

$$\bar{\mathsf{d}}_\mathsf{v} = \sum_{i=1}^{\mathsf{d}_\mathsf{v}^{\max}} i\Lambda_i \quad \bar{\mathsf{d}}_\mathsf{c} = \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i\mathrm{P}_i \tag{3.11}$$

the average VN and CN degrees, respectively. Note that $n\bar{\mathsf{d}}_{\mathsf{v}} = m\bar{\mathsf{d}}_{\mathsf{c}}$ represents the total number of edges. We define

$$\xi = \frac{m}{n} = \frac{\bar{\mathsf{d}}_{\mathsf{v}}}{\bar{\mathsf{d}}_{\mathsf{c}}}. \tag{3.12}$$

The rate of the LDPC code can be lower bounded as

$$R \geq 1 - \frac{m}{n} \tag{3.13}$$

where equality holds if $\boldsymbol{H}$ is full rank.

An unstructured binary irregular LDPC code ensemble $\mathscr{C}_n^{\lambda,\rho}$ ($\mathscr{C}_n^{\Lambda,\mathrm{P}}$) is the set of all binary LDPC codes with block length n and degree distributions $\lambda(x)$ and $\rho(x)$ ($\Lambda(x)$ and $\mathrm{P}(x)$).

An LDPC code is called $(\mathsf{d}_{\mathsf{v}}, \mathsf{d}_{\mathsf{c}})$ regular if all the VNs have the same degree d_{v} and all the CNs have the same degree d_{c}, i.e.,

$$\Lambda(x) = x^{\mathsf{d}_{\mathsf{v}}}, \quad \mathrm{P}(x) = x^{\mathsf{d}_{\mathsf{c}}}. \tag{3.14}$$

We denote by $\mathscr{C}_n^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$ the binary regular LDPC code ensemble, which is the set of all binary LDPC codes with block length n, VN degree d_{v} and CN degree d_{c}.

3.2.2 Binary Protograph-Based LDPC Codes

A protograph $\mathsf{P} = (\mathcal{V}^{\mathsf{P}}, \mathcal{C}^{\mathsf{P}}, \mathscr{E}^{\mathsf{P}})$ is a small Tanner graph consisting of n_{P} VNs, m_{P} CNs and e edges forming the sets $\mathcal{V}^{\mathsf{P}}, \mathcal{C}^{\mathsf{P}}$ and $\mathscr{E}^{\mathsf{P}}$, respectively. It can be defined by an $m_{\mathsf{P}} \times n_{\mathsf{P}}$ base matrix $\boldsymbol{B} = [b_{i,j}]$, where $b_{i,j}$ is the number of edges connecting $\mathsf{v}_j^{\mathsf{P}}$ to $\mathsf{c}_i^{\mathsf{P}}$. Each VN/CN/edge in a protograph defines a VN/CN/edge type. We denote by $\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}$ ($\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$) the set of edges in the protograph connected to $\mathsf{v}_j^{\mathsf{P}}$ ($\mathsf{c}_i^{\mathsf{P}}$). The degree $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ of $\mathsf{v}_j^{\mathsf{P}}$ ($\mathsf{d}_{\mathsf{c}_\mathsf{i}}$ of $\mathsf{c}_i^{\mathsf{P}}$) is then equal to $|\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}|$ ($|\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}|$). The bipartite graph G of an LDPC code can be derived by lifting the protograph. In particular, the protograph is copied Q times (where Q is referred to as the lifting factor), and the edges of the protograph copies are permuted under the following constraint: if an edge connects a type-$\mathsf{v}_j^{\mathsf{P}}$ VN to a type-$\mathsf{c}_i^{\mathsf{P}}$ CN in P, after permutation the edge should connect one of the Q type-v_j VN copies with one of the Q type-c_i CN copies in G. In the following, we will denote by $\mathcal{V}$ and $\mathcal{C}$ the sets of CNs and VNs in G, respectively. The lifted graph G defines the $m \times n$ parity-check matrix $\boldsymbol{H}$, where $m = m_{\mathsf{P}}Q$ and $n = n_{\mathsf{P}}Q$. To distinguish the VNs and CNs in the protograph from those in the lifted graph, we use the subscript P. A protograph-based LDPC code ensemble $\mathscr{C}_n^{\mathsf{P}}$ is the set of length-n LDPC codes whose bipartite graph G is obtained by lifting P.

3.3 Protograph-Based Spatially Coupled LDPC Codes

Spatially coupled low-density parity-check code (SC-LDPC) codes are known to show a *threshold saturation effect* [74] that allows approaching the bit-wise maximum a posteriori (MAP) decoding threshold of the underlying block code with (unquantized) belief propagation (BP) decoding.

We consider protograph-based SC-LDPC codes with base matrix in the form

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_0 & & & \\ \boldsymbol{B}_1 & \boldsymbol{B}_0 & & \\ \vdots & \boldsymbol{B}_1 & \boldsymbol{B}_0 & \\ \boldsymbol{B}_\mu & \vdots & \boldsymbol{B}_1 & \\ & \boldsymbol{B}_\mu & \vdots & \ddots \\ & & \boldsymbol{B}_\mu & \ddots \\ & & & \ddots \end{pmatrix} \tag{3.15}$$

where μ denotes the syndrome former memory of the SC-LDPC code. The protograph in (3.15) is then lifted by a factor of Q to obtain the final parity-check matrix $\boldsymbol{H}$.

For practical operation, the SC-LDPC code is commonly terminated after a number of S spatial positions. Due to this termination, a rate loss occurs that vanishes for large S. The resulting code rate is

$$R = 1 - \frac{\mu + S}{S} \frac{m_{\mathsf{P}}^{\mathrm{SC}}}{n_{\mathsf{P}}^{\mathrm{SC}}} = 1 - \left(1 + \frac{\mu}{S}\right) \frac{m_{\mathsf{P}}^{\mathrm{SC}}}{n_{\mathsf{P}}^{\mathrm{SC}}} \tag{3.16}$$

where the base matrices $\boldsymbol{B}_0, \ldots, \boldsymbol{B}_\mu$ have dimensions $m_{\mathsf{P}}^{\mathrm{SC}} \times n_{\mathsf{P}}^{\mathrm{SC}}$. The overall size of the matrix $\boldsymbol{B}$ is $m_{\mathsf{P}} \times n_{\mathsf{P}}$ where $m_{\mathsf{P}} = (\mu + S) m_{\mathsf{P}}^{\mathrm{SC}}$ and $n_{\mathsf{P}} = S n_{\mathsf{P}}^{\mathrm{SC}}$.

3.4 Non-Binary LDPC Codes

3.4.1 Non-Binary Unstructured LDPC Codes

Non-binary LDPC codes are linear block codes characterized by an $m \times n$ sparse parity-check matrix $\boldsymbol{H}$ with coefficients in $\mathbb{F}_q$. The parity-check matrix can be represented by a Tanner graph $G = (\mathcal{V} \cup \mathcal{C}, \mathscr{E})$ consisting of n VNs corresponding to codeword symbols and m CNs corresponding to parity checks. The set $\mathscr{E}$ of edges contains the elements e_{ij},

$$
\boldsymbol{H} = \begin{bmatrix} \alpha & 1 & 0 & \alpha^2 & 1 & 0 & 0 \\ 1 & 0 & \alpha & 1 & 0 & \alpha^2 & 0 \\ 0 & \alpha^2 & \alpha & 1 & 0 & 0 & \alpha \end{bmatrix}
$$

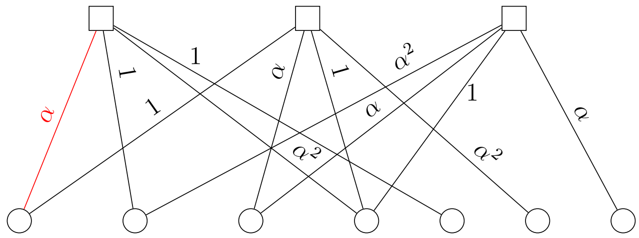


Figure 3.1: Tanner Graph and its corresponding parity-check matrix of a 4-ary LDPC code.

where e_{ij} is an edge between VN $\mathsf{v}_j \in \mathcal{V}$ and CN $\mathsf{c}_i \in \mathcal{C}$. Note that e_{ij} belongs to the set $\mathcal{E}$ if and only if the parity-check matrix element $h_{ij} \neq 0$. The edge label associated to the edge connecting v_j and c_i is denoted by h_{ij}, with $h_{ij} \in \mathbb{F}_q \setminus \{0\}$. Fig. 3.1 shows the Tanner graph of a simple 4-ary LDPC code.

The definitions and notation of (maximum) node degrees, degree distributions, neighboring nodes and ξ are the same as the binary LDPC codes in Section 3.2.

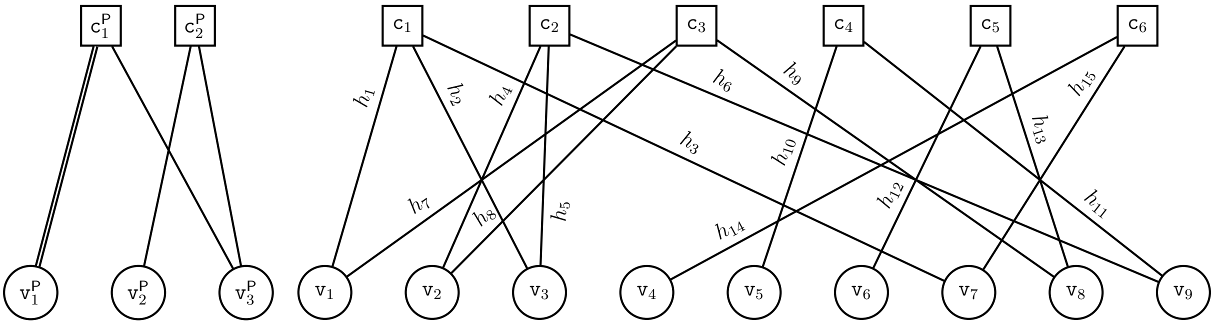
An unstructured irregular q-ary LDPC code ensemble $\mathscr{C}_{q,n}^{\lambda,\rho}$ ($\mathscr{C}_{q,n}^{\Lambda,\mathrm{P}}$) is the set of all q-ary LDPC codes with block length n and degree distributions $\lambda(x)$ and $\rho(x)$ ($\Lambda(x)$ and $\mathrm{P}(x)$) and edge labels uniformly chosen in $\mathbb{F}_q \setminus \{0\}$. Further, we denote by $\mathscr{C}_{q,n}^{\mathsf{d_v},\mathsf{d_c}}$ the regular LDPC code ensemble, which is the set of all q-ary LDPC codes with block length n, VN degree $\mathsf{d_v}$, CN degree $\mathsf{d_c}$ and edge labels uniformly chosen in $\mathbb{F}_q \setminus \{0\}$.

3.4.2 Non-Binary Protograph-Based LDPC Codes

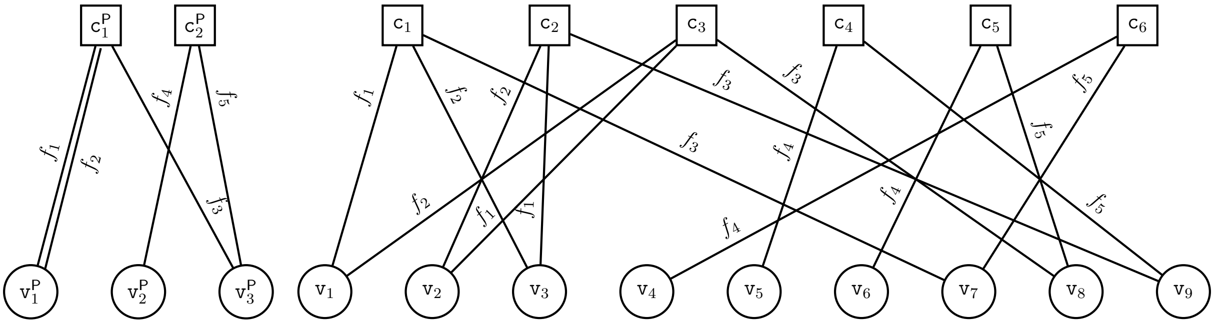
Similar to the binary case, non-binary protograph-based LDPC are obtained from a protograph. The code Tanner graph is obtained by lifting the protograph. The lifted graph has Q VNs of type $\mathsf{v}_j^{\mathsf{P}} \in \mathcal{V}^{\mathsf{P}}$, Q CNs of type $\mathsf{c}_i^{\mathsf{P}} \in \mathcal{C}^{\mathsf{P}}$ and Q edges of type $g \in \mathscr{E}^{\mathsf{P}}$. Upon labelling its edges with elements from the order-q finite field $\mathbb{F}_q$, the Tanner graph defines the $m \times n$ parity-check matrix $\boldsymbol{H}$ of a non-binary LDPC code over $\mathbb{F}_q$, where $m = m_{\mathsf{P}} Q$ and $n = n_{\mathsf{P}} Q$. In [38], two classes of non-binary protograph LDPC code ensembles were introduced. The main difference between the two classes deals with the way the edge labels are assigned, as summarized next.

Unconstrained Non-Binary Protograph-Based LDPC Code Ensembles

An unconstrained non-binary protograph-based (U-NBPB) code ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ is defined by the set of length-n LDPC codes over $\mathbb{F}_q$ whose Tanner graph is obtained by first lifting P, followed by labeling each edge in the obtained Tanner graph with an arbitrary element from $\mathbb{F}_q \setminus \{0\} = \{\alpha^0, \alpha^1, \ldots, \alpha^{q-2}\}$, where α is a primitive element of $\mathbb{F}_q$. An example of



(a) The original unlabeled protograph and an example of a U-NBPB code construction.



(b) The original scaled protograph and an example of a C-NBPB code construction.

Figure 3.2: Different non-binary protograph based code constructions with $Q = 3$.

an U-NBPB code is shown in Fig. 3.2(a), where $Q = 3$, the CNs $\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$ are of type $\mathsf{c}_1^{\mathsf{P}}$, $\mathsf{c}_4, \mathsf{c}_5, \mathsf{c}_6$ are of type $\mathsf{c}_2^{\mathsf{P}}$, the VNs $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ are of type $\mathsf{v}_1^{\mathsf{P}}$, $\mathsf{v}_4, \mathsf{v}_5, \mathsf{v}_6$ are of type $\mathsf{v}_2^{\mathsf{P}}$, $\mathsf{v}_7, \mathsf{v}_8, \mathsf{v}_9$ are of type $\mathsf{v}_3^{\mathsf{P}}$ and $h_1, \ldots, h_{15}$ are the edge labels.

Constrained Non-Binary Protograph-Based LDPC Code Ensembles

A constrained non-binary protograph-based (C-NBPB) code ensemble is defined by the set of length-n LDPC codes over $\mathbb{F}_q$ whose Tanner graph is obtained by first assigning a label to each edge in the protograph P, followed by lifting the protograph. By definition, in the Tanner graph of a code from a C-NBPB code ensemble, all edges of the same type share the same label. More specifically, consider $\boldsymbol{f} = (f_g)_{g \in \mathscr{E}}$ where $f_g \in \mathbb{F}_q \setminus \{0\}$ is the label of edge g in P. The set of length-n LDPC codes over $\mathbb{F}_q$ obtained from lifting P while preserving the edge labels specified by $\boldsymbol{f}$ is the C-NBPB code ensemble $\mathscr{C}_{q,n}^{\mathsf{P};\mathsf{c}}(\boldsymbol{f})$. An example of an U-NBPB code is shown in Fig. 3.2(a), where $Q = 3$ and $f_1, \ldots, f_5$ are the edge labels.

Note that when the field size is $q = 2$, both U-NBPB and C-NBPB constructions reduce to the binary case presented in Section 3.2.2.

3.5 GLDPC Codes

GLDPC codes introduced in [12], are a class of LDPC codes where the CNs represent more general codes than the SPC codes in LDPC codes. The Tanner graph of a GLDPC code is a bipartite graph $G = (\mathcal{V} \cup \mathcal{C}, \mathscr{E})$ consisting of n VNs and m CNs. The set $\mathscr{E}$ of edges contains the elements e_{ij}, where e_{ij} is an edge between VN $\mathtt{v}_j \in \mathcal{V}$ and CN $\mathtt{c}_i \in \mathcal{C}$. The sets $\mathcal{N}(\mathtt{v}_j)$ and $\mathcal{N}(\mathtt{c}_i)$ denote the neighbors of VN $\mathtt{v}_j$ and CN $\mathtt{c}_i$, respectively. The degree of a VN (CN) $\mathtt{v}_j$ ($\mathtt{c}_i$) is the cardinality of the set $\mathcal{N}(\mathtt{v}_j)$ ($\mathcal{N}(\mathtt{c}_i)$). A CN of type τ is an $(n_\tau, k_\tau, \mathsf{d}_{\min,\tau})$ linear block code $\mathscr{C}_\tau$. A CN is called a super check node if it is associated with a linear block code different than the SPC code. Further, we assume that $\mathsf{d}_{\min,\tau} \geq 3$ for super CNs and we denote by

$$t_\tau = \left\lfloor \frac{\mathsf{d}_{\min,\tau} - 1}{2} \right\rfloor$$

the component code decoding radius, i.e., its guaranteed error correction capability under BDD.

The node-oriented VN degree distribution is denoted by $\boldsymbol{\Lambda} = (\Lambda_j)_{j=1,\ldots,\mathsf{d}_\mathtt{v}^{\max}}$, where Λ_j corresponds to the fraction of VNs with degree j and $\mathsf{d}_\mathtt{v}^{\max}$ corresponds to the maximum VN degree. Similarly, the CN-type degree distribution is denoted by $\mathbf{P} = (\mathrm{P}_\tau)_{\tau=1,\ldots,n_c}$, where P_τ corresponds to the fraction of CNs of type τ and n_c is the number of CN types. The edge oriented VN degree distribution polynomial is defined as

$$\lambda(x) = \sum_i \lambda_i x^{i-1} \tag{3.17}$$

where λ_i is the fraction of edges connected to VNs of degree i. The CN-type degree distribution polynomial is given by

$$\rho(x) = \sum_{\tau=1}^{n_c} \rho_\tau x^{n_\tau - 1} \tag{3.18}$$

where ρ_τ is the fraction of edges connected to CNs of type τ.

We denote by

$$\bar{\mathsf{d}}_\mathtt{v} = \sum_{j=1}^{\mathsf{d}_\mathtt{v}^{\max}} j\Lambda_j, \quad \bar{\mathsf{d}}_\mathtt{c} = \sum_{\tau=1}^{n_c} n_\tau \mathrm{P}_\tau \tag{3.19}$$

the average VN and CN degree, respectively. Here again, we define

$$\xi = \frac{m}{n} = \frac{\bar{\mathsf{d}}_\mathsf{v}}{\bar{\mathsf{d}}_\mathsf{c}}. \tag{3.20}$$

The number of parity-check equations for a GLDPC code is

$$m_0 = m \sum_{\tau=1}^{n_c} \mathrm{P}_\tau (n_\tau - k_\tau) = \frac{m}{\int_0^1 \rho(x)\,\mathrm{d}x} \sum_{\tau=1}^{n_c} \rho_\tau \left(1 - \frac{k_\tau}{n_\tau}\right). \tag{3.21}$$

We denote by $\mathbf{\Pi}$ the $m \times n$ adjacency matrix of a GLDPC Tanner graph. In order to obtain the $m_0 \times n$ parity-check matrix $\boldsymbol{H}$ of a GLDPC code, for each row of $\mathbf{\Pi}$ the ones are replaced by the columns of the parity-check matrix of the corresponding component code and the zeros by zero column vectors [75]. The rate of the GLDPC code can be lower bounded as

$$R \geq 1 - \frac{m_0}{n} \tag{3.22}$$

where equality holds if $\boldsymbol{H}$ is full rank.

A GLDPC code is called regular if all the VNs have the same degree d_v and all the CNs are of the same type, i.e., all CNs are associated with the same linear block code $\mathscr{C}$ of length d_c.

An unstructured irregular GLDPC code ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ is the set of all GLDPC codes with block length n, defined by a Tanner graph with node-oriented degree distributions $\mathbf{\Lambda}$ and $\mathbf{P}$ (with a specified set of CN types).

3.6 Enumeration Methods for Analyzing LDPC Codes

In this section, we briefly describe the methods used in this thesis to enumerate (elementary) trapping and (fully) absorbing sets. The first approach is based on random matrix enumeration and the second one follows the generating functions methodology.

3.6.1 Random Matrix Enumeration

Trapping and (fully) absorbing sets for binary LDPC codes (Chapter 6) impose a certain structure and row/column weights for the parity-check matrix. Thus, the asymptotic trapping and (fully) absorbing sets can be obtained by enumerating matrices with prescribed row and column weight profiles. The following Theorem on the number of binary matrices with specific weight distribution properties will be useful to derive the enumerators.

Theorem 3.1. Let $\mathcal{H}_{m,n}^{\boldsymbol{R},\boldsymbol{L}}$ be the set of all $m \times n$ binary matrices with row weight vector $\boldsymbol{R} = (R_1, \ldots, R_m)$ and column weight vector $\boldsymbol{L} = (L_1, \ldots, L_n)$, where R_i, $1 \leq i \leq m$, is the weight of the i-th row and L_j, $1 \leq j \leq n$, is the weight of the j-th column. The cardinality of $\mathcal{H}_{m,n}^{\boldsymbol{R},\boldsymbol{L}}$ for constant ratio $\xi = m/n$ and $\max\{\max_i R_i, \max_j L_j\} \leq (\ln(n))^{1/4-\epsilon}$, $\epsilon > 0$, as $n \to \infty$ is given by [76], [77]

$$|\mathcal{H}_{m,n}^{\boldsymbol{R},\boldsymbol{L}}| = \frac{f!}{\prod\limits_{j=1}^{n} L_j! \prod\limits_{i=1}^{m} R_i!} \exp\left[-\frac{1}{2f^2}\left(\sum_{i=1}^{m} R_i(R_i-1)\right)\left(\sum_{j=1}^{n} L_j(L_j-1)\right)\right] \times (1 + o(n^{-1+\delta})) \tag{3.23}$$

and for $\delta > 0$, with $f = \sum\limits_{j=1}^{n} L_j = \sum\limits_{i=1}^{m} R_i$.

3.6.2 Generating Function Approach

The random matrix enumeration technique (Theorem 3.1) can only be applied to unstructured binary LDPC codes. Therefore, we present a more general method to obtain these enumerators for binary/non-binary and generalized LDPC code ensembles. The method is based on generating functions, previously adopted to study the distance spectrum and the stopping set distributions of (generalized) binary LDPC code ensembles [9, 46, 78–80]. The generating function approach is general and we can enumerate several graphical structures by defining the appropriate generating functions. In particular, for the enumeration of trapping and (elementary) absorbing sets, we need to impose VN and CN conditions. Considering (a, b) TSs/ASs/elementary absorbing sets (EASs), the a VNs in the set must satisfy the VN condition according to the corresponding definitions in Chapters 6, 7 and 8. Moreover, given the set of edges, we need to obtain exactly b unsatisfied CNs. Finally, we need to consider all possible edge permutations according to edge types. The average number of VN/CN/edge sets satisfying a specific condition is the coefficient of suitably-defined generating function. After deriving the finite-length enumerators, we obtain the normalized logarithmic asymptotic distributions by using the following Lemmas.

Lemma 3.1. (Hayman formula for multivariate polynomials [79, Corollary 16]) Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ and let $p(\boldsymbol{x})$ be a multivariate polynomial with $p(\boldsymbol{0}) \neq 0$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d)$ where $0 \leq \beta_t \leq 1$ and $\beta_t n$ is an integer for all $t \in \{1, 2, \ldots, d\}$.

Then we have as $n \to \infty$

$$\text{coeff}\left((p(\boldsymbol{x}))^n, \boldsymbol{x}^{n\boldsymbol{\beta}}\right) \doteq \exp\left\{n\left[\ln p(\boldsymbol{x}) - \sum_{t=1}^{d} \beta_t \ln x_t\right]\right\}$$

where $\text{coeff}\left(p(\boldsymbol{x})^n, \boldsymbol{x}^{n\boldsymbol{\beta}}\right)$ represents the coefficient of $\boldsymbol{x}^{n\boldsymbol{\beta}}$ in the polynomial $p(\boldsymbol{x})^n$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ and $x_1, x_2, \ldots, x_d$ are the unique positive solutions to

$$x_t \frac{\partial p(\boldsymbol{x})}{\partial x_t} = \beta_t p(\boldsymbol{x}), \qquad \forall t \in \{1, 2, \ldots, d\}.$$

Lemma 3.2. Suppose $0 \leq \beta_t \leq \kappa$ and $\beta_t n$ is an integer $\forall t \in \{1, 2, \ldots, d\}$ with $\sum_{t=1}^{d} \beta_t = 1$. We have [81, Chapter 11]

$$\binom{n}{\beta_1 n, \beta_2 n, \ldots, \beta_d n} \doteq \exp\left\{nH\left(\beta_1, \beta_2, \ldots, \beta_d\right)\right\}.$$

3.7 Iterative Message Passing Decoding

3.7.1 Sum Product Algorithm for Binary LDPC Codes

The sum product algorithm (SPA), also called BP decoder, was introduced by Gallager [9]. It is a soft decision decoding algorithm that can approach symbol-wise MAP decoding performance and provide an estimate of the logarithmic APP ratio

$$L_j^{\text{app}} = \ln\left(\frac{\Pr\{X_j = +1|\boldsymbol{y}\}}{\Pr\{X_j = -1|\boldsymbol{y}\}}\right), \quad j = 1, 2, \ldots, n \tag{3.24}$$

where $\Pr\{X_j = +1|\boldsymbol{y}\}$ is the probability that the j-th codeword bit is $+1$, given the received sequence $\boldsymbol{y}$. To estimate L_j^{app}, the VN decoders (repetition SISO decoders) and the CN decoders (SPC SISO decoders) exchange messages iteratively.

Let

$$L_j = \ln\left(\frac{p_{Y|X}(y_j|+1)}{p_{Y|X}(y_j|-1)}\right) \tag{3.25}$$

be the channel log-likelihood ratio (LLR) associated with the VN $\mathtt{v}_j$. For biAWGN channel, we have

$$L_j = \frac{2}{\sigma^2} y_j. \tag{3.26}$$

We denote by $L_{\mathtt{v}_j \to \mathtt{c}_i}^{(\ell)}$ the message sent by the VN $\mathtt{v}_j$ to the CN $\mathtt{c}_i$ at the ℓ-th iteration. Similarly, $L_{\mathtt{c}_i \to \mathtt{v}_j}^{(\ell)}$ is the message sent by $\mathtt{c}_i$ to $\mathtt{v}_j$.

Algorithm 1: Sum-Product of Binary LDPC Codes over a biAWGN

for $j = 1, 2, \ldots, n$ **do**
  for $\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)$ **do**
    $L^{(0)}_{\mathtt{v}_j \to \mathtt{c}_i} = L_j$
  end
end
for $\ell = 1, \ldots, \ell_{max}$ **do**
  for $i = 1, 2, \ldots, m$ **do**
    for $\mathtt{v}_j \in \mathcal{N}(\mathtt{c}_i)$ **do**
      $L^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = 2 \tanh^{-1}\left(\prod_{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j} \tanh\left(\frac{L^{(\ell-1)}_{\mathtt{v}_{j'} \to \mathtt{c}_i}}{2} \right) \right)$
    end
  end
  for $j = 1, 2, \ldots, n$ **do**
    for $\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)$ **do**
      $L^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i} = L_j + \sum_{\mathtt{c}_{i'} \in \mathcal{N}(\mathtt{v}_j) \setminus \mathtt{c}_i} L^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j}$
    end
  end
  for $j = 1, 2, \ldots, n$ **do**
    $L^{(\ell)}_{j,\mathrm{app}} = L_j + \sum_{\mathtt{c}_{i'} \in \mathcal{N}(\mathtt{v}_j)} L^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j}$
    $L^{(\ell)}_{j,\mathrm{app}} \underset{\hat{x}_j = -1}{\overset{\hat{x}_j = +1}{\gtrless}} 0$
  end
end

Algorithm 1 describes the iterative message passing decoding over a biAWGN channel. The decoder stops if the maximum number of iterations is reached or $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ is a valid codeword.

Note that, for the min-sum decoder, one replaces the expression in the CN message update by

$$L^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = \left(\prod_{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j} \operatorname{sign}\left(L^{(\ell-1)}_{\mathtt{v}_{j'} \to \mathtt{c}_i} \right) \right) \min_{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j} |L^{(\ell-1)}_{\mathtt{v}_{j'} \to \mathtt{c}_i}|. \tag{3.27}$$

3.7.2 Sum Product Algorithm for Non-Binary LDPC Codes

The non-binary version of the SPA (Algorithm 1) can decode non-binary LDPC codes over $\mathbb{F}_q = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{q-2}\}$, where α is a primitive element of $\mathbb{F}_q$. We consider a probability domain SPA. The exchanged check and variable node messages are q-ary probability vectors. Let $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i} = (m^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}(0), \ldots, m^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}(\alpha^{q-2}))$ be the message sent from the VN $\mathtt{v}_j$ to its neighboring VN $\mathtt{c}_i$ at the ℓ-th iteration. The entry $m^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}(a)$ for $a \in \mathbb{F}_q$ represents the probability that the codeword symbol associated with the message takes the value a. Similarly, $\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = (m^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}(0), \ldots, m^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}(\alpha^{q-2}))$ is the message sent from $\mathtt{c}_i$ to $\mathtt{v}_j$ at the ℓ-th iteration. Further, let $M^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}$ ($M^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}$) be the RV of the codeword symbol associated with the message sent from $\mathtt{v}_j$ to $\mathtt{c}_i$ ($\mathtt{c}_i$ to $\mathtt{v}_j$), i.e., $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}$ ($\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}$) is the PMFs of $M^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}$ ($M^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}$). Initially, each VN sends to its neighboring CNs a q-ary vector with the symbol probabilities given the corresponding channel observation. Formally, for all $\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)$, we have

$$\boldsymbol{m}^{(0)}_{\mathtt{v}_j \to \mathtt{c}_i} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}_j}. \tag{3.28}$$

Each CN represents a non-binary SPC code. Thus, for $a \in \mathbb{F}_q$ we have

$$m^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}(a) = \Pr\left\{ \sum_{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j} h_{j'i} M^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i} = -h_{ji} a \right\}. \tag{3.29}$$

The PMF of $h_{j'i} M^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i}$ is obtained by permuting the entries (except the first one) of $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i}$. We write $(\boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i})^{\pi} = \boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i} \boldsymbol{\Pi}_{\mathtt{v}_{j'}, \mathtt{c}_i}$, where $\boldsymbol{\Pi}_{\mathtt{v}_{j'}, \mathtt{c}_i}$ is a $q \times q$ permutation matrix associated to $h_{j'i}$. Under the independence assumption, we have

$$\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = \underset{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j}{\circledast} (\boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i})^{\pi} \tag{3.30}$$

where $\circledast$ is the convolution in $\mathbb{F}_q$. For non-binary codes over binary extension fields, the discrete convolution becomes a componentwise multiplication using the Hadamard transform, yielding

$$\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = \mathcal{H}\left\{ \prod_{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j} \mathcal{H}\left\{ (\boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i})^{\pi} \right\} \right\}. \tag{3.31}$$

Each VN computes

$$\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}_j} \odot \left(\bigodot_{\mathtt{c}_{i'} \in \mathcal{N}(\mathtt{v}_j) \setminus \mathtt{c}_i} (\boldsymbol{m}^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j})^{\pi^{-1}} \right) \tag{3.32}$$

where $(\boldsymbol{m}^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j})^{\pi^{-1}} = (\boldsymbol{m}^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j})\boldsymbol{\Pi}^{-1}_{\mathtt{v}_{j'},\mathtt{c}_i}$ is a reverse permutation of $(\boldsymbol{m}^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j})$ and $\boldsymbol{\Pi}^{-1}_{\mathtt{v}_{j'},\mathtt{c}_i}$ is a $q \times q$ reverse permutation matrix associated to $h_{ji'}$. All multiplications are componentwise. Note that it is necessary to normalize the elements of $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}$ to sum to 1. The estimation of the codeword symbol associated to $\mathtt{v}_j$ is the symbol that maximizes

$$\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j,\mathrm{app}} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}_j} \odot \left(\bigodot_{\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)} (\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j})^{\pi^{-1}} \right). \tag{3.33}$$

The SPA in the probability domain is summarized in Algorithm 2.

3.7.3 Parallel Bit Flipping Algorithm for Binary GLDPC Codes

The parallel bit flipping (PBF) decoding [82] is similar to the algorithm proposed in [83, 84] for expander codes. It is closely related to the one introduced in [2] for decoding a class of spatially-coupled GLDPC codes and to the one used in [85] to decode GLDPC codes based on Reed-Solomon and BCH component codes, and it tightly follows the formulation of bit flipping decoding of GLDPC codes outlined in [12, 82].

For the PBF decoder, we transmit the all-zero codeword over a binary symmetric channel (BSC). We denote by $\hat{c}^{(\ell)}_{\mathtt{v}_j}$ the estimate of the codeword bit associated with the VN $\mathtt{v}_j$ at the ℓ-th iteration and $\mathsf{n}^{(\mathrm{f})}_{\mathtt{v}_j}$ the number of flip messages that the VN $\mathtt{v}_j$ receives from its neighboring CNs. Initially, the estimate of a VN is its channel observation, i.e., $\hat{c}^{(0)}_{\mathtt{v}_j} = m^{\mathrm{ch}}_{\mathtt{v}_j}$. At the ℓ-th iteration, each VN sends its estimate $\hat{c}^{(\ell)}_{\mathtt{v}_j}$ to its neighboring CNs. Each super CN performs BDD on the received messages from the VNs and sends then flip messages to the VNs whose values differ from the decoded vector. Special care is needed for SPC component codes: Here, we follow the policy of flipping the component code decoder input at the output whenever the SPC constraint is not satisfied, i.e., an unsatisfied single parity check node sends flip messages to all its neighboring VNs. For a VN $\mathtt{v}_j$ of degree $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$, if it receives strictly more than $\mathsf{d}_{\mathsf{v}_\mathsf{j}}/2$ flip messages, it flips its estimate, i.e., $\hat{c}^{(\ell+1)}_{\mathtt{v}_j} = 1 - \hat{c}^{(\ell)}_{\mathtt{v}_j}$. A VN is called correct if the corresponding estimate is zero, and it is called corrupt if it is one. The VNs and CNs exchange messages iteratively until a maximum number of iterations is reached or a valid codeword is found. Algorithm 3 shows the steps performed in each decoding iteration, where $\ell_{\max}$ is the maximum number of iterations.

Algorithm 2: Sum-Product Algorithm for Non-Binary LDPC Codes

for $j = 1, 2, \ldots, n$ **do**
    for $\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)$ **do**
        $\boldsymbol{m}^{(0)}_{\mathtt{v}_j \to \mathtt{c}_i} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}_j}$.
    end
end
for $\ell = 1, \ldots, \ell_{max}$ **do**
    for $i = 1, 2, \ldots, m$ **do**
        for $\mathtt{v}_j \in \mathcal{N}(\mathtt{c}_i)$ **do**
            $\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j} = \underset{\mathtt{v}_{j'} \in \mathcal{N}(\mathtt{c}_i) \setminus \mathtt{v}_j}{\circledast} \left(\boldsymbol{m}^{(\ell)}_{\mathtt{v}_{j'} \to \mathtt{c}_i}\right)^{\pi}$
        end
    end
    for $j = 1, 2, \ldots, n$ **do**
        for $\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)$ **do**
            $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}_j} \odot \left(\underset{\mathtt{c}_{i'} \in \mathcal{N}(\mathtt{v}_i) \setminus \mathtt{c}_i}{\odot} \left(\boldsymbol{m}^{(\ell)}_{\mathtt{c}_{i'} \to \mathtt{v}_j}\right)^{\pi^{-1}} \right)$
            Normalize $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j \to \mathtt{c}_i}$
        end
    end
    for $j = 1, 2, \ldots, n$ **do**
        $\boldsymbol{m}^{(\ell)}_{\mathtt{v}_j,\mathrm{app}} = \boldsymbol{m}^{\mathrm{ch}}_{\mathtt{v}} \odot \left(\underset{\mathtt{c}_i \in \mathcal{N}(\mathtt{v}_j)}{\odot} \left(\boldsymbol{m}^{(\ell)}_{\mathtt{c}_i \to \mathtt{v}_j}\right)^{\pi^{-1}} \right)$
        $\hat{c}_{\mathtt{v}_j} = \underset{a \in \mathbb{F}_q}{\mathrm{argmax}}\, m^{(\ell)}_{\mathtt{v}_j,\mathrm{app}}(a)$
    end
end

Algorithm 3: Parallel Bit Flipping Algorithm.

```
for j = 1, 2, ..., n do
    \hat{c}^{(0)}_{v_j} = m^{ch}_{v_j}
end
for \ell = 1, ..., \ell_{max} do
    for j = 1, 2, ..., n do
        for c_i \in \mathcal{N}(v_j) do
            m^{(\ell)}_{v_j \to c_i} = \hat{c}^{(\ell-1)}_{v_j}
        end
    end
    for i = 1, 2, ..., m do
        if c_i is an unsatisfied SPC then
            for v_j \in \mathcal{N}(c_i) do
                n^{(f)}_{v_j} = n^{(f)}_{v_j} + 1
            end
        end
        if c_i is a super CN then
            Perform BBD (with output \boldsymbol{z})
            for v_j \in \mathcal{N}(c_i) do
                if z_{v_j} \neq \hat{c}^{(\ell)}_{v_j} then
                    n^{(f)}_{v_j} = n^{(f)}_{v_j} + 1
                end
            end
        end
    end
    for j = 1, 2, ..., n do
        if n^{(f)}_{v_j} > d_{v_j}/2 then
            \hat{c}^{(\ell)}_{v_j} = 1 - \hat{c}^{(\ell-1)}_{v_j}
        end
    end
end
```

4

Quantized Decoding Algorithms for LDPC Codes

The deployment of high throughput communication links [42, 86] is motivating a revived interest in low-complexity, high-speed channel code decoders. Recently, attention has been devoted to the design and analysis of iterative decoders where the messages exchanged within the decoder are coarsely quantized. The study of low-complexity message passing algorithms for LDPC codes originates from the work by Gallager [9] who introduced two decoding algorithms, known as Gallager A and Gallager B, where the variable and check nodes exchange binary messages. By introducing erasures, the performance of these algorithms is improved [33]. Finite-alphabet iterative decoders were also studied, for instance, in [13, 15, 50, 67, 87]. While coarse message quantization reduces the amount of information exchanged within the decoder, the decoding complexity can also be reduced by employing simplified update rules at the CNs. Examples are the min-sum decoder [16, 88] and some of its variations (see, e.g., [17, 18, 89]), that limit the losses due to the min-approximation at the CNs by introducing simple corrections.

Non-binary LDPC codes show an outstanding error correction capability, outperforming their binary counterparts [11]. Nevertheless, the complexity of the BP decoder for these codes is very high, and several works considered reduced-complexity decoding algorithms for non-binary LDPC codes over the biAWGN channel [19–22, 90] and the QSC [23–27, 91]. Majority logic based algorithms were considered in [28–30, 92]. In [20], an extension of the min-sum algorithm to non-binary fields was presented.

In this chapter, we analyze and design reduced complexity decoding algorithms for binary and non-binary LDPC codes over different channel models.

4.1 Binary LDPC Codes

In this section, we analyze and design quantized min-sum decoders for binary LDPC codes over the biAWGN channel. At the CNs, we use the standard min-approximation rule. In contrast to the quantized min-sum (QMS) algorithm [14], the VN decoder converts all incoming messages to LLRs by modeling the extrinsic channel as a DMC, extending the approach introduced for binary message passing decoding in [13] to the case where messages are represented by b bits. The transition probabilities of the extrinsic DMCs are derived via DE analysis, which we develop for unstructured irregular LDPC ensembles. Because the VN inbound messages are matched to the reliability of the underlying extrinsic DMC, we refer to the proposed algorithm as MQMS decoding [55].

4.1.1 Extrinsic Channels

Consider a binary-input M-ary output DMC with input alphabet $\mathcal{X} = \{-1, +1\}$ and output alphabet $\mathcal{Z} = \{-(M-1)/2, -(M-3)/2, \ldots, 0, \ldots, (M-1)/2\}$, where $M = 2^b - 1$ and b is a positive integer. For a generic channel output z, LLRs can be obtained as

$$L(z) = \ln\left[\frac{P_{Z|X}(z|+1)}{P_{Z|X}(z|-1)}\right]. \tag{4.1}$$

If the channel satisfies the symmetry constraint

$$P_{Z|X}(-z|+1) = P_{Z|X}(z|-1)$$

for all $z \in \mathcal{Z}$, we have

$$L(z) = \mathrm{sign}(z)\mathsf{D}_{|z|} \tag{4.2}$$

where $\forall a \in \mathcal{Z}, a > 0$

$$\mathsf{D}_a := \ln\left[\frac{P_{Z|X}(a|+1)}{P_{Z|X}(-a|+1)}\right] \tag{4.3}$$

and where by convention the sign(x) function takes on the value 0 for $x = 0$. We refer to $\mathsf{D}_{|z|}$ as the *reliability* of z. The decomposition (4.2) will be instrumental to developing a message-passing decoding algorithm for LDPC codes. In particular, we focus on a decoding algorithm that exchanges quantized messages. In this case, a message sent from a CN to a VN can be modeled as the observation of the RV X after transmission over a binary-input

M-ary output discrete memoryless extrinsic channel [93, Fig. 3], where M is the number of message quantization levels. While the transition probabilities of the extrinsic channel are in general unknown, accurate estimates can be obtained via DE analysis, as suggested in [13]. This observation will be used to derive the MQMS decoding algorithm.

4.1.2 Quantization

Throughout the paper, we consider uniform quantization. We denote by $\mathsf{f} : \mathbb{R} \to \mathcal{M}$ the quantization function of the exchanged messages, where the quantized message alphabet is $\mathcal{M} = \{-S\Delta, -(S-1)\Delta, \dots, S\Delta\}$. The function f is a b-bit uniform quantizer with step size Δ and $2^b - 1$ quantization levels. Formally, we have

$$\mathsf{f}(x) := \operatorname{sign}(x)\Delta \cdot \min\left\{ \left\lfloor \frac{|x|}{\Delta} + \frac{1}{2} \right\rfloor, S \right\} \tag{4.4}$$

where $S = 2^{b-1} - 1$.

For the channel output, we consider two cases: unquantized channel outputs and quantized channel outputs. For the latter case, the biAWGN channel output is quantized using a b_0-bit uniform quantizer with step size Δ_0, where b_0 and Δ_0 may, in general, differ from the corresponding parameters for the message quantization. The quantized channel output alphabet is $\mathcal{M}_0 = \{-S_0\Delta_0, -(S_0-1)\Delta_0, \dots, S_0\Delta_0\}$ with $S_0 = 2^{b_0-1} - 1$, and the quantized version of y is denoted as m_{ch}.

4.1.3 Matched Quantized Min-Sum Decoding

We denote by $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ the message sent from CN $\mathtt{c}$ to its neighboring VN $\mathtt{v}$. Similarly, $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is the message sent from VN $\mathtt{v}$ to CN $\mathtt{c}$ at the ℓ-th iteration.

Unquantized Channel Output

Each VN computes the LLR of the corresponding channel output

$$L_{\text{ch}}(y) = \frac{2}{\sigma^2} y \tag{4.5}$$

and the VN passes a b-bit quantized value to its neighboring CNs. Thus, $\forall \mathtt{c} \in \mathcal{N}(\mathtt{v})$ we have

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = \mathsf{f}(L_{\text{ch}}(y)) \tag{4.6}$$

where f is defined in (4.4) and we choose Δ to minimize the iterative decoding threshold.

The min update rule is performed at the CNs. We have

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = \min_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} \left|m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}}\right| \prod_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} \operatorname{sign}\left(m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}}\right). \tag{4.7}$$

At the ℓ-th iteration, each VN converts its channel message and the incoming CN messages to LLRs. The sum of these LLRs is then quantized into a b-bit message. Formally, we have

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \mathsf{f}\left(L_{\mathrm{ch}}(y) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} L_{\mathrm{ex}}\left(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right)\right) \tag{4.8}$$

where

$$L_{\mathrm{ex}}\left(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right) := \operatorname{sign}\left(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right) \mathsf{D}^{(\ell)}_{\left|m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right|}. \tag{4.9}$$

The final hard decision at each VN is

$$\hat{x}^{(\ell)}_{\mathtt{v}} = \operatorname{sign}\left(L_{\mathrm{ch}}(y) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})} L_{\mathrm{ex}}\left(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right)\right). \tag{4.10}$$

Note that the reliability of $m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}$ depends on the iteration number and is in general unknown. In fact, the transition probabilities of the underlying extrinsic DMCs are not known. As proposed in [13], their values can be estimated via Monte Carlo simulations, or via DE analysis. The latter approach provides accurate results for moderate to large block lengths, as shown in [13, 50]. We hence follow this direction and use the DE presented in Section 4.1.4 to estimate the message reliability at each iteration. For the special case of $b = 2$, we will obtain the TMP decoder that we introduced in [50].

Quantized Channel Output

If the channel output is quantized as described in Section 4.1.2, we replace $L_{\mathrm{ch}}(y)$ in (4.8) and (4.10) by

$$L_{\mathrm{ch}}(m_{\mathrm{ch}}) = \operatorname{sign}(m_{\mathrm{ch}})\mathsf{D}_{|m_{\mathrm{ch}}|}.$$

We choose Δ and Δ_0 to minimize the decoding threshold. As mentioned in Sec. 4.1.1, the decoder's communication channel can be modeled as a binary-input $|\mathcal{M}_0|$-ary output DMC that satisfies the symmetry condition. The value of $\mathsf{D}_{|m_{\mathrm{ch}}|}$ can then be computed from (4.3) by using the transition probabilities of the quantized communication channel.

4.1.4 Density Evolution Analysis

We provide a DE analysis of the MQMS algorithm for unstructured LDPC code ensembles. Due to symmetry, we may assume that the all-zeros codeword is transmitted. Let $M^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ be the RV associated to VN to CN messages at the ℓ-th iteration. Similarly, $M^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ represents the RV associated to CN to VN messages. We denote by $p_i^{(\ell)}$ the probability that $M^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ takes the value Δi, with $i \in \{-S, -(S-1), \ldots, S\}$. Similarly, we denote by $q_i^{(\ell)}$ the probability that $M^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ takes the value Δi. In the following, $\ell_{\max}$ denotes the maximum number of iterations. In the limit of $n \to \infty$, the evolution of the message distributions can be tracked as follows.

1. **Initialization.** Conditioned on $X = +1$, the channel LLRs are Gaussian RVs with mean $\mu_{\text{ch}} = 4RE_{\mathsf{b}}/N_0$ and variance $\sigma^2_{\text{ch}} = 2\mu_{\text{ch}}$. Therefore, we have

$$p_i^{(0)} = \begin{cases} Q\left(\frac{(S-\frac{1}{2})\Delta+\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) & \text{if } i=-S \\ Q\left(\frac{(S-\frac{1}{2})\Delta-\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) & \text{if } i=S \\ Q\left(\frac{(i-\frac{1}{2})\Delta-\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) - Q\left(\frac{(i+\frac{1}{2})\Delta-\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) & \text{otherwise} \end{cases} \tag{4.11}$$

while if the channel output is quantized we have

$$p_i^{(0)} = \sum_{m_{\text{ch}}:\mathsf{f}(L_{\text{ch}}(m_{\text{ch}}))=\Delta i} P_{M_{\text{ch}}|X}(m_{\text{ch}}|+1). \tag{4.12}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. For all $j \in \{1, \ldots, 2^{b-1}\}$, we define $\Phi_j^{(\ell)}$ and $\Psi_j^{(\ell)}$ as

$$\Phi_j^{(\ell)} := \Pr\left\{M^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \geq \Delta j\right\} \tag{4.13}$$

$$\Psi_j^{(\ell)} := \Pr\left\{M^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \leq -\Delta j\right\}. \tag{4.14}$$

The probabilities $q_i^{(\ell)}$ can be computed as

$$q_i^{(\ell)} = \begin{cases} \frac{1}{2}\Big[\rho\left(\Phi_i^{(\ell-1)} + \Psi_i^{(\ell-1)}\right) + \rho\left(\Phi_i^{(\ell-1)} - \Psi_i^{(\ell-1)}\right) & \text{if } i>0 \\ \quad -\rho\left(\Phi_{i+1}^{(\ell-1)} + \Psi_{i+1}^{(\ell-1)}\right) - \rho\left(\Phi_{i+1}^{(\ell-1)} - \Psi_{i+1}^{(\ell-1)}\right)\Big] & \\ 1-\rho\left(1-p_0^{(\ell-1)}\right) & \text{if } i=0 \\ \frac{1}{2}\Big[\rho\left(\Phi_{-i}^{(\ell-1)} + \Psi_{-i}^{(\ell-1)}\right) - \rho\left(\Phi_{-i}^{(\ell-1)} - \Psi_{-i}^{(\ell-1)}\right) & \text{if } i<0. \\ \quad -\rho\left(\Phi_{-i+1}^{(\ell-1)} + \Psi_{-i+1}^{(\ell-1)}\right) + \rho\left(\Phi_{-i+1}^{(\ell-1)} - \Psi_{-i+1}^{(\ell-1)}\right)\Big] & \end{cases} \tag{4.15}$$

Variable to check update. For $i \in \{-S, -(S-1), \ldots, S\}$, the probabilities $p_i^{(\ell)}$ for the unquantized channel output are given by

$$p_i^{(\ell)} = \begin{cases} \sum_d \lambda_d \sum_{l_\text{in}} \Pr\left\{L_\text{in}^{(\ell)} = l_\text{in}\right\} Q\left(\frac{(S-\frac{1}{2})\Delta + l_\text{in} + \mu_\text{ch}}{\sigma_\text{ch}}\right) & \text{if } i = -S \\ \sum_d \lambda_d \sum_{l_\text{in}} \Pr\left\{L_\text{in}^{(\ell)} = l_\text{in}\right\} Q\left(\frac{(S-\frac{1}{2})\Delta - l_\text{in} - \mu_\text{ch}}{\sigma_\text{ch}}\right) & \text{if } i = S \\ \sum_d \lambda_d \sum_{l_\text{in}} \Pr\left\{L_\text{in}^{(\ell)} = l_\text{in}\right\} \left[Q\left(\frac{(i-\frac{1}{2})\Delta - l_\text{in} - \mu_\text{ch}}{\sigma_\text{ch}}\right)\right. & \text{else} \\ \left. -Q\left(\frac{(i+\frac{1}{2})\Delta - l_\text{in} - \mu_\text{ch}}{\sigma_\text{ch}}\right)\right] & \end{cases} \tag{4.16}$$

while for the quantized output, we have

$$p_i^{(\ell)} = \sum_d \lambda_d \sum_{m_\text{ch}} P_{M_\text{ch}|X}(m_\text{ch}| + 1) \sum_{l_\text{in}:\mathsf{f}(L_\text{ch}(m_\text{ch}) + l_\text{in}) = \Delta i} \Pr\left\{L_\text{in}^{(\ell)} = l_\text{in}\right\} \tag{4.17}$$

where $L_\text{in}^{(\ell)}$ is a RV associated to the sum of the LLRs of the $d-1$ incoming CN messages at the ℓ-th iteration. We have

$$\Pr\left\{L_\text{in}^{(\ell)} = l_\text{in}\right\} = \sum_{\boldsymbol{v}} \binom{d-1}{v_{-S}, \ldots, v_S} \prod_{i=-S}^{S} \left(q_i^{(\ell)}\right)^{v_i} \tag{4.18}$$

where the sum is over all integer vectors $\boldsymbol{v}$ for which

$$\sum_{i=-S}^{S} v_i = d - 1 \tag{4.19}$$

$$\sum_{i=1}^{S} (v_i - v_{-i}) \mathsf{D}_{\Delta i}^{(\ell)} = l_\text{in} \tag{4.20}$$

where

$$\mathsf{D}_{\Delta i}^{(\ell)} := \ln\left(\frac{q_i^{(\ell)}}{q_{-i}^{(\ell)}}\right). \tag{4.21}$$

Note that the vector entry v_i represents the number of incoming CN messages with value Δi.

The ensemble iterative decoding threshold $(E_\mathsf{b}/N_0)^\star$ is defined as the minimum E_b/N_0

Table 4.1: Decoding thresholds $(E_\mathsf{b}/N_0)^\star$[dB] of MQMS for quantized and unquantized channel output and for QMS.

$(\mathsf{d_v},\mathsf{d_c})$	b	MQMS (unquant. channel)	b_0	MQMS (quant. channel)	QMS
(3,6)	2	1.85	2	2.39	2.66
			3	1.9	2.66
			4	1.86	2.58
	3	1.32	3	1.45	1.8
			4	1.34	1.8
	4	1.21	3	1.35	1.72
			4	1.24	1.65
	5	1.18	5	1.19	1.62
(4,8)	2	2.11	2	2.71	2.78
			3	2.22	2.4
			4	2.11	2.43
	3	1.73	3	1.85	2.17
			4	1.76	2.12
	4	1.65	3	1.77	2.14
			4	1.68	2.08
	5	1.63	5	1.64	2.06

for which $\lim_{\ell\to\infty} P_e^{(\ell)} = 0$ as $n \to \infty$, where

$$P_e^{(\ell)} = \sum_{i=-S}^{0} p_i^{(\ell)}. \tag{4.22}$$

A first set of results deals with the asymptotic performance of MQMS decoding. Table 4.1 compares the iterative decoding thresholds of MQMS for both quantized and unquantized channel outputs and QMS [14] for $(\mathsf{d_v},\mathsf{d_c})$ regular LDPC ensembles and different values of b and b_0. MQMS decoding largely outperforms QMS, with gains of up to 0.7 dB. Remarkably, for $b = b_0 = 5$ the MQMS thresholds are within 0.1 dB of the unquantized belief propagation thresholds (which are at $(E_\mathsf{b}/N_0)^\star \approx 1.1$ dB for the regular $(3,6)$ ensemble, and at $(E_\mathsf{b}/N_0)^\star \approx 1.58$ dB for the regular $(4,8)$ ensemble).

Based on the DE analysis, we designed a set of optimized irregular ensembles with various rates. For the design, we chose a MQMS decoder with $b = 4$ and unquantized channel output. We set the maximum VN degree to $\mathsf{d}_\mathsf{v}^{\max} = 20$. The optimized degree distributions, obtained via differential evolution are provided in Table 4.2.

We next consider the performance for rate 4/5 and 7/8 codes, designed for a MQMS decoder and unquantized channel outputs, where we set $b = 4, \mathsf{d}_\mathsf{v}^{\max} = 15, \ell_{\max} = 30$. The

Table 4.2: Thresholds of optimized degree distributions for the MQMS decoder for unquantized channel with $b = 4$ and quantized channel with $b = b_0 = 4$.

R	$\lambda(x)$	$\rho(x)$	$(E_\mathsf{b}/N_0)^\star$ [dB]	$(E_\mathsf{b}/N_0)^\star$ [dB] $b_0 = 4$	$(E_\mathsf{b}/N_0)_\mathsf{Sh}$ [dB]
2/3	$0.0317x + 0.489x^2 + 0.0374x^9 + 0.4419x^{19}$	$0.328x^{13} + 0.672x^{14}$	1.47	1.5	1.06
3/4	$0.0313x + 0.463x^2 + 0.0058x^9 + 0.4999x^{19}$	$0.5336x^{19} + 0.4664x^{20}$	1.96	2	1.62
4/5	$0.4961x^2 + 0.0051x^9 + 0.4988x^{19}$	$0.7907x^{25} + 0.2093x^{26}$	2.34	2.37	2.04
5/6	$0.0205x + 0.4646x^2 + 0.0534x^9 + 0.4616x^{19}$	$0.9926x^{30} + 0.0074x^{31}$	2.63	2.66	2.36
7/8	$0.4789x^2 + 0.0021x^4 + 0.032x^9 + 0.487x^{19}$	$0.3752x^{41} + 0.6248x^{42}$	3.08	3.11	2.85
9/10	$0.4442x^2 + 0.0403x^3 + 0.0025x^9 + 0.513x^{19}$	$0.6604x^{53} + 0.3396x^{54}$	3.42	3.44	3.2

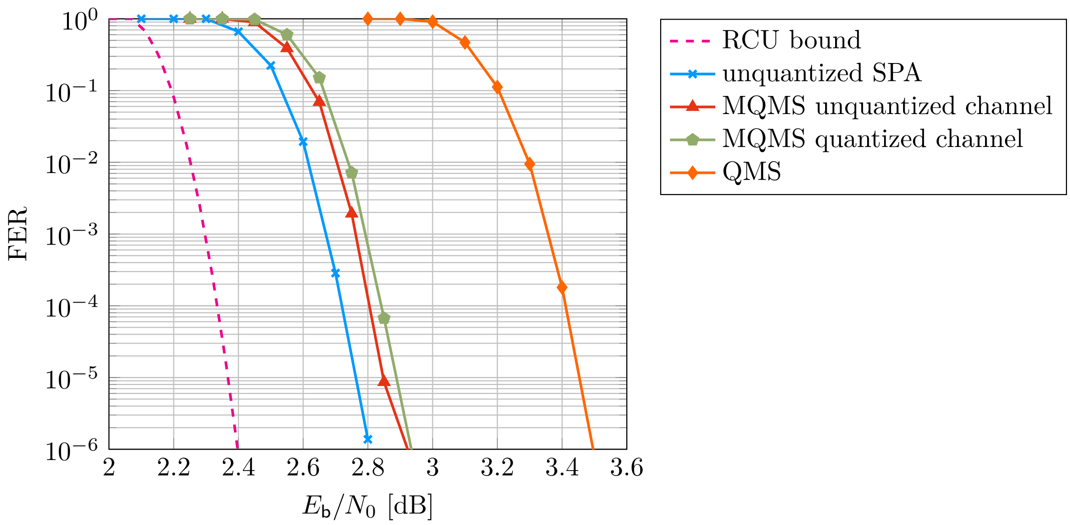


Figure 4.1: FER versus E_b/N_0 [dB] for unquantized SPA MQMS for unquantized and quantized channel output, QMS and RCU bound for $R = 4/5$.

codes have a block length $n = 20000$ bits and their graphs were designed via the progressive edge-growth (PEG) algorithm [94]. The simulation results are shown in Fig. 4.1 and Fig. 4.2 in terms of FER versus E_b/N_0. As a reference, we provide the simulation results of the optimized codes for MQMS under unquantized BP decoding, MQMS for both 4 bit quantized and unquantized channel output and QMS with $b = b_0 = 4$, as well as the random coding union bound (RCU) of [95]. Observe that the MQMS algorithm outperforms the QMS decoder although they both use the same CN update rule. Admittedly, the VN update rule of MQMS is more complex than the one of the plain QMS decoder: An open question is whether the VN update rule in (4.8) can be efficiently implemented in approximate form (e.g., via look-up tables) without compromising the performance of the MQMS algorithm.

Remark 4.1. The cardinality of the message alphabet is $2^b - 1$, i.e., we are not taking full advantage of the b bits. To have a message alphabet of cardinality 2^b, one can replace the

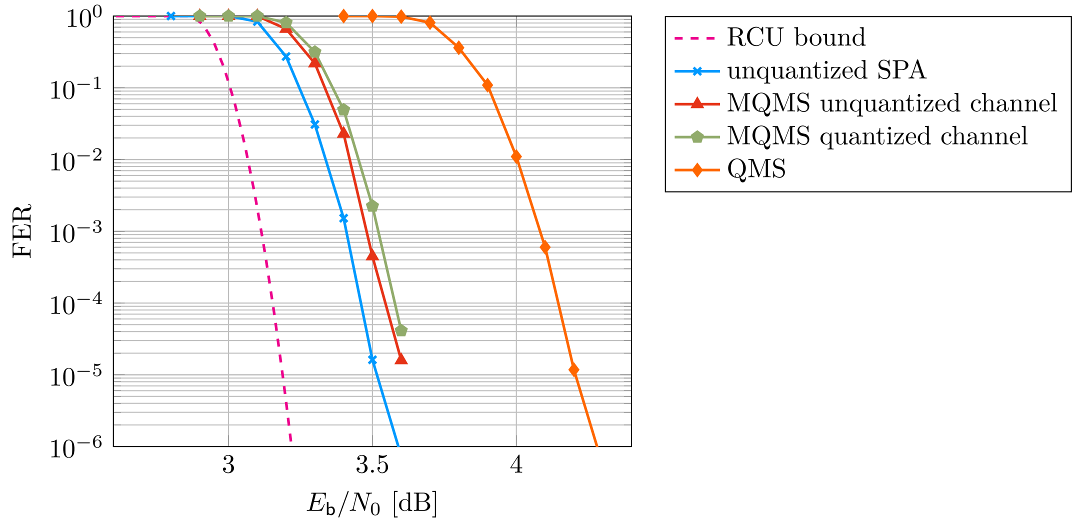


Figure 4.2: FER versus E_b/N_0 [dB] for unquantized SPA MQMS for unquantized and quantized channel output, QMS and RCU bound for $R = 7/8$.

quantization function in (4.4) with

$$\mathsf{f}(x) := \operatorname{sign}(x)\Delta \cdot \min\left\{\left\lfloor \frac{|x|}{\Delta} \right\rfloor + \frac{1}{2}, 2^{b-1} - \frac{1}{2}\right\}. \tag{4.23}$$

Using $\mathsf{f}(x)$ in (4.23) instead of (4.4), we obtain a remarkable gain for $b = 2$. For $b \geq 3$, the thresholds using the two quantization functions nearly coincide. In fact, for $b = 2$, we obtain the quaternary message passing (QMP) that we presented and analyzed in [67], [57]. We investigate next the gains of QMP over TMP (MQMS with $b = 2$) in terms of the iterative decoding threshold. For both algorithms we obtain individually optimized ensembles for rates $R \in \{1/2, 2/3, 3/4, 4/5, 7/8, 9/10\}$, where we restrict the maximum VN degree to 20. Fig. 4.3 depicts the obtained iterative decoding thresholds of the optimized degree distributions under TMP and QMP. QMP decoding improves TMP decoding especially for low rates. For $R = 1/2$, the decoding threshold improves by 0.2 dB as compared to TMP.

We also provide in Fig. 4.4 the iterative decoding thresholds for protograph-based SC-LDPC codes. We follow the approach of [96] for code ensembles and window decoding. For this, we apply the protograph-based DE analysis in [50] and [67] for TMP and QMP. We consider the protograph matrix $\boldsymbol{B}_{[1:W,1:W]}$ that has been derived from (3.15) for a decoding window size of W with $\mu + 1 \leq W \leq L$. The notation $\boldsymbol{B}_{[1:W,1:W]}$ refers to the block matrix of size $W \times W$ that is formed from the first W block rows and W block

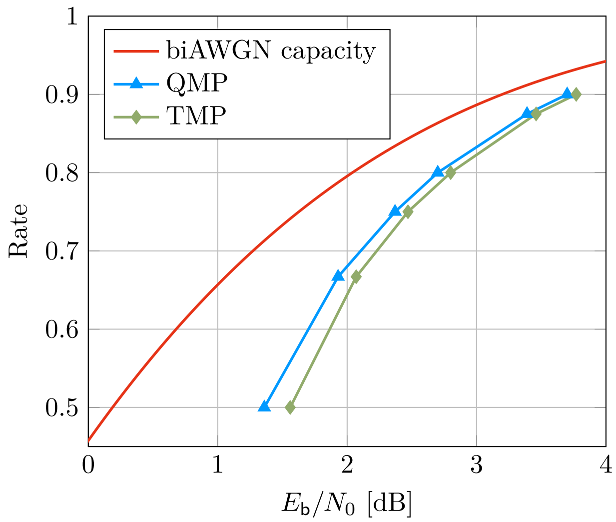


Figure 4.3: Decoding thresholds of optimized LDPC code ensembles under TMP and QMP.

columns of $\boldsymbol{B}$. For instance, for $\mu = 2$ and $W = 4$ we have

$$\boldsymbol{B}_{[1:4,1:4]} = \begin{pmatrix} \boldsymbol{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{B}_1 & \boldsymbol{B}_0 & \mathbf{0} & \mathbf{0} \\ \boldsymbol{B}_2 & \boldsymbol{B}_1 & \boldsymbol{B}_0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B}_2 & \boldsymbol{B}_1 & \boldsymbol{B}_0 \end{pmatrix}. \tag{4.24}$$

Convergence of the window decoder is declared when the probability of decoding error for the VNs in the first block column is (approximately) zero. We consider (asymptotically) regular, protograph-based SC-LDPC codes with VN degrees $\mathsf{d_v} = 6$ and $\mathsf{d_c} \in \{12, 18, 24, 30, 36, 48, 60\}$. The submatrices $\boldsymbol{B}_i$ in (3.15) are given by

$$\boldsymbol{B}_i = \underbrace{(1 \quad 1 \quad \ldots \quad 1)}_{\mathsf{d_c}}, \qquad i = 0, \ldots, \mu, \tag{4.25}$$

where $\mu = \mathsf{d_v} - 1$.

4.2 Non-Binary LDPC Codes

In this section, we introduce and analyze decoding algorithms for q-ary LDPC codes. We start with the SMP decoder over the QSC introduced in [91]. We analyze the performance of SMP for protograph-based SC-LDPC in [51]. We adopt the SMP to the QEC and to

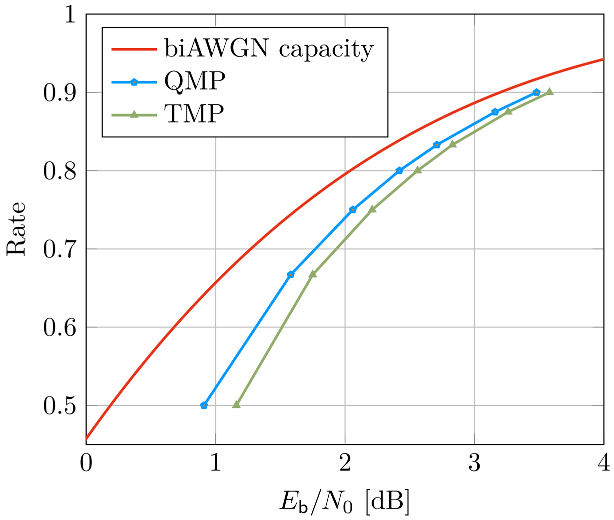


Figure 4.4: Decoding thresholds of spatially coupled LDPC code ensembles under TMP and QMP.

the AWGN and Poisson channels with orthogonal modulations [58, 60]. Further, we extend the SMP to the SRLMP and RSMP decoders [56, 59]. We denote by $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ the message sent from CN $\mathtt{c}$ to its neighboring VN $\mathtt{v}$. Similarly, $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is the message sent from VN $\mathtt{v}$ to CN $\mathtt{c}$ at the ℓ-th iteration.

4.2.1 Log-Likelihood Vector

For a given channel output y of a DMC with input alphabet $\mathcal{X} = \mathbb{F}_q$, we introduce the log-likelihood vector, also referred to as $\boldsymbol{L}$-vector,

$$\boldsymbol{L}(y) = [L_0(y), L_1(y), \ldots, L_{\alpha^{q-2}}(y)] \tag{4.26}$$

whose elements are

$$L_u(y) = \ln P(y|u) \quad \forall u \in \mathbb{F}_q. \tag{4.27}$$

The $\boldsymbol{L}$-vector will be instrumental to the design of message passing decoding algorithms for non-binary LDPC codes.

4.2.2 Q-ary Symmetric Channel

SMP

In this section, we describe the proposed SMP algorithm [91] in detail, assuming transmission over the QSC. An exchanged message between a check and a variable node is a symbol from $\mathbb{F}_q$. We have $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \in \mathcal{M}_{\text{SMP}} = \mathbb{F}_q$.

Each VN sends its channel observation y to its neighboring CNs

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = y. \tag{4.28}$$

Consider a CN $\mathtt{c}$ and a VN $\mathtt{v}$ connected to it. The CN $\mathtt{c}$ computes the symbol that satisfies the parity check equation given the incoming VN messages. Formally,

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = -h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}\to\mathtt{c}} \tag{4.29}$$

where the multiplication and the sum in (4.29) are performed over $\mathbb{F}_q$, $h_{\mathtt{v},\mathtt{c}}$ is the edge label associated to the edge connecting $\mathtt{v}$ and $\mathtt{c}$ and $h^{-1}_{\mathtt{v},\mathtt{c}}$ is the inverse of $h_{\mathtt{v},\mathtt{c}}$ in $\mathbb{F}_q$.

At the ℓ-th iteration, each VN computes

$$\begin{aligned} \boldsymbol{L}^{(\ell)}_{\mathsf{ex}} &= \left[L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \ldots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} \boldsymbol{L}\left(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}\right). \end{aligned} \tag{4.30}$$

The outgoing VN message is the $\mathbb{F}_q$ symbol with the maximum entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$, i.e.,

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \operatorname*{argmax}_{u\in\mathbb{F}_q} L^{(\ell)}_{\mathsf{ex},u}. \tag{4.31}$$

Whenever multiple maximizing arguments exist, the arg max function outputs one of them uniformly at random. The VN operation can be interpreted as if the CNs and the channel would *vote* for the value of the code symbol associated to the VN. The VN assigns different weights to the CN and channel votes and selects the symbol with the highest score.

In (4.30), the $\boldsymbol{L}$-vector $\boldsymbol{L}(y)$ corresponding to the QSC channel observation is obtained from (2.18) and (4.27). Moreover, we model each CN to VN message as an observation of the symbol X (associated to $\mathtt{v}$) at the output of an *extrinsic channel* with input and output alphabets $\mathcal{X} = \mathcal{Z} = \mathbb{F}_q$. The transition probabilities of the extrinsic channel are unknown in general. It was shown in [13, 50] that, for moderate to large block lengths, these probabilities can be accurately estimated via the DE analysis. They are then used to compute the $\boldsymbol{L}$-vectors of the CN messages in (4.26) and (4.27).

To estimate its codeword symbol, each VN computes

$$\begin{aligned}\boldsymbol{L}^{(\ell)}_{\mathsf{app}} =& \left[L^{(\ell)}_{\mathsf{app},0}, L^{(\ell)}_{\mathsf{app},1}, \ldots, L^{(\ell)}_{\mathsf{app},\alpha^{q-2}}\right] \\ =& \boldsymbol{L}\left(y\right) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathtt{v})} \boldsymbol{L}\left(m^{(\ell)}_{\mathsf{c}' \to \mathtt{v}}\right).\end{aligned} \tag{4.32}$$

The final decision is

$$\hat{x}^{(\ell)} = \underset{u \in \mathbb{F}_q}{\operatorname{argmax}}\, L^{(\ell)}_{\mathsf{app},u}. \tag{4.33}$$

We present now a DE for SMP for non-binary LDPC codes over a QSC with error probability ϵ. We partition the message alphabet $\mathcal{M}_{\text{SMP}}$ into 2 disjoint sets $\mathcal{I}_0 = \{0\}$ and $\mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$ where $|\mathcal{I}_0| = 1$, $|\mathcal{I}_1| = q-1$. Due to symmetry, the messages in the same set have the same probability. Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration and $s^{(\ell)}_{\mathcal{I}_k}$ the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1\}$. The ensemble iterative decoding threshold $\epsilon^\star$ is defined as the maximum ϵ for which $p^{(\ell)}_{\mathcal{I}_0} \to 1$ as $\ell \to \infty$. In the limit of $n \to \infty$, the DE analysis can be summarized in the following steps.

1. **Initialization.**

$$p^{(0)}_{\mathcal{I}_0} = 1 - \epsilon \tag{4.34}$$

$$p^{(0)}_{\mathcal{I}_1} = \epsilon. \tag{4.35}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   Check to variable update. We have

$$s^{(\ell)}_{\mathcal{I}_0} = \frac{1}{q}\left[1 + (q-1)\rho\left(\frac{q p^{(\ell-1)}_{\mathcal{I}_0} - 1}{q-1}\right)\right] \tag{4.36}$$

$$s^{(\ell)}_{\mathcal{I}_1} = 1 - s^{(\ell)}_{\mathcal{I}_0}. \tag{4.37}$$

   Variable to check update. The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathbb{F}_q$ and transition probabilities

$$P(z|u) = \begin{cases} s^{(\ell)}_{\mathcal{I}_0} & \text{if } z = u \\ \frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1} & \text{if } z = e, \quad e \in \mathbb{F}_q \setminus \{u\}. \end{cases} \tag{4.38}$$

Consider now the VN to CN messages. Define the random vector $\boldsymbol{F}^{(\ell)} = \left(F_0^{(\ell)}, \ldots, F_{\alpha^{q-2}}^{(\ell)}\right)$ where $F_u^{(\ell)}$, for $u \in \mathbb{F}_q$ denotes the RV associated to the number of incoming CN messages to a degree d VN that are equal to u at the ℓ-th iteration. Let $\boldsymbol{f}^{(\ell)}$ be the realization of $\boldsymbol{F}^{(\ell)}$. The entries of $\boldsymbol{L}\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right)$ in (4.30) are given by

$$L_u\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right) = \ln\left(P(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}|u)\right) \tag{4.39}$$

where $m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} \in \mathbb{F}_q, u \in \mathbb{F}_q$ and $P(z|u)$ can be computed from (4.36), (4.37) and (4.38) $\forall z \in \mathbb{F}_q$. Hence, the elements $L_{\mathsf{ex},u}^{(\ell)}$ of the aggregated extrinsic $\boldsymbol{L}$-vector in (4.30) are related to $f_u^{(\ell)}$ and the channel observation y by

$$L_{\mathsf{ex},u}^{(\ell)} = \mathsf{D}^{(\ell)} f_u^{(\ell)} + \mathsf{D}_{\mathrm{ch}}\delta_{uy} + K \quad \forall u \in \mathbb{F}_q \tag{4.40}$$

where δ_{ij} is the Kronecker delta function and

$$\mathsf{D}_{\mathrm{ch}} = \ln(1-\epsilon) - \ln\left(\frac{\epsilon}{q-1}\right) \tag{4.41}$$

$$\mathsf{D}^{(\ell)} = \ln(s_{\mathcal{I}_0}^{(\ell)}) - \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) \tag{4.42}$$

$$K = \ln\left(\frac{\epsilon}{q-1}\right) + (d-1)\ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right). \tag{4.43}$$

Note that K in (4.43) can be ignored in the VN update rule since it is independent of the symbol u. We obtain

$$p_{\mathcal{I}_0}^{(\ell)} = \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} \frac{\mathbb{I}\left(0 \in \mathcal{L}_{\mathsf{ex}}^{(\ell)}\right)}{|\mathcal{L}_{\mathsf{ex}}^{(\ell)}|} \tag{4.44}$$

$$p_{\mathcal{I}_1}^{(\ell)} = 1 - p_{\mathcal{I}_0}^{(\ell)} \tag{4.45}$$

where the inner sum is over all length q integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries are non-negative and sum to $d-1$ and

$$\mathcal{L}_{\mathsf{ex}}^{(\ell)} = \left\{u \in \mathbb{F}_q \middle| L_{\mathsf{ex},u}^{(\ell)} = \max_{a\in\mathbb{F}_q} L_{\mathsf{ex},a}^{(\ell)}\right\} \tag{4.46}$$

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} = \binom{d-1}{f_0^{(\ell)}, \ldots, f_{\alpha^{q-2}}^{(\ell)}} (s_{\mathcal{I}_0}^{(\ell)})^{f_0^{(\ell)}} \left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right)^{d-1-f_0^{(\ell)}}. \tag{4.47}$$

SRLMP

In this section, we introduce an extension of the SMP algorithm for transmission over a QSC. An exchanged message between a check and a variable node can be an erasure or a list of symbols from $\mathbb{F}_q$ of size at most Γ, i.e., the message alphabet is $\mathcal{M}_\Gamma = \{\mathsf{E}\} \cup \mathcal{O}_\Gamma$, where $\mathcal{O}_\Gamma$ contains all possible sets of symbols in $\mathbb{F}_q$ of size less than or equal to Γ and $\{\mathsf{E}\}$ corresponds to an erasure. The cardinality of the message alphabet is

$$|\mathcal{M}_\Gamma| = \sum_{i=0}^{\Gamma} \binom{q}{i}.$$

For $\Gamma = 1$, we call the decoder SEMP.

Initially, each VN sends its channel observation to its neighboring CNs, i.e.,

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = y \tag{4.48}$$

where y is the channel observation associated to VN $\mathtt{v}$.

Consider a CN $\mathtt{c}$ and a VN $\mathtt{v}$ connected to it. If all of the incoming messages to $\mathtt{c}$ from the other neighboring VNs are not empty, $\mathtt{c}$ computes the set of all symbols that satisfy the parity check equation given the received VN messages. Formally, it computes

$$\mathcal{U}^{(\ell)}_{\mathtt{v},\mathtt{c}} = -h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}' \in \mathcal{N}(\mathtt{c}) \setminus \mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}}. \tag{4.49}$$

The multiplication in (4.49) is performed element-wise over $\mathbb{F}_q$ and the sum is over sets of symbols. The sum over two sets $\mathcal{A}$ and $\mathcal{B}$ is defined as the Minkowski sum, i.e.,

$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}. \tag{4.50}$$

If the size of $\mathcal{U}^{(\ell)}_{\mathtt{v},\mathtt{c}}$ is larger than Γ or $\mathtt{c}$ receives at least one erasure from its neighboring VNs, then $\mathtt{c}$ sends an erasure to $\mathtt{v}$, otherwise it sends the set $\mathcal{U}^{(\ell)}_{\mathtt{v},\mathtt{c}}$. Formally, we write

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = \begin{cases} \mathcal{U}^{(\ell)}_{\mathtt{v},\mathtt{c}} & \text{if } m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} \neq \mathsf{E}\ \forall \mathtt{v}' \in \mathcal{N}(\mathtt{c}) \setminus \mathtt{v} \text{ and } |\mathcal{U}^{(\ell)}_{\mathtt{v},\mathtt{c}}| \leq \Gamma \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.51}$$

Algorithm 4: VN Update Rule.

1 Initialize the set $\mathcal{T} = \emptyset$
2 Find one symbol $a \in \mathbb{F}_q$ with $L^{(\ell)}_{\mathsf{ex},a} = \max_{u \in \mathbb{F}_q \setminus \mathcal{T}} L^{(\ell)}_{\mathsf{ex},u}$
3 Update the set $\mathcal{T} = \mathcal{T} \cup \{a\}$
4 **if** $L^{(\ell)}_{\mathsf{ex},e} > L^{(\ell)}_{\mathsf{ex},u} + \Delta^{(\ell)}$ $\forall e \in \mathcal{T}$ *and* $\forall u \in \mathbb{F}_q \setminus \mathcal{T}$ **then**
5 $\quad m^{(\ell)}_{\mathsf{v}\to\mathsf{c}} = \mathcal{T}$
6 **else**
7 $\quad$ **if** $|\mathcal{T}| < \Gamma$ **then**
8 $\quad\quad$ return to 2
9 $\quad$ **else**
10 $\quad\quad m^{(\ell)}_{\mathsf{v}\to\mathsf{c}} = \mathsf{E}$
11 $\quad$ **end**
12 **end**

At the ℓ-th iteration, each VN computes

$$\begin{aligned} \boldsymbol{L}^{(\ell)}_{\mathsf{ex}} &= \left[L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \dots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathsf{v}) \setminus \mathsf{c}} \boldsymbol{L}\left(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}\right). \end{aligned} \tag{4.52}$$

The outgoing VN message is then obtained by applying Algorithm 4. For $\Gamma = 1$, Algorithm 4 simplifies to

$$m^{(\ell)}_{\mathsf{v}\to\mathsf{c}} = \begin{cases} \{a\} & \text{if } \exists a \in \mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta^{(\ell)} \ \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathsf{E} & \text{otherwise} \end{cases} \tag{4.53}$$

and for $\Gamma = 2$

$$m^{(\ell)}_{\mathsf{v}\to\mathsf{c}} = \begin{cases} \{a\} & \text{if } \exists a \in \mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta^{(\ell)} \ \ \forall u \in \mathbb{F}_q \setminus \{a\} \\ \{a,e\} & \text{if } \exists a, e \in \mathbb{F}_q : |L^{(\ell)}_{\mathsf{ex},a} - L^{(\ell)}_{\mathsf{ex},e}| \le \Delta^{(\ell)} \text{ and } L^{(\ell)}_{\mathsf{ex},a}, L^{(\ell)}_{\mathsf{ex},e} > \\ & L^{(\ell)}_{\mathsf{ex},u} + \Delta^{(\ell)} \ \forall u \in \mathbb{F}_q \setminus \{a,e\} \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.54}$$

In (4.52), the $\boldsymbol{L}$-vector $\boldsymbol{L}(y)$ corresponding to the channel observation is obtained from (4.27) using the transition probabilities of the QSC communication channel given in (2.18). Further, we model each CN to VN message as an observation of the symbol X (associated to v) at the output of an *extrinsic channel* with input alphabet $\mathcal{X} = \mathbb{F}_q$ and output alphabet

$\mathcal{Z} = \mathcal{M}_\Gamma$. The transition probabilities of the extrinsic channel can be estimated via DE and are used to compute the $\boldsymbol{L}$-vectors of the CN messages as shown in (4.26) and (4.27). The parameters $\Delta^{(\ell)}$ are chosen to maximize the iterative decoding threshold and are thus subject of optimization. They can be chosen for each iteration individually or kept constant over the iterations. In the latter case, one can compute the iterative decoding thresholds obtained for several values of Δ and choose the best one.

To estimate its codeword symbol, each variable node computes

$$\begin{aligned}\boldsymbol{L}_{\mathsf{app}}^{(\ell)} &= \left[L_{\mathsf{app},0}^{(\ell)}, L_{\mathsf{app},1}^{(\ell)}, \ldots, L_{\mathsf{app},\alpha^{q-2}}^{(\ell)}\right] \\ &= \boldsymbol{L}\left(y\right) + \sum_{\mathsf{c}'\in\mathcal{N}(\mathsf{v})} \boldsymbol{L}\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right).\end{aligned} \tag{4.55}$$

We have

$$\hat{x}^{(\ell)} = \underset{u\in\mathbb{F}_q}{\operatorname{argmax}}\, L_{\mathsf{app},u}^{(\ell)}. \tag{4.56}$$

In (4.56), if multiple maximizing arguments exist we choose one of them uniformly at random.

Note that for $\Gamma = 1$, the SRLMP is similar to the SMP but SRLMP includes an additional erasure.

We present next a DE analysis for non-binary irregular LDPC codes under SRLMP with $\Gamma = 1, 2$.

Density Evolution Analysis for SRLMP with $\Gamma = 1$

For $\Gamma = 1$, the cardinality of the message alphabet is $|\mathcal{M}_1| = q + 1$. In the DE, the probabilities of VN to CN and CN to VN messages are tracked as iterations progress and we consider the limit as $n \to \infty$. Due to symmetry and under the all-zero codeword assumption, we can partition the message alphabet $\mathcal{M}_1$ into 3 disjoint sets $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ such that the messages in the same set have the same probability. We have

$$\mathcal{I}_0 = \{\{0\}\} \tag{4.57}$$

$$\mathcal{I}_1 = \{\{a\} : a \in \mathbb{F}_q \setminus \{0\}\} \tag{4.58}$$

$$\mathcal{I}_2 = \{\mathsf{E}\}. \tag{4.59}$$

Note that $|\mathcal{I}_0| = |\mathcal{I}_2| = 1$, $|\mathcal{I}_1| = q-1$. Let $p_{\mathcal{I}_k}^{(\ell)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration, i.e., a VN to CN message takes the value $a \in \mathcal{I}_k$ with probability $p_{\mathcal{I}_k}^{(\ell)}/|\mathcal{I}_k|$. Similarly $s_{\mathcal{I}_k}^{(\ell)}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1,2\}$. The ensemble iterative decoding threshold $\epsilon^\star$ is defined as the maximum ϵ for which $p_{\mathcal{I}_0}^{(\ell)} \to 1$ as $\ell \to \infty$. In the limit of $n \to \infty$, the DE analysis can be summarized in the following steps.

1. **Initialization.** Initially, we have

$$p_{\mathcal{I}_0}^{(0)} = 1-\epsilon \tag{4.60}$$
$$p_{\mathcal{I}_1}^{(0)} = \epsilon \tag{4.61}$$
$$p_{\mathcal{I}_2}^{(0)} = 0. \tag{4.62}$$

2. **For** $\ell = 1, 2, \dots, \ell_{\max}$

Check to variable update. We have

$$s_{\mathcal{I}_0}^{(\ell)} = \frac{1}{q}\left[\rho\left(1-p_{\mathcal{I}_2}^{(\ell-1)}\right) + (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.63}$$
$$s_{\mathcal{I}_1}^{(\ell)} = \frac{q-1}{q}\left[\rho\left(1-p_{\mathcal{I}_2}^{(\ell-1)}\right) - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.64}$$
$$s_{\mathcal{I}_2}^{(\ell)} = 1-\rho\left(1-p_{\mathcal{I}_2}^{(\ell-1)}\right). \tag{4.65}$$

The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathcal{M}_1$ and transition probabilities

$$P(z|u) = \begin{cases} s_{\mathcal{I}_0}^{(\ell)} & \text{if } z = \{u\} \\ \frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1} & \text{if } z = \{e\} \qquad e \in \mathbb{F}_q \setminus \{u\} \\ s_{\mathcal{I}_2}^{(\ell)} & \text{if } z = \mathsf{E}. \end{cases} \tag{4.66}$$

Note that for $\Gamma = 1$, the $\boldsymbol{L}$-vectors of the CN messages in (4.52) can be computed from (4.27) and (4.63)-(4.66).

Variable to check update. Consider now the VN to CN messages. We define the random vector $\boldsymbol{F}^{(\ell)} = \left(F_{\{0\}}^{(\ell)}, \dots, F_{\{\alpha^{q-2}\}}^{(\ell)}, F_{\mathsf{E}}^{(\ell)}\right)$, where $F_a^{(\ell)}$ denotes the RV associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_1$ at the ℓ-th iteration, and $f_a^{(\ell)}$ is its realization. The entries of $\boldsymbol{L}\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right)$ in (4.52) are

$$L_u\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right) = \ln\left(P(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}|u)\right) \tag{4.67}$$

where $m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} \in \mathcal{M}_1, u \in \mathbb{F}_q$ and $P(z|u)$ is given in (4.66) $\forall z \in \mathcal{M}_1$. Hence, the elements $L_{\mathsf{ex},u}^{(\ell)}$ of the extrinsic $\boldsymbol{L}$-vector in (4.52) are

$$L_{\mathsf{ex},u}^{(\ell)} = \mathsf{D}_1^{(\ell)} f_{\{u\}}^{(\ell)} + \mathsf{D}_{\mathrm{ch}} \delta_{uy} + K_1 \tag{4.68}$$

$$K_1 = \ln\left(\frac{\epsilon}{q-1}\right) + f_{\mathsf{E}}^{(\ell)} \ln(s_{\mathcal{I}_2}^{(\ell)}) + (d-1-f_{\mathsf{E}}^{(\ell)}) \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) \tag{4.69}$$

$$\mathsf{D}_{\mathrm{ch}} = \ln(1-\epsilon) - \ln\left(\frac{\epsilon}{q-1}\right) \tag{4.70}$$

$$\mathsf{D}_1^{(\ell)} = \ln(s_{\mathcal{I}_0}^{(\ell)}) - \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) \tag{4.71}$$

and δ_{ij} is the Kronecker delta function. Note that K_1 in (4.69) is independent of u. Thus, it can be ignored when computing the extrinsic $\boldsymbol{L}$-vector. We obtain

$$p_{\mathcal{I}_0}^{(\ell)} = \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} \times \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}) \tag{4.72}$$

$$p_{\mathcal{I}_2}^{(\ell)} = \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} \times \left[1 - \sum_{a\in\mathbb{F}_q} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)})\right] \tag{4.73}$$

$$p_{\mathcal{I}_1}^{(\ell)} = 1 - p_{\mathcal{I}_0}^{(\ell)} - p_{\mathcal{I}_2}^{(\ell)} \tag{4.74}$$

where $\mathbb{I}(\mathcal{A})$ is an indicator function and the inner sum is over all length $q+1$ non-negative integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries sum to $d-1$ and

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} = \binom{d-1}{f_{\{0\}}^{(\ell)}, \ldots, f_{\mathsf{E}}^{(\ell)}} \prod_{k=0}^{2} \left(\frac{s_{\mathcal{I}_k}^{(\ell)}}{|\mathcal{I}_k|}\right)^{f_{\mathcal{I}_k}^{(\ell)}} \tag{4.75}$$

$$f_{\mathcal{I}_k}^{(\ell)} = \sum_{a\in\mathcal{I}_k} f_a^{(\ell)} \quad \forall k \in \{0,1,2\}. \tag{4.76}$$

Density Evolution Analysis for SRLMP with $\Gamma = 2$

This section gives a DE analysis for SRLMP with maximum list size $\Gamma = 2$. For $\Gamma = 2$, the cardinality of the message alphabet is $|\mathcal{M}_2| = 1 + q + \binom{q}{2}$. Due to symmetry and under the all-zero codeword assumption, we can partition the message alphabet $\mathcal{M}_2$ into 5 disjoint sets $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ such that the messages in the same set have the same probability. We have $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ as defined in (4.57)-(4.59) and

$$\mathcal{I}_3 = \{\{0,a\} : a \in \mathbb{F}_q \setminus \{0\}\} \tag{4.77}$$
$$\mathcal{I}_4 = \{\{a,e\} : a, e \in \mathbb{F}_q \setminus \{0\} \text{ and } a \neq e\}. \tag{4.78}$$

Note that $|\mathcal{I}_0| = |\mathcal{I}_2| = 1$, $|\mathcal{I}_1| = |\mathcal{I}_3| = q-1$ and $|\mathcal{I}_4| = \binom{q-1}{2}$. Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration. Similarly $s^{(\ell)}_{\mathcal{I}_k}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1,2,3,4\}$.

1. **Initialization.** Initially, we have

$$p^{(0)}_{\mathcal{I}_0} = 1-\epsilon \tag{4.79}$$
$$p^{(0)}_{\mathcal{I}_1} = \epsilon \tag{4.80}$$
$$p^{(0)}_{\mathcal{I}_2} = p^{(0)}_{\mathcal{I}_3} = p^{(0)}_{\mathcal{I}_4} = 0. \tag{4.81}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. For the CN to VN messages, $s^{(\ell)}_{\mathcal{I}_0}$, $s^{(\ell)}_{\mathcal{I}_1}$ are given in (4.63), (4.64), respectively and

$$\begin{aligned} s^{(\ell)}_{\mathcal{I}_3} =& \frac{q-1}{q}\Bigg[2\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} + p^{(\ell-1)}_{\mathcal{I}_1} + \frac{p^{(\ell-1)}_{\mathcal{I}_3}}{q-1} + \frac{p^{(\ell-1)}_{\mathcal{I}_4}}{q-1}\right) - 2\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} + p^{(\ell-1)}_{\mathcal{I}_1}\right) \\ &+(q-2)\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} - \frac{p^{(\ell-1)}_{\mathcal{I}_1}}{q-1} + \frac{p^{(\ell-1)}_{\mathcal{I}_3}}{q-1} - \frac{2p^{(\ell-1)}_{\mathcal{I}_4}}{(q-1)(q-2)}\right) \\ &-(q-2)\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} - \frac{p^{(\ell-1)}_{\mathcal{I}_1}}{q-1}\right)\Bigg] \end{aligned} \tag{4.82}$$

$$\begin{aligned} s^{(\ell)}_{\mathcal{I}_4} =& \frac{(q-1)(q-2)}{q}\Bigg[\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} + p^{(\ell-1)}_{\mathcal{I}_1} + \frac{p^{(\ell-1)}_{\mathcal{I}_3}}{q-1} + \frac{p^{(\ell-1)}_{\mathcal{I}_4}}{q-1}\right) \\ &-\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} + p^{(\ell-1)}_{\mathcal{I}_1}\right) - \rho\left(p^{(\ell-1)}_{\mathcal{I}_0} - \frac{p^{(\ell-1)}_{\mathcal{I}_1}}{q-1} + \frac{p^{(\ell-1)}_{\mathcal{I}_3}}{q-1} - \frac{2p^{(\ell-1)}_{\mathcal{I}_4}}{(q-1)(q-2)}\right) \\ &+\rho\left(p^{(\ell-1)}_{\mathcal{I}_0} - \frac{p^{(\ell-1)}_{\mathcal{I}_1}}{q-1}\right)\Bigg] \end{aligned} \tag{4.83}$$

$$s^{(\ell)}_{\mathcal{I}_2} = 1 - s^{(\ell)}_{\mathcal{I}_0} - s^{(\ell)}_{\mathcal{I}_1} - s^{(\ell)}_{\mathcal{I}_3} - s^{(\ell)}_{\mathcal{I}_4}. \tag{4.84}$$

In this case, the extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathcal{M}_2$ and transition probabilities

$$P(z|u) = \begin{cases} s_{\mathcal{I}_0}^{(\ell)} & \text{if } z = \{u\} \\ \frac{s_{\mathcal{I}_1}^{(\ell)}}{|\mathcal{I}_1|} & \text{if } z = \{e\} \quad e \in \mathbb{F}_q \setminus \{u\} \\ s_{\mathcal{I}_2}^{(\ell)} & \text{if } z = \mathsf{E} \\ \frac{s_{\mathcal{I}_3}^{(\ell)}}{|\mathcal{I}_3|} & \text{if } z = \{u, e\} \quad e \in \mathbb{F}_q \setminus \{u\} \\ \frac{s_{\mathcal{I}_4}^{(\ell)}}{|\mathcal{I}_4|} & \text{if } z = \{a, e\} \quad a, e \in \mathbb{F}_q \setminus \{u\}. \end{cases} \tag{4.85}$$

Note that for $\Gamma = 2$, the $\boldsymbol{L}$-vectors of the CN messages in (4.52) can be computed from (4.27), (4.63), (4.64) and (4.82)-(4.85).

Variable to check update. Consider now the VN to CN messages. We extend the random vector $\boldsymbol{F}^{(\ell)}$ to $\boldsymbol{F}^{(\ell)} = \left(F_{\{0\}}^{(\ell)}, \ldots, F_{\{\alpha^{q-2}\}}^{(\ell)}, F_{\{0,1\}}^{(\ell)}, \ldots, F_{\{\alpha^{q-3}, \alpha^{q-2}\}}^{(\ell)}, F_{\mathsf{E}}^{(\ell)}\right)$ where $F_a^{(\ell)}$ denotes the RV associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_2$ at the ℓ-th iteration. The entries of $\boldsymbol{L}\left(m_{\mathsf{c}' \to \mathsf{v}}^{(\ell)}\right)$ in (4.52) are given by

$$L_u\left(m_{\mathsf{c}' \to \mathsf{v}}^{(\ell)}\right) = \ln\left(P(m_{\mathsf{c}' \to \mathsf{v}}^{(\ell)}|u)\right) \tag{4.86}$$

where $m_{\mathsf{c}' \to \mathsf{v}}^{(\ell)} \in \mathcal{M}_2, u \in \mathbb{F}_q$ and $P(z|u)$ is given in (4.85) $\forall z \in \mathcal{M}_2$. Hence, the entries $L_{\mathsf{ex},u}^{(\ell)}$ of the aggregated extrinsic $\boldsymbol{L}$-vector in (4.52) are related to $f_u^{(\ell)}$ and the channel observation y by

$$L_{\mathsf{ex},u}^{(\ell)} = \mathsf{D}_1^{(\ell)} f_{\{u\}}^{(\ell)} + \mathsf{D}_2^{(\ell)} \sum_{a \in \mathbb{F}_q \setminus \{u\}} f_{\{u,a\}}^{(\ell)} + \mathsf{D}_{\text{ch}} \delta_{uy} + K_2 \tag{4.87}$$

where D_{ch} and $\mathsf{D}_1^{(\ell)}$ are given in (4.70) and (4.71) and we have

$$\mathsf{D}_2^{(\ell)} = \ln\left(\frac{s_{\mathcal{I}_3}^{(\ell)}}{|\mathcal{I}_3|}\right) - \ln\left(\frac{s_{\mathcal{I}_4}^{(\ell)}}{|\mathcal{I}_4|}\right) \tag{4.88}$$

$$\begin{aligned} K_2 = &\ln\left(\frac{\epsilon}{q-1}\right) + \sum_{a \in \mathbb{F}_q} f_{\{a\}}^{(\ell)} \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{|\mathcal{I}_1|}\right) \\ &+ \sum_{a,e \in \mathbb{F}_q} f_{\{a,e\}}^{(\ell)} \ln\left(\frac{s_{\mathcal{I}_4}^{(\ell)}}{|\mathcal{I}_4|}\right) + f_{\mathsf{E}}^{(\ell)} \ln(s_{\mathcal{I}_2}^{(\ell)}). \end{aligned} \tag{4.89}$$

Note that K_2 in (4.89) can be ignored in the VN update rule since it is independent of u. We obtain

$$\begin{aligned} p_{\mathcal{I}_0}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \prod_{u\in\mathbb{F}_q\backslash\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}) \end{aligned} \quad (4.90)$$

$$\begin{aligned} p_{\mathcal{I}_1}^{(\ell)} =& \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\backslash\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \\ & \prod_{u\in\mathbb{F}_q\backslash\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}) \end{aligned} \quad (4.91)$$

$$\begin{aligned} p_{\mathcal{I}_3}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \sum_{a\in\mathbb{F}_q\backslash\{0\}} \mathbb{I}\left(|L_{\mathsf{ex},0}^{(\ell)} - L_{\mathsf{ex},a}^{(\ell)}| \le \Delta^{(\ell)}\right) \times \\ & \prod_{u\in\mathbb{F}_q\backslash\{0,a\}} \mathbb{I}\left(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}\right) \mathbb{I}\left(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}\right) \end{aligned} \quad (4.92)$$

$$\begin{aligned} p_{\mathcal{I}_4}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \sum_{a,e\in\mathbb{F}_q\backslash\{0\},a\neq e} \mathbb{I}\left(|L_{\mathsf{ex},a}^{(\ell)} - L_{\mathsf{ex},e}^{(\ell)}| \le \Delta^{(\ell)}\right) \times \\ & \prod_{u\in\mathbb{F}_q\backslash\{a,e\}} \mathbb{I}\left(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}\right) \mathbb{I}\left(L_{\mathsf{ex},e}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta^{(\ell)}\right) \end{aligned} \quad (4.93)$$

$$p_{\mathcal{I}_2}^{(\ell)} = 1 - p_{\mathcal{I}_0}^{(\ell)} - p_{\mathcal{I}_1}^{(\ell)} - p_{\mathcal{I}_3}^{(\ell)} - p_{\mathcal{I}_4}^{(\ell)} \quad (4.94)$$

where the inner sum is over all length $|\mathcal{M}_2|$ non-negative integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries sum to $d-1$

$$\Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} = \binom{d-1}{f_{\{0\}}^{(\ell)},\ldots,f_{\mathsf{E}}^{(\ell)}} \prod_{k=0}^{4} \left(\frac{s_{\mathcal{I}_k}^{(\ell)}}{|\mathcal{I}_k|}\right)^{f_{\mathcal{I}_k}^{(\ell)}} \quad (4.95)$$

$$f_{\mathcal{I}_k}^{(\ell)} = \sum_{a\in\mathcal{I}_k} f_a^{(\ell)} \quad \forall k\in\{0,\ldots,4\}. \quad (4.96)$$

We investigate the asymptotic performance of SRLMP with maximum list size 1 and 2 obtained by DE. Table 4.3 shows the iterative decoding thresholds of SRLMP for $(3,5)$ regular ensemble and various values of q. For the sake of comparison, we provide the belief propagation thresholds $\epsilon^{\star}_{\mathsf{BP}}$, the Shannon limit ϵ_{Sh} and the thresholds of the SMP decoder. By comparing the thresholds for $\Gamma = 1$ with the SMP ones, we see that significant gains are obtained if erasures are allowed in the decoding

Table 4.3: Decoding thresholds $\epsilon^\star$ of the $(3, 5)$ regular LDPC code ensemble

q	[27] $\Gamma = 1$	[27] $\Gamma = 2$	SMP [91]	SRLMP $\Gamma = 1$	SRLMP $\Gamma = 2$	$\epsilon^\star_{\mathsf{BP}}$	ϵ_{Sh}
2	0.061	-	0.061	0.0975	–	0.113	0.146
4	0.092	0.153	0.122	0.1283	0.1632	0.196	0.248
8	0.093	0.186	0.133	0.1430	0.1918	0.254	0.319
16	0.094	0.2	0.138	0, 1627	0.2057	0.296	0.370
32	-	-	0.140	0.1906	0.2163	0.328	0.4086
64	-	-	0.141	0.2153	0.2209	0.352	0.4369

algorithm. Increasing Γ improves the threshold but this comes at the cost of an increasing complexity. We believe that increasing Γ further will significantly increase the decoding complexity and will not achieve significant gains compared to the case of $\Gamma = 2$.

Note that the SRLMP outperforms the decoding algorithm in [27] for the same maximum list size. Since the CN update rule of both decoders is the same, the gain is probably due to the VN update rule which is more complex for the case of the SRLMP decoder. In fact, the VNs in [27] compute the sum of binary vectors, whereas, here the incoming messages are converted to $\boldsymbol{L}$-vectors before summation. To check the finite-length performance under SRLMP, we consider the performance of a regular $(3, 5)$ code where we set the maximum number of iterations to $\ell_{\max} = 50$. The code has a block length $n = 60000$ and its Tanner graph is obtained via the PEG algorithm [94]. Finite-length simulation results for $\Gamma = 1$ and $\Gamma = 2$ are shown in Fig. 4.5 in terms of symbol error rate (SER) versus the QSC error probability ϵ. We keep $\Delta^{(\ell)}$ constant over the iterations and use $\Delta^{(\ell)} = 1$ for $\Gamma = 1$ and $\Delta^{(\ell)} = 1.25$ for $\Gamma = 2$. As a reference, we provide the simulation results under the SMP decoder [91] and under the decoding algorithm in [27] for $\Gamma = 1$.

RSMP

We introduce now a message passing algorithm for q-ary LDPC codes over the QSC, which we dub RSMP. To decrease the data flow, instead of passing a list of symbols as in SRLMP, the exchanged messages are symbols from $\mathbb{F}_q$ together with their reliability scores from $\{\mathtt{H}, \mathtt{L}\}$ for 1-bit RSMP and $\{\mathtt{vH}, \mathtt{H}, \mathtt{L}, \mathtt{vL}\}$ for 2-bit RSMP. We improve the performance of SMP by including reliability scores in the decoding.

1-bit RSMP

An exchanged message between a check and a variable node is a symbol from $\mathbb{F}_q$ together

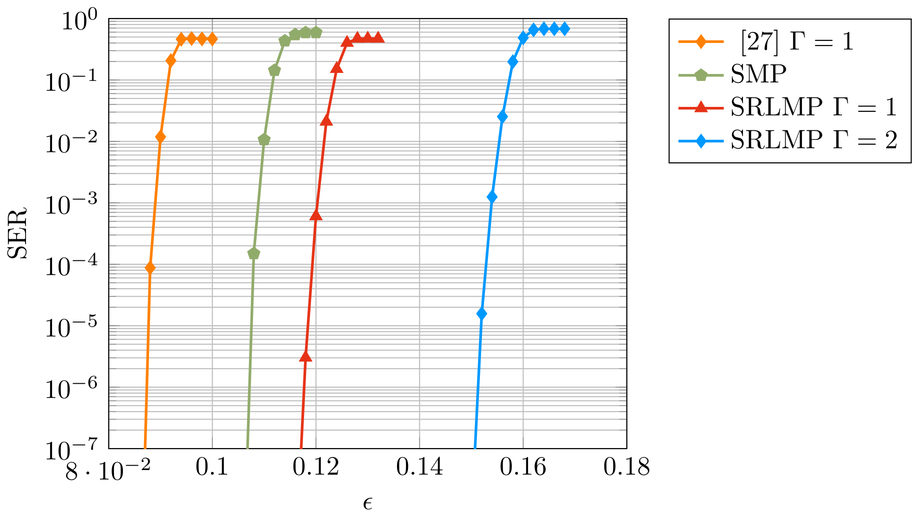


Figure 4.5: SER versus channel error probability ϵ for 4-ary regular $(3, 5)$ LDPC code with $n = 60000$.

with its reliability score from $\{\mathtt{H}, \mathtt{L}\}$, where $\mathtt{H}$ and $\mathtt{L}$ correspond to symbols with high and low reliability, respectively. We denote by $(m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, r^{(\ell)}_{\mathtt{c}\to\mathtt{v}})$ the message sent from CN $\mathtt{c}$ to its neighboring VN $\mathtt{v}$. Similarly, $(m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}, r^{(\ell)}_{\mathtt{v}\to\mathtt{c}})$ is the message sent from VN $\mathtt{v}$ to CN $\mathtt{c}$ at the ℓ-th iteration. We have $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \in \mathbb{F}_q$ and $r^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, r^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \in \{\mathtt{H}, \mathtt{L}\}$.

Initially, each VN sends its channel observation y to its neighboring CNs

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = y. \tag{4.97}$$

The reliability score of $m^{(0)}_{\mathtt{v}\to\mathtt{c}}$ is

$$r^{(0)}_{\mathtt{v}\to\mathtt{c}} = \begin{cases} \mathtt{H} & \text{if } \mathsf{D}_{\text{ch}} > \Delta \\ \mathtt{L} & \text{otherwise} \end{cases} \tag{4.98}$$

where

$$\mathsf{D}_{\text{ch}} = \ln(1-\epsilon) - \ln\left(\frac{\epsilon}{q-1}\right). \tag{4.99}$$

The real-valued parameter Δ is chosen to maximize the iterative decoding threshold and can be chosen for each iteration individually. In this work, we keep Δ constant over the iterations, i.e., we compute the iterative decoding thresholds for several values of Δ and choose the best one.

Consider a CN $\mathtt{c}$ and a VN $\mathtt{v}$ connected to it. The CN $\mathtt{c}$ computes the symbol that satisfies the parity check equation given the incoming VN messages. We assign to the outgoing symbol from $\mathtt{c}$ the reliability score $\mathtt{L}$ if any incoming symbols from the other neighboring VNs has low reliability and the reliability score $\mathtt{H}$ otherwise. Formally, the outgoing message is $(m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, r^{(\ell)}_{\mathtt{c}\to\mathtt{v}})$ with

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = -\, h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} \tag{4.100}$$

and the reliability score of $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ is

$$r^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = \begin{cases} \mathtt{H} & \text{if } r^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} = \mathtt{H} \quad \forall \mathtt{v}' \in \mathcal{N}(\mathtt{c}) \setminus \mathtt{v} \\ \mathtt{L} & \text{otherwise.} \end{cases} \tag{4.101}$$

The multiplication and the sum in (4.100) are performed over $\mathbb{F}_q$ and $h^{-1}_{\mathtt{v},\mathtt{c}}$ is the inverse of $h_{\mathtt{v},\mathtt{c}}$ in $\mathbb{F}_q$.

At the ℓ-th iteration, each VN computes

$$\begin{aligned} \boldsymbol{L}^{(\ell)}_{\mathsf{ex}} &= \left[L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \dots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} \boldsymbol{L}\left((m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}, r^{(\ell)}_{\mathtt{c}'\to\mathtt{v}})\right). \end{aligned} \tag{4.102}$$

Then, the VN determines the $\mathbb{F}_q$ symbol with the maximum entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$. The outgoing symbol has high reliability if its corresponding entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ is greater by Δ than each of the other entries. Formally, the VN sends $(m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}, r^{(\ell)}_{\mathtt{v}\to\mathtt{c}})$ with

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \operatorname*{argmax}_{u\in\mathbb{F}_q} L^{(\ell)}_{\mathsf{ex},u} \tag{4.103}$$

and the reliability score of $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is

$$r^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \begin{cases} \mathtt{H} & \text{if } \exists a \in \mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta \quad \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathtt{L} & \text{otherwise.} \end{cases} \tag{4.104}$$

In (4.103), if multiple maximizing arguments exist the arg max function outputs one of them uniformly at random.

To estimate its codeword symbol each VN computes

$$\begin{aligned}\boldsymbol{L}_{\mathsf{app}}^{(\ell)} &= \left[L_{\mathsf{app},0}^{(\ell)}, L_{\mathsf{app},1}^{(\ell)}, \dots, L_{\mathsf{app},\alpha^{q-2}}^{(\ell)}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathsf{v})} \boldsymbol{L}\left((m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}, r_{\mathsf{c}'\to\mathsf{v}}^{(\ell)})\right).\end{aligned} \tag{4.105}$$

The final decision is

$$\hat{x}^{(\ell)} = \operatorname*{argmax}_{u \in \mathbb{F}_q} L_{\mathsf{app},u}^{(\ell)}. \tag{4.106}$$

Note that we can easily include erasures in the decoding algorithm. We observed that both decoding algorithms (with and without erasures) have similar performance.

Remark 4.2. The complexity of a message passing decoding algorithm can be studied from 2 perspectives: the cost of the arithmetic operations and the decoder data flow. The internal decoder data flow, defined as the number of bits that are processed in each iteration, scales linearly in the number of bits that represent the exchanged CN and VN messages [42]. This work targets this second complexity, i.e., the reduction of the internal data flow. The exchanged messages in BP decoder are $(q-1)$-ary real valued vectors, whereas for RSMP the exchanged messages are symbols from $\mathbb{F}_q$ together with a reliability score from $\{\mathtt{H}, \mathtt{L}\}$. This approach substantially reduces the number of bits needed to represent the exchanged CN and VN messages and therefore the decoder data flow.

Density Evolution for 1-bit RSMP

This section provides a DE analysis for RSMP with 1-bit reliability for non-binary irregular LDPC code ensembles. In the DE, the probabilities of VN to CN and CN to VN messages are tracked as iterations progress. Due to symmetry and under the all-zero codeword assumption, we can partition $\mathbb{F}_q \times \{\mathtt{H}, \mathtt{L}\}$ into the following 4 disjoint sets

$$\mathcal{I}_0 = \{(0, \mathtt{H})\} \tag{4.107}$$

$$\mathcal{I}_1 = \{(a, \mathtt{H}) : a \in \mathbb{F}_q \setminus \{0\}\} \tag{4.108}$$

$$\mathcal{I}_2 = \{(0, \mathtt{L})\} \tag{4.109}$$

$$\mathcal{I}_3 = \{(a, \mathtt{L}) : a \in \mathbb{F}_q \setminus \{0\}\} \tag{4.110}$$

where $(u, \mathtt{H})$ denotes a high-reliability symbol u and $(u, \mathtt{L})$ denotes a low-reliability symbol $u \in \mathbb{F}_q$. Note that $|\mathcal{I}_0| = |\mathcal{I}_2| = 1$, $|\mathcal{I}_1| = |\mathcal{I}_3| = q-1$. Let $p_{\mathcal{I}_k}^{(\ell)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration. That means a VN to CN symbol takes the value $a \in \mathbb{F}_q$ and has the reliability score $r \in \{\mathtt{H}, \mathtt{L}\}$ with probability $p_{\mathcal{I}_k}^{(\ell)}/|\mathcal{I}_k|$ if

$(a,r) \in \mathcal{I}_k$. Similarly $s_{\mathcal{I}_k}^{(\ell)}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1,2,3\}$. The iterative decoding threshold $\epsilon^\star$ is defined as the maximum channel error probability such that $p_{\mathcal{I}_0}^{(\ell)} \to 1$ as $\ell \to \infty$.

1. **Initialization.** Initially, we have

$$p_{\mathcal{I}_0}^{(0)} = \mathbb{I}(\mathsf{D}_{\text{ch}} > \Delta)(1-\epsilon) \tag{4.111}$$
$$p_{\mathcal{I}_1}^{(0)} = \mathbb{I}(\mathsf{D}_{\text{ch}} > \Delta)\epsilon \tag{4.112}$$
$$p_{\mathcal{I}_2}^{(0)} = \mathbb{I}(\mathsf{D}_{\text{ch}} \leq \Delta)(1-\epsilon) \tag{4.113}$$
$$p_{\mathcal{I}_3}^{(0)} = \mathbb{I}(\mathsf{D}_{\text{ch}} \leq \Delta)\epsilon \tag{4.114}$$

where $\mathbb{I}(\mathcal{A})$ is an indicator function that takes the value 1 if the proposition $\mathcal{A}$ is true and 0 otherwise.

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. For the CN to VN messages, we have

$$s_{\mathcal{I}_0}^{(\ell)} = \frac{1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) + (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.115}$$

$$s_{\mathcal{I}_1}^{(\ell)} = \frac{q-1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.116}$$

$$\begin{aligned} s_{\mathcal{I}_2}^{(\ell)} =& \frac{1}{q}\left[1 - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) - (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right. \\ &\left. +(q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right)\right] \end{aligned} \tag{4.117}$$

$$\begin{aligned} s_{\mathcal{I}_3}^{(\ell)} =& \frac{q-1}{q}\left[1 - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) + \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right. \\ &\left. -\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right)\right]. \end{aligned} \tag{4.118}$$

Variable to check update. The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathbb{F}_q \times \{\mathtt{H}, \mathtt{L}\}$ and transition probabilities

$$P(z|u) = \begin{cases} s_{\mathcal{I}_0}^{(\ell)} & \text{if } z = (u, \mathtt{H}) \\ \frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{H}) \quad e \in \mathbb{F}_q \setminus \{u\} \\ s_{\mathcal{I}_2}^{(\ell)} & \text{if } z = (u, \mathtt{L}) \\ \frac{s_{\mathcal{I}_3}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{L}) \quad e \in \mathbb{F}_q \setminus \{u\}. \end{cases} \tag{4.119}$$

Consider now the VN to CN messages. Define the random vector $\boldsymbol{F}^{(\ell)} = \left(F_{(0,\mathtt{H})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{H})}^{(\ell)}, F_{(0,\mathtt{L})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{L})}^{(\ell)}\right)$ where $F_{(u,r)}^{(\ell)}$, for $u \in \mathbb{F}_q$ and $r \in \{\mathtt{H}, \mathtt{L}\}$, denotes the RV associated to the number of incoming CN messages to a degree d VN that are equal to (u, r) at the ℓ-th iteration. Let $\boldsymbol{f}^{(\ell)}$ be the realization of $\boldsymbol{F}^{(\ell)}$. The entries of $\boldsymbol{L}\left((m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}, r_{\mathsf{c}'\to\mathsf{v}}^{(\ell)})\right)$ in (4.102) are given by

$$L_u\left((m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}, r_{\mathsf{c}'\to\mathsf{v}}^{(\ell)})\right) = \ln\left(P((m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}, r_{\mathsf{c}'\to\mathsf{v}}^{(\ell)})|u)\right) \tag{4.120}$$

where $m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} \in \mathbb{F}_q, r_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} \in \{\mathtt{H}, \mathtt{L}\}, u \in \mathbb{F}_q$ and $P(z|u)$ can be computed from (4.115), (4.116), (4.117), (4.118) and (4.119) $\forall z \in \mathbb{F}_q \times \{\mathtt{H}, \mathtt{L}\}$. Hence, the elements $L_{\mathsf{ex},u}^{(\ell)}$ of the aggregated extrinsic $\boldsymbol{L}$-vector in (4.102) are related to $f_u^{(\ell)}$ and the channel observation y by

$$L_{\mathsf{ex},u}^{(\ell)} = \mathsf{D}_{\mathtt{H}}^{(\ell)} f_{(u,\mathtt{H})}^{(\ell)} + \mathsf{D}_{\mathtt{L}}^{(\ell)} f_{(u,\mathtt{L})}^{(\ell)} + \mathsf{D}_{\text{ch}} \delta_{uy} + K \quad \forall u \in \mathbb{F}_q \tag{4.121}$$

where δ_{ij} is the Kronecker delta function, D_{ch} is given in (4.70) and we have

$$\mathsf{D}_{\mathtt{H}}^{(\ell)} = \ln(s_{\mathcal{I}_0}^{(\ell)}) - \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) \tag{4.122}$$

$$\mathsf{D}_{\mathtt{L}}^{(\ell)} = \ln(s_{\mathcal{I}_2}^{(\ell)}) - \ln\left(\frac{s_{\mathcal{I}_3}^{(\ell)}}{q-1}\right) \tag{4.123}$$

$$K_1 = \ln\left(\frac{\epsilon}{q-1}\right) + \sum_{a\in\mathbb{F}_q} f_{(a,\mathtt{H})}^{(\ell)} \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) + \sum_{a\in\mathbb{F}_q} f_{(a,\mathtt{L})}^{(\ell)} \ln\left(\frac{s_{\mathcal{I}_3}^{(\ell)}}{q-1}\right). \tag{4.124}$$

Note that K_1 in (4.124) can be ignored in the VN update rule since it is independent of the symbol u. We obtain

$$p_{\mathcal{I}_0}^{(\ell)} = \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta) \quad (4.125)$$

$$p_{\mathcal{I}_1}^{(\ell)} = \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \times \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta) \quad (4.126)$$

$$p_{\mathcal{I}_2}^{(\ell)} = \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \left[\mathbb{I}(\mathcal{S}_0\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}) + \frac{\mathbb{I}(0\in\mathcal{U})}{|\mathcal{U}|}\right] \quad (4.127)$$

$$p_{\mathcal{I}_3}^{(\ell)} = \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \left[\mathbb{I}(\mathcal{S}_a\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}) + \frac{\mathbb{I}(a\in\mathcal{U})}{|\mathcal{U}|}\right] \quad (4.128)$$

where the inner sum is over all length $2q$ integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries are non-negative and sum to $d-1$. For all $u \in \mathbb{F}_q$, we have

$$\mathcal{S}_u = \{e\in\mathbb{F}_q : L_{\mathsf{ex},u}^{(\ell)} - \Delta \leq L_{\mathsf{ex},e}^{(\ell)} < L_{\mathsf{ex},u}^{(\ell)}\} \quad (4.129)$$

$$\mathcal{U} = \{e\in\mathbb{F}_q : L_{\mathsf{ex},e}^{(\ell)} = \max_{u\in\mathbb{F}_q} L_{\mathsf{ex},u}^{(\ell)}\} \quad (4.130)$$

$$\Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} = \binom{d-1}{f_{(0,\mathtt{H})}^{(\ell)},\ldots,f_{(\alpha^{q-2},\mathtt{L})}^{(\ell)}} \prod_{k=0}^{3} \left(\frac{s_{\mathcal{I}_k}^{(\ell)}}{|\mathcal{I}_k|}\right)^{f_{\mathcal{I}_k}^{(\ell)}} \quad (4.131)$$

$$f_{\mathcal{I}_k}^{(\ell)} = \sum_{(a,r)\in\mathcal{I}_k} f_{(a,r)}^{(\ell)} \quad \forall k\in\{0,\ldots,3\}. \quad (4.132)$$

Tables 4.4 and 4.5 compare the iterative decoding thresholds $\epsilon^\star$ of 1-bit RSMP, SMP, SRLMP (for maximum list size $\Gamma=1$ and $\Gamma=2$) and BP decoding $\epsilon_{\mathsf{BP}}^\star$ for $(4,8)$ and $(3,4)$ regular ensembles and several q values. The tables also give the Shannon limit ϵ_{Sh} and the thresholds of the list message passing algorithm in [27] for maximum list size $\Gamma=1$ and $\Gamma=2$. Observe that 1-bit RSMP outperforms SMP decoding. This gain is due to including reliability scores in the decoding process. For 1-bit RSMP, the alphabet size of the messages is $2q$ which is much smaller than the alphabet size of SRLMP and the list

Table 4.4: Decoding thresholds $\epsilon^\star$ of the $(4,8)$ regular LDPC code ensembles

q	SMP [91]	SRLMP $\Gamma=1$	SRLMP $\Gamma=2$	1-bit RSMP	$\epsilon^\star_{\mathsf{BP}}$	ϵ_{Sh}
2	0.0516	0.0656	-	0.0687	0.076	0.110
4	0.0814	0.0923	0.1075	0.1041	0.134	0.189
8	0.1064	0.1151	0.1332	0.1321	0.175	0.247
16	0.137	0.1389	0.1533	0.1481	0.204	0.2897
32	0.1636	0.1636	0.1673	0.1697	0.226	0.3217
64	0.1758	0.1758	0.1758	0.1866	0.241	0.3462

Table 4.5: Decoding thresholds $\epsilon^\star$ of the $(3,4)$ regular LDPC code ensemble

q	SMP [91]	[27] $\Gamma=1$	[27] $\Gamma=2$	SRLMP $\Gamma=1$	SRLMP $\Gamma=2$	1-bit RSMP	$\epsilon^\star_{\mathsf{BP}}$	ϵ_{Sh}
2	0.1069	0.106	-	0.1439	–	0.1448	0.167	0.2145
4	0.1724	0.123	0.222	0.1842	0.2390	0.2213	0.280	0.3546
8	0.1867	0.124	0.269	0.2096	0.2790	0.2791	0.355	0.4480
16	0.1930	0.120	0.287	0.2481	0.2977	0.3138	0.407	0.5120
32	0.1960	-	-	0.2893	0.3110	0.3382	0.444	0.5570
64	0.1974	-	-	0.3128	0.3175	0.354	0.475	0.5894

message passing [27] for maximum list size 2, which is equal to $1+q(q+1)/2$. Remarkably, for some values of q and degree distributions, 1-bit RSMP outperforms both SRLMP and the algorithm in [27] for maximum list size 2 and with reduced complexity and data flow.

To check the finite-length performance under 1-bit RSMP, we consider the performance of a regular $(4,8)$ code where we set the maximum number of iterations $\ell_{\max} = 50$. The code has a block length $n = 12000$ and its Tanner graph is obtained via the PEG algorithm [94] and edge labels uniformly chosen in $\mathbb{F}_q \setminus \{0\}$. Finite-length simulation results for $q \in \{2,4,8,16\}$ are shown in Fig. 4.6 in terms of FER versus the QSC error probability ϵ. We use $\Delta = 1.6$ for $q = 2$ and 8, $\Delta = 1.5$ for $q = 4$ and $\Delta = 1.8$ for $q = 16$. The parameters $\mathsf{D}_{\mathtt{H}}^{(\ell)}, \mathsf{D}_{\mathtt{L}}^{(\ell)}$ are not provided but are obtained as a byproduct of DE analysis. As a reference, we provide the simulation results under SMP decoding [91].

2-bit RSMP

We extend the 1-bit RSMP by using 2 bits for the reliability, i.e., an exchanged message between a check and a variable node is a symbol from $\mathbb{F}_q$ together with its reliability score from $\{\mathtt{vH}, \mathtt{H}, \mathtt{L}, \mathtt{vL}\}$, where vH, H, L, vL correspond to symbols with very high, high, low and very low reliability, respectively. We sort the reliabilities as $\mathtt{vL} < \mathtt{L} < \mathtt{H} < \mathtt{vH}$. We introduce three real-valued parameters Δ_1, Δ_2 and Δ_3. These parameters are chosen to maximize the iterative decoding threshold and can be chosen for each iteration individually. In this work,

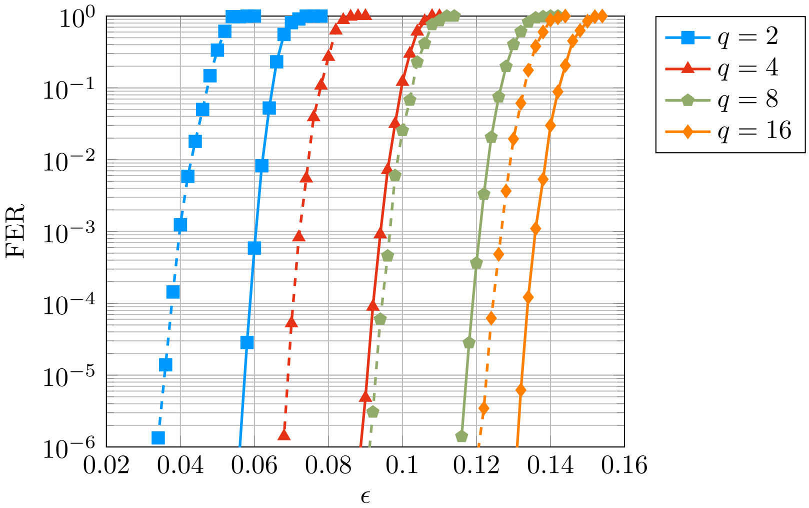


Figure 4.6: FER versus channel error probability ϵ for regular $(4, 8)$ LDPC codes with $n = 12000$ for SMP (dashed lines) and 1-bit RSMP (solid lines).

we keep them constant over the iterations.

Initially, each VN sends its channel observation y to its neighboring CNs

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = y. \tag{4.133}$$

The reliability score of $m^{(0)}_{\mathtt{v}\to\mathtt{c}}$ is

$$r^{(0)}_{\mathtt{v}\to\mathtt{c}} = \begin{cases} \mathtt{vH} & \text{if } \mathsf{D}_{\text{ch}} > \Delta_3 \\ \mathtt{H} & \text{if } \Delta_2 < \mathsf{D}_{\text{ch}} \leq \Delta_3 \\ \mathtt{L} & \text{if } \Delta_1 < \mathsf{D}_{\text{ch}} \leq \Delta_2 \\ \mathtt{vL} & \text{otherwise} \end{cases} \tag{4.134}$$

where

$$\mathsf{D}_{\text{ch}} = \ln(1 - \epsilon) - \ln\left(\frac{\epsilon}{q-1}\right). \tag{4.135}$$

Consider a CN $\mathtt{c}$ and a VN $\mathtt{v}$ connected to it. The CN $\mathtt{c}$ computes the symbol that satisfies the parity check equation given the incoming VN messages. We assign to the outgoing symbol from $\mathtt{c}$ the lowest reliability score of the incoming symbols from the other neighboring VNs. Formally, the outgoing message is $(m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, r^{(\ell)}_{\mathtt{c}\to\mathtt{v}})$ with

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = -\,h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} \tag{4.136}$$

and the reliability score of $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ is

$$r^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = \min_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} r^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}}. \tag{4.137}$$

The multiplication and the sum in (4.136) are performed over $\mathbb{F}_q$ and $h^{-1}_{\mathtt{v},\mathtt{c}}$ is the inverse of $h_{\mathtt{v},\mathtt{c}}$ in $\mathbb{F}_q$.

At the ℓ-th iteration, each VN computes

$$\begin{aligned}\boldsymbol{L}^{(\ell)}_{\mathsf{ex}} &= \left[L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \ldots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} \boldsymbol{L}\left((m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}, r^{(\ell)}_{\mathtt{c}'\to\mathtt{v}})\right).\end{aligned} \tag{4.138}$$

Then, the VN determines the $\mathbb{F}_q$ symbol with the maximum entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$. The outgoing symbol has high reliability if its corresponding entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ is greater by Δ than each of the other entries. Formally, the VN sends $(m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}, r^{(\ell)}_{\mathtt{v}\to\mathtt{c}})$ with

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \operatorname*{argmax}_{u\in\mathbb{F}_q} L^{(\ell)}_{\mathsf{ex},u} \tag{4.139}$$

and the reliability score of $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is

$$r^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \begin{cases} \mathtt{vH} & \text{if } \exists a\in\mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_3 \quad \forall u\in\mathbb{F}_q\setminus\{a\} \\ \mathtt{H} & \text{if } \nexists a\in\mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_3 \quad \forall u\in\mathbb{F}_q\setminus\{a\}\ \& \\ & \exists a\in\mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_2 \quad \forall u\in\mathbb{F}_q\setminus\{a\} \\ \mathtt{L} & \text{if } \nexists a\in\mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_2 \quad \forall u\in\mathbb{F}_q\setminus\{a\}\ \& \\ & \exists a\in\mathbb{F}_q : L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_1 \quad \forall u\in\mathbb{F}_q\setminus\{a\} \\ \mathtt{vL} & \text{otherwise.} \end{cases} \tag{4.140}$$

In (4.139), if multiple maximizing arguments exist the arg max function outputs one of them uniformly at random.

To estimate its codeword symbol each VN computes

$$\begin{aligned}\boldsymbol{L}_{\mathsf{app}}^{(\ell)} &= \left[L_{\mathsf{app},0}^{(\ell)}, L_{\mathsf{app},1}^{(\ell)}, \ldots, L_{\mathsf{app},\alpha^{q-2}}^{(\ell)}\right] \\ &= \boldsymbol{L}(y) + \sum_{\mathtt{c}' \in \mathcal{N}(\mathtt{v})} \boldsymbol{L}\left((m_{\mathtt{c}' \to \mathtt{v}}^{(\ell)}, r_{\mathtt{c}' \to \mathtt{v}}^{(\ell)})\right). \end{aligned} \quad (4.141)$$

The final decision is

$$\hat{x}^{(\ell)} = \operatorname*{argmax}_{u \in \mathbb{F}_q} L_{\mathsf{app},u}^{(\ell)}. \quad (4.142)$$

Density Evolution for 2-bit RSMP

We provide a DE analysis for RSMP with 2-bit reliability for non-binary irregular LDPC code ensembles. We partition $\mathbb{F}_q \times \{\mathtt{H}, \mathtt{L}\}$ into the following 8 disjoint sets

$$\mathcal{I}_0 = \{(0, \mathtt{vH})\} \quad (4.143)$$
$$\mathcal{I}_1 = \{(a, \mathtt{vH}) : a \in \mathbb{F}_q \setminus \{0\}\} \quad (4.144)$$
$$\mathcal{I}_2 = \{(0, \mathtt{H})\} \quad (4.145)$$
$$\mathcal{I}_3 = \{(a, \mathtt{H}) : a \in \mathbb{F}_q \setminus \{0\}\} \quad (4.146)$$
$$\mathcal{I}_4 = \{(0, \mathtt{L})\} \quad (4.147)$$
$$\mathcal{I}_5 = \{(a, \mathtt{L}) : a \in \mathbb{F}_q \setminus \{0\}\} \quad (4.148)$$
$$\mathcal{I}_6 = \{(0, \mathtt{vL})\} \quad (4.149)$$
$$\mathcal{I}_7 = \{(a, \mathtt{vL}) : a \in \mathbb{F}_q \setminus \{0\}\} \quad (4.150)$$

where $(u, \mathtt{vH}), (u, \mathtt{H}), (u, \mathtt{L}), (u, \mathtt{vL})$ denote a very high, high, low, very low reliable symbol $u \in \mathbb{F}_q$, respectively. Note that $|\mathcal{I}_0| = |\mathcal{I}_2| = |\mathcal{I}_4| = |\mathcal{I}_6| = 1$, $|\mathcal{I}_1| = |\mathcal{I}_3| = |\mathcal{I}_5| = |\mathcal{I}_7| = q - 1$.

Let $p_{\mathcal{I}_k}^{(\ell)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration. Similarly $s_{\mathcal{I}_k}^{(\ell)}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0, 1, \ldots, 7\}$.

1. **Initialization.** Initially, we have

$$p_{\mathcal{I}_0}^{(0)} = \mathbb{I}(\mathsf{D}_{\mathrm{ch}} > \Delta_3)(1 - \epsilon) \quad (4.151)$$
$$p_{\mathcal{I}_1}^{(0)} = \mathbb{I}(\mathsf{D}_{\mathrm{ch}} > \Delta_3)\epsilon \quad (4.152)$$
$$p_{\mathcal{I}_2}^{(0)} = \mathbb{I}(\Delta_2 < \mathsf{D}_{\mathrm{ch}} \leq \Delta_3)(1 - \epsilon) \quad (4.153)$$
$$p_{\mathcal{I}_3}^{(0)} = \mathbb{I}(\Delta_2 < \mathsf{D}_{\mathrm{ch}} \leq \Delta_3)\epsilon \quad (4.154)$$

$$p_{\mathcal{I}_4}^{(0)} = \mathbb{I}(\Delta_1 < \mathsf{D}_{\mathrm{ch}} \leq \Delta_2)(1-\epsilon) \tag{4.155}$$
$$p_{\mathcal{I}_5}^{(0)} = \mathbb{I}(\Delta_1 < \mathsf{D}_{\mathrm{ch}} \leq \Delta_2)\epsilon \tag{4.156}$$
$$p_{\mathcal{I}_6}^{(0)} = \mathbb{I}(\mathsf{D}_{\mathrm{ch}} \leq \Delta_1)(1-\epsilon) \tag{4.157}$$
$$p_{\mathcal{I}_7}^{(0)} = \mathbb{I}(\mathsf{D}_{\mathrm{ch}} \leq \Delta_1)\epsilon. \tag{4.158}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. For the CN to VN messages, we have

$$s_{\mathcal{I}_0}^{(\ell)} = \frac{1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) + (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.159}$$

$$s_{\mathcal{I}_1}^{(\ell)} = \frac{q-1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.160}$$

$$\begin{aligned} s_{\mathcal{I}_2}^{(\ell)} = &\frac{1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}\right) - (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right. \\ &\left. -\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) + (q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right)\right] \end{aligned} \tag{4.161}$$

$$\begin{aligned} s_{\mathcal{I}_3}^{(\ell)} = &\frac{q-1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}\right) + \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right. \\ &\left. -\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)}\right) - \rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right)\right] \end{aligned} \tag{4.162}$$

$$\begin{aligned} s_{\mathcal{I}_4}^{(\ell)} = &\frac{1}{q}\left[\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}\right)\right. \\ &-(q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right) \\ &-\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}\right) \\ &\left. +(q-1)\rho\left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}}{q-1}\right)\right] \end{aligned} \tag{4.163}$$

$$\begin{aligned}
s_{\mathcal{I}_5}^{(\ell)} =& \frac{q-1}{q} \Big[\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}\right) \\
&+\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}}{q-1}\right) \\
&-\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)}\right) \\
&-\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}}{q-1}\right)\Bigg]
\end{aligned} \tag{4.164}$$

$$\begin{aligned}
s_{\mathcal{I}_6}^{(\ell)} =& \frac{1}{q} \Bigg[1 - (q-1)\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}}{q-1}\right) + \\
&(q-1)\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_6}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)} + p_{\mathcal{I}_7}^{(\ell-1)}}{q-1}\right) \\
&-\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}\right)\Bigg]
\end{aligned} \tag{4.165}$$

$$\begin{aligned}
s_{\mathcal{I}_7}^{(\ell)} =& \frac{q-1}{q} \Bigg[1 + \rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}}{q-1}\right) \\
&-\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)}\right) \\
&-\rho \left(p_{\mathcal{I}_0}^{(\ell-1)} + p_{\mathcal{I}_2}^{(\ell-1)} + p_{\mathcal{I}_4}^{(\ell-1)} + p_{\mathcal{I}_6}^{(\ell-1)} - \frac{p_{\mathcal{I}_1}^{(\ell-1)} + p_{\mathcal{I}_3}^{(\ell-1)} + p_{\mathcal{I}_5}^{(\ell-1)} + p_{\mathcal{I}_7}^{(\ell-1)}}{q-1}\right)\Bigg].
\end{aligned} \tag{4.166}$$

Variable to check update. The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathbb{F}_q \times \{\mathtt{H}, \mathtt{L}\}$ and transition probabilities

$$P(z|u) = \begin{cases}
s_{\mathcal{I}_0}^{(\ell)} & \text{if } z = (u, \mathtt{vH}) \\
\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{vH}) \quad e \in \mathbb{F}_q \setminus \{u\} \\
s_{\mathcal{I}_2}^{(\ell)} & \text{if } z = (u, \mathtt{H}) \\
\frac{s_{\mathcal{I}_3}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{H}) \quad e \in \mathbb{F}_q \setminus \{u\} \\
s_{\mathcal{I}_4}^{(\ell)} & \text{if } z = (u, \mathtt{L}) \\
\frac{s_{\mathcal{I}_5}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{L}) \quad e \in \mathbb{F}_q \setminus \{u\} \\
s_{\mathcal{I}_6}^{(\ell)} & \text{if } z = (u, \mathtt{vL}) \\
\frac{s_{\mathcal{I}_7}^{(\ell)}}{q-1} & \text{if } z = (e, \mathtt{vL}) \quad e \in \mathbb{F}_q \setminus \{u\}.
\end{cases} \tag{4.167}$$

Consider now the VN to CN messages. Define the random vector $\boldsymbol{F}^{(\ell)} = \left(F_{(0,\mathtt{vH})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{vH})}^{(\ell)}, F_{(0,\mathtt{H})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{H})}^{(\ell)}, F_{(0,\mathtt{L})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{L})}^{(\ell)}, F_{(0,\mathtt{vL})}^{(\ell)}, \ldots, F_{(\alpha^{q-2},\mathtt{vL})}^{(\ell)}\right)$

where $F^{(\ell)}_{(u,r)}$, for $u \in \mathbb{F}_q$ and $r \in \{\texttt{vH}, \texttt{H}, \texttt{L}, \texttt{vL}\}$, denotes the RV associated to the number of incoming CN messages to a degree d VN that are equal to (u, r) at the ℓ-th iteration. Let $\boldsymbol{f}^{(\ell)}$ be the realization of $\boldsymbol{F}^{(\ell)}$. The entries of $\boldsymbol{L}\left((m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}, r^{(\ell)}_{\mathsf{c}'\to\mathsf{v}})\right)$ in (4.138) are given by

$$L_u\left((m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}, r^{(\ell)}_{\mathsf{c}'\to\mathsf{v}})\right) = \ln\left(P((m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}, r^{(\ell)}_{\mathsf{c}'\to\mathsf{v}})|u)\right) \tag{4.168}$$

where $m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} \in \mathbb{F}_q, r^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} \in \{\texttt{vH}, \texttt{H}, \texttt{L}, \texttt{vL}\}, u \in \mathbb{F}_q$ and $P(z|u)$ can be computed from (4.159)-(4.166) and (4.167) $\forall z \in \mathbb{F}_q \times \{\texttt{vH}, \texttt{H}, \texttt{L}, \texttt{vL}\}$. Hence, the elements $L^{(\ell)}_{\mathsf{ex},u}$ of the aggregated extrinsic $\boldsymbol{L}$-vector in (4.138) are related to $f^{(\ell)}_u$ and the channel observation y by

$$\begin{aligned} L^{(\ell)}_{\mathsf{ex},u} =& \mathsf{D}^{(\ell)}_{\texttt{vH}} f^{(\ell)}_{(u,\texttt{vH})} + \mathsf{D}^{(\ell)}_{\texttt{H}} f^{(\ell)}_{(u,\texttt{H})} + \mathsf{D}^{(\ell)}_{\texttt{L}} f^{(\ell)}_{(u,\texttt{L})} + \mathsf{D}^{(\ell)}_{\texttt{vL}} f^{(\ell)}_{(u,\texttt{vL})} \\ &+ \mathsf{D}_{\text{ch}}\delta_{uy} + K_2 \quad \forall u \in \mathbb{F}_q \end{aligned} \tag{4.169}$$

where D_{ch} is given in (4.70) and we have

$$\mathsf{D}^{(\ell)}_{\texttt{vH}} = \ln(s^{(\ell)}_{\mathcal{I}_0}) - \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\right) \tag{4.170}$$

$$\mathsf{D}^{(\ell)}_{\texttt{H}} = \ln(s^{(\ell)}_{\mathcal{I}_2}) - \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_3}}{q-1}\right) \tag{4.171}$$

$$\mathsf{D}^{(\ell)}_{\texttt{L}} = \ln(s^{(\ell)}_{\mathcal{I}_4}) - \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_5}}{q-1}\right) \tag{4.172}$$

$$\mathsf{D}^{(\ell)}_{\texttt{vL}} = \ln(s^{(\ell)}_{\mathcal{I}_6}) - \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_7}}{q-1}\right) \tag{4.173}$$

$$\begin{aligned} K_2 =& \ln\left(\frac{\epsilon}{q-1}\right) + \sum_{a\in\mathbb{F}_q} f^{(\ell)}_{(a,\texttt{vH})} \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\right) + \sum_{a\in\mathbb{F}_q} f^{(\ell)}_{(a,\texttt{H})} \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_3}}{q-1}\right) \\ &+ \sum_{a\in\mathbb{F}_q} f^{(\ell)}_{(a,\texttt{L})} \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_5}}{q-1}\right) + \sum_{a\in\mathbb{F}_q} f^{(\ell)}_{(a,\texttt{vL})} \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_7}}{q-1}\right). \end{aligned} \tag{4.174}$$

Note that K_2 in (4.174) can be ignored in the VN update rule since it is independent of the symbol u. We obtain

$$\begin{aligned} p^{(\ell)}_{\mathcal{I}_0} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L^{(\ell)}_{\mathsf{ex},0} > L^{(\ell)}_{\mathsf{ex},u} + \Delta_3) \end{aligned} \tag{4.175}$$

$$\begin{aligned} p_{\mathcal{I}_1}^{(\ell)} =& \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \times \\ & \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta_3) \end{aligned} \tag{4.176}$$

$$\begin{aligned} p_{\mathcal{I}_2}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \mathbb{I}(\mathcal{A}_0\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta_2) \end{aligned} \tag{4.177}$$

$$\begin{aligned} p_{\mathcal{I}_3}^{(\ell)} =& \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \mathbb{I}(\mathcal{A}_a\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta_2) \end{aligned} \tag{4.178}$$

$$\begin{aligned} p_{\mathcal{I}_4}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \mathbb{I}(\mathcal{B}_0\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta_1) \end{aligned} \tag{4.179}$$

$$\begin{aligned} p_{\mathcal{I}_5}^{(\ell)} =& \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \mathbb{I}(\mathcal{B}_a\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}+\Delta_1) \end{aligned} \tag{4.180}$$

$$\begin{aligned} p_{\mathcal{I}_6}^{(\ell)} =& \sum_d \lambda_d \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \left[\mathbb{I}(\mathcal{D}_0\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{0\}} \mathbb{I}(L_{\mathsf{ex},0}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}) + \frac{\mathbb{I}(0\in\mathcal{U})}{|\mathcal{U}|}\right] \end{aligned} \tag{4.181}$$

$$\begin{aligned} p_{\mathcal{I}_7}^{(\ell)} =& \sum_d \lambda_d \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y\in\mathbb{F}_q} \Pr\{Y=y|X=0\} \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\} \times \\ & \left[\mathbb{I}(\mathcal{D}_a\neq\emptyset) \prod_{u\in\mathbb{F}_q\setminus\{a\}} \mathbb{I}(L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)}) + \frac{\mathbb{I}(a\in\mathcal{U})}{|\mathcal{U}|}\right] \end{aligned} \tag{4.182}$$

where the inner sum is over all length $4q$ integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries are non-negative and sum to $d-1$. For all $u \in \mathbb{F}_q$, we have

$$\mathcal{A}_u =\{e\in\mathbb{F}_q : L_{\mathsf{ex},u}^{(\ell)}-\Delta_3 \le L_{\mathsf{ex},e}^{(\ell)} < L_{\mathsf{ex},u}^{(\ell)}-\Delta_2\} \tag{4.183}$$

$$\mathcal{B}_u =\{e\in\mathbb{F}_q : L_{\mathsf{ex},u}^{(\ell)}-\Delta_2 \le L_{\mathsf{ex},e}^{(\ell)} < L_{\mathsf{ex},u}^{(\ell)}-\Delta_1\} \tag{4.184}$$

$$\mathcal{D}_u =\{e\in\mathbb{F}_q : L_{\mathsf{ex},u}^{(\ell)}-\Delta_1 \le L_{\mathsf{ex},e}^{(\ell)} < L_{\mathsf{ex},u}^{(\ell)}\} \tag{4.185}$$

$$\mathcal{U} =\{e\in\mathbb{F}_q : L_{\mathsf{ex},e}^{(\ell)} = \max_{u\in\mathbb{F}_q} L_{\mathsf{ex},u}^{(\ell)}\} \tag{4.186}$$

Table 4.6: Decoding thresholds $\epsilon^\star$ of the $(3,6)$ regular LDPC code ensembles

q	SMP [91]	[27] $\Gamma=1$	[27] $\Gamma=2$	SRLMP $\Gamma=1$	SRLMP $\Gamma=2$	1-bit RSMP	2-bit RSMP	$\epsilon^\star_{\mathsf{BP}}$	ϵ_{Sh}
2	0.0395	0.039	-	0.0707	-	0.0741	0.0801	0.084	0.110
4	0.0890	0.072	0.111	0.0946	0.1203	0.1102	0.1159	0.149	0.189
8	0.1039	0.073	0.137	0.1086	0.1411	0.1390	0.1429	0.196	0.247
16	0.1075	0.075	0.148	0.122	0.1517	0.1676	0.1677	0.231	0.2897
32	0.1092	-	-	0.1387	0.1560	0.1814	0.1814	0.26	0.3217
64	0.1101	-	-	0.1576	0.1585	0.1915	0.1915	0.279	0.3462

$$\Pr\left\{\boldsymbol{F}^{(\ell)}=\boldsymbol{f}^{(\ell)}|X=0\right\}=\binom{d-1}{f^{(\ell)}_{(0,\mathtt{vH})},\ldots,f^{(\ell)}_{(\alpha^{q-2},\mathtt{vL})}}\prod_{k=0}^{7}\left(\frac{s^{(\ell)}_{\mathcal{I}_k}}{|\mathcal{I}_k|}\right)^{f^{(\ell)}_{\mathcal{I}_k}} \tag{4.187}$$

$$f^{(\ell)}_{\mathcal{I}_k}=\sum_{(a,r)\in\mathcal{I}_k}f^{(\ell)}_{(a,r)}\quad\forall k\in\{0,\ldots,7\}. \tag{4.188}$$

Tables 4.6 compares the iterative decoding thresholds $\epsilon^\star$ of 1- and 2-bit RSMP, SMP, SRLMP (for maximum list size $\Gamma=1$ and $\Gamma=2$) and BP decoding $\epsilon^\star_{\mathsf{BP}}$ for $(3,6)$ regular ensemble and several q values. The tables also give the Shannon limit ϵ_{Sh} and the thresholds of the list message passing algorithm in [27] for maximum list size $\Gamma=1$ and $\Gamma=2$. Observe that 2-bit RSMP outperforms the 1-bit RSMP. This gain is due to using one bit more for the reliability scores. We see that for some values of q, the 1- and 2-bit RSMP algorithms outperform both SRLMP and the algorithm in [27] for maximum list size 2 and with reduced complexity and data flow.

4.2.3 Q-ary Erasure Channel

We extend the SMP and SEMP (SRLMP with $\Gamma=1$) to a QEC with erasure probability ϵ. In this case, the entries of the channel $\boldsymbol{L}$-vector are

$$L_a(y)=\begin{cases}\ln(1-\epsilon) & a=y\\ \ln(\epsilon) & y=\mathsf{E}\\ -\infty & \text{otherwise.}\end{cases} \tag{4.189}$$

SMP

Initially, each VN sends the symbol that maximizes $\boldsymbol{L}(y)$, i.e.,

$$m^{(0)}_{\mathsf{v}\to\mathsf{c}} = \operatorname*{argmax}_{a\in\mathbb{F}_q} L_a(y). \tag{4.190}$$

If $y \in \mathbb{F}_q$, we have $m^{(0)}_{\mathsf{v}\to\mathsf{c}} = y$ and if y is erased, then $m^{(0)}_{\mathsf{v}\to\mathsf{c}}$ is an $\mathbb{F}_q$ symbol chosen uniformly at random.

The CN update is the same as (4.29).

At the ℓ-th iteration, each VN computes (4.30). The outgoing VN message is the $\mathbb{F}_q$ symbol with the maximum entry in $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$. We obtain

$$m^{(\ell)}_{\mathsf{v}\to\mathsf{c}} = \begin{cases} y & \text{if } y \in \mathbb{F}_q \\ \operatorname*{argmax}_{u\in\mathbb{F}_q} |\mathsf{c}' \in \mathcal{N}(\mathtt{v}) \setminus \mathsf{c} : m^{(\ell)}_{\mathsf{c}'\to\mathtt{v}} = u| & \text{if } y = \mathsf{E}. \end{cases} \tag{4.191}$$

Whenever multiple maximizing arguments exist, the arg max function outputs one of them uniformly at random.

The transition probabilities of the extrinsic channel are estimated via the DE analysis. They are then used to compute the $\boldsymbol{L}$-vectors of the CN messages in (4.26) and (4.27). To estimate its codeword symbol each VN computes

$$\begin{aligned} \boldsymbol{L}^{(\ell)}_{\mathsf{app}} &= \left[L^{(\ell)}_{\mathsf{app},0}, L^{(\ell)}_{\mathsf{app},1}, \ldots, L^{(\ell)}_{\mathsf{app},\alpha^{q-2}}\right] \\ &= \boldsymbol{L}\left(y\right) + \sum_{\mathsf{c}'\in\mathcal{N}(\mathtt{v})} \boldsymbol{L}\left(m^{(\ell)}_{\mathsf{c}'\to\mathtt{v}}\right). \end{aligned} \tag{4.192}$$

For the QEC, the final decision is

$$\hat{x}^{(\ell)} = \begin{cases} y & \text{if } y \in \mathbb{F}_q \\ \operatorname*{argmax}_{u\in\mathbb{F}_q} |\mathsf{c}' \in \mathcal{N}(\mathtt{v}) : m^{(\ell)}_{\mathsf{c}'\to\mathtt{v}} = u| & y = \mathsf{E}. \end{cases} \tag{4.193}$$

Density Evolution SMP over QEC

We present now a DE for SMP for non-binary LDPC codes over a QEC with erasure probability ϵ. We partition the message alphabet $\mathcal{M}_{\text{SMP}} = \mathbb{F}_q$ into 2 disjoint sets $\mathcal{I}_0 = \{0\}$ and $\mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$. Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration and $s^{(\ell)}_{\mathcal{I}_k}$ the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0, 1\}$. The ensemble iterative decoding threshold $\epsilon^\star$ is defined as the maximum channel erasure probability ϵ for which $p^{(\ell)}_{\mathcal{I}_0} \to 1$ as $\ell \to \infty$. In the limit of $n \to \infty$, the DE analysis can be summarized in the following steps.

1. **Initialization.**

$$p^{(0)}_{\mathcal{I}_0} = 1 - \frac{q-1}{q}\epsilon \tag{4.194}$$

$$p^{(0)}_{\mathcal{I}_1} = \frac{q-1}{q}\epsilon. \tag{4.195}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. We have (4.36) and (4.37).

Variable to check update. The extrinsic channel has input alphabet $\mathcal{X} = \mathbb{F}_q$, output alphabet $\mathcal{Z} = \mathbb{F}_q$ and transition probabilities in (4.38).

Consider now the VN to CN messages. We use the random vector $\boldsymbol{F}^{(\ell)} = \left(F^{(\ell)}_0, \ldots, F^{(\ell)}_{\alpha^{q-2}}\right)$ where $F^{(\ell)}_u$, for $u \in \mathbb{F}_q$ denotes the RV associated to the number of incoming CN messages to a degree d VN that are equal to u at the ℓ-th iteration. Let $\boldsymbol{f}^{(\ell)}$ be the realization of $\boldsymbol{F}^{(\ell)}$. We have

$$p^{(\ell)}_{\mathcal{I}_0} = 1 - \epsilon + \epsilon \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | X = 0\right\} \frac{\mathbb{I}\left(0 \in \mathcal{F}^{(\ell)}\right)}{|\mathcal{F}^{(\ell)}|} \tag{4.196}$$

$$p^{(\ell)}_{\mathcal{I}_1} = 1 - p^{(\ell)}_{\mathcal{I}_0} \tag{4.197}$$

where the inner sum is over all length q integer vectors $\boldsymbol{f}^{(\ell)}$ whose entries are non-negative and sum to $d-1$ and

$$\mathcal{F}^{(\ell)} = \left\{ u \in \mathbb{F}_q \middle| f^{(\ell)}_u = \max_{a \in \mathbb{F}_q} f^{(\ell)}_a \right\} \tag{4.198}$$

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | X = 0\right\} = \binom{d-1}{f^{(\ell)}_0, \ldots, f^{(\ell)}_{\alpha^{q-2}}} (s^{(\ell)}_{\mathcal{I}_0})^{f^{(\ell)}_0} \left(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\right)^{d-1-f^{(\ell)}_0}. \tag{4.199}$$

SEMP

At the beginning, the message from VN v to a neighboring CN c is

$$m^{(0)}_{\mathsf{v}\to\mathsf{c}} = \begin{cases} y & \text{if } y \in \mathbb{F}_q \\ \mathsf{E} & \text{if } y = \mathsf{E}. \end{cases} \tag{4.200}$$

At the ℓ-th iteration, CN $\mathtt{c}$ sends to a neighboring VN $\mathtt{v}$ the message

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = \begin{cases} -h^{-1}_{\mathtt{v},\mathtt{c}} \sum\limits_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} & \text{if } m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} \neq \mathsf{E}\ \forall \mathtt{v}' \in \mathcal{N}(\mathtt{c})\setminus\mathtt{v} \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.201}$$

For the QEC, the outgoing VN message is

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \begin{cases} \mathsf{E} & y = \mathsf{E}\ \&\ \forall \mathtt{c}' \in \mathcal{N}(\mathtt{v})\setminus\mathtt{c} : m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}} = \mathsf{E} \\ y & \text{otherwise.} \end{cases} \tag{4.202}$$

The final decision is

$$\hat{m}^{(\ell)}_{\mathtt{v}} = \begin{cases} \mathsf{E} & y = \mathsf{E}\ \&\ \forall \mathtt{c}' \in \mathcal{N}(\mathtt{v}) : m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}} = \mathsf{E} \\ y & \text{otherwise.} \end{cases} \tag{4.203}$$

Density Evolution SEMP over QEC

We present now a DE for SEMP over the QEC. Due to symmetry and under the all-zero codeword assumption, we can partition the message alphabet $\mathcal{M}_1$ into 3 disjoint sets $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ such that the messages in the same set have the same probability. We have (4.57)-(4.59). Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration, i.e., a VN to CN message takes the value $a \in \mathcal{I}_k$ with probability $p^{(\ell)}_{\mathcal{I}_k}/|\mathcal{I}_k|$. Similarly $s^{(\ell)}_{\mathcal{I}_k}$ is the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1,2\}$. The ensemble iterative decoding threshold $\epsilon^\star$ is defined as the maximum ϵ for which $p^{(\ell)}_{\mathcal{I}_0} \to 1$ as $\ell \to \infty$. In the limit of $n \to \infty$, the DE analysis can be summarized in the following steps.

1. **Initialization.** Initially, we have

$$p^{(0)}_{\mathcal{I}_0} = 1 - \epsilon \tag{4.204}$$
$$p^{(0)}_{\mathcal{I}_1} = 0 \tag{4.205}$$
$$p^{(0)}_{\mathcal{I}_2} = \epsilon. \tag{4.206}$$

2. **For** $\ell = 1, 2, \dots, \ell_{\max}$

Check to variable update. We have (4.63)-(4.65).

Variable to check update. Consider now the VN to CN messages. We obtain

$$p_{\mathcal{I}_0}^{(\ell)} = 1 - \epsilon + \epsilon \left(1 - \lambda \left(s_{\mathcal{I}_2}^{(\ell)}\right)\right) \tag{4.207}$$

$$p_{\mathcal{I}_2}^{(\ell)} = \epsilon \lambda \left(s_{\mathcal{I}_2}^{(\ell)}\right) \tag{4.208}$$

$$p_{\mathcal{I}_1}^{(\ell)} = 0. \tag{4.209}$$

4.2.4 AWGN Channel with PPM

We extend the SMP to AWGN channels with orthogonal modulations where the field order q and the modulation order are equal. This makes non-binary LDPC codes a natural choice since each q-ary modulation symbol is in one-to-one correspondence with a q-ary code symbol. We aim to show that non-binary LDPC codes with low-complexity decoding algorithms are favorable for certain coded-modulation scenarios.

Let the channel message be represented by a length-q $\boldsymbol{L}$-vector

$$\boldsymbol{L}(\boldsymbol{y}) = \left(L_0(\boldsymbol{y}), L_1(\boldsymbol{y}), \ldots, L_{\alpha^{q-2}}(\boldsymbol{y})\right) \tag{4.210}$$

with $L_a(\boldsymbol{y}) = \ln\left(p(\boldsymbol{y}|a)\right)$. Decoding proceeds as follows. Initially, Each VN computes the channel $\boldsymbol{L}$-vector defined in (4.210) and sends the symbol with the highest L-value. Since

$$L_a(\boldsymbol{y}) = \frac{y_a}{\sigma^2} - \frac{q}{2}\ln(2\pi\sigma^2) - \frac{\|\boldsymbol{y}\|^2 + 1}{2\sigma^2} \quad \forall a \in \mathbb{F}_q \tag{4.211}$$

finding the maximum of the length-q vector $\boldsymbol{L}(\boldsymbol{y})$ is equivalent to finding the maximum of $\boldsymbol{y}$. Hence, we have

$$m_{\mathsf{v}\to\mathsf{c}}^{(0)} = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, L_a(\boldsymbol{y}) = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, y_a. \tag{4.212}$$

The complexity of the initialization step scales as $\mathcal{O}(nq)$. The performed operation is finding a maximum.

The CN update is the same as (4.29). From [97] the CN operation can be implemented with $2\mathsf{d}_\mathsf{c} - 1$ q-ary additions and $2\mathsf{d}_\mathsf{c}$ q-ary multiplications. If q-ary additions/multiplications are implemented using elementary operations the complexity may depend on q. For instance, the sum of two q-ary symbols can be performed by $\log_2 q$ binary XOR operations. Thus, the complexity scales as $\mathcal{O}(m\mathsf{d}_\mathsf{c} g(q))$, where $g(\cdot)$ is an implementation dependent cost.

At the ℓ-th iteration, each VN computes

$$\begin{aligned}\boldsymbol{L}_{\mathsf{ex}}^{(\ell)} &= \left[L_{\mathsf{ex},0}^{(\ell)}, L_{\mathsf{ex},1}^{(\ell)}, \ldots, L_{\mathsf{ex},\alpha^{q-2}}^{(\ell)}\right] \\ &= \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathtt{v}) \setminus \mathsf{c}} \boldsymbol{L}\left(m_{\mathsf{c}' \to \mathtt{v}}^{(\ell)}\right).\end{aligned} \tag{4.213}$$

The outgoing VN message is the $\mathbb{F}_q$ symbol with the maximum entry in $\boldsymbol{L}_{\mathsf{ex}}^{(\ell)}$, i.e.,

$$m_{\mathtt{v}\to\mathsf{c}}^{(\ell)} = \underset{u \in \mathbb{F}_q}{\operatorname{argmax}}\, L_{\mathsf{ex},u}^{(\ell)}. \tag{4.214}$$

The final decision is

$$\hat{x}^{(\ell)} = \underset{u \in \mathbb{F}_q}{\operatorname{argmax}}\, L_{\mathsf{app},u}^{(\ell)} \tag{4.215}$$

where

$$\begin{aligned}\boldsymbol{L}_{\mathsf{app}}^{(\ell)} &= \left[L_{\mathsf{app},0}^{(\ell)}, L_{\mathsf{app},1}^{(\ell)}, \ldots, L_{\mathsf{app},\alpha^{q-2}}^{(\ell)}\right] \\ &= \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathtt{v})} \boldsymbol{L}\left(m_{\mathsf{c}' \to \mathtt{v}}^{(\ell)}\right).\end{aligned} \tag{4.216}$$

The following Lemma will be useful for the complexity analysis of the decoder.

Lemma 4.1. Suppose the $w^{(i)}$, $i = 1, 2, \ldots, d$, are observations of a QSC with $1 - \epsilon > \frac{\epsilon}{q-1}$. When summing d $\boldsymbol{L}$-vectors $\boldsymbol{L}\left(w^{(i)}\right)$, the elements of the sum with indices $\mathcal{I} = \bigcup_{i=1}^{d} w^{(i)}$, where $|\mathcal{I}| \leq \min(d, q)$, all have values greater than $d \ln \frac{\epsilon}{q-1}$ and thus contain the maximum value of the sum.

Proof. Observe that each $\boldsymbol{L}\left(w^{(i)}\right)$ has a single maximum with index $a = w^{(i)}$ and that all other entries are $\ln(\epsilon/(q-1))$. ■

The complexity of the VN operation scales as $\mathcal{O}(n\mathsf{d}_\mathsf{v})$: as shown in [97] all d_v extrinsic messages can be computed efficiently from the sum $\boldsymbol{L}_{\text{tot}}^{(\ell)}$ of *all* incoming messages $\sum_{\mathsf{c}' \in \mathcal{N}(\mathtt{v})} \boldsymbol{L}\left(m_{\mathsf{c}' \to \mathtt{v}}^{(\ell)}\right)$ and $\boldsymbol{L}(\boldsymbol{y})$. Let the entry with index a be the maximum of $\boldsymbol{L}(\boldsymbol{y})$ computed in (4.212).[1] By Lemma 4.1 the largest values of $\boldsymbol{L}_{\text{tot}}^{(\ell)}$ will be in $\mathcal{I} \cup a$, where $|\mathcal{I} \cup a| \leq \min(\mathsf{d}_\mathsf{v} + 1, q) \leq \mathsf{d}_\mathsf{v} + 1$. This step requires $\mathcal{O}(\mathsf{d}_\mathsf{v})$ additions of floating-point numbers. The identification of the (two) largest values of $\boldsymbol{L}_{\text{tot}}^{(\ell)}$ requires d_v steps. Then, the extrinsic messages and their maximum can be obtained from $\boldsymbol{L}_{\text{tot}}^{(\ell)}$ with d_v additional operations (subtractions, comparisons) and we have an overall complexity scaling of $\mathcal{O}(n\mathsf{d}_\mathsf{v})$.

[1] For an unquantized AWGN channel $\boldsymbol{L}(\boldsymbol{y})$ has a unique maximum with probability one.

Density Evolution Analysis

We discuss DE analysis for SMP for non-binary irregular LDPC code ensembles over the AWGN channel with PPM modulation. Due to symmetry, we can assume the all-zero codeword was transmitted. We consider again the two disjoint sets $\mathcal{I}_0 = \{0\}$ and $\mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$. Due to symmetry, the messages in the same set have the same probability. Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration and $s^{(\ell)}_{\mathcal{I}_k}$ the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0, 1\}$. The ensemble iterative decoding threshold $(E_\mathsf{b}/N_0)^\star$ is defined as the minimum E_b/N_0 for which $p^{(\ell)}_{\mathcal{I}_0} \to 1$ as $\ell \to \infty$. In the limit of $n \to \infty$, the DE analysis can be summarized in the following steps.

1. **Initialization.** Define the random vector

$$\boldsymbol{Z}_a = Y_a \mathbf{1}_{q-1} - \boldsymbol{Y}_{[a]} \tag{4.217}$$

for $a \in \mathbb{F}_q$, with $\boldsymbol{Y}_{[a]}$ being the random vector $\boldsymbol{Y}$ of channel observations without the entry Y_a and $\mathbf{1}_{q-1}$ the length-$(q-1)$ all-one vector. Conditioned on the transmission of the all-zero codeword, $\boldsymbol{Y}$ is a Gaussian random vector with mean $\boldsymbol{\mu}_{\boldsymbol{Y}} = (1, 0, \ldots, 0)$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \sigma^2 \boldsymbol{I}_q$, where $\boldsymbol{I}_q$ is the size q identity matrix. Thus, $\boldsymbol{Z}_a$ is a Gaussian random vector with mean

$$\boldsymbol{\mu}_{\boldsymbol{Z}_a} = \begin{cases} \mathbf{1}_{q-1} & a = 0 \\ (-1, 0, \ldots, 0) & a \in \mathbb{F}_q \setminus \{0\} \end{cases} \tag{4.218}$$

and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Z}_a}$ with entries

$$(\boldsymbol{\Sigma}_{\boldsymbol{Z}_a})_{i,j} = \begin{cases} 2\sigma^2 & i = j \\ \sigma^2 & \text{otherwise.} \end{cases} \tag{4.219}$$

The parameters of $\boldsymbol{Z}_a$ $\forall a \in \mathbb{F}_q \setminus \{0\}$ do not depend on a and thus take the same value. Therefore, $\Pr\{\boldsymbol{Z}_a > \mathbf{0}\}$ is the same $\forall a \in \mathbb{F}_q \setminus \{0\}$. We have

$$p^{(0)}_{\mathcal{I}_0} = \Pr\{\boldsymbol{Z}_0 > \mathbf{0}\} \tag{4.220}$$

$$p^{(0)}_{\mathcal{I}_1} = 1 - p^{(0)}_{\mathcal{I}_0}. \tag{4.221}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. For the CN messages, we again have $s_{\mathcal{I}_0}^{(0)}$ and $s_{\mathcal{I}_1}^{(0)}$ in (4.36) and (4.37). The extrinsic channel is a QSC with error probability $1 - s_{\mathcal{I}_0}^{(0)}$.

Variable to check update. We again use the random vector $\boldsymbol{F}^{(\ell)} = \left(F_0^{(\ell)}, \ldots, F_{\alpha^{q-2}}^{(\ell)}\right)$ where $F_u^{(\ell)}$, for $u \in \mathbb{F}_q$ denotes the RV associated to the number of incoming CN messages to a degree d VN that are equal to u at the ℓ-th iteration. Let $\boldsymbol{f}^{(\ell)}$ be the realization of $\boldsymbol{F}^{(\ell)}$. The entries of $\boldsymbol{L}\left(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right)$ in (4.213) are given by

$$L_{\mathsf{ex},u}^{(\ell)} = \mathsf{D}^{(\ell)} f_u^{(\ell)} + \frac{y_a}{\sigma^2} + K \quad \forall u \in \mathbb{F}_q \tag{4.222}$$

where δ_{ij} is the Kronecker delta function and

$$\mathsf{D}^{(\ell)} = \ln(s_{\mathcal{I}_0}^{(\ell)}) - \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) \tag{4.223}$$

$$K = -\frac{q}{2}\ln(2\pi\sigma^2) - \frac{\|\boldsymbol{y}\|^2 + 1}{2\sigma^2} + (d-1)\ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right). \tag{4.224}$$

Note that K in (4.224) can be ignored in the VN update rule since it is independent of the symbol u. We obtain

$$\begin{aligned} p_{\mathcal{I}_0}^{(\ell)} &= \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} \Pr\left\{\underset{u\in\mathbb{F}_q}{\operatorname{argmax}}\, L_{\text{ex},a}^{(\ell)} = a \middle| \boldsymbol{f}^{(\ell)}\right\} \\ &= \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} \Pr\left\{\boldsymbol{Z}_0 > \sigma^2 \mathsf{D}^{(\ell)}(\boldsymbol{f}_{[0]}^{(\ell)} - f_0^{(\ell)} \mathbf{1}_{q-1})\right\} \end{aligned} \tag{4.225}$$

where $\boldsymbol{f}_{[0]}^{(\ell)}$ is the vector $\boldsymbol{f}^{(\ell)}$ without its entry $f_0^{(\ell)}$, $\boldsymbol{Z}_0$ is defined in (4.217) and the sum is over integer vectors $\boldsymbol{f}^{(\ell)}$ for which one has $0 \le f_u^{(\ell)} \le d-1 \, \forall u \in \mathbb{F}_q$ and $\sum_{u\in\mathbb{F}_q} f_u^{(\ell)} = d-1$ and

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} = \binom{d-1}{f_0^{(\ell)}, \ldots, f_{\alpha^{q-2}}^{(\ell)}} (s_{\mathcal{I}_0}^{(\ell)})^{f_0^{(\ell)}} \left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right)^{d-1-f_0^{(\ell)}}. \tag{4.226}$$

Stability Condition

We next derive the stability condition for SMP decoder over AWGN channels with PPM. The stability analysis examines the convergence of the probability $p_{\mathcal{I}_1}^{(\ell)}$ to zero under the assumption that it is close to the fixed point $p_{\mathcal{I}_1}^{\star} = 0$. Note that $s_{\mathcal{I}_1}^{(\ell)} \to 0$ as $p_{\mathcal{I}_1}^{(\ell)} \to 0$. Thus,

$\mathsf{D}^{(\ell)} \to \infty$ and

$$p_{\mathcal{I}_1}^{(\ell)} = \sum_d \lambda_d \sum_{a\in\mathbb{F}_q} \left[\sum_{\boldsymbol{f}^{(\ell)}\in\mathcal{F}_{1,a}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} + \sum_{\boldsymbol{f}^{(\ell)}\in\mathcal{F}_{2,a}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} \Pr\left\{\operatorname*{argmax}_{e\in\mathcal{S}} Y_e = a\right\}\right] \tag{4.227}$$

where $\mathcal{F}_{1,a}$ is the set of all integer vectors $\boldsymbol{f}^{(\ell)}$ for which one has $\sum_{u\in\mathbb{F}_q} f_u^{(\ell)} = d-1$ and $0 \le f_u^{(\ell)} < f_a^{(\ell)} \, \forall u \in \mathbb{F}_q \setminus \{a\}$. $\mathcal{F}_{2,a}$ is the set of all integer vectors $\boldsymbol{f}^{(\ell)}$ for which one has $\sum_{u\in\mathbb{F}_q} f_u^{(\ell)} = d-1$ and $0 \le f_u^{(\ell)} < f_a^{(\ell)} \, \forall u \in \mathbb{F}_q \setminus \mathcal{S}_a$ where

$$\mathcal{S}_a = \{b \in \mathbb{F}_q | f_b^{(\ell)} = f_a^{(\ell)}\} \tag{4.228}$$

and $|\mathcal{S}_a| > 1$. Recall that for any $a \in \mathbb{F}_q \setminus \{0\}$, we have

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} = \binom{d-1}{f_0^{(\ell)}, \ldots, f_{\alpha^{q-2}}^{(\ell)}} (1 - s_{\mathcal{I}_0}^{(\ell)})^{f_0^{(\ell)}} \left(\frac{s_{\mathcal{I}_0}^{(\ell)}}{q-1}\right)^{d-1-f_0^{(\ell)}}. \tag{4.229}$$

We obtain

$$\lim_{s_{\mathcal{I}_1}^{(\ell)}\to 0} \frac{\mathrm{d}p_{\mathcal{I}_1}^{(\ell)}}{\mathrm{d}s_{\mathcal{I}_1}^{(\ell)}} = \lambda_2 + 2\lambda_3 Q\left(\frac{1}{\sqrt{2\sigma^2}}\right). \tag{4.230}$$

Furthermore, we have

$$s_{\mathcal{I}_1}^{(\ell)} = \frac{q-1}{q}\left[1 - \rho\left(1 - \frac{q p_{\mathcal{I}_1}^{(\ell-1)}}{q-1}\right)\right] \tag{4.231}$$

and

$$\lim_{p_{\mathcal{I}_1}^{(\ell-1)}\to 0} \frac{\mathrm{d}s_{\mathcal{I}_1}^{(\ell)}}{p_{\mathcal{I}_1}^{(\ell-1)}} = \rho'(1). \tag{4.232}$$

The first order Taylor expansions via (4.230), (4.232) yield

$$p_{\mathcal{I}_1}^{(\ell)} = \rho'(1)\left[\lambda_2 + 2\lambda_3 Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right] p_{\mathcal{I}_1}^{(\ell-1)}. \tag{4.233}$$

Table 4.7: Thresholds $(E_{\mathsf{b}}/N_0)^{\star}$ of $R = 1/2$ LDPC code ensembles under SMP. Shannon limit $(E_{\mathsf{b}}/N_0)_{\mathsf{Sh}}$ as a reference. Ensembles with constraints on low-degree nodes are marked with $\bullet$.

q	$\lambda(x)$	$\rho(x)$		$(E_{\mathsf{b}}/N_0)^{\star}$ [dB]	$(E_{\mathsf{b}}/N_0)_{\mathsf{Sh}}$ [dB]
4	$0.0187x + 0.5597x^2 + 0.003x^3 + 0.4186x^{11}$	$0.3358x^7 + 0.6642x^8$		3.22	1.06
	$0.3699x^2 + 0.2799x^3 + 0.3502x^{11}$	$0.0084x^7 + 0.9916x^8$	$\bullet$	3.40	
8	$0.0465x + 0.5735x^2 + 0.38x^{11}$	$0.859x^7 + 0.141x^8$		2.19	0.08
	$0.4407x^2 + 0.2652x^3 + 0.2941x^{11}$	$0.5576x^7 + 0.4424x^8$	$\bullet$	2.39	
16	$0.0595x + 0.5868x^2 + 0.3537x^{11}$	$0.1351x^6 + 0.8649x^7$		1.63	-0.47
	$0.5199x^2 + 0.1436x^3 + 0.3365x^{11}$	$0.5407x^7 + 0.4593x^8$	$\bullet$	1.82	
32	$0.0582x + 0.6141x^2 + 0.3277x^{11}$	$0.311x^6 + 0.689x^7$		1.32	-0.80
	$0.55x^2 + 0.1424x^3 + 0.3076x^{11}$	$0.8045x^7 + 0.1955x^8$	$\bullet$	1.48	

The stability condition is fulfilled if and only if

$$\rho'(1)\left[\lambda_2 + 2\lambda_3 Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right] < 1. \tag{4.234}$$

Remark 4.3. The fraction of edges connected to degree 2 *and* 3 VNs impacts the stability condition for SMP decoding. Thus, certain degree distributions optimized for unquantized BP (see, e.g., [98]) might be unsuitable for SMP due to their large number of degree 2 and 3 VNs.

Iterative Decoding Thresholds

The DE analysis suggests an optimization algorithm to find rate $R = 1/2$ irregular LDPC ensembles with 'good' thresholds for $q \in \{8, 16, 32\}$. We restrict the maximum VN degree to 12 and perform two optimizations: one without further constraints and one with constraints on the degree two and three VNs. Threshold results are depicted in Table 4.7 and show a gap of at least 2.1 dB with respect to the Shannon limit for various q. Thresholds of q-ary LDPC codes under full BP decoding in [98] show gaps of only 0.2 dB, i.e., the simple SMP decoder yields a loss of around 1.9 dB. Interestingly, for binary LDPC codes with orthogonal modulations and bit-interleaved coded modulation (BICM) (no iterative detection) the gap to coded modulation capacity is comparable or even larger. For instance, for $q = 16$ the gap is 1.8 dB [98, Fig. 1].

Monte Carlo Simulations

We designed three codes with $q \in \{8, 16, 32\}$, $n = 10^4$ (in $\mathbb{F}_q$ symbols), and $R = 1/2$ based on the constraint degree distribution pairs from Table 4.7. Figure 4.7 shows the FER versus E_{b}/N_0 of q-ary PPM allowing a maximum of 50 decoding iterations. Observe that the waterfall performance is predicted well by the DE analysis. In addition, we provide the performance of three $R = 1/2$ binary accumulate-repeat-4-jagged-accumulate (AR4JA)

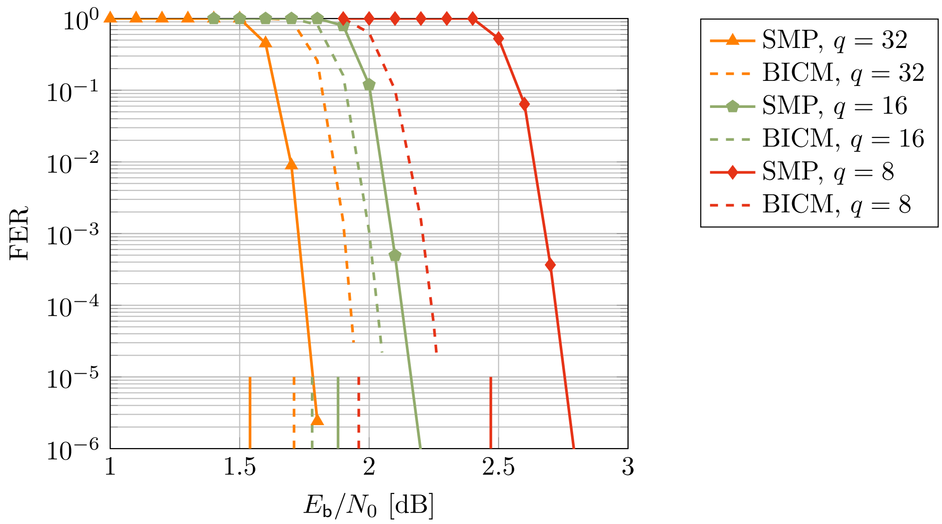


Figure 4.7: FER versus E_b/N_0 of rate 1/2 codes under SMP and BP with BICM. Respective thresholds are indicated by vertical lines.

LDPC codes assuming a BICM setting and q-ary PPM for $q \in \{8, 16, 32\}$. The AR4JA protograph was taken from [99] and expanded to obtain block lengths (in bits) of $n \log_2 q$. For $q \geq 16$ the performance of the non-binary codes under SMP decoding is competitive and for $q = 32$ they outperform the AR4JA codes with BICM by almost 0.2 dB.

Complexity

Despite the gap to capacity in Table 4.7, SMP decoding might be a good choice when low-complexity decoding is targeted. First, a comparison of the algorithmic complexity of the SMP decoder and a binary LDPC decoder with BICM is given in Table 4.8. For the binary decoder, the initialization step requires computing symbol-wise probabilities, followed by a marginalization to obtain bit-wise LLRs. The CN operations in the binary decoder follow the approximate min* rule [100]. The VN operations consist of summing up LLRs. Table 4.8 indicates that the algorithmic complexity of SMP is competitive with binary BP, but a fair comparison is difficult due to the different types of operations. E.g., the approximate min* rule and the SMP CN operations can be implemented by look-up tables. Then, an elementary operation is a look-up with $g(q) = 1$ and complexity is reduced by a factor of $\log_2 q$ w.r.t. the binary decoder. Other implementations may change the picture. Second, an important figure for implementation is the data flow in the decoder [42]. For SMP we need $\log_2 q$ bits to represent a symbol, for binary BP typically 4 to 5 bits to represent an LLR. Since the binary Tanner graph has $\log_2 q$ times more nodes the data flow of the SMP decoder will be lower by a factor of 4 to 5 (for the same average node degrees). Overall, the algorithmic complexity/data flow of an SMP decoder is highly competitive

Table 4.8: Complexity scaling of algorithmic operations.

	SMP	BP with BICM
Init	$\mathcal{O}(nq)$ maximization	$\mathcal{O}(nq\log_2 q)$ sum of floats
CN	$\mathcal{O}(m\mathsf{d_c}g(q))$ elementary operations	$\mathcal{O}(m\mathsf{d_c}\log_2 q)$ box-plus/multiplications
VN	$\mathcal{O}(n\mathsf{d_v})$ addition of floats/maximization	$\mathcal{O}(n\mathsf{d_v}\log_2 q)$ addition of floats

w.r.t. that of a binary decoder with BICM. However, only a hardware implementation will give final insights.

4.2.5 Poisson Channel with PPM

In this section, we adapt the SMP and SEMP (SRLMP with $\Gamma = 1$) decoders to Poisson channels with orthogonal (PPM) modulations where the field size q matches the modulation order. We develop a DE analysis for the two different decoders which allows to design code ensembles with optimized iterative decoding thresholds.

We consider again the log-likelihood vector ($\boldsymbol{L}$-vector)

$$\boldsymbol{L}(\boldsymbol{y}) = [L_0(\boldsymbol{y}), L_1(\boldsymbol{y}), \ldots, L_{\alpha^{q-2}}(\boldsymbol{y})] \tag{4.235}$$

with elements (dubbed L-values)

$$L_a(\boldsymbol{y}) = \ln\left(P(\boldsymbol{y}|a)\right) \quad \forall a \in \mathbb{F}_q. \tag{4.236}$$

SMP

At the beginning, the VN v computes the channel $\boldsymbol{L}$-vector and sends the symbol which has the maximum L-value to all its neighbors. From (2.28), (4.235) and (4.236), the channel $\boldsymbol{L}$-vector is

$$\begin{aligned}\boldsymbol{L}(\boldsymbol{y}) =& [L_0(\boldsymbol{y}), L_1(\boldsymbol{y}), \ldots, L_{\alpha^{q-2}}(\boldsymbol{y})]\\ L_a(\boldsymbol{y}) =& K y_a - q n_{\mathsf{b}} - n_{\mathsf{s}} + \sum_{u\in\mathbb{F}_q}\left(y_u \ln(n_{\mathsf{b}}) - \ln(y_u!)\right) \quad \forall a \in \mathbb{F}_q\end{aligned} \tag{4.237}$$

where $K = \ln\left(1+\frac{n_\mathsf{s}}{n_\mathsf{b}}\right)$. The outgoing VN message is computed as

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = \operatorname*{argmax}_{a\in\mathbb{F}_q} L_a(\boldsymbol{y}) = \operatorname*{argmax}_{a\in\mathbb{F}_q} y_a. \tag{4.238}$$

The message from CN $\mathtt{c}$ to a neighboring VN $\mathtt{v}$ is obtained by determining the symbol that satisfies the parity-check equation given the incoming messages from all other neighbors. The outgoing CN message at the ℓ-th iteration is

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = -h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}} \tag{4.239}$$

where the multiplication and the sum are performed over $\mathbb{F}_q$, $h_{\mathtt{v},\mathtt{c}}$ is a parity-check matrix element and $h^{-1}_{\mathtt{v},\mathtt{c}}$ its inverse.

Each VN computes

$$\begin{aligned}\boldsymbol{L}^{(\ell)}_{\mathsf{ex}} =& [L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \ldots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}]\\ =& \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} \boldsymbol{L}(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}})\end{aligned} \tag{4.240}$$

where $\boldsymbol{L}(\boldsymbol{y})$ is calculated according to (4.237). Further, we model each CN to VN message as an observation of the symbol X (associated to $\mathtt{v}$) at the output of an *extrinsic* QSC whose crossover probability is obtained via DE analysis. The crossover probability is used to obtain $\boldsymbol{L}(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}})$. A VN passes the symbol that maximizes $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ to its neighboring CNs, i.e.,

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \operatorname*{argmax}_{a\in\mathbb{F}_q} L^{(\ell)}_{\mathsf{ex},a}. \tag{4.241}$$

Each VN estimates the value of the respective codeword symbol as

$$\hat{m}^{(\ell)}_{\mathtt{v}} = \operatorname*{argmax}_{a\in\mathbb{F}_q} L^{(\ell)}_{\mathsf{app},a} \tag{4.242}$$

$$\begin{aligned}\boldsymbol{L}^{(\ell)}_{\mathsf{app}} =& [L^{(\ell)}_{\mathsf{app},0}, L^{(\ell)}_{\mathsf{app},1}, \ldots, L^{(\ell)}_{\mathsf{app},\alpha^{q-2}}]\\ =& \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})} \boldsymbol{L}(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}).\end{aligned} \tag{4.243}$$

We remark that in (4.238), (4.241) and (4.242), whenever multiple maximizing arguments exist, we choose one of them uniformly at random.

Density Evolution Analysis

We provide a DE analysis for the SMP decoder over Poisson channels with orthogonal modulations. In particular, we are interested in the iterative decoding threshold of non-binary irregular LDPC code ensembles. For the analysis, we use the all-zero codeword assumption since both the channel and decoder fulfill the symmetry conditions [101], [102]

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\mathsf{P}_a) = P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}^{+a}|\mathsf{P}_0) \tag{4.244}$$

$$\boldsymbol{y}^{+a} = (y_a, y_{a+1}, \ldots, y_{\alpha^{q-2}+a}) \tag{4.245}$$

$$P(m|a) = P(m+a|0) \tag{4.246}$$

where the sum is over $\mathbb{F}_q$.

We partition the message alphabet $\mathcal{M}_{\text{SMP}}$ into 2 disjoint sets $\mathcal{I}_0 = \{0\}, \mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$ where $|\mathcal{I}_0| = 1$, $|\mathcal{I}_1| = q-1$. Due to symmetry, the messages in the same set have the same probability. Let $p_{\mathcal{I}_k}^{(\ell)}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration and $s_{\mathcal{I}_k}^{(\ell)}$ the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1\}$. The ensemble iterative decoding threshold $\gamma^\star$ is defined as the minimum γ for which $p_{\mathcal{I}_0}^{(\ell)} \to 1$ as $\ell \to \infty$. DE proceeds as follows.

1. **Initialization.** Under the all-zero codeword assumption, the elements of $\boldsymbol{Y}$ are Poisson distributed with expectation

$$\mathbb{E}[Y_u] = \begin{cases} n_{\mathsf{s}} + n_{\mathsf{b}} & u = 0 \\ n_{\mathsf{b}} & \text{otherwise.} \end{cases} \tag{4.247}$$

   We have

$$p_{\mathcal{I}_0}^{(0)} = \exp\left(-(n_{\mathsf{s}} + q n_{\mathsf{b}})\right) \sum_{y=0}^{\infty} \frac{(n_{\mathsf{s}}+n_{\mathsf{b}})^y}{y!} \sum_{t=0}^{q-1} \binom{q-1}{t} \frac{1}{t+1} \left(\frac{n_{\mathsf{b}}^y}{y!}\right)^t \left(\sum_{i=0}^{y-1} \frac{n_{\mathsf{b}}^i}{i!}\right)^{q-1-t} \tag{4.248}$$

$$p_{\mathcal{I}_1}^{(0)} = 1 - p_{\mathcal{I}_0}^{(0)}. \tag{4.249}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   Check to variable update. We have

$$s_{\mathcal{I}_0}^{(\ell)} = \frac{1}{q}\left[1 + (q-1)\rho\left(\frac{q \cdot p_{\mathcal{I}_0}^{(\ell-1)} - 1}{q-1}\right)\right] \tag{4.250}$$

$$s_{\mathcal{I}_1}^{(\ell)} = 1 - s_{\mathcal{I}_0}^{(\ell)}. \tag{4.251}$$

   The extrinsic channel is a QSC with error probability $s_{\mathcal{I}_1}^{(\ell)}$.

   Variable to check update. We use the random vector $\boldsymbol{F}^{(\ell)} = (F_0^{(\ell)}, \ldots, F_{\alpha^{q-2}}^{(\ell)})$, where $F_a^{(\ell)}$ denotes the RV associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_{\mathrm{SMP}}$ at the ℓ-th iteration, and $f_a\ell$ is its realization. The entries of $\boldsymbol{L}(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}})$ in (4.240) are

$$L_u\left(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}\right) = \ln\left(P(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}|u)\right) \tag{4.252}$$

$$P(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}|u) = \begin{cases} s^{(\ell)}_{\mathcal{I}_0} & \text{if } m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} = u \\ \frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1} & \text{if } m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} \neq u. \end{cases} \tag{4.253}$$

The elements of $\boldsymbol{L}^{(\ell)}_{\mathrm{ex}}$ in (4.241) are

$$L^{(\ell)}_{\mathrm{ex},a} = K y_a + \mathsf{D}^{(\ell)} f^{(\ell)}_a + w_1 \tag{4.254}$$

$$\mathsf{D}^{(\ell)} = \ln(s^{(\ell)}_{\mathcal{I}_0}) - \ln\left(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\right) \tag{4.255}$$

$$w_1 = \sum_{u\in\mathbb{F}_q} (y_u \ln(n_{\mathsf{b}}) - \ln(y_u!)) - q n_{\mathsf{b}} - n_{\mathsf{s}} + (d-1)\ln\left(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\right). \tag{4.256}$$

Note that w_1 in (4.256) is independent of a. It can thus be ignored when computing $\boldsymbol{L}^{(\ell)}_{\mathrm{ex}}$. We obtain

$$\begin{aligned} p^{(\ell)}_{\mathcal{I}_0} =& \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}|\boldsymbol{X} = \mathsf{P}_0\right\} \exp(-(n_{\mathsf{s}}+n_{\mathsf{b}})) \sum_{y=0}^{\infty} \frac{(n_{\mathsf{s}}+n_{\mathsf{b}})^y}{y!} \\ & \sum_{t=0}^{q-1} \frac{1}{t+1} \sum_{\mathcal{S}_t} \left(\prod_{a\in\mathcal{S}_t} \Pr\left\{Y_a = y + \mathsf{D}^{(\ell)}\frac{f^{(\ell)}_0 - f^{(\ell)}_a}{K}|\boldsymbol{X} = \mathsf{P}_0\right\}\right) \times \\ & \left(\prod_{a\in\mathbb{F}_q\setminus\{0,\mathcal{S}_t\}} \Pr\left\{Y_a < y + \mathsf{D}^{(\ell)}\frac{f^{(\ell)}_0 - f^{(\ell)}_a}{K}|\boldsymbol{X} = \mathsf{P}_0\right\}\right) \end{aligned} \tag{4.257}$$

$$p^{(\ell)}_{\mathcal{I}_1} = 1 - p^{(\ell)}_{\mathcal{I}_0} \tag{4.258}$$

where $\mathcal{S}_t$ is a subset of $\mathbb{F}_q \setminus \{0\}$ of size t and $\forall a \in \mathbb{F}_q \setminus \{0\}$, we have

$$\Pr\{Y_a = y_a|\boldsymbol{X} = \mathsf{P}_0\} = \begin{cases} \exp(-n_{\mathsf{b}})\frac{n_{\mathsf{b}}^{y_a}}{y_a!} & y_a \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.259}$$

Further, the second sum is over integer vectors $\boldsymbol{f}^{(\ell)}$ for which we have $0 \leq f^{(\ell)}_u \leq$

$d-1\,\forall u \in \mathbb{F}_q$, $\sum_{u\in\mathbb{F}_q} f_u^{(\ell)} = d-1$, and

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} = \binom{d-1}{f_0^{(\ell)}, \ldots, f_{\alpha^{q-2}}^{(\ell)}} (s_{\mathcal{I}_0}^{(\ell)})^{f_0^{(\ell)}} \left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right)^{d-1-f_0^{(\ell)}}. \tag{4.260}$$

SEMP

For the SEMP, we introduce a real-valued parameter Δ, which is chosen to maximize the iterative decoding threshold. In this work, we keep Δ constant over all iterations (but in principle, one could allow Δ to vary over iterations). The message alphabet is $\mathcal{M}_1 = \mathbb{F}_q \cup \{\mathsf{E}\}$, where E corresponds to an erasure denoting complete uncertainty about the respective symbol value.

At the beginning, the message from VN $\mathtt{v}$ to a neighboring CN $\mathtt{c}$ is

$$m_{\mathtt{v}\to\mathtt{c}}^{(0)} = \begin{cases} a & \text{if } \exists a \in \mathbb{F}_q \text{ with } y_a > y_u + \frac{\Delta}{K} \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.261}$$

At the ℓ-th iteration, CN $\mathtt{c}$ sends to a neighboring VN $\mathtt{v}$ the message

$$m_{\mathtt{c}\to\mathtt{v}}^{(\ell)} = \begin{cases} -h_{\mathtt{v},\mathtt{c}}^{-1} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m_{\mathtt{v}'\to\mathtt{c}}^{(\ell-1)} & \text{if } m_{\mathtt{v}'\to\mathtt{c}}^{(\ell-1)} \neq \mathsf{E}\ \forall \mathtt{v}' \in \mathcal{N}(\mathtt{c}) \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.262}$$

The message from VN $\mathtt{v}$ to CN $\mathtt{c}$ is obtained by first computing $\boldsymbol{L}_{\mathsf{ex}}^{(\ell)}$ defined in (4.240). $\boldsymbol{L}(\boldsymbol{y})$ is calculated according to (4.237) and, for SEMP, the extrinsic channel is a QEEC whose error and erasure probabilities can be estimated via DE analysis. The error and erasure probabilities are used to obtain $\boldsymbol{L}(m_{\mathtt{c}'\to\mathtt{v}}^{(\ell)})$ from (2.20), (4.26) and (4.27). Second, for the outgoing message we pick

$$m_{\mathtt{v}\to\mathtt{c}}^{(\ell)} = \begin{cases} a & \text{if } \exists a \in \mathbb{F}_q \text{ with } \ L_{\mathsf{ex},a}^{(\ell)} > L_{\mathsf{ex},u}^{(\ell)} + \Delta\ \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathsf{E} & \text{otherwise.} \end{cases} \tag{4.263}$$

Each VN computes $\boldsymbol{L}_{\mathsf{app}}^{(\ell)}$ defined in (4.243) by using the error and erasure probabilities of the extrinsic QEEC. The final decision is

$$\hat{m}_{\mathtt{v}}^{(\ell)} = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, L_{\mathsf{app},a}^{(\ell)}. \tag{4.264}$$

Density Evolution Analysis

We partition the message alphabet $\mathcal{M}_1$ into 3 disjoint sets such that the messages in the same set have the same probability. We have $\mathcal{I}_0 = \{0\}, \mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$ and $\mathcal{I}_2 = \{\mathsf{E}\}$.

1. **Initialization.** We have

$$p_{\mathcal{I}_0}^{(0)} = \sum_{y=0}^{\infty} \Pr\{Y_0 = y | \boldsymbol{X} = \mathsf{P}_0\} \prod_{u\in\mathbb{F}_q\setminus\{0\}} \Pr\{Y_u < y - \Delta/K | \boldsymbol{X} = \mathsf{P}_0\} \tag{4.265}$$

$$p_{\mathcal{I}_1}^{(0)} = \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y=0}^{\infty} \Pr\{Y_a = y | \boldsymbol{X} = \mathsf{P}_0\} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \Pr\{Y_u < y - \Delta/K | \boldsymbol{X} = \mathsf{P}_0\} \tag{4.266}$$

$$p_{\mathcal{I}_2}^{(0)} = 1 - p_{\mathcal{I}_0}^{(0)} - p_{\mathcal{I}_1}^{(0)} \tag{4.267}$$

where for $y \in \mathbb{N}_0$ and $a \in \mathbb{F}_q$

$$\Pr\{Y_a = y | \boldsymbol{X} = \mathsf{P}_0\} = \begin{cases} \exp(-(n_\mathsf{s} + n_\mathsf{b})) \frac{(n_\mathsf{s}+n_\mathsf{b})^y}{y!} & a = 0 \\ \exp(-n_\mathsf{b}) \frac{n_\mathsf{b}^y}{y!} & a \in \mathbb{F}_q \setminus \{0\} \end{cases} \tag{4.268}$$

$$\Pr\{Y_a < w | \boldsymbol{X} = \mathsf{P}_0\} = \sum_{j=0}^{\lceil w \rceil - 1} \Pr\{Y_a = j | \boldsymbol{X} = \mathsf{P}_0\}. \tag{4.269}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

Check to variable update. We have

$$s_{\mathcal{I}_0}^{(\ell)} = \frac{1}{q} \left[\rho(1 - p_{\mathcal{I}_2}^{(\ell-1)}) + (q-1)\rho \left(\frac{q \cdot p_{\mathcal{I}_0}^{(\ell-1)} - 1 + p_{\mathcal{I}_2}^{(\ell-1)}}{q-1} \right) \right] \tag{4.270}$$

$$s_{\mathcal{I}_2}^{(\ell)} = 1 - \rho(1 - p_{\mathcal{I}_2}^{(\ell-1)}) \tag{4.271}$$

$$s_{\mathcal{I}_1}^{(\ell)} = 1 - s_{\mathcal{I}_0}^{(\ell)} - s_{\mathcal{I}_2}^{(\ell)}. \tag{4.272}$$

Variable to check update. We extend the random vector $\boldsymbol{F}^{(\ell)}$ to $\boldsymbol{F}^{(\ell)} = (F_0^{(\ell)}, \ldots, F_{\alpha^{q-2}}^{(\ell)}, F_\mathsf{E}^{(\ell)})$, where $F_a^{(\ell)}$ denotes the RV associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_1$ at the ℓ-th iteration, and $f_a^{(\ell)}$ is its realization. The entries of $\boldsymbol{L}(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)})$ in (4.240) are

$$L_u(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}) = \ln\left(P(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} | u)\right) \tag{4.273}$$

$$P(m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}|u) = \begin{cases} s_{\mathcal{I}_0}^{(\ell)} & \text{if } m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} = u \\ \frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1} & \text{if } m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} \in \mathbb{F}_q \setminus \{u\} \\ s_{\mathcal{I}_2}^{(\ell)} & \text{if } m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)} = \mathsf{E}. \end{cases} \tag{4.274}$$

The elements of $\boldsymbol{L}_{\text{ex}}^{(\ell)}$ in (4.241) are

$$L_{\text{ex},a}^{(\ell)} = K y_a + \mathsf{D}^{(\ell)} f_a^{(\ell)} + w_2 \tag{4.275}$$

$$\begin{aligned} w_2 = &\sum_{u\in\mathbb{F}_q} (y_u \ln(n_{\mathsf{b}}) - \ln(y_u!)) + f_{\mathsf{E}}^{(\ell)} \ln(s_{\mathcal{I}_2}^{(\ell)}) \\ &+ (d-1-f_{\mathsf{E}}^{(\ell)}) \ln\left(\frac{s_{\mathcal{I}_1}^{(\ell)}}{q-1}\right) - q n_{\mathsf{b}} - n_{\mathsf{s}}. \end{aligned} \tag{4.276}$$

where $\mathsf{D}^{(\ell)}$ is defined in (4.255). Note that w_2 in (4.276) is independent of a. It can thus be ignored when computing $\boldsymbol{L}_{\text{ex}}^{(\ell)}$. We obtain

$$\begin{aligned} p_{\mathcal{I}_0}^{(\ell)} = &\sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} \sum_{y=0}^{\infty} \Pr\left\{Y_0 = y | \boldsymbol{X} = \mathsf{P}_0\right\} \\ &\prod_{u\in\mathbb{F}_q\setminus\{0\}} \Pr\left\{Y_u < y + \frac{\mathsf{D}^{(\ell)}(f_0^{(\ell)} - f_u^{(\ell)}) - \Delta}{K} \Big| \boldsymbol{X} = \mathsf{P}_0\right\} \end{aligned} \tag{4.277}$$

$$\begin{aligned} p_{\mathcal{I}_1}^{(\ell)} = &\sum_d \lambda_d \sum_{\substack{a\in\mathbb{F}_q\setminus\{0\} \\ \boldsymbol{f}^{(\ell)}}} \Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} \sum_{y=0}^{\infty} \Pr\left\{Y_a = y | \boldsymbol{X} = \mathsf{P}_0\right\} \\ &\prod_{u\in\mathbb{F}_q\setminus\{a\}} \Pr\left\{Y_u < y + \frac{\mathsf{D}^{(\ell)}(f_a^{(\ell)} - f_u^{(\ell)}) - \Delta}{K} \Big| \boldsymbol{X} = \mathsf{P}_0\right\} \end{aligned} \tag{4.278}$$

$$p_{\mathcal{I}_2}^{(\ell)} = 1 - p_{\mathcal{I}_0}^{(\ell)} - p_{\mathcal{I}_1}^{(\ell)} \tag{4.279}$$

where the second sum is over integer vectors $\boldsymbol{f}^{(\ell)}$ for which $0 \le f_u^{(\ell)} \le d-1$ for all $u \in \mathcal{M}_1$ and $\sum_{u\in\mathcal{M}_1} f_u^{(\ell)} = d-1$ and

$$\Pr\left\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)} | \boldsymbol{X} = \mathsf{P}_0\right\} = \binom{d-1}{f_0^{(\ell)}, \ldots, f_{\mathsf{E}}^{(\ell)}} \prod_{k=0}^{2} \left(\frac{s_{\mathcal{I}_k}^{(\ell)}}{|\mathcal{I}_k|}\right)^{f_{\mathcal{I}_k}^{(\ell)}} \tag{4.280}$$

$$f_{\mathcal{I}_k}^{(\ell)} = \sum_{a\in\mathcal{I}_k} f_a^{(\ell)} \quad \forall k \in \{0, \ldots, 2\}. \tag{4.281}$$

Table 4.9: Threshold $\gamma^\star$ of $R = 1/2$ LDPC ensembles under SMP/SEMP for $n_\mathsf{b} = 0.1$. Ensembles optimized for the surrogate QEC are marked with $\bullet$. As references: Shannon limit γ_Sh and threshold $\gamma^\star_\mathsf{lit}$ of non-binary LDPC ensemble under BP from [105, Example 1].

	Decoder	q	$\lambda(x)$	$\rho(x)$	$\gamma^\star$[dB]	γ_Sh[dB]	$\gamma^\star_\mathsf{lit}$[dB]
	SMP	4	$0.2486x^2 + 0.4556x^3 + 0.2958x^{11}$	$0.9633x^8 + 0.0367x^9$	-3.48	-5.6	-5.42
	SEMP		$0.7328x^2 + 0.012x^3 + 0.2552x^{11}$	$0.5188x^6 + 0.4812x^7$	-4.14		
$\bullet$	SMP		$0.1979x^2 + 0.7769x^3 + 0.0252x^{11}$	$0.3441x^6 + 0.6559x^7$	-3.42		
$\bullet$	SEMP		$0.7555x^2 + 0.0018x^3 + 0.2427x^{11}$	$0.6301x^6 + 0.3699x^7$	-4.13		
	SMP	8	$0.3344x^2 + 0.3334x^3 + 0.3322x^{11}$	$0.0103x^7 + 0.9897x^8$	-6.2	-8.35	-8.07
	SEMP		$0.595x^2 + 0.0029x^3 + 0.4021x^{11}$	$0.3721x^7 + 0.6279x^8$	-6.53		
	SMP	16	$0.3691x^2 + 0.2812x^3 + 0.3497x^{11}$	$0.0089x^7 + 0.9911x^8$	-8.88	-11.02	-10.73
	SEMP	16	$0.60421x^2 + 0.0093x^3 + 0.3865x^{11}$	$0.4938x^7 + 0.5062x^8$	-8.99		
	SMP	32	$0.4711x^2 + 0.1276x^3 + 0.4013x^{11}$	$0.0128x^7 + 0.9779x^8 + 0.0093x^9$	-11.56	-13.59	-13.37
	SEMP	32	$0.7068x^2 + 0.0044x^3 + 0.2888x^{11}$	$0.3014x^6 + 0.6986x^7$	-11.69		

Table 4.10: Threshold $\gamma^\star$ of $R = 1/2$ LDPC code ensembles under SMP/SEMP for $n_\mathsf{b} = 0.002$. Ensembles optimized for the surrogate QEC are marked with $\bullet$. As references: Shannon limit γ_Sh and threshold $\gamma^\star_\mathsf{lit}$ of non-binary LDPC ensemble under BP from [105, Example 1].

	Decoder	q	$\lambda(x)$	$\rho(x)$	$\gamma^\star$[dB]	γ_Sh[dB]	$\gamma^\star_\mathsf{lit}$[dB]
	SMP	4	$0.2055x^2 + 0.6953x^3 + 0.0992x^{11}$	$0.0246x^6 + 0.9648x^7 + 0.0106x^8$	-4.68	-7.45	-7.23
	SEMP		$0.6871x^2 + 0.3129x^{11}$	$0.143x^6 + 0.857x^7$	-6.3		
$\bullet$	SMP		$0.1979x^2 + 0.7769x^3 + 0.0252x^{11}$	$0.3441x^6 + 0.6559x^7$	-4.68		
$\bullet$	SEMP		$0.7555x^2 + 0.0018x^3 + 0.2427x^{11}$	$0.6301x^6 + 0.3699x^7$	-6.3		
	SMP	8	$0.2152x^2 + 0.5352x^3 + 0.2496x^{11}$	$0.1481x^7 + 0.8519x^8$	-7.61	-10.38	-10.19
	SEMP		$0.7083x^2 + 0.0093x^4 + 0.2824x^{11}$	$0.3348x^6 + 0.6652x^7$	-9.1		
	SMP	16	$0.2284x^2 + 0.4866x^3 + 0.285x^{11}$	$0.969x^8 + 0.031x^9$	-10.59	-13.3	-13.1
	SEMP		$0.6285x^2 + 0.0095x^3 + 0.362x^{11}$	$0.7132x^7 + 0.2868x^8$	-11.9		
	SMP	32	$0.2456x^2 + 0.4206x^3 + 0.3338x^{11}$	$0.6673x^8 + 0.3327x^9$	-13.57	-16.24	-15.99
	SEMP		$0.5868x^2 + 0.0359x^3 + 0.3773x^{11}$	$0.4963x^7 + 0.5037x^8$	-14.6		

Surrogate Erasure Channel

For $n_\mathsf{b} = 0$, the Poisson PPM channel can be modeled as a QEC with erasure probability $\epsilon = \exp(-n_\mathsf{s})$ [103, 104]. Thus, for low n_b, we may rely on a simplified DE analysis on a surrogate QEC to find optimized ensembles under SMP and SEMP decoding for the Poisson PPM channel. The derivation of DE for SMP and SEMP on the QEC is shown in Section 4.2.3.

With the help of DE, we designed optimized rate $R = 1/2$ irregular LDPC code ensembles for $q \in \{4, 8, 16, 32\}$, $n_\mathsf{b} \in \{0.002, 0.1\}$ for both SMP and SEMP decoding. The maximum VN degree was restricted to 12 and the number of iterations to 50. The optimized degree distributions are provided in Tables 4.9 and 4.10. SEMP shows visible gains over SMP for small values of q (e.g. $> 0.6\,$dB for $q = 4$), while for $q = 32$ the iterative decoding

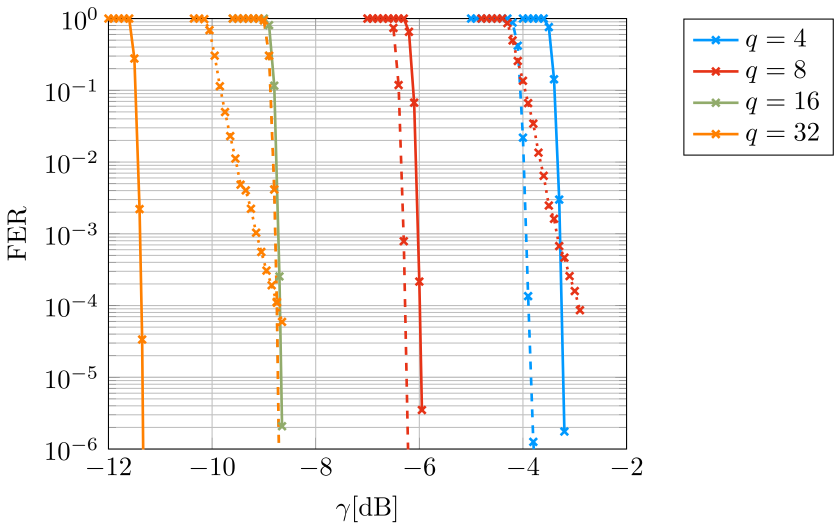


Figure 4.8: FER versus γ of rate 1/2 optimized codes via DE under SMP (solid lines) and SEMP (dashed lines) for $n_{\mathsf{b}} = 0.1$. As a reference: performance of ARJA code from [99] under SMP for $q = 8$ and $q = 32$ (dotted lines).

thresholds nearly coincide. A comparison with the Shannon limit reveals an increasing gap for increasing q, ranging from 1.2 dB for $q = 4$ to 1.6 dB for $q = 32$ in Table 4.10. As a comparison with the literature, we provide iterative decoding thresholds of non-binary LDPC code ensembles under BP decoding [105, Example 1], which show an almost constant gap of 0.3 dB to the Shannon limit. We also observe from Tables 4.9 and 4.10 that the gap to the Shannon limit increases as n_{b} increases, e.g., from 1.3 dB for $n_{\mathsf{b}} = 0.002$ to 1.8 dB for $n_{\mathsf{b}} = 0.1$ in case of $q = 8$ and SEMP. The complexity analysis of SMP was provided in [58], showing that SMP decoding might be a good choice when low-complexity decoding is targeted. Finally, DE on a surrogate QEC yields ensembles with similar thresholds as DE on the Poisson PPM channel for $n_{\mathsf{b}} \in \{0.002, 0.1\}$, confirming the validity of a surrogate QEC code design.

For completeness, simulation results with $n_{\mathsf{b}} = 0.1$ and a maximum of 50 decoding iterations are shown in Fig. 4.8 for $q \in \{4, 8, 16, 32\}$. All codes have a block length $n = 10^4$ in q-ary symbols. The obtained FERs closely follow the predicted thresholds. To illustrate the need for a tailored code design, we also simulated an off-the-shelf AR4JA code from [99] for $q \in \{8, 32\}$. The performance under SEMP is close to the one under SMP and therefore is removed from the Figure. Under SMP, the codes show a significant loss compared to an optimized design.

5

Quantized Decoding Algorithms for GLDPC Codes

GLDPC codes, introduced in [12], are a class of LDPC codes where the CNs represent more general codes than the SPC codes in standard LDPC codes. The codes associated to the CNs can be any linear block code and will be referred to as component codes. GLDPC codes offer a trade-off between error floor and waterfall performance due to their good distance and trapping set properties and the powerful block codes used at the CNs (compared to SPC codes employed by the CNs of LDPC codes). This comes at the cost of increasing decoding complexity, especially if optimum SISO decoding is performed at each CN. Several works studied reducing the decoding complexity of GLDPC and product-like codes [2, 31, 32, 106–108]. In [2], it was shown that iterative hard decision decoding using extrinsic BDD at the component codes of spatially coupled GLDPC codes, where all VNs have degree 2, can approach capacity at high rates. Exchanging binary messages between VNs and CNs is particularly attractive for high-throughput applications since it allows reducing the internal decoder data flow. The method was extended in [107, 109] such that the VNs exploit the channel reliabilities while the exchanged messages are still binary. In particular, the BMP decoding algorithms introduced in [107, 109] make use of BDD decoding at the check nodes and follow the approach in [2] to ensure the exchange of extrinsic messages.

In this chapter, we analyze the performance of GLDPC codes under BMP and TMP decoding. At the CNs, the binary and ternary messages are obtained by either using BDD or optimum APP SISO decoding. In the latter case, the component decoder soft-output (i.e., the extrinsic likelihood ratios) is mapped to messages from the desired binary/ternary alphabet. The analysis is limited to GLDPC codes whose CN component codes admit a simple trellis representation, hence enabling the analysis under APP decoding at the CNs. The results help to shed light on the performance loss incurred by BDD of the component codes, in the context of BMP [107, 109] and TMP decoding. When applying BDD at the CNs, we use two approaches following [2] and [1, 65] to make the decoding rule extrinsic – a prerequisite to perform the DE analysis. Besides providing asymptotic decoding thresholds for GLDPC code ensembles, the DE analysis for BMP and TMP decoding plays an additional role, as suggested in [13]. In fact, the VN messages are obtained by combing the channel LLR with a weighted sum of the incoming CN messages and quantizing the result to obtain a binary or ternary message: the weighting factors used at the VNs can be estimated via DE.

We focus on GLDPC codes with CNs based on (extended) Hamming codes. For this class of codes, we show that under BMP decoding, BDD at the CNs yields almost the same performance as optimum APP CN processing, while under TMP decoding the loss incurred by the sub-optimum BDD at the CNs is within 0.7 dB, when compared with APP decoding at the CNs. This observation, together with the low decoding complexity entailed by BDD, suggests that the use BDD within BMP/TMP decoders can provide an excellent trade-off between decoding complexity and coding gain.

5.1 Extrinsic Channel

The messages exchanged between check and variable nodes in an iterative decoder can be modeled as observations of a symmetric discrete memoryless channel, with channel input given by the codeword symbol associated with the message. When considering an extrinsic channel, the channel input X takes values in the input alphabet $\mathcal{X} = \{-1, +1\}$. For a BMP decoder, the extrinsic channel is a BSC with output alphabet $\mathcal{Z} = \{-1, +1\}$ and crossover probability θ. The channel LLR of this BSC is

$$L(z) = \ln\left[\frac{\Pr\{Z = z|X = +1\}}{\Pr\{Z = z|X = -1\}}\right] = \underbrace{\ln\left(\frac{1-\theta}{\theta}\right)}_{\mathsf{D}(\theta)} \cdot z \tag{5.1}$$

where $\mathsf{D}(\theta)$ is referred to as the channel reliability. For TMP, the extrinsic channel is a binary error and erasure channel (BEEC) with output alphabet $\mathcal{Z} = \{-1, 0, +1\}$, where 0 corresponds to an erasure. Let θ and ϵ be the respective error and erasure probabilities of this channel. The channel LLR of the BEEC is

$$L(z) = \ln\left[\frac{\Pr\{Z = z|X = +1\}}{\Pr\{Z = z|X = -1\}}\right] = \underbrace{\ln\left(\frac{1-\theta-\epsilon}{\theta}\right)}_{\mathsf{D}(\theta,\epsilon)} \cdot z \tag{5.2}$$

where $\mathsf{D}(\theta, \epsilon)$ is referred to as channel reliability.

5.2 Decoding Algorithms

We next describe the decoding algorithms that will be used in the analysis. We will consider two types of local decoders at the CNs, i.e., optimum APP decoding and BDD. In both cases, we assume the messages exchanged between check and variable nodes belong either to a binary set (BMP decoding) or to a ternary set (TMP), while the observations at the output of the communication channel will be unquantized. Under BDD decoding at the CNs, we consider an extrinsic message passing approach [1, 2, 65]. Let $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ be the message sent from CN $\mathtt{c}$ to its neighboring VN $\mathtt{v}$ at the ℓ-th iteration. Similarly, $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is the message sent from VN $\mathtt{v}$ to CN $\mathtt{c}$ at the ℓ-th iteration. For BMP the exchanged messages between VNs and CNs are binary, i.e., $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \in \mathcal{M}_{\text{BMP}} \triangleq \{-1, +1\}$ whereas for TMP the exchanged messages are ternary, i.e., $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}, m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} \in \mathcal{M}_{\text{TMP}} \triangleq \{-1, 0, +1\}$. An erased message indicates complete uncertainty about the respective bit.

5.2.1 APP SISO Algorithm at the Check Nodes

Each VN computes the LLR

$$L_{\text{ch}} = \frac{2}{\sigma^2} y \tag{5.3}$$

for the corresponding channel observation and passes a quantized value to its neighboring CNs. In particular, for all $\mathtt{c} \in \mathcal{N}(\mathtt{v})$ we have

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = m^{\text{ch}}_{\mathtt{v}} \tag{5.4}$$

where $m^{\text{ch}}_{\mathtt{v}} = \Psi(L_{\text{ch}})$ and the quantization function Ψ is defined as

$$\Psi(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{5.5}$$

for BMP and

$$\Psi(x) = \begin{cases} +1 & x \geq \mathsf{T} \\ 0 & -\mathsf{T} < x < \mathsf{T} \\ -1 & x \leq -\mathsf{T} \end{cases} \tag{5.6}$$

for TMP. In (5.5), if $x = 0$ we choose randomly between -1 and $+1$. We choose the real-valued parameter T in (5.6) to minimize the iterative decoding threshold. It can be chosen for each iteration individually. In this work, we keep T constant over the iterations since we observed that, for the codes considered in the numerical results, optimizing T across iterations does not yield any tangible performance gain.

Each CN computes the extrinsic likelihood ratio vector $\boldsymbol{L}^e_{\mathsf{c}}$ by using any APP SISO algorithm to obtain extrinsic likelihood ratios where the incoming VN messages represent the received sequence and are modeled as observations of a BSC with crossover probability $p_{-1}^{(\ell-1)}$ for BMP and a BEEC with error and erasure probabilities $p_{-1}^{(\ell-1)}$ and $p_0^{(\ell-1)}$, respectively, for TMP. The probabilities $p_{-1}^{(\ell)}$ and $p_0^{(\ell)}$ are the error and erasure probabilities of the VN messages at the ℓ-th iteration and they can be estimated via the DE analysis discussed in Section 5.3 as proposed in [13]. The CN c sends to its neighboring VN v

$$m^{(\ell)}_{\mathsf{c}\to\mathsf{v}} = \Phi\left(L^e_{\mathsf{c},j}\right) \tag{5.7}$$

where j is the codeword bit position assigned to v in the code of c and

$$\Phi(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{5.8}$$

for TMP [1] and $\Phi(x) = \Psi(x)$ given in (5.5) for BMP.

Each VN converts the channel output and the incoming CN messages to L-values and

[1] Note that, due to the discrete nature of the extrinsic channel at the CN input, the output of the APP decoder may indeed yield a zero value. The quantization function (5.8) may be replaced by a function of the form (5.6) where the output 0 represents an interval rather than the value $x = 0$. We empirically verified , for the codes under investigation, that using an interval does not improve performance.

passes the quantization of the result to its neighboring CNs. We have

$$m_{\mathsf{v}\to\mathsf{c}}^{(\ell)} = \Psi\left(L_{\mathrm{ch}} + \mathsf{D}^{(\ell)} \sum_{\mathsf{c}'\in\mathcal{N}(\mathsf{v})\setminus\mathsf{c}} m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right) \tag{5.9}$$

where L_{ch} is given in (5.3) and Ψ is defined in (5.5) for BMP and in (5.6) for TMP.

To estimate the corresponding codeword bit, each VN computes

$$\hat{m}_{\mathsf{v}}^{(\ell)} = \Psi\left(L_{\mathrm{ch}} + \mathsf{D}^{(\ell)} \sum_{\mathsf{c}'\in\mathcal{N}(\mathsf{v})} m_{\mathsf{c}'\to\mathsf{v}}^{(\ell)}\right) \tag{5.10}$$

where Ψ is defined in (5.5).

The value of $\mathsf{D}^{(\ell)}$ in (5.9) and (5.10) can be estimated via DE analysis (Section 5.3). For BMP

$$\mathsf{D}^{(\ell)} = \ln\left(\frac{1-q_{-1}^{(\ell)}}{q_{-1}^{(\ell)}}\right) \tag{5.11}$$

where $q_{-1}^{(\ell)}$ is the error probability of the CN output messages at the ℓ-th iteration. For TMP, we have

$$\mathsf{D}^{(\ell)} = \ln\left(\frac{1-q_0^{(\ell)}-q_{-1}^{(\ell)}}{q_{-1}^{(\ell)}}\right) \tag{5.12}$$

where $q_0^{(\ell)}$ and $q_{-1}^{(\ell)}$ are respectively the erasure and error probabilities of the CN output messages at the ℓ-th iteration.

5.2.2 Bounded Distance Decoding at the Check Nodes

Similarly, the VN to CN messages are initialized by (5.4). Let j be the codeword bit position assigned to v in the component code of c. Let $\boldsymbol{m}_{\mathsf{c}}$ be the length n_τ vector containing the incoming messages to the CN c from the other neighboring VNs and an erasure in its j-th entry. This way the decoder passes extrinsic messages, which makes the DE analysis possible. Note that this method is different than the one in [2]. For BMP, $\boldsymbol{m}_{\mathsf{c}}$ contains exactly one erasure (the j-th entry). Thus, if there exists a codeword $\hat{\boldsymbol{c}} \in \mathscr{C}$ with $2\mathsf{d}_{\mathrm{H}}(\hat{\boldsymbol{c}}, \boldsymbol{m}_{\mathsf{c}}) \leq \mathsf{d}_{\min,\tau} - 2$ then the outgoing message from c to its neighboring VN v is the j-th entry of $\hat{\boldsymbol{c}}$, i.e., $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = \hat{c}_j$ otherwise the CN outputs $+1$ or -1 uniformly at random. For TMP, let v be the number of erased messages the CN c receives from the other VNs. If there exists a codeword $\hat{\boldsymbol{c}} \in \mathscr{C}$ with $2\mathsf{d}_{\mathrm{H}}(\hat{\boldsymbol{c}}, \boldsymbol{m}_{\mathsf{c}}) + v \leq \mathsf{d}_{\min,\tau} - 2$ [2] then the outgoing message from c to its neighboring VN v is $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = \hat{c}_j$ otherwise the CN returns an erasure, i.e., $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = 0$.

To compare, we follow the method in [2], where the j-th entry of $\boldsymbol{m}_{\mathsf{c}}$ is equal to $m_{\mathsf{v}}^{\mathrm{ch}}$. In this case, $\boldsymbol{m}_{\mathsf{c}}$ does not contain any erasure for BMP. Thus, if there exists a codeword $\hat{\boldsymbol{c}} \in \mathscr{C}$ with $\mathsf{d}_{\mathrm{H}}(\hat{\boldsymbol{c}}, \boldsymbol{m}_{\mathsf{c}}) \leq \lfloor (\mathsf{d}_{\min,\tau} - 1)/2 \rfloor$ then the outgoing message from c to its neighboring VN v is the j-th entry of $\hat{\boldsymbol{c}}$, i.e., $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = \hat{c}_j$ otherwise the CN outputs $+1$ or -1 uniformly at random. For TMP, let v be the number of erasures in $\boldsymbol{m}_{\mathsf{c}}$. If there exists a codeword $\hat{\boldsymbol{c}} \in \mathscr{C}$ with $2\mathsf{d}_{\mathrm{H}}(\hat{\boldsymbol{c}}, \boldsymbol{m}_{\mathsf{c}}) + v \leq \mathsf{d}_{\min,\tau} - 1$ then the outgoing message from c to its neighboring VN v is $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = \hat{c}_j$ otherwise the CN returns an erasure, i.e., $m_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = 0$. The VN update rule and the codeword bit estimation are the same as in (5.9) and (5.10).

5.3 Density Evolution Analysis

Under BMP and TMP decoding, the DE analysis plays a twofold role: On one hand, it gives the asymptotic decoding threshold achievable by a specified GLDPC code ensemble; on the other hand, it estimates the error/erasure probabilities of the messages that are required to compute the reliability terms in (5.1) and in (5.2). We provide a DE analysis for both BMP and TMP under the assumption that the all-ones codeword is transmitted. Let $p_0^{(\ell)}$ and $p_{-1}^{(\ell)}$ be the erasure and error probabilities of VN messages at the ℓ-th iteration. Similarly, $q_0^{(\ell)}$ and $q_{-1}^{(\ell)}$ are the erasure and error probabilities of CN messages. Note that, while the threshold definitions below require an infinite number of iterations, for the numerical analysis we limited the number to $\ell_{\max}$.

5.3.1 APP SISO Algorithm at the Check Nodes

Consider the DE analysis of BMP and TMP decoding under optimum APP decoding at the CNs. Analyzing the error probability is complex, and we therefore adopt a hybrid approach. We analytically evaluate the evolution of the error (and erasure) probabilities at the VNs output, while the Monte Carlo method is used to estimate the evolution at the CNs. The analysis is outlined next.

[2] The Hamming distance $\mathsf{d}_{\mathrm{H}}(\hat{\boldsymbol{c}}, \boldsymbol{m}_{\mathsf{c}})$ is here defined as the number of positions where $\boldsymbol{m}_{\mathsf{c}}$ is not erased, and it is different from $\hat{\boldsymbol{c}}$.

Density Evolution Analysis for BMP

1. **Initialization.**

   Under the all-ones codeword assumption, the channel LLRs are Gaussian RVs with mean $\mu_{\text{ch}} = 4RE_{\mathsf{b}}/N_0$ and variance $\sigma^2_{\text{ch}} = 2\mu_{\text{ch}}$. Hence, recalling (5.7), we have

$$p_{-1}^{(0)} = \Pr\{L_{\text{ch}} < 0\} = Q\left(\frac{\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) \tag{5.13}$$

   where $Q(x)$ is the Gaussian Q function.

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   CN to VN update We can obtain $q_{-1}^{(\ell)}$ via Monte-Carlo simulation.

   VN to CN update

   We have

$$p_{-1}^{(\ell)} = \sum_d \lambda_d \sum_{u=0}^{d-1} \binom{d-1}{u} (1 - q_{-1}^{(\ell)})^{d-1-u} (q_{-1}^{(\ell)})^u Q\left(\frac{\mathsf{D}^{(\ell)}(d-1-2u) + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) \tag{5.14}$$

   where u is the number of incoming CN messages equal to -1 to a degree d VN and $\mathsf{D}^{(\ell)}$ is defined in (5.11). Note that (5.14) evaluates the probability that the sum of the channel LLR with the VN input messages (scaled by their corresponding reliability $\mathsf{D}^{(\ell)}$) used to compute an extrinsic estimate, lies in the wrong decision region.

Density Evolution Analysis for TMP

1. **Initialization.**

   Recalling (5.7), we have

$$p_0^{(0)} = \Pr\{-\mathsf{T} < L_{\text{ch}} < \mathsf{T}\} = Q\left(\frac{-\mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) - Q\left(\frac{\mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) \tag{5.15}$$

$$p_{-1}^{(0)} = \Pr\{L_{\text{ch}} \leq -\mathsf{T}\} = Q\left(\frac{\mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right). \tag{5.16}$$

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   CN to VN update We can obtain $q_{-1}^{(\ell)}$ and $q_0^{(\ell)}$ via Monte-Carlo simulation.

   VN to CN update

We have

$$
\begin{aligned}
p_{-1}^{(\ell)} = & \sum_d \lambda_d \sum_{u=0}^{d-1} \sum_{v=0}^{d-1-u} \binom{d-1}{u, v, d-1-u-v} (q_{-1}^{(\ell)})^u (q_0^{(\ell)})^v \times \\
& (1 - q_0^{(\ell)} - q_{-1}^{(\ell)})^{d-1-u-v} Q\left(\frac{\mathsf{D}^{(\ell)}(d-1-v-2u) + \mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right)
\end{aligned}
\quad (5.17)
$$

$$
\begin{aligned}
p_{0}^{(\ell)} = & \sum_d \lambda_d \sum_{u=0}^{d-1} \sum_{v=0}^{d-1-u} \binom{d-1}{u, v, d-1-u-v} (q_{-1}^{(\ell)})^u (q_0^{(\ell)})^v \times \\
& (1 - q_0^{(\ell)} - q_{-1}^{(\ell)})^{d-1-u-v} \left[Q\left(\frac{\mathsf{D}^{(\ell)}(d-1-v-2u) - \mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right)\right. \\
& \left. -Q\left(\frac{\mathsf{D}^{(\ell)}(d-1-v-2u) + \mathsf{T} + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right)\right]
\end{aligned}
\quad (5.18)
$$

where u and v are the number of -1 and 0, respectively, that a degree d VN receives at the ℓ-th iteration and $\mathsf{D}^{(\ell)}$ is given in (5.12). Note that (5.17) evaluates the probability that the sum of the channel LLR with the VN input messages (scaled by their corresponding reliability $\mathsf{D}^{(\ell)}$) used to compute an extrinsic estimate, lies on the wrong decision region. Similarly, (5.18) computes the probability that the sum results in a value within the erasure range $(-\mathsf{T}, +\mathsf{T})$.

The ensemble iterative decoding threshold $(E_\mathsf{b}/N_0)^\star$ is defined as the minimum E_b/N_0 for which $p_0^{(\ell)}, p_{-1}^{(\ell)} \to 0$ as $\ell \to \infty$.

5.3.2 Bounded Distance Decoding at the Check Nodes Following [1]

In contrast to APP decoding, iterative decoding with BDD at the CNs permits an exact analysis of the evolution of the error (and erasure) probabilities. The analysis is outlined next for both BMP and TMP decoding.

Density Evolution Analysis for BMP

1. **Initialization.**

   $p_{-1}^{(0)}$ is given in (5.13).

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   CN to VN update

We have $\mathcal{M}_{\text{BMP}} = \{-1, +1\}$ and the input of the decoder contains exactly one erasure (its j-th entry). Let $\mathsf{O}_{-1,u}$ ($\mathsf{O}_{+1,u}$) be the probability that there exists a codeword $\boldsymbol{c} \in \mathscr{C}_\tau$ with $2\mathsf{d}_{\text{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) \leq \mathsf{d}_{\min,\tau} - 2$ that has a -1 ($+1$) in a randomly chosen position and there are u errors in the other $n_\tau - 1$ positions. We have

$$q_{-1}^{(\ell)} = \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau - 1} \binom{n_\tau - 1}{u} (1 - p_{-1}^{(\ell-1)})^{n_\tau - 1 - u} (p_{-1}^{(\ell-1)})^u \frac{1}{2} \left(1 + \mathsf{O}_{-1,u} - \mathsf{O}_{+1,u}\right) \quad (5.19)$$

with $h = u - \delta + 2j$ and A_h is the cardinality of codewords of weight h in $\mathscr{C}$ and

$$\mathsf{O}_{-1,u} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-2}{2} \rfloor} \sum_{j=0}^{\delta} \frac{h+1}{n_\tau} \mathsf{A}_{h+1} \mathsf{I}_h & \lceil \frac{\mathsf{d}_{\min,\tau}-1}{2} \rceil \leq u \leq n_\tau - 1 \\ 0 & 0 \leq u \leq \lfloor \frac{\mathsf{d}_{\min,\tau}-2}{2} \rfloor \end{cases} \quad (5.20)$$

$$\mathsf{O}_{+1,u} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-2}{2} \rfloor} \sum_{j=0}^{\delta} \frac{n_\tau - h}{n_\tau} \mathsf{A}_{h} \mathsf{I}_h & \lceil \frac{\mathsf{d}_{\min,\tau}-1}{2} \rceil \leq u \leq n_\tau - 1 \\ 1 & 0 \leq u \leq \lfloor \frac{\mathsf{d}_{\min,\tau}-2}{2} \rfloor \end{cases} \quad (5.21)$$

where

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau - h - 1}{\delta - j}}{\binom{n_\tau - 1}{u}}. \quad (5.22)$$

We briefly explain the derivation of $\mathsf{O}_{-1,u}$ and $\mathsf{O}_{+1,u}$ in Appendix 5.6.1.

VN to CN update

$p_{-1}^{(\ell)}$ is given in (5.14).

Density Evolution Analysis for TMP

1. **Initialization.**

   $p_0^{(0)}$ and $p_{-1}^{(0)}$ are obtained from (5.15) and (5.16).

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

   CN to VN update

   Let $\mathsf{O}_{-1,u,v}(\mathsf{O}_{+1,u,v})$ be the probability that a randomly chosen bit is decoded incorrectly (correctly) when it was initially erased and there are u errors and v erasures in the other $n_\tau - 1$ positions. We have

$$q_{-1}^{(\ell)} = \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times (p_0^{(\ell-1)})^v (p_{-1}^{(\ell-1)})^u \mathsf{O}_{-1,u,v} \tag{5.23}$$

$$q_{0}^{(\ell)} = \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times (p_0^{(\ell-1)})^v (p_{-1}^{(\ell-1)})^u (1 - \mathsf{O}_{-1,u,v} - \mathsf{O}_{+1,u,v}) \tag{5.24}$$

with $h = u + 2j_1 + j_2 - \delta$ and

$$\mathsf{O}_{-1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-2}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{h+1}{n_\tau} \mathsf{A}_{h+1} \mathsf{F}_h & 2u + v \geq \mathsf{d}_{\min,\tau} - 1 \\ 0 & 2u + v \leq \mathsf{d}_{\min,\tau} - 2 \end{cases} \tag{5.25}$$

$$\mathsf{O}_{+1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-2}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{n_\tau-h}{n_\tau} \mathsf{A}_{h} \mathsf{F}_h & 2u + v \geq \mathsf{d}_{\min,\tau} - 1 \\ 1 & 2u + v \leq \mathsf{d}_{\min,\tau} - 2 \end{cases} \tag{5.26}$$

where

$$\mathsf{F}_h = \frac{\binom{h}{j_1, j_2, h-j_1-j_2} \binom{n_\tau-h-1}{n_\tau-u-v-1-j_1, \delta-j_1, v-j_2}}{\binom{n_\tau-1}{u, v, n_\tau-1-u-v}}. \tag{5.27}$$

We clarify the derivation of $\mathsf{O}_{-1,u,v}$ and $\mathsf{O}_{+1,u,v}$ in Appendix 5.6.2.

VN to CN update

$p_{-1}^{(\ell)}$ and $p_0^{(\ell)}$ and are given in (5.17) and (5.18).

5.3.3 Bounded Distance Decoding at the Check Nodes Following [2]

Density Evolution Analysis for BMP

1. **Initialization.**

   $p_{-1}^{(0)}$ is given in (5.13).

2. **For** $\ell = 1, 2, \ldots, \ell_{\max}$

CN to VN update Let $\mathsf{K}_{-1,u}$ ($\mathsf{K}_{+1,u}$) be the probability that there exists a codeword $\boldsymbol{c} \in \mathscr{C}_\tau$ with $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) \leq \mathsf{d}_{\min,\tau} - 1$ that has a -1 ($+1$) in a randomly chosen position and there are u errors in the other $n_\tau - 1$ positions. We have

$$q_{-1}^{(\ell)} = p_{-1}^{(0)}\beta^{(\ell)} + (1 - p_{-1}^{(0)})\alpha^{(\ell)} \tag{5.28}$$

where

$$\begin{aligned}\alpha^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = -1 | M_{\mathsf{v}}^{\mathrm{ch}} = +1\} \\ &= \frac{1}{2}\sum_{\tau=1}^{n_c}\rho_\tau \sum_{u=0}^{n_\tau-1}\binom{n_\tau-1}{u}(p_{-1}^{(\ell-1)})^u(1-p_{-1}^{(\ell-1)})^{n_\tau-1-u}(1+\mathsf{W}_{-1,u}-\mathsf{W}_{+1,u})\end{aligned} \tag{5.29}$$

$$\begin{aligned}\beta^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = -1 | M_{\mathsf{v}}^{\mathrm{ch}} = -1\} \\ &= \frac{1}{2}\sum_{\tau=1}^{n_c}\rho_\tau \sum_{u=0}^{n_\tau-1}\binom{n_\tau-1}{u}(p_{-1}^{(\ell-1)})^u(1-p_{-1}^{(\ell-1)})^{n_\tau-1-u}(1+\mathsf{K}_{-1,u}-\mathsf{K}_{+1,u})\end{aligned} \tag{5.30}$$

and

$$\mathsf{K}_{-1,u} = \begin{cases}\sum_{\delta=0}^{t_\tau}\sum_{j=0}^{\delta}\frac{h+1}{n_\tau}\mathsf{A}_{h+1}\mathsf{I}_h & t_\tau \leq u \leq n_\tau - 1\\ 0 & 0 \leq u \leq t_\tau - 1\end{cases} \tag{5.31}$$

$$\mathsf{K}_{+1,u} = \begin{cases}\sum_{\delta=1}^{t_\tau}\sum_{j=0}^{\delta-1}\frac{n_\tau-h}{n_\tau}\mathsf{A}_{h}\mathsf{P}_h & t_\tau \leq u \leq n_\tau - 1\\ 1 & 0 \leq u \leq t_\tau - 1\end{cases} \tag{5.32}$$

$$\mathsf{W}_{-1,u} = \begin{cases}\sum_{\delta=1}^{t_\tau}\sum_{j=0}^{\delta-1}\frac{h+1}{n_\tau}\mathsf{A}_{h+1}\mathsf{P}_h & t_\tau + 1 \leq u \leq n_\tau - 1\\ 0 & 0 \leq u \leq t_\tau\end{cases} \tag{5.33}$$

$$\mathsf{W}_{+1,u} = \begin{cases}\sum_{\delta=0}^{t_\tau}\sum_{j=0}^{\delta}\frac{n_\tau-h}{n_\tau}\mathsf{A}_{h}\mathsf{I}_h & t_\tau + 1 \leq u \leq n_\tau - 1\\ 1 & 0 \leq u \leq t_\tau\end{cases} \tag{5.34}$$

where

$$\mathsf{P}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j-1}}{\binom{n_\tau-1}{u}} \tag{5.35}$$

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j}}{\binom{n_\tau-1}{u}} \tag{5.36}$$

with $h = u - \delta + 2j + 1$ in (5.32), (5.33) and $h = u - \delta + 2j$ in (5.31), (5.34), (5.35), (5.36) and A_h is the cardinality of codewords of weight h in $\mathscr{C}$. We briefly explain the derivation of $\mathsf{K}_{-1,u}, \mathsf{K}_{+1,u}, \mathsf{W}_{-1,u}$ and $\mathsf{W}_{+1,u}$ in Appendix 5.6.3.

VN to CN update

To calculate $p_{-1}^{(\ell)}$, we must determine the distribution of $L_{\text{ch}} + \mathsf{D}^{(\ell)} \sum_{\mathtt{c}' \in \mathcal{N}(\mathtt{v}) \setminus \mathtt{c}} m_{\mathtt{c}' \to \mathtt{v}}^{(\ell)}$. Note that the CN messages and the channel LLR are statistically dependent since $m_{\mathtt{v}}^{\text{ch}} = \Psi(L_{\text{ch}})$ is used to compute the outgoing CN messages. Let $M_{\mathtt{c}\to\mathtt{v}}^{(\ell)}$ be the RV associated to CN to VN messages at the ℓ-th iteration. Similarly, $M_{\mathtt{v}}^{\text{ch}}$ is the RV associated to the quantized channel LLR. Let z be the number of incoming CN messages equal to $+1$ to a degree d VN. We obtain

$$\begin{aligned}
p_{-1}^{(\ell)} =& \sum_d \lambda_d \sum_{z=0}^{d-1} \binom{d-1}{z} \Big[\Pr\{L_{\text{ch}} < \min\{0, -\mathsf{D}^{(\ell)}(2z-d+1)\}\} \times \\
& (1-\beta^{(\ell)})^z (\beta^{(\ell)})^{d-1-z} + \Pr\{0 < L_{\text{ch}} < -\mathsf{D}^{(\ell)}(2z-d+1)\} \times \\
& (1-\alpha^{(\ell)})^z (\alpha^{(\ell)})^{d-1-z} \Big] \\
=& \sum_d \lambda_d \left(\sum_{z=\lceil \frac{d-1}{2} \rceil}^{d-1} \binom{d-1}{z} Q\left(\frac{\mathsf{D}^{(\ell)}(2z-d+1) + \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) (1-\beta^{(\ell)})^z \times \right. \\
& (\beta^{(\ell)})^{d-1-z} + \sum_{z=0}^{\lfloor \frac{d-2}{2} \rfloor} \binom{d-1}{z} \left[\left(Q\left(\frac{\mathsf{D}^{(\ell)}(2z-d+1) + \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) - p_{-1}^{(0)} \right) \times \right. \\
& \left. \left. (1-\alpha^{(\ell)})^z (\alpha^{(\ell)})^{d-1-z} + p_{-1}^{(0)} (1-\beta^{(\ell)})^z (\beta^{(\ell)})^{d-1-z} \right] \right)
\end{aligned} \tag{5.37}$$

where $\alpha^{(\ell)}, \beta^{(\ell)}$ are defined in (5.29) and (5.30) and $\mathsf{D}^{(\ell)}$ is defined in (5.11).

Density Evolution Analysis for TMP

1. **Initialization.**

   $p_0^{(0)}$ and $p_{-1}^{(0)}$ are obtained from (5.15) and (5.16).

2. **For** $\ell = 1, 2, \dots, \ell_{\max}$

   CN to VN update

   Let $\mathsf{K}_{-1,u,v}(\mathsf{K}_{+1,u,v})$ be the probability that a randomly chosen bit is decoded incorrectly (correctly) when it was initially in error and there are u errors and v erasures in the other $n_\tau - 1$ positions. Similarly, $\mathsf{W}_{-1,u,v}(\mathsf{W}_{+1,u,v})$ is the probability that a randomly chosen bit is decoded incorrectly (correctly) when it was initially correct and there are

u errors and v erasures in the other $n_\tau - 1$ positions. $\mathsf{O}_{-1,u,v}(\mathsf{O}_{+1,u,v})$ is the probability that a randomly chosen bit is decoded incorrectly (correctly) when it was initially erased and there are u errors and v erasures in the other $n_\tau - 1$ positions. We have

$$q_{-1}^{(\ell)} = p_{-1}^{(0)}\beta_{-1}^{(\ell)} + (1 - p_{-1}^{(0)} - p_0^{(0)})\alpha_{-1}^{(\ell)} + p_0^{(0)}\gamma_{-1}^{(\ell)} \tag{5.38}$$

$$q_{0}^{(\ell)} = p_{-1}^{(0)}\beta_{0}^{(\ell)} + (1 - p_{-1}^{(0)} - p_0^{(0)})\alpha_{0}^{(\ell)} + p_0^{(0)}\gamma_{0}^{(\ell)} \tag{5.39}$$

where

$$\begin{aligned}\alpha_{-1}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = -1 | M_{\mathsf{v}}^{\text{ch}} = +1\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times \\ &\quad (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v \mathsf{W}_{-1,u,v}\end{aligned} \tag{5.40}$$

$$\begin{aligned}\beta_{-1}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = -1 | M_{\mathsf{v}}^{\text{ch}} = -1\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times \\ &\quad (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v \mathsf{K}_{-1,u,v}\end{aligned} \tag{5.41}$$

$$\begin{aligned}\gamma_{-1}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = -1 | M_{\mathsf{v}}^{\text{ch}} = 0\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times \\ &\quad (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v \mathsf{O}_{-1,u,v}\end{aligned} \tag{5.42}$$

$$\begin{aligned}\alpha_{0}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = 0 | M_{\mathsf{v}}^{\text{ch}} = +1\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} \times \\ &\quad (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v (1 - \mathsf{W}_{-1,u,v} - \mathsf{W}_{+1,u,v})\end{aligned} \tag{5.43}$$

$$\begin{aligned}\beta_{0}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = 0 | M_{\mathsf{v}}^{\text{ch}} = -1\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v \times \\ &\quad (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} (1 - \mathsf{K}_{-1,u,v} - \mathsf{K}_{+1,u,v})\end{aligned} \tag{5.44}$$

$$\begin{aligned}\gamma_{0}^{(\ell)} &= \Pr\{M_{\mathsf{c}\to\mathsf{v}}^{(\ell)} = 0 | M_{\mathsf{v}}^{\text{ch}} = 0\} \\ &= \sum_{\tau=1}^{n_c} \rho_\tau \sum_{u=0}^{n_\tau-1} \sum_{v=0}^{n_\tau-1-u} \binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v} (p_{-1}^{(\ell-1)})^u (p_0^{(\ell-1)})^v \times \\ &\quad (1 - p_{-1}^{(\ell-1)} - p_0^{(\ell-1)})^{n_\tau-1-u-v} (1 - \mathsf{O}_{-1,u,v} - \mathsf{O}_{+1,u,v}).\end{aligned} \tag{5.45}$$

We have for all $0 \le u, v, u+v \le n_\tau - 1$

$$\mathsf{K}_{-1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-1}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{h+1}{n_\tau} \mathsf{A}_{h+1} \mathsf{F}_h & 2u+v \ge \mathsf{d}_{\min,\tau} - 2 \\ 0 & 2u+v \le \mathsf{d}_{\min,\tau} - 3 \end{cases} \tag{5.46}$$

$$\mathsf{K}_{+1,u,v} = \begin{cases} \sum_{\delta=1}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-1}{2} \rfloor} \sum_{j_1=0}^{\delta-1} \sum_{j_2=0}^{v} \frac{n_\tau-h}{n_\tau} \mathsf{A}_{h} \mathsf{S}_h & 2u+v \ge \mathsf{d}_{\min,\tau} - 2 \\ 1 & 2u+v \le \mathsf{d}_{\min,\tau} - 3 \end{cases} \tag{5.47}$$

$$\mathsf{W}_{-1,u,v} = \begin{cases} \sum_{\delta=1}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-1}{2} \rfloor} \sum_{j_1=0}^{\delta-1} \sum_{j_2=0}^{v} \frac{h+1}{n_\tau} \mathsf{A}_{h+1} \mathsf{S}_h & 2u+v \ge \mathsf{d}_{\min,\tau} \\ 0 & 2u+v \le \mathsf{d}_{\min,\tau} - 1 \end{cases} \tag{5.48}$$

$$\mathsf{W}_{+1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-1}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{n_\tau-h}{n_\tau} \mathsf{A}_{h} \mathsf{F}_h & 2u+v \ge \mathsf{d}_{\min,\tau} \\ 1 & 2u+v \le \mathsf{d}_{\min,\tau} - 1 \end{cases} \tag{5.49}$$

$$\mathsf{O}_{-1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-2}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{h+1}{n_\tau} \mathsf{A}_{h+1} \mathsf{F}_h & 2u+v \ge \mathsf{d}_{\min,\tau} - 1 \\ 0 & 2u+v \le \mathsf{d}_{\min,\tau} - 2 \end{cases} \tag{5.50}$$

$$\mathsf{O}_{+1,u,v} = \begin{cases} \sum_{\delta=0}^{\lfloor \frac{\mathsf{d}_{\min,\tau}-v-2}{2} \rfloor} \sum_{j_1=0}^{\delta} \sum_{j_2=0}^{v} \frac{n_\tau-h}{n_\tau} \mathsf{A}_{h} \mathsf{F}_h & 2u+v \ge \mathsf{d}_{\min,\tau} - 1 \\ 1 & 2u+v \le \mathsf{d}_{\min,\tau} - 2 \end{cases} \tag{5.51}$$

where

$$\mathsf{F}_h = \frac{\binom{h}{j_1, j_2, h-j_1-j_2} \binom{n_\tau-h-1}{n_\tau-u-v-1-j_1, \delta-j_1, v-j_2}}{\binom{n_\tau-1}{u, v, n_\tau-1-u-v}} \tag{5.52}$$

$$\mathsf{S}_h = \frac{\binom{h}{j_1, j_2, h-j_1-j_2} \binom{n_\tau-h-1}{n_\tau-u-v-1-j_1, \delta-1-j_1, v-j_2}}{\binom{n_\tau-1}{u, v, n_\tau-1-u-v}} \tag{5.53}$$

where in (5.46), (5.49), (5.50), (5.51) $h = u + 2j_1 + j_2 - \delta$ and in (5.47), (5.48), $h = u + 2j_1 + j_2 + 1 - \delta$. We clarify the derivation of $\mathsf{K}_{-1,u,v}, \mathsf{K}_{+1,u,v}, \mathsf{W}_{-1,u,v}, \mathsf{W}_{+1,u,v}, \mathsf{O}_{-1,u,v}, \mathsf{O}_{+1,u,v}$ in Appendix 5.6.4.

VN to CN update

Recalling (5.6), for TMP $M_{\mathtt{v}}^{\mathrm{ch}} = \Psi(L_{\mathrm{ch}})$ takes value from $\{-1, 0, +1\}$. Let z and

w be, respectively, the number of incoming CN messages equal to $+1$ and -1 to a degree d VN. We obtain

$$\begin{aligned} p_{-1}^{(\ell)} = \sum_d \lambda_d \sum_{z=0}^{d-1} \sum_{w=0}^{d-1-z} \binom{d-1}{z, w, d-1-z-w} \Big[(\alpha_0^{(\ell)})^{d-1-w-z} (\alpha_{-1}^{(\ell)})^{w} \times \\ (1-\alpha_{-1}^{(\ell)}-\alpha_0^{(\ell)})^{z} \Pr\{\mathsf{T} \le L_{\text{ch}} \le -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\} + (\gamma_0^{(\ell)})^{d-1-w-z} \times \\ (\gamma_{-1}^{(\ell)})^{w}(1-\gamma_{-1}^{(\ell)}-\gamma_0^{(\ell)})^{z} \Pr\{L_{\text{ch}} \le -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w) \cap -\mathsf{T} < L_{\text{ch}} < \mathsf{T}\} + \\ (\beta_{-1}^{(\ell)})^{w}(\beta_0^{(\ell)})^{d-1-w-z}(1-\beta_{-1}^{(\ell)}-\beta_0^{(\ell)})^{z} \\ \Pr\{L_{\text{ch}} \le \min\{-\mathsf{T}, -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\}\}\Big] \end{aligned} \tag{5.54}$$

$$\begin{aligned} p_{0}^{(\ell)} = \sum_d \lambda_d \sum_{z=0}^{d-1} \sum_{w=0}^{d-1-z} \binom{d-1}{z, w, d-1-z-w} \Big[(\alpha_0^{(\ell)})^{d-1-w-z} (\alpha_{-1}^{(\ell)})^{w} \times \\ (1-\alpha_{-1}^{(\ell)}-\alpha_0^{(\ell)})^{z} \\ \Pr\{\mathsf{T} \le L_{\text{ch}} \cap -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w) < L_{\text{ch}} < \mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\} \\ + (\gamma_{-1}^{(\ell)})^{w}(\gamma_0^{(\ell)})^{d-1-w-z}(1-\gamma_{-1}^{(\ell)}-\gamma_0^{(\ell)})^{z} \times \\ \Pr\{\max\{-\mathsf{T}, -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\} < L_{\text{ch}} < \min\{\mathsf{T}, \mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\}\} \\ + (\beta_0^{(\ell)})^{d-1-w-z}(\beta_{-1}^{(\ell)})^{w}(1-\beta_{-1}^{(\ell)}-\beta_0^{(\ell)})^{z} \times \\ \Pr\{L_{\text{ch}} \le -\mathsf{T} \cap -\mathsf{T}-\mathsf{D}^{(\ell)}(z-w) < L_{\text{ch}} < \mathsf{T}-\mathsf{D}^{(\ell)}(z-w)\}\Big] \end{aligned} \tag{5.55}$$

where $\alpha_{-1}^{(\ell)}, \beta_{-1}^{(\ell)}, \gamma_{-1}^{(\ell)}, \alpha_0^{(\ell)}, \beta_0^{(\ell)}, \gamma_0^{(\ell)}$ are defined in (5.40)-(5.45) and $\mathsf{D}^{(\ell)}$ is defined in (5.12).

5.4 Stability Analysis

We study the convergence of the error and erasure probabilities to zero assuming that they are sufficiently small. In particular, we derive the stability condition for BMP and TMP decoding when BDD decoding is applied at the CNs. Recalling that the stability condition provides a necessary condition to achieve arbitrarily small error probabilities, its evaluation can be used to verify the suitability of a given degree distribution [110]. The analysis is presented next.

5.4.1 Bounded Distance Decoding at the Check Nodes Following [1]

Stability Condition for BMP

We determine the evolution of $p_{-1}^{(\ell)}$ over one iteration when we are close to the fixed point $p_1^\star = 0$. From (5.19), we have

$$\lim_{p_{-1}^{(\ell-1)} \to 0} \frac{\mathrm{d}q_{-1}^{(\ell)}}{\mathrm{d}p_{-1}^{(\ell-1)}} = \frac{1}{2} \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)[\mathsf{O}_{-1,1} - \mathsf{O}_{+1,1} - \mathsf{O}_{-1,0} + \mathsf{O}_{+1,0}]. \tag{5.56}$$

Note that if $p_{-1}^{(\ell)} \to p_1^\star$, we have $q_{-1}^{(\ell)} \to 0$ and $\mathsf{D}^{(\ell)} \to \infty$. Thus, for small error probabilities, we have

$$\begin{aligned} p_{-1}^{(\ell)} = \sum_d \lambda_d \Bigg[& \sum_{z=\lfloor \frac{d-1}{2} \rfloor + 1}^{d-1} \binom{d-1}{z} (q_{-1}^{(\ell)})^z (1 - q_{-1}^{(\ell)})^{d-1-z} \\ & + Q\left(\frac{\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) \binom{d-1}{\frac{d-1}{2}} (q_{-1}^{(\ell)})^{\frac{d-1}{2}} (1 - q_{-1}^{(\ell)})^{\frac{d-1}{2}} \Bigg]. \end{aligned} \tag{5.57}$$

We obtain

$$\lim_{q_{-1}^{(\ell)} \to 0} \frac{\mathrm{d}p_{-1}^{(\ell)}}{\mathrm{d}q_{-1}^{(\ell)}} = \lambda_2 + 2\lambda_3 Q\left(\frac{\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right). \tag{5.58}$$

The first order Taylor expansions via (5.56) and (5.58) yield $p_{-1}^{(\ell)} = \Omega p_{-1}^{(\ell-1)}$ where

$$\Omega = \frac{1}{2} \left(\lambda_2 + 2\lambda_3 Q\left(\frac{\mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) \right) \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)[\mathsf{O}_{-1,1} - \mathsf{O}_{+1,1} - \mathsf{O}_{-1,0} + \mathsf{O}_{+1,0}]. \tag{5.59}$$

The stability condition is satisfied if and only if $\Omega < 1$. Clearly, $\Omega = 0$ if $\forall \tau \in \{1, \ldots, n_c\}$ we have $\mathsf{d}_{\min,\tau} \geq 4$, which implies that the stability condition is fulfilled.

Stability Condition for TMP

Deriving the stability condition under TMP decoding is slightly more complicated than in the BMP case. The reason is that, while for BMP decoding it suffices to study the linearization of the two DE equations in a single variable, under TMP decoding the DE analysis entails two recursions per VN/CN step, each involving two variables. We proceed by defining the vectors $\boldsymbol{p}^{(\ell)} = [p_{-1}^{(\ell)}, p_0^{(\ell)}]^T$, $\boldsymbol{p}^{(\ell)} = [q_{-1}^{(\ell)}, q_0^{(\ell)}]^T$. We determine the evolution of

$\boldsymbol{p}^{(\ell)}$ over one iteration when we are close to the fixed point $\boldsymbol{p}^\star = \mathbf{0}$. From (5.23), we have

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial q_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{-1,1,0} - \mathsf{O}_{-1,0,0}) := \kappa_1 \tag{5.60}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial q_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{-1,0,1} - \mathsf{O}_{-1,0,0}) := \kappa_2 \tag{5.61}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial q_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{-1,0,0} + \mathsf{O}_{0,0,0} - \mathsf{O}_{-1,1,0} - \mathsf{O}_{0,1,0}) := \kappa_3 \tag{5.62}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial q_{0}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{-1,0,0} + \mathsf{O}_{0,0,0} - \mathsf{O}_{-1,0,1} - \mathsf{O}_{0,0,1}) := \kappa_4. \tag{5.63}$$

As $\mathsf{D}^{(\ell)} \to \infty$, we have

$$\begin{aligned} p_{-1}^{(\ell)} =& \sum_d \lambda_d \Bigg(\sum_{u=0}^{d-1} \sum_{v=d-2u}^{d-1-u} (q_{-1}^{(\ell)})^u (1 - q_{-1}^{(\ell)} - q_0^{(\ell)})^{d-1-u-v} (q_0^{(\ell)})^v \\ &+ \sum_{u=0}^{\lfloor \frac{d-1}{2} \rfloor} Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \binom{d-1}{u, u, d-1-2u} (q_{-1}^{(\ell)})^u (q_0^{(\ell)})^{d-1-2u} (1 - q_{-1}^{(\ell)} - q_0^{(\ell)})^u \Bigg) \end{aligned} \tag{5.64}$$

$$\begin{aligned} p_{0}^{(\ell)} =& \sum_d \lambda_d \sum_{u=0}^{\lfloor \frac{d-1}{2} \rfloor} \left[Q\left(\frac{\mu_{\text{ch}} - T}{\sigma_{\text{ch}}}\right) - Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \right] \times \\ & \binom{d-1}{u, u, d-1-2u} (q_{-1}^{(\ell)})^u (q_0^{(\ell)})^{d-1-2u} (1 - q_{-1}^{(\ell)} - q_0^{(\ell)})^u. \end{aligned} \tag{5.65}$$

Following the same steps as the case of BMP, we obtain

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \left(\lambda_2 + 2\lambda_3 Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right)\right) \kappa_1 + \lambda_2 Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \kappa_3 \tag{5.66}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = \left(\lambda_2 + 2\lambda_3 Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right)\right) \kappa_2 + \lambda_2 Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \kappa_4 \tag{5.67}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \left[Q\left(\frac{\mu_{\text{ch}} - T}{\sigma_{\text{ch}}}\right) - Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \right] (2\lambda_3 \kappa_1 + \lambda_2 \kappa_3) \tag{5.68}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = \left[Q\left(\frac{\mu_{\text{ch}} - T}{\sigma_{\text{ch}}}\right) - Q\left(\frac{\mu_{\text{ch}} + T}{\sigma_{\text{ch}}}\right) \right] (2\lambda_3 \kappa_2 + \lambda_2 \kappa_4). \tag{5.69}$$

Define the matrix

$$\boldsymbol{J} := \begin{bmatrix} \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} & \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} \\ \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} & \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} \end{bmatrix}. \tag{5.70}$$

Let Ω be the spectral radius of $\boldsymbol{J}$, i.e., the largest magnitude of its eigenvalues. The stability condition is satisfied if and only if $\Omega < 1$.

It can be easily verified that if $\forall s = 1, \dots, n_c$ $\mathsf{d}_{\min,\tau} \geq 4$, $\boldsymbol{J} = \boldsymbol{0}_{2\times 2}$. As a result, if all CN types have minimum distance larger or equal than 4, then the stability condition is satisfied.

5.4.2 Bounded Distance Decoding at the Check Nodes Following [2]

Stability Condition for BMP

We determine the evolution of $p_{-1}^{(\ell)}$ over one iteration when we are close to the fixed point $p_{-1}^{\star} = 0$.

From (5.29) and (5.30), we have

$$\lim_{p_{-1}^{(\ell-1)}\to 0} \frac{\mathrm{d}\alpha^{(\ell)}}{\mathrm{d}p_{-1}^{(\ell-1)}} = \frac{1}{2}\sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{W}_{-1,1} - \mathsf{W}_{+1,1} - \mathsf{W}_{-1,0} + \mathsf{W}_{+1,0}) \tag{5.71}$$

$$\lim_{p_{-1}^{(\ell-1)}\to 0} \frac{\mathrm{d}\beta^{(\ell)}}{\mathrm{d}p_{-1}^{(\ell-1)}} = \frac{1}{2}\sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{K}_{-1,1} - \mathsf{K}_{+1,1} - \mathsf{K}_{-1,0} + \mathsf{K}_{+1,0}). \tag{5.72}$$

Note that if $p_{-1}^{(\ell)} \to p_{-1}^{\star}$, we have $\alpha^{(\ell)}, \beta^{(\ell)} \to 0$. Recalling (5.11) and (5.28), we get $\mathsf{D}^{(\ell)} \to \infty$. Thus, for small error probabilities, we have

$$\begin{aligned} p_{-1}^{(\ell)} = \sum_d \lambda_d \Bigg(& p_{-1}^{(0)} \binom{d-1}{\frac{d-1}{2}} (\beta^{(\ell)})^{\frac{d-1}{2}} (1-\beta^{(\ell)})^{\frac{d-1}{2}} \\ & + \sum_{z=0}^{\lfloor \frac{d-2}{2} \rfloor} \binom{d-1}{z} \left[(1-p_{-1}^{(0)})(\alpha^{(\ell)})^{d-1-z}(1-\alpha^{(\ell)})^z + p_{-1}^{(0)}(\beta^{(\ell)})^{d-1-z}(1-\beta^{(\ell)})^z \right] \Bigg). \end{aligned} \tag{5.73}$$

We obtain

$$\lim_{\alpha^{(\ell)}\to 0} \frac{\partial p_{-1}^{(\ell)}}{\partial \alpha^{(\ell)}} = \lambda_2 (1 - p_{-1}^{(0)}), \quad \lim_{\beta^{(\ell)}\to 0} \frac{\partial p_{-1}^{(\ell)}}{\partial \beta^{(\ell)}} = p_{-1}^{(0)} (\lambda_2 + 2\lambda_3) \tag{5.74}$$

where $p_{-1}^{(0)}$ is given in (5.13). Since

$$\frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \frac{\partial p_{-1}^{(\ell)}}{\partial \alpha^{(\ell)}} \frac{\mathrm{d}\alpha^{(\ell)}}{\mathrm{d}p_{-1}^{(\ell-1)}} + \frac{\partial p_{-1}^{(\ell)}}{\partial \beta^{(\ell)}} \frac{\mathrm{d}\beta^{(\ell)}}{\mathrm{d}p_{-1}^{(\ell-1)}} \tag{5.75}$$

we have

$$\lim_{p_{-1}^{(\ell-1)} \to 0} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \Omega \tag{5.76}$$

where

$$\begin{aligned}\Omega = &\frac{1}{2} \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1) \Big[(\mathsf{W}_{-1,1} - \mathsf{W}_{+1,1} - \mathsf{W}_{-1,0} + \mathsf{W}_{+1,0}) \lambda_2 (1 - p_{-1}^{(0)}) \\ &+ p_{-1}^{(0)} (\mathsf{K}_{-1,1} - \mathsf{K}_{+1,1} - \mathsf{K}_{-1,0} + \mathsf{K}_{+1,0})(\lambda_2 + 2\lambda_3) \Big] .\end{aligned} \tag{5.77}$$

The stability condition is satisfied if and only if $\Omega < 1$. Clearly from (5.31)-(5.34), $\Omega = 0$ if $\forall \tau \in \{1, \dots, n_c\} \quad t_\tau \geq 2$ which implies that the stability condition is fulfilled.

Stability Condition for TMP

We define the vector $\boldsymbol{p}^{(\ell)} = [p_{-1}^{(\ell)}, p_0^{(\ell)}]^T$. We determine the evolution of $\boldsymbol{p}^{(\ell)}$ over one iteration when we are close to the fixed point $\boldsymbol{p}^\star = \mathbf{0}$. From (5.40)-(5.45), we have

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \alpha_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{W}_{-1,1,0} - \mathsf{W}_{-1,0,0}) \tag{5.78}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \beta_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{K}_{-1,1,0} - \mathsf{K}_{-1,0,0}) \tag{5.79}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \gamma_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{-1,1,0} - \mathsf{O}_{-1,0,0}) \tag{5.80}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \alpha_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{W}_{0,1,0} - \mathsf{W}_{0,0,0}) \tag{5.81}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \beta_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{K}_{0,1,0} - \mathsf{K}_{0,0,0}) \tag{5.82}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \gamma_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{O}_{0,1,0} - \mathsf{O}_{0,0,0}) \tag{5.83}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)} \to \mathbf{0}} \frac{\partial \alpha_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)(\mathsf{W}_{-1,0,1} - \mathsf{W}_{-1,0,0}) \tag{5.84}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial\beta_{-1}^{(\ell)}}{\partial p_0^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)(\mathsf{K}_{-1,0,1}-\mathsf{K}_{-1,0,0}) \tag{5.85}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial\gamma_{-1}^{(\ell)}}{\partial p_0^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)(\mathsf{O}_{-1,0,1}-\mathsf{O}_{-1,0,0}) \tag{5.86}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial\alpha_{0}^{(\ell)}}{\partial p_0^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)(\mathsf{W}_{0,0,1}-\mathsf{W}_{0,0,0}) \tag{5.87}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial\beta_{0}^{(\ell)}}{\partial p_0^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)(\mathsf{K}_{0,0,1}-\mathsf{K}_{0,0,0}) \tag{5.88}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial\gamma_{0}^{(\ell)}}{\partial p_0^{(\ell-1)}} = \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)(\mathsf{O}_{0,0,1}-\mathsf{O}_{0,0,0}). \tag{5.89}$$

As $\mathsf{D}^{(\ell)}\to\infty$, we have

$$\begin{aligned} p_{-1}^{(\ell)} =& \sum_d \lambda_d \Bigg(\sum_{z=0}^{d-1}\sum_{w=z+1}^{d-1-z} \binom{d-1}{z,w,d-1-z-w} \Big[(1-p_0^{(0)}-p_{-1}^{(0)})(\alpha_0^{(\ell)})^{d-1-z-w}\times \\ & (1-\alpha_{-1}^{(\ell)}-\alpha_0^{(\ell)})^z + p_{-1}^{(0)}(\beta_0^{(\ell)})^{d-1-z-w}(\beta_{-1}^{(\ell)})^w(1-\beta_{-1}^{(\ell)}-\beta_0^{(\ell)})^z \\ & +p_0^{(0)}(\gamma_0^{(\ell)})^{d-1-z-w}(\gamma_{-1}^{(\ell)})^w(1-\gamma_{-1}^{(\ell)}-\gamma_0^{(\ell)})^z\Big] \\ & +p_{-1}^{(0)} \sum_{z=0}^{\lfloor\frac{d-1}{2}\rfloor} \binom{d-1}{z,z,d-1-2z} (\beta_{-1}^{(\ell)})^z(\beta_0^{(\ell)})^{d-1-2z}(1-\beta_{-1}^{(\ell)}-\beta_0^{(\ell)})^z \Bigg) \end{aligned} \tag{5.90}$$

$$p_0^{(\ell)} = p_{-1}^{(0)}\sum_d \lambda_d \sum_{z=0}^{\lfloor\frac{d-1}{2}\rfloor} \binom{d-1}{z,z,d-1-2z} (\gamma_{-1}^{(\ell)})^z(\gamma_0^{(\ell)})^{d-1-2z}(1-\gamma_{-1}^{(\ell)}-\gamma_0^{(\ell)})^z. \tag{5.91}$$

Following the same steps as the case of BMP, we obtain

$$\begin{aligned} \lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} =& \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)\Big[\Big((1-p_{-1}^{(0)}-p_0^{(0)})(\mathsf{W}_{-1,1,0}-\mathsf{W}_{-1,0,0}) + \\ & p_{-1}^{(0)}(\mathsf{K}_{-1,1,0}-\mathsf{K}_{-1,0,0}+\mathsf{K}_{0,1,0}-\mathsf{K}_{0,0,0}) + p_0^{(0)}(\mathsf{O}_{-1,1,0}-\mathsf{O}_{-1,0,0})\Big)\lambda_2 \\ & +2p_{-1}^{(0)}\lambda_3(\mathsf{K}_{-1,1,0}-\mathsf{K}_{-1,0,0})\Big] \end{aligned} \tag{5.92}$$

$$\begin{aligned} \lim_{\boldsymbol{p}^{(\ell-1)}\to\mathbf{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} =& \sum_{\tau=1}^{n_c} \rho_\tau(n_\tau-1)\Big[\Big((1-p_{-1}^{(0)}-p_0^{(0)})(\mathsf{W}_{-1,0,1}-\mathsf{W}_{-1,0,0}) + \\ & p_0^{(0)}(\mathsf{O}_{-1,0,1}-\mathsf{O}_{-1,0,0}) + p_{-1}^{(0)}(\mathsf{K}_{-1,0,1}-\mathsf{K}_{-1,0,0}+\mathsf{K}_{0,0,1}-\mathsf{K}_{0,0,0})\Big)\lambda_2 \\ & +2p_{-1}^{(0)}\lambda_3(\mathsf{K}_{-1,0,1}-\mathsf{K}_{-1,0,0})\Big] \end{aligned} \tag{5.93}$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_0^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} = p_0^{(0)} \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)\left[(\mathsf{O}_{0,1,0} - \mathsf{O}_{0,0,0})\lambda_2 + 2\lambda_3(\mathsf{O}_{-1,1,0} - \mathsf{O}_{-1,0,0})\right] \quad (5.94)$$

$$\lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_0^{(\ell)}}{\partial p_{0}^{(\ell-1)}} = p_0^{(0)} \sum_{\tau=1}^{n_c} \rho_\tau (n_\tau - 1)\left[(\mathsf{O}_{0,0,1} - \mathsf{O}_{0,0,0})\lambda_2 + 2\lambda_3(\mathsf{O}_{-1,0,1} - \mathsf{O}_{-1,0,0})\right] \quad (5.95)$$

where $p_{-1}^{(0)}$ and $p_0^{(0)}$ are given in (5.15) and (5.16).

Define the matrix

$$\boldsymbol{J} := \begin{bmatrix} \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} & \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{-1}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} \\ \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{-1}^{(\ell-1)}} & \lim_{\boldsymbol{p}^{(\ell-1)}\to\boldsymbol{0}} \frac{\partial p_{0}^{(\ell)}}{\partial p_{0}^{(\ell-1)}} \end{bmatrix}. \quad (5.96)$$

Let Ω be the spectral radius of $\boldsymbol{J}$, i.e., the largest magnitude of its eigenvalues. The stability condition is satisfied if and only if $\Omega < 1$.

It can be verified that if $\forall s = 1, \dots, n_c$ $\mathsf{d}_{\min,\tau} \geq 5$ then $\boldsymbol{J} = \mathbf{0}_{2\times 2}$. As a result, if all CN types have minimum distance larger or equal than 5, then the stability condition is satisfied.

5.5 Numerical Results

We provide two examples where two GLDPC code ensembles are considered. For the examples, both the iterative decoding thresholds and finite-length BER simulation results are given over the biAWGN channel.

Example 5.1. Consider the rate $R = 0.625$ regular GLDPC ensemble with VN degree $\mathsf{d}_\mathsf{v} = 2$, where all the CNs correspond to the $(32, 26)$ extended Hamming code. Using the derived DE, we obtained the iterative decoding thresholds of this ensemble under BMP and TMP where we set $\ell_{\max} = 200$ and applied different decoding methods at the CNs. We observed that for the APP SISO decoder at the CNs, the decoding threshold for TMP improves by 1.82 dB as compared to BMP. For extrinsic BDD [1, 65], the gain of TMP is 1.18 dB. We designed a length $n = 8000$ code from this ensemble via the PEG algorithm [94]. For the simulations, we set the maximum number of iterations to $\ell_{\max} = 50$. The simulation results for BMP (dashed lines) and TMP (solid lines) are depicted in Fig. 5.1 in terms of BER versus E_b/N_0. The results are provided by employing APP SISO decoding (blue lines), BDD, where we replace the j-th entry with an erasure (green lines) [1, 65] and BDD following [2] (red lines) at the CNs, where BDD is implemented in the extrinsic message passing setting. The waterfall performance of the different decoders is in agreement

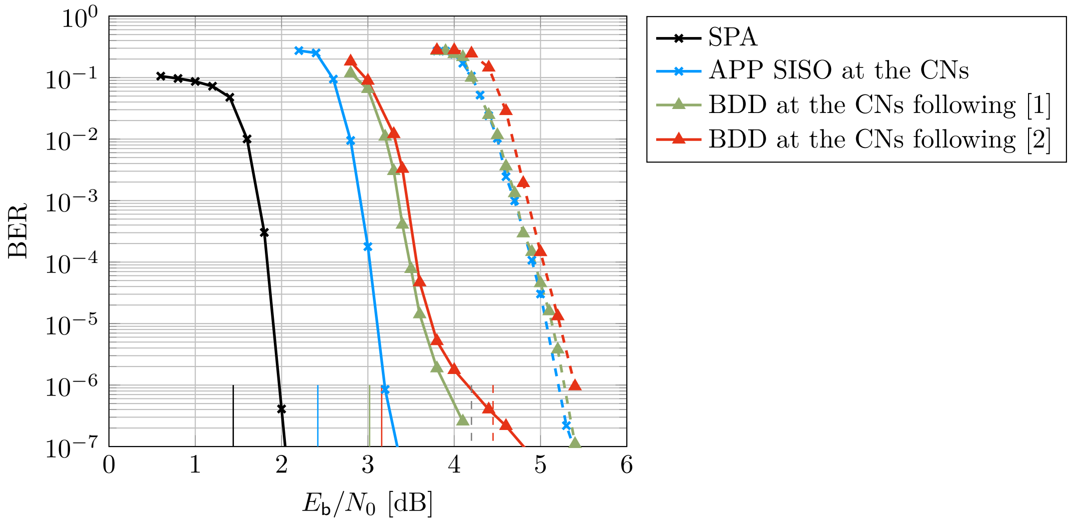


Figure 5.1: BER versus E_b/N_0 for unquantized SPA, TMP (solid lines) and BMP (dashed lines) with APP SISO algorithm, with extrinsic BDD following [1, 65] and with extrinsic BDD following [2] at the check nodes for the GLDPC code in Example 5.1. The corresponding iterative decoding thresholds are provided as vertical lines, where for the case of BMP decoding the thresholds under the APP SISO algorithm and under BDD coincide (gray, dashed line).

with the thresholds derived via DE. We observe that in this case, when using BDD at the CNs, it is better to replace the j-th entry with an erasure than with the channel message (following the method in [2]). The performance gap between the TMP decoder employing optimum APP decoding at the CNs and the one using BDD is around 0.6 dB at a BER$= 10^{-4}$, in good agreement with the DE results. Remarkably, under BMP decoding the performances obtained through APP SISO decoding and BDD are indistinguishable. The result is confirmed, in the asymptotic setting, by the DE analysis: The two algorithms yield identical numerical values for the iterative decoding threshold, up to the second decimal digit. The result may point to the fact that, under BMP decoding (and for certain selection of component codes), optimum APP decoding and BDD at the CNs deliver a similar performance. If proven to be applicable to GLDPC codes based on a wide class of component codes, this result may have some important consequences for the design of BMP decoders for GLDPC and product-like codes.[3]

Example 5.2. Consider the rate $R = 3/8$ regular GLDPC ensemble with VN degree $\mathsf{d_v} = 2$, where all the CNs correspond to the $(16, 11)$ extended Hamming code. We designed a length $n = 8000$ code from this ensemble via the PEG algorithm [94]. For the simulations,

[3]The peculiar behavior observed under BMP decoding can be subject of further investigations, targeting longer and more powerful component codes such as those adopted in optical communication systems [42].

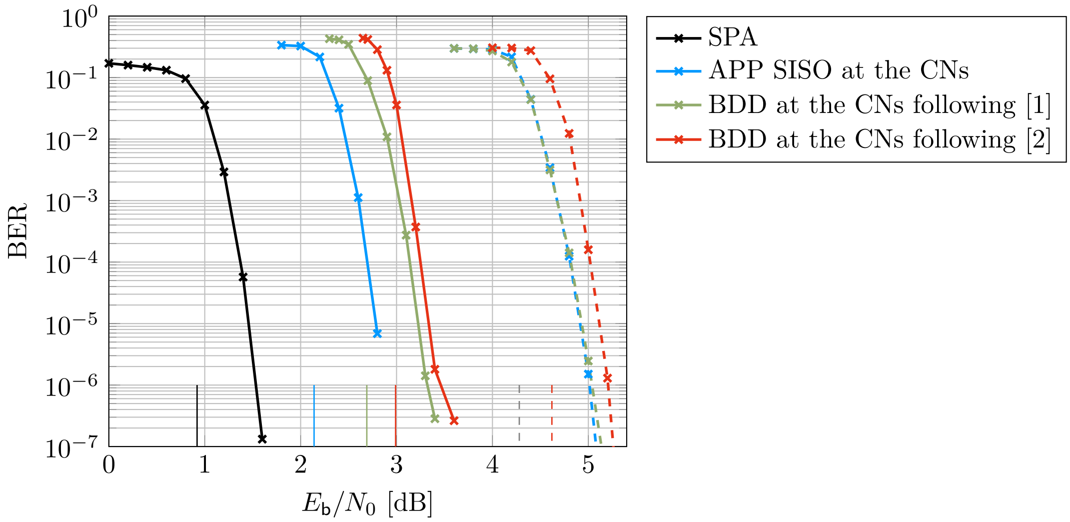


Figure 5.2: BER versus E_b/N_0 for unquantized SPA, TMP (solid lines) and BMP (dashed lines) with APP SISO algorithm, with extrinsic BDD following [1, 65] and with extrinsic BDD following [2] at the check nodes for the GLDPC code in Example 5.2. The corresponding iterative decoding thresholds are provided as vertical lines, where for the case of BMP decoding the thresholds under the APP SISO algorithm and under BDD coincide (gray, dashed line).

we set the maximum number of iterations to $\ell_{\max} = 50$. The simulation results for TMP are depicted in Fig. 5.2 in terms of BER versus E_b/N_0. The iterative decoding thresholds $(E_b/N_0)^\star$ [dB] are also depicted in the figure. It can be observed that the asymptotic DE analysis correctly predicts the finite-length waterfall performance. Here again, the APP SISO decoding and BDD at the CNs yield similar performance for BMP. Moreover, some gain can be achieved if we replace the j-th entry with an erasure [1, 65] rather than the channel message [2] for BDD at the CNs.

5.6 Appendices

5.6.1 Derivation of $\mathsf{O}_{-1,u}$ and $\mathsf{O}_{+1,u}$ in (5.20) and (5.21)

We clarify briefly the derivation of $\mathsf{O}_{-1,u}$ and $\mathsf{O}_{+1,u}$. Note that the number of errors in the received sequence is u and the number of erasures is 1. If $2u \le \mathsf{d}_{\min,\tau} - 2$, the decoder can correct the errors and erasures ($\hat{\boldsymbol{c}}$ is the all-ones vector). Thus $\mathsf{O}_{-1,u} = 0$ and $\mathsf{O}_{+1,u} = 1$. $\mathsf{O}_{-1,u}$ is the probability that given a codeword $\boldsymbol{c} \in \mathscr{C}_\tau$ of a given weight, the erased bit in the input vector corresponds to an entry where $\boldsymbol{c}$ is -1 and the other u (-1s) of the input vector are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \le \mathsf{d}_{\min,\tau} - 2$. Consider the codewords of weight

$h+1$, the number of which is A_{h+1}. The probability that the randomly selected initially erased bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$ codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_\mathsf{c}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_\mathsf{c}$ has $u-h+j$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$. The number of possibilities is $\binom{h}{h-j}\binom{n_\tau-h-1}{u-h+j}$. The probability that this occurs is given by

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j}}{\binom{n_\tau-1}{u}} \tag{5.97}$$

where $\delta := \mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) = u - h + 2j$. By summing over $0 \le \delta \le \lfloor (\mathsf{d}_{\min,\tau} - 2)/2 \rfloor$ and j, we obtain $\mathsf{O}_{-1,u}$ in (5.20).

Consider now $\mathsf{O}_{+1,u}$, which is the probability that, given a codeword $\boldsymbol{c} \in \mathscr{C}_\tau$ of a given weight, the erased bit in the input vector corresponds to an entry where $\boldsymbol{c}$ is $+1$ and the other u (-1s) of the input vector are placed such that $2\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \le \mathsf{d}_{\min,\tau} - 2$. Consider the codewords of weight h, the number of which is A_h. The probability that the randomly selected initially erased bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau - h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_\mathsf{c}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_\mathsf{c}$ has $u-(h-j)$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$.The probability is given by

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j}}{\binom{n_\tau-1}{u}} \tag{5.98}$$

where $\delta := \mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) = u - h + 2j$. By summing over $0 \le \delta \le \lfloor (\mathsf{d}_{\min,\tau} - 2)/2 \rfloor$ and j, we obtain $\mathsf{O}_{+1,u}$ in (5.21).

5.6.2 Derivation of $\mathsf{O}_{-1,u,v}$ and $\mathsf{O}_{+1,u,v}$ in (5.25) and (5.26)

We next clarify the derivation of $\mathsf{O}_{-1,u,v}$ and $\mathsf{O}_{+1,u,v}$. If $2u + v \le \mathsf{d}_{\min,\tau} - 2$, then the number of erasures in the received sequence is $v+1$ and the number of errors is u. Hence, the decoder can correct the errors and erasures. Thus, $\mathsf{O}_{-1,u,v} = 0$ and $\mathsf{O}_{+1,u,v} = 1$ for $2u + v \le \mathsf{d}_{\min,\tau} - 2$. $\mathsf{O}_{-1,u,v}$ is the probability that, given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to -1. The other (-1s) and erasures of $\boldsymbol{m}_\mathsf{c}$ are placed such that $2\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) + v \le \mathsf{d}_{\min,\tau} - 2$. Consider the codewords of weight $h+1$. The probability that the erased bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$

codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j_1-j_2$ (-1s) and j_2 erasures in $h-j_1-j_2$ and j_2 out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-h+j_1+j_2$ (-1s) in $u-(h-j_1-j_2)$ and $v-j_2$ erasures in $v-j_2$ out of the $n_\tau-h-1$ positions where $\boldsymbol{c}$ is $+1$. The number of possibilities is

$$\binom{h}{j_1, j_2, h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1, u-h+j_1+j_2, v-j_2}.$$

The probability is

$$\mathsf{F}_h = \frac{\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,\delta-j_1,v-j_2}}{\binom{n_\tau-1}{u,v,n_\tau-1-u-v}} \tag{5.99}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u-h+2j_1+j_2$. Summing over $0 \leq \delta \leq \lfloor(\mathsf{d}_{\min,\tau}-2-v)/2\rfloor$, j_1 and j_2 completes the proof.

Consider now $\mathsf{O}_{+1,u,v}$. If $2u+v \leq \mathsf{d}_{\min,\tau}-2$, then the number of erasures in the received sequence is $v+1$ and the number of errors is u. Hence, the decoder can correct the errors and erasures. $\mathsf{O}_{+1,u,v}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to $+1$. The other (-1s) and erasures of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}})+v \leq \mathsf{d}_{\min,\tau}-2$. Consider the codewords of weight h. The probability that the erased bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau-h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j_1-j_2$ (-1s) and j_2 erasures in $h-j_1-j_2$ and j_2 out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-h+j_1+j_2$ (-1s) in $u-(h-j_1-j_2)$ and $v-j_2$ erasures in $v-j_2$ out of the $n_\tau-h-1$ positions where $\boldsymbol{c}$ is $+1$ (one entry is already fixed). The number of possibilities is $\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,u-h+j_1+j_2,v-j_2}$ and the probability is

$$\mathsf{F}_h = \frac{\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,\delta-j_1,v-j_2}}{\binom{n_\tau-1}{u,v,n_\tau-1-u-v}} \tag{5.100}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u-h+2j_1+j_2$. Summing over $0 \leq \delta \leq \lfloor(\mathsf{d}_{\min,\tau}-2-v)/2\rfloor$, j_1 and j_2 completes the proof.

5.6.3 Derivation of $\mathsf{K}_{-1,u}, \mathsf{K}_{+1,u}, \mathsf{W}_{-1,u}$ and $\mathsf{W}_{+1,u}$ in (5.31)-(5.34)

We clarify briefly the derivation of $\mathsf{K}_{-1,u}, \mathsf{K}_{+1,u}, \mathsf{W}_{-1,u}, \mathsf{W}_{+1,u}$ and $\mathsf{W}_{+1,u}$. If $u \leq t_\tau-1$, then the decoder can correct all errors and the estimated codeword is the all-ones vector. Hence $\mathsf{K}_{-1,u}=0$ and $\mathsf{K}_{+1,u}=1$. $\mathsf{K}_{-1,u}$ is the probability that there exists a codeword

$\boldsymbol{c}$ with $\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \leq t_\tau$ that has a -1 in the randomly chosen position, where $\boldsymbol{m}_\mathsf{c}$ is the received sequence and $\boldsymbol{m}_\mathsf{c}$ has a -1 in the randomly chosen position. Thus, $\mathsf{K}_{-1,u}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to -1. The other (-1s) of $\boldsymbol{m}_\mathsf{c}$ are placed such that $\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \leq t_\tau$. Consider the codewords of weight $h+1$, the number of which is A_{h+1}. The probability that the randomly selected initially in error bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$ codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_\mathsf{c}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_\mathsf{c}$ has $u-(h-j)$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$. The number of possibilities is $\binom{h}{h-j}\binom{n_\tau-h-1}{u-h+j}$. The probability is

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j}}{\binom{n_\tau-1}{u}}$$

where $\delta := \mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) = u - h + 2j$. By summing over $0 \leq \delta \leq t_\tau$ and j, we obtain $\mathsf{K}_{-1,u}$ in (5.31). Consider now $\mathsf{K}_{+1,u}$, which is the probability that there exists a codeword $\boldsymbol{c}$ with $\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \leq t_\tau$ that has a $+1$ in the randomly chosen position and $\boldsymbol{m}_\mathsf{c}$ has a -1 in the randomly chosen position. Thus, $\mathsf{K}_{+1,u}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to $+1$. The other (-1s) of $\boldsymbol{m}_\mathsf{c}$ are placed such that $\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \leq t_\tau$. Consider the codewords of weight h, the number of which is A_h. The probability that the randomly selected initially in error bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau - h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_\mathsf{c}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_\mathsf{c}$ has $u-(h-j)$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$ (one entry is already fixed). The number of possibilities is $\binom{h}{h-j}\binom{n_\tau-h-1}{u-h+j}$ and probability is

$$\mathsf{P}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j-1}}{\binom{n_\tau-1}{u}}$$

where $\delta := \mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) = u - h + 2j + 1$. By summing over $1 \leq \delta \leq t_\tau$ and j, we obtain $\mathsf{K}_{+1,u}$ in (5.32).

If $u \leq t_\tau$, then the decoder can correct all errors and the estimated codeword is the all-ones vector. Hence $\mathsf{W}_{-1,u} = 0$ and $\mathsf{W}_{+1,u} = 1$. $\mathsf{W}_{-1,u}$ is the probability that there exists a codeword $\boldsymbol{c}$ with $\mathsf{d}_\mathrm{H}(\boldsymbol{c}, \boldsymbol{m}_\mathsf{c}) \leq t_\tau$ that has a -1 in the randomly chosen position and $\boldsymbol{m}_\mathsf{c}$ has a $+1$ in the randomly chosen position . Thus, $\mathsf{W}_{-1,u}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to -1. The other (-1s) of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) \le t_\tau$. Consider the codewords of weight $h+1$, the number of which is A_{h+1}. The probability that the randomly selected initially correct bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$ codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-(h-j)$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$. The number of possibilities is $\binom{h}{h-j}\binom{n_\tau-h-1}{u-h+j}$. The probability is

$$\mathsf{P}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j-1}}{\binom{n_\tau-1}{u}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u-h+2j+1$. By summing over $1 \le \delta \le t_\tau$ and j, we obtain $\mathsf{W}_{-1,u}$ in (5.33).

Consider now $\mathsf{W}_{+1,u}$ which is the probability that there exists a codeword $\boldsymbol{c}$ with $\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) \le t_\tau$ that has a $+1$ in the randomly chosen position and $\boldsymbol{m}_{\mathsf{c}}$ has a $+1$ in the randomly chosen position. Thus, $\mathsf{W}_{+1,u}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ is equal to $+1$. The other (-1s) of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) \le t_\tau$. Consider the codewords of weight h, the number of which is A_h. The probability that the randomly selected initially correct bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau - h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j$ (-1s) in $h-j$ out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-(h-j)$ (-1s) in $u-(h-j)$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$ (one entry is already fixed). The number of possibilities is $\binom{h}{h-j}\binom{n_\tau-h-1}{u-h+j}$. The probability is

$$\mathsf{I}_h = \frac{\binom{h}{h-j}\binom{n_\tau-h-1}{\delta-j}}{\binom{n_\tau-1}{u}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u-h+2j$. By summing over $0 \le \delta \le t_\tau$ and j, we obtain $\mathsf{W}_{+1,u}$ in (5.34).

5.6.4 Derivation of $\mathsf{K}_{-1,u,v}, \mathsf{K}_{+1,u,v}, \mathsf{W}_{-1,u,v}, \mathsf{W}_{+1,u,v}, \mathsf{O}_{-1,u,v}, \mathsf{O}_{+1,u,v}$ in (5.46)-(5.51)

If $2u+v \le \mathsf{d}_{\min,\tau}-3$, the number of erasures in the received sequence is v and number of errors is $u+1 \le \lfloor(\mathsf{d}_{\min,\tau}-1-v)/2\rfloor$. Hence, the decoder can correct the errors and erasures. Thus, $\mathsf{K}_{-1,u,v}=0$ and $\mathsf{K}_{+1,u,v}=1$ for $2u+v \le \mathsf{d}_{\min,\tau}-3$. $\mathsf{K}_{-1,u,v}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ equals -1. The other (-1s) and erasures of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c},\boldsymbol{m}_{\mathsf{c}})+v \le \mathsf{d}_{\min,\tau}-1$. Consider the codewords of weight $h+1$. The probability that the randomly selected initially in error bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$ codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j_1-j_2$ (-1s) in $h-j_1-j_2$ and j_2 erasures in j_2 out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-(h-j_1-j_2)$ (-1s) in $u-(h-j_1-j_2)$ and $v-j_2$ erasures in $v-j_2$ out of the $n_\tau-h-1$ positions where $\boldsymbol{c}$ is $+1$. The number of possibilities is $\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,u-h+j_1+j_2,v-j_2}$. The probability is

$$\mathsf{F}_h = \frac{\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,\delta-j_1,v-j_2}}{\binom{n_\tau-1}{u,v,n_\tau-1-u-v}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c},\boldsymbol{m}_{\mathsf{c}}) = u-h+2j_1+j_2$. Summing over $0 \le \delta \le \lfloor(\mathsf{d}_{\min,\tau}-1-v)/2\rfloor$, j_1 and j_2 completes the proof.

$\mathsf{K}_{+1,u,v}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ equals $+1$. The other (-1s) and erasures of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c},\boldsymbol{m}_{\mathsf{c}})+v \le \mathsf{d}_{\min,\tau}-1$. Consider the codewords of weight h. The probability that the randomly selected initially in error bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau-h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h-j_1-j_2$ (-1s) in $h-j_1-j_2$ and j_2 erasures in j_2 out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u-(h-j_1-j_2)$ (-1s) in $u-(h-j_1-j_2)$ and $v-j_2$ erasures in $v-j_2$ out of the $n_\tau-h-1$ positions where $\boldsymbol{c}$ is $+1$ (one entry is already fixed). The number of possibilities is $\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,u-h+j_1+j_2,v-j_2}$. The probability is

$$\mathsf{S}_h = \frac{\binom{h}{j_1,j_2,h-j_1-j_2}\binom{n_\tau-h-1}{n_\tau-u-v-1-j_1,\delta-1-j_1,v-j_2}}{\binom{n_\tau-1}{u,v,n_\tau-1-u-v}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c},\boldsymbol{m}_{\mathsf{c}}) = u-h+2j_1+j_2+1$. Summing over $1 \le \delta \le \lfloor(\mathsf{d}_{\min,\tau}-1-v)/2\rfloor$, j_1 and j_2 completes the proof.

Consider now $\mathsf{W}_{-1,u,v}$ and $\mathsf{W}_{+1,u,v}$. If $2u+v \le \mathsf{d}_{\min,\tau}-1$, the number of erasures in the received sequence is v, and the number of errors is $u \le \lfloor (\mathsf{d}_{\min,\tau} - 1 - v)/2 \rfloor$. Hence, the decoder can correct the errors and erasures. Thus $\mathsf{W}_{-1,u,v} = 0$ and $\mathsf{W}_{+1,u,v} = 1$. $\mathsf{W}_{-1,u,v}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ equals -1. The other (-1s) and erasures of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) + v \le \mathsf{d}_{\min,\tau} - 1$. Consider the codewords of weight $h+1$. The probability that the randomly selected initially correct bit is chosen among the codeword bit positions that are -1 is $(h+1)/n_\tau$. For a given weight $h+1$ codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h - j_1 - j_2$ (-1s) in $h - j_1 - j_2$ and j_2 erasures in j_2 out of the h entries where $\boldsymbol{c}$ is -1 (one entry is already fixed). Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u - (h - j_1 - j_2)$ (-1s) in $u - (h - j_1 - j_2)$ and $v - j_2$ erasures in $v - j_2$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$. The probability is

$$\mathsf{S}_h = \frac{\binom{h}{j_1, j_2, h-j_1-j_2}\binom{n_\tau - h - 1}{n_\tau - u - v - 1 - j_1, \delta - 1 - j_1, v - j_2}}{\binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u - h + 2j_1 + j_2 + 1$. Summing over $1 \le \delta \le \lfloor (\mathsf{d}_{\min,\tau} - 1 - v)/2 \rfloor$, j_1 and j_2 completes the proof.

$\mathsf{W}_{+1,u,v}$ is the probability that given a codeword $\boldsymbol{c}$ of a given weight, the randomly selected bit for the received sequence corresponds to an entry where $\boldsymbol{c}$ equals $+1$. The other (-1s) and erasures of $\boldsymbol{m}_{\mathsf{c}}$ are placed such that $2\mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) + v \le \mathsf{d}_{\min,\tau} - 1$. Consider the codewords of weight h. The probability that the randomly selected initially correct bit is chosen among the codeword bit positions that are $+1$ is $(n_\tau - h)/n_\tau$. For a given weight h codeword $\boldsymbol{c}$, suppose $\boldsymbol{m}_{\mathsf{c}}$ has $h - j_1 - j_2$ (-1s) in $h - j_1 - j_2$ and j_2 erasures in j_2 out of the h entries where $\boldsymbol{c}$ is -1. Thus, $\boldsymbol{m}_{\mathsf{c}}$ has $u - (h - j_1 - j_2)$ (-1s) in $u - (h - j_1 - j_2)$ and $v - j_2$ erasures in $v - j_2$ out of the $n_\tau - h - 1$ positions where $\boldsymbol{c}$ is $+1$. The probability is

$$\mathsf{F}_h = \frac{\binom{h}{j_1, j_2, h-j_1-j_2}\binom{n_\tau - h - 1}{n_\tau - u - v - 1 - j_1, \delta - j_1, v - j_2}}{\binom{n_\tau - 1}{u, v, n_\tau - 1 - u - v}}$$

where $\delta := \mathsf{d}_{\mathrm{H}}(\boldsymbol{c}, \boldsymbol{m}_{\mathsf{c}}) = u - h + 2j_1 + j_2$. Summing over $0 \le \delta \le \lfloor (\mathsf{d}_{\min,\tau} - 1 - v)/2 \rfloor$, j_1 and j_2 completes the proof.

The derivations of $\mathsf{O}_{-1,u,v}$ and $\mathsf{O}_{+1,u,v}$ are given in Appendix 5.6.2.

6

Trapping and Absorbing Set Enumerators for Binary LDPC Code Ensembles

Trapping sets [35, 36] and (fully) absorbing sets [37, 111] play a fundamental role in the error floor performance (under iterative decoding) of LDPC codes [112–114] especially for quantized decoders, such as the algorithms in Chapter 4. Enumerating the trapping sets of a specific LDPC code graph is a formidable problem (see, e.g., [43–45]). The difficulty can be circumvented by analyzing the average trapping set enumerator of an LDPC code ensemble, rather than analyzing a specific code. This is reasonable if the weight and trapping set enumerators of a code drawn uniformly at random from the ensemble are close to the average enumerators of the ensemble with high probability. The author of [115, 116] derived an asymptotic lower bound on the probability that the weight and stopping set enumerators of a random code from the binary LDPC code ensemble is close to the ensemble average. The approach relies on the second moment method using the variances of the weight and stopping set distributions. We extended in [64] the method in [115], to the weight and trapping set distributions of non-binary LDPC code ensembles. Following [117], we derived upper bounds on the typical minimum distance and the relative minimum Δ-trapping set sizes for binary and non-binary regular LDPC code ensembles. In [48], a characterization of the (asymptotic) trapping set properties of regular/irregular unstructured LDPC ensembles was obtained based on random matrix enumeration methods. In this chapter, we derive the finite-length and asymptotic (elementary) trapping and (fully) absorbing set enumerators for binary unstructured and protograph-based LDPC codes. Numerical results illustrate how the proposed enumeration technique can be used to estimate the error floor performance for LDPC codes.

6.1 Preliminaries

Let $G = (\mathcal{V} \cup \mathcal{C}, \mathscr{E})$ be a Tanner graph of a binary LDPC code, where $\mathcal{V}$ ($\mathcal{C}$) is the set of VNs (CNs) and $\mathscr{E}$ is the set of edges. A VN is called correct if the corresponding value is zero and it is called corrupt if it is one. Consider a set $\mathcal{I} \subseteq \mathcal{V}$ of corrupt VNs. We denote by $\mathcal{N}(\mathcal{I})$ the set of its neighboring CNs. Further, we denote by $\mathcal{U}(\mathcal{I})$ the set of CNs in $\mathcal{N}(\mathcal{I})$ that are connected to $\mathcal{I}$ an odd number of times (unsatisfied CNs) and $\mathcal{S}(\mathcal{I})$ the set of CNs in $\mathcal{N}(\mathcal{I})$ that are connected to $\mathcal{I}$ an even number of times (satisfied CNs).

Definition 6.1 (Trapping set)**.** An (a, b) TS $\mathcal{T}_{a,b}$ is set $\mathcal{I}$ of a VNs such that $\mathcal{U}(\mathcal{I})$ contains b CNs [48].

Definition 6.2 (Elementary trapping set)**.** An elementary trapping set (ETS) $\mathcal{T}^{\mathrm{E}}_{a,b}$ is a TS where each CN $\mathsf{c} \in \mathcal{S}(\mathcal{I})$ is connected to two VNs in $\mathcal{I}$ and each CN $\mathsf{c} \in \mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$.

Definition 6.3 (Absorbing set)**.** An (a, b) AS $\mathcal{A}_{a,b}$ is a trapping set with the additional property that each VN $\mathsf{v} \in \mathcal{I}$ has strictly fewer neighboring CNs from $\mathcal{U}(\mathcal{I})$ than from $\mathcal{S}(\mathcal{I})$ [37].

Definition 6.4 (Fully absorbing set)**.** An (a, b) FAS $\mathcal{F}_{a,b}$ is a trapping set with the additional property that each VN $\mathsf{v} \in \mathcal{V}$ has strictly fewer neighboring CNs from $\mathcal{U}(\mathcal{I})$ than from $\mathcal{C} \setminus \mathcal{U}(\mathcal{I})$ [37].

Definition 6.5 (Elementary (fully) absorbing set)**.** An EAS $\mathcal{A}^{\mathrm{E}}_{a,b}$ (elementary fully absorbing set (EFAS) $\mathcal{F}^{\mathrm{E}}_{a,b}$) is an AS (FAS) where each CN $\mathsf{c} \in \mathcal{S}(\mathcal{I})$ is connected to two VNs in $\mathcal{I}$ and each CN $\mathsf{c} \in \mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$ [37].

The normalized logarithmic asymptotic distribution of (elementary) trapping or (fully) absorbing sets for an LDPC code ensemble for $a = \theta n$ and $b = \gamma n$ is defined by

$$G(\theta, \gamma) := \lim_{n\to\infty} \frac{1}{n} \ln \left(\mathsf{E}(\theta n, \gamma n)\right) \tag{6.1}$$

where $\mathsf{E}(\theta n, \gamma n)$ is the average number of $(\theta n, \gamma n)$ (elementary) trapping or (fully) absorbing sets in the Tanner graph of a random code from the ensemble.

Definition 6.6 (Relative minimum Δ-trapping set size)**.** For a fixed ratio $\Delta = b/a$, the second zero crossing of the normalized logarithmic asymptotic distribution of trapping sets (the first one is zero), if it exists, is called the *relative minimum Δ-trapping set size* that we denote by $\theta^{\star}_{\mathrm{TS}}$ [49].

Analogously to the *relative minimum Δ-trapping set size* [49], the *relative minimum Δ-absorbing set size* was introduced in [37].

Definition 6.7 (Relative minimum Δ-absorbing set size)**.** For a fixed ratio $\Delta = b/a$, the second zero crossing of the normalized logarithmic asymptotic distribution of absorbing sets (the first one is zero), if it exists, is called the *relative minimum Δ-absorbing set size* that we denote by $\theta^{\star}_{\mathrm{AS}}$.

Analogously to the *relative minimum Δ-trapping set size* [49], the *relative minimum Δ-fully absorbing set size* was introduced in [37].

Definition 6.8 (Relative minimum Δ-fully absorbing set size)**.** For a fixed ratio $\Delta = b/a$, the second zero crossing of the normalized logarithmic asymptotic distribution of fully absorbing sets (the first one is zero), if it exists, is called the *relative minimum Δ-fully absorbing set size* that we denote by $\theta^{\star}_{\mathrm{FAS}}$.

In the following, we derive the finite-length and the normalized logarithmic asymptotic (elementary) trapping and (fully) absorbing set enumerators for binary LDPC code ensembles.

6.2 Trapping and Absorbing Set Enumerators for Unstructured Ensembles

In this section, we derive the finite-length (elementary) trapping and (fully) absorbing set enumerators for binary unstructured LDPC codes and we present an analytical method for evaluating the normalized logarithmic asymptotic distributions of (elementary) trapping and (fully) absorbing sets. First, we briefly review the random matrix enumeration approach, which was applied in [48] to obtain the asymptotic enumerators for (elementary) TSs and stopping sets (SSs) for binary irregular LDPC code ensembles and in [37, 77] for the weight distribution and the (elementary) (fully) ASs of regular LDPC code ensembles, respectively. We first follow this approach and extend the analysis to obtain the (elementary) AS and FAS enumerators of irregular LDPC code ensembles. Then, we provide alternative derivations of the (elementary) trapping and (fully) absorbing set enumerators for binary unstructured LDPC codes. The alternative derivation relies on the generating function approach, already adopted to analyze weight and stopping set enumerators of unstructured (generalized) LDPC ensembles [46, 78–80, 118].

6.2.1 Review of the Existing Approach

In this section, we review the TS and ETS asymptotic enumerators derived in [48]. We extended the approach to obtain (elementary) AS and FAS enumerators for irregular LDPC code ensembles.

The parity-check matrix of each code from $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ contains $\Lambda_j n$ columns of weight j and $\mathrm{P}_i m$ rows of weight i. From Theorem 3.1, the cardinality of the set containing all $m \times n$ binary matrices with these row and column weights is

$$|\mathcal{H}_n^{\Lambda,\mathrm{P}}| = \frac{f!}{\prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} (i!)^{\mathrm{P}_i m} \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} (j!)^{\Lambda_j n}} \exp\left(- \frac{mn \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i(i-1)\mathrm{P}_i \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j(j-1)\Lambda_j}{2f^2} \right) \times \left(1 + o(n^{-1+\delta})\right) \tag{6.2}$$

for $\delta > 0$, with $f = n\bar{\mathsf{d}}_\mathsf{v} = m\bar{\mathsf{d}}_\mathsf{c}$.

Trapping Set Distribution

In this section, we review the derivation of the asymptotic distribution of TSs for the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ for $a = \theta n$ and $b = \gamma n$ from [48].

Theorem 6.1. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs is

$$G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j H_\mathsf{b}\left(\frac{\theta_j^\star}{\Lambda_j}\right) + \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)\star}}{\xi \mathrm{P}_i}, \dots, \frac{\tilde{\alpha}_i^{(i)\star}}{\xi \mathrm{P}_i}\right) + \sum_{j=0}^{i} \tilde{\alpha}_j^{(i)\star} \ln \binom{i}{j} \right] - \bar{\mathsf{d}}_\mathsf{v} H_\mathsf{b}\left(\frac{\tilde{\theta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}\right) \tag{6.3}$$

where

$$\tilde{\theta}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i A_3 \mathrm{P}_i \frac{(1+A_3)^{i-1}(1+A_2) + (1-A_3)^{i-1}(1-A_2)}{(1+A_3)^i(1+A_2) - (1-A_3)^i(1-A_2)} \tag{6.4}$$

$$\theta_j^\star = \Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j}, \quad \forall j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\} \tag{6.5}$$

$$\tilde{\alpha}_j^{(i)\star} = \begin{cases} \frac{2\xi \mathrm{P}_i \binom{i}{j} A_2 A_3^j}{(1+A_3)^i(1+A_2)-(1-A_3)^i(1-A_2)} & j \text{ is even} \\ \frac{2\xi \mathrm{P}_i \binom{i}{j} A_3^j}{(1+A_3)^i(1+A_2)-(1-A_3)^i(1-A_2)} & j \text{ is odd} \end{cases} \quad \forall i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}, j \in \{0, \ldots, i\} \tag{6.6}$$

and A_1, A_2, A_3 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j \Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j} = \tilde{\theta}^\star \tag{6.7}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j} = \theta \tag{6.8}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{(1+A_3)^i - (1-A_3)^i}{(1+A_3)^i(1+A_2) - (1-A_3)^i(1-A_2)} = \frac{\gamma}{\xi}. \tag{6.9}$$

Elementary Trapping Set Distribution

In this section, we review the derivation of the asymptotic distribution of ETSs for the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ for $a = \theta n$ and $b = \gamma n$ from [48].

Theorem 6.2. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ETSs is

$$\begin{aligned} G_{\mathrm{ETS}}^{\Lambda,\mathrm{P}}(\theta, \gamma) = & \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j H_\mathsf{b}\left(\frac{\theta_j^\star}{\Lambda_j}\right) + \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)\star}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_1^{(i)\star}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_2^{(i)\star}}{\xi \mathrm{P}_i}\right) + \sum_{j=0}^{2} \tilde{\alpha}_j^{(i)\star} \ln \binom{i}{j}\right] \\ & - \bar{\mathsf{d}}_\mathsf{v} H_\mathsf{b}\left(\frac{\tilde{\theta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}\right) \end{aligned} \tag{6.10}$$

where

$$\tilde{\theta}^\star = \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \xi \mathrm{P}_i \frac{i(i-1) A_3^2 A_2}{A_2 + i A_3 + \binom{i}{2} A_3^2 A_2} + \gamma \tag{6.11}$$

$$\theta_j^\star = \Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j}, \quad \forall j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\} \tag{6.12}$$

for all $i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}$

$$\tilde{\alpha}_0^{(i)\star} = \frac{\xi \mathrm{P}_i A_2}{A_2 + i A_3 + \binom{i}{2} A_3^2 A_2} \tag{6.13}$$

$$\tilde{\alpha}_1^{(i)\star} = \frac{i\xi \mathrm{P}_i A_3}{A_2 + iA_3 + \binom{i}{2} A_3^2 A_2} \tag{6.14}$$

$$\tilde{\alpha}_2^{(i)\star} = \frac{\binom{i}{2} \xi \mathrm{P}_i A_3^2 A_2}{A_2 + iA_3 + \binom{i}{2} A_3^2 A_2} \tag{6.15}$$

and A_1, A_2, A_3 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j\Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j} = \tilde{\theta}^\star \tag{6.16}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{(\tilde{\theta}^\star)^j}{(\tilde{\theta}^\star)^j + A_1 A_3^j (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j} = \theta \tag{6.17}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{iA_3}{A_2 + iA_3 + \binom{i}{2} A_3^2 A_2} = \frac{\gamma}{\xi}. \tag{6.18}$$

Absorbing Set Distribution

The authors of [37] derived the asymptotic distribution of ASs for regular LDPC code ensembles. We extended in [52] the analysis to irregular LDPC code ensembles.

Theorem 6.3. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs is

$$\begin{aligned} G_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = &\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\Lambda_j H_\mathsf{b}\left(\frac{\theta_j^\star}{\Lambda_j}\right) + \theta_j^\star H\left(\frac{\tilde{\beta}^{(j)\star}_{\lfloor \frac{j}{2} \rfloor+1}}{\theta_j^\star}, \ldots, \frac{\tilde{\beta}_j^{(j)\star}}{\theta_j^\star}\right) + \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)\star} \ln \binom{j}{l} \right] \\ &+ \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)\star}}{\xi \mathrm{P}_i}, \ldots, \frac{\tilde{\alpha}_i^{(i)\star}}{\xi \mathrm{P}_i}\right) + \sum_{j=0}^{i} \tilde{\alpha}_j^{(i)\star} \ln \binom{i}{j} \right] \\ &- \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde{\beta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\tilde{\theta}^\star - \tilde{\beta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}\right) \end{aligned} \tag{6.19}$$

where

$$\tilde{\theta}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i \mathrm{P}_i A_3 \frac{A_2 A_4 \left[(1+A_3A_4)^{i-1} - (1-A_3A_4)^{i-1}\right] + (1+A_3)^{i-1} + (1-A_3)^{i-1}}{A_2[(1+A_3A_4)^i + (1-A_3A_4)^i] + (1+A_3)^i - (1-A_3)^i} \tag{6.20}$$

$$\theta_j^\star = \Lambda_j \frac{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l}{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l + (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j A_1 A_3^j} \tag{6.21}$$

$$\tilde{\alpha}_j^{(i)\star} = \begin{cases} \frac{2\xi \mathrm{P}_i \binom{i}{j} A_2 A_3^j A_4^j}{A_2(1+A_3A_4)^i + A_2(1-A_3A_4)^i + (1+A_3)^i - (1-A_3)^i} & j \text{ is even} \\ \frac{2\xi \mathrm{P}_i \binom{i}{j} A_3^j}{A_2(1+A_3A_4)^i + A_2(1-A_3A_4)^i + (1+A_3)^i - (1-A_3)^i} & j \text{ is odd} \end{cases} \tag{6.22}$$

$$\tilde{\beta}_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l (\tilde{\theta}^\star - \tilde{\beta}^\star)^j}{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l'=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l'}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^{l'} + (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j A_1 A_3^j} \tag{6.23}$$

$$\tilde{\beta}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i \mathrm{P}_i \frac{A_2 A_3 A_4 \left[(1+A_3A_4)^{i-1} - (1-A_3A_4)^{i-1}\right]}{A_2(1+A_3A_4)^i + A_2(1-A_3A_4)^i + (1+A_3)^i - (1-A_3)^i} \tag{6.24}$$

where (6.21) holds for all $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}$, (6.22) holds for all $i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}, j \in \{0, \ldots, i\}$ and (6.23) holds for all $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}, l \in \{\lfloor j/2 \rfloor + 1, \ldots, j\}$ and A_1, A_2, A_3, A_4 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j \Lambda_j \frac{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l}{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l + (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j A_1 A_3^j} = \tilde{\theta}^\star \tag{6.25}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l}{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l + (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j A_1 A_3^j} = \theta \tag{6.26}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{(1+A_3)^i - (1-A_3)^i}{A_2(1+A_3A_4)^i + A_2(1-A_3A_4)^i + (1+A_3)^i - (1-A_3)^i} = \frac{\gamma}{\xi} \tag{6.27}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} \Lambda_j l \frac{\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l (\tilde{\theta}^\star - \tilde{\beta}^\star)^j}{(\tilde{\theta}^\star - \tilde{\beta}^\star)^j \sum\limits_{l'=\lfloor \frac{j}{2} \rfloor + 1}^{j} \binom{j}{l'}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^{l'} + (\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star)^j A_1 A_3^j} = \tilde{\beta}^\star. \tag{6.28}$$

Proof. We write the transpose of the parity-check matrix as

$$\boldsymbol{H}^\mathsf{T} = \left[\begin{array}{c|c} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \hline \multicolumn{2}{c}{\boldsymbol{M}_3} \end{array}\right] \tag{6.29}$$

where $\boldsymbol{M}_1$ is an $a \times (m-b)$ binary matrix representing the subgraph of the Tanner graph containing the VNs in $\mathcal{A}_{a,b}$ ($\mathcal{A}_{a,b}^\mathrm{E}$) and the CNs that are connected to $\mathcal{A}_{a,b}$ ($\mathcal{A}_{a,b}^\mathrm{E}$) an even number of times (including zero), $\boldsymbol{M}_2$ is an $a \times b$ binary matrix corresponding to the subgraph of the Tanner graph containing the VNs in $\mathcal{A}_{a,b}$ ($\mathcal{A}_{a,b}^\mathrm{E}$) and the CNs that are connected to $\mathcal{A}_{a,b}$ ($\mathcal{A}^{\mathrm{E}}_{a,b}$) an odd number of times, and $\boldsymbol{M}_3$ is an $(n-a)\times m$ binary matrix representing the remainder of the Tanner graph [37]. Note that the columns of $\boldsymbol{M}_1$ have even weights and the ones of $\boldsymbol{M}_2$ have odd weights. We use $\boldsymbol{\theta}=(\theta_1,\dots,\theta_{\mathsf{d}_\mathsf{v}^{\max}})$, where $n\theta_j$ represents the number of VNs of degree j in $\mathcal{A}_{a,b}$ ($\mathcal{A}^{\mathrm{E}}_{a,b}$), i.e., the number of rows of weight j in the submatrix $[\boldsymbol{M}_1\,|\,\boldsymbol{M}_2]$. Note that $\sum_j \theta_j=\theta$. We use for $i\in\{1,\dots,\mathsf{d}_\mathsf{c}^{\max}\}$, the vector $\boldsymbol{\alpha}^{(i)}=(\alpha_0^{(i)},\dots,\alpha_i^{(i)})$, where $\alpha_j^{(i)}$ is the number of columns in $\boldsymbol{H}^\mathsf{T}$ of weight i whose first a entries sum to j, where $j\in\{0,1,2\}$ for $\mathcal{A}^{\mathrm{E}}_{a,b}$ and $j\in\{0,\dots,i\}$ for $\mathcal{A}_{a,b}$. Clearly, it holds for all $i\in\{1,\dots,\mathsf{d}_\mathsf{c}^{\max}\}$,

$$\sum_{j=0}^{i}\alpha_j^{(i)}=n\xi\mathrm{P}_i. \tag{6.30}$$

Since there are b CNs that are connected an odd number of times to the VNs in $\mathcal{T}_{a,b}$, we have

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}}\sum_{\substack{j=0\\ j\text{ is odd}}}^{i}\alpha_j^{(i)}=b. \tag{6.31}$$

Similarly, we introduce for $j\in\{1,\dots,\mathsf{d}_\mathsf{v}^{\max}\}$, $\boldsymbol{\beta}^{(j)}=(\beta^{(j)}_{\lfloor\frac{j}{2}\rfloor+1},\dots,\beta^{(j)}_j)$, where $\beta_l^{(j)}$ represents the number of rows in $[\boldsymbol{M}_1\,|\,\boldsymbol{M}_2]$ of weight j whose first $m-b$ entries sum to $l\in\{\lfloor j/2\rfloor+1,\dots,j\}$. It holds for all $j\in\{1,\dots,\mathsf{d}_\mathsf{v}^{\max}\}$

$$\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\beta_l^{(j)}=n\theta_j. \tag{6.32}$$

We define $\mathcal{M}_l$ as the set of binary matrices with the same weight vectors as $\boldsymbol{M}_l$ for $l=1,2,3$ and the set $\mathcal{M}$ containing all $n\times m$ binary matrices with the structure shown in (6.29) and where $\boldsymbol{M}_l\in\mathcal{M}_l$ for $l=1,2,3$.

Consider the matrix $\boldsymbol{M}_1$. It contains, for each $j\in\{1,\dots,\mathsf{d}_\mathsf{v}^{\max}\}$, $\beta_l^{(j)}$ rows of weight $l\in\{\lfloor j/2\rfloor+1,\dots,j\}$ and, for each $i\in\{1,\dots,\mathsf{d}_\mathsf{c}^{\max}\}$, $\alpha_j^{(i)}$ columns of weight j where $j\in\{0,\dots,i\}$ and j is even. The number of ones in the matrix $\boldsymbol{M}_1$ is

$$f_1=\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}}\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}l\beta_l^{(j)}=\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}}\sum_{\substack{j=0\\ j\text{ is even}}}^{i}j\alpha_j^{(i)}. \tag{6.33}$$

From Theorem 3.1, the cardinality of $\mathcal{M}_1$, for $\delta_1 > 0$, is

$$|\mathcal{M}_1| = \frac{f_1!}{\prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \prod_{l=\lfloor \frac{j}{2} \rfloor +1}^{j} (l!)^{\beta_l^{(j)}} \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \prod_{\substack{j=0 \\ j \text{ is even}}}^{i} (j!)^{\alpha_j^{(i)}}} (1 + o(n^{-1+\delta_1})) \times \\ \exp\left[-\frac{1}{2f_1^2} \left(\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor +1}^{j} (l-1) l \beta_l^{(j)} \right) \left(\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{\substack{j=0 \\ j \text{ is even}}}^{i} (j-1) j \alpha_j^{(i)} \right) \right]. \tag{6.34}$$

Consider now the matrix $\boldsymbol{M}_2$. For each $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}$, there are $\beta_l^{(j)}$ rows of weight $j - l$ and all columns have an odd weight. The number of ones in $\boldsymbol{M}_2$ is

$$f_2 = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor +1}^{j} (j-l) \beta_l^{(j)} = \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{\substack{j=0 \\ j \text{ is odd}}}^{i} j \alpha_j^{(i)}. \tag{6.35}$$

Note that $f_1 + f_2$ is the total number of ones in the submatrix $[\boldsymbol{M}_1 \,|\, \boldsymbol{M}_2]$, which is equal to

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{j=0}^{i} j \alpha_j^{(i)} = n\tilde{\theta} \tag{6.36}$$

where $\tilde{\theta} = \sum_j j\theta_j$. The cardinality of $\mathcal{M}_2$, for $\delta_2 > 0$, is then

$$|\mathcal{M}_2| = \frac{(n\tilde{\theta} - f_1)!}{\prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \prod_{l=\lfloor \frac{j}{2} \rfloor +1}^{j} ((j-l)!)^{\beta_l^{(j)}} \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \prod_{\substack{j=0 \\ j \text{ is odd}}}^{i} ((j!)^{\alpha_j^{(i)}}} (1 + o(n^{-1+\delta_2})) \times \\ \exp\left[\frac{-1}{2(n\tilde{\theta} - f_1)^2} \left(\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor +1}^{j} (j-l-1)(j-l) \beta_l^{(j)} \right) \left(\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{\substack{j=0 \\ j \text{ is odd}}}^{i} (j-1) j \alpha_j^{(i)} \right) \right]. \tag{6.37}$$

The matrix $\boldsymbol{M}_3$ has $n(\Lambda_j - \theta_j)$ rows of weight j for each $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}$ and $\alpha_j^{(i)}$ columns of weight $i - j$, where $j \in \{0, \ldots, i\}$ and $i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}$. The number of ones in $\boldsymbol{M}_3$ is

$$f_3 = n \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j(\Lambda_j - \theta_j) = \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{j=0}^{i} \alpha_j^{(i)} (i - j) = n\bar{\mathsf{d}}_\mathsf{v} - n\tilde{\theta}. \tag{6.38}$$

From Theorem 3.1, the cardinality of $\mathcal{M}_3$, for $\delta_3 > 0$, is

$$|\mathcal{M}_3| = \frac{(n\bar{\mathsf{d}}_{\mathsf{v}} - n\tilde{\theta})!}{\prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} (j!)^{n(\Lambda_j - \theta_j)} \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \prod_{j=0}^{i} ((i-j)!)^{\alpha_j^{(i)}}} (1 + o(n^{-1+\delta_3})) \times$$
$$\exp\left[-\frac{1}{2n(\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta})^2} \left(\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} (j-1)j(\Lambda_j - \theta_j)\right) \left(\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \sum_{j=0}^{i} (i-j-1)(i-j)\alpha_j^{(i)}\right)\right]. \tag{6.39}$$

The cardinality of $\mathcal{M}$ can be expressed as

$$|\mathcal{M}| = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \binom{n\xi \mathrm{P}_i}{\alpha_0^{(i)}, \ldots, \alpha_i^{(i)}} \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \binom{n\theta_j}{\beta_{\lfloor \frac{j}{2} \rfloor + 1}^{(j)}, \ldots, \beta_j^{(j)}} |\mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3| \tag{6.40}$$

where the sum is over the vectors $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(\mathsf{d}_{\mathsf{c}}^{\max})})$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(\mathsf{d}_{\mathsf{v}}^{\max})})$ that satisfy (6.30)-(6.33) and (6.36).

The average number of size (a, b) ASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(a, b) = \sum_{\boldsymbol{\theta}} \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \binom{n\Lambda_j}{n\theta_j} \frac{|\mathcal{M}|}{|\mathcal{H}_n^{\Lambda,\mathrm{P}}|} \tag{6.41}$$

where the sum is over the vectors $\boldsymbol{\theta}$ satisfying $\sum_j n\theta_j = a$ and $|\mathcal{H}_n^{\Lambda,\mathrm{P}}|, |\mathcal{M}|$ are given in (6.2) and (6.40), respectively.

Let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/n, \tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}/n$. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs is then

$$G_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(\theta, \gamma) = \max_{\boldsymbol{\theta}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}} S(\boldsymbol{\theta}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \tag{6.42}$$

under the constraints

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \theta_j = \theta \tag{6.43}$$

$$\sum_{j=0}^{i} \tilde{\alpha}_j^{(i)} = \xi \mathrm{P}_i, \quad \forall i \in \{1, \ldots, \mathsf{d}_{\mathsf{c}}^{\max}\} \tag{6.44}$$

$$\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \sum_{\substack{j=0 \\ j \text{ is odd}}}^{i} \tilde{\alpha}_j^{(i)} = \gamma \tag{6.45}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{j=0}^{i} j\tilde{\alpha}_j^{(i)} = \tilde{\theta} \tag{6.46}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{\substack{j=0 \\ j \text{ is even}}}^{i} j\tilde{\alpha}_j^{(i)} = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} l\tilde{\beta}_l^{(j)} \tag{6.47}$$

$$\sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)} = \theta_j, \quad \forall j \in \{1, \dots, \mathsf{d}_\mathsf{v}^{\max}\} \tag{6.48}$$

and where

$$\begin{aligned} S(\boldsymbol{\theta}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = &\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)}}{\xi \mathrm{P}_i}, \dots, \frac{\tilde{\alpha}_i^{(i)}}{\xi \mathrm{P}_i} \right) + \sum_{j=0}^{i} \tilde{\alpha}_j^{(i)} \ln \binom{i}{j} \right] \\ &+ \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\Lambda_j H_\mathsf{b}\left(\frac{\theta_j}{\Lambda_j} \right) + \theta_j H\left(\frac{\tilde{\beta}_{\lfloor \frac{j}{2} \rfloor+1}^{(j)}}{\theta_j}, \dots, \frac{\tilde{\beta}_j^{(j)}}{\theta_j} \right) + \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)} \ln \binom{j}{l} \right] \\ &- \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde{\beta}}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\tilde{\theta} - \tilde{\beta}}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}}{\bar{\mathsf{d}}_\mathsf{v}} \right) \end{aligned} \tag{6.49}$$

$$\tilde{\beta} = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} l\tilde{\beta}_l^{(j)}. \tag{6.50}$$

By using the Lagrangian multiplier method, we obtain (6.19)-(6.28). ■

Elementary Absorbing Set Distribution

We derive next the asymptotic EAS enumerator for irregular binary LDPC code ensembles.

Theorem 6.4. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs is

$$\begin{aligned} G_{\mathrm{EAS}}^{\Lambda,\mathrm{P}}(\theta, \gamma) = &\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\Lambda_j H_\mathsf{b}\left(\frac{\theta_j^\star}{\Lambda_j} \right) + \theta_j^\star H\left(\frac{\tilde{\beta}_{\lfloor \frac{j}{2} \rfloor+1}^{(j)\star}}{\theta_j^\star}, \dots, \frac{\tilde{\beta}_j^{(j)\star}}{\theta_j^\star} \right) + \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)\star} \ln \binom{j}{l} \right] \\ &+ \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)\star}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_1^{(i)\star}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_2^{(i)\star}}{\xi \mathrm{P}_i} \right) + \sum_{j=0}^{2} \tilde{\alpha}_j^{(i)\star} \ln \binom{i}{j} \right] \\ &- \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde{\theta}^\star - \gamma}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\gamma}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star}{\bar{\mathsf{d}}_\mathsf{v}} \right) \end{aligned} \tag{6.51}$$

where

$$\tilde{\theta}^{\star} = \xi \sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \mathrm{P}_i \frac{iA_3 + i(i-1)A_3^2A_4^2A_2}{A_2 + iA_3 + \binom{i}{2}A_3^2A_4^2A_2} \tag{6.52}$$

$$\theta_j^{\star} = \Lambda_j \frac{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l}}{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l} + (\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta}^{\star})^j A_1 A_3^j}, \quad \forall j \in \{1, \dots, \mathsf{d}_{\mathsf{v}}^{\max}\} \tag{6.53}$$

$$\tilde{\alpha}_0^{(i)\star} = \xi \mathrm{P}_i \frac{A_2}{A_2 + iA_3 + \binom{i}{2}A_3^2A_4^2A_2} \tag{6.54}$$

$$\tilde{\alpha}_1^{(i)\star} = \xi \mathrm{P}_i \frac{iA_3}{A_2 + iA_3 + \binom{i}{2}A_3^2A_4^2A_2} \tag{6.55}$$

$$\tilde{\alpha}_2^{(i)\star} = \xi \mathrm{P}_i \frac{\binom{i}{2}A_3^2A_4^2A_2}{A_2 + iA_3 + \binom{i}{2}A_3^2A_4^2A_2} \tag{6.56}$$

$$\tilde{\beta}_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l}\frac{1}{A_4^l}(\tilde{\theta}^{\star}-\gamma)^j}{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'}\frac{1}{A_4^{l'}} + (\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta}^{\star})^j A_1 A_3^j} \tag{6.57}$$

where (6.57) holds for all $j \in \{1, \dots, \mathsf{d}_{\mathsf{v}}^{\max}\}, l \in \left\{\lfloor \frac{j}{2} \rfloor + 1, \dots, j\right\}$ and A_1, A_2, A_3, A_4 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} j\Lambda_j \frac{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l}}{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l} + (\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta}^{\star})^j A_1 A_3^j} = \tilde{\theta}^{\star} \tag{6.58}$$

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \Lambda_j \frac{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l}}{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l} + (\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta}^{\star})^j A_1 A_3^j} = \theta \tag{6.59}$$

$$\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \mathrm{P}_i \frac{iA_3}{A_2 + iA_3 + \binom{i}{2}A_3^2A_4^2A_2} = \frac{\gamma}{\xi} \tag{6.60}$$

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \Lambda_j l \frac{\binom{j}{l}\frac{1}{A_4^l}(\tilde{\theta}^{\star}-\gamma)^j}{(\tilde{\theta}^{\star}-\gamma)^j \sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'}\frac{1}{A_4^{l'}} + (\bar{\mathsf{d}}_{\mathsf{v}} - \tilde{\theta}^{\star})^j A_1 A_3^j} = \tilde{\theta}^{\star} - \gamma. \tag{6.61}$$

Proof. We use the same notation of the AS enumerator. Since for EASs each unsatisfied CNs is connected to exactly one VN in $\mathcal{A}^{\mathrm{E}}_{a,b}$ and each satisfied CNs is connected zero or two VN in $\mathcal{A}^{\mathrm{E}}_{a,b}$, we have for all $i \in \{1, \ldots, \mathsf{d}_{\mathsf{c}}^{\max}\}$,

$$\alpha_0^{(i)} + \alpha_1^{(i)} + \alpha_2^{(i)} = n\xi \mathrm{P}_i \tag{6.62}$$

$$\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \alpha_1^{(i)} = b. \tag{6.63}$$

The matrix $\boldsymbol{M}_1$ contains, for each $j \in \{1, \ldots, \mathsf{d}_{\mathsf{v}}^{\max}\}$, $\beta_l^{(j)}$ rows of weight $l \in \{\lfloor j/2 \rfloor + 1, \ldots, j\}$ and, for each $i \in \{1, \ldots, \mathsf{d}_{\mathsf{c}}^{\max}\}$, $\alpha_0^{(i)}$ columns of weight 0 and $\alpha_2^{(i)}$ columns of weight 2. The number of ones in the matrix $\boldsymbol{M}_1$ is

$$f_1 = \sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} l \beta_l^{(j)} = 2 \sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \alpha_2^{(i)}. \tag{6.64}$$

From Theorem 3.1, the cardinality of $\mathcal{M}_1$, for $\delta_1 > 0$, is

$$\begin{aligned} |\mathcal{M}_1| =& \frac{f_1!}{\prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \prod_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} (l!)^{\beta_l^{(j)}} \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} (2)^{\alpha_2^{(i)}}} (1 + o(n^{-1+\delta_1})) \times \\ & \exp\left[-\frac{1}{f_1^2} \left(\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} (l-1) l \beta_l^{(j)} \right) \left(\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \alpha_2^{(i)} \right) \right]. \end{aligned} \tag{6.65}$$

Consider now the matrix $\boldsymbol{M}_2$. For each $j \in \{1, \ldots, \mathsf{d}_{\mathsf{v}}^{\max}\}$, there are $\beta_l^{(j)}$ rows of weight $j - l$ and all columns have weight 1. The number of ones in $\boldsymbol{M}_2$ is

$$f_2 = \sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor + 1}^{j} (j - l) \beta_l^{(j)} = \sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \alpha_1^{(i)} = b. \tag{6.66}$$

Note that $f_1 + f_2$ is the total number of ones in the submatrix $[\boldsymbol{M}_1 \,|\, \boldsymbol{M}_2]$, which is equal to

$$\sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \alpha_1^{(i)} + 2\alpha_2^{(i)} = n\tilde{\theta} \tag{6.67}$$

where $\tilde{\theta} = \sum_j j\theta_j$.

The cardinalities of $\mathcal{M}_2$, for $\delta_2 > 0$, is then

$$|\mathcal{M}_2| = \frac{b!}{\prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \prod_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} ((j-l)!)^{\beta_l^{(j)}}} (1 + o(n^{-1+\delta_2})). \tag{6.68}$$

The matrix $\boldsymbol{M}_3$ has $n(\Lambda_j - \theta_j)$ rows of weight j for each $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}$ and $\alpha_j^{(i)}$ columns of weight $i - j$, where $j \in \{0, 1, 2\}$ and $i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}$. The number of ones in $\boldsymbol{M}_3$ is

$$f_3 = n \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j(\Lambda_j - \theta_j) = \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{j=0}^{i} \alpha_j^{(i)} (i-j) = n\bar{\mathsf{d}}_\mathsf{v} - n\tilde{\theta}. \tag{6.69}$$

From Theorem 3.1, the cardinality of $\mathcal{M}_3$, for $\delta_3 > 0$, is

$$\begin{aligned} |\mathcal{M}_3| = & \frac{(n\bar{\mathsf{d}}_\mathsf{v} - n\tilde{\theta})!}{\prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} (j!)^{n(\Lambda_j-\theta_j)} \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \prod_{j=0}^{2} ((i-j)!)^{\alpha_j^{(i)}}} (1 + o(n^{-1+\delta_3})) \times \\ & \exp\left[\frac{-1}{2n(\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta})^2} \left(\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} (j-1)j(\Lambda_j - \theta_j) \right) \left(\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \sum_{j=0}^{2} (i-j-1)(i-j)\alpha_j^{(i)} \right) \right]. \end{aligned} \tag{6.70}$$

The cardinality of $\mathcal{M}$ can be expressed as

$$|\mathcal{M}| = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \binom{n\xi \mathrm{P}_i}{\alpha_0^{(i)}, \alpha_1^{(i)}, \alpha_2^{(i)}} \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \binom{n\theta_j}{\beta_{\lfloor \frac{j}{2} \rfloor+1}^{(j)}, \ldots, \beta_j^{(j)}} |\mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3| \tag{6.71}$$

where the sum is over the vectors $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(\mathsf{d}_\mathsf{c}^{\max})})$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(\mathsf{d}_\mathsf{v}^{\max})})$ that satisfy (6.32), (6.62)-(6.64) and (6.67).

The average number of size (a, b) EASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{EAS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{\boldsymbol{\theta}} \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \binom{n\Lambda_j}{n\theta_j} \frac{|\mathcal{M}|}{|\mathcal{H}_n^{\Lambda,\mathrm{P}}|} \tag{6.72}$$

where the sum is over the vectors $\boldsymbol{\theta}$ satisfying $\sum_j n\theta_j = a$ and $|\mathcal{H}_n^{\Lambda,\mathrm{P}}|$, $|\mathcal{M}|$ are given in (6.2) and (6.71), respectively.

Let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/n, \tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}/n$. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs is then

$$G^{\Lambda,\mathrm{P}}_{\mathrm{EAS}}(\theta,\gamma) = \max_{\boldsymbol{\theta},\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}} S(\boldsymbol{\theta},\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}) \tag{6.73}$$

under the constraints

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \theta_j = \theta \tag{6.74}$$

$$\tilde{\alpha}_0^{(i)} + \tilde{\alpha}_1^{(i)} + \tilde{\alpha}_2^{(i)} = \xi \mathrm{P}_i, \quad \forall i \in \{1,\ldots,\mathsf{d}_\mathsf{c}^{\max}\} \tag{6.75}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \tilde{\alpha}_1^{(i)} = \gamma \tag{6.76}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \tilde{\alpha}_1^{(i)} + 2\tilde{\alpha}_2^{(i)} = \tilde{\theta} \tag{6.77}$$

$$2\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \tilde{\alpha}_2^{(i)} = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} l \tilde{\beta}_l^{(j)} \tag{6.78}$$

$$\sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)} = \theta_j, \quad \forall j \in \{1,\ldots,\mathsf{d}_\mathsf{v}^{\max}\} \tag{6.79}$$

and where

$$\begin{aligned} S(\boldsymbol{\theta},\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}) = &\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_1^{(i)}}{\xi \mathrm{P}_i}, \frac{\tilde{\alpha}_2^{(i)}}{\xi \mathrm{P}_i}\right) + \sum_{j=0}^{2} \tilde{\alpha}_j^{(i)} \ln \binom{i}{j}\right] + \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\Lambda_j H_\mathsf{b}\left(\frac{\theta_j}{\Lambda_j}\right)\right. \\ &\left. + \theta_j H\left(\frac{\tilde{\beta}^{(j)}_{\lfloor \frac{j}{2} \rfloor+1}}{\theta_j}, \ldots, \frac{\tilde{\beta}_j^{(j)}}{\theta_j}\right) + \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \tilde{\beta}_l^{(j)} \ln \binom{j}{l}\right] \\ &- \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde{\theta}-\gamma}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\gamma}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}}{\bar{\mathsf{d}}_\mathsf{v}}\right). \end{aligned} \tag{6.80}$$

By using the Lagrangian multiplier method, we obtain (6.51)-(6.61). ■

Fully Absorbing Set Distribution

The asymptotic distribution of FASs for regular LDPC code ensembles was derived in [37]. We extend now the analysis to irregular code ensembles.

Theorem 6.5. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs is

$$\begin{aligned} G_{\mathrm{FAS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Bigg[& \theta_j^\star H\left(\frac{\tilde{\beta}_{\lfloor \frac{j}{2} \rfloor+1}^{(j)\star}}{\theta_j^\star}, \ldots, \frac{\tilde{\beta}_j^{(j)\star}}{\theta_j^\star}\right) + (\Lambda_j - \theta_j^\star) H\left(\frac{\tilde{\kappa}_{\lfloor \frac{j}{2} \rfloor+1}^{(j)\star}}{\Lambda_j - \theta_j^\star}, \ldots, \frac{\tilde{\kappa}_j^{(j)\star}}{\Lambda_j - \theta_j^\star}\right) \\ & + \Lambda_j H_\mathsf{b}\left(\frac{\theta_j^\star}{\Lambda_j}\right) + \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \left(\tilde{\beta}_l^{(j)\star} \ln\binom{j}{l} + \tilde{\kappa}_l^{(j)\star} \ln\binom{j}{l}\right)\Bigg] \\ & + \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde{\alpha}_0^{(i)\star}}{\xi \mathrm{P}_i}, \ldots, \frac{\tilde{\alpha}_i^{(i)\star}}{\xi \mathrm{P}_i}\right) + \sum_{j=0}^{i} \tilde{\alpha}_j^{(i)\star} \ln\binom{i}{j}\right] \\ & - \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde{\beta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\tilde{\theta}^\star - \tilde{\beta}^\star}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\tilde{\kappa}^\star}{\bar{\mathsf{d}}_\mathsf{v}}, \frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star}{\bar{\mathsf{d}}_\mathsf{v}}\right) \end{aligned} \tag{6.81}$$

where

$$\tilde{\theta}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i \mathrm{P}_i A_3 \frac{A_2 A_4 \left[(A_5+A_3A_4)^{i-1} - (A_5-A_3A_4)^{i-1}\right] + (A_5+A_3)^{i-1} + (A_5-A_3)^{i-1}}{A_2[(A_5+A_3A_4)^i + (A_5-A_3A_4)^i] + (A_5+A_3)^i - (A_5-A_3)^i} \tag{6.82}$$

$$\theta_j^\star = \Lambda_j \frac{\sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l}{\sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l + \left(\frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star}{\tilde{\theta}^\star - \tilde{\beta}^\star}\right)^j A_1 A_3^j \sum_{l=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l} \left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star)}\right)^l} \tag{6.83}$$

$$\tilde{\alpha}_j^{(i)\star} = \begin{cases} \frac{2\xi \mathrm{P}_i \binom{i}{j} A_2 A_3^j A_4^j A_5^{i-j}}{A_2(A_5+A_3A_4)^i + A_2(A_5-A_3A_4)^i + (A_5+A_3)^i - (A_5-A_3)^i} & j \text{ is even} \\ \frac{2\xi \mathrm{P}_i \binom{i}{j} A_3^j A_5^{i-j}}{A_2(A_5+A_3A_4)^i + A_2(A_5-A_3A_4)^i + (A_5+A_3)^i - (A_5-A_3)^i} & j \text{ is odd} \end{cases} \tag{6.84}$$

$$\tilde{\beta}_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^l}{\sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^{l'} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star}{\tilde{\theta}^\star - \tilde{\beta}^\star}\right)^j A_1 A_3^j \sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'} \left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star)}\right)^{l'}} \tag{6.85}$$

$$\kappa_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l} \left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star)}\right)^l \left(\frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star}{\tilde{\theta}^\star - \tilde{\beta}^\star}\right)^j A_1 A_3^j}{\sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'} \left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star - \tilde{\beta}^\star)}\right)^{l'} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star}{\tilde{\theta}^\star - \tilde{\beta}^\star}\right)^j A_1 A_3^j \sum_{l'=\lfloor \frac{j}{2} \rfloor+1}^{j} \binom{j}{l'} \left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v} - \tilde{\theta}^\star - \tilde{\kappa}^\star)}\right)^{l'}} \tag{6.86}$$

$$\tilde{\beta}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i \mathrm{P}_i \frac{A_2 A_3 A_4 \left[(A_5 + A_3A_4)^{i-1} - (A_5 - A_3A_4)^{i-1}\right]}{A_2(A_5 + A_3A_4)^i + A_2(A_5 - A_3A_4)^i + (A_5 + A_3)^i - (A_5 - A_3)^i} \tag{6.87}$$

$$\tilde{\kappa}^\star = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} i \mathrm{P}_i \frac{A_2 A_5 \left[\right](A_5 + A_3A_4)^{i-1} + (A_5 - A_3A_4)^{i-1}]}{A_2(A_5 + A_3A_4)^i + A_2(A_5 - A_3A_4)^i + (A_5 + A_3)^i - (A_5 - A_3)^i} \tag{6.88}$$

where (6.83) holds for all $j \in \{1, \ldots, \mathsf{d}_\mathsf{v}^{\max}\}$, (6.84) holds for all $i \in \{1, \ldots, \mathsf{d}_\mathsf{c}^{\max}\}, j \in$

$\{0,\ldots,i\}$, (6.85) and (6.86) hold for all $j \in \{1,\ldots,\mathsf{d}_\mathsf{v}^{\max}\}, l \in \left\{\lfloor\frac{j}{2}\rfloor+1,\ldots,j\right\}$ and A_1, A_2, A_3, A_4, A_5 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \frac{j\Lambda_j \sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^l}{\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^l+\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\tilde{\beta}^\star}\right)^j A_1A_3^j\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l} = \tilde{\theta}^\star \tag{6.89}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^l}{\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^l+\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\tilde{\beta}^\star}\right)^j A_1A_3^j\sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l} = \theta \tag{6.90}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{(A_5+A_3)^i-(A_5-A_3)^i}{A_2(A_5+A_3A_4)^i+A_2(A_5-A_3A_4)^i+(A_5+A_3)^i-(A_5-A_3)^i} = \frac{\gamma}{\xi} \tag{6.91}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j} \frac{\Lambda_j l\binom{j}{l}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^l}{\sum_{l'=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^{l'}+\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\tilde{\beta}^\star}\right)^j A_1A_3^j\sum_{l'=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^{l'}} = \tilde{\beta}^\star \tag{6.92}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor\frac{j}{2}\rfloor+1}^{j} \frac{\Lambda_j l\binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\tilde{\beta}^\star}\right)^j A_1A_3^j}{\sum_{l'=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde{\beta}^\star}{A_4(\tilde{\theta}^\star-\tilde{\beta}^\star)}\right)^{l'}+\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\tilde{\beta}^\star}\right)^j A_1A_3^j\sum_{l'=\lfloor\frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^{l'}} = \tilde{\kappa}^\star. \tag{6.93}$$

The proof is omitted since it is similar to the proof of Theorem 6.3.

Elementary Fully Absorbing Set Distribution

The asymptotic distribution of EFASs for regular LDPC code ensembles was derived in [37]. We extend the derivation to irregular code ensembles.

Theorem 6.6. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EFASs is

$$\begin{aligned} G^{\Lambda,\mathrm{P}}_{\mathrm{EFAS}}(\theta,\gamma) = & \sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Bigg[\theta_j^\star H\left(\frac{\tilde\beta^{(j)\star}_{\lfloor \frac{j}{2}\rfloor+1}}{\theta_j^\star},\dots,\frac{\tilde\beta^{(j)\star}_{j}}{\theta_j^\star}\right) + (\Lambda_j-\theta_j^\star) H\left(\frac{\tilde\kappa^{(j)\star}_{\lfloor \frac{j}{2}\rfloor+1}}{\Lambda_j-\theta_j^\star},\dots,\frac{\tilde\kappa^{(j)\star}_{j}}{\Lambda_j-\theta_j^\star}\right) \\ & + \Lambda_j H_{\mathsf{b}}\left(\frac{\theta_j^\star}{\Lambda_j}\right) + \sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j}\left(\tilde\beta_l^{(j)\star}\ln\binom{j}{l} + \tilde\kappa_l^{(j)\star}\ln\binom{j}{l}\right)\Bigg] \\ & + \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\xi \mathrm{P}_i H\left(\frac{\tilde\alpha_0^{(i)\star}}{\xi\mathrm{P}_i},\frac{\tilde\alpha_1^{(i)\star}}{\xi\mathrm{P}_i},\frac{\tilde\alpha_2^{(i)\star}}{\xi\mathrm{P}_i}\right) + \sum_{j=0}^{2}\tilde\alpha_j^{(i)\star}\ln\binom{i}{j}\right] \\ & - \bar{\mathsf{d}}_\mathsf{v} H\left(\frac{\tilde\theta^\star-\gamma}{\bar{\mathsf{d}}_\mathsf{v}},\frac{\gamma}{\bar{\mathsf{d}}_\mathsf{v}},\frac{\tilde\kappa^\star}{\bar{\mathsf{d}}_\mathsf{v}},\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star}{\bar{\mathsf{d}}_\mathsf{v}}\right) \end{aligned} \tag{6.94}$$

where

$$\tilde\theta^\star = \xi\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{iA_3A_5^{i-1} + i(i-1)A_3^2A_5^{i-2}A_4^2A_2}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} \tag{6.95}$$

$$\theta_j^\star = \Lambda_j \frac{\sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\frac{1}{A_4^l}}{\sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\frac{1}{A_4^l} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star}{\tilde\theta^\star-\gamma}\right)^j A_1A_3^j \sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l}\left(\frac{\tilde\kappa^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star)}\right)^l} \tag{6.96}$$

$$\tilde\alpha_0^{(i)\star} = \xi\mathrm{P}_i\frac{A_2A_5^i}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} \tag{6.97}$$

$$\tilde\alpha_1^{(i)\star} = \xi\mathrm{P}_i\frac{iA_3A_5^{i-1}}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} \tag{6.98}$$

$$\tilde\alpha_2^{(i)\star} = \xi\mathrm{P}_i\frac{\binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} \tag{6.99}$$

$$\tilde\beta_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l}\frac{1}{A_4^l}}{\sum_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\frac{1}{A_4^{l'}} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star}{\tilde\theta^\star-\gamma}\right)^j A_1A_3^j \sum_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde\kappa^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star)}\right)^{l'}} \tag{6.100}$$

$$\kappa_l^{(j)\star} = \Lambda_j \frac{\binom{j}{l}\left(\frac{\tilde\kappa^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star)}\right)^l\left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star}{\tilde\theta^\star-\gamma}\right)^j A_1A_3^j}{\sum_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\frac{1}{A_4^{l'}} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star}{\tilde\theta^\star-\gamma}\right)^j A_1A_3^j \sum_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j}\binom{j}{l'}\left(\frac{\tilde\kappa^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde\theta^\star-\tilde\kappa^\star)}\right)^{l'}} \tag{6.101}$$

$$\tilde\kappa^\star = \xi\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} A_2\mathrm{P}_i \frac{iA_5^i + (i-2)\binom{i}{2}A_3^2A_4^2A_5^{i-2}}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} \tag{6.102}$$

where (6.96) holds for all $j \in \{1,\dots,\mathsf{d}_\mathsf{v}^{\max}\}$, (6.98) and (6.99) hold for all $j \in \{1,\dots,\mathsf{d}_\mathsf{v}^{\max}\}, l \in \left\{\lfloor \frac{j}{2} \rfloor + 1,\dots,j\right\}$ and A_1, A_2, A_3, A_4, A_5 are the positive roots of

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} j\Lambda_j \frac{\sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l}}{\sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\gamma}\right)^j A_1 A_3^j \sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l} = \tilde{\theta}^\star \tag{6.103}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{\sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l}}{\sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\frac{1}{A_4^l} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\gamma}\right)^j A_1 A_3^j \sum\limits_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l} = \theta \tag{6.104}$$

$$\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{iA_3A_5^{i-1}}{A_2A_5^i + iA_3A_5^{i-1} + \binom{i}{2}A_3^2A_4^2A_5^{i-2}A_2} = \frac{\gamma}{\xi} \tag{6.105}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} \frac{l\Lambda_j \binom{j}{l}\frac{1}{A_4^l}}{\sum\limits_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l'}\frac{1}{A_4^{l'}} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\gamma}\right)^j A_1 A_3^j \sum\limits_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l'}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^{l'}} = \tilde{\theta}^\star - \gamma \tag{6.106}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \sum_{l=\lfloor \frac{j}{2}\rfloor+1}^{j} l\Lambda_j \frac{\binom{j}{l}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^l \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\gamma}\right)^j A_1A_3^j}{\sum\limits_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l'}\frac{1}{A_4^{l'}} + \left(\frac{\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star}{\tilde{\theta}^\star-\gamma}\right)^j A_1 A_3^j \sum\limits_{l'=\lfloor \frac{j}{2}\rfloor+1}^{j} \binom{j}{l'}\left(\frac{\tilde{\kappa}^\star}{A_5(\bar{\mathsf{d}}_\mathsf{v}-\tilde{\theta}^\star-\tilde{\kappa}^\star)}\right)^{l'}} = \tilde{\kappa}^\star. \tag{6.107}$$

The proof is similar to the one of Theorem 6.4.

6.2.2 The Generating Function Approach

The approach in Section 6.2.1 based on random matrix enumeration can be applied only to unstructured binary LDPC codes. Therefore, we present an alternative derivation of the average enumerators of (elementary) trapping and (fully) absorbing sets for binary irregular LDPC code ensembles using the generating function methodology, previously adopted to study the distance spectrum and the stopping set distributions of (generalized) binary LDPC code ensembles [9, 46, 78–80]. The generating function approach is general and we can enumerate several graphical structures by defining the appropriate generating functions. For instance, we derive the (elementary) trapping and (fully) absorbing set enumerators of GLDPC codes or the trapping and (elementary) absorbing set enumerators of non-binary LDPC codes using generating functions in the next chapters.

In the following, we derive the finite-length (elementary) trapping and (fully) absorbing set enumerators for binary unstructured LDPC code ensembles using the generating function methodology presented in Section 3.6. We develop an analytical method to evaluate the normalized logarithmic asymptotic distributions of (elementary) trapping and (fully) absorbing sets. Further, we derive the asymptotic approximations for the small-sized trapping sets cases.

Trapping and Elementary Trapping Set Distributions

We derived the finite-length and asymptotic distribution of TSs for irregular LDPC code ensembles in [53].

Lemma 6.1. The average number of size (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{w} \frac{\operatorname{coeff}\left(g(x,y)^n, x^w y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{w}} \operatorname{coeff}\left(f(t,s)^n, t^a s^w\right) \tag{6.108}$$

where we introduced the generating functions

$$f(t,s) = \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} (1 + t s^j)^{\Lambda_j} \tag{6.109}$$

$$g(x,y) = \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \left[\frac{(1+x)^i + (1-x)^i}{2} + y \frac{(1+x)^i - (1-x)^i}{2} \right]^{\xi \mathrm{P}_i} . \tag{6.110}$$

Proof. Consider the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities and assign the value 1 to each VN in the set. We denote by $\alpha(a, w)$ the number of ways to choose a VNs such that exactly w edges emanate from them. Its generating function is $\sum_{a,w} \alpha(a,w) t^a s^w$. Consider a single VN of degree j. This generating function is $1 + ts^j$ because we can either skip this VN or include it in the set $\mathcal{I}$. If we skip the VN, then we will get 0 nodes and 0 edges and this gives us the term 1 corresponding to $t^0 s^0$. If we choose the VN, then we get 1 VN and j edges and this gives us $t^1 s^j$. By considering all possible VN degrees, and since we have $\Lambda_j n$ VNs of degree j and for each VN we can decide to include it in $\mathcal{I}$ or not, we obtain $f(t, s)^n$. Thus, we have

$$\alpha(a,w) = \operatorname{coeff}\left(f(t,s)^n, t^a s^w\right).$$

Let $\beta(b, w)$ be the number of ways to choose w edges such that exactly b CNs each have an odd number of sockets and the other CNs each have an even number of check sockets.

Its generating function is $\sum_{b,w}\beta(b,w)y^b x^w$. Consider a CN of degree i. The generating function of a degree i CN with an even number of connections to the VN in $\mathcal{I}$ is

$$g_{\mathsf{c}}(x,y) := y^0 \sum_{l \text{ is even}} \binom{i}{l} x^l = \frac{1}{2}\left[(1+x)^i + (1-x)^i\right].$$

If the CN is connected an odd number of times to the VN in $\mathcal{I}$, then its generating function is

$$g_{\bar{\mathsf{c}}}(x,y) := y^1 \sum_{l \text{ is odd}} \binom{i}{l} x^l = \frac{1}{2} y\left[(1+x)^i - (1-x)^i\right].$$

Considering all CN degrees and that there are $\xi \mathrm{P}_i n$ of degree i, we obtain

$$\beta(b,w) = \operatorname{coeff}\left(g(x,y)^n, x^w y^b\right).$$

Let Z_1 be a RV indicating the number of edges emanating from the set $\mathcal{I}$. Further, let Z_2 be a RV that is equal to 1 if there are exactly b CNs each connected an odd number of times to $\mathcal{I}$ and the other CNs each have an even number (including zero) of connections to $\mathcal{I}$, and to 0 otherwise. Thus, we have

$$\mathsf{E}_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(a,b) = \binom{n}{a} \Pr\{Z_2 = 1\} \tag{6.111}$$

and

$$\begin{aligned} \Pr\{Z_2 = 1\} &= \sum_w \Pr\{Z_1 = w\} \Pr\{Z_2 = 1 | Z_1 = w\} \\ &= \sum_w \frac{\operatorname{coeff}\left(f(t,s)^n, t^a s^w\right)}{\binom{n}{a}} \frac{\operatorname{coeff}\left(g(x,y)^n, x^w y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{w}}. \end{aligned} \tag{6.112}$$

■

The exact average number of size (a,b) TSs derived in Lemma 6.1 for a finite block length n is extremely complex to compute for large n. As $n \to \infty$, one can use the Hayman formula in Lemma 3.1 to derive the normalized logarithmic asymptotic distribution of TSs.

Theorem 6.7. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs is

$$G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = -\bar{\mathsf{d}}_{\mathsf{v}} \ln(1+xs) - \theta \ln(t) - \gamma \ln(y) + \ln\left(f(t,s)\right) + \ln\left(g(x,y)\right) \tag{6.113}$$

where t, s, x, y are the unique positive solutions of

$$t\frac{\partial \ln f(t,s)}{\partial t} = \theta \tag{6.114}$$
$$s\frac{\partial \ln f(t,s)}{\partial s} = x\frac{\partial \ln g(x,y)}{\partial x} = \tilde{w}^\star \tag{6.115}$$
$$y\frac{\partial \ln g(x,y)}{\partial y} = \gamma \tag{6.116}$$

where $f(t,s)$ and $g(x,y)$ are defined in (6.109) and (6.110) respectively and

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v}\frac{xs}{1+xs}. \tag{6.117}$$

The proof can be found in Appendix 6.4.1.

To determine $\theta^\star_{\text{TS}}$ we add another equation to the system of equations of Theorem 6.7, namely $G^{\Lambda,\text{P}}_{\text{TS}}(\theta, \Delta\theta) = 0$ with $\theta > 0$.

Note that to compute the finite-length and the asymptotic distribution of ETSs, in (6.108) and (6.113) we must replace the generating function of (6.110) with

$$g(x,y) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[1 + \binom{i}{2}x^2 + ixy\right]^{\xi \text{P}_i}. \tag{6.118}$$

We briefly explain how to derive $g(x,y)$ in (6.118). For an ETS, a satisfied CN of degree i is connected zero or 2 times to VNs in $\mathcal{I}$. The corresponding generating function is

$$g_\mathsf{c}(x,y) := y^0 \left[1 + \binom{i}{2}x^2\right].$$

Each CN in $\mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$. The corresponding generating function is $g_{\bar{\mathsf{c}}}(x,y) := iyx$. Considering all CNs degrees and that there are $\xi \text{P}_i n$ of degree i, we obtain $g(x,y)$ in (6.118).

The following Lemma will be useful to analyze $G^{\Lambda,\text{P}}_{\text{TS}}(\theta, \Delta\theta)$ for small θ and fixed Δ.

Lemma 6.2. For a fixed $\Delta = \gamma/\theta$, the derivative in θ of $G^{\Lambda,\text{P}}_{\text{TS}}(\theta, \Delta\theta)$ is

$$\frac{\mathrm{d}G^{\Lambda,\text{P}}_{\text{TS}}(\theta, \Delta\theta)}{\mathrm{d}\theta} = -\ln(t) - \Delta \ln(y) \tag{6.119}$$

where for each θ, the values of t and y are given by the solution of the system of equations (6.114)-(6.116).

Proof. Note that the solutions of the system of equations in (6.114)-(6.116) are implicit functions of θ. From (6.113) and (6.117), we obtain

$$\begin{aligned}\frac{\mathrm{d}G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)}{\mathrm{d}\theta} =& -\ln(t) - \Delta\ln(y) + \frac{\mathrm{d}t}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial t}}{f(t,s)} - \frac{\theta}{t}\right] + \frac{\mathrm{d}s}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial s}}{f(t,s)} - \frac{\tilde{w}^{\star}}{s}\right] \\ &+ \frac{\mathrm{d}x}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial x}}{g(x,y)} - \frac{\tilde{w}^{\star}}{x}\right] + \frac{\mathrm{d}y}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial y}}{g(x,y)} - \frac{\Delta\theta}{y}\right].\end{aligned} \tag{6.120}$$

The terms in the square brackets in (6.120) are equal to zero due to (6.114)-(6.116). This establishes the result of Lemma 6.2. ■

Consider now the case of small θ and $\gamma = \Delta\theta$. We obtain a closed form expression of $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$, which we introduce in the following corollary.

Corollary 6.1. For a fixed $\Delta = \gamma/\theta$ and small θ, we have

$$\begin{aligned}G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta) =& \theta\left[\frac{\mathsf{d}_{\mathsf{v}}^{\min} - 2 - \Delta}{2}(\ln(\theta) - 1) + \ln\left(\frac{(\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}}}{\Delta^{\Delta}}\right)\right. \\ &\left. - \frac{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta}{2}\ln\left(\frac{\bar{\mathsf{d}}_{\mathsf{v}}\bar{\mathsf{d}}_{\mathsf{c}}(\mathsf{d}_{\mathsf{v}}^{\min} - \Delta)}{\mathrm{P}''(1)}\right)\right] + o(\theta)\end{aligned} \tag{6.121}$$

where $\mathsf{d}_{\mathsf{v}}^{\min}$ is the minimum VN degree and $\mathrm{P}''(x)$ is the second derivative of $\mathrm{P}(x)$. The proof is provided in Appendix 6.4.2.

Note that we obtain exactly the same expression for ETSs.

Note that a positive $\theta_{\mathrm{TS}}^{\star}$ exists whenever the derivative of $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$ is negative as $\theta \to 0$. Thus, by substituting (6.284) and (6.285) in (6.120) we find that a positive $\theta_{\mathrm{TS}}^{\star}$ exists whenever $\mathsf{d}_{\mathsf{v}}^{\min} > 2 + \Delta$ or $\mathsf{d}_{\mathsf{v}}^{\min} = 2 + \Delta$ and

$$\frac{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}}(\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}\mathrm{P}''(1)}{2\bar{\mathsf{d}}_{\mathsf{v}}\bar{\mathsf{d}}_{\mathsf{c}}(\mathsf{d}_{\mathsf{v}}^{\min} - 2)^{\mathsf{d}_{\mathsf{v}}^{\min}-2}} < 1. \tag{6.122}$$

If $\Delta = 0$ and $\mathsf{d}_{\mathsf{v}}^{\min} = 2$, we obtain the inequality $\lambda_2\rho'(1) < 1$ in [79] for the existence of the typical minimum distance of binary LDPC codes, where $\lambda(x), \rho(x)$ are the edge-oriented degree distribution polynomials and $\rho'(x)$ is the derivative of $\rho(x)$.

If the relative minimum Δ-trapping set size is small enough, then we can use Corollary 6.1 to approximate it. Through numerical simulations, we observed that the relative minimum Δ-trapping set size is small for small VN degrees or high CN degrees as observed in [48]. We need to determine θ such that $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta) = 0$ with $\theta > 0$. By neglecting the term $o(\theta)$, we obtain

$$\theta^{\star}_{\mathrm{TS}} \approx \exp(1) \left(\frac{\bar{\mathsf{d}}_{\mathsf{c}}\bar{\mathsf{d}}_{\mathsf{v}}(\mathsf{d}_{\mathsf{v}}^{\min} - \Delta)}{\mathrm{P}''(1)} \right)^{\frac{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta}{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta-2}} \left(\frac{\Delta^{\Delta}}{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}}(\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}} \right)^{\frac{2}{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta-2}}. \tag{6.123}$$

The approximation of the relative minimum Δ-trapping set size given in (6.123) is accurate when $\theta^{\star}_{\mathrm{TS}}$ is sufficiently small (for the case of small VN degrees or high CN degrees as observed in [48]) and does not need solving the system of equations given in Theorem 6.7.

For the regular ensemble $\mathscr{C}_n^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$, the expressions in Lemma 6.1 and Theorem 6.7 can be simplified as follows.

Lemma 6.3. The average number of size (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}(a,b) = \binom{n}{a} \frac{\mathrm{coeff}\left(g(x,y)^n, x^{a\mathsf{d}_{\mathsf{v}}}y^b\right)}{\binom{n\mathsf{d}_{\mathsf{v}}}{a\mathsf{d}_{\mathsf{v}}}} \tag{6.124}$$

where

$$g(x,y) = \left[\frac{(1+x)^{\mathsf{d}_{\mathsf{c}}} + (1-x)^{\mathsf{d}_{\mathsf{c}}}}{2} + y\frac{(1+x)^{\mathsf{d}_{\mathsf{c}}} - (1-x)^{\mathsf{d}_{\mathsf{c}}}}{2} \right]^{\xi}. \tag{6.125}$$

Proof. The Lemma can be proved from Lemma 6.1. Note that for a regular code, all VNs have degree d_{v}. Therefore, w in (6.108) is equal to $a\mathsf{d}_{\mathsf{v}}$. Moreover, the number of ways to choose a VNs such that exactly $a\mathsf{d}_{\mathsf{v}}$ edges emanate from them is equal to $\binom{n}{a}$. Further, the generating function $g(x, y)$ in (6.125) can be obtained from the one in (6.110) by taking $\mathrm{P}(x) = x^{\mathsf{d}_{\mathsf{c}}}$. ■

Theorem 6.8. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the ensemble $\mathscr{C}_n^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$ is

$$G_{\mathrm{TS}}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}(\theta,\gamma) = -(\mathsf{d}_{\mathsf{v}} - 1)H_{\mathsf{b}}(\theta) - \gamma\ln(y) - \theta\mathsf{d}_{\mathsf{v}}\ln(x) + \ln\left(g(x,y)\right) \tag{6.126}$$

where

$$y = \frac{\gamma}{\xi - \gamma} \frac{(1+x)^{\mathsf{d}_{\mathsf{c}}} + (1-x)^{\mathsf{d}_{\mathsf{c}}}}{(1+x)^{\mathsf{d}_{\mathsf{c}}} - (1-x)^{\mathsf{d}_{\mathsf{c}}}} \tag{6.127}$$

and x is the unique positive solution of

$$x\frac{\partial \ln g(x,y)}{\partial x} = \theta \mathsf{d_v} \tag{6.128}$$

where $g(x,y)$ is defined in (6.125). The proof can be found in Appendix 6.4.3.

Note that to evaluate the normalized logarithmic asymptotic distribution of TSs for unstructured LDPC code ensembles, one needs to solve 4 equations (with the same number of unknowns) as shown in Theorem 6.7. For the regular case the number of equations reduces to one equation.

To compute the finite-length distribution of ETSs for regular ensembles, we simply need to replace in (6.124) the generating function $g(x,y)$ given in (6.125) with

$$g(x,y) = \left[1 + \binom{\mathsf{d_c}}{2}x^2 + \mathsf{d_c}xy\right]^{\xi}. \tag{6.129}$$

To obtain $g(x,y)$ in (6.129), we simply need to take $\mathrm{P}(x) = x^{\mathsf{d_c}}$ in (6.118).

Due to the simplicity of the generating function $g(x,y)$ in this case, we can obtain a closed form expression of the normalized asymptotic distribution of $(\theta n, \gamma n)$ ETSs for the ensemble $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$:

$$G_{\mathrm{ETS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = -\left(\mathsf{d_v}-1\right)H_{\mathsf{b}}(\theta) - \gamma\ln(y) - \theta\mathsf{d_v}\ln(x) + \ln\left(g(x,y)\right) \tag{6.130}$$

where $g(x,y)$ is defined in (6.129) and

$$x = \sqrt{\frac{2(\theta\mathsf{d_v}-\gamma)}{\mathsf{d_c}(\mathsf{d_c}-1)(2\xi-\gamma-\theta\mathsf{d_v})}} \tag{6.131}$$

$$y = \frac{\gamma}{\xi-\gamma}\frac{1+\binom{\mathsf{d_c}}{2}x^2}{\mathsf{d_c}x}. \tag{6.132}$$

Proof. The proof is similar to the one of TSs. We need only replace in (6.126) the generating function $g(x,y)$ given in (6.125) with the one in (6.129), where x, y are the unique positive solutions of (6.287) and (6.288). Substituting (6.129) in (6.287) and (6.288), and with some manipulations, we obtain x, y in (6.131) and (6.132). ■

Example 6.1. We consider regular $(\mathsf{d_v},\mathsf{d_c})$ binary LDPC code ensembles. Fig. 6.1 illustrates a comparison between the exact values of the relative minimum Δ-trapping set sizes of some regular $(\mathsf{d_v},\mathsf{d_c})$ LDPC codes for $\Delta = 0.1$ (solid lines) and their corresponding approximations

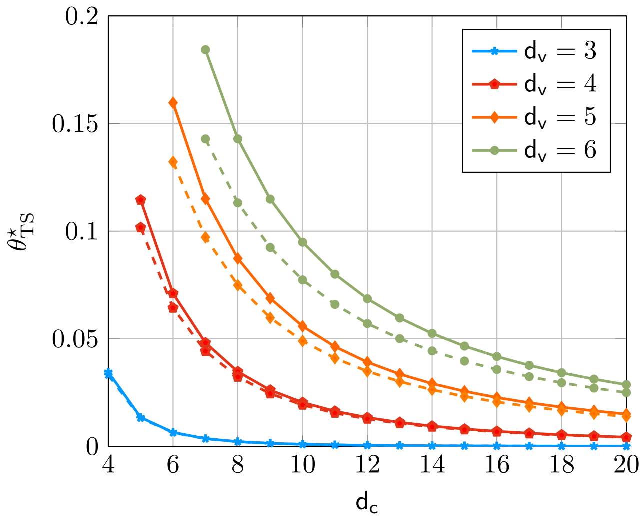


Figure 6.1: Exact values of the relative minimum Δ-trapping set size (solid lines) and the corresponding approximation obtained from (6.123) (dashed lines) for some regular $(\mathsf{d_v}, \mathsf{d_c})$ LDPC codes for $\Delta = 0.1$.

obtained from (6.123) (dashed lines). We see that the approximation is good for small values of $\mathsf{d_v}$ and large values of $\mathsf{d_c}$ since for this case, the relative minimum Δ-trapping set sizes are small. We also observe that, for the same CN degree, increasing the VN degree improves the relative minimum Δ-trapping set size and for fixed VN degree, the TS properties improves with decreasing CN degree.

Absorbing Set Distribution

The following Lemma presents the finite-length AS enumerator for unstructured binary LDPC codes and we develop an analytical method for evaluating the normalized logarithmic asymptotic distribution of ASs.

Lemma 6.4. The average number of size (a, b) ASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{e,w} \frac{\operatorname{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}} \operatorname{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^w\right) \tag{6.133}$$

$$f(t,s_1,s_2) = \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[1 + t \sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1}\right]^{\Lambda_j} \tag{6.134}$$

and

$$g(x_1, x_2, y) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[\frac{(1+x_1)^i + (1-x_1)^i}{2} + y \frac{(1+x_2)^i - (1-x_2)^i}{2} \right]^{\xi \mathrm{P}_i} . \tag{6.135}$$

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set. The CNs that are connected to the VNs in $\mathcal{I}$ an even number (including zero) of times are satisfied and the ones connected an odd number of times are unsatisfied. We have two type of edges. Edges of the first type emanate from satisfied CNs and the edges of the second type emanate from unsatisfied CNs. We denote by $\alpha(a, e, w)$ the number of ways to choose a VNs such that exactly e type 1 edges and w type 2 edges emanate from them and each of the VNs in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges. The corresponding generating function is $\sum_{a,e,w} \alpha(a, e, w) t^a s_1^e s_2^w$. Consider a VN $\mathtt{v}$ of degree j. Let $j - j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to $\mathtt{v}$. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it, then we obtain 0 nodes and 0 type 1 and type 2 edges. If we choose it, then we will have 1 node, $j - j_1$ type 1 edges and j_1 type 2 edges where $j_1 \in \{0, 1, \dots, \lfloor (j-1)/2 \rfloor\}$ (since each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges). Considering all possible VN degrees and that there are $\Lambda_j n$ of degree j, we obtain $f(t, s_1, s_2)^n$ and we have

$$\alpha(a, e, w) = \operatorname{coeff}\left(f(t, s_1, s_2)^n, t^a s_1^e s_2^w \right).$$

Let $\beta(b, e, w)$ be the number of ways to choose e type 1 edges and w type 2 edges such that there are exactly b unsatisfied CNs. The corresponding generating function is $\sum_{b,e,w} \beta(b, e, w) y^b x_1^e x_2^w$. Consider a CN of degree i. If it is connected an even number (including zero) of times to the VNs in $\mathcal{I}$, then its generating function is

$$g_{\mathsf{c}}(x_1, y) := y^0 \sum_{l \text{ is even}} \binom{i}{l} x_1^l = \frac{1}{2} \left[(1+x_1)^i + (1-x_1)^i \right]$$

and if it is connected an odd number of times to the VNs in $\mathcal{I}$, its generating function is

$$g_{\bar{\mathsf{c}}}(x_2, y) := y^1 \sum_{l \text{ is odd}} \binom{i}{l} x_2^l = \frac{y}{2} \left[(1+x_2)^i - (1-x_2)^i \right].$$

Considering all CN degrees and that there are $\xi \mathrm{P}_i n$ of degree i, we obtain

$$\beta(b,e,w) = \operatorname{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^w y^b\right).$$

We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities. Let Z_1 and Z_2 be two RVs indicating, respectively, the number of type 1 and type 2 edges emanating from $\mathcal{I}$, where each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges. Further, let Z_3 be a RV that is equal to 1 if there are exactly b unsatisfied CNs, and is equal to 0 otherwise. We have

$$\mathsf{E}_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(a,b) = \binom{n}{a} \Pr\{Z_3 = 1\} \tag{6.136}$$

and

$$\begin{aligned}
\Pr\{Z_3=1\} &= \sum_{e,w} \Pr\{Z_1=e, Z_2=w\} \Pr\{Z_3=1|Z_1=e,Z_2=w\} \\
&= \sum_{e,w} \frac{\operatorname{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^w\right)}{\binom{n}{a}} \frac{\operatorname{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}}.
\end{aligned} \tag{6.137}$$

■

We derive the normalized logarithmic asymptotic distribution of ASs for binary codes in the following Theorem.

Theorem 6.9. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ ensemble is

$$\begin{aligned}
G_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = &- \bar{\mathsf{d}}_\mathsf{v} \ln(1 + x_1 s_1 + x_2 s_2) - \theta \ln(t) - \gamma \ln(y) \\
&+ \ln\left(g(x_1,x_2,y)\right) + \ln\left(f(t,s_1,s_2)\right)
\end{aligned} \tag{6.138}$$

where t, s_1, s_2, x_1, x_2, y are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{6.139}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2,y)}{\partial x_1} = \tilde{e}^\star \tag{6.140}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2,y)}{\partial x_2} = \tilde{w}^\star \tag{6.141}$$

$$y\frac{\partial \ln g(x_1,x_2,y)}{\partial y} = \gamma \tag{6.142}$$

where $f(t, s_1, s_2)$ and $g(x_1, x_2, y)$ are defined in (6.134) and (6.135) respectively and

$$\tilde{e}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{x_1 s_1}{1 + x_1 s_1 + x_2 s_2} \tag{6.143}$$

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{x_2 s_2}{1 + x_1 s_1 + x_2 s_2}. \tag{6.144}$$

The proof follows the same steps as the one of Theorem 6.7.

To determine $\theta^\star_{\text{AS}}$ we add another equation to the system of equations of Theorem 6.9, namely $G^{\Lambda,\text{P}}_{\text{AS}}(\theta, \Delta\theta) = 0$ with $0 < \theta \leq 1$.

Similar to the TS case, the expressions in Lemma 6.4 and Theorem 6.9 can be simplified for regular ensembles.

Lemma 6.5. The average number of size (a, b) ASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ is

$$\mathsf{E}^{\mathsf{d_v},\mathsf{d_c}}_{\text{AS}}(a, b) = \sum_e \binom{m}{b} \frac{\text{coeff}\left(g_1(x_1)^{m-b}, x_1^e\right) \text{coeff}\left(g_2(x_2)^b, x_2^{a\mathsf{d_v}-e}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{e}} \times \binom{n}{a} \text{coeff}\left(f(s)^a, s^{a\mathsf{d_v}-e}\right) \tag{6.145}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor \frac{\mathsf{d_v}-1}{2} \rfloor} \binom{\mathsf{d_v}}{j_1} s^{j_1} \tag{6.146}$$

$$g_1(x_1) = \frac{1}{2}\left[(1 + x_1)^{\mathsf{d_c}} + (1 - x_1)^{\mathsf{d_c}}\right] \tag{6.147}$$

$$g_2(x_2) = \frac{1}{2}\left[(1 + x_2)^{\mathsf{d_c}} - (1 - x_2)^{\mathsf{d_c}}\right]. \tag{6.148}$$

Proof. The Lemma can be proved similarly to Lemma 6.4. Note that for a regular code, all VNs have degree $\mathsf{d_v}$. Therefore, $e + w$ in (6.133) is equal to $a\mathsf{d_v}$. The generating function of a VN in $\mathcal{I}$ is given by $f(s)$, since there are a VNs in $\mathcal{I}$ and there are $\binom{n}{a}$ ways to chose the a VNs, the number of ways to choose a VNs such that exactly e type 1 edges and $a\mathsf{d_v} - e$ type 2 edges emanate from them and each of the VNs in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges is

$$\binom{n}{a} \text{coeff}\left(f(s)^a, s^{a\mathsf{d_v}-e}\right).$$

Further, $g_1(x_1)$ in (6.147) is the generating function of a satisfied CN and $g_2(x_2)$ in (6.148) is the generating function of an unsatisfied one. Since we have $m-b$ satisfied CNs and b unsatisfied ones, the number of ways to chose e type 1 edges and $a\mathsf{d_v}-e$ type 2 edges such that there are exactly b unsatisfied CNs is

$$\binom{m}{b} \operatorname{coeff}\left(g_1(x_1)^{m-b}, x_1^e\right) \operatorname{coeff}\left(g_2(x_2)^b, x_2^{a\mathsf{d_v}-e}\right).$$

■

We show now that to compute the normalized logarithmic asymptotic distribution of ASs for regular codes, one needs to solve 3 equations instead of 6 for the irregular case.

Theorem 6.10. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ ensemble is

$$\begin{aligned} G_{\mathrm{AS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) =& -\theta\mathsf{d_v}\ln(x_1+x_2 s) - (\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) + \theta\ln\left(f(s)\right) \\ &+ (\xi-\gamma)\ln\left(g_1(x_1)\right) + \gamma\ln\left(g_2(x_2)\right) + \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) \end{aligned} \tag{6.149}$$

and s, x_1, x_2 are the unique positive solutions of

$$\theta s\frac{\mathrm{d}\ln f(s)}{\mathrm{d}s} = \gamma x_2 \frac{\mathrm{d}\ln g_2(x_2)}{\mathrm{d}x_2} = (\theta\mathsf{d_v} - \tilde{e}^\star) \tag{6.150}$$

$$(\xi-\gamma)x_1\frac{\mathrm{d}\ln g_1(x_1)}{\mathrm{d}x_1} = \tilde{e}^\star \tag{6.151}$$

where $f(s)$, $g_1(x_1)$ and $g_2(x_2)$ are defined in (6.146), (6.147) and (6.148) respectively and

$$\tilde{e}^\star = \theta\mathsf{d_v}\frac{x_1}{x_1+x_2 s}. \tag{6.152}$$

The proof follows the same steps as the one of Theorem 6.8.

Elementary Absorbing Set Distribution

The following Lemma gives the EAS enumerator for binary LDPC codes.

Lemma 6.6. The average number of size (a,b) EASs in the Tanner graph of a code drawn uniformly at random from the LDPC ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{EAS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_e \frac{\operatorname{coeff}\left(g(x_1,x_2)^n, x_1^e x_2^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+b}\binom{e+b}{b}} \operatorname{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^b\right) \tag{6.153}$$

where $f(t, s_1, s_2)$ is defined in (6.134) and

$$g(x_1, x_2) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[1 + \binom{i}{2} x_1^2 + i x_2\right]^{\xi \mathrm{P}_i} . \tag{6.154}$$

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set. The edges connected to a VN $\mathtt{v}$ are assigned the binary value chosen for $\mathtt{v}$. We have 2 types of edges connected to the VNs in $\mathcal{I}$. Edges of the first type emanate from satisfied CNs and the edges of the second type emanate from unsatisfied CNs. Let $\beta(b, e)$ be the number of ways to choose e type 1 edges such that there are exactly b unsatisfied CNs and each of the satisfied CNs is connected to 0 or 2 VNs in $\mathcal{I}$ and each of the unsatisfied CNs is connected to exactly one VNs in $\mathcal{I}$. In that case, since we have b unsatisfied CNs and each of them is connected to exactly one VNs in $\mathcal{I}$, we have b type 2 edges. The corresponding generating function is $\sum_{b,e} \beta(b, e) x_1^e x_2^b$. The generating function of a satisfied CN of degree i, which is connected to 0 or 2 VNs is $\mathcal{I}$ is $g_\mathsf{c}(x_1) := 1 + \binom{i}{2} x_1^2$ and if it is unsatisfied and connected to only one VN in $\mathcal{I}$, its generating function is $g_{\bar{\mathsf{c}}}(x_2) := i x_2$. Considering all CN degrees and that there are $\xi \mathrm{P}_i n$ of degree i, we obtain

$$\beta(b, e) = \mathrm{coeff}\left(g(x_1, x_2)^n, x_1^e x_2^b\right) .$$

We denote by $\alpha(a, e, b)$ the number of ways to choose a VNs such that exactly e type 1 edges and b type 2 edges emanate from them and each of the VNs in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges. The corresponding generating function is $\sum_{a,e,b} \alpha(a, e, b) t^a s_1^e s_2^b$. Consider a VN $\mathtt{v}$ of degree j. Let $j - j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to $\mathtt{v}$. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it, then we obtain 0 nodes and 0 type 1 and type 2 edges. If we choose it, then we will have 1 node, $j - j_1$ type 1 edges and j_1 type 2 edges where $j_1 \in \{0, 1, \ldots, \lfloor (j-1)/2 \rfloor\}$ (since each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges). Considering all possible VN degrees, the generating function is $f(t, s_1, s_2)^n$ and we have

$$\alpha(a, e, b) = \mathrm{coeff}\left(f(t, s_1, s_2)^n, t^a s_1^e s_2^b\right) .$$

We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities. Let Z_1 and Z_2 be two RVs indicating, respectively, the number of type 1 and type 2 edges emanating from $\mathcal{I}$, where each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges. Further, let Z_3 be a RV that is equal to 1 if there are exactly b unsatisfied CNs, and is equal to 0 otherwise. We have

$$\mathsf{E}^{\Lambda,\mathrm{P}}_{\mathrm{EAS}}(a,b) = \binom{n}{a} \Pr\{Z_3 = 1\} \tag{6.155}$$

and

$$\begin{aligned} \Pr\{Z_3 = 1\} =& \sum_e \Pr\{Z_1 = e, Z_2 = b\} \Pr\{Z_3 = 1 | Z_1 = e, Z_2 = b\} \\ =& \sum_e \frac{\mathrm{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^b\right)}{\binom{n}{a}} \frac{\mathrm{coeff}\left(g(x_1,x_2)^n, x_1^e x_2^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+b}\binom{e+b}{b}}. \end{aligned} \tag{6.156}$$

■

Next, we analyze the normalized logarithmic asymptotic distribution of EAS and present an efficient way to compute it.

Theorem 6.11. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the ensemble is

$$\begin{aligned} G^{\Lambda,\mathrm{P}}_{\mathrm{EAS}}(\theta,\gamma) =& -\bar{\mathsf{d}}_{\mathsf{v}} \ln(\bar{\mathsf{d}}_{\mathsf{v}}) + (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\ln(\bar{\mathsf{d}}_{\mathsf{v}} - \gamma) - \theta \ln(t) - (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\ln(1 + x_1 s_1) \\ &+ \ln\left(g(x_1,x_2)\right) + \ln\left(f(t,s_1,s_2)\right) - \gamma \ln(x_2 s_2) + \gamma \ln(\gamma) \end{aligned} \tag{6.157}$$

where t, s_1, s_2, x_1, x_2 are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{6.158}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2)}{\partial x_1} = \tilde{e}^\star \tag{6.159}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2)}{\partial x_2} = \gamma \tag{6.160}$$

and where $f(t,s_1,s_2)$ and $g(x_1,x_2)$ are defined in (6.134) and (6.154) respectively and

$$\tilde{e}^\star = (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\frac{x_1 s_1}{1 + x_1 s_1}. \tag{6.161}$$

We now derive the EAS finite-length and asymptotic enumerators for the regular ensembles.

Lemma 6.7. The average number of size (a,b) EASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ is

$$\mathsf{E}_{\mathrm{EAS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) = \binom{m}{b}\binom{n}{a}\frac{\mathsf{d_c}^{\,b}\operatorname{coeff}\left(g(x)^{m-b}, x^{a\mathsf{d_v}-b}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{b}}\operatorname{coeff}\left(f(s)^a, s^b\right) \tag{6.162}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor\frac{\mathsf{d_v}-1}{2}\rfloor}\binom{\mathsf{d_v}}{j_1}s^{j_1} \tag{6.163}$$

$$g(x) = 1 + \binom{\mathsf{d_c}}{2}x_1^2. \tag{6.164}$$

Proof. For a regular code, all VNs have degree $\mathsf{d_v}$. Therefore, e in (6.153) is equal to $a\mathsf{d_v} - b$. Further, $g(x)$ in (6.164) is the generating function of a satisfied CN. Since each unsatisfied CN is connected to exactly one type 2 edge, the number of ways to chose $a\mathsf{d_v} - b$ type 1 and b type 2 edges such that there are exactly b unsatisfied CNs is $\binom{m}{b}\mathsf{d_c}^{\,b}\operatorname{coeff}\left(g(x)^{m-b}, x^{a\mathsf{d_v}-b}\right)$. The generating function $f(s)$ in (6.163) is the same as for the AS case. ■

We show now that for the computation of the normalized logarithmic asymptotic distribution of EASs for regular codes, one needs to solve one equation compared to 5 for the irregular case.

Theorem 6.12. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ ensemble is

$$\begin{aligned} G_{\mathrm{EAS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = &-(\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) - \mathsf{d_v}\theta H_{\mathsf{b}}\left(\frac{\gamma}{\theta\mathsf{d_v}}\right) + \gamma\ln(\mathsf{d_c}) - \gamma\ln(s) \\ &+ (\xi-\gamma)\ln\left(g(x)\right) + \theta\ln\left(f(s)\right) - (\theta\mathsf{d_v}-\gamma)\ln(x) + \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) \end{aligned} \tag{6.165}$$

where

$$x = \sqrt{\frac{2(\theta\mathsf{d_v}-\gamma)}{\mathsf{d_c}(\mathsf{d_c}-1)(2\xi-\theta\mathsf{d_v}-\gamma)}} \tag{6.166}$$

and s is the unique positive solution of

$$\theta s\frac{\mathrm{d}\ln f(s)}{\mathrm{d}s} = \gamma \tag{6.167}$$

where $f(s)$ is defined in (6.163).

Fully Absorbing Set Distribution

Lemma 6.8. The average number of size (a,b) FASs in the Tanner graph of a code drawn uniformly at random from the LDPC ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{FAS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{e,w,l} \frac{\mathrm{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^w x_3^l y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+w}\binom{e+w}{e}\binom{n\bar{\mathsf{d}}_{\mathsf{v}}-e-w}{l}} \times \mathrm{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^w s_3^l\right) \tag{6.168}$$

where

$$f(t,s_1,s_2,s_3) = \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \left[\sum_{j_1=0}^{\lfloor\frac{j-1}{2}\rfloor} \binom{j}{j_1} s_3^{j_1} + t \sum_{j_1=0}^{\lfloor\frac{j-1}{2}\rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1}\right]^{\Lambda_j} \tag{6.169}$$

$$g(x_1,x_2,x_3,y) = \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \left[\frac{(1+x_1)^i + (1-x_1)^i}{2} + y\frac{(x_3+x_2)^i - (x_3-x_2)^i}{2}\right]^{\xi \mathrm{P}_i}. \tag{6.170}$$

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble. Let $\mathcal{I}$ be a set of VNs. We assign to each of these VNs the value 1. We have 3 types of edges. Edges of the first type are connected to the VNs in $\mathcal{I}$ and satisfied CNs, the edges of the second type are the ones connected to the VNs in $\mathcal{I}$ and unsatisfied CNs and the edges of the third type are connected to the VNs in $\mathcal{V} \setminus \mathcal{I}$ and satisfied CNs. We denote by $\alpha(a,e,w,l)$ the number of ways to choose a VNs such that exactly e type 1 edges, w type 2 edges emanate from them and l type 3 edges emanate from the other VNs and each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. The corresponding generating function is $\sum_{a,e,w,l} \alpha(a,e,w,l) t^a s_1^e s_2^w s_3^l$. Consider a VN v of degree j. Let $j-j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to v. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it we obtain 0 nodes and 0 type 1 and type 2 edges and j_1 type 3 edges where $j_1 \in \{0,1,\ldots,\lfloor(j-1)/2\rfloor\}$. If we choose it, we will have 1 node, $j-j_1$ type 1 edges and j_1 type 2 edges where $j_1 \in \{0,1,\ldots,\lfloor(j-1)/2\rfloor\}$. Considering all possible VN degrees, the generating function is $f(t,s_1,s_2,s_3)^n$. Thus, we have

$$\alpha(a,e,w,l) = \mathrm{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^w s_3^l\right).$$

Let $\beta(b,e,w,l)$ be the number of ways to choose e type 1 edges, w type 2 edges and l type 3 edges such that there are exactly b unsatisfied CNs. The corresponding generating function is $\sum_{b,e,w,l} \beta(b,e,w,l) y^b x_1^e x_2^w x_3^l$. Consider a CN of degree i. If it is connected an even number (including zero) of times to the VNs in $\mathcal{I}$, then its generating function is

$$g_{\mathsf{c}}(x_1, y) := y^0 \sum_{\substack{j=0 \\ j \text{ is even}}}^{i} \binom{i}{j} x_1^j = \frac{1}{2}\left[(1+x_1)^i + (1-x_1)^i\right]$$

and if it is connected an odd number of times to the VNs in $\mathcal{I}$, its generating function is

$$g_{\bar{\mathsf{c}}}(x_2, x_3, y) := y^1 \sum_{\substack{j=0 \\ j \text{ is odd}}}^{i} \binom{i}{j} x_2^j x_3^{i-j} = \frac{y}{2}\left[(x_3+x_2)^i - (x_3-x_2)^i\right].$$

Considering all CN degrees and that there are $\xi \mathrm{P}_i n$ of degree i, we obtain

$$\beta(b,e,w,l) = \operatorname{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^w x_3^l y^b\right).$$

We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities. Let Z_1, Z_2 and Z_3 be three RVs indicating, respectively, the number of type 1, type 2 and type 3 edges, where each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. Further, let Z_4 be a RV that is equal to 1 if there are exactly b unsatisfied CNs and each of the other CNs is satisfied, and to 0 otherwise. Thus, we have

$$\mathsf{E}_{\mathrm{FAS}}^{\Lambda,\mathrm{P}}(a,b) = \binom{n}{a} \Pr\{Z_4 = 1\} \tag{6.171}$$

and

$$\begin{aligned}\Pr\{Z_4=1\} &= \sum_{e,w,l} \Pr\{Z_1=e, Z_2=w, Z_3=l\} \Pr\{Z_4=1 | Z_1=e, Z_2=w, Z_3=l\} \\ &= \sum_{e,w,l} \frac{\operatorname{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^w s_3^l\right)}{\binom{n}{a}} \times \\ &\quad \frac{\operatorname{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^w x_3^l y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+w}\binom{e+w}{e}\binom{n\bar{\mathsf{d}}_{\mathsf{v}}-e-w}{l}}.\end{aligned} \tag{6.172}$$

■

Theorem 6.13. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs for the $\mathscr{C}_n^{\Lambda,\mathrm{P}}$

ensemble is

$$G_{\mathrm{FAS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = -\bar{\mathsf{d}}_{\mathsf{v}}\ln(1+x_1s_1+x_2s_2+x_3s_3) - \theta\ln(t) - \gamma\ln(y) + \ln\left(g(x_1,x_2,x_3,y)\right) + \ln\left(f(t,s_1,s_2,s_3)\right) \tag{6.173}$$

where $t, s_1, s_2, s_3, x_1, x_2, x_3, y$ are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial t} = \theta \tag{6.174}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_1} = \tilde{e}^\star \tag{6.175}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_2} = \tilde{w}^\star \tag{6.176}$$

$$s_3\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_3} = x_3\frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_3} = \tilde{l}^\star \tag{6.177}$$

$$y\frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial y} = \gamma \tag{6.178}$$

where $f(t,s_1,s_2,s_3)$ and $g(x_1,x_2,x_3,y)$ are defined in (6.169) and (6.170) respectively and

$$\tilde{e}^\star = \bar{\mathsf{d}}_{\mathsf{v}}\frac{x_1s_1}{1+x_1s_1+x_2s_2+x_3s_3} \tag{6.179}$$

$$\tilde{w}^\star = \bar{\mathsf{d}}_{\mathsf{v}}\frac{x_2s_2}{1+x_1s_1+x_2s_2+x_3s_3} \tag{6.180}$$

$$\tilde{l}^\star = \bar{\mathsf{d}}_{\mathsf{v}}\frac{x_3s_3}{1+x_1s_1+x_2s_2+x_3s_3}. \tag{6.181}$$

To determine $\theta^\star_{\mathrm{FAS}}$ we add another equation to the system of equations of Theorem 6.13, namely

$$G_{\mathrm{FAS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta) = 0 \tag{6.182}$$

with $0 < \theta \leq 1$.

The next Lemma presents the FAS enumerator for the regular $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$.

Lemma 6.9. The average number of size (a,b) FASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ is

$$\mathsf{E}_{\mathrm{FAS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) = \sum_e \binom{m}{b}\binom{n}{a}\frac{\mathrm{coeff}\left(g_1(x_1)^{m-b}, x_1^e\right)\mathrm{coeff}\left(g_2(x_2)^b, x_2^{a\mathsf{d_v}-e}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{e}\binom{(n-a)\mathsf{d_v}}{b\mathsf{d_c}-a\mathsf{d_v}+e}}\times \mathrm{coeff}\left(f(s_1)^a, s_1^{a\mathsf{d_v}-e}\right)\mathrm{coeff}\left(f(s_2)^{n-a}, s_2^{b\mathsf{d_c}-a\mathsf{d_v}+e}\right) \tag{6.183}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor \frac{\mathsf{d_v}-1}{2} \rfloor} \binom{\mathsf{d_v}}{j_1} s^{j_1} \tag{6.184}$$

$$g_1(x_1) = \frac{1}{2}\left[(1+x_1)^{\mathsf{d_c}} + (1-x_1)^{\mathsf{d_c}}\right] \tag{6.185}$$

$$g_2(x_2) = \frac{1}{2}\left[(1+x_2)^{\mathsf{d_c}} - (1-x_2)^{\mathsf{d_c}}\right]. \tag{6.186}$$

Proof. For a regular code, all VNs have degree $\mathsf{d_v}$. Therefore, $e + w$ in (6.168) is equal to $a\mathsf{d_v}$. Moreover, all CNs have degree $\mathsf{d_c}$. Thus, $w + l$ in (6.168) is equal to $b\mathsf{d_c}$. The generating function of a VN is given by $f(s_1)$. The number of ways to choose a VNs such that exactly e type 1 edges and $a\mathsf{d_v} - e$ type 2 edges emanate from them and $b\mathsf{d_c} - a\mathsf{d_v} + e$ type 3 edges emanate from the remaining VNs is

$$\binom{n}{a} \operatorname{coeff}\left(f(s_1)^a, s_1^{a\mathsf{d_v}-e}\right) \operatorname{coeff}\left(f(s_2)^{n-a}, s_2^{b\mathsf{d_c}-a\mathsf{d_v}+e}\right).$$

Further, $g_1(x_1)$ and $g_2(x_2)$ are the generating functions of a satisfied and unsatisfied CN, respectively. The number of ways to chose e type 1 edges, $a\mathsf{d_v} - e$ type 2 and $b\mathsf{d_c} - a\mathsf{d_v} + e$ type 3 edges such that there are exactly b unsatisfied CNs is

$$\binom{m}{b} \operatorname{coeff}\left(g_1(x_1)^{m-b}, x_1^e\right) \operatorname{coeff}\left(g_2(x_2)^b, x_2^{a\mathsf{d_v}-e}\right).$$

■

We show now that for the computation of the normalized logarithmic asymptotic distribution of FASs for regular codes, one needs to solve 4 equations instead of 8 for the irregular case.

Theorem 6.14. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs for the $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ ensemble is

$$\begin{aligned} G_{\mathrm{FAS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = &(\xi-\gamma)\ln\left(g_1(x_1)\right) + \gamma\ln\left(g_2(x_2)\right) + \theta\ln\left(f(s_1)\right) \\ &+ (1-\theta)\ln\left(f(s_2)\right) - (\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) - \theta\mathsf{d_v}H_{\mathsf{b}}\left(\frac{\tilde{e}^\star}{\theta\mathsf{d_v}}\right) - \tilde{e}^\star\ln(x_1) \\ &- (\theta\mathsf{d_v}-\tilde{e}^\star)\ln(x_2 s_1) - (\gamma\mathsf{d_c}-\theta\mathsf{d_v}+\tilde{e}^\star)\ln(s_2) \\ &+ \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) - (1-\theta)\mathsf{d_v}H_{\mathsf{b}}\left(\frac{\gamma\mathsf{d_c}-\theta\mathsf{d_v}+\tilde{e}^\star}{(1-\theta)\mathsf{d_v}}\right) \end{aligned} \tag{6.187}$$

where s_1, s_3, x_1, x_2 are the unique positive solutions of

$$\theta s_1 \frac{\mathrm{d}\ln f(s_1)}{\mathrm{d}s_1} = \gamma x_2 \frac{\mathrm{d}\ln g_2(x_2)}{\mathrm{d}x_2} = (\theta \mathsf{d}_\mathsf{v} - \tilde{e}^\star) \tag{6.188}$$

$$(1-\theta) s_2 \frac{\mathrm{d}\ln f(s_2)}{\mathrm{d}s_2} = (\gamma \mathsf{d}_\mathsf{c} - \theta \mathsf{d}_\mathsf{v} + \tilde{e}^\star) \tag{6.189}$$

$$(\xi - \gamma) x_1 \frac{\mathrm{d}\ln g_1(x_1)}{\mathrm{d}x_1} = \tilde{e}^\star \tag{6.190}$$

and where $f(s_1), g_1(x_1)$ and $g_2(x_2)$ are defined in (6.184)-(6.186) respectively and

$$\tilde{e}^\star = \frac{-x_2 s_1 (\gamma \mathsf{d}_\mathsf{c} - \theta \mathsf{d}_\mathsf{v}) - x_1 s_2 (\theta \mathsf{d}_\mathsf{v} + \mathsf{d}_\mathsf{v} - \gamma \mathsf{d}_\mathsf{c}) + \sqrt{C}}{2(x_2 s_1 - x_1 s_2)} \tag{6.191}$$

$$\begin{aligned} C = & \left(x_2 s_1 (\gamma \mathsf{d}_\mathsf{c} - \theta \mathsf{d}_\mathsf{v}) + x_1 s_2 (\theta \mathsf{d}_\mathsf{v} + \mathsf{d}_\mathsf{v} - \gamma \mathsf{d}_\mathsf{c})\right)^2 \\ & + 4(x_2 s_1 - x_1 s_2)(\mathsf{d}_\mathsf{v} - \gamma \mathsf{d}_\mathsf{c}) x_1 s_2 \theta \mathsf{d}_\mathsf{v}. \end{aligned} \tag{6.192}$$

Elementary Fully Absorbing Set Distribution

Lemma 6.10. The average number of size (a, b) EFASs in the Tanner graph of a code drawn uniformly at random from LDPC ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{EFAS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{e,l} \frac{\mathrm{coeff}\left(g(x_1,x_2,x_3)^n, x_1^e x_2^b x_3^l\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+b}\binom{e+b}{b}\binom{n\bar{\mathsf{d}}_\mathsf{v}-e-b}{l}} \mathrm{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^b s_3^l\right) \tag{6.193}$$

where

$$f(t,s_1,s_2,s_3) = \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_3^{j_1} + t \sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1} \right]^{\Lambda_j} \tag{6.194}$$

$$g(x_1,x_2,x_3) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \left[1 + \binom{i}{2} x_1^2 + i x_2 x_3^{i-1} \right]^{\xi \mathrm{P}_i}. \tag{6.195}$$

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble. Let $\mathcal{I}$ be a set of VNs. We assign to each of these VNs the value 1. We have 3 types of edges. Edges of the first type are connected to the VNs in $\mathcal{I}$ and satisfied CNs, the edges of the second type are the ones connected to the VNs in $\mathcal{I}$ and unsatisfied CNs and the edges of the third type are connected to the VNs in $\mathcal{V} \setminus \mathcal{I}$ and satisfied CNs. We denote by $\alpha(a, e, b, l)$ the number of ways to choose a VNs such that exactly e type 1 edges, b type 2 edges emanate from them and l type 3 edges emanate from the other VNs and each VN in

$\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. The corresponding generating function is $\sum_{a,e,b,l} \alpha(a,e,b,l) t^a s_1^e s_2^b s_3^l$. Consider a VN $\mathtt{v}$ of degree j. Let $j - j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to $\mathtt{v}$. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it we obtain 0 nodes and 0 type 1 and type 2 edges and j_1 type 3 edges where $j_1 \in \{0, 1, \ldots, \lfloor (j-1)/2 \rfloor\}$. If we choose it, we will have 1 node, $j - j_1$ type 1 edges and j_1 type 2 edges where $j_1 \in \{0, 1, \ldots, \lfloor (j-1)/2 \rfloor\}$. Considering all possible VN degrees, the generating function is $f(t, s_1, s_2, s_3)^n$. Thus, we have

$$\alpha(a,e,b,l) = \text{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^b s_3^l\right).$$

Let $\beta(b,e,l)$ be the number of ways to choose e type 1 edges, b type 2 edges and l type 3 edges such that there are exactly b unsatisfied CNs and each of the satisfied CNs is connected to 0 or 2 VNs in $\mathcal{I}$ and each of the unsatisfied CNs is connected to exactly one VNs in $\mathcal{I}$. The corresponding generating function is $\sum_{b,e,l} \beta(b,e,l) x_1^e x_2^b x_3^l$. Consider a CN of degree i. If it is connected to 0 or 2 VNs in $\mathcal{I}$, then its generating function is $1 + \binom{i}{2} x_1^2$ and if it is connected to exactly one VN in $\mathcal{I}$, its generating function is $i x_2 x_3^{i-1}$. Considering all CN degrees and that there are $\xi \mathrm{P}_i n$ of degree i, we obtain

$$\beta(b,e,l) = \text{coeff}\left(g(x_1,x_2,x_3)^n, x_1^e x_2^b x_3^l\right).$$

We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities. Let Z_1, Z_2 and Z_3 be three RVs indicating, respectively, the number of type 1, type 2 and type 3 edges, where each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. Further, let Z_4 be a RV that is equal to 1 if there are exactly b unsatisfied CNs and each of the other CNs is satisfied, and to 0 otherwise. Thus, we have

$$\mathsf{E}_{\mathrm{EFAS}}^{\Lambda,\mathrm{P}}(a,b) = \binom{n}{a} \Pr\{Z_4 = 1\} \tag{6.196}$$

and

$$\begin{aligned} \Pr\{Z_4 = 1\} &= \sum_{e,l} \Pr\{Z_1 = e, Z_2 = w, Z_3 = l\} \Pr\{Z_4 = 1 | Z_1 = e, Z_2 = b, Z_3 = l\} \\ &= \sum_{e,l} \frac{\text{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^b s_3^l\right)}{\binom{n}{a}} \frac{\text{coeff}\left(g(x_1,x_2,x_3)^n, x_1^e x_2^b x_3^l\right)}{\binom{n\bar{\mathsf{d}}_\mathtt{v}}{e+b}\binom{e+b}{e}\binom{n\bar{\mathsf{d}}_\mathtt{v}-e-b}{l}}. \end{aligned} \tag{6.197}$$

■

Theorem 6.15. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EFASs for the $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ ensemble is

$$\begin{aligned} G_{\mathrm{EFAS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) =&(\bar{\mathsf{d}}_{\mathsf{v}}-\gamma)\ln(\bar{\mathsf{d}}_{\mathsf{v}}-\gamma)-\bar{\mathsf{d}}_{\mathsf{v}}\ln(\bar{\mathsf{d}}_{\mathsf{v}})-(\bar{\mathsf{d}}_{\mathsf{v}}-\gamma)\ln(1+x_1s_1+x_3s_3)\\ &+\gamma\ln(\gamma)-\theta\ln(t)-\gamma\ln(x_2s_2)+\ln\left(g(x_1,x_2,x_3)\right)\\ &+\ln\left(f(t,s_1,s_2,s_3)\right) \end{aligned} \tag{6.198}$$

where $t, s_1, s_2, s_3, x_1, x_2, x_3$ are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial t} =\theta \tag{6.199}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_1} =x_1\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_1} = \tilde{e}^\star \tag{6.200}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_2} =x_2\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_2} = \gamma \tag{6.201}$$

$$s_3\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_3} =x_3\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_3} = \tilde{l}^\star \tag{6.202}$$

where $f(t, s_1, s_2, s_3)$ and $g(x_1, x_2, x_3)$ are defined in (6.194) and (6.195) respectively and

$$\tilde{e}^\star =(\bar{\mathsf{d}}_{\mathsf{v}}-\gamma)\frac{x_1s_1}{1+x_1s_1+x_3s_3} \tag{6.203}$$

$$\tilde{l}^\star =(\bar{\mathsf{d}}_{\mathsf{v}}-\gamma)\frac{x_3s_3}{1+x_1s_1+x_3s_3}. \tag{6.204}$$

We present in the next Lemma, the EFAS enumerator for the regular $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ ensemble.

Lemma 6.11. The average number of size (a, b) EFASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ is

$$\begin{aligned} \mathsf{E}_{\mathrm{EFAS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) =&\binom{m}{b}\binom{n}{a}\frac{\mathsf{d_c}^b\operatorname{coeff}\left(g(x)^{m-b},x^{a\mathsf{d_v}-b}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{b}\binom{(n-a)\mathsf{d_v}}{b(\mathsf{d_c}-1)}}\times\\ &\operatorname{coeff}\left(f(s_1)^a,s_1^b\right)\operatorname{coeff}\left(f(s_2)^{n-a},s_2^{b(\mathsf{d_c}-1)}\right) \end{aligned} \tag{6.205}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor\frac{\mathsf{d_v}-1}{2}\rfloor}\binom{\mathsf{d_v}}{j_1}s^{j_1} \tag{6.206}$$

$$g(x) =1 + \binom{\mathsf{d_c}}{2} x^2. \tag{6.207}$$

Proof. For a regular code, all VNs have degree $\mathsf{d_v}$. Therefore, $e + b$ in (6.193) is equal to $a\mathsf{d_v}$. Moreover, all CNs have degree $\mathsf{d_c}$. Thus, $b + l$ in (6.193) is equal to $b\mathsf{d_c}$. Since each unsatisfied CN is connected to exactly one type 2 edge, the number of ways to chose $a\mathsf{d_v} - b$ type 1, b type 2 edges and $b(\mathsf{d_c} - 1)$ type 3 edges such that there are exactly b unsatisfied CNs is $\binom{m}{b}\mathsf{d_c}^b \operatorname{coeff}\left(g(x)^{m-b}, x^{a\mathsf{d_v}-b}\right)$. The generating function $f(s)$ is the same as for the FAS case. ■

We show that for the computation of the normalized logarithmic asymptotic distribution of EFASs for regular codes, one needs to solve 2 equations instead of 7 for the irregular case.

Theorem 6.16. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EFASs for the $\mathscr{C}_n^{\mathsf{d_v},\mathsf{d_c}}$ ensemble is

$$\begin{aligned} G^{\mathsf{d_v},\mathsf{d_c}}_{\mathrm{EFAS}}(\theta, \gamma) =&(\xi - \gamma) \ln\left(g(x)\right) + \theta \ln\left(f(s_1)\right) + (1-\theta)\ln\left(f(s_2)\right) - \gamma\ln(s_1) \\ &- (\mathsf{d_v} - 1)H_{\mathsf{b}}(\theta) - \theta\mathsf{d_v}H_{\mathsf{b}}\left(\frac{\gamma}{\theta\mathsf{d_v}}\right) - (1-\theta)\mathsf{d_v}H_{\mathsf{b}}\left(\frac{\gamma(\mathsf{d_c}-1)}{(1-\theta)\mathsf{d_v}}\right) \\ &- (\theta\mathsf{d_v} - \gamma)\ln(x) - \gamma(\mathsf{d_c}-1)\ln(s_2) + \gamma\ln(\mathsf{d_c}) \end{aligned} \tag{6.208}$$

where

$$x = \sqrt{\frac{2(\theta\mathsf{d_v} - \gamma)}{\mathsf{d_c}(\mathsf{d_c}-1)(2\xi - \theta\mathsf{d_v} - \gamma)}} \tag{6.209}$$

and s_1, s_2 are the unique positive solutions of

$$\theta s_1 \frac{\mathrm{d}\ln f(s_1)}{\mathrm{d}s_1} =\gamma \tag{6.210}$$

$$(1-\theta)s_2 \frac{\mathrm{d}\ln f(s_2)}{\mathrm{d}s_2} =\gamma(\mathsf{d_c} - 1) \tag{6.211}$$

where $f(s)$ is defined in (6.206).

Example 6.2. Consider the regular $(3, 6)$ ensemble. We evaluate the normalized logarithmic asymptotic distributions of TSs and the corresponding approximations derived in Corollary 6.1. Fig. 6.2 compares the exact values of the normalized logarithmic asymptotic distributions of TSs, obtained from Theorem 6.7, and the approximations obtained from Corollary 6.1 for this ensemble. Observe that the approximations are accurate for small values of θ. Table 6.1 compares the exact values of the relative minimum Δ-trapping set

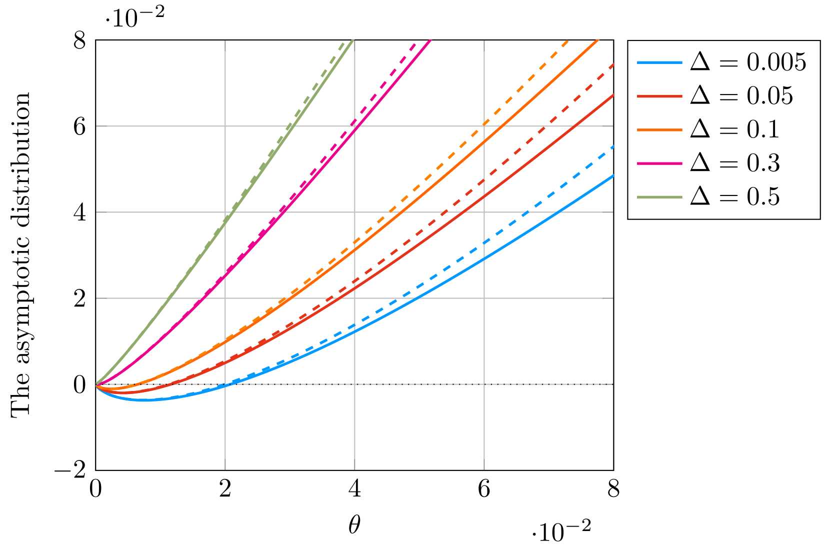


Figure 6.2: Normalized logarithmic asymptotic distributions of trapping sets of the regular $(3, 6)$ ensemble for several values of Δ. The dashed lines denote the corresponding approximations obtained from Corollary 6.1.

sizes and the approximations obtained from (6.123) for the regular $(3, 6)$ ensemble for different values of Δ. Observe that the values obtained from (6.123), which we derived by analyzing the asymptotic distributions of TSs for small θ are good approximations of the relative minimum Δ-trapping set sizes.

For this ensemble, we compare the exact computation of $\ln(\mathsf{E}^{\Lambda,\mathrm{P}}_{\mathrm{ETS}})/n$ and $\ln(\mathsf{E}^{\Lambda,\mathrm{P}}_{\mathrm{TS}})/n$ obtained from (6.108) and the asymptotic result according to Theorem 6.7. The normalized logarithmic distribution of (elementary) trapping sets for the asymptotic case (computed according to Theorem 6.7) and for $n \in \{50, 100, 200, 400\}$ (calculated from (6.108)) for a fixed ratio $\Delta = 0.5$ are depicted in Fig. 6.3. We observe that the asymptotic results give a good approximation for normalized logarithmic distributions even for short codes. As expected, for increasing n, the exact normalized logarithmic distributions approach the normalized logarithmic asymptotic distributions.

The expected distribution of $(2b, b)$ trapping and (fully) absorbing sets of the ensemble $\mathscr{C}^{3,6}_{900}$ derived in (6.124), (6.145) and (6.183) are depicted in Fig. 6.4.

Next, we use our theoretical results to estimate the error floor of the regular ensemble $\mathscr{C}^{3,6}_{6000}$. We picked 100 random codes from this ensemble, and simulated their performance under Gallager B [9] decoding over a BSC obtained by hard-quantizing the output of a biAWGN channel. The performance of the codes is provided in Fig. 6.5 in terms of BER versus E_{b}/N_0[dB]. In Fig. 6.5 an analytic estimate of the average error probability at large signal-to-noise ratios is given. The estimate is based on Eq. 1 in [43], where we considered

Table 6.1: The exact values of the relative minimum Δ-trapping set sizes $\theta^{\star}_{\mathrm{TS}}$ and their corresponding approximations obtained from (6.123) for the regular $(3, 6)$ ensemble

Δ	$\theta^{\star}_{\mathrm{TS}}$	$\theta^{\star}_{\mathrm{TS}}$ from (6.123)
0.001	0.02225844	0.02131029
0.005	0.02079887	0.01996448
0.050	0.01160803	0.01132319
0.100	0.00650212	0.00640276
0.150	0.00363893	0.00360421
0.200	0.00198389	0.00197234
0.250	0.00103484	0.00103132
0.300	5.07516e-04	5.06560-04
0.350	2.29400e-04	2.29179e-04
0.400	9.32314e-05	9.31898e-05

the dominant $(2, 2)$ FAS. As multiplicity of $(2, 2)$ FASs, we employed the average ensemble enumerator from (6.183). We observe that the codes provide an error floor performance that is in accordance with the estimated average error probability derived with the proposed analysis. Similar results have been observed for other blocklengths and quantized decoders.

Example 6.3. Consider the rate 1/2 LDPC ensemble with $\Lambda(x) = 0.5x^3 + 0.5x^4$, $\mathrm{P}(x) = x^7$. We evaluate the asymptotic distributions of trapping and (fully) absorbing sets according to Theorem 6.7, Theorem 6.9 and Theorem 6.13. The normalized logarithmic asymptotic distributions of TSs, ASs and FASs of this ensemble for fixed ratio $\Delta \in \{0.005, 0.05, 0.1, 0.3, 0.5\}$ are depicted in Fig. 6.6. We see that the gap between the normalized logarithmic asymptotic distributions of TSs, ASs and FASs vanishes for small θ. Further, we evaluate the asymptotic distributions of the elementary sets. Fig. 6.7 depicts the normalized logarithmic asymptotic distributions of ETSs, EASs and EFASs. We observe that normalized logarithmic asymptotic distributions of TSs and ETSs (ASs and EASs) are approximately equal and the gap grows slightly when θ increases.

6.3 Trapping and Absorbing Set Enumerators for Protograph-Based Ensembles

In this section, we derive the average finite-length (elementary) trapping and (fully) absorbing set enumerators for binary protograph-based LDPC codes and we present an analytical method to evaluate the normalized logarithmic asymptotic distributions of these

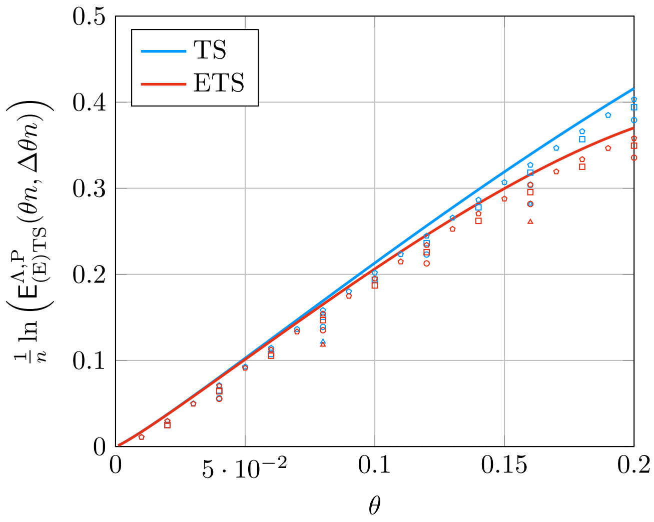


Figure 6.3: Normalized logarithmic asymptotic distributions of trapping and elementary trapping sets of the regular (3, 6) LDPC ensemble for $\Delta = 0.5$ (solid line). The triangles, dots, squares and pentagons are the exact normalized logarithmic trapping and elementary trapping set distributions for $n = 50, 100, 200, 400$, respectively.

sets.

6.3.1 Trapping and Elementary Trapping Set Distributions

Define the VN weight vector $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \dots, \epsilon_{n_\mathsf{P}}]$, where ϵ_j is the number of VNs of type $\mathtt{v}_j$ in $\mathcal{T}_{a,b}$. Clearly we have $0 \le \epsilon_j \le Q$ for all $j \in \{1, \dots, n_\mathsf{P}\}$ and

$$\sum_{j=1}^{n_\mathsf{P}} \epsilon_j = a. \tag{6.212}$$

Similarly, define the edge weight vector $\boldsymbol{w}(\boldsymbol{\epsilon}) = (w_g)_{g \in \mathscr{E}^\mathsf{P}}$ where w_g is the number of edges of type g in $\mathcal{T}_{a,b}$. The VN and edge weight vectors are related: for a given $\boldsymbol{\epsilon}$, we have $w_g = \epsilon_j$ if $g \in \mathscr{E}^\mathsf{P}_{\mathtt{v}_j}$.

Lemma 6.12. The average number of size (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^\mathsf{P}$ is

$$\mathsf{E}^\mathsf{P}_\mathrm{TS}(a,b) = \sum_{\boldsymbol{\epsilon}} \frac{\mathrm{coeff}\left(\prod_{i=1}^{m_\mathsf{P}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right)}{\prod_{j=1}^{n_\mathsf{P}} \binom{Q}{\epsilon_j}^{\mathsf{d}_{\mathsf{v}_j}-1}} \tag{6.213}$$

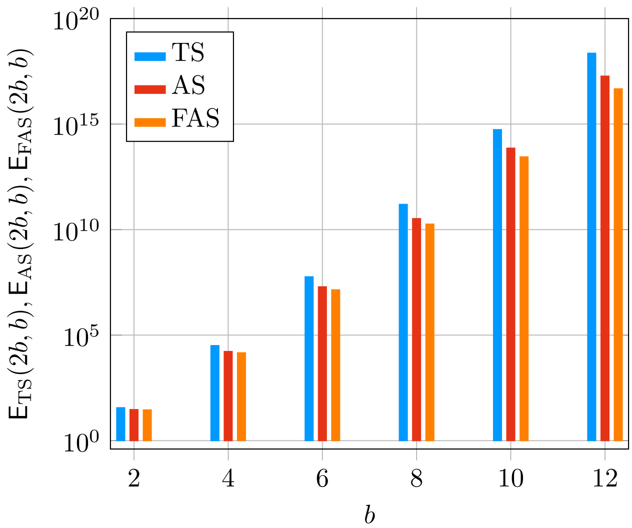


Figure 6.4: Expected distributions of $(2b, b)$ trapping and (fully) absorbing sets of the ensemble $\mathscr{C}_{900}^{3,6}$.

where

$$A_i(\boldsymbol{x}_i, y) = \frac{1}{2}\left[\prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}(1+x_g) + \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}(1-x_g)\right] + \frac{1}{2}\left[\prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}(1+x_g) - \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}(1-x_g)\right] y \quad (6.214)$$

and where $\boldsymbol{x} = (x_g)_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_i = (x_g)_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}$, y and $x_g, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}$ are dummy variables.

Proof. Consider the Tanner graph of a code drawn randomly from $\mathscr{C}_n^{\mathsf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set. For a given $\boldsymbol{\epsilon}$, each $\mathsf{v}_j \in \mathcal{V}^{\mathsf{P}}$ has ϵ_j replicas in $\mathcal{T}_{a,b}$. Since there are Q copies of each VN type in the lifted graph, the number of VN sets with weight vector $\boldsymbol{\epsilon}$ is

$$N_{\mathsf{v}}(\boldsymbol{\epsilon}) = \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}. \quad (6.215)$$

Since $w_g = \epsilon_j$ if $g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}$, the number of edge sets with weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ is

$$N_e(\boldsymbol{w}(\boldsymbol{\epsilon})) = \prod_{g\in\mathscr{E}^{\mathsf{P}}} \binom{Q}{w_g} = \prod_{j=1}^{n_{\mathsf{P}}} \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}} \binom{Q}{\epsilon_j} = \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}^{\mathsf{d}_{\mathsf{v}_j}}. \quad (6.216)$$

Let $N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))$ be the number of configurations with edge weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ that give

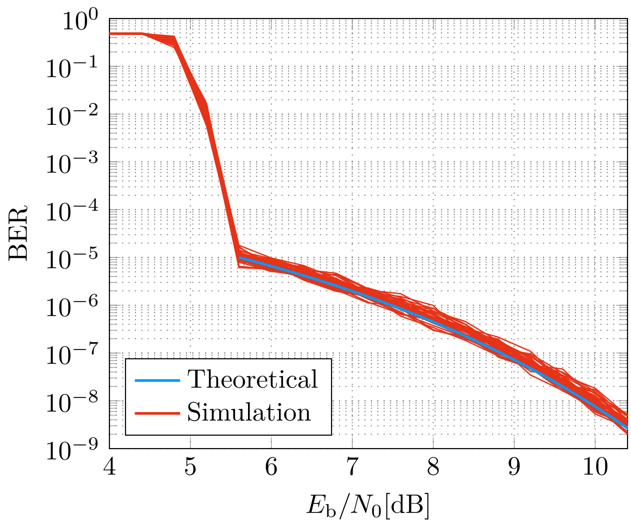


Figure 6.5: BER versus E_{b}/N_0[dB] for random codes drawn from the code ensemble $\mathscr{C}_{6000}^{3,6}$ under Gallager B decoding and the predicted average performance (error floor).

exactly b unsatisfied CNs. Its generating function is

$$\sum_{b,\boldsymbol{w}(\boldsymbol{\epsilon})} N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) y^b \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})}.$$

Recall that a CN is satisfied if it is connected an even number of times (including zero) to $\mathcal{I}$, and it is unsatisfied otherwise. Consider a CN of type c_i. The number of configurations for which the CN is satisfied is tracked by the generating function

$$g_{\mathsf{c}}(\boldsymbol{x}_i, y) := y^0 \sum_{\substack{\boldsymbol{c}\in\{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} \\ \mathrm{w_H}(\boldsymbol{c}) \text{ is even}}} \boldsymbol{x}_i^{\boldsymbol{c}} = \frac{1}{2}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1+x_g) + \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1-x_g)\right]. \tag{6.217}$$

The number of configurations for which the CN is unsatisfied is tracked by the generating function

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y) := y^1 \sum_{\substack{\boldsymbol{c}\in\{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} \\ \mathrm{w_H}(\boldsymbol{c}) \text{ is odd}}} \boldsymbol{x}_i^{\boldsymbol{c}} = \frac{1}{2}y\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1+x_g) - \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1-x_g)\right]. \tag{6.218}$$

In the expressions above, the exponent of the dummy variable y is used to track the status of the CN (i.e., the exponent is zero if the CN is satisfied, one otherwise). The sum $g_{\mathsf{c}}(\boldsymbol{x}_i, y) + g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y)$ yields $A_i(\boldsymbol{x}_i, y)$. Considering all CN types, and that there are Q CNs

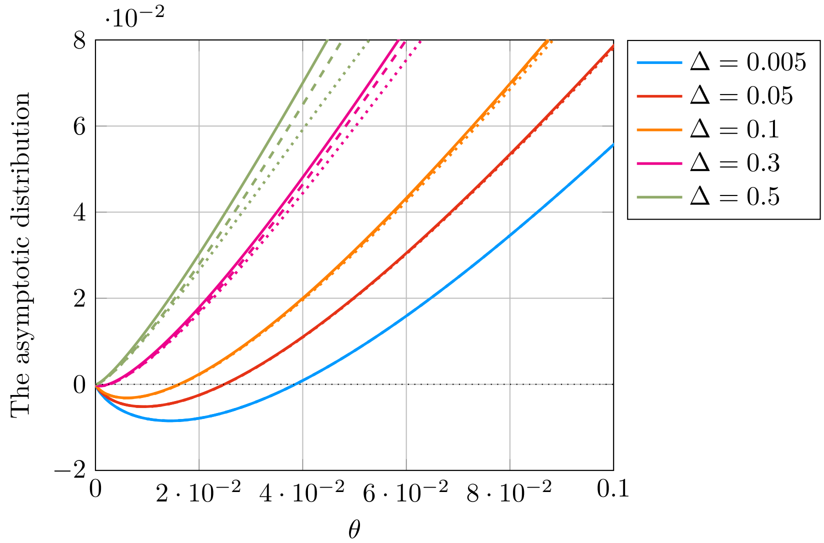


Figure 6.6: Normalized logarithmic asymptotic distributions of trapping sets (solid lines), absorbing sets (dashed lines) and fully absorbing sets (dotted lines) of the ensemble in Example 6.3.

of each type, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) = \operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right). \tag{6.219}$$

Using

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{P}}(a, b) = \sum_{\boldsymbol{\epsilon}} \frac{N_{\mathsf{v}}(\boldsymbol{\epsilon}) N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))}{N_e(\boldsymbol{w}(\boldsymbol{\epsilon}))} \tag{6.220}$$

completes the proof. ■

Lemma 6.12 provides the average number of size (a, b) TSs for a finite block length n. In the asymptotic case, we analyze the normalized logarithmic asymptotic distribution of TSs for the ensemble $\mathscr{C}_n^{\mathsf{P}}$ for $a = \theta n$ and $b = \gamma n$. The normalized logarithmic asymptotic distribution of TSs is a useful tool to analyze and design LDPC codes with good TS properties and can be computed efficiently. In particular, the analysis of the normalized logarithmic asymptotic distribution of TSs for a given U-NBPB LDPC code ensemble allows to determine if the expected number of TSs with size θn, with θ small, goes exponentially fast to zero, providing insights on the TS properties of the ensemble.

Theorem 6.17. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the ensemble

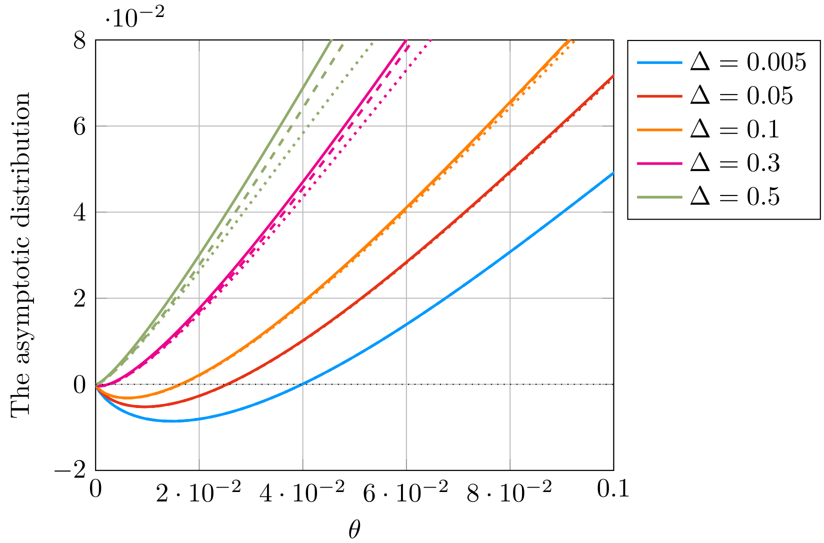


Figure 6.7: Normalized logarithmic asymptotic distributions of elementary trapping sets (solid lines), elementary absorbing sets (dashed lines) and elementary fully absorbing sets (dotted lines) of the ensemble in Example 6.3.

$\mathscr{C}_n^{\mathsf{P}}$ is

$$G_{\mathrm{TS}}^{\mathsf{P}}(\theta,\gamma) = \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i, y) - \gamma \ln y - \sum_{j=1}^{n_{\mathsf{P}}}\left[\frac{\mathsf{d}_{\mathsf{v}_j}-1}{n_{\mathsf{P}}} H_{\mathsf{b}}(n_{\mathsf{P}}\tilde{\epsilon}_j^\star) + \tilde{\epsilon}_j^\star \sum_{g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \ln x_g\right]. \quad (6.221)$$

The values x_g for $g \in \mathscr{E}^{\mathsf{P}}$, the value y and $\tilde{\epsilon}_j^\star$ for $j \in \{1,\ldots,n_{\mathsf{P}}\}$ are the unique positive solutions of

$$x_g \frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_g} = n_{\mathsf{P}} \tilde{w}_g^\star \quad (6.222)$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y} = n_{\mathsf{P}} \gamma \quad (6.223)$$

$$(\mathsf{d}_{\mathsf{v}_j} - 1) \ln\left(\frac{n_{\mathsf{P}}\tilde{\epsilon}_j^\star}{1 - n_{\mathsf{P}}\tilde{\epsilon}_j^\star}\right) = \sum_{g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \ln x_g + \mu \quad (6.224)$$

where (6.222) is valid for all $i \in \{1,\ldots,m_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$, μ is chosen to satisfy $\sum_j \tilde{\epsilon}_j^\star = \theta$ and $A_i(\boldsymbol{x}_i, y)$ is defined in (6.214), and $\tilde{w}_g^\star = \tilde{\epsilon}_j^\star$ if $g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}$. The proof of the Theorem can be found in Appendix 6.4.4.

Note that to determine the relative minimum Δ-trapping set size $\theta_{\mathrm{TS}}^\star$, we may add another equation to the system of Theorem 6.17, namely $G_{\mathrm{TS}}^{\mathsf{P}}(\theta, \Delta\theta) = 0$ with $0 < \theta \leq 1$.

Note that for computing the normalized asymptotic distribution of ETSs, we simply need to replace $A_i(\boldsymbol{x}_i, y)$ given in (6.214) with

$$A_i(\boldsymbol{x}_i, y) = 1 + \sum_{g_1, g_2 \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i} : g_1 \neq g_2} x_{g_1} x_{g_2} + y \sum_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g. \tag{6.225}$$

Theorem 6.17 shows that the evaluation of $G^{\mathsf{P}}_{\mathrm{TS}}(\theta, \gamma)$ and $G^{\mathsf{P}}_{\mathrm{ETS}}(\theta, \gamma)$ requires solving $e + n_{\mathsf{P}} + 2$ equations in $e + n_{\mathsf{P}} + 2$ variables: x_g (e variables), $\tilde{\epsilon}^{\star}_j$ (n_{P} variables), y (one variable) and μ (one variable). The following Lemma follows the approach of [119], and it can reduce the dimension of the system of equations by exploiting symmetries in the protograph.

Lemma 6.13. Let u, v be two edges in $\mathscr{E}^{\mathsf{P}}$. If u and v are connected to the same VN-CN pair in the protograph, then $x_u = x_v$.

Proof. Consider two edges u and v that connect c_i to v_j. Note that in this case $\tilde{w}^{\star}_u = \tilde{w}^{\star}_v = \tilde{\epsilon}^{\star}_j$. It is clear that the function $A_i(\boldsymbol{x}_i, y)$ in (6.214) is symmetric in the variables x_g, $g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}$. We have

$$\left.\frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_u}\right|_{\substack{x_u=\kappa \\ x_v=\beta}} = \left.\frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_v}\right|_{\substack{x_u=\beta \\ x_v=\kappa}} \tag{6.226}$$

$$\left.\frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y}\right|_{\substack{x_u=\kappa \\ x_v=\beta}} = \left.\frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y}\right|_{\substack{x_u=\beta \\ x_v=\kappa}}. \tag{6.227}$$

Thus, for the system of equations in Theorem 6.17, if there is a solution with $x_u = \kappa, x_v = \beta$ then another solution exists with $x_u = \beta, x_v = \kappa$ (all the other variables being unchanged). Since the solutions $x_g, g \in \mathscr{E}^{\mathsf{P}}$ are unique, we have $\kappa = \beta$. ∎

Remark 6.1. To avoid high error floors caused by trapping sets with specific $\Delta = b/a$, the relative minimum Δ-trapping set size of the code ensemble should satisfy $\theta^{\star}_{\mathrm{TS}} > 0$. In order to design code ensembles with good waterfall and error floor performance under a certain decoding algorithm, one can choose a threshold value $\theta_{\mathrm{th}} > 0$ of the relative minimum Δ-trapping set size and add the following step in the differential evolution [120]: evaluate the relative minimum Δ-trapping set size of the code ensemble and hold the base matrix only if $\theta^{\star}_{\mathrm{TS}} > \theta_{\mathrm{th}}$.

6.3.2 Absorbing and Elementary Absorbing Set Distributions

In this section, we derive the average finite-length (elementary) absorbing set enumerators for binary protograph-based LDPC codes and we present an analytical method for evaluating the normalized logarithmic asymptotic distributions of these sets.

Lemma 6.14. The average number of size (a, b) ASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{P}}(a,b) = \sum_{\boldsymbol{e},\boldsymbol{w}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}}(\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b\right)}{\prod_{g\in\mathscr{E}^{\mathsf{P}}} \binom{Q}{e_g+w_g}\binom{e_g+w_g}{e_g}} \times \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}}(\boldsymbol{s}^{(2)})^{\boldsymbol{w}}\right) \tag{6.228}$$

with

$$B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) = 1 + t \sum_{\boldsymbol{r}^{(j)}\in\mathcal{R}_j} (\boldsymbol{s}_j^{(1)})^{\mathbf{1}_{\mathsf{d}_{\mathsf{v}_\mathsf{j}}} - \boldsymbol{r}^{(j)}} (\boldsymbol{s}_j^{(2)})^{\boldsymbol{r}^{(j)}} \tag{6.229}$$

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = \frac{1}{2}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 + x_g^{(1)}) + \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 - x_g^{(1)})\right] + \frac{1}{2} y \left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 + x_g^{(2)}) - \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 - x_g^{(2)})\right] \tag{6.230}$$

and where $\mathbf{1}_{\mathsf{d}_{\mathsf{v}_\mathsf{j}}}$ is the length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ all-ones vector, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ and Hamming weight $\leq \lfloor(\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2\rfloor$, and $\boldsymbol{s}^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{s}_j^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}}$, $\boldsymbol{x}^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_i^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$, y, t and $s_g^{(o)}, x_g^{(o)}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}, o = 1, 2$ are dummy variables [54].

Proof. As before, define the edge weight vectors $\boldsymbol{e} = (e_g)_{g\in\mathscr{E}^{\mathsf{P}}}$ and $\boldsymbol{w} = (w_g)_{g\in\mathscr{E}^{\mathsf{P}}}$ where e_g is the number of edges of type g in $\mathcal{A}_{a,b}$ emanating from satisfied CNs and w_g represents the number of edges of type g in $\mathcal{A}_{a,b}$ emanating from unsatisfied CNs. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value one to each VN in the set. We denote by $N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w})$ the number of configurations with edge weight vectors $\boldsymbol{e}, \boldsymbol{w}$ that give exactly b unsatisfied CNs. Its generating function is

$$\sum_{b,\boldsymbol{e},\boldsymbol{w}} N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}) y^b (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}}.$$

Consider a CN of type c_i. The number of configurations for which the CN is satisfied is tracked by the generating function

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := y^0 \sum_{\substack{\boldsymbol{c} \in \{0,1\}^{\mathsf{d}_{\mathsf{c}_\mathsf{i}}} \\ \mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) \text{ is even}}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} = \frac{1}{2} \left[\prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 + x_g^{(1)}) + \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 - x_g^{(1)}) \right]. \tag{6.231}$$

The number of configurations for which the CN is unsatisfied is tracked by the generating function

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y) := y^1 \sum_{\substack{\boldsymbol{c} \in \{0,1\}^{\mathsf{d}_{\mathsf{c}_\mathsf{i}}} \\ \mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) \text{ is odd}}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{c}} = \frac{1}{2} y \left[\prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 + x_g^{(2)}) - \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 - x_g^{(2)}) \right]. \tag{6.232}$$

The sum $g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) + g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y)$ yields $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$. Considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff} \left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b \right) \tag{6.233}$$

where $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ is defined in (6.230). We denote by $N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w})$ the number of configurations with a VNs and edge weight vectors $\boldsymbol{e}, \boldsymbol{w}$ such that each of these VNs is connected to strictly fewer unsatisfied CNs than satisfied CNs. The corresponding generating function is

$$\sum_{a, \boldsymbol{e}, \boldsymbol{w}} N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}}.$$

Consider a VN of type v_j. Let $\boldsymbol{r}^{(j)} = (r_g^{(j)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$ be a length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ binary vector with $r_g^{(j)} = 1$ if the type g edge emanates from an unsatisfied CN and $r_g^{(j)} = 0$ otherwise. Note that if the VN of type v_j belongs to $\mathcal{I}$, the Hamming weight of $\boldsymbol{r}^{(j)}$ should satisfy $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) = \sum_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}} r_g^{(j)} \leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$. We can either include this VN in $\mathcal{I}$ or not. If we skip it we obtain the zero-degree term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ corresponding to zero VNs and zero edges. If we include it in the set, we will have one node, $\mathsf{d}_{\mathsf{v}_\mathsf{j}} - \mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges emanating from satisfied CNs and $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges emanating from unsatisfied CNs with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$. Considering all possible binary vectors $\boldsymbol{r}^{(j)}$, we obtain the second term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$. Taking into account all possible VN types and that there are Q VNs of each type, we get

$$N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff} \left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} \right) \tag{6.234}$$

where $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ is defined in (6.229). The number of edge sets with weight vectors $\boldsymbol{e}$ and $\boldsymbol{w}$ is

$$N_e(\boldsymbol{e}, \boldsymbol{w}) = \prod_{g \in \mathscr{E}^{\mathsf{P}}} \binom{Q}{e_g + w_g} \binom{e_g + w_g}{e_g}. \tag{6.235}$$

Noting that

$$\mathsf{E}^{\mathsf{P}}_{\mathrm{AS}}(a, b) = \sum_{\boldsymbol{e}, \boldsymbol{w}} \frac{N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w})}{N_e(\boldsymbol{e}, \boldsymbol{w})} \tag{6.236}$$

completes the proof. ■

Next, we analyze the normalized logarithmic asymptotic distribution of ASs for the protograph-based LDPC code ensemble.

The next Theorem presents a simple way to compute the normalized logarithmic asymptotic distribution of ASs for the ensemble $\mathscr{C}_n^{\mathsf{P}}$.

Theorem 6.18. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\begin{aligned} G^{\mathsf{P}}_{\mathrm{AS}}(\theta, \gamma) =& \frac{1}{n_{\mathsf{P}}} \sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) + \frac{1}{n_{\mathsf{P}}} \sum_{j=1}^{n_{\mathsf{P}}} \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) \\ & - \gamma \ln y - \theta \ln t - \frac{1}{n_{\mathsf{P}}} \sum_{g \in \mathscr{E}^{\mathsf{P}}} \ln \left(1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)}\right). \end{aligned} \tag{6.237}$$

The values t, $s_g^{(1)}$, $s_g^{(2)}$, $x_g^{(1)}$, $x_g^{(2)}$, for $g \in \mathscr{E}^{\mathsf{P}}$ and the value y are the unique positive solutions of

$$t \frac{\partial \ln \prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial t} = n_{\mathsf{P}} \theta \tag{6.238}$$

$$s_g^{(1)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(1)}} = x_g^{(1)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(1)}} = n_{\mathsf{P}} \tilde{e}_g^{\star} \tag{6.239}$$

$$s_g^{(2)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(2)}} = x_g^{(2)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(2)}} = n_{\mathsf{P}} \tilde{w}_g^{\star} \tag{6.240}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{6.241}$$

where (6.239) and (6.240) are for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, j \in \{1, \ldots, n_{\mathsf{P}}\}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j} \cap \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}$, and $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ are defined in (6.229) and (6.230), respectively and

$$\tilde{e}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(1)} s_g^{(1)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)}} \tag{6.242}$$

$$\tilde{w}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(2)} s_g^{(2)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)}}. \tag{6.243}$$

The proof of the Theorem can be found in Appendix 6.4.5.

To determine $\theta_{\text{AS}}^{\star}$, we add another equation to the system of equations of Theorem 6.18, namely $G_{\text{AS}}^{\mathsf{P}}(\theta, \Delta\theta) = 0$ with $0 < \theta \leq 1$.

The result can be easily extended to enumerate EASs. In fact, for computing the finite-length and the asymptotic distribution of EASs, we simply need to replace in (6.228) and (6.237) the generating function $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ given in (6.230) with

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = 1 + \sum_{g,g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g \neq g'} x_g^{(1)} x_{g'}^{(1)} + y \sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} x_g^{(2)}. \tag{6.244}$$

We briefly explain the derivation of $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ in (6.244). For EASs, each satisfied CN is connected to zero or two VNs from $\mathcal{I}$ and each unsatisfied CN is connected to exactly one VN from $\mathcal{I}$. Consider a CN of type c_i. If it is connected to zero or two VNs from $\mathcal{I}$, the number of configurations can be tracked by the generating function

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := y^0 \sum_{\boldsymbol{c} \in \{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} : \mathrm{w_H}(\boldsymbol{c}) \in \{0,2\}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} = 1 + \sum_{g,g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g \neq g'} x_g^{(1)} x_{g'}^{(1)}. \tag{6.245}$$

If the CN is connected to exactly one VN from $\mathcal{I}$ then its generating function is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y) := y^1 \sum_{\boldsymbol{c} \in \{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} : \mathrm{w_H}(\boldsymbol{c}) = 1} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{c}} = y \sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} x_g^{(2)}. \tag{6.246}$$

We can see from Theorem 6.18 that the evaluation of $G_{\text{AS}}^{\mathsf{P}}(\theta, \gamma)$ and $G_{\text{EAS}}^{\mathsf{P}}(\theta, \gamma)$ requires solving $4e + 2$ equations in $4e + 2$ variables: $s_g^{(1)}, s_g^{(2)}, x_g^{(1)}, x_g^{(2)}$ ($4e$ variables), y (one variable) and t (one variable). The following Lemma, also based on the approach of [119], is similar to Lemma 6.13 and can reduce the dimension of the system of equations.

Lemma 6.15. Let u, v be two edges in $\mathscr{E}^{\mathsf{P}}$. If u and v are connected to the same VN-CN pair in the protograph, then for all $o \in \{1, 2\}$, $s_u^{(o)} = s_v^{(o)}$ and $x_u^{(o)} = x_v^{(o)}$.

Proof. Consider two edges u and v which connect c_i to v_j. We define $\boldsymbol{z} := [s_u^{(1)},\ s_v^{(1)},\ s_u^{(2)},\ s_v^{(2)},\ x_u^{(1)},\ x_v^{(1)},\ x_u^{(2)},\ x_v^{(2)}]$, $\boldsymbol{z}_1 := [\chi_1,\ \psi_1,\ \chi_2,\ \psi_2,\ \kappa_1,\ \beta_1,\ \kappa_2,\ \beta_2]$ and $\boldsymbol{z}_2 := [\psi_1, \chi_1, \psi_2, \chi_2, \beta_1, \kappa_1, \beta_2, \kappa_2]$. It is clear that $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ in (6.229) is symmetric in the variables $s_g^{(1)}, s_g^{(2)}$, $g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}$ and the functions $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ in (6.230) and (6.244) are symmetric in the variables $x_g^{(1)}, x_g^{(2)}$, $g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$. Thus, for the system of equations in Theorem

6.9, if there is a solution with $\boldsymbol{z} = \boldsymbol{z}_1$ then another solution exists with $\boldsymbol{z} = \boldsymbol{z}_2$ (all the other variables being unchanged). Since the solutions $s_g^{(1)}$, $s_g^{(2)}$, $x_g^{(1)}$, $x_g^{(2)}$, $g \in \mathscr{E}^{\mathsf{P}}$ are unique, we have $\boldsymbol{z}_1 = \boldsymbol{z}_2$, i.e., $\psi_1 = \chi_1, \psi_2 = \chi_2, \kappa_1 = \beta_1, \kappa_2 = \beta_2$. ■

6.3.3 Fully and Elementary Fully Absorbing Set Distributions

We derived in [61] the average finite-length fully absorbing and elementary fully absorbing set enumerators for binary protograph-based LDPC codes and we present an analytical method for evaluating the normalized logarithmic asymptotic distributions of these sets.

Lemma 6.16. The average number of (a, b) FASs in the graph G of a code drawn randomly from the code ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\mathsf{E}^{\mathsf{P}}_{\mathrm{FAS}}(a,b) = \sum_{\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}} \frac{N_{\mathsf{v}}(a,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}) N_{\mathsf{c}}(b,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l})}{\prod_{g\in\mathscr{E}^{\mathsf{P}}} \binom{Q}{e_g+w_g}\binom{e_g+w_g}{e_g}\binom{Q-e_g-w_g}{l_g}} \tag{6.247}$$

with

$$N_{\mathsf{v}}(a,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}) = \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j\left(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)}\right)^Q, t^a \left(\boldsymbol{s}^{(1)}\right)^{\boldsymbol{e}} \left(\boldsymbol{s}^{(2)}\right)^{\boldsymbol{w}} \left(\boldsymbol{s}^{(3)}\right)^{\boldsymbol{l}}\right) \tag{6.248}$$

$$N_{\mathsf{c}}(b,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}) = \operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i\left(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y\right)^Q, \left(\boldsymbol{x}^{(1)}\right)^{\boldsymbol{e}} \left(\boldsymbol{x}^{(2)}\right)^{\boldsymbol{w}} \left(\boldsymbol{x}^{(3)}\right)^{\boldsymbol{l}} y^b\right) \tag{6.249}$$

$$\begin{aligned} A_i\left(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y\right) = & \frac{1}{2}\left[\prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(1 + x_g^{(1)}\right) + \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(1 - x_g^{(1)}\right)\right] \\ & + \frac{1}{2} y \left[\prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(x_g^{(3)} + x_g^{(2)}\right) - \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(x_g^{(3)} - x_g^{(2)}\right)\right] \end{aligned} \tag{6.250}$$

$$B_j\left(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)}\right) = \sum_{\boldsymbol{r}^{(j)}\in\mathcal{R}_j} \left(\boldsymbol{s}_j^{(3)}\right)^{\boldsymbol{r}^{(j)}} + t \sum_{\boldsymbol{r}^{(j)}\in\mathcal{R}_j} \left(\boldsymbol{s}_j^{(1)}\right)^{\boldsymbol{1}_j - \boldsymbol{r}^{(j)}} \left(\boldsymbol{s}_j^{(2)}\right)^{\boldsymbol{r}^{(j)}} \tag{6.251}$$

where $\boldsymbol{1}_j$ is the length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ all-ones vector, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ and Hamming weight lower than or equal to $\lfloor(\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2\rfloor$, and $\boldsymbol{s}^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{s}_j^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{v}_\mathsf{j}}}$, $\boldsymbol{x}^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_i^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}$, y, t and $s_g^{(o)}, x_g^{(o)}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}, o = 1, 2, 3$ are dummy variables.

Proof. Consider the graph G of a code drawn uniformly at random from the ensemble. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set. The edges connected to a VN $\mathtt{v}$ are assigned the binary value chosen for $\mathtt{v}$. We introduce the edge weight vectors $\boldsymbol{e} = (e_g)_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{w} = (w_g)_{g\in\mathscr{E}^{\mathsf{P}}}$ and $\boldsymbol{l} = (l_g)_{g\in\mathscr{E}^{\mathsf{P}}}$. Here, e_g is the number of type-g edges adjacent to VNs in $\mathcal{F}_{a,b}$ that are connected to satisfied CNs, w_g is the number of type-g edges adjacent to VNs in $\mathcal{F}_{a,b}$ that are connected to unsatisfied CNs, and l_g is the number of type-g edges adjacent to VNs in $\mathcal{V} \setminus \mathcal{F}_{a,b}$ that are connected to unsatisfied CNs.

Let $N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})$ be the number of configurations with edge set weight vectors $(\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})$ that give exactly b unsatisfied CNs. Its generating function is

$$F_{\mathsf{c}}\left(y, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}\right) := \sum_{b,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}} N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) y^b (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} (\boldsymbol{x}^{(3)})^{\boldsymbol{l}}$$

where the dummy variable y is used to track the number of unsatisfied CNs. Recall that a CN is satisfied if it is connected an even number of times (including zero) to $\mathcal{I}$, and it is unsatisfied otherwise. Consider now a CN of type c_i. If it is satisfied, then its generating function is

$$g_{\mathsf{c}}\left(y, \boldsymbol{x}_i^{(1)}\right) := y^0 \sum_{\boldsymbol{c}\in\{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} : \mathrm{w_H}(\boldsymbol{c}) \text{ is even}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} = \frac{1}{2} \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 + x_g^{(1)}) + \frac{1}{2} \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 - x_g^{(1)})$$

If the CN is unsatisfied then its generating function is

$$g_{\bar{\mathsf{c}}}\left(y, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}\right) := y^1 \sum_{\boldsymbol{c}\in\{0,1\}^{\mathsf{d}_{\mathsf{c}_i}} : \mathrm{w_H}(\boldsymbol{c}) \text{ odd}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{c}} (\boldsymbol{x}_i^{(3)})^{\mathbf{1}_{\mathsf{d}_{\mathsf{c}_i}} - \boldsymbol{c}} = \frac{y}{2} \left[\prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (x_g^{(3)} + x_g^{(2)}) - \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (x_g^{(3)} - x_g^{(2)}) \right]$$

where $\mathbf{1}_{\mathsf{d}_{\mathsf{c}_i}}$ is the length $\mathsf{d}_{\mathsf{c}_i}$ all-ones vector and $\mathrm{w_H}(\boldsymbol{c})$ is the Hamming weight of $\boldsymbol{c}$. The overall generating function for a type-c_i CN is hence $g_{\mathsf{c}}\left(y, \boldsymbol{x}_i^{(1)}\right) + g_{\bar{\mathsf{c}}}\left(y, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}\right)$ as in (6.250). By noting that $F_{\mathsf{c}}\left(y, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}\right)$ can be obtained by raising (6.250) to the power of Q (lifting factor) and by multiplying the result for $i = 1, \ldots, m_{\mathsf{P}}$, we obtain (6.249). Let $N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})$ be the number of configurations with a VNs and edge set weight vectors $(\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})$ such that each VN is connected to strictly fewer unsatisfied CNs than satisfied CNs. Its generating function is

$$F_{\mathsf{v}}\left(t, \boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}, \boldsymbol{s}^{(3)}\right) := \sum_{a,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}} N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} (\boldsymbol{s}^{(3)})^{\boldsymbol{l}}.$$

Consider a VN of type v_j. We are interested in computing the corresponding generating function $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$. Let $\boldsymbol{r}^{(j)} = \left(r_g^{(j)}\right)_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$ be a length $\mathsf{d}_{\mathsf{v}_j}$ binary vector with $r_g^{(j)} = 1$ if the type g edge emanates from an unsatisfied CN and $r_g^{(j)} = 0$ otherwise. Note that for each VN of type v_j, the vector $\boldsymbol{r}^{(j)}$ should satisfy $\mathrm{w_H}(\boldsymbol{r}^{(j)}) \leq \lfloor (\mathsf{d}_{\mathsf{v}_j} - 1)/2 \rfloor$. We can either include this VN type in $\mathcal{I}$ or not. If we skip it, we obtain the first term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$

corresponding to zero VNs and $\mathsf{w}_\mathrm{H}(\boldsymbol{r}^{(j)}) \leq \lfloor(\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2\rfloor$ edges emanating from unsatisfied CNs and VNs outside $\mathcal{I}$. If we include it in $\mathcal{I}$, we will have 1 node, $\mathsf{d}_{\mathsf{v}_\mathsf{j}} - \mathsf{w}_\mathrm{H}(\boldsymbol{r}^{(j)})$ edges emanating from satisfied CNs and $\mathsf{w}_\mathrm{H}(\boldsymbol{r}^{(j)})$ edges emanating from unsatisfied CNs with $\mathsf{w}_\mathrm{H}(\boldsymbol{r}^{(j)}) \leq \lfloor(\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2\rfloor$. Considering all binary vectors $\boldsymbol{r}^{(j)}$, we obtain the second term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$. $F_\mathsf{v}\,(t, \boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}, \boldsymbol{s}^{(3)})$ is obtained by raising (6.251) to the power Q and by multiplying the result for $j = 1, \ldots, n_\mathsf{P}$, yielding (6.248).

The number of edge sets with weight vectors $\boldsymbol{e}$, $\boldsymbol{w}$ and $\boldsymbol{l}$ is

$$N_e(\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) = \prod_{g \in \mathscr{E}^\mathsf{P}} \binom{Q}{e_g + w_g} \binom{e_g + w_g}{e_g} \binom{Q - e_g - w_g}{l_g}.$$

The proof is completed by substituting these expressions in

$$\mathsf{E}^\mathsf{P}_\mathrm{FAS}(a, b) = \sum_{\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}} \frac{N_\mathsf{v}(a, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) N_\mathsf{c}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})}{N_e(\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})}. \tag{6.252}$$

■

Lemma 6.16 yields the average number of (a, b) FASs for finite n. In the asymptotic case, the following Theorem provides the normalized logarithmic asymptotic distribution of FASs for the ensemble $\mathscr{C}_n^\mathsf{P}$ for $a = \theta n$ and $b = \gamma n$.

Theorem 6.19. The normalized logarithmic asymptotic distribution of $(\theta n, \gamma n)$ FASs for the ensemble $\mathscr{C}_n^\mathsf{P}$ is

$$\begin{aligned} G^\mathsf{P}_\mathrm{FAS}(\theta, \gamma) =& \frac{1}{n_\mathsf{P}} \sum_{i=1}^{m_\mathsf{P}} \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) + \frac{1}{n_\mathsf{P}} \sum_{j=1}^{n_\mathsf{P}} \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)}) \\ &- \theta \ln(t) - \gamma \ln(y) - \frac{1}{n_\mathsf{P}} \sum_{g \in \mathscr{E}^\mathsf{P}} \ln\left(1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)} + x_g^{(3)} s_g^{(3)}\right) \end{aligned} \tag{6.253}$$

where the values t, $s_g^{(1)}, s_g^{(2)}, s_g^{(3)}, x_g^{(1)}, x_g^{(2)}, x_g^{(3)}$ for $g \in \mathscr{E}^\mathsf{P}$ and the value y are the unique positive solutions of

$$t \frac{\partial \ln \prod_{j=1}^{n_\mathsf{P}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})}{\partial t} = n_\mathsf{P} \theta \tag{6.254}$$

$$s_g^{(1)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})}{\partial s_g^{(1)}} = x_g^{(1)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)}{\partial x_g^{(1)}} = n_\mathsf{P} \tilde{e}_g^\star \tag{6.255}$$

$$s_g^{(2)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})}{\partial s_g^{(2)}} = x_g^{(2)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)}{\partial x_g^{(2)}} = n_\mathsf{P} \tilde{w}_g^\star \tag{6.256}$$

$$s_g^{(3)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})}{\partial s_g^{(3)}} = x_g^{(3)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)}{\partial x_g^{(3)}} = n_{\mathsf{P}} \tilde{l}_g^{\star} \tag{6.257}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{6.258}$$

where (6.255)-(6.257) are for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, j \in \{1, \ldots, n_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}} \cap \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$, and $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$ and $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)$ are defined in (6.251) and (6.250), respectively and

$$\tilde{e}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(1)} s_g^{(1)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)} + x_g^{(3)} s_g^{(3)}} \tag{6.259}$$

$$\tilde{w}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(2)} s_g^{(2)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)} + x_g^{(3)} s_g^{(3)}} \tag{6.260}$$

$$\tilde{l}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(3)} s_g^{(3)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)} + x_g^{(3)} s_g^{(3)}}. \tag{6.261}$$

The proof is similar to the one of Theorem 6.4.5.

To determine $\theta_{\mathrm{FAS}}^{\star}$ we add another equation to the system of equations of Theorem 6.19, namely $G_{\mathrm{FAS}}^{\mathsf{P}}(\theta, \Delta\theta) = 0$ with $0 < \theta \leq 1$.

Observe that the computation of the asymptotic distribution of EFASs follows similar steps. In particular, it suffices to replace $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)$ given in (6.250) with

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) = 1 + \sum_{g, g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g' \neq g} x_g^{(1)} x_{g'}^{(1)} + y \sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} x_g^{(2)} \prod_{g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g' \neq g} x_{g'}^{(3)}. \tag{6.262}$$

Remark 6.2. To evaluate the normalized asymptotic distribution of (elementary) FASs, one needs to solve a system of $6e + 2$ equations with $6e + 2$ variables. The solution is complex for protographs with several edges. Fortunately, protographs often have parallel edges between pairs of nodes, yielding symmetries in the equations. This observation was used in [119] to reduce the dimension of the system of equations needed to evaluate the weight spectral shape of protograph-based ensembles. The principle is applied to the FAS enumeration through the following Corollary (the proof is omitted, since it follows closely the derivation of Lemma 6.15).

Lemma 6.17. Denote by u, v two parallel edges in the protograph. Then $s_u^{(\ell)} = s_v^{(\ell)}$ and $x_u^{(\ell)} = x_v^{(\ell)}$ for $\ell \in \{1, 2, 3\}$.

Example 6.4. We next give an example applying the technique described in Section 6.3.1, Section 6.3.2 and Section 6.3.3 to the analysis of the protograph-based ensemble with the base

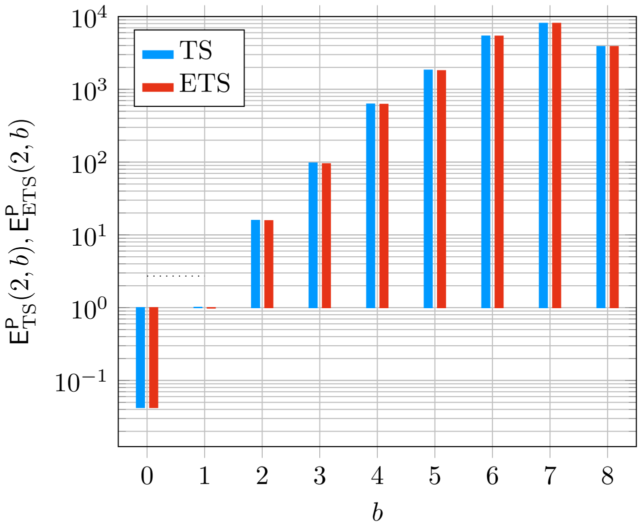


Figure 6.8: Expected distributions of $(2, b)$ trapping and elementary trapping sets of the ensemble in Example 6.4 for $Q = 100$.

matrix $\boldsymbol{B} = [\,3\;\;4\,]$. This base matrix was introduced in [121] to design short, low error floor LDPC codes for satellite telecommand links. The expected distributions of $(2, b)$ trapping and elementary trapping sets for $Q = 100$ are shown in Fig. 6.8. We evaluate the expressions of the normalized logarithmic asymptotic distribution of trapping and (fully) absorbing sets from Theorem 6.17, Theorem 6.18 and Theorem 6.19 for $\Delta \in \{0.005, 0.05, 0.1, 0.2, 0.3, 0.4\}$. The results are shown in Fig. 6.9. Fig. 6.10 depicts the normalized logarithmic asymptotic distributions of ETSs, EASs and EFASs. Observe that the relative minimum Δ-trapping and (fully) absorbing set size decreases as Δ increases. For instance, for $\Delta = 0.005$, we have $\theta^{\star}_{\mathrm{FAS}} = 0.040129$ and for $\Delta = 0.2$, $\theta^{\star}_{\mathrm{FAS}} = 0.008448$. Since the protograph has $e = 7$ edges, we need to solve a system of $6e + 2 = 44$ equations in 44 unknowns to compute the normalized asymptotic distribution of (elementary) fully absorbing sets. Thanks to Lemma 6.17, the number of equations/unknowns reduces to 14.

6.4 Appendices

6.4.1 Proof of Theorem 6.7

From Lemma 3.1, we have

$$\mathrm{coeff}\left(f(t,s)^n, t^{n\theta}s^{n\tilde{w}}\right) \doteq \exp\left\{n\left[\ln(f(t,s)) - \theta\ln(t) - \tilde{w}\ln(s)\right]\right\} \tag{6.263}$$
$$\mathrm{coeff}\left(g(x,y)^n, x^{n\tilde{w}}y^{n\gamma}\right) \doteq \exp\left\{n\left[\ln(g(x,y)) - \gamma\ln(y) - \tilde{w}\ln(x)\right]\right\} \tag{6.264}$$

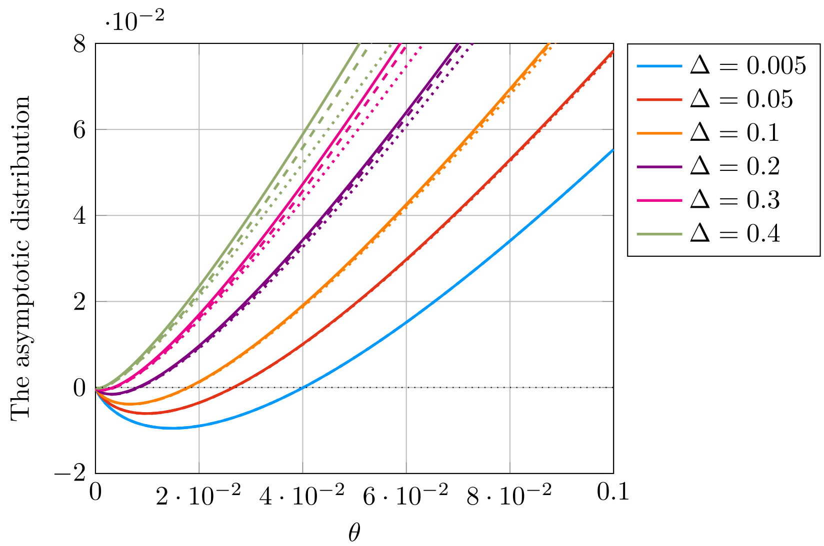


Figure 6.9: Normalized logarithmic asymptotic distributions of trapping sets (solid lines), absorbing sets (dashed lines) and fully absorbing sets (dotted lines) for the code ensemble in Example 6.4.

where $\tilde{w} = w/n$ and t, s, x, y are the unique positive solutions of (6.114)-(6.116) with $\tilde{w}^\star$ replaced by $\tilde{w}$.

Lemma 3.2 gives

$$\binom{n\bar{\mathsf{d}}_\mathsf{v}}{n\tilde{w}} \doteq \exp\left\{n\bar{\mathsf{d}}_\mathsf{v} H_\mathsf{b}\left(\frac{\tilde{w}}{\bar{\mathsf{d}}_\mathsf{v}}\right)\right\} \tag{6.265}$$

and from (6.263), (6.264) and (6.265), we have

$$\mathsf{E}_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) \doteq \sum_{\tilde{w}} \exp(nS(\tilde{w})) \tag{6.266}$$

with

$$S(\tilde{w}) = -\bar{\mathsf{d}}_\mathsf{v} H_\mathsf{b}\left(\frac{\tilde{w}}{\bar{\mathsf{d}}_\mathsf{v}}\right) + \ln(f(t,s)) - \theta\ln(t) - \tilde{w}\ln(xs) + \ln(g(x,y)) - \gamma\ln(y). \tag{6.267}$$

Thus, we have $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = \max S(\tilde{w})$. By setting the derivative of $S(\tilde{w})$ in $\tilde{w}$ to zero, we obtain

$$\tilde{w}^\star = \operatorname*{argmax}_{\tilde{w}} S(\tilde{w}) = \bar{\mathsf{d}}_\mathsf{v}\frac{xs}{1+xs}. \tag{6.268}$$

By substituting (6.268) in (6.267), we obtain (6.113)-(6.117) as desired.

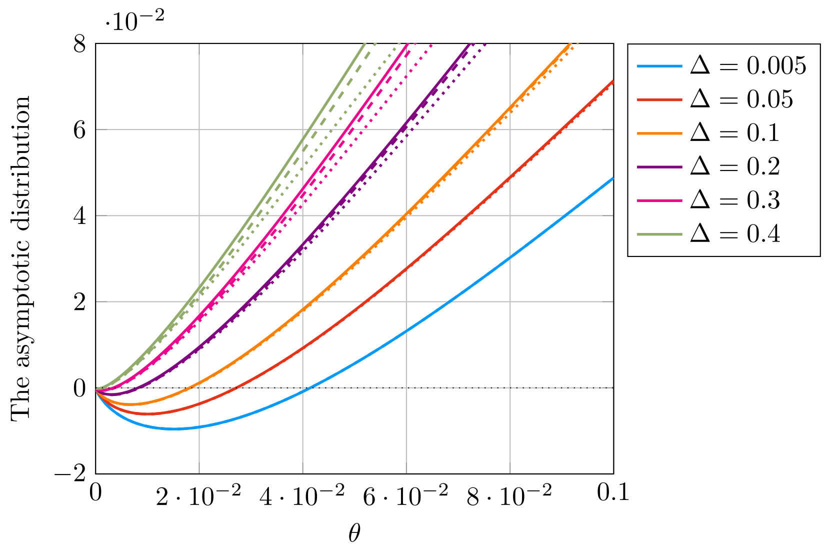


Figure 6.10: Normalized logarithmic asymptotic distributions of elementary trapping sets (solid lines), elementary absorbing sets (dashed lines) and elementary fully absorbing sets (dotted lines) for the code ensemble in Example 6.4.

6.4.2 Proof of Corollary 6.1

The proof is based on obtaining expressions for t, s, x, y in terms of $\tilde{w}^{\star}$, and for $\tilde{w}^{\star}$ in terms of θ. Consider (6.114)-(6.116) when $\theta \to 0$ and $\gamma = \Delta\theta$. These equations can be rewritten as

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \Lambda_j \frac{ts^j}{1+ts^j} = \theta \tag{6.269}$$

$$\sum_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \Lambda_j \frac{jts^j}{1+ts^j} = \tilde{w}^{\star} \tag{6.270}$$

$$\xi \sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \mathrm{P}_i \frac{\sum_{\substack{k=2 \\ k \text{ is even}}}^{i} \binom{i}{k} k x^k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} \binom{i}{k} k x^k y}{1 + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i} \binom{i}{k} x^k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} \binom{i}{k} x^k y} = \tilde{w}^{\star} \tag{6.271}$$

$$\xi \sum_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \mathrm{P}_i \frac{\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} \binom{i}{k} x^k y}{1 + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i} \binom{i}{k} x^k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} \binom{i}{k} x^k y} = \Delta\theta. \tag{6.272}$$

From (6.269) and (6.270), we see that $\mathsf{d}_\mathsf{v}^{\min}\theta \le \tilde{w}^\star \le \mathsf{d}_\mathsf{v}^{\max}\theta$. Thus, we have

$$\lim_{\theta\to 0} \tilde{w}^\star = 0 \tag{6.273}$$

and the notations $o(\theta)$ and $o(\tilde{w}^\star)$ are equivalent, i.e., for any function f, $f = o(\theta) \Longleftrightarrow f = o(\tilde{w}^\star)$. Therefore, we will use $o(\theta)$ and $o(\tilde{w}^\star)$ interchangeably. The left hand side of (6.272) is also $o(1)$, i.e., for some odd k we have $x^k y = o(1)$ and for all other k we have $x^k y = o(\theta)$. Thus, we have

$$\Delta\theta(1 + o(1)) = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i ixy = \bar{\mathsf{d}}_\mathsf{v} xy. \tag{6.274}$$

Because of (6.273), the left hand side of (6.271) must be $o(1)$, i.e., $x^k = o(1)$ for some k, $x^k = o(\tilde{w}^\star)$ for the other k, $x^k y = o(1)$ for some k and $x^k y = o(\tilde{w}^\star)$ for the other k. The left hand side of (6.271) is dominated by the terms corresponding to $k = 1, 2$. We have

$$\begin{aligned} \tilde{w}^\star(1 + o(1)) =& \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \left[i(i-1)x^2 + ixy \right] \\ =& \xi \mathrm{P}''(1)x^2 + \bar{\mathsf{d}}_\mathsf{v} xy. \end{aligned} \tag{6.275}$$

From (6.274) and (6.275), we obtain

$$x = \sqrt{\frac{\tilde{w}^\star - \Delta\theta}{\xi \mathrm{P}''(1)}}(1 + o(1)). \tag{6.276}$$

Substituting (6.274) into (6.117), we obtain

$$s = \tilde{w}^\star \sqrt{\frac{\mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(\tilde{w}^\star - \Delta\theta)}}(1 + o(1)). \tag{6.277}$$

Thus, (6.270) can be written as

$$\tilde{w}^\star(1 + o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}} \mathsf{d}_\mathsf{v}^{\min} t s^{\mathsf{d}_\mathsf{v}^{\min}} \tag{6.278}$$

and we have

$$t = \frac{1}{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}} \mathsf{d}_\mathsf{v}^{\min} (\tilde{w}^\star)^{\mathsf{d}_\mathsf{v}^{\min}-1}} \left(\frac{(\tilde{w}^\star - \Delta\theta)\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}}{\mathrm{P}''(1)} \right)^{\frac{\mathsf{d}_\mathsf{v}^{\min}}{2}} (1 + o(1)). \tag{6.279}$$

Similarly, from (6.269), we have

$$\theta(1+o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}} t s^{\mathsf{d}_\mathsf{v}^{\min}} \tag{6.280}$$

and

$$\tilde{w}^\star = \mathsf{d}_\mathsf{v}^{\min}\theta(1+o(1)). \tag{6.281}$$

Substituting (6.281) into (6.276), (6.277) and (6.279), we obtain

$$x = \sqrt{\theta\frac{\mathsf{d}_\mathsf{v}^{\min}-\Delta}{\xi \mathrm{P}''(1)}}(1+o(1)) \tag{6.282}$$

$$s = \mathsf{d}_\mathsf{v}^{\min}\sqrt{\theta\frac{\mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(\mathsf{d}_\mathsf{v}^{\min}-\Delta)}}(1+o(1)) \tag{6.283}$$

$$t = \frac{\theta^{\frac{2-\mathsf{d}_\mathsf{v}^{\min}}{2}}}{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}}(\mathsf{d}_\mathsf{v}^{\min})^{\mathsf{d}_\mathsf{v}^{\min}}}\left(\frac{(\mathsf{d}_\mathsf{v}^{\min}-\Delta)\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}}{\mathrm{P}''(1)}\right)^{\frac{\mathsf{d}_\mathsf{v}^{\min}}{2}}(1+o(1)). \tag{6.284}$$

From (6.274) and (6.282), we have

$$y = \Delta\sqrt{\frac{\theta \mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(\mathsf{d}_\mathsf{v}^{\min}-\Delta)}}(1+o(1)). \tag{6.285}$$

By substituting (6.282)-(6.285) into (6.113) and by using the Taylor series of $\ln(1+x)$ at $x=0$, we obtain (6.121).

6.4.3 Proof of Theorem 6.8

From Lemma 3.1, we have

$$\operatorname{coeff}\left(g(x,y)^n, x^{n\theta\mathsf{d}_\mathsf{v}}y^{n\gamma}\right) \doteq \exp\left\{n\left[\ln(g(x,y)) - \gamma\ln(y) - \theta\mathsf{d}_\mathsf{v}\ln(x)\right]\right\} \tag{6.286}$$

where x, y are the unique positive solutions of

$$x\frac{\partial \ln g(x,y)}{\partial x} = \theta\mathsf{d}_\mathsf{v} \tag{6.287}$$

$$y\frac{\partial \ln g(x,y)}{\partial y} = \gamma. \tag{6.288}$$

We obtain from (6.288)

$$y = \frac{\gamma}{\xi - \gamma}\frac{(1+x)^{\mathsf{d_c}} + (1-x)^{\mathsf{d_c}}}{(1+x)^{\mathsf{d_c}} - (1-x)^{\mathsf{d_c}}}. \tag{6.289}$$

Lemma 3.2 gives

$$\binom{n}{n\theta} \doteq \exp\left\{nH_{\mathsf{b}}\left(\theta\right)\right\} \tag{6.290}$$

$$\binom{n\mathsf{d_v}}{n\theta\mathsf{d_v}} \doteq \exp\left\{n\mathsf{d_v}H_{\mathsf{b}}\left(\theta\right)\right\}. \tag{6.291}$$

From (6.286) and (6.289)-(6.291), we obtain (6.126)-(6.128).

6.4.4 Proof of Theorem 6.17

From Lemma 3.1, and recalling that $Q = n/n_{\mathsf{P}}$ we have

$$\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^{\frac{n}{n_{\mathsf{P}}}}, \boldsymbol{x}^{n\tilde{\boldsymbol{w}}(\tilde{\boldsymbol{\theta}})}y^{\gamma n}\right) \doteq \exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i, y) - \gamma \ln y - \sum_{g\in\mathscr{E}^{\mathsf{P}}}\tilde{w}_g \ln x_g\right]\right\}$$

where $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}/n$, $\tilde{\boldsymbol{w}}(\tilde{\boldsymbol{\epsilon}}) = \boldsymbol{w}(\boldsymbol{\epsilon})/n$ and y and x_g for $g \in \mathscr{E}^{\mathsf{P}}$ are the unique positive solutions of (6.222) and (6.223) if we replace $\tilde{w}_g^\star$ by $\tilde{w}_g$. We obtain

$$\prod_{j=1}^{n_{\mathsf{P}}}\binom{Q}{\epsilon_j}^{\mathsf{d_{v_j}}-1} \doteq \exp\left\{n\sum_{j=1}^{n_{\mathsf{P}}}(\mathsf{d_{v_j}}-1)\frac{1}{n_{\mathsf{P}}}H_{\mathsf{b}}(n_{\mathsf{P}}\tilde{\epsilon}_j)\right\}.$$

Thus, we have

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{P}}(\theta n, \gamma n) \doteq \sum_{\tilde{\boldsymbol{\epsilon}}}\exp(nS(\tilde{\boldsymbol{\epsilon}}))$$

where

$$S(\tilde{\boldsymbol{\epsilon}}) = \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i, y) - \sum_{g\in\mathscr{E}^{\mathsf{P}}}\tilde{w}_g \ln x_g - \gamma\ln y - \frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}(\mathsf{d_{v_j}}-1)H_{\mathsf{b}}(n_{\mathsf{P}}\tilde{\epsilon}_j).$$

Hence, we have $G_{\mathrm{TS}}^{\mathsf{P}}(\theta,\gamma) = \max S(\tilde{\boldsymbol{\epsilon}})$ subject to the constraint $\sum_j \tilde{\epsilon}_j = \theta$ obtained from (6.212) (see e.g., [78, 119, 122]). Using Lagrangian multipliers, we obtain the entries of $\tilde{\boldsymbol{\epsilon}}^\star = \operatorname{argmax} S(\tilde{\boldsymbol{\epsilon}})$ in (6.224).

6.4.5 Proof of Theorem 6.18

From Lemma 3.1, we have

$$\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)^{\frac{n}{n_{\mathsf{P}}}},(\boldsymbol{x}_i^{(1)})^{n\tilde{\boldsymbol{e}}}(\boldsymbol{x}_i^{(2)})^{n\tilde{\boldsymbol{w}}}y^{\gamma n}\right)\doteq\exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)-\gamma\ln y\right.\right.$$
$$\left.\left.-\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left(\tilde{e}_g\ln x_g^{(1)}+\tilde{w}_g\ln x_g^{(2)}\right)\right]\right\} \tag{6.292}$$

$$\operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})^{\frac{n}{n_{\mathsf{P}}}},(\boldsymbol{s}_j^{(1)})^{n\tilde{\boldsymbol{e}}}(\boldsymbol{s}_j^{(2)})^{n\tilde{\boldsymbol{w}}}t^{\theta n}\right)\doteq\exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})-\theta\ln t\right.\right.$$
$$\left.\left.-\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left(\tilde{e}_g\ln s_g^{(1)}+\tilde{w}_g\ln s_g^{(2)}\right)\right]\right\} \tag{6.293}$$

where $\tilde{\boldsymbol{e}}=\boldsymbol{e}/n$, $\tilde{\boldsymbol{w}}=\boldsymbol{w}/n$, t and y and $x_g^{(1)},x_g^{(2)},s_g^{(1)},s_g^{(2)}$ $\forall g\in\mathscr{E}^{\mathsf{P}}$ are the unique positive solutions of (6.238)-(6.241) if we replace $\tilde{e}_g^\star$ and $\tilde{w}_g^\star$ by $\tilde{e}_g$ and $\tilde{w}_g$. We obtain

$$\prod_{g\in\mathscr{E}^{\mathsf{P}}}\binom{Q}{n(\tilde{e}_g+\tilde{w}_g)}\binom{n(\tilde{e}_g+\tilde{w}_g)}{n\tilde{e}_g}\doteq\exp\left\{n\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left[\frac{1}{n_{\mathsf{P}}}H_{\mathsf{b}}(n_{\mathsf{P}}(\tilde{e}_g+\tilde{w}_g))\right.\right.$$
$$\left.\left.+(\tilde{e}_g+\tilde{w}_g)H_{\mathsf{b}}\left(\frac{\tilde{e}_g}{\tilde{e}_g+\tilde{w}_g}\right)\right]\right\}. \tag{6.294}$$

From (6.292), (6.293) and (6.294), we have

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{P}}(\theta n,\gamma n)\doteq\sum_{\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}}}\exp(nS(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})) \tag{6.295}$$

where

$$\begin{aligned}S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})=&\frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})+\frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)-\gamma\ln y-\theta\ln t\\&-\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left[\tilde{e}_g\ln(x_g^{(1)}s_g^{(1)})+\tilde{w}_g\ln(x_g^{(2)}s_g^{(2)})+\frac{1}{n_{\mathsf{P}}}H_{\mathsf{b}}(n_{\mathsf{P}}(\tilde{e}_g+\tilde{w}_g))\right.\\&\left.+(\tilde{e}_g+\tilde{w}_g)H_{\mathsf{b}}\left(\frac{\tilde{e}_g}{\tilde{e}_g+\tilde{w}_g}\right)\right].\end{aligned} \tag{6.296}$$

This implies $G_{\mathrm{AS}}^{\mathsf{P}}(\theta,\gamma)=\max S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})$. By computing the partial derivatives of $S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})$, we obtain the vector pair $(\tilde{\boldsymbol{e}}^\star,\tilde{\boldsymbol{w}}^\star)=\operatorname{argmax} S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})$ in (6.242) and (6.243).

7

Trapping and Absorbing Set Enumerators for Non-Binary LDPC Code Ensembles

In this chapter, we study TSs and ASs of non-binary unstructured and protograph-based LDPC codes. Our interest in these codes stems from their applications: Besides their excellent performance for short-packet wireless links [38, 98, 123–125], non-binary LDPC codes have been suggested for NAND flash memories [126–128] and code-based public key cryptosystems [129] which require low-complexity decoders. Hence, hard decision decoders for non-binary LDPC codes have been studied, either in the form of symbol flipping algorithms [130], majority-logic decoding [30], and hard (e.g., symbol) message passing decoding [27, 56, 91]. The authors of [40] presented an algorithm to reduce the number of ASs in the Tanner graph of a non-binary code by changing the edge weights. The results showed the effect of ASs in the error floor performance of non-binary LDPC codes. We provide numerical evidence that they are a major contributor to the error probability under certain hard-decision passing decoding algorithms [27, 91] that represent the non-binary equivalent of the Gallager B algorithm.

We published the distribution of TSs and (elementary) ASs for non-binary unstructured and protograph-based LDPC code ensembles in [62, 63].

7.1 Preliminaries

Suppose we assign non-zero symbols to the VNs composing a set $\mathcal{I}$, and the zero symbol to the VNs outside $\mathcal{I}$. Let $\mathcal{N}(\mathcal{I})$ be the set of the neighboring CNs of $\mathcal{I}$. Furthermore, let $\mathcal{U}(\mathcal{I})$ be the set of unsatisfied CNs in $\mathcal{N}(\mathcal{I})$ and $\mathcal{S}(\mathcal{I})$ the set of satisfied CNs in $\mathcal{N}(\mathcal{I})$. Following [38–40, 48], we define TSs and (elementary) ASs.

Definition 7.1 (Non-binary trapping set)**.** An (a, b) TS $\mathcal{T}_{a,b}$ is a set $\mathcal{I}$ of a VNs such that $\mathcal{U}(\mathcal{I})$ contains b CNs [38].

Definition 7.2 (Non-binary absorbing set)**.** An (a, b) AS $\mathcal{A}_{a,b}$ is a trapping set with the additional property that each VN $\mathsf{v} \in \mathcal{I}$ has strictly fewer neighbors from $\mathcal{U}(\mathcal{I})$ than from $\mathcal{S}(\mathcal{I})$ [39, 40].

Definition 7.3 (Non-binary elementary absorbing set)**.** An EAS $\mathcal{A}^{\mathrm{E}}_{a,b}$ is an AS where each CN $\mathsf{c} \in \mathcal{S}(\mathcal{I})$ is connected to two VNs in $\mathcal{I}$ and each CN $\mathsf{c} \in \mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$ [39, 40].

7.2 Trapping and Absorbing Set Enumerators for Unstructured Ensembles

7.2.1 Trapping Set Distribution

We next derive the finite-length TS enumerator for non-binary LDPC codes and develop an analytical method to evaluate the normalized logarithmic asymptotic distribution of TSs. Further, we derive the asymptotic approximation for the small-sized trapping set case.

Lemma 7.1. The average number of size (a, b) TSs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}^{\Lambda,\mathrm{P}}_{q,n}$ is

$$\mathsf{E}^{\Lambda,\mathrm{P}}_{\mathrm{TS}}(a,b) = \sum_{w} \frac{\operatorname{coeff}\left(g(x,y)^n, x^w y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{w}(q-1)^w} \operatorname{coeff}\left(f(t,s)^n, t^a s^w\right) \tag{7.1}$$

where

$$f(t,s) = \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \left(1+(q-1)ts^j\right)^{\Lambda_j} \tag{7.2}$$

$$g(x,y) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \Bigg[\frac{(1+(q-1)x)^i + (q-1)(1-x)^i}{q} + y(q-1)\frac{(1+(q-1)x)^i - (1-x)^i}{q} \Bigg]^{\xi \mathrm{P}_i}. \tag{7.3}$$

Note that for binary LDPC codes, we obtain (6.108)-(6.110). The proof is analogous to the proof of Lemma 6.1.

We next present a simple way to compute the normalized logarithmic asymptotic distribution of TSs for the ensemble $\mathscr{C}_{q,n}^{\Lambda,\mathrm{P}}$.

Theorem 7.1. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the q-ary irregular ensemble is

$$G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = -\bar{\mathsf{d}}_\mathsf{v} \ln(1+(q-1)xs) - \theta \ln(t) - \gamma \ln(y) + \ln(f(t,s)) + \ln(g(x,y)) \tag{7.4}$$

where t, s, x, y are the unique positive solutions of

$$t\frac{\partial \ln f(t,s)}{\partial t} = \theta \tag{7.5}$$

$$s\frac{\partial \ln f(t,s)}{\partial s} = x\frac{\partial \ln g(x,y)}{\partial x} = \tilde{w}^\star \tag{7.6}$$

$$y\frac{\partial \ln g(x,y)}{\partial y} = \gamma \tag{7.7}$$

where $f(t,s)$ and $g(x,y)$ are defined in (7.2) and (7.3), respectively, and

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{(q-1)xs}{1+(q-1)xs}. \tag{7.8}$$

The proof of Theorem 7.1 is omitted since it is similar to the one of Theorem 6.7.

The following Lemma will be useful to analyse $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$ for small θ and fixed Δ.

Lemma 7.2. For a fixed $\Delta = \gamma/\theta$, the derivative in θ of $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$ is

$$\frac{\mathrm{d}G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)}{\mathrm{d}\theta} = -\ln(t) - \Delta \ln(y) \tag{7.9}$$

where for each θ, the values of t and y are given by the solution of the system of equations (7.5)-(7.7).

Proof. Note that the solutions of the system of equations in (7.5)-(7.7) are implicit functions of θ. From (7.4) and (7.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)}{\mathrm{d}\theta} &= -\ln(t) - \Delta\ln(y) + \frac{\mathrm{d}t}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial t}}{f(t,s)} - \frac{\theta}{t}\right] + \frac{\mathrm{d}s}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial s}}{f(t,s)} - \frac{\tilde{w}^\star}{s}\right] \\ &+ \frac{\mathrm{d}x}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial x}}{g(x,y)} - \frac{\tilde{w}^\star}{x}\right] + \frac{\mathrm{d}y}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial y}}{g(x,y)} - \frac{\Delta\theta}{y}\right].\end{aligned} \tag{7.10}$$

The terms in the square brackets in (7.10) are equal to zero due to (7.5)-(7.7). ■

Consider now the case of small θ and $\gamma = \Delta\theta$. We obtain a closed form expression of $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$, which we introduce in the following corollary.

Corollary 7.1. For a fixed $\Delta = \gamma/\theta$ and small θ, we have

$$\begin{aligned}G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta) =& \theta\left[\frac{\mathsf{d}_\mathsf{v}^{\min} - 2 - \Delta}{2}(\ln(\theta) - 1 - \ln(q-1)) + \ln\left(\frac{(\mathsf{d}_\mathsf{v}^{\min})^{\mathsf{d}_\mathsf{v}^{\min}}\Lambda_{\mathsf{d}_\mathsf{v}^{\min}}}{\Delta^\Delta}\right)\right. \\ &\left. - \frac{\mathsf{d}_\mathsf{v}^{\min} - \Delta}{2}\ln\left(\frac{\bar{\mathsf{d}}_\mathsf{v}\bar{\mathsf{d}}_\mathsf{c}(\mathsf{d}_\mathsf{v}^{\min} - \Delta)}{\mathrm{P}''(1)}\right)\right] + o(\theta)\end{aligned} \tag{7.11}$$

where $\mathsf{d}_\mathsf{v}^{\min}$ is the minimum VN degree and $\mathrm{P}''(x)$ is the second derivative of $\mathrm{P}(x)$. The proof is provided in Appendix 7.4.1.

If $\Delta = 0$ and $\mathsf{d}_\mathsf{v}^{\min} = 2$, we obtain equation (14) in [98], which is an approximation of the growth rate of a non-binary code ensemble for the case of small-weight codewords. This is expected since an $(a, 0)$ TS is a codeword (all CNs are satisfied).

Note that a positive $\theta_{\mathrm{TS}}^\star$ (as defined in Definition 6.6) exists whenever the derivative of $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta)$ is negative as $\theta \to 0$. Thus, by substituting (7.117) and (7.118) in (7.10) we find that a positive $\theta_{\mathrm{TS}}^\star$ exists whenever $\mathsf{d}_\mathsf{v}^{\min} > 2 + \Delta$ or $\mathsf{d}_\mathsf{v}^{\min} = 2 + \Delta$ and

$$\frac{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}}(\mathsf{d}_\mathsf{v}^{\min})^{\mathsf{d}_\mathsf{v}^{\min}}\mathrm{P}''(1)}{2\bar{\mathsf{d}}_\mathsf{v}\bar{\mathsf{d}}_\mathsf{c}(\mathsf{d}_\mathsf{v}^{\min} - 2)^{\mathsf{d}_\mathsf{v}^{\min}-2}} < 1. \tag{7.12}$$

If the relative minimum Δ-trapping set size is small enough, then we can use Corollary 7.1 to approximate it. Through numerical simulations, we observed that the relative minimum Δ-trapping set size is small for the case of small VN degrees or high CN degrees as observed in [48]. We need to determine θ such that $G_{\mathrm{TS}}^{\Lambda,\mathrm{P}}(\theta,\Delta\theta) = 0$ with $\theta > 0$. By neglecting the term $o(\theta)$, we obtain

$$\theta^{\star}_{\mathrm{TS}} \approx (q-1)\exp(1)\left(\frac{\bar{\mathsf{d}}_{\mathsf{c}}\bar{\mathsf{d}}_{\mathsf{v}}(\mathsf{d}_{\mathsf{v}}^{\min}-\Delta)}{\mathrm{P}''(1)}\right)^{\frac{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta}{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta-2}}\left(\frac{\Delta^{\Delta}}{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}}(\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}}\right)^{\frac{2}{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta-2}}. \tag{7.13}$$

For $q=2$, we obtain (6.123). The approximation of the relative minimum Δ-trapping set size given in (7.13) is accurate when $\theta^{\star}_{\mathrm{TS}}$ is sufficiently small (for the case of small VN degrees or high CN degrees as observed in [48]) and does not need solving the system of equations given in Theorem 7.1.

For the regular ensemble $\mathscr{C}_{q,n}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$, the expressions in Lemma 7.1 and Theorem 7.1 can be simplified as follows.

Lemma 7.3. The average number of size (a,b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_{q,n}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}(a,b) = \binom{n}{a}\frac{\operatorname{coeff}\left(g(x,y)^{n}, x^{a\mathsf{d}_{\mathsf{v}}}y^{b}\right)}{\binom{n\mathsf{d}_{\mathsf{v}}}{a\mathsf{d}_{\mathsf{v}}}(q-1)^{a(\mathsf{d}_{\mathsf{v}}-1)}} \tag{7.14}$$

where

$$\begin{aligned} g(x,y) = \Bigg[& \frac{(1+(q-1)x)^{\mathsf{d}_{\mathsf{c}}}+(q-1)(1-x)^{\mathsf{d}_{\mathsf{c}}}}{q} \\ & +y(q-1)\frac{(1+(q-1)x)^{\mathsf{d}_{\mathsf{c}}}-(1-x)^{\mathsf{d}_{\mathsf{c}}}}{q}\Bigg]^{\xi}. \end{aligned} \tag{7.15}$$

Theorem 7.2. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the ensemble $\mathscr{C}_{q,n}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}$ is

$$\begin{aligned} G_{\mathrm{TS}}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}(\theta,\gamma) = & -(\mathsf{d}_{\mathsf{v}}-1)H_{\mathsf{b}}(\theta)-\gamma\ln(y)-\theta\mathsf{d}_{\mathsf{v}}\ln(x)+\ln\left(g(x,y)\right) \\ & -\theta(\mathsf{d}_{\mathsf{v}}-1)\ln(q-1) \end{aligned} \tag{7.16}$$

where

$$y = \frac{\gamma}{(q-1)(\xi-\gamma)}\frac{(1+(q-1)x)^{\mathsf{d}_{\mathsf{c}}}+(q-1)(1-x)^{\mathsf{d}_{\mathsf{c}}}}{(1+(q-1)x)^{\mathsf{d}_{\mathsf{c}}}-(1-x)^{\mathsf{d}_{\mathsf{c}}}} \tag{7.17}$$

and x is the unique positive solution of

$$x\frac{\partial g(x,y)}{\partial x} = \theta\mathsf{d}_{\mathsf{v}}g(x,y) \tag{7.18}$$

where $g(x,y)$ is defined in (7.15). The proof is omitted since it is similar to the one of

Theorem 6.8.

7.2.2 Absorbing Set Distribution

The following Lemma presents the finite-length AS enumerator for non-binary LDPC codes. Moreover, we develop an analytical method to evaluate the normalized logarithmic asymptotic distribution of ASs.

Lemma 7.4. The average number of size (a, b) ASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_{q,n}^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{e,w} \frac{\operatorname{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}(q-1)^{e+w}} \operatorname{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^w\right) \tag{7.19}$$

where

$$f(t,s_1,s_2) = \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[1 + (q-1)t \sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1}\right]^{\Lambda_j} \tag{7.20}$$

$$\begin{aligned} g(x_1,x_2,y) = \prod_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \Bigg[& \frac{(1+(q-1)x_1)^i + (q-1)(1-x_1)^i}{q} \\ & + y(q-1)\frac{(1+(q-1)x_2)^i - (1-x_2)^i}{q} \Bigg]^{\xi \mathrm{P}_i}. \end{aligned} \tag{7.21}$$

The proof is analogous to the proof of Lemma 6.4.

We derive the normalized logarithmic asymptotic distribution of ASs for non-binary codes in the following Theorem.

Theorem 7.3. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the ensemble $\mathscr{C}_{q,n}^{\Lambda,\mathrm{P}}$ is

$$\begin{aligned} G_{\mathrm{AS}}^{\Lambda,\mathrm{P}}(\theta,\gamma) = & -\bar{\mathsf{d}}_\mathsf{v} \ln(1 + (q-1)(x_1 s_1 + x_2 s_2)) - \theta \ln(t) - \gamma \ln(y) \\ & + \ln\left(g(x_1,x_2,y)\right) + \ln\left(f(t,s_1,s_2)\right) \end{aligned} \tag{7.22}$$

where t, s_1, s_2, x_1, x_2, y are the unique positive solutions of

$$t \frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{7.23}$$

$$s_1 \frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1 \frac{\partial \ln g(x_1,x_2,y)}{\partial x_1} = \tilde{e}^\star \tag{7.24}$$

$$s_2 \frac{\partial \ln f(t, s_1, s_2)}{\partial s_2} = x_2 \frac{\partial \ln g(x_1, x_2, y)}{\partial x_2} = \tilde{w}^\star \quad (7.25)$$

$$y \frac{\partial \ln g(x_1, x_2, y)}{\partial y} = \gamma \quad (7.26)$$

where $f(t, s_1, s_2)$ and $g(x_1, x_2, y)$ are defined in (7.20) and (7.21) respectively and

$$\tilde{e}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{(q-1)x_1 s_1}{1+(q-1)(x_1 s_1 + x_2 s_2)} \quad (7.27)$$

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{(q-1)x_2 s_2}{1+(q-1)(x_1 s_1 + x_2 s_2)}. \quad (7.28)$$

The proof is omitted since it is similar to the one of Theorem 6.7.

Similar to the TS case, the expressions in Lemma 7.4 and Theorem 7.3 can be simplified for regular ensembles.

Lemma 7.5. The average number of size (a, b) ASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_{q,n}^{\mathsf{d_v},\mathsf{d_c}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) = \sum_e \binom{m}{b} \frac{\mathrm{coeff}\left(g_1(x_1)^{m-b}, x_1^e\right) \mathrm{coeff}\left(g_2(x_2)^b, x_2^{a\mathsf{d_v}-e}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{e}(q-1)^{a(\mathsf{d_v}-1)}} \times \binom{n}{a} \mathrm{coeff}\left(f(s)^a, s^{a\mathsf{d_v}-e}\right) \quad (7.29)$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor \frac{\mathsf{d_v}-1}{2} \rfloor} \binom{\mathsf{d_v}}{j_1} s^{j_1} \quad (7.30)$$

$$g_1(x_1) = \frac{1}{q}\left[(1+(q-1)x_1)^{\mathsf{d_c}} + (q-1)(1-x_1)^{\mathsf{d_c}}\right] \quad (7.31)$$

$$g_2(x_2) = \frac{q-1}{q}\left[(1+(q-1)x_2)^{\mathsf{d_c}} - (1-x_2)^{\mathsf{d_c}}\right]. \quad (7.32)$$

We show now that to compute the normalized logarithmic asymptotic distribution of ASs for q-ary regular codes, one needs to solve 3 equations instead of 6 for the irregular case.

Theorem 7.4. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the $\mathscr{C}_{q,n}^{\mathsf{d_v},\mathsf{d_c}}$

ensemble is

$$G_{\mathrm{AS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = -\,\theta \mathsf{d_v} \ln(x_1 + x_2 s) + (\xi - \gamma)\ln(g_1(x_1)) + \gamma \ln(g_2(x_2)) + \theta \ln(f(s)) - (\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) - \theta(\mathsf{d_v}-1)\ln(q-1) + \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) \tag{7.33}$$

where s, x_1, x_2 are the unique positive solutions of

$$\theta s \frac{\mathrm{d}\ln f(s)}{\mathrm{d}s} = \gamma x_2 \frac{\mathrm{d}\ln g_2(x_2)}{\mathrm{d}x_2} = (\theta \mathsf{d_v} - \tilde{e}^\star) \tag{7.34}$$

$$(\xi - \gamma) x_1 \frac{\mathrm{d}\ln g_1(x_1)}{\mathrm{d}x_1} = \tilde{e}^\star \tag{7.35}$$

where $f(s), g_1(x_1)$ and $g_2(x_2)$ are defined in (7.30), (7.31) and (7.32) respectively and

$$\tilde{e}^\star = \theta \mathsf{d_v} \frac{x_1}{x_1 + x_2 s}. \tag{7.36}$$

The proof is similar to the one of Theorem 6.8.

7.2.3 Elementary Absorbing Set Distribution

The following Lemma gives the EAS enumerator for non-binary LDPC codes.

Lemma 7.6. The average number of size (a,b) EASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_{q,n}^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{EAS}}^{\Lambda,\mathrm{P}}(a,b) = \sum_{e} \frac{\operatorname{coeff}\left(g(x_1,x_2)^n, x_1^e x_2^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+b}\binom{e+b}{b}(q-1)^{b+e}} \operatorname{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^b\right) \tag{7.37}$$

where $f(t,s_1,s_2)$ is defined in (7.20) and

$$g(x_1,x_2) = \prod_{i=1}^{\mathsf{d}_{\mathsf{c}}^{\max}} \left[1 + \binom{i}{2}(q-1)x_1^2 + i(q-1)x_2\right]^{\xi \mathrm{P}_i}. \tag{7.38}$$

The proof is similar to the proof of Lemma 6.6.

Next, we analyze the normalized logarithmic asymptotic distribution of EAS and present an efficient way to compute it.

Theorem 7.5. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the ensemble is

$$\begin{aligned} G^{\Lambda,\mathrm{P}}_{\mathrm{EAS}}(\theta,\gamma) =& (\bar{\mathsf{d}}_\mathsf{v}-\gamma)\ln(\bar{\mathsf{d}}_\mathsf{v}-\gamma) - \bar{\mathsf{d}}_\mathsf{v}\ln(\bar{\mathsf{d}}_\mathsf{v}) - (\bar{\mathsf{d}}_\mathsf{v}-\gamma)\ln(1+(q-1)x_1 s_1) \\ &+ \ln\left(g(x_1,x_2)\right) + \ln\left(f(t,s_1,s_2)\right) - \gamma\ln((q-1)x_2 s_2) \\ &+ \gamma\ln(\gamma) - \theta\ln(t) \end{aligned} \tag{7.39}$$

where t, s_1, s_2, x_1, x_2 are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{7.40}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2)}{\partial x_1} = \tilde{e}^\star \tag{7.41}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2)}{\partial x_2} = \gamma \tag{7.42}$$

and where $f(t, s_1, s_2)$ and $g(x_1, x_2)$ are defined in (7.20) and (7.38), respectively, and

$$\tilde{e}^\star = \frac{(\bar{\mathsf{d}}_\mathsf{v}-\gamma)(q-1)x_1 s_1}{1+(q-1)x_1 s_1}. \tag{7.43}$$

We next consider the EAS finite-length and asymptotic enumerator for the regular ensembles.

Lemma 7.7. The average number of size (a, b) EASs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}^{\mathsf{d}_\mathsf{v},\mathsf{d}_\mathsf{c}}_{q,n}$ is

$$\mathsf{E}^{\mathsf{d}_\mathsf{v},\mathsf{d}_\mathsf{c}}_{\mathrm{EAS}}(a,b) = \binom{m}{b}\binom{n}{a}\frac{{\mathsf{d}_\mathsf{c}}^b \operatorname{coeff}\left(g(x)^{m-b}, x^{a\mathsf{d}_\mathsf{v}-b}\right)}{\binom{n\mathsf{d}_\mathsf{v}}{a\mathsf{d}_\mathsf{v}}\binom{a\mathsf{d}_\mathsf{v}}{b}(q-1)^{a(dv-1)-b}} \operatorname{coeff}\left(f(s)^a, s^b\right) \tag{7.44}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor\frac{\mathsf{d}_\mathsf{v}-1}{2}\rfloor}\binom{\mathsf{d}_\mathsf{v}}{j_1}s^{j_1} \tag{7.45}$$

and

$$g(x) = 1+(q-1)\binom{\mathsf{d}_\mathsf{c}}{2}x^2. \tag{7.46}$$

The Lemma can be proved from Lemma 7.6.

We show now that to compute the normalized logarithmic asymptotic distribution of EASs for regular codes, one must solve one equation rather than five as for the irregular case.

Theorem 7.6. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the $\mathscr{C}_{q,n}^{\mathsf{d_v},\mathsf{d_c}}$ ensemble is

$$\begin{aligned} G_{\mathrm{EAS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) =& -(\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) - \mathsf{d_v}\theta H_{\mathsf{b}}\left(\frac{\gamma}{\theta \mathsf{d_v}}\right) + \gamma\ln(\mathsf{d_c}) - \gamma\ln(s) \\ & + (\xi-\gamma)\ln\left(g(x)\right) + \theta\ln\left(f(s)\right) - (\theta\mathsf{d_v}-\gamma)\ln(x) + \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) \\ & - (\theta(\mathsf{d_v}-1)-\gamma)\ln(q-1) \end{aligned} \tag{7.47}$$

where

$$x = \sqrt{\frac{2(\theta\mathsf{d_v}-\gamma)}{\mathsf{d_c}(\mathsf{d_c}-1)(2\xi-\theta\mathsf{d_v}-\gamma)(q-1)}} \tag{7.48}$$

and s is the unique positive solution of

$$\theta s \frac{\mathrm{d}\ln f(s)}{\mathrm{d}s} = \gamma \tag{7.49}$$

where $f(s)$ is defined in (7.45).

Example 7.1. Consider the rate 1/2 LDPC ensemble with $\Lambda(x) = 0.5x^4 + 0.5x^5$, $\mathrm{P}(x) = x^9$. We evaluate the asymptotic distributions of TSs, ASs and EASs according to Theorems 7.1, 7.3 and 7.5. The normalized logarithmic asymptotic distributions of TSs, ASs and EASs of this ensemble for fixed ratio $\Delta = 0.1$ and $q \in \{2,4,8,16,32,64\}$ are depicted in Fig. 7.1. We see that the gap between the normalized logarithmic asymptotic distributions of TSs and ASs vanishes for small θ. Moreover, the gap between the AS and EAS enumerators increases with increasing q and θ. We also observe that the trapping and absorbing set properties of the ensemble improve with increasing the field order q. For instance $\theta_{\mathrm{TS}}^{\star} = 0.086818$ for $q = 4$ and $\theta_{\mathrm{TS}}^{\star} = 0.166643$ for $q = 32$.

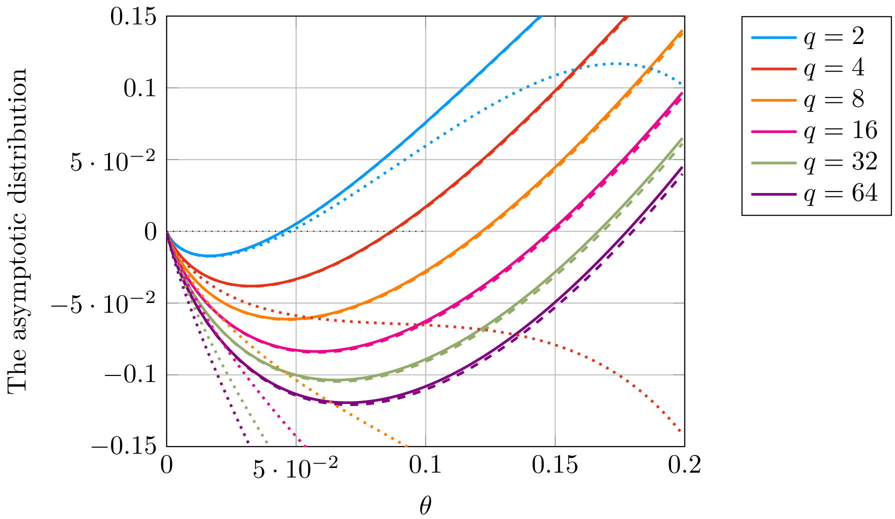


Figure 7.1: Normalized logarithmic asymptotic distributions of trapping sets (solid lines), absorbing sets (dashed lines) and elementary absorbing sets (dotted lines) of the ensemble in Example 7.1.

7.3 Trapping and Absorbing Set Enumerators for Protograph-Based Ensembles

7.3.1 Trapping Set Distribution

In this section, we derive the average finite-length TS enumerator for non-binary LDPC codes from the U-NBPB and C-NBPB code ensembles and we present an analytical method to evaluate the normalized logarithmic asymptotic distribution of TSs.

Unconstrained Protograph-Based LDPC Codes

Define the VN weight vector $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_{n_\mathsf{P}}]$, where ϵ_j is the number of VNs of type $\mathtt{v}_j$ in $\mathcal{T}_{a,b}$. Clearly we have $0 \leq \epsilon_j \leq Q$ for all $j \in \{1, \ldots, n_\mathsf{P}\}$ and

$$\sum_{j=1}^{n_\mathsf{P}} \epsilon_j = a. \tag{7.50}$$

Similarly, define the edge weight vector $\boldsymbol{w}(\boldsymbol{\epsilon}) = (w_g)_{g \in \mathscr{E}^\mathsf{P}}$ where w_g is the number of edges of type g in $\mathcal{T}_{a,b}$. The VN and edge weight vectors are related: for a given $\boldsymbol{\epsilon}$, we have $w_g = \epsilon_j$ if $g \in \mathscr{E}^\mathsf{P}_{\mathtt{v}_j}$.

Lemma 7.8. The average number of size (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{P},u}(a,b) = \sum_{\boldsymbol{\epsilon}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right)}{\prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}^{\mathsf{d}_{\mathsf{v_j}}-1} (q-1)^{\epsilon_j(\mathsf{d}_{\mathsf{v_j}}-1)}} \tag{7.51}$$

where

$$\begin{aligned} A_i(\boldsymbol{x}_i, y) =& \frac{1}{q}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1+(q-1)x_g) + (q-1)\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g)\right] \\ &+ \frac{q-1}{q} y \left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1+(q-1)x_g) - \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g)\right] \end{aligned} \tag{7.52}$$

and where $\boldsymbol{x} = (x_g)_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_i = (x_g)_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$, y and $x_g, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$ are dummy variables. The proof is analogous to the one of Lemma 6.12.

Lemma 7.8 provides the average number of size (a,b) TSs for a finite block length n. In the asymptotic case, we analyze the normalized logarithmic asymptotic distribution of TSs for the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ for $a = \theta n$ and $b = \gamma n$.

The normalized logarithmic asymptotic distribution of TSs is a useful tool to analyze and design LDPC codes with good TS properties and can be computed efficiently. In particular, the analysis of the normalized logarithmic asymptotic distribution of TSs for a given U-NBPB LDPC code ensemble allows to determine if the expected number of TSs with size θn, with θ small, goes exponentially fast to zero, providing insights on the TS properties of the ensemble.

Theorem 7.7. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ is

$$\begin{aligned} G_{\mathrm{TS}}^{\mathsf{P},u}(\theta,\gamma) =& -\gamma \ln y + \frac{1}{n_{\mathsf{P}}} \sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i, y) \\ &- \sum_{j=1}^{n_{\mathsf{P}}} \left[\frac{\mathsf{d}_{\mathsf{v_j}}-1}{n_{\mathsf{P}}} H_{\mathsf{b}}(n_{\mathsf{P}}\tilde{\epsilon}_j^{\star}) + (\mathsf{d}_{\mathsf{v_j}}-1)\tilde{\epsilon}_j^{\star}\ln(q-1) + \tilde{\epsilon}_j^{\star} \sum_{g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \ln x_g\right]. \end{aligned} \tag{7.53}$$

The values x_g for $g \in \mathscr{E}^{\mathsf{P}}$, the value y and $\tilde{\epsilon}_j^{\star}$ for $j \in \{1, \ldots, n_{\mathsf{P}}\}$ are the unique positive solutions of

$$x_g \frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_g} = n_{\mathsf{P}} \tilde{w}_g^{\star} \tag{7.54}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{7.55}$$

$$(\mathsf{d}_{\mathsf{v_j}} - 1) \ln \left(\frac{n_{\mathsf{P}} \tilde{\epsilon}_j^{\star}}{(q-1)(1 - n_{\mathsf{P}} \tilde{\epsilon}_j^{\star})} \right) = \sum_{g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \ln x_g + \mu \tag{7.56}$$

where (7.54) is valid for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$, μ is chosen to satisfy $\sum_j \tilde{\epsilon}_j^{\star} = \theta$ and $A_i(\boldsymbol{x}_i, y)$ is defined in (7.52), and $\tilde{w}_g^{\star} = \tilde{\epsilon}_j^{\star}$ if $g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}$. The proof of Theorem 7.7 is similar to the one of Theorem 6.17.

Theorem 7.7 shows that the evaluation of $G_{\text{TS}}^{\mathsf{P},u}(\theta, \gamma)$ requires solving $e + n_{\mathsf{P}} + 2$ equations in $e + n_{\mathsf{P}} + 2$ variables: x_g (e variables), $\tilde{\epsilon}_j^{\star}$ (n_{P} variables), y (one variable) and μ (one variable). The following Lemma follows the approach of [119] to reduce the dimension of the system of equations by exploiting symmetries in the protograph.

Lemma 7.9. Let u, v be two edges in $\mathscr{E}^{\mathsf{P}}$. If u and v are connected to the same VN-CN pair in the protograph, then $x_u = x_v$.

Proof. Consider two edges u and v that connect c_i to v_j. Note that in this case $\tilde{w}_u^{\star} = \tilde{w}_v^{\star} = \tilde{\epsilon}_j^{\star}$. It is clear that $A_i(\boldsymbol{x}_i, y)$ in (7.52) is symmetric in the variables x_g, $g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$. We have

$$\left. \frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_u} \right|_{\substack{x_u = \kappa \\ x_v = \beta}} = \left. \frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_v} \right|_{\substack{x_u = \beta \\ x_v = \kappa}} \tag{7.57}$$

$$\left. \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y} \right|_{\substack{x_u = \kappa \\ x_v = \beta}} = \left. \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y} \right|_{\substack{x_u = \beta \\ x_v = \kappa}}. \tag{7.58}$$

Thus, for the system of equations in Theorem 7.7, if there is a solution with $x_u = \kappa, x_v = \beta$ then another solution exists with $x_u = \beta, x_v = \kappa$ (all the other variables being unchanged). Since the solutions $x_g, g \in \mathscr{E}^{\mathsf{P}}$ are unique, we have $\kappa = \beta$. ■

Next, we extend the results to C-NBPB code ensembles.

Constrained Protograph-Based LDPC Codes

Define the VN *frequency weight vector* $\boldsymbol{\epsilon} = [\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_{n_{\mathsf{P}}}]$ with $\boldsymbol{\epsilon_j} = [\epsilon_{j,0}, \ldots, \epsilon_{j,q-2}]$, where $\epsilon_{j,\ell}$ is the number of times the symbol α^{ℓ} occurs in the VNs of type v_j in $\mathcal{T}_{a,b}$, $j \in \{1, \ldots, n_{\mathsf{P}}\}, \ell \in$

$\{0,\dots,q-2\}$. Obviously, we have

$$\sum_{j=1}^{n_\mathsf{P}}\sum_{\ell=0}^{q-2}\epsilon_{j,\ell}=a. \tag{7.59}$$

Define also the *edge frequency weight vector* $\boldsymbol{w}(\boldsymbol{\epsilon})=(\boldsymbol{w}_g)_{g\in\mathscr{E}^\mathsf{P}}$ with $\boldsymbol{w}_g=[w_{g,0},\dots,w_{g,q-2}]$ where $w_{g,\ell}$ is the number of times the symbol α^ℓ occurs in the edges of type g in $\mathcal{T}_{a,b}$. For a given $\boldsymbol{\epsilon}$, we have for all $\ell\in\{0,\dots,q-2\}$, $w_{g,\ell}=\epsilon_{j,\ell}$ if $g\in\mathscr{E}^\mathsf{P}_{\mathsf{v}_j}$. We define next the composition vector weight enumerator function (CVWEF) of a q-ary linear code which we will use to derive the TS and AS enumerators of C-NBPB code ensembles.

Definition 7.4 (Composition vector weight enumerator function)**.** Let $\mathcal{P}$ be an (n,k) linear code over $\mathbb{F}_q=\{0,\alpha^0,\alpha^1,\dots,\alpha^{q-2}\}$ where α is a primitive element of $\mathbb{F}_q$. The CVWEF is

$$W_\mathcal{P}(\boldsymbol{x})=\sum_{\boldsymbol{c}\in\mathcal{P}}\boldsymbol{x}^{\boldsymbol{\phi}(\boldsymbol{c})} \tag{7.60}$$

where $\boldsymbol{x}=[\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_n]$, $\boldsymbol{x}_i=[x_{i,0},x_{i,1},\dots,x_{i,q-2}]$, and $x_{i,\ell}$, $i\in\{1,2,\dots,n\}$, $\ell\in\{0,1,\dots,q-2\}$ are dummy variables. Moreover, $\boldsymbol{\phi}(\boldsymbol{c})=[\boldsymbol{\phi}_1(c_1),\boldsymbol{\phi}_2(c_2),\dots,\boldsymbol{\phi}_n(c_n)]$, $\boldsymbol{\phi}_i(c_i)=[\boldsymbol{\phi}_{i,0}(c_i),\boldsymbol{\phi}_{i,1}(c_i),\dots,\boldsymbol{\phi}_{i,q-2}(c_i)]$ where for all $\ell=0,1,\dots,q-2$, $i=1,2,\dots,n$ we have $\phi_{i,\ell}(c_i)=1$ if $c_i=\alpha^\ell$, $\phi_{i,\ell}(c_i)=0$ otherwise.

The following Theorem is an adapted version of the MacWilliams identity [131] and will be useful to derive the TS and AS enumerators of C-NBPB code ensembles.

Theorem 7.8. Let $\mathcal{P}$ be an (n,k) linear code over $\mathbb{F}_q$, where $q=p^s$, p is a prime number and s is a positive integer, with CVWEF $W_\mathcal{P}(\boldsymbol{x})$. The CVWEF of its dual code $\mathcal{P}^\perp$ is

$$W_{\mathcal{P}^\perp}(\boldsymbol{x})=\frac{1}{q^k}W_\mathcal{P}(\boldsymbol{x}')\prod_{i=1}^{n}\left(1+\sum_{\ell=0}^{q-2}x_{i,\ell}\right) \tag{7.61}$$

where $\boldsymbol{x}'=[\boldsymbol{x}'_1,\boldsymbol{x}'_2,\dots,\boldsymbol{x}'_n]$, $\boldsymbol{x}'_i=[x'_{i,0},x'_{i,1},\dots,x'_{i,q-2}]$ and

$$x'_{i,\ell}=\frac{1+\sum_{\ell'=0}^{q-2}\chi(\alpha^{\ell+\ell'})x_{i,\ell'}}{1+\sum_{\ell'=0}^{q-2}x_{i,\ell'}} \tag{7.62}$$

for $\ell=0,1,\dots,q-2$, $i=1,2,\dots,n$, and $\chi(u)$ is a non-trivial character of $\mathbb{F}_q$. The proof of Theorem 7.8 can be found in Appendix 7.4.2.

In the next Lemma, we derive the finite-length TS enumerator for C-NBPB code ensembles.

Lemma 7.10. The average number of size (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},c}(\boldsymbol{f})$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{P},c}(a,b) = \sum_{\boldsymbol{\epsilon}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right)}{\prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{Q - \sum_{\ell=0}^{q-2} \epsilon_{j,\ell}, \epsilon_{j,0}, \epsilon_{j,1}, \dots, \epsilon_{j,q-2}}^{\mathsf{d}_{\mathsf{v}_\mathsf{j}}-1}} \tag{7.63}$$

where

$$A_i(\boldsymbol{x}_i, y) = W_{\mathcal{P}_i}(\boldsymbol{x}_i) + y\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}\left(1+\sum_{\ell=0}^{q-2} x_{g,\ell}\right) - W_{\mathcal{P}_i}(\boldsymbol{x}_i)\right] \tag{7.64}$$

and where $\mathcal{P}_i$ is the codebook of c_i, $\boldsymbol{x} = (\boldsymbol{x}_g)_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_g = [x_{g,0}, \dots, x_{g,q-2}]$, $\boldsymbol{x}_i = (\boldsymbol{x}_g)_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$, y and $x_{g,\ell}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}, \ell \in \{0, \dots, q-2\}$ are dummy variables and

$$W_{\mathcal{P}_i}(\boldsymbol{x}_i) = \frac{1}{q}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}\left(1+\sum_{\ell=0}^{q-2} x_{g,\ell}\right) + \sum_{\ell=0}^{q-2}\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}\left(1+\sum_{\ell'=0}^{q-2}\chi\left(f_g\alpha^{\ell+\ell'}\right)x_{g,\ell'}\right)\right]. \tag{7.65}$$

Proof. First, note that for the edge labels $\boldsymbol{f} = (f_g)_{g\in\mathscr{E}^{\mathsf{P}}}$, the codewords $\boldsymbol{c} \in \mathcal{P}_i$ satisfy $\boldsymbol{c}^T\boldsymbol{f}_i = 0$, where $\boldsymbol{f}_i = (f_g)_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$. The dual code $\mathcal{P}_i^{\perp}$ of $\mathcal{P}_i$ is characterized by its generator matrix $\boldsymbol{G}_i = \boldsymbol{f}_i$ and CVWEF that is equal to $1 + \sum_l \boldsymbol{x}^{\phi(\alpha^l \boldsymbol{f}_i)}$. By applying Theorem 7.8 to $\mathcal{P}_i$, we obtain (7.65). Consider now the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},c}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign a non-zero symbol from $\mathbb{F}_q$ to each VN in the set. The edges connected to a VN v are assigned the non-zero symbol chosen for v. For a given $\boldsymbol{\epsilon}$, each $\mathsf{v}_j \in \mathcal{V}$ has $\epsilon_{j,\ell}$ replicas in $\mathcal{T}_{a,b}$ with VN symbol value α^{ℓ}. Since there are Q copies of each VN type in the lifted graph, the number of VN sets with frequency weight vector $\boldsymbol{\epsilon}$ is

$$N_{\mathsf{v}}(\boldsymbol{\epsilon}) = \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{Q - \sum_{\ell=0}^{q-2} \epsilon_{j,\ell}, \epsilon_{j,0}, \epsilon_{j,1}, \dots, \epsilon_{j,q-2}}. \tag{7.66}$$

Similarly, the number of edge sets with frequency weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ is

$$\begin{aligned} N_e(\boldsymbol{w}(\boldsymbol{\epsilon})) &= \prod_{g\in\mathscr{E}^{\mathsf{P}}} \binom{Q}{Q-\sum_{\ell=0}^{q-2} w_{g,\ell}, w_{g,0}, w_{g,1}, \ldots, w_{g,q-2}} \\ &= \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{Q-\sum_{\ell=0}^{q-2} \epsilon_{j,\ell}, \epsilon_{j,0}, \epsilon_{j,1}, \ldots, \epsilon_{j,q-2}}^{\mathsf{d}_{\mathsf{v}_\mathsf{j}}}. \end{aligned} \tag{7.67}$$

Let $N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))$ be the number of configurations with edge set frequency weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ that give exactly b unsatisfied CNs. We introduce the corresponding generating function

$$\sum_{b,\boldsymbol{w}(\boldsymbol{\epsilon})} N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) y^b \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})}.$$

Consider a CN of type c_i. The number of configurations for which the CN is satisfied is tracked by the generating function

$$g_{\mathsf{c}}(\boldsymbol{x}_i, y) := y^0 W_{\mathcal{P}_i}(\boldsymbol{x}_i)$$

while the number of configurations for which the CN is unsatisfied is tracked by the generating function

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y) := y^1 \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(1 + \sum_{\ell=0}^{q-2} x_{g,\ell}\right) - y^1 W_{\mathcal{P}_i}(\boldsymbol{x}_i).$$

The sum $g_{\mathsf{c}}(\boldsymbol{x}_i, y) + g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y)$ yields $A_i(\boldsymbol{x}_i, y)$. Considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) = \operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right). \tag{7.68}$$

Using

$$\mathsf{E}^{\mathsf{P},c}_{\mathrm{TS}}(a,b) = \sum_{\boldsymbol{\epsilon}} \frac{N_{\mathsf{v}}(\boldsymbol{\epsilon}) N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))}{N_e(\boldsymbol{w}(\boldsymbol{\epsilon}))} \tag{7.69}$$

completes the proof. ■

Note that the exact average number of size (a, b) TSs derived in Lemma 7.10 for a finite block length n is extremely complex to compute for large n. Next, we analyze the normalized logarithmic asymptotic distribution of TSs.

Theorem 7.9. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the ensemble $\mathscr{C}^{\mathsf{P},c}_{q,n}(\boldsymbol{f})$ is

$$
\begin{aligned}
G^{\mathsf{P},c}_{\mathrm{TS}}(\theta,\gamma) = -\sum_{j=1}^{n_{\mathsf{P}}} \Bigg[\frac{\mathsf{d}_{\mathsf{v_j}}-1}{n_{\mathsf{P}}} H\left(1-n_{\mathsf{P}}\sum_{\ell=0}^{q-2}\tilde{\epsilon}^{\star}_{j,\ell}, n_{\mathsf{P}}\tilde{\epsilon}^{\star}_{j,0},\dots,n_{\mathsf{P}}\tilde{\epsilon}^{\star}_{j,q-2}\right) \\
+\sum_{\ell=0}^{q-2}\tilde{\epsilon}^{\star}_{j,\ell}\sum_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}\ln x_{g,\ell}\Bigg] + \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i,y) - \gamma\ln y.
\end{aligned}
\tag{7.70}
$$

The values $x_{g,\ell}$ for $g \in \mathscr{E}^{\mathsf{P}}$, the value y and $\tilde{\epsilon}^{\star}_{j,\ell}$ for $j \in \{1,\dots,n_{\mathsf{P}}\}, \ell \in \{0,1,\dots,q-2\}$ are the unique positive solutions of

$$x_{g,\ell}\frac{\partial \ln A_i(\boldsymbol{x}_i,y)}{\partial x_{g,\ell}} = n_{\mathsf{P}}\tilde{w}^{\star}_{g,\ell} \tag{7.71}$$

$$y\frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i,y)}{\partial y} = n_{\mathsf{P}}\gamma \tag{7.72}$$

$$(\mathsf{d}_{\mathsf{v_j}}-1)\ln\left(\frac{n_{\mathsf{P}}\tilde{\epsilon}^{\star}_{j,\ell}}{1-n_{\mathsf{P}}\sum_{\ell'=0}^{q-2}\tilde{\epsilon}^{\star}_{j,\ell'}}\right) = \sum_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}\ln x_{g,\ell}+\mu \tag{7.73}$$

where (7.71) is valid $\forall i \in \{1,2,\dots,m_{\mathsf{P}}\}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}, \ell \in \{0,\dots,q-2\}$, μ is chosen to satisfy $\sum_{j=1}^{n_{\mathsf{P}}}\sum_{\ell=0}^{q-2}\tilde{\epsilon}^{\star}_{j,\ell} = \theta$, $A_i(\boldsymbol{x}_i,y)$ is defined in (7.64), and $\tilde{w}^{\star}_{g,\ell} = \tilde{\epsilon}^{\star}_{j,\ell}$ if $g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}$. The proof of the Theorem can be found in Appendix 7.4.3.

7.3.2 Absorbing and Elementary Absorbing Set Distribution

In this section, we extend the analysis developed for TSs to evaluate the average finite-length AS and EAS enumerators for non-binary LDPC codes from the U-NBPB and C-NBPB code ensembles, and we present an analytical method to evaluate the normalized logarithmic asymptotic distributions of ASs and EASs.

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Lemma 7.11. The average number of size (a,b) ASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ is

$$\begin{aligned}\mathsf{E}_{\mathrm{AS}}^{\mathsf{P},u}(a,b) =& \sum_{\boldsymbol{e},\boldsymbol{w}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}}(\boldsymbol{x}^{(2)})^{\boldsymbol{w}}y^b\right)}{\prod_{g\in\mathscr{E}^{\mathsf{P}}}\binom{Q}{e_g+w_g}\binom{e_g+w_g}{e_g}(q-1)^{e_g+w_g}}\times \\ & \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})^Q, t^a(\boldsymbol{s}^{(1)})^{\boldsymbol{e}}(\boldsymbol{s}^{(2)})^{\boldsymbol{w}}\right)\end{aligned} \tag{7.74}$$

with

$$B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)}) = 1+(q-1)t\sum_{\boldsymbol{r}^{(j)}\in\mathcal{R}_j}(\boldsymbol{s}_j^{(1)})^{\mathbf{1}_{\mathsf{d}_{\mathsf{v}_j}}-\boldsymbol{r}^{(j)}}(\boldsymbol{s}_j^{(2)})^{\boldsymbol{r}^{(j)}} \tag{7.75}$$

where

$$\begin{aligned}A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y) =& \frac{1}{q}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1+(q-1)x_g^{(1)}) + (q-1)\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g^{(1)})\right] \\ &+\frac{q-1}{q}y\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1+(q-1)x_g^{(2)}) - \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g^{(2)})\right]\end{aligned} \tag{7.76}$$

and where $\mathbf{1}_{\mathsf{d}_{\mathsf{v}_j}}$ is the length $\mathsf{d}_{\mathsf{v}_j}$ all-ones vector, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v}_j}$ and Hamming weight $\le \lfloor(\mathsf{d}_{\mathsf{v}_j}-1)/2\rfloor$, and $\boldsymbol{s}^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{s}_j^{(o)} = (s_g^{(o)})_{g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}}$, $\boldsymbol{x}^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_i^{(o)} = (x_g^{(o)})_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$, y, t and $s_g^{(o)}, x_g^{(o)}, g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}, o=1,2$ are dummy variables [54]. The proof is similar to the one of Lemma 6.14. It is hence omitted.

Next, we analyze the normalized logarithmic asymptotic distribution of ASs for the U-NBPB LDPC ensemble. The next Theorem presents a simple way to compute the normalized logarithmic asymptotic distribution of ASs for the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$.

Theorem 7.10. The normalized asymptotic distribution of $(\theta n,\gamma n)$ ASs for the ensemble $\mathscr{C}_{q,n}^{\mathsf{P},\mathrm{u}}$ is

$$\begin{aligned}G_{\mathrm{AS}}^{\mathsf{P},u}(\theta,\gamma) =& \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y) + \frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)}) \\ &-\gamma\ln y - \theta\ln t - \frac{1}{n_{\mathsf{P}}}\sum_{g\in\mathscr{E}^{\mathsf{P}}}\ln\left(1+(q-1)\left(x_g^{(1)}s_g^{(1)}+x_g^{(2)}s_g^{(2)}\right)\right).\end{aligned} \tag{7.77}$$

The values t, $s_g^{(1)}$, $s_g^{(2)}$, $x_g^{(1)}$, $x_g^{(2)}$, for $g \in \mathscr{E}^{\mathsf{P}}$ and the value y are the unique positive solutions of

$$t \frac{\partial \ln \prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial t} = n_{\mathsf{P}} \theta \tag{7.78}$$

$$s_g^{(1)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(1)}} = x_g^{(1)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(1)}} = n_{\mathsf{P}} \tilde{e}_g^{\star} \tag{7.79}$$

$$s_g^{(2)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(2)}} = x_g^{(2)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(2)}} = n_{\mathsf{P}} \tilde{w}_g^{\star} \tag{7.80}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{7.81}$$

where (7.79) and (7.80) are for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, j \in \{1, \ldots, n_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}} \cap \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$, $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ and $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ are defined in (7.75) and (7.76), respectively, and

$$\tilde{e}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{(q-1) x_g^{(1)} s_g^{(1)}}{1 + (q-1)(x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)})} \tag{7.82}$$

$$\tilde{w}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{(q-1) x_g^{(2)} s_g^{(2)}}{1 + (q-1)(x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)})}. \tag{7.83}$$

The proof of Theorem 7.10 is omitted since it is similar to the one of Theorem 6.17.

The result can be easily extended to enumerate the EASs. In fact, to compute the finite-length and the asymptotic distribution of EASs, we simply need to replace in (7.74) and (7.77) the generating function $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ given in (7.76) with

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = 1 + (q-1) \sum_{g, g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g \neq g'} x_g^{(1)} x_{g'}^{(1)} + (q-1) y \sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} x_g^{(2)}. \tag{7.84}$$

We briefly explain the derivation of $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ in (7.84). For EASs, each satisfied CN is connected to zero or two VNs from $\mathcal{I}$ and each unsatisfied CN is connected to exactly one VN from $\mathcal{I}$. Consider a CN of type c_i. If it is satisfied and connected to zero or two VNs from $\mathcal{I}$, the number of configurations can be tracked by the generating function $g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := y^0 \sum_{\boldsymbol{c}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{p}(\boldsymbol{c})}$ where the sum is over all $\boldsymbol{c} \in \mathbb{F}_q^{\mathsf{d}_{\mathsf{c}_i}}$ such that $\mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) \in \{0, 2\}$ and $\sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} c_g = 0$, and $\boldsymbol{p}(\boldsymbol{c}) = (p_g)_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$ with $p_g = 1$ if $c_g \neq 0$ and $p_g = 0$ otherwise, yielding

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) = 1 + (q-1) \sum_{g, g' \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} : g \neq g'} x_g^{(1)} x_{g'}^{(1)}.$$

If the CN is unsatisfied and connected to exactly one VN from $\mathcal{I}$ then its generating function is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y) := y^1 \sum_{\boldsymbol{c}\in\mathbb{F}_q^{\mathsf{d_{c_i}}}:\mathrm{w_H}(\boldsymbol{c})=1} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{p}(\boldsymbol{c})} = (q-1)y \sum_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g^{(2)}.$$

We can see from Theorem 7.10 that to evaluate $G^{\mathsf{P},u}_{\mathrm{AS}}(\theta,\gamma)$ and $G^{\mathsf{P},u}_{\mathrm{EAS}}(\theta,\gamma)$ one must solve $4e+2$ equations in $4e+2$ variables: $s_g^{(1)}, s_g^{(2)}, x_g^{(1)}, x_g^{(2)}$ ($4e$ variables), y (one variable) and t (one variable). The following Lemma, also based on the approach of [119], is similar to Lemma 7.9 and can reduce the dimension of the system of equations.

Lemma 7.12. Let u, v be two edges in $\mathscr{E}^{\mathsf{P}}$. If u and v are connected to the same VN-CN pair in the protograph, then for all $o \in \{1,2\}$ we have $s_u^{(o)} = s_v^{(o)}$ and $x_u^{(o)} = x_v^{(o)}$.

Proof. Consider two edges u and v which connect c_i to v_j. We define $\boldsymbol{z} := [s_u^{(1)},\ s_v^{(1)},\ s_u^{(2)},\ s_v^{(2)},\ x_u^{(1)},\ x_v^{(1)},\ x_u^{(2)},\ x_v^{(2)}]$, $\boldsymbol{z}_1 := [\chi_1,\ \psi_1,\ \chi_2,\ \psi_2,\ \kappa_1,\ \beta_1,\ \kappa_2,\ \beta_2]$ and $\boldsymbol{z}_2 := [\psi_1,\ \chi_1,\ \psi_2,\ \chi_2,\ \beta_1,\ \kappa_1,\ \beta_2,\ \kappa_2]$. It is clear that $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ in (7.75) is symmetric in the variables $s_g^{(1)}, s_g^{(2)}$, $g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}$ and the functions $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ in (7.76) and (7.84) are symmetric in the variables $x_g^{(1)}, x_g^{(2)}$, $g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}$. Thus, for the system of equations in Theorem 7.10, if there is a solution with $\boldsymbol{z} = \boldsymbol{z}_1$ then another solution exists with $\boldsymbol{z} = \boldsymbol{z}_2$ (all the other variables being unchanged). Since the solutions $s_g^{(1)}$, $s_g^{(2)}$, $x_g^{(1)}$, $x_g^{(2)}, g \in \mathscr{E}^{\mathsf{P}}$ are unique, we have $\boldsymbol{z}_1 = \boldsymbol{z}_2$, i.e., $\psi_1 = \chi_1, \psi_2 = \chi_2, \kappa_1 = \beta_1, \kappa_2 = \beta_2$. ■

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Lemma 7.13. The average number of size (a,b) ASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}^{\mathsf{P},\mathsf{c}}_{q,n}(\boldsymbol{f})$ is

$$\mathsf{E}^{\mathsf{P},c}_{\mathrm{AS}}(a,b) = \sum_{\boldsymbol{e},\boldsymbol{w}} \frac{\mathrm{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b\right)}{\prod_{g\in\mathscr{E}^{\mathsf{P}}} \binom{Q}{Q-\sum_{\ell=0}^{q-2}(e_{g,\ell}+w_{g,\ell}), e_{g,0}+w_{g,0},\dots,e_{g,q-2}+w_{g,q-2}} \prod_{\ell=0}^{q-2}\binom{e_{g,\ell}+w_{g,\ell}}{e_{g,\ell}}} \times \mathrm{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}}\right) \tag{7.85}$$

where

$$B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) = 1 + t \sum_{\ell=0}^{q-2} \sum_{\boldsymbol{r}^{(j)}\in\mathcal{R}_j} \prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}} (s_{g,\ell}^{(1)})^{1-r_g^{(j)}} (s_{g,\ell}^{(2)})^{r_g^{(j)}} \tag{7.86}$$

and

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(1)}) + y \left[\prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(1 + \sum_{\ell=0}^{q-2} x_{g,\ell}^{(2)} \right) - W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(2)}) \right] \tag{7.87}$$

and where $\boldsymbol{r}^{(j)} = (r_g^{(j)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ and Hamming weight $\leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$, $W_{\mathcal{P}_i}(\boldsymbol{x}_i)$ is given in (7.65), $\boldsymbol{s}^{(o)} = (\boldsymbol{s}_g^{(o)})_{g \in \mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{s}_g^{(o)} = [s_{g,0}^{(o)}, \ldots, s_{g,q-2}^{(o)}]$, $\boldsymbol{s}_j^{(o)} = (\boldsymbol{s}_g^{(o)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$, $\boldsymbol{x}^{(o)} = (\boldsymbol{x}_g^{(o)})_{g \in \mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{x}_g^{(o)} = [x_{g,0}^{(o)}, \ldots, x_{g,q-2}^{(o)}]$, $\boldsymbol{x}_i^{(o)} = (\boldsymbol{x}_g^{(o)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}$, y, t and $s_{g,\ell}^{(o)}, x_{g,\ell}^{(o)}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}, o = 1, 2$, $\ell = 0, \ldots, q-2$ are dummy variables.

Proof. Define the edge frequency weight vectors $\boldsymbol{e} = (\boldsymbol{e}_g)_{g \in \mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{e}_g = [e_{g,0}, \ldots, e_{g,q-2}]$ and $\boldsymbol{w} = (\boldsymbol{w}_g)_{g \in \mathscr{E}^{\mathsf{P}}}$, $\boldsymbol{w}_g = [w_{g,0}, \ldots, w_{g,q-2}]$ where $e_{g,\ell}$ represents the number of times the symbol α^ℓ occurs in the edges of type g in $\mathcal{A}_{a,b}$ emanating from satisfied CNs in $\mathcal{A}_{a,b}$ and $w_{g,\ell}$ is the number of times the symbol α^ℓ occurs in the edges of type g in $\mathcal{A}_{a,b}$ emanating from unsatisfied CNs. We randomly choose a set $\mathcal{I}$ of a VNs and assign a non-zero symbol from $\mathbb{F}_q$ to each VN in the set. The edges connected to a VN v are assigned the non-zero symbol chosen for v. We denote by $N_\mathsf{c}(b, \boldsymbol{e}, \boldsymbol{w})$ the number of configurations with edge weight vectors $\boldsymbol{e}, \boldsymbol{w}$ that give exactly b unsatisfied CNs. Its generating function is

$$\sum_{b, \boldsymbol{e}, \boldsymbol{w}} N_\mathsf{c}(b, \boldsymbol{e}, \boldsymbol{w}) y^b (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}}.$$

Consider a CN of type c_i. The number of configurations for which the CN is satisfied is tracked by the generating function

$$g_\mathsf{c}(\boldsymbol{x}_i^{(1)}, y) := y^0 W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(1)})$$

while the number of configurations for which the CN is unsatisfied is tracked by

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y) := y^1 \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} \left(1 + \sum_{\ell=0}^{q-2} x_{g,\ell}^{(2)} \right) - y^1 W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(2)}).$$

Recalling that the sum $g_\mathsf{c}(\boldsymbol{x}_i^{(1)}, y) + g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y)$ yields $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$, and considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_\mathsf{c}(b, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff}\left(\prod_{i=1}^{m_\mathsf{P}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b \right) \tag{7.88}$$

where $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ is defined in (7.87).

Let $N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w})$ be the number of configurations with a VNs and edge weight vectors $\boldsymbol{e}, \boldsymbol{w}$ such that each of these VNs is connected to strictly fewer unsatisfied CNs than satisfied CNs. Its generating function is

$$\sum_{a,\boldsymbol{e},\boldsymbol{w}} N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}}.$$

Consider a VN of type v_j. Let $\boldsymbol{r}^{(j)} = (r_g^{(j)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$ be a length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ binary vector with $r_g^{(j)} = 1$ if the type g edge emanates from an unsatisfied CN and $r_g^{(j)} = 0$ otherwise. Note that if the VN of type v_j belongs to $\mathcal{I}$, the Hamming weight of $\boldsymbol{r}^{(j)}$ should satisfy $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) = \sum_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}} r_g^{(j)} \leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$. We can either include this VN in $\mathcal{I}$ or not. If we skip it we obtain the zero-degree term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ corresponding to zero VNs and zero edges. If we include it in the set, we will have one node, $\mathsf{d}_{\mathsf{v}_\mathsf{j}} - \mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges emanating from satisfied CNs and $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges emanating from unsatisfied CNs with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$. Considering all possible non-zero symbols we can assign to the VN and all possible binary vectors $\boldsymbol{r}^{(j)}$, we obtain the second term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$. Taking into account all possible VN types and that there are Q VNs of each type, we obtain

$$N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} \right) \tag{7.89}$$

where $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ is defined in (7.86). The number of edge sets with frequency weight vectors $\boldsymbol{e}$ and $\boldsymbol{w}$ is

$$N_e(\boldsymbol{e}, \boldsymbol{w}) = \prod_{g \in \mathscr{E}^{\mathsf{P}}} \binom{Q}{Q - \sum_{\ell=0}^{q-2}(e_{g,\ell} + w_{g,\ell}), e_{g,0} + w_{g,0}, \ldots, e_{g,q-2} + w_{g,q-2}} \times \prod_{\ell=0}^{q-2} \binom{e_{g,\ell} + w_{g,\ell}}{e_{g,\ell}}. \tag{7.90}$$

Noting that

$$\mathsf{E}^{\mathsf{P},c}_{\mathrm{AS}}(a, b) = \sum_{\boldsymbol{e},\boldsymbol{w}} \frac{N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w})}{N_e(\boldsymbol{e}, \boldsymbol{w})} \tag{7.91}$$

completes the proof. ■

We remark that in this case the exact average number of size (a, b) ASs for a finite block length n is extremely complex to compute for large n.

Theorem 7.11. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the ensemble

$\mathscr{C}_{q,n}^{\mathsf{P},\mathsf{c}}(\boldsymbol{f})$ is

$$\begin{aligned}G_{\mathrm{AS}}^{\mathsf{P},c}(\theta,\gamma) =& \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y) - \theta\ln t + \frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)}) \\ & - \gamma\ln y - \frac{1}{n_{\mathsf{P}}}\sum_{g\in\mathscr{E}^{\mathsf{P}}}\ln\left(1+\sum_{\ell=0}^{q-2}\left(x_{g,\ell}^{(1)}s_{g,\ell}^{(1)} + x_{g,\ell}^{(2)}s_{g,\ell}^{(2)}\right)\right).\end{aligned} \tag{7.92}$$

The values t, $s_{g,\ell}^{(1)}$, $s_{g,\ell}^{(2)}$, $x_{g,\ell}^{(1)}$, $x_{g,\ell}^{(2)}$, for $g\in\mathscr{E}^{\mathsf{P}}$, $\ell\in\{0,\ldots,q-2\}$ and the value y are the unique positive solutions of

$$t\frac{\partial\ln\prod_{j=1}^{n_{\mathsf{P}}}B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})}{\partial t} = n_{\mathsf{P}}\theta \tag{7.93}$$

$$s_{g,\ell}^{(1)}\frac{\partial\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})}{\partial s_{g,\ell}^{(1)}} = x_{g,\ell}^{(1)}\frac{\partial\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)}{\partial x_{g,\ell}^{(1)}} = n_{\mathsf{P}}\tilde{e}_{g,\ell}^{\star} \tag{7.94}$$

$$s_{g,\ell}^{(2)}\frac{\partial\ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})}{\partial s_{g,\ell}^{(2)}} = x_{g,\ell}^{(2)}\frac{\partial\ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)}{\partial x_{g,\ell}^{(2)}} = n_{\mathsf{P}}\tilde{w}_{g,\ell}^{\star} \tag{7.95}$$

$$y\frac{\partial\ln\prod_{i=1}^{m_{\mathsf{P}}}A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)}{\partial y} = n_{\mathsf{P}}\gamma \tag{7.96}$$

where (7.94) and (7.95) are for all $i\in\{1,\ldots,m_{\mathsf{P}}\}, j\in\{1,\ldots,n_{\mathsf{P}}\}, g\in\mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}\cap\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}, \ell\in\{0,\ldots,q-2\}$, $B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)})$ and $A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)$ are defined in (7.86) and (7.87), respectively, and

$$\tilde{e}_{g,\ell}^{\star} = \frac{1}{n_{\mathsf{P}}}\frac{x_{g,\ell}^{(1)}s_{g,\ell}^{(1)}}{1+\sum_{\ell'=0}^{q-2}\left(x_{g,\ell'}^{(1)}s_{g,\ell'}^{(1)}+x_{g,\ell'}^{(2)}s_{g,\ell'}^{(2)}\right)} \tag{7.97}$$

$$\tilde{w}_{g,\ell}^{\star} = \frac{1}{n_{\mathsf{P}}}\frac{x_{g,\ell}^{(2)}s_{g,\ell}^{(2)}}{1+\sum_{\ell'=0}^{q-2}\left(x_{g,\ell'}^{(1)}s_{g,\ell'}^{(1)}+x_{g,\ell'}^{(2)}s_{g,\ell'}^{(2)}\right)}. \tag{7.98}$$

The proof can be found in Appendix 7.4.4.

Note that for computing the normalized asymptotic distribution of EASs, we simply need to replace in (7.85) and (7.92) the generating function $A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y)$ given in (7.87) with

$$A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},y) = 1 + \sum_{g,g'\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}:g\neq g'}\ \sum_{\ell,\ell':f_g\alpha^{\ell}+f_{g'}\alpha^{\ell'}=0} x_{g,\ell}^{(1)}x_{g',\ell'}^{(1)} + y\sum_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}\sum_{\ell=0}^{q-2}x_{g,\ell}^{(2)}. \tag{7.99}$$

Table 7.1: The error profiles for the hard decision decoder for the QSC crossover probability $\epsilon = 0.004$.

	(1,3)	(2,2)	(2,3)	(2,4)	(2,5)	(3,2)	(3,3)	(3,4)	(3,5)
TS	43	736	8	8	2	21	8	1	3
AS	0	736	0	0	0	0	4	0	0

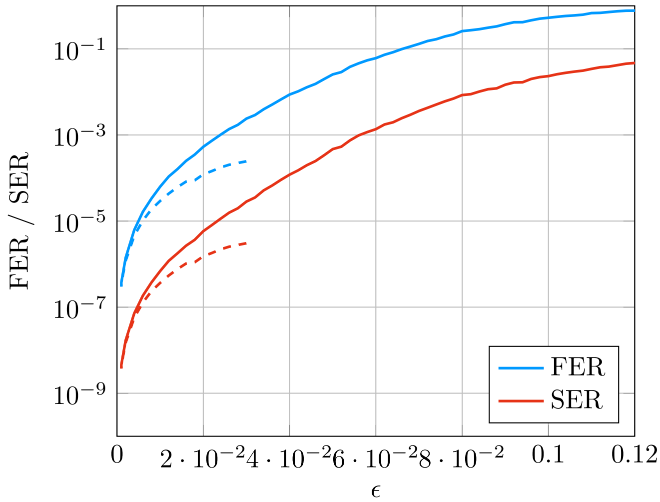

Figure 7.2: FER and SER versus the QSC crossover probability ϵ. The dashed lines represent the contribution of the dominant $(2, 2)$ TS to the FER and SER.

Example 7.2. We consider an LDPC code from the U-NBPB code ensemble defined by the protograph base matrix $\boldsymbol{B} = [3 \;\; 3]$ for $q = 8$ and $Q = 80$. We present some experimental results to validate the effect of the TSs and ASs (as defined in Definitions 7.1 and 7.2) on the performance of non-binary protograph-based LDPC codes under the hard message passing decoding algorithm of [27] with list size one, which is similar to the SMP decoder in Chapter 4. We transmit the all-zero codeword over a QSC with error probability ϵ, where the channel alphabet cardinality is matched to the field order. For each channel realization leading to a decoding failure, we check if the subgraph containing the corrupted VNs and their neighboring CNs is a TS or AS. In this case, we determine its size. We collected 1000 frame errors at channel crossover probability $\epsilon = 0.004$. Table 7.1 shows the obtained error profiles, i.e., the number of occurrences of specific TSs and ASs. Simulation results of the considered code are shown in Fig. 7.2 in terms of FER and SER versus the QSC crossover probability ϵ. The dashed lines represent the contribution of the dominant $(2, 2)$ TS to the FER and SER. Note that for small ϵ, the FER and SER are dominated by the $(2, 2)$ TS.

Example 7.3. Consider the protograph with the base matrix $\boldsymbol{B} = [2 \;\; 2]$. We evaluate the TS distribution of the U-NBPB and C-NBPB ensembles from (7.51) and (7.63). The

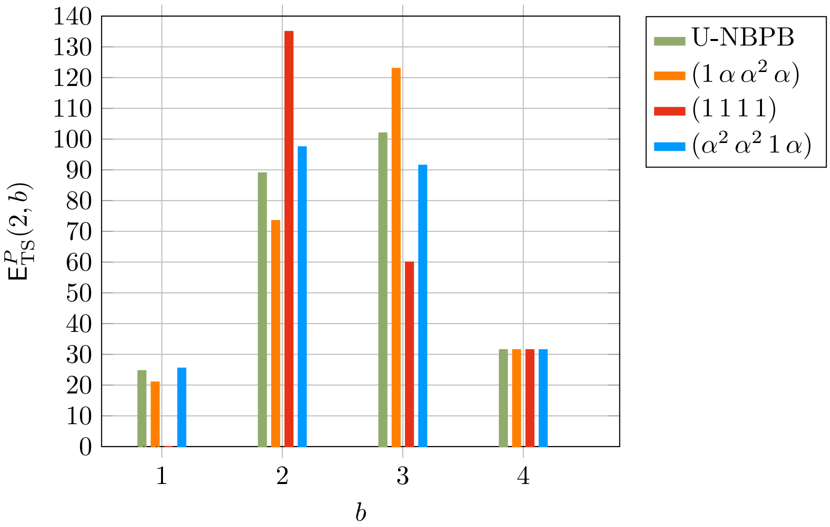

Figure 7.3: Average number of size $(2, b)$ TSs for the ensemble in Example 7.3 for $q = 4$ and $Q = 4$.

Table 7.2: The relative minimum Δ-trapping and absorbing set sizes for the ensemble in Example 7.4 for $\Delta = 0.1$.

q	$\theta^{\star}_{\mathrm{TS}}$ $(\boldsymbol{B}_1)$	$\theta^{\star}_{\mathrm{AS}}$ $(\boldsymbol{B}_1)$	$\theta^{\star}_{\mathrm{TS}}$ $(\boldsymbol{B}_2)$	$\theta^{\star}_{\mathrm{AS}}$ $(\boldsymbol{B}_2)$
4	0.014875	0.015016	0.068831	0.069397
8	0.023894	0.024167	0.098416	0.099534
16	0.031946	0.032384	0.121417	0.123159
32	0.037808	0.038421	0.137429	0.139788
64	0.040997	0.041772	0.147011	0.149935
128	0.041660	0.042566	0.151134	0.154555

average number of size $(2, b)$ TSs of the U-NBPB and C-NBPB ensembles for $q = 4$, $Q = 4$ and different edge labels is depicted in Fig. 7.3. Observe that with a good edge weight assignment, the C-NBPB ensemble can have fewer small sized TSs. Thus, we can obtain a C-NBPB ensemble with better TS properties by carefully choosing the edge weights. For instance, if the size $(2, 2)$ TS dominates the error floor performance of the code under a specific decoding algorithm, then the edge label $(1, \alpha, \alpha^2, \alpha)$ would be a better choice than the other edge labels since it has fewer $(2, 2)$ TSs. Otherwise, the edge label $(1, 1, 1, 1)$ would be the best choice.

Example 7.4. Consider the U-NBPB code ensembles obtained from the base matrices $\boldsymbol{B}_1 = [3\ \ 3]$ and $\boldsymbol{B}_2 = [4\ \ 4]$. For both ensembles, we evaluate the expressions of the normalized logarithmic asymptotic distribution of TSs, ASs and EASs from Theorems 7.7 and 7.10 for a fixed ratio $\Delta = 0.1$. The results are shown in Fig. 7.4 and 7.5. Observe that

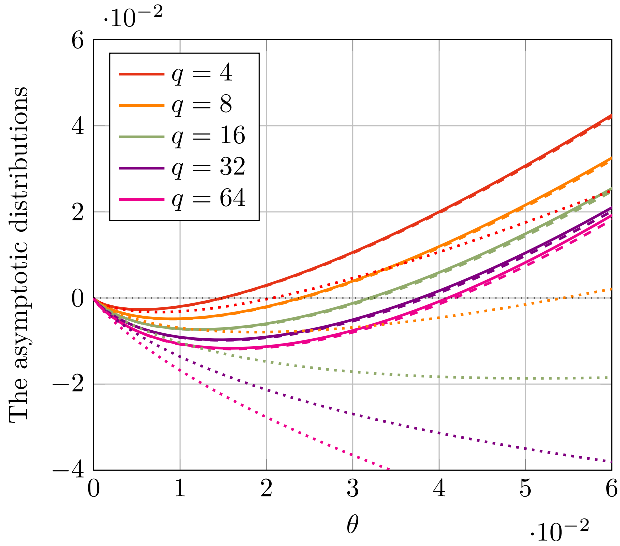

Figure 7.4: Normalized logarithmic asymptotic distributions of TSs (solid lines), ASs (dashed lines) and EASs (dotted lines) of the LDPC ensemble with base matrix $\boldsymbol{B}_1$ in Example 7.4 for $\Delta = 0.1$.

Table 7.3: The relative minimum Δ-trapping and absorbing set sizes for the ensemble in Example 7.4 for $q = 4$.

Δ	$\theta^{\star}_{\mathrm{TS}}$ $(\boldsymbol{B}_1)$	$\theta^{\star}_{\mathrm{AS}}$ $(\boldsymbol{B}_1)$	$\theta^{\star}_{\mathrm{TS}}$ $(\boldsymbol{B}_2)$	$\theta^{\star}_{\mathrm{AS}}$ $(\boldsymbol{B}_2)$
0.005	0.043007	0.043008	0.112292	0.112294
0.05	0.025367	0.025428	0.087155	0.087349
0.15	0.008667	0.008857	0.055246	0.056222
0.2	0.004904	0.005105	0.044668	0.0460343
0.3	0.001342	0.001487	0.029372	0.031380
0.4	0.000261	0.000325	0.019166	0.021595

the ensemble obtained from $\boldsymbol{B}_2$ has better TS and AS properties than the one with $\boldsymbol{B}_1$. For instance, we can see in Table. 7.2 that for $q = 4$ and $\Delta = 0.1$, we have $\theta^{\star}_{\mathrm{TS}} = 0.068831$ while for the first ensemble, we have $\theta^{\star}_{\mathrm{TS}} = 0.014875$. A comparison of the relative minimum Δ-trapping and absorbing set sizes of these ensembles for $q = 4$ and different values of Δ is shown in Table. 7.3. We see that for a fixed rate, increasing the VN degrees improves the TS properties of the ensemble and increases the relative minimum trapping and absorbing sizes (for the same Δ and q). This matches the observation made in [48].

We generate length $N = 6400$ codes from the ensembles characterized by $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ for $q = 8$. Fig. 7.6 shows the performance of these codes under the algorithm introduced in [27] with list size one. We observe that the code obtained from $\boldsymbol{B}_2$ has a better error floor performance. This is due to the better TS properties of the $\boldsymbol{B}_2$ base matrix that we

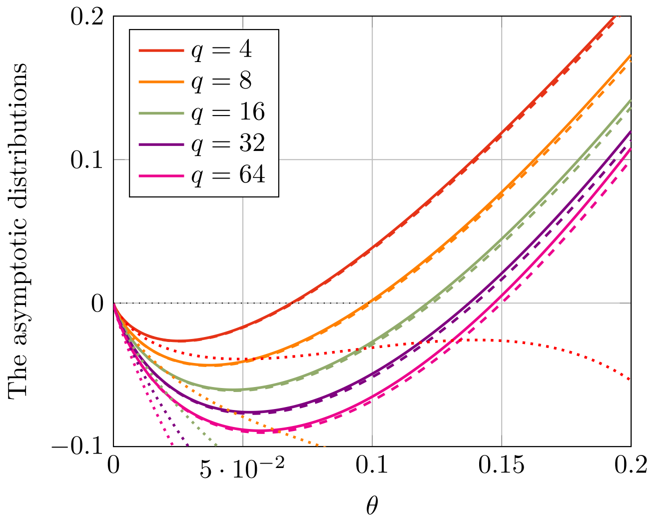

Figure 7.5: Normalized logarithmic asymptotic distributions of TSs (solid lines), ASs (dashed lines) and EASs (dotted lines) of the LDPC ensemble with base matrix $\boldsymbol{B}_2$ in Example 7.4 for $\Delta = 0.1$.

observed in the analysis.

7.4 Appendices

7.4.1 Proof of Corollary 7.1

The proof is based on obtaining expressions for t, s, x, y in terms of $\tilde{w}^\star$, and for $\tilde{w}^\star$ in terms of θ. Consider (7.5)-(7.7) when $\theta \to 0$ and $\gamma = \Delta\theta$. These equations can be rewritten as

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{(q-1)ts^j}{1+(q-1)ts^j} = \theta \quad (7.100)$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{j(q-1)ts^j}{1+(q-1)ts^j} = \tilde{w}^\star \quad (7.101)$$

$$\xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{\sum_{k=2}^{i} u_k k x^k + \sum_{k=1}^{i} c_k k x^k y}{1 + \sum_{k=2}^{i} u_k x^k + \sum_{k=1}^{i} c_k x^k y} = \tilde{w}^\star \quad (7.102)$$

$$\xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i \frac{\sum_{k=1}^{i} c_k x^k y}{1 + \sum_{k=2}^{i} u_k x^k + \sum_{k=1}^{i} c_k x^k y} = \Delta\theta \quad (7.103)$$

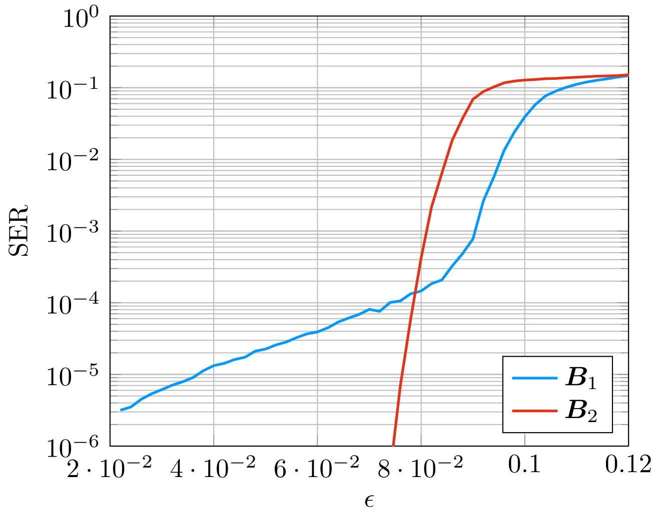

Figure 7.6: SER versus the QSC crossover probability ϵ for 8-ary codes drawn from the ensembles in Example 7.4 with $N = 6400$.

where

$$u_k = \binom{i}{k} \frac{(-1)^k(q-1) + (q-1)^k}{q} \tag{7.104}$$

$$c_k = \binom{i}{k} (q-1) \frac{(q-1)^k - (-1)^k}{q}. \tag{7.105}$$

From (7.100) and (7.101), we see that $\mathsf{d}_\mathsf{v}^{\min}\theta \leq \tilde{w}^\star \leq \mathsf{d}_\mathsf{v}^{\max}\theta$. Thus, we have

$$\lim_{\theta \to 0} \tilde{w}^\star = 0 \tag{7.106}$$

and the notations $o(\theta)$ and $o(\tilde{w}^\star)$ are equivalent, i.e., for any function f, $f = o(\theta) \iff f = o(\tilde{w}^\star)$. Therefore, we will use $o(\theta)$ and $o(\tilde{w}^\star)$ interchangeably. The left hand side of (7.103) is also $o(1)$, i.e., for some odd k we have $x^k y = o(1)$ and for all other k we have $x^k y = o(\theta)$. Thus, we have

$$\Delta\theta(1 + o(1)) = \xi \sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i i (q-1) x y = \bar{\mathsf{d}}_\mathsf{v} (q-1) x y. \tag{7.107}$$

Because of (7.106), the left hand side of (7.102) must be $o(1)$, i.e., $x^k = o(1)$ for some k, $x^k = o(\tilde{w}^\star)$ for the other k, $x^k y = o(1)$ for some k and $x^k y = o(\tilde{w}^\star)$ for the other k. The left hand side of (7.102) is dominated by the terms corresponding to $k = 1, 2$. We have

$$\begin{aligned}\tilde{w}^\star(1+o(1)) =& \xi(q-1)\sum_{i=1}^{\mathsf{d}_\mathsf{c}^{\max}} \mathrm{P}_i\left[i(i-1)x^2 + ixy\right]\\ =& \xi\mathrm{P}''(1)(q-1)x^2 + \bar{\mathsf{d}}_\mathsf{v}(q-1)xy.\end{aligned} \tag{7.108}$$

From (7.107) and (7.108), we obtain

$$x = \sqrt{\frac{\tilde{w}^\star - \Delta\theta}{\xi(q-1)\mathrm{P}''(1)}}(1+o(1)). \tag{7.109}$$

Substituting (7.107) into (7.8), we obtain

$$s = \tilde{w}^\star\sqrt{\frac{\mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(q-1)(\tilde{w}^\star - \Delta\theta)}}(1+o(1)). \tag{7.110}$$

Thus, (7.101) can be written as

$$\tilde{w}^\star(1+o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}}\mathsf{d}_\mathsf{v}^{\min}(q-1)ts^{\mathsf{d}_\mathsf{v}^{\min}} \tag{7.111}$$

and we have

$$t = \frac{(q-1)^{\frac{\mathsf{d}_\mathsf{v}^{\min}-2}{2}}}{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}}\mathsf{d}_\mathsf{v}^{\min}(\tilde{w}^\star)^{\mathsf{d}_\mathsf{v}^{\min}-1}}\left(\frac{(\tilde{w}^\star - \Delta\theta)\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}}{\mathrm{P}''(1)}\right)^{\frac{\mathsf{d}_\mathsf{v}^{\min}}{2}}(1+o(1)). \tag{7.112}$$

Similarly, from (7.100), we have

$$\theta(1+o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}}(q-1)ts^{\mathsf{d}_\mathsf{v}^{\min}} \tag{7.113}$$

and

$$\tilde{w}^\star = \mathsf{d}_\mathsf{v}^{\min}\theta(1+o(1)). \tag{7.114}$$

Substituting (7.114) into (7.109), (7.110) and (7.112), we obtain

$$x = \sqrt{\theta\frac{\mathsf{d}_\mathsf{v}^{\min} - \Delta}{\xi(q-1)\mathrm{P}''(1)}}(1+o(1)) \tag{7.115}$$

$$s = \mathsf{d}_\mathsf{v}^{\min}\sqrt{\theta\frac{\mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(q-1)(\mathsf{d}_\mathsf{v}^{\min}-\Delta)}}(1+o(1)) \tag{7.116}$$

$$t = \frac{(q-1)^{\frac{\mathsf{d}_\mathsf{v}^{\min}-2}{2}}\theta^{\frac{2-\mathsf{d}_\mathsf{v}^{\min}}{2}}}{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}}(\mathsf{d}_\mathsf{v}^{\min})^{\mathsf{d}_\mathsf{v}^{\min}}}\left(\frac{(\mathsf{d}_\mathsf{v}^{\min}-\Delta)\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}}{\mathrm{P}''(1)}\right)^{\frac{\mathsf{d}_\mathsf{v}^{\min}}{2}}(1+o(1)). \tag{7.117}$$

From (7.107) and (7.115), we have

$$y = \Delta\sqrt{\frac{\theta\mathrm{P}''(1)}{\bar{\mathsf{d}}_\mathsf{c}\bar{\mathsf{d}}_\mathsf{v}(q-1)(\mathsf{d}_\mathsf{v}^{\min}-\Delta)}}(1+o(1)). \tag{7.118}$$

By substituting (7.115)-(7.118) into (7.4) and by using the Taylor series of $\ln(1+x)$ at $x=0$, we obtain (7.11).

7.4.2 Proof of Theorem 7.8

Let $\mathcal{P}$ be an (n,k) linear code over $\mathbb{F}_q$ and ξ be a non-trivial character of $\mathbb{F}_q$. We have [131]

$$\chi(u+o) = \chi(u)\chi(o) \quad \forall u,o \in \mathbb{F}_q. \tag{7.119}$$

Define the function

$$g(\boldsymbol{c}) = \sum_{\boldsymbol{v}\in\mathbb{F}_q^n}\chi(\boldsymbol{c}^T\boldsymbol{v})\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)}.$$

We have

$$\begin{aligned}
\sum_{\boldsymbol{c}\in\mathcal{P}}g(\boldsymbol{c}) &= \sum_{\boldsymbol{c}\in\mathcal{P}}\sum_{\boldsymbol{v}\in\mathbb{F}_q^n}\chi(\boldsymbol{c}^T\boldsymbol{v})\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)} = \sum_{\boldsymbol{v}\in\mathbb{F}_q^n}\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)}\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{c}^T\boldsymbol{v}) \\
&= \sum_{\boldsymbol{v}\in\mathcal{P}^\perp}\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)}\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(0) + \sum_{\boldsymbol{v}\notin\mathcal{P}^\perp}\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)}\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{c}^T\boldsymbol{v}) \\
&= |\mathcal{P}|W_{\mathcal{P}^\perp}(\boldsymbol{x}) + \sum_{\boldsymbol{v}\notin\mathcal{P}^\perp}\prod_{i=1}^{n}\prod_{\ell=0}^{q-2}x_{i,\ell}^{\phi_{i,\ell}(v_i)}\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{c}^T\boldsymbol{v}).
\end{aligned}$$

For $\boldsymbol{v}\notin\mathcal{P}^\perp$, $\exists\boldsymbol{e}\in\mathcal{P}, u\in\mathbb{F}_q$ with $\boldsymbol{v}^T\boldsymbol{e}\neq 0$ and $\chi(u\boldsymbol{v}^T\boldsymbol{e})\neq 1$. Thus, from (7.119) we obtain

$$\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{c}^T\boldsymbol{v}) = \sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{v}^T(\boldsymbol{c}+u\boldsymbol{e})) = \chi(u\boldsymbol{v}^T\boldsymbol{e})\sum_{\boldsymbol{c}\in\mathcal{P}}\chi(\boldsymbol{v}^T\boldsymbol{c}).$$

Since $\chi(u\boldsymbol{v}^T\boldsymbol{e}) \neq 1$ and $\chi(u\boldsymbol{v}^T\boldsymbol{e}) \neq 0$, $\sum_{\boldsymbol{c}\in\mathcal{P}} \chi(\boldsymbol{c}^T\boldsymbol{v}) = 0$ and as a result $\sum_{\boldsymbol{c}\in\mathcal{P}} g(\boldsymbol{c}) = |\mathcal{P}| W_{\mathcal{P}^\perp}(\boldsymbol{x})$. Using (7.119), we have

$$\begin{aligned} g(\boldsymbol{c}) &= \sum_{\boldsymbol{v}\in\mathbb{F}_q^n} \chi(\boldsymbol{c}^T\boldsymbol{v}) \prod_{i=1}^{n} \prod_{\ell=0}^{q-2} x_{i,\ell}^{\phi_{i,\ell}(v_i)} \\ &= \sum_{\boldsymbol{v}\in\mathbb{F}_q^n} \prod_{i=1}^{n} \chi(c_i \cdot v_i) \prod_{\ell=0}^{q-2} x_{i,\ell}^{\phi_{i,\ell}(v_i)} \\ &= \prod_{i=1}^{n} \sum_{v_i\in\mathbb{F}_q} \chi(c_i \cdot v_i) \prod_{\ell=0}^{q-2} x_{i,\ell}^{\phi_{i,\ell}(v_i)} \end{aligned} \tag{7.120}$$

with

$$\sum_{v_i\in\mathbb{F}_q} \chi(c_i \cdot v_i) \prod_{\ell=0}^{q-2} x_{i,\ell}^{\phi_{i,\ell}(v_i)} = \begin{cases} 1 + \sum_{\ell=0}^{q-2} x_{i,\ell} & \text{if } c_i = 0 \\ 1 + \sum_{\ell=0}^{q-2} \chi(c_i \cdot \alpha^\ell) x_{i,\ell} & \text{otherwise.} \end{cases} \tag{7.121}$$

We obtain

$$g(\boldsymbol{c}) = \prod_{i=1}^{n} \left(1 + \sum_{\ell=0}^{q-2} x_{i,\ell}\right) \prod_{\ell'=0}^{q-2} \left(\frac{1 + \sum_{\ell=0}^{q-2} \chi(\alpha^{\ell+\ell'}) x_{i,\ell}}{1 + \sum_{\ell=0}^{q-2} x_{i,\ell}} \right)^{\phi_{i,\ell'}(c_i)} \tag{7.122}$$

and therefore

$$\begin{aligned} W_{\mathcal{P}^\perp}(\boldsymbol{x}) &= \frac{1}{|\mathcal{P}|} \sum_{\boldsymbol{c}\in\mathcal{P}} g(\boldsymbol{c}) \\ &= \frac{1}{q^k} \left[\prod_{i=1}^{n} \left(1 + \sum_{\ell=0}^{q-2} x_{i,\ell}\right) \right] \sum_{\boldsymbol{c}\in\mathcal{P}} \prod_{i=1}^{n} \prod_{\ell'=0}^{q-2} \left(\frac{1 + \sum_{\ell=0}^{q-2} \chi(\alpha^{\ell+\ell'}) x_{i,\ell}}{1 + \sum_{\ell=0}^{q-2} x_{i,\ell}} \right)^{\phi_{i,\ell'}(c_i)} \\ &= \frac{1}{q^k} \left[\prod_{i=1}^{n} \left(1 + \sum_{\ell=0}^{q-2} x_{i,\ell}\right) \right] W_{\mathcal{P}}(\boldsymbol{x}') \end{aligned} \tag{7.123}$$

where $\boldsymbol{x}' = [\boldsymbol{x}'_1, \boldsymbol{x}'_2, \ldots, \boldsymbol{x}'_n]$, $\boldsymbol{x}'_i = [x'_{i,0}, x'_{i,1}, \ldots, x'_{i,q-2}]$,

$$x'_{i,\ell} = \frac{1 + \sum_{\ell'=0}^{q-2} \chi(\alpha^{\ell+\ell'}) x_{i,\ell'}}{1 + \sum_{\ell'=0}^{q-2} x_{i,\ell'}}, \quad \ell = 0, 1, \ldots, q-2,\ i = 1, 2, \ldots, n. \tag{7.124}$$

7.4.3 Proof of Theorem 7.9

From Lemma 3.1, we obtain

$$\mathrm{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^{\frac{n}{n_{\mathsf{P}}}}, \boldsymbol{x}^{n\tilde{\boldsymbol{w}}(\tilde{\boldsymbol{\epsilon}})} y^{\gamma n}\right) \doteq \exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i, y) \right.\right.$$
$$\left.\left. -\gamma \ln y - \sum_{g\in\mathscr{E}^{\mathsf{P}}} \sum_{\ell=0}^{q-2} \tilde{w}_{g,\ell} \ln x_{g,\ell}\right]\right\}$$

where $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}/n$, $\tilde{\boldsymbol{w}}(\tilde{\boldsymbol{\epsilon}}) = \boldsymbol{w}(\boldsymbol{\epsilon})/n$, y and $x_{g,\ell}$ for $g \in \mathscr{E}^{\mathsf{P}}, \ell \in \{0, \ldots, q-2\}$ are the unique positive solutions of (7.71) and (7.72) if we replace $\tilde{w}^{\star}_{g,\ell}$ by $\tilde{w}_{g,\ell}$. We have

$$\prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{Q - \sum_{\ell=0}^{q-2} n\tilde{\epsilon}_{j,\ell}, n\tilde{\epsilon}_{j,0}, \ldots, n\tilde{\epsilon}_{j,q-2}}^{\mathsf{d}_{\mathsf{v_j}}-1}$$
$$\doteq \exp\left\{n \sum_{j=1}^{n_{\mathsf{P}}} \frac{\mathsf{d}_{\mathsf{v_j}} - 1}{n_{\mathsf{P}}} H\left(1 - n_{\mathsf{P}} \sum_{\ell=0}^{q-2} \tilde{\epsilon}_{j,\ell}, n_{\mathsf{P}}\tilde{\epsilon}_{j,0}, \ldots, n_{\mathsf{P}}\tilde{\epsilon}_{j,q-2}\right)\right\}$$

and also

$$\mathsf{E}^{\mathsf{P},c}_{\mathrm{TS}}(\theta n, \gamma n) \doteq \sum_{\tilde{\boldsymbol{\epsilon}}} \exp(nS(\tilde{\boldsymbol{\epsilon}}))$$

with

$$S(\tilde{\boldsymbol{\epsilon}}) = \frac{1}{n_{\mathsf{P}}} \sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i, y) - \sum_{g\in\mathscr{E}^{\mathsf{P}}} \sum_{\ell=0}^{q-2} \tilde{w}_{g,\ell} \ln x_{g,\ell} - \gamma \ln y$$
$$- \sum_{j=1}^{n_{\mathsf{P}}} \frac{\mathsf{d}_{\mathsf{v_j}} - 1}{n_{\mathsf{P}}} H\left(1 - n_{\mathsf{P}} \sum_{\ell=0}^{q-2} \tilde{\epsilon}_{j,\ell}, n_{\mathsf{P}}\tilde{\epsilon}_{j,0}, \ldots, n_{\mathsf{P}}\tilde{\epsilon}_{j,q-2}\right).$$

Thus, we have $G^{\mathsf{P},c}_{\mathrm{TS}}(\theta, \gamma) = \max S(\tilde{\boldsymbol{\epsilon}})$ where the maximization is subject to the constraint $\sum_{j,\ell} \tilde{\epsilon}_{j,\ell} = \theta$ obtained from (7.59). Using Lagrangian multipliers, we obtain $\tilde{\boldsymbol{\epsilon}}^{\star} = \operatorname{argmax} S(\tilde{\boldsymbol{\epsilon}})$ in (7.73).

7.4.4 Proof of Theorem 7.11

From Lemma 3.1, we have

$$\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^{\frac{n}{n_{\mathsf{P}}}}, (\boldsymbol{x}_i^{(1)})^{n\tilde{\boldsymbol{e}}}(\boldsymbol{x}_i^{(2)})^{n\tilde{\boldsymbol{w}}} y^{\gamma n}\right) \doteq$$
$$\exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) - \gamma \ln y - \sum_{g\in\mathscr{E}^{\mathsf{P}}}\sum_{\ell=0}^{q-2}\left(\tilde{e}_{g,\ell}\ln x_{g,\ell}^{(1)} + \tilde{w}_{g,\ell}\ln x_{g,\ell}^{(2)}\right)\right]\right\}$$

$$\operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^{\frac{n}{n_{\mathsf{P}}}}, (\boldsymbol{s}_j^{(1)})^{n\tilde{\boldsymbol{e}}}(\boldsymbol{s}_j^{(2)})^{n\tilde{\boldsymbol{w}}} t^{\theta n}\right) \doteq$$
$$\exp\left\{n\left[\frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}} \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) - \theta \ln t - \sum_{g\in\mathscr{E}^{\mathsf{P}}}\sum_{\ell=0}^{q-2}\left(\tilde{e}_{g,\ell}\ln s_{g,\ell}^{(1)} + \tilde{w}_{g,\ell}\ln s_{g,\ell}^{(2)}\right)\right]\right\}$$

where $\tilde{\boldsymbol{e}} = \boldsymbol{e}/n$, $\tilde{\boldsymbol{w}} = \boldsymbol{w}/n$, t and y and $x_{g,\ell}^{(1)}, x_{g,\ell}^{(2)}, s_{g,\ell}^{(1)}, s_{g,\ell}^{(2)}$ $\forall i \in \{1,\ldots,m_{\mathsf{P}}\}, \forall j \in \{1,\ldots,n_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}} \cap \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}, \ell \in \{0,\ldots,q-2\}$ are the unique positive solutions of (7.93)-(7.96) if we replace $\tilde{e}_{g,\ell}^{\star}$ and $\tilde{w}_{g,\ell}^{\star}$ by $\tilde{e}_{g,\ell}$ and $\tilde{w}_{g,\ell}$. We obtain

$$\prod_{g\in\mathscr{E}^{\mathsf{P}}}\binom{Q}{Q-\sum_{\ell=0}^{q-2} n(\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell}), n(\tilde{e}_{g,0}+\tilde{w}_{g,0}),\ldots,n(\tilde{e}_{g,q-2}+\tilde{w}_{g,q-2})}\prod_{\ell=0}^{q-2}\binom{n(\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell})}{n\tilde{e}_{g,\ell}} \doteq$$
$$\exp\left\{n\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left[\frac{1}{n_{\mathsf{P}}}H\left(1-n_{\mathsf{P}}\sum_{\ell=0}^{q-2}(\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell}), n_{\mathsf{P}}(\tilde{e}_{g,0}+\tilde{w}_{g,0}),\ldots,n_{\mathsf{P}}(\tilde{e}_{g,q-2}+\tilde{w}_{g,q-2})\right)\right.\right.$$
$$\left.\left.+\sum_{\ell=0}^{q-2}(\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell})H_{\mathsf{b}}\left(\frac{\tilde{e}_{g,\ell}}{\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell}}\right)\right]\right\}.$$

Thus, we have

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{P},c}(\theta n, \gamma n) \doteq \sum_{\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}}}\exp(nS(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}}))$$

where

$$S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}}) = \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}}\ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) - \gamma\ln y - \theta\ln t + \frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}}\ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$$
$$-\sum_{g\in\mathscr{E}^{\mathsf{P}}}\left[\frac{1}{n_{\mathsf{P}}}H\left(1-n_{\mathsf{P}}\sum_{\ell=0}^{q-2}(\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell}), n_{\mathsf{P}}(\tilde{e}_{g,0}+\tilde{w}_{g,0}),\ldots,n_{\mathsf{P}}(\tilde{e}_{g,q-2}+\tilde{w}_{g,q-2})\right)\right.$$
$$\left.+\sum_{\ell=0}^{q-2}\left(\tilde{e}_{g,\ell}\ln(x_{g,\ell}^{(1)}s_{g,\ell}^{(1)}) + \tilde{w}_{g,\ell}\ln(x_{g,\ell}^{(2)}s_{g,\ell}^{(2)}) + (\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell})H_{\mathsf{b}}\left(\frac{\tilde{e}_{g,\ell}}{\tilde{e}_{g,\ell}+\tilde{w}_{g,\ell}}\right)\right)\right].$$

We obtain $G_{\mathrm{AS}}^{\mathsf{P},c}(\theta,\gamma) = \max S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})$ and the vector pair $(\tilde{\boldsymbol{e}}^{\star},\tilde{\boldsymbol{w}}^{\star}) = \operatorname{argmax} S(\tilde{\boldsymbol{e}},\tilde{\boldsymbol{w}})$ in (7.98).

8

Trapping and Absorbing Set Enumerators for Binary GLDPC Code Ensembles

In this chapter, we propose new definitions of a TS, AS and FAS for GLDPC codes. The definitions stem from the definitions for LDPC codes and are based on a reference PBF decoder. We consider the PBF algorithm since it has low-complexity and is suitable for high-throughput applications. In particular, hard decision decoders for GLDPC-like codes, e.g., product codes and staircase codes, with bounded distance decoding at the CNs are currently considered as a baseline approach for very high speed fiber-optic communications; see [2, 42].

We start from a broad definition of TSs that includes sets that may be resolved by the PBF algorithm. The definition is then sharpened, yielding a simple definition of ASs and FASs. The latter sets cannot be corrected by the PBF algorithm. If all CNs are SPC codes, we recover the definitions of TSs and (fully) ASs of binary LDPC codes [35–37]. We use generating functions to derive the distribution of (elementary) TSs, ASs, and FASs for irregular GLDPC code ensembles. We present a numerical technique to evaluate the normalized logarithmic asymptotic distributions of these sets, which requires solving a system of equations, and we derive asymptotic approximations for small-sized TSs. Simulation results confirm the stability of FASs under the PBF algorithm and show the impact of TSs and (fully) ASs on the performance of a GLDPC code. The proposed enumeration technique is used to estimate the error floor performance for GLDPC codes.

8.1 Preliminaries

We assign the value 1 to each VN in a set $\mathcal{I}$ and 0 to the VNs outside $\mathcal{I}$. We denote by $\mathcal{N}(\mathcal{I})$ the set of the neighboring CNs of $\mathcal{I}$. Further, we denote by $\mathcal{U}(\mathcal{I})$ the set of unsatisfied CNs in $\mathcal{N}(\mathcal{I})$ and $\mathcal{S}(\mathcal{I})$ the set of satisfied CNs in $\mathcal{N}(\mathcal{I})$. A CN in $\mathcal{N}(\mathcal{I})$ is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0. In the following, we propose definitions of TSs and (fully) ASs for GLDPC codes that generalize the ones of LDPC codes [37, 48, 114].

Definition 8.1 (GLDPC trapping set)**.** An (a, b) TS $\mathcal{T}_{a,b}$ is a set $\mathcal{I}$ of a VNs such that $\mathcal{U}(\mathcal{I})$ contains b CNs.

It was observed in [36] that the error floor performance is dominated by small TSs where CNs are connected to only one or two VNs. These sets were referred to as ETS. Analogously, we propose a definition of ETSs for GLDPC codes.

Definition 8.2 (GLDPC elementary trapping set)**.** An (a, b) ETS $\mathcal{T}^{\mathrm{E}}_{a,b}$ is an (a, b) TS where $\forall \tau \in \{1, 2, \ldots, n_c\}$ each CN of type τ in $\mathcal{S}(\mathcal{I})$ is connected to $\mathsf{d}_{\min,\tau}$ VNs in $\mathcal{I}$ and each CN in $\mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$.

An example of a $(3, 1)$ ETS of a simple GLDPC code is given in Fig. 8.1. We have $\boldsymbol{\Lambda} = [0, 6/7, 1/7]$, $\mathbf{P} = [2/3, 1/3]$, the CNs $\mathsf{c}_1, \mathsf{c}_3$ of type 1, are $(4, 3)$ SPC codes and c_2 is a $(7, 4)$ Hamming code. We assign the value 1 to the set $\mathcal{I} = \{\mathsf{v}_3, \mathsf{v}_5, \mathsf{v}_7\}$. Note that $\boldsymbol{c}^{(1)} = [0, 0, 1, 0]$ is not a valid codeword of the SPC code. However, $\boldsymbol{c}^{(2)} = [0, 0, 1, 0, 1, 0, 1]$ and $\boldsymbol{c}^{(3)} = [0, 1, 0, 1]$ are valid codewords of the Hamming and SPC codes, respectively. Thus, $\mathcal{U}(\mathcal{I}) = \{\mathsf{c}_1\}$ and $\mathcal{S}(\mathcal{I}) = \{\mathsf{c}_2, \mathsf{c}_3\}$. Moreover, c_3, which is of type 1, is connected to $\mathsf{d}_{\min,1} = 2$ VNs in $\mathcal{I}$ and c_2, of type 2, is connected to $\mathsf{d}_{\min,2} = 3$ VNs in $\mathcal{I}$. The unsatisfied CN c_1 is connected to one VN in $\mathcal{I}$. Note that $|\mathcal{I}| = 3$ and $|\mathcal{U}(\mathcal{I})| = 1$.

Note that some TSs and ETSs do not necessarily cause a decoding failure: For example, suppose that each CN in $\mathcal{U}(\mathcal{I})$ is connected to a number of erroneous variable nodes that is within the component code error correction capability, and that each VN in the (E)TS is connected to more unsatisfied CNs than satisfied ones. The error pattern would be resolvable in this case. The connection between (E)TS and decoding failures is difficult to analyze in general, and it may require using numerical methods to obtain accurate performance predictions; see [36, 48, 132]. The merit of the definitions above is to extend in a natural way the ones that are commonly accepted for LDPC codes and to yield to the following stricter definitions of AS and FAS. The latter, in particular, provides a rigorous description of combinatorial structures that lead the PBF (Algorithm 3) to fail. Let $\mathsf{n}^{(\mathrm{f})}_{\mathsf{v}}$ be the number of flip messages that the VN v receives from its neighboring CNs.

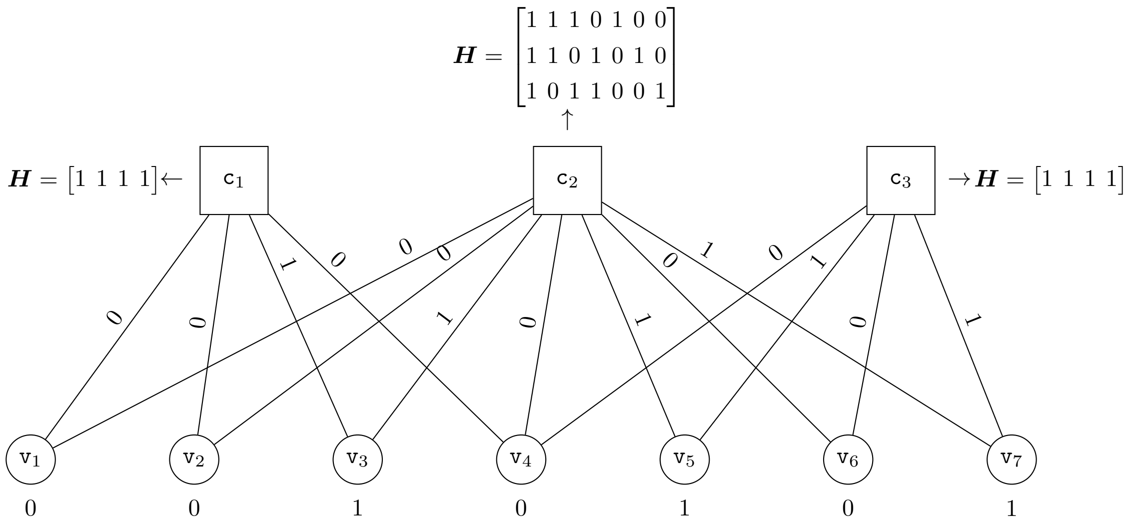

Figure 8.1: Example of an ETS for a GLDPC code.

Definition 8.3 (GLDPC absorbing set)**.** An (a, b) AS $\mathcal{A}_{a,b}$ is an (a, b) TS with the additional property that for each VN $\mathsf{v} \in \mathcal{I}$, we have $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$.

Definition 8.4 (GLDPC fully absorbing set)**.** An (a, b) FAS $\mathcal{F}_{a,b}$ is an (a, b) TS with the additional property that for each VN v in the Tanner graph $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$.

Remark 8.1. Consider a set $\mathcal{I}$ of corrupt VNs, where each VN v in the Tanner graph receives $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$ flip messages. Since for the PBF algorithm, each VN flips its estimate only if $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} > \mathsf{d}_{\mathsf{v}}/2$, no VN will flip its estimate. Thus, according to Definition 8.4, the VNs in a FAS cannot be corrected by the PBF algorithm.

Definition 8.5 (GLDPC elementary (fully) absorbing set)**.** An EAS $\mathcal{A}_{a,b}^{\mathrm{E}}$ (EFAS $\mathcal{F}_{a,b}^{\mathrm{E}}$) is an AS (FAS) where $\forall \tau \in \{1, 2 \ldots, n_c\}$ each CN of type τ in $\mathcal{S}(\mathcal{I})$ is connected to $\mathsf{d}_{\min,\tau}$ VNs in $\mathcal{I}$ and each CN in $\mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$.

The normalized logarithmic asymptotic distribution of (elementary) TSs or (fully) ASs for a GLDPC code ensemble for $a = \theta n$ and $b = \gamma n$ is defined by

$$G(\theta, \gamma) := \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathsf{E}(\theta n, \gamma n)\right) \tag{8.1}$$

where $\mathsf{E}(\theta n, \gamma n)$ is the average number of $(\theta n, \gamma n)$ (elementary) TSs or (fully) ASs in the Tanner graph of a random code from the ensemble.

Definition 8.6 (Relative minimum Δ-trapping/ absorbing/ fully absorbing set size)**.** For fixed ratio $\Delta = b/a = \gamma/\theta$, the second zero crossing of $G_{\mathrm{TS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta, \Delta\theta)$/ $G_{\mathrm{AS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta, \Delta\theta)$/

$G_{\mathrm{FAS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta, \Delta\theta)$ (the first one is zero), if it exists, is called the *relative minimum* Δ-*TS/ AS/ FAS size* that we denote by $\theta^{\star}_{\mathrm{TS}}$/ $\theta^{\star}_{\mathrm{AS}}$/ $\theta^{\star}_{\mathrm{FAS}}$.

8.2 Trapping and Absorbing Set Enumerators for Unstructured Ensembles

For GLDPC codes, we cannot identify a TS only from the underlying topological structure. For instance, unlike binary LDPC codes, we cannot determine if a CN is satisfied or not by only checking the number of edges connected to it like binary LDPC codes. Thus, the methods in [37, 48] relying on random matrix enumeration techniques cannot be trivially extended to GLDPC codes.

In this section, we derive the (elementary) TS and (fully) AS enumerators for unstructured GLDPC code ensembles and we present an analytical method for evaluating the normalized logarithmic asymptotic distributions of (elementary) TSs and (fully) ASs.

8.2.1 Trapping and Elementary Trapping Set Distributions

We derive next the TS and ETS enumerators for unstructured GLDPC code ensembles.

Lemma 8.1. The average number of (a, b) TSs in the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathbf{\Lambda},\mathbf{P}}(a, b) = \sum_{w} \frac{\mathrm{coeff}\left(g(x,y)^n, x^w y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{w}} \mathrm{coeff}\left(f(t,s)^n, t^a s^w\right) \tag{8.2}$$

where

$$f(t, s) = \prod_{j=1}^{\mathsf{d}_{\mathsf{v}}^{\max}} \left[1 + ts^j\right]^{\Lambda_j} \tag{8.3}$$

$$g(x, y) = \prod_{\tau=1}^{n_c} \left[W^{(\tau)}(x) + y\left[(1+x)^{n_\tau} - W^{(\tau)}(x)\right]\right]^{\xi \mathrm{P}_\tau} \tag{8.4}$$

and

$$W^{(\tau)}(x) = \sum_{\boldsymbol{c} \in \mathscr{C}_\tau} x^{\mathrm{w_H}(\boldsymbol{c})} = \sum_{h=1}^{n_\tau} W_h^{(\tau)} x^h \tag{8.5}$$

is the WEF of $\mathscr{C}_\tau$, $\mathsf{w}_\mathrm{H}(\boldsymbol{c})$ is the Hamming weight of $\boldsymbol{c}$ and $W_h^{(\tau)}$ is the number of codewords of Hamming weight h in $\mathscr{C}_\tau$.

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\boldsymbol{\Lambda},\mathbf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. We denote by $\alpha(a,w)$ the number of ways to choose a VNs such that exactly w edges emanate from them. Its generating function is $\sum_{a,w}\alpha(a,w)t^a s^w$. Consider a single VN of degree j. This generating function is $1+ts^j$ because we can either skip this VN or include it in the set $\mathcal{I}$. If we skip the VN, then we will get 0 nodes and 0 edges and this gives us the term 1 corresponding to t^0s^0. If we choose the VN, then we get 1 VN and j edges and this gives us t^1s^j. By considering all possible VN degrees, and since we have $\Lambda_j n$ VNs of degree j and for each VN we can decide to include it in $\mathcal{I}$ or not, the generating function is $f(t,s)^n$. Thus, we have

$$\alpha(a,w) = \mathrm{coeff}\left(f(t,s)^n, t^a s^w\right).$$

Let $\beta(b,w)$ be the number of ways to choose w edges such that exactly b CNs are unsatisfied. Its generating function is $\sum_{b,w}\beta(b,w)y^b x^w$. A CN is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0, and it is unsatisfied otherwise. Consider a CN of type τ. If it is satisfied, then its generating function is

$$g_{\mathsf{c}}(x,y) := y^0 \sum_{\boldsymbol{c}\in\mathscr{C}_\tau} x^{\mathsf{w}_\mathrm{H}(\boldsymbol{c})} = W^{(\tau)}(x)$$

and if it is unsatisfied, then its generating function is

$$g_{\bar{\mathsf{c}}}(x,y) := y^1\left[(1+x)^{n_\tau} - W^{(\tau)}(x)\right].$$

Considering all types of CNs and that there are $\xi \mathrm{P}_\tau n$ CNs of type τ, we obtain

$$\beta(b,w) = \mathrm{coeff}\left(g(x,y)^n, x^w y^b\right).$$

Note that for an LDPC code where all CNs are SPC codes, $g(x,y)$ in (8.4) simplifies to (6.110). Let Z_1 be a RV indicating the number of edges emanating from the set $\mathcal{I}$. Further, let Z_2 be a RV that is equal to 1 if there are exactly b unsatisfied CNs. Thus, we have

$$\mathsf{E}_{\mathrm{TS}}^{\boldsymbol{\Lambda},\mathbf{P}}(a,b) = \binom{n}{a}\Pr\{Z_2 = 1\} \tag{8.6}$$

and

$$\begin{aligned}\Pr\{Z_2 = 1\} &= \sum_w \Pr\{Z_1 = w\} \Pr\{Z_2 = 1 | Z_1 = w\} \\ &= \sum_w \frac{\operatorname{coeff}\left(f(t,s)^n, t^a s^w\right)}{\binom{n}{a}} \frac{\operatorname{coeff}\left(g(x,y)^n, x^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{w}}.\end{aligned} \tag{8.7}$$

■

Lemma 8.1 characterizes the exact average number of (a, b) TSs for block length n. In the asymptotic case, we analyze the normalized logarithmic asymptotic distribution of TSs for the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ for $a = \theta n$ and $b = \gamma n$. This distribution is a useful tool to analyze and design GLDPC codes with good TS properties and can be computed efficiently. In particular, the analysis allows to determine if the expected number of TSs with size θn, with θ small, goes exponentially fast to zero, hence providing insights on the TS properties of the ensemble. We next present a simple way to compute the distribution.

Theorem 8.1. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs is

$$G_{\mathrm{TS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta, \gamma) = -\bar{\mathsf{d}}_\mathsf{v} \ln(1 + xs) - \theta \ln(t) - \gamma \ln(y) + \ln\left(f(t,s)\right) + \ln\left(g(x,y)\right) \tag{8.8}$$

where t, s, x, y are the unique positive solutions of

$$t\frac{\partial \ln f(t,s)}{\partial t} = \theta \tag{8.9}$$

$$s\frac{\partial \ln f(t,s)}{\partial s} = x\frac{\partial \ln g(x,y)}{\partial x} = \tilde{w}^\star \tag{8.10}$$

$$y\frac{\partial \ln g(x,y)}{\partial y} = \gamma \tag{8.11}$$

where $f(t, s)$ and $g(x, y)$ are defined in (8.3) and (8.4), respectively, and

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{xs}{1 + xs}. \tag{8.12}$$

The proof of Theorem 8.1 is omitted since it is similar to the one of Theorem 6.7.

Note that to compute the normalized distribution of ETSs, we need to replace $g(x, y)$ given in (8.4) with

$$g(x,y) = \prod_{\tau=1}^{n_c} \left[1 + W_{\mathsf{d}_{\min,\tau}}^{(\tau)} x^{\mathsf{d}_{\min,\tau}} + y n_\tau x\right]^{\xi \mathrm{P}_\tau}. \tag{8.13}$$

We briefly explain the derivation of $g(x, y)$ in (8.13). For an ETS, a satisfied CN of type τ, which recognizes a valid local codeword, is connected zero or $\mathsf{d}_{\min,\tau}$ times to VNs in $\mathcal{I}$. The corresponding generating function is

$$g_{\mathsf{c}}(x,y) := y^0 \left[1 + W^{(\tau)}_{\mathsf{d}_{\min,\tau}} x^{\mathsf{d}_{\min,\tau}}\right].$$

Each CN in $\mathcal{U}(\mathcal{I})$ is connected to exactly one VN in $\mathcal{I}$. The corresponding generating function is $g_{\bar{\mathsf{c}}}(x,y) := y n_\tau x$. Considering all types of CNs, we obtain $g(x,y)$ in (8.13).

We introduce next a Lemma that will be useful in analyzing the behavior of $G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta,\Delta\theta)$ for small θ and fixed Δ.

Lemma 8.2. For a fixed $\Delta = \gamma/\theta$, the derivative in θ of $G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta,\Delta\theta)$ is

$$\frac{\mathrm{d}G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta,\Delta\theta)}{\mathrm{d}\theta} = -\ln(t) - \Delta\ln(y) \tag{8.14}$$

where for each θ, the values of t and y are given by the solution of the system of equations (8.9)-(8.11).

Proof. Note that the solutions of the system of equations in (8.9)-(8.11) are implicit functions of θ. From (8.12) and (8.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}G^{\mathbf{\Lambda},\mathrm{P}}_{\mathrm{TS}}(\theta,\Delta\theta)}{\mathrm{d}\theta} =& -\ln(t) - \Delta\ln(y) + \frac{\mathrm{d}t}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial t}}{f(t,s)} - \frac{\theta}{t}\right] + \frac{\mathrm{d}s}{\mathrm{d}\theta}\left[\frac{\frac{\partial f(t,s)}{\partial s}}{f(t,s)} - \frac{\tilde{w}^\star}{s}\right]\\ &+ \frac{\mathrm{d}x}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial x}}{g(x,y)} - \frac{\tilde{w}^\star}{x}\right] + \frac{\mathrm{d}y}{\mathrm{d}\theta}\left[\frac{\frac{\partial g(x,y)}{\partial y}}{g(x,y)} - \frac{\Delta\theta}{y}\right].\end{aligned} \tag{8.15}$$

The terms in the brackets are equal to zeros due to (8.9)-(8.11). ∎

Consider now small θ and $\gamma = \Delta\theta$. We obtain a closed form expression of $G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta,\Delta\theta)$, which we introduce in the following corollary proved in Appendix 8.4.1.

Corollary 8.1. Let $\mathsf{d}_{\mathsf{v}}^{\min}$ be the minimum VN degree and r the smallest minimum distance over all CNs. For fixed $\Delta = \gamma/\theta$ and $\theta \to 0$, we have

$$\begin{aligned}G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta,\Delta\theta) =& \theta\left[\frac{(r-1)(\mathsf{d}_{\mathsf{v}}^{\min}-\Delta)-r}{r}(\ln(\theta)-1) - \frac{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta}{r}\ln(\bar{\mathsf{d}}_{\mathsf{c}})\right.\\ &- \frac{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta}{r}\ln\left(\frac{\mathsf{d}_{\mathsf{v}}^{\min}-\Delta}{r\sum_\tau \mathrm{P}_\tau W^{(\tau)}_r}\right) - \frac{(r-1)(\mathsf{d}_{\mathsf{v}}^{\min}-\Delta)}{r}\ln(\bar{\mathsf{d}}_{\mathsf{v}})\\ &\left.+ \ln\left(\frac{(\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}}}{\Delta^\Delta}\right)\right] + o(\theta).\end{aligned} \tag{8.16}$$

Note that a positive $\theta^{\star}_{\mathrm{TS}}$ exists whenever the derivative of $G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta, \Delta\theta)$ is negative as $\theta \to 0$. Thus, by substituting (8.174) and (8.175) in (8.15) we find that a positive $\theta^{\star}_{\mathrm{TS}}$ exists whenever $\mathsf{d}_{\mathsf{v}}^{\min} > \frac{r}{r-1} + \Delta$ or $\mathsf{d}_{\mathsf{v}}^{\min} = \frac{r}{r-1} + \Delta$ and

$$\frac{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}} (\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}}{\bar{\mathsf{d}}_{\mathsf{v}} (\bar{\mathsf{d}}_{\mathsf{c}})^{\frac{1}{r-1}}} \left((r-1) \sum_{\tau} \mathrm{P}_{\tau} W_r^{(\tau)} \right)^{\frac{1}{r-1}} \left(\mathsf{d}_{\mathsf{v}}^{\min} - \frac{r}{r-1} \right)^{\frac{r}{r-1} - \mathsf{d}_{\mathsf{v}}^{\min}} < 1. \tag{8.17}$$

If the relative minimum Δ-TS size is small enough, then we can use Corollary 8.1 to approximate it. Numerical simulations show that the relative minimum Δ-TS size is small for the case of small VN degrees or high CN degrees, as observed in [48], especially if the CNs are SPC nodes or super CNs with small minimum distance. We now only need to determine θ such that $G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{TS}}(\theta, \Delta\theta) = 0$ with $0 < \theta \leq 1$. By neglecting the term $o(\theta)$, we have

$$\begin{aligned} \theta^{\star}_{\mathrm{TS}} \approx \exp(1) \left(\frac{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta}{r \sum_{\tau} \mathrm{P}_{\tau} W_r^{(\tau)}} \right)^{\frac{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta}{(r-1)(\mathsf{d}_{\mathsf{v}}^{\min} - \Delta) - r}} \times \\ \left(\frac{\Delta^{\Delta} \bar{\mathsf{d}}_{\mathsf{v}}^{-\frac{(r-1)(\mathsf{d}_{\mathsf{v}}^{\min} - \Delta)}{r}} \bar{\mathsf{d}}_{\mathsf{c}}^{-\frac{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta}{r}}}{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}} (\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}} \right)^{\frac{r}{(r-1)(\mathsf{d}_{\mathsf{v}}^{\min} - \Delta) - r}} . \end{aligned} \tag{8.18}$$

Note that if all CNs are associated with SPC codes, the expression reduces to (6.123). The approximation (8.18) is accurate when $\theta^{\star}_{\mathrm{TS}}$ is sufficiently small and does not require solving the system of equations in Theorem 8.1.

For the regular ensemble, Lemma 8.1 and Theorem 8.1 simplify as follows.

Lemma 8.3. The average number of (a, b) TSs in the Tanner graph of a code drawn randomly from the regular ensemble with variable node degree d_{v} and where all the CNs are associated with the linear code $\mathscr{C}$ of length d_{c} and WEF $W(x)$ is

$$\mathsf{E}^{\mathsf{d}_{\mathsf{v}},\mathsf{d}_{\mathsf{c}}}_{\mathrm{TS}}(a, b) = \binom{n}{a} \frac{\mathrm{coeff}\left(g(x,y)^n, x^{a\mathsf{d}_{\mathsf{v}}} y^b \right)}{\binom{n\mathsf{d}_{\mathsf{v}}}{a\mathsf{d}_{\mathsf{v}}}} \tag{8.19}$$

where

$$g(x, y) = \left[W(x) + y \left((1+x)^{\mathsf{d}_{\mathsf{c}}} - W(x) \right) \right]^{\xi} . \tag{8.20}$$

Proof. The Lemma can be proved from Lemma 8.1. Note that w in (8.2) is equal to $a\mathsf{d}_{\mathsf{v}}$. Moreover, the number of ways to choose a VNs such that exactly $a\mathsf{d}_{\mathsf{v}}$ edges emanate from them is equal to $\binom{n}{a}$. Further, the generating function $g(x, y)$ in (8.20) can be obtained from the one in (8.4) by taking $n_c = \tau = 1, \mathrm{P}_\tau = 1$. ■

Theorem 8.2. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs for the regular ensemble is

$$G_{\mathrm{TS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta, \gamma) = -(\mathsf{d_v} - 1)H_{\mathsf{b}}(\theta) - \gamma \ln(y) - \theta \mathsf{d_v} \ln(x) + \ln(g(x, y)) \tag{8.21}$$

where

$$y = \frac{\gamma}{\xi - \gamma} \frac{W(x)}{(1 + x)^{\mathsf{d_c}} - W(x)} \tag{8.22}$$

and x is the unique positive solution of

$$x \frac{\partial \ln g(x, y)}{\partial x} = \theta \mathsf{d_v} \tag{8.23}$$

where $g(x, y)$ is defined in (8.20). The proof is omitted since it is similar to the one of Theorem 6.8.

To compute the distribution of ETSs for regular ensembles, we need to replace in (8.19) the generating function $g(x, y)$ given in (8.20) with

$$g(x, y) = \left[1 + W_{\mathsf{d}_{\min}} x^{\mathsf{d}_{\min}} + \mathsf{d_c} x y\right]^{\xi} \tag{8.24}$$

where $\mathsf{d}_{\min}$ is the minimum distance of $\mathscr{C}$.

Due to the simplicity of $g(x, y)$ in this case, we can obtain a closed form expression of the normalized asymptotic distribution of $(\theta n, \gamma n)$ ETSs for the regular ensemble:

$$G_{\mathrm{ETS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta, \gamma) = -(\mathsf{d_v} - 1)H_{\mathsf{b}}(\theta) - \gamma \ln(y) - \theta \mathsf{d_v} \ln(x) + \ln(g(x, y)) \tag{8.25}$$

where

$$x = \left(\frac{\theta \mathsf{d_v} - \gamma}{W_{\mathsf{d}_{\min}} \left(\xi \mathsf{d}_{\min} - \gamma(\mathsf{d}_{\min} - 1) - \theta \mathsf{d_v}\right)}\right)^{\frac{1}{\mathsf{d}_{\min}}} \tag{8.26}$$

$$y = \frac{\gamma}{\xi - \gamma} \frac{1 + W_{\mathsf{d}_{\min}} x^{\mathsf{d}_{\min}}}{\mathsf{d_c} x}. \tag{8.27}$$

Proof. The proof is similar to the one of TSs. We need only replace in (8.21) the generating function $g(x, y)$ given in (8.20) with the one in (8.24), where x, y are the unique positive solutions of

$$x\frac{\partial \ln g(x,y)}{\partial x} = \theta \mathsf{d}_\mathsf{v} \tag{8.28}$$

$$y\frac{\partial \ln g(x,y)}{\partial y} = \gamma. \tag{8.29}$$

Substituting (8.24) in (8.28), (8.29) and with some manipulations, we obtain x, y in (8.26) and (8.27). ■

8.2.2 Absorbing Set Distribution

Similar to TSs, an AS cannot be identified only from its underlying topological structure since we cannot determine if a CN is satisfied or not by only checking the number of edges connected to it. Moreover, even if the constraints imposed by a CN are not satisfied, the node will not necessarily send flip messages in the bit flipping algorithm, as would happen for binary LDPC codes. The generating functions used for the AS enumerator capture the behavior of the bit flipping decoder with BDD at the CNs. The approach can also be used for other hard decision decoding algorithms by deriving a generating function that enumerates the outgoing flip messages.

In this section, we derive the AS enumerator for GLDPC codes and we develop an analytical method for evaluating the normalized logarithmic asymptotic distribution of ASs.

Lemma 8.4. The average number of (a, b) ASs in the Tanner graph of a code drawn randomly from the irregular GLDPC ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\mathbf{\Lambda},\mathbf{P}}(a,b) = \sum_{e,w} \frac{\mathrm{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}} \times \mathrm{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^w\right) \tag{8.30}$$

where

$$f(t,s_1,s_2) = \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[1 + t\sum_{j_1=0}^{\lfloor\frac{j-1}{2}\rfloor}\binom{j}{j_1} s_1^{j-j_1} s_2^{j_1}\right]^{\Lambda_j} \tag{8.31}$$

$$g(x_1,x_2,y) = \prod_{\tau=1}^{n_c} A_\tau(x_1,x_2,y)^{\xi \mathrm{P}_\tau} \tag{8.32}$$

and

$$A_\tau(x_1,x_2,y)=\begin{cases}\frac{1}{2}\left[(1+x_1)^{n_\tau}+(1-x_1)^{n_\tau}\right]+y\frac{1}{2}\left[(1+x_2)^{n_\tau}-(1-x_2)^{n_\tau}\right] & \mathscr{C}_\tau \text{ is SPC}\\ W^{(\tau)}(x_1)+y\left[\sum_{k=1}^{n_\tau}\sum_{\delta=1}^{t_\tau}\sum_{j=0}^{\min(\delta,n_\tau-k)}\binom{h}{h-j}\binom{n_\tau-h}{\delta-j}\times\right. & \text{else}\\ \left.(x_1^{h-j}x_2^{k-h+j}-x_1^k)W_h^{(\tau)}+(1+x_1)^{n_\tau}-W^{(\tau)}(x_1)\right] & \end{cases} \tag{8.33}$$

where $h=k-\delta+2j$.

Proof. Consider the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. We have 2 types of edges connected to the VNs in $\mathcal{I}$. Edges of the first type do not carry flip messages and the edges of the second type are the ones carrying flip messages. Compared to the TS analysis, we distinguish between the edges connected to $\mathcal{I}$ which carry flip messages and the edges not carrying flip messages. This lets us include the additional VN constraints imposed by the AS, namely that for each of the VNs in $\mathcal{I}$, we have $\mathsf{n}_\mathsf{v}^{(\mathrm{f})}<\mathsf{d}_\mathsf{v}/2$. Let $\alpha(a,e,w)$ be the number of ways to choose a VNs such that exactly e type 1 edges and w type 2 edges emanate from them and for each of these VNs $\mathsf{n}_\mathsf{v}^{(\mathrm{f})}<\mathsf{d}_\mathsf{v}/2$. The corresponding generating function is $\sum_{a,e,w}\alpha(a,e,w)t^a s_1^e s_2^w$. Consider a VN v of degree j. Let $j-j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to v. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it, then we obtain 0 nodes and 0 type 1 and type 2 edges. If we choose it, then we will have 1 node, $j-j_1$ type 1 edges and j_1 type 2 edges where $j_1\in\{0,1,\ldots,\lfloor(j-1)/2\rfloor\}$ (since for the VNs in $\mathcal{I}$ we have $\mathsf{n}_\mathsf{v}^{(\mathrm{f})}<\mathsf{d}_\mathsf{v}/2$). Considering all possible VN degrees, the generating function is $f(t,s_1,s_2)^n$. Thus, we have

$$\alpha(a,e,w)=\operatorname{coeff}\left(f(t,s_1,s_2)^n,t^a s_1^e s_2^w\right).$$

Let $\beta(b,e,w)$ be the number of ways to choose e type 1 edges and w type 2 edges such that there are exactly b unsatisfied CNs. The corresponding generating function is $\sum_{b,e,w}\beta(b,e,w)y^b x_1^e x_2^w$. We clarify briefly the derivation of $A_\tau(x_1,x_2,y)$ for super CNs. A CN of type τ is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0. In that case, the super CN doesn't send flip messages to its neighboring VNs in $\mathcal{I}$, i.e., the number of type 1 edges connected to that CN is equal to the weight of the codeword and the number of type 2 edges is 0. The generating function of a satisfied CN of type τ is then

$$g_{\mathsf{c}}(x_1, y) := \sum_{k=0}^{n_\tau} W_k^{(\tau)} x_1^k = W^{(\tau)}(x_1). \tag{8.34}$$

For an unsatisfied super CN of type τ, the received vector $\boldsymbol{z}$ is not a valid codeword. Given $\boldsymbol{z} \in \mathbb{F}_2^{n_\tau} \setminus \mathscr{C}_\tau$ of weight k, the decoded vector $\hat{\boldsymbol{c}}$ is a codeword if $\exists \boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) \leq t_\tau$. Consider codewords of weight h, the number of which is $W_h^{(\tau)}$. For a given $\boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) = h$, assume $\boldsymbol{z}$ has $h - j$ ones in $h - j$ out of the h entries where $\boldsymbol{c}$ is 1. Thus, $\boldsymbol{z}$ has $k - (h - j)$ ones in $k - (h - j)$ out of the $n_\tau - h$ positions where $\boldsymbol{c}$ is zero. The number of possibilities is $\binom{h}{h-j}\binom{n_\tau - h}{k-h+j}$. Note that the number of type 1 edges is the number of positions where both $\boldsymbol{c}$ and $\boldsymbol{z}$ are 1 which is $h - j$, the number of type 2 edges is the number of positions where $\boldsymbol{c}$ is zero and $\boldsymbol{z}$ is 1 which is $k - h + j$ and $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) = k - h + 2j =: \delta$. We need $1 \leq \delta \leq t_\tau$ so that the decoded vector is $\boldsymbol{c}$ ($\delta \geq 1$ since $\boldsymbol{z}$ is not a valid codeword). By summing over $\delta = 1, \ldots, t_\tau$, $j = 0, \ldots, \min(\delta, n_\tau - k)$ and over all possible weights that $\boldsymbol{z}$ can have we obtain the generating function

$$y \sum_{k=1}^{n_\tau} \sum_{\delta=1}^{t_\tau} \sum_{j=0}^{\min(\delta, n_\tau - k)} \binom{h}{h-j} \binom{n_\tau - h}{\delta - j} x_1^{h-j} x_2^{k-h+j} W_h^{(\tau)}. \tag{8.35}$$

Consider now the case of $\boldsymbol{z} \in \mathbb{F}_2^{n_\tau} \setminus \mathscr{C}_\tau$ of weight k such that $\nexists \boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) \leq t_\tau$, the decoded vector is $\boldsymbol{z}$ and the super CN will not send any flip messages to its neighboring VNs in $\mathcal{I}$, i.e., the number of type 1 edges connected that CN is equal to the weight of the codeword and the number of type 2 edges is 0. We obtain the generating function

$$y \sum_{k=1}^{n_\tau} \left[\binom{n_\tau}{k} - W_k^{(\tau)} - \sum_{\delta=1}^{t_\tau} \sum_{j=0}^{\min(\delta, n_\tau - k)} \binom{h}{h-j} \binom{n_\tau - h}{\delta - j} W_h^{(\tau)} \right] x_1^k. \tag{8.36}$$

From (8.34), (8.35) and (8.36) we obtain (8.33) for a super CN. Consider now an SPC node. An SPC node is satisfied if it is connected an even number of times (including zero) to $\mathcal{I}$. In this case, it doesn't send any flip messages. The generating function of a satisfied SPC node is

$$g_{\mathsf{c}}(x_1, y) := y^0 \sum_{\substack{0 \leq i \leq n_\tau \\ i \text{ is even}}} \binom{n_\tau}{i} x_1^i = \frac{1}{2} \left[(1 + x_1)^{n_\tau} + (1 - x_1)^{n_\tau} \right].$$

The SPC node is unsatisfied if it is connected an even number of times to $\mathcal{I}$. In this case, it sends flip messages to all its neighboring VNs. The generating function of an unsatisfied

SPC node is

$$g_{\bar{c}}(x_2, y) := y^1 \sum_{\substack{0 \le i \le n_\tau \\ i \text{ is odd}}} \binom{n_\tau}{i} x_2^i = \frac{y}{2}\left[(1+x_2)^{n_\tau} - (1-x_2)^{n_\tau}\right].$$

Note that for an LDPC code, where all CNs are SPC codes, $g(x_1, x_2, y)$ in (8.32) simplifies to (6.135). We have

$$\beta(b, e, w) = \text{coeff}\left(g(x_1, x_2, y)^n, x_1^e x_2^w y^b\right).$$

Let Z_1, Z_2 be two RVs indicating, respectively, the number of type 1 and type 2 edges emanating from $\mathcal{I}$, where each VN in $\mathcal{I}$ is connected to fewer type 2 edges than to type 1 edges. Further, let Z_3 be a RV that is equal to 1 if there are exactly b unsatisfied CNs and each of the other CNs is satisfied, and to 0 otherwise. Thus, we have

$$\mathsf{E}_{\text{AS}}^{\mathbf{\Lambda},\mathbf{P}}(a, b) = \binom{n}{a} \Pr\{Z_3 = 1\} \tag{8.37}$$

and

$$\begin{aligned} \Pr\{Z_3 = 1\} &= \sum_{e,w} \Pr\{Z_1 = e, Z_2 = w\} \times \Pr\{Z_3 = 1 | Z_1 = e, Z_2 = w\} \\ &= \sum_{e,w} \frac{\text{coeff}\left(f(t, s_1, s_2)^n, t^a s_1^e s_2^w\right)}{\binom{n}{a}} \times \frac{\text{coeff}\left(g(x_1, x_2, y)^n, x_1^e x_2^w y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}}. \end{aligned} \tag{8.38}$$

■

Next, we analyze the normalized logarithmic asymptotic distribution of ASs and present an efficient way to compute it.

The exact average number of (a, b) ASs derived in Lemma 8.4 is difficult to compute for large block length n. As $n \to \infty$, one can use the Hayman Formula in Lemma 3.1 to derive the normalized logarithmic asymptotic distribution of ASs for the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ for $a = \theta n$ and $b = \gamma n$ as shown in Theorem 8.3.

Theorem 8.3. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs is

$$\begin{aligned} G_{\text{AS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta, \gamma) &= \ln(f(t, s_1, s_2)) - \theta \ln(t) - \gamma \ln(y) + \ln(g(x_1, x_2, y)) \\ &\quad - \bar{\mathsf{d}}_\mathsf{v} \ln(1 + x_1 s_1 + x_2 s_2) \end{aligned} \tag{8.39}$$

where t, s_1, s_2, x_1, x_2, y are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{8.40}$$
$$s_1\frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2,y)}{\partial x_1} = \tilde{e}^\star \tag{8.41}$$
$$s_2\frac{\partial \ln f(t,s_1,s_2)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2,y)}{\partial x_2} = \tilde{w}^\star \tag{8.42}$$
$$y\frac{\partial \ln g(x_1,x_2,y)}{\partial y} = \gamma \tag{8.43}$$

where $f(t, s_1, s_1)$ and $g(x_1, x_2, y)$ are defined in (8.31) and (8.32), respectively and

$$\tilde{e}^\star = \bar{\mathsf{d}}_\mathsf{v}\frac{x_1 s_1}{1 + x_1 s_1 + x_2 s_2} \tag{8.44}$$
$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v}\frac{x_2 s_2}{1 + x_1 s_1 + x_2 s_2}. \tag{8.45}$$

Similar to the TS case, the expressions in Lemma 8.4 and Theorem 8.3 can be simplified for regular ensembles. We consider the case where all the CNs are super CNs and are associated with the linear code $\mathscr{C}$ of length $\mathsf{d_c}$, error correcting capability t and WEF $W(x)$. The case of SPC CNs is presented in Lemma 6.5 and Theorem 6.10.

Lemma 8.5. The average number of (a, b) ASs in the Tanner graph of a code drawn uniformly at random from the regular ensemble is

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) = \sum_e \binom{n}{a} \frac{\mathrm{coeff}\left(g(x_1,x_2,y)^n, x_1^e x_2^{a\mathsf{d_v}-e} y^b\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{e}} \mathrm{coeff}\left(f(s)^a, s^{a\mathsf{d_v}-e}\right) \tag{8.46}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor\frac{\mathsf{d_v}-1}{2}\rfloor} \binom{\mathsf{d_v}}{j_1} s^{j_1} \tag{8.47}$$
$$\begin{aligned} g(x_1,x_2,y) = \Bigg[& W(x_1) + y\left(\sum_{k=1}^{\mathsf{d_c}}\sum_{\delta=1}^{t}\sum_{j=0}^{\min(\delta,\mathsf{d_c}-k)}\binom{h}{h-j}\binom{\mathsf{d_c}-h}{\delta-j}W_h \times \right. \\ & \left. (x_1^{h-j}x_2^{k-h+j} - x_1^k) + (1+x_1)^{\mathsf{d_c}} - W(x_1)\right)\Bigg]^\xi \end{aligned} \tag{8.48}$$

where $h = k - 2\delta + 2j$.

The Lemma can be proved using Lemma 8.4.

We show now that to compute the normalized logarithmic asymptotic distribution of

ASs for regular codes, one must solve 3 equations instead of 6 for the irregular case.

Theorem 8.4. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs for the regular ensemble is

$$\begin{aligned} G_{\mathrm{AS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = &-\theta \mathsf{d_v} \ln(x_1 + x_2 s) - \gamma \ln(y) \\ &+ \ln\left(g(x_1,x_2,y)\right) + \theta \ln\left(f(s)\right) - (\mathsf{d_v}-1) H_{\mathsf{b}}(\theta) \end{aligned} \tag{8.49}$$

where

$$y = \frac{\gamma}{\xi-\gamma} \frac{W(x_1)}{\sum_{k=1}^{\mathsf{d_c}} \sum_{\delta=1}^{t} \sum_{j=0}^{\min(\delta,\mathsf{d_c}-k)} \binom{h}{h-j}\binom{\mathsf{d_c}-h}{\delta-j}(x_1^{h-j}x_2^{k-h+j} - x_1^k)W_h + (1+x_1)^{\mathsf{d_c}} - W(x_1)} \tag{8.50}$$

and s_1, x_1, x_2 are the unique positive solutions of

$$\theta s \frac{\mathrm{d} \ln f(s)}{\mathrm{d}s} = x_2 \frac{\partial \ln g(x_1,x_2,y)}{\partial x_2} = (\theta \mathsf{d_v} - \tilde{e}^\star) \tag{8.51}$$

$$x_1 \frac{\partial \ln g(x_1,x_2,y)}{\partial x_1} = \tilde{e}^\star g(x_1,x_2,y) \tag{8.52}$$

where $f(s)$ and $g(x_1,x_2,y)$ are defined in (8.47) and (8.48), respectively, and

$$\tilde{e}^\star = \theta \mathsf{d_v} \frac{x_1}{x_1 + x_2 s}. \tag{8.53}$$

The proof follows the same steps of the proof of Theorem 6.8.

8.2.3 Elementary Absorbing Set Distribution

The following Lemma gives the EAS enumerator for GLDPC code ensembles.

Lemma 8.6. The average number of (a,b) EASs in the Tanner graph of a code drawn uniformly at random from the LDPC ensemble $\mathscr{C}_n^{\Lambda,\mathrm{P}}$ is

$$\mathsf{E}_{\mathrm{EAS}}^{\mathbf{\Lambda},\mathbf{P}}(a,b) = \sum_e \frac{\mathrm{coeff}\left(g(x_1,x_2)^n, x_1^e x_2^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+b}\binom{e+b}{b}} \mathrm{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^b\right) \tag{8.54}$$

where $f(t,s_1,s_2)$ is defined in (8.31) and

$$g(x_1,x_2) = \prod_{\tau=1}^{n_c} A_\tau(x_1,x_2)^{\xi \mathrm{P}_\tau} \tag{8.55}$$

and

$$A_\tau(x_1, x_2) = \begin{cases} 1 + \binom{n_\tau}{2} x_1^2 + n_\tau x_2 & \text{if } \mathscr{C}_\tau \text{ is an SPC code} \\ 1 + W^{(\tau)}_{\mathsf{d}_{\min,\tau}} x_1^{\mathsf{d}_{\min,\tau}} + n_\tau x_2 & \text{otherwise.} \end{cases} \tag{8.56}$$

Proof. Consider the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. As for ASs, we have 2 types of edges connected to the VNs in $\mathcal{I}$. Edges of the first type don't carry flip messages and the edges of the second type are the ones carrying flip messages. For EASs an unsatisfied super CN of type τ is connected to only one VN in $\mathcal{I}$, considering the received vector $\boldsymbol{z}$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{z}) = 1$, since $t_\tau \geq 1$, the output of the BDD is the all-zeros vector. Thus, the super CN sends a flip message to the VN in $\mathcal{I}$. The edge connecting the VN from $\mathcal{I}$ and the super CN is then of type 2. The same holds for SPC CNs. Since we have b unsatisfied CNs and each of them is connected to exactly one VNs in $\mathcal{I}$, we have b type 2 edges. Let $\alpha(a, e, b)$ be the number of ways to choose a VNs such that exactly e type 1 edges and b type 2 edges emanate from them and for each of these VNs $\mathsf{n}_\mathsf{v}^{(\mathsf{f})} < \mathsf{d}_\mathsf{v}/2$. Similar to the AS case, we obtain

$$\alpha(a, e, b) = \operatorname{coeff}\left(f(t, s_1, s_2)^n, t^a s_1^e s_2^b\right).$$

Let $\beta(b, e)$ be the number of ways to choose e type 1 edges such that there are exactly b unsatisfied CNs connected each to exactly one VN from $\mathcal{I}$ and where all satisfied CNs of type τ are connected to $\mathsf{d}_{\min,\tau}$ VNs from $\mathcal{I}$. The corresponding generating function is $\sum_{e,b} \beta(e, b) x_1^e x_2^b$. We clarify briefly the derivation of $A_\tau(x_1, x_2)$. Each satisfied CN is connected to $\mathsf{d}_{\min,\tau}$ VN from $\mathcal{I}$. The generating function of a satisfied CN of type τ is then

$$g_\mathsf{c}(x_1) := 1 + W^{(\tau)}_{\mathsf{d}_{\min,\tau}} x_1^{\mathsf{d}_{\min,\tau}}.$$

For a degree d_c unsatisfied super CN of type τ, the received vector $\boldsymbol{z}$ is not a valid codeword and has weight 1. Since $t_\tau \geq 1$, the super CN corrects the error and sends a flip message to the VN from $\mathcal{I}$. The same holds for SPC CNs. We obtain the generating function $g_{\bar{\mathsf{c}}}(x_2) := n_\tau x_2$ for an unsatisfied type τ CN. Note that for an LDPC code, where all CNs are SPC codes, $g(x_1, x_2)$ in (8.55) simplifies to (6.154). We have

$$\beta(e, b) = \operatorname{coeff}\left(g(x_1, x_2)^n, x_1^e x_2^b\right).$$

We randomly choose a set $\mathcal{I}$ of a VNs with a uniform distribution over all $\binom{n}{a}$ possibilities. Let Z_1 and Z_2 be two RVs indicating, respectively, the number of type 1 and type 2 edges emanating from $\mathcal{I}$, where each VN in $\mathcal{I}$ is connected to strictly fewer type 2 edges than type 1 edges. Further, let Z_3 be a RV that is equal to 1 if there are exactly b unsatisfied CNs, and is equal to 0 otherwise. We have

$$\mathsf{E}_{\mathrm{EAS}}^{\mathbf{\Lambda},\mathbf{P}}(a,b) = \binom{n}{a} \Pr\{Z_3 = 1\} \tag{8.57}$$

and

$$\begin{aligned} \Pr\{Z_3 = 1\} &= \sum_e \Pr\{Z_1 = e, Z_2 = b\} \Pr\{Z_3 = 1 | Z_1 = e, Z_2 = b\} \\ &= \sum_e \frac{\mathrm{coeff}\left(f(t,s_1,s_2)^n, t^a s_1^e s_2^b\right)}{\binom{n}{a}} \frac{\mathrm{coeff}\left(g(x_1,x_2)^n, x_1^e x_2^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+b}\binom{e+b}{b}}. \end{aligned} \tag{8.58}$$

■

Next, we analyze the normalized logarithmic asymptotic distribution of EAS and present an efficient way to compute it.

Theorem 8.5. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the ensemble is

$$\begin{aligned} G_{\mathrm{EAS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta,\gamma) = &- \bar{\mathsf{d}}_{\mathsf{v}} \ln(\bar{\mathsf{d}}_{\mathsf{v}}) + (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\ln(\bar{\mathsf{d}}_{\mathsf{v}} - \gamma) - \theta \ln(t) - (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\ln(1 + x_1 s_1) \\ &+ \ln\left(g(x_1,x_2)\right) + \ln\left(f(t,s_1,s_2)\right) - \gamma \ln(x_2 s_2) + \gamma \ln(\gamma) \end{aligned} \tag{8.59}$$

where t, s_1, s_2, x_1, x_2 are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2)}{\partial t} = \theta \tag{8.60}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2)}{\partial x_1} = \tilde{e}^\star \tag{8.61}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2)}{\partial x_2} = \gamma \tag{8.62}$$

and where $f(t,s_1,s_2)$ and $g(x_1,x_2)$ are defined in (8.31) and (8.55), respectively, and

$$\tilde{e}^\star = (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\frac{x_1 s_1}{1 + x_1 s_1}. \tag{8.63}$$

We now derive the EAS finite-length and asymptotic distributions for the regular ensembles. Suppose all the CNs are super CNs and are associated with the linear code $\mathscr{C}$ of length $\mathsf{d_c}$, minimum distance $\mathsf{d}_{\min}$ and WEF $W(x)$. The case of SPC CNs is presented in Lemma 6.7 and Theorem 6.12.

Lemma 8.7. The average number of (a,b) EASs in the Tanner graph of a code drawn uniformly at random from the regular ensemble is

$$\mathsf{E}_{\mathrm{EAS}}^{\mathsf{d_v},\mathsf{d_c}}(a,b) = \binom{m}{b}\binom{n}{a}\frac{\mathsf{d_c}^{b}\,\mathrm{coeff}\left(g(x)^{m-b}, x^{a\mathsf{d_v}-b}\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{b}}\,\mathrm{coeff}\left(f(s)^{a}, s^{b}\right) \tag{8.64}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor\frac{\mathsf{d_v}-1}{2}\rfloor}\binom{\mathsf{d_v}}{j_1}s^{j_1} \tag{8.65}$$

$$g(x) = 1 + W_{\mathsf{d}_{\min}}x^{\mathsf{d}_{\min}}. \tag{8.66}$$

The Lemma can be easily derived from Lemma 8.6.

We show now that to compute the normalized logarithmic asymptotic distribution of EASs for regular codes, one must solve one equation compared to 5 for the irregular case.

Theorem 8.6. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EASs for the regular GLDPC ensemble is

$$\begin{aligned} G_{\mathrm{EAS}}^{\mathsf{d_v},\mathsf{d_c}}(\theta,\gamma) = & -(\mathsf{d_v}-1)H_{\mathsf{b}}(\theta) - \mathsf{d_v}\theta H_{\mathsf{b}}\left(\frac{\gamma}{\theta\mathsf{d_v}}\right) + \gamma\ln(\mathsf{d_c}) - \gamma\ln(s) \\ & + (\xi-\gamma)\ln\left(g(x)\right) + \theta\ln\left(f(s)\right) - \left(\theta\mathsf{d_v}-\gamma\right)\ln(x) + \xi H_{\mathsf{b}}\left(\frac{\gamma}{\xi}\right) \end{aligned} \tag{8.67}$$

where

$$x = \left(\frac{\theta\mathsf{d_v}-\gamma}{W_{\mathsf{d}_{\min}}(\xi\mathsf{d}_{\min}-\theta\mathsf{d_v}-\gamma(\mathsf{d}_{\min}-1))}\right)^{\frac{1}{\mathsf{d}_{\min}}} \tag{8.68}$$

and s is the unique positive solution of

$$\theta s\frac{\mathrm{d}\ln f(s)}{\mathrm{d}s} = \gamma \tag{8.69}$$

where $f(s)$ is defined in (8.65). The proof of the Theorem is similar to the one of Theorem 6.8.

8.2.4 Fully Absorbing Set Distribution

In this section, we derive the FAS enumerator for GLDPC codes and we present an analytical method for evaluating the normalized logarithmic asymptotic distribution of FASs.

Lemma 8.8. The average number of (a,b) FASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$ is

$$\mathsf{E}_{\mathrm{FAS}}^{\mathbf{\Lambda},\mathbf{P}}(a,b) = \sum_{e,w,l} \frac{\mathrm{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^w x_3^l y^b\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+w}\binom{e+w}{e}\binom{n\bar{\mathsf{d}}_\mathsf{v}-e-w}{l}} \times \mathrm{coeff}(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^w s_3^l) \tag{8.70}$$

where

$$f(t,s_1,s_2,s_3) = \prod_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \left[\sum_{j_1=0}^{\lfloor\frac{j-1}{2}\rfloor} \binom{j}{j_1} s_3^{j_1} + t \sum_{j_1=0}^{\lfloor\frac{j-1}{2}\rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1}\right]^{\Lambda_j} \tag{8.71}$$

$$g(x_1,x_2,x_3,y) = \prod_{\tau=1}^{n_c} A_\tau(x_1,x_2,x_3,y)^{\xi \mathrm{P}_\tau} \tag{8.72}$$

and

$$A_\tau(x_1,x_2,x_3,y) = \begin{cases} \frac{(1+x_1)^{n_\tau}+(1-x_1)^{n_\tau}}{2} + y\frac{(x_3+x_2)^{n_\tau}-(x_3-x_2)^{n_\tau}}{2} & \mathscr{C}_\tau \text{ is SPC} \\ W^{(\tau)}(x_1) + y\left[\sum_{k=1}^{n_\tau}\sum_{\delta=1}^{t_\tau}\sum_{j=0}^{\min(\delta,n_\tau-k)} \binom{h}{h-j}\binom{n_\tau-h}{\delta-j}\times\right. & \text{otherwise} \\ \left.(x_1^{h-j}x_2^{k-h+j}x_3^j - x_1^k)W_h^{(\tau)} + (1+x_1)^{n_\tau} - W^{(\tau)}(x_1)\right] & \end{cases} \tag{8.73}$$

where $h = k - \delta + 2j$.

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. We have 3 types of edges. Edges of the first type are connected to the VNs in $\mathcal{I}$ and don't carry flip messages, the edges of the second type are the ones connected to the VNs in $\mathcal{I}$ and carrying flip messages and the edges of the third type are connected to the VNs in $\mathcal{V}\setminus\mathcal{I}$ and carry flip messages. Compared to ASs, we differentiate between the edges connected to the VNs in $\mathcal{V}\setminus\mathcal{I}$ that carry flip messages and the ones not carrying flip messages. This is needed to include the additional constraint on the VNs in $\mathcal{V}\setminus\mathcal{I}$ imposed by the FAS. Let $\alpha(a,e,w,l)$ be the number of ways to choose a VNs such that exactly e type 1 edges, w type 2 edges emanate from them and l type 3 edges emanate from the other VNs and for each VNs $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$. The corresponding generating function is $\sum_{a,e,w,l} \alpha(a,e,w,l) t^a s_1^e s_2^w s_3^l$. Consider a VN v of degree j. Let $j - j_1$ and j_1 be, respectively, the number of type 1 and 2 edges connected to v. Again, we can either include this VN in $\mathcal{I}$ or not. If we skip it, then we obtain 0 nodes and 0 type 1 and type 2 edges and j_1 type 3 edges where $j_1 \in \{0, 1, \ldots, \lfloor (j-1)/2 \rfloor\}$ (since for the VNs in $\mathcal{V} \setminus \mathcal{I}$ we have $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$). If we choose it, then we will have 1 node, $j - j_1$ type 1 edges and j_1 type 2 edges where $j_1 \in \{0, 1, \ldots, \lfloor (j-1)/2 \rfloor\}$ (since for the VNs in $\mathcal{I}$ we have $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$). Considering all possible VN degrees, the generating function is $f(t, s_1, s_2, s_3)^n$. Thus, we have

$$\alpha(a,e,w,l) = \mathrm{coeff}\left(f(t, s_1, s_2, s_3)^n, t^a s_1^e s_2^w s_3^l \right).$$

Let $\beta(b,e,w,l)$ be the number of ways to choose e type 1 edges, w type 2 edges and l type 3 edges such that there are exactly b unsatisfied CNs. The corresponding generating function is $\sum_{b,e,w,l} \beta(b,e,w,l) y^b x_1^e x_2^w x_3^l$. We clarify briefly the derivation of $A_\tau(x_1, x_2, x_3, y)$ for super CNs. A CN of type τ is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0. In that case, the CN doesn't send flip messages to its neighboring VNs in $\mathcal{I}$, i.e., the number of type 1 edges connected that CN is equal to the weight of the codeword and the number of type 2 and 3 edges is 0. The generating function of a satisfied CN of type τ is then

$$g_{\mathsf{c}}(x_1, y) := \sum_{k=0}^{n_\tau} W_k^{(\tau)} x_1^k = W^{(\tau)}(x_1). \tag{8.74}$$

For an unsatisfied super CN of type τ, the received vector $\boldsymbol{z}$ is not a valid codeword. Given $\boldsymbol{z} \in \mathbb{F}_2^{n_\tau} \setminus \mathscr{C}_\tau$ of weight k, the decoded vector $\hat{\boldsymbol{c}}$ is a codeword if $\exists \boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) \le t_\tau$. Consider codewords of weight h, the number of which is $W_h^{(\tau)}$. For a given $\boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) = h$, assume $\boldsymbol{z}$ has $h - j$ ones in $h - j$ out of the h entries where $\boldsymbol{c}$ is 1. Thus, $\boldsymbol{z}$ has $k - (h - j)$ ones in $k - (h - j)$ out of the $n_\tau - h$ positions where $\boldsymbol{c}$ is zero. The number of possibilities is $\binom{h}{h-j}\binom{n_\tau - h}{k-h+j}$. Note that the number of type 1 edges is the number of positions where both $\boldsymbol{c}$ and $\boldsymbol{z}$ are 1 which is $h - j$, the number of type 2 edges is the number of positions where $\boldsymbol{c}$ is zero and $\boldsymbol{z}$ is 1 which is $k - h + j$ and the number of type 3 edges is the number of positions where $\boldsymbol{c}$ is 1 and $\boldsymbol{z}$ is 0 which is j and $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) = k - h + 2j =: \delta$. We require $1 \le \delta \le t_\tau$ so that the decoded vector is $\boldsymbol{c}$ ($\delta \ge 1$ since $\boldsymbol{z}$ is not a valid codeword). By summing over $\delta = 1, \ldots, t_\tau$, $j = 0, \ldots, \min(\delta, n_\tau - k)$ and over all possible weights that

$\boldsymbol{z}$ can have we obtain the generating function

$$y \sum_{k=1}^{n_\tau} \sum_{\delta=1}^{t_\tau} \sum_{j=0}^{\min(\delta, n_\tau - k)} \binom{h}{h-j} \binom{n_\tau - h}{\delta - j} x_1^{h-j} x_2^{k-h+j} x_3^j W_h^{(\tau)}. \tag{8.75}$$

Consider now the case of $\boldsymbol{z} \in \mathbb{F}_2^{n_\tau} \setminus \mathscr{C}_\tau$ of weight k such that $\nexists \boldsymbol{c} \in \mathscr{C}_\tau$ with $\mathsf{d}_{\mathrm{H}}(\boldsymbol{z}, \boldsymbol{c}) \leq t_\tau$. The decoded vector in that case is $\boldsymbol{z}$ and the super CN will not send any flip messages to its neighboring VNs in $\mathcal{I}$, i.e., the number of type 1 edges connected that CN is equal to the weight of the codeword. Moreover, the number of type 2 and 3 edges is 0 and we obtain the generating function

$$y \sum_{k=1}^{n_\tau} \left[\binom{n_\tau}{k} - W_k^{(\tau)} - \sum_{\delta=1}^{t_\tau} \sum_{j=0}^{\min(\delta, n_\tau - k)} \binom{h}{h-j} \binom{n_\tau - h}{\delta - j} W_h^{(\tau)} \right] x_1^k. \tag{8.76}$$

Consider now an SPC node. An SPC node is satisfied if it is connected an even number of times (including zero) to $\mathcal{I}$. In this case, it doesn't send any flip messages. The generating function of a satisfied SPC node is

$$g_{\mathsf{c}}(x_1, y) := y^0 \sum_{\substack{0 \leq i \leq n_\tau \\ i \text{ is even}}} \binom{n_\tau}{i} x_1^i = \frac{1}{2} \left[(1 + x_1)^{n_\tau} + (1 - x_1)^{n_\tau} \right].$$

The SPC node is unsatisfied if it is connected an even number of times to $\mathcal{I}$. In this case, it sends flip messages to all its neighboring VNs. The generating function of an unsatisfied SPC node is

$$g_{\bar{\mathsf{c}}}(x_2, x_3, y) := y^1 \sum_{\substack{0 \leq i \leq n_\tau \\ i \text{ is odd}}} \binom{n_\tau}{i} x_2^i x_3^{n_\tau - i} = \frac{y}{2} \left[(x_3 + x_2)^{n_\tau} - (x_3 - x_2)^{n_\tau} \right].$$

Note that for an LDPC code, where all CNs are SPC codes, $g(x_1, x_2, x_3, y)$ in (8.72) simplifies to (6.170). We have

$$\beta(b, e, w, l) = \operatorname{coeff}\left(g(x_1, x_2, x_3, y)^n, x_1^e x_2^w x_3^l y^b \right).$$

Let Z_1, Z_2 and Z_3 be three RVs indicating, respectively, the number of type 1, type 2, and type 3 edges, where each VN in $\mathcal{I}$ is connected to strictly less type 2 edges than to type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. Further, let Z_4 be a RV that is equal to 1 if there are exactly b unsatisfied CNs and each of the other CNs is satisfied, and to 0 otherwise. We have

$$\mathsf{E}_{\mathrm{FAS}}^{\mathbf{\Lambda},\mathbf{P}}(a,b) = \binom{n}{a} \Pr\{Z_4 = 1\} \tag{8.77}$$

and

$$\begin{aligned}
\Pr\{Z_4 = 1\} &= \sum_{e,w,l} \Pr\{Z_1 = e, Z_2 = w, Z_3 = l\} \Pr\{Z_4 = 1 | Z_1 = e, Z_2 = w, Z_3 = l\} \\
&= \sum_{e,w,l} \frac{\operatorname{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^w s_3^l\right)}{\binom{n}{a}} \times \\
&\quad \frac{\operatorname{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^w x_3^l y^b\right)}{\binom{n\bar{\mathsf{d}}_{\mathsf{v}}}{e+w}\binom{e+w}{e}\binom{n\bar{\mathsf{d}}_{\mathsf{v}}-e-w}{l}}.
\end{aligned} \tag{8.78}$$

■

Similar to TSs and ASs, we study next the normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs and present an efficient way to compute it.

Theorem 8.7. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs is

$$\begin{aligned}
G_{\mathrm{FAS}}^{\mathbf{\Lambda},\mathbf{P}}(\theta,\gamma) =& -\bar{\mathsf{d}}_{\mathsf{v}} \ln(1 + x_1 s_1 + x_2 s_2 + x_3 s_3) - \theta \ln(t) + \ln\left(g(x_1,x_2,x_3,y)\right) \\
& - \gamma \ln(y) + \ln\left(f(t,s_1,s_2,s_3)\right)
\end{aligned} \tag{8.79}$$

where $t, s_1, s_2, s_3, x_1, x_2, x_3, y$ are the unique positive solutions of

$$t \frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial t} = \theta \tag{8.80}$$

$$s_1 \frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_1} = x_1 \frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_1} = \tilde{e}^{\star} \tag{8.81}$$

$$s_2 \frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_2} = x_2 \frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_2} = \tilde{w}^{\star} \tag{8.82}$$

$$s_3 \frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_3} = x_3 \frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial x_3} = \tilde{l}^{\star} \tag{8.83}$$

$$y \frac{\partial \ln g(x_1,x_2,x_3,y)}{\partial y} = \gamma \tag{8.84}$$

where $f(t,s_1,s_1,s_3)$ and $g(x_1,x_2,x_3)$ are defined in (8.71) and (8.72), respectively, and

$$\tilde{e}^{\star} = \bar{\mathsf{d}}_{\mathsf{v}} \frac{x_1 s_1}{1 + x_1 s_1 + x_2 s_2 + x_3 s_3} \tag{8.85}$$

$$\tilde{w}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{x_2 s_2}{1 + x_1 s_1 + x_2 s_2 + x_3 s_3} \tag{8.86}$$

$$\tilde{l}^\star = \bar{\mathsf{d}}_\mathsf{v} \frac{x_3 s_3}{1 + x_1 s_1 + x_2 s_2 + x_3 s_3}. \tag{8.87}$$

The proof of the Theorem is similar to the one of Theorem 6.7.

Similar to the TS and AS cases, the expressions in Lemma 8.8 and Theorem 8.7 can be simplified for regular ensembles. We consider the case where all the CNs are super CNs and are associated with the linear code $\mathscr{C}$ of length $\mathsf{d_c}$, error correcting capability t and WEF $W(x)$. The case of SPC CNs is presented in Lemma 6.9 and Theorem 6.14.

Lemma 8.9. The average number of (a, b) FASs in the Tanner graph of a code drawn uniformly at random from the regular ensemble is

$$\begin{aligned} \mathsf{E}^{\mathsf{d_v},\mathsf{d_c}}_{\mathrm{FAS}}(a,b) = \sum_{e,l} & \frac{\operatorname{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^{a\mathsf{d_v}-e} x_3^l y^b\right)}{\binom{n\mathsf{d_v}}{a\mathsf{d_v}}\binom{a\mathsf{d_v}}{e}\binom{(n-a)\mathsf{d_v}}{l}} \\ & \binom{n}{a} \operatorname{coeff}\left(f(s_1)^a, s_1^{a\mathsf{d_v}-e}\right) \operatorname{coeff}\left(f(s_2)^{n-a}, s_2^l\right) \end{aligned} \tag{8.88}$$

where

$$f(s) = \sum_{j_1=0}^{\lfloor \frac{\mathsf{d_v}-1}{2} \rfloor} \binom{\mathsf{d_v}}{j_1} s^{j_1} \tag{8.89}$$

and

$$\begin{aligned} g(x_1,x_2,x_3,y) = \Bigg[W(x_1) + y \left(\sum_{k=1}^{\mathsf{d_c}} \sum_{\delta=1}^{t} \sum_{j=0}^{\min(\delta,\mathsf{d_c}-k)} \binom{h}{h-j} \binom{\mathsf{d_c}-h}{\delta-j} W_h \times \right. \\ \left. (x_1^{h-j} x_2^{k-h+j} x_3^j - x_1^k) + (1+x_1)^{\mathsf{d_c}} - W(x_1) \right) \Bigg]^\xi \end{aligned} \tag{8.90}$$

where $h = k - 2\delta + 2j$. The Lemma can be proved using Lemma 8.8.

We show now that to compute the normalized logarithmic asymptotic distribution of FASs for regular codes, one must solve 5 equations instead of 8 for the irregular case.

Theorem 8.8. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs for the regular ensemble is

$$\begin{aligned} G^{\mathsf{d_v},\mathsf{d_c}}_{\mathrm{FAS}}(\theta,\gamma) = & -\theta \mathsf{d_v} \ln(x_1 + x_2 s_1) - \gamma \ln(y) - (\mathsf{d_v} - 1) H_\mathsf{b}(\theta) + \ln\left(g(x_1,x_2,x_3,y)\right) \\ & + \theta \ln\left(f(s_1)\right) + (1-\theta)\ln\left(f(s_2)\right) - (1-\theta)\mathsf{d_v} \ln(1 + x_3 s_2) \end{aligned} \tag{8.91}$$

where

$$y = \frac{\gamma}{\xi - \gamma} \frac{W(x_1)}{\sum_{k=1}^{\mathsf{d_c}} \sum_{\delta=1}^{t} \sum_{j=0}^{\min(\delta, \mathsf{d_c}-k)} \binom{h}{h-j}\binom{\mathsf{d_c}-h}{\delta-j} (x_1^{h-j} x_2^{k-h+j} x_3^j - x_1^k) W_h + (1+x_1)^{\mathsf{d_c}} - W(x_1)}$$

and s_1, s_2, x_1, x_2, x_3 are the unique positive solutions of

$$\theta s_1 \frac{\mathrm{d} \ln f(s_1)}{\mathrm{d} s_1} = x_2 \frac{\partial \ln g(x_1, x_2, x_3, y)}{\partial x_2} = (\theta \mathsf{d_v} - \tilde{e}^\star) \tag{8.92}$$

$$(1-\theta) s_2 \frac{\mathrm{d} \ln f(s_2)}{\mathrm{d} s_2} = x_3 \frac{\partial \ln g(x_1, x_2, x_3, y)}{\partial x_3} = \tilde{l}^\star \tag{8.93}$$

$$x_1 \frac{\partial \ln g(x_1, x_2, x_3, y)}{\partial x_1} = \tilde{e}^\star \tag{8.94}$$

where $f(s_1)$ and $g(x_1, x_2, x_3, y)$ are defined in (8.89) and (8.90), respectively, and

$$\tilde{e}^\star = \theta \mathsf{d_v} \frac{x_1}{x_1 + x_2 s_1} \tag{8.95}$$

$$\tilde{l}^\star = (1-\theta) \mathsf{d_v} \frac{x_3 s_2}{1 + x_3 s_2}. \tag{8.96}$$

The proof follows the same steps of the proof of Theorem 6.8.

8.2.5 Elementary Fully Absorbing Set Distribution

In this section, we derive the finite-length and asymptotic EFAS enumerators for GLDPC codes.

Lemma 8.10. The average number of (a, b) EFASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda}, \mathbf{P}}$ is

$$\mathsf{E}_{\mathrm{EFAS}}^{\mathbf{\Lambda}, \mathbf{P}}(a, b) = \sum_{e,l} \frac{\mathrm{coeff}\left(g(x_1, x_2, x_3)^n, x_1^e x_2^b x_3^l\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+b}\binom{e+b}{e}\binom{n\bar{\mathsf{d}}_\mathsf{v}-e-b}{l}} \mathrm{coeff}(f(t, s_1, s_2, s_3)^n, t^a s_1^e s_2^b s_3^l) \tag{8.97}$$

where

$$f(t, s_1, s_2, s_3) = \prod_{j=1}^{\mathsf{d_v^{max}}} \left[\sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_3^{j_1} + t \sum_{j_1=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j}{j_1} s_1^{j-j_1} s_2^{j_1} \right]^{\Lambda_j} \tag{8.98}$$

$$g(x_1, x_2, x_3) = \prod_{\tau=1}^{n_c} A_\tau(x_1, x_2, x_3)^{\xi \mathrm{P}_\tau} \tag{8.99}$$

and

$$A_\tau(x_1, x_2, x_3) = \begin{cases} 1 + \binom{n_\tau}{2} x_1^2 + n_\tau x_2 x_3^{n_\tau - 1} & \mathscr{C}_\tau \text{ is SPC} \\ 1 + W^{(\tau)}_{\mathsf{d}_{\min,\tau}} x_1^{\mathsf{d}_{\min,\tau}} + n_\tau x_2 & \text{otherwise.} \end{cases} \tag{8.100}$$

Note that, using BDD, for a received sequence $\boldsymbol{z}$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{z}) = 1$ and $t_\tau \geq 1$, we have $\hat{\boldsymbol{c}} = \mathbf{0}$. That means, super check nodes never send flip messages to VNs outside $\mathcal{I}$. Thus, if all CNs are super check nodes, the condition that for each VN outside $\mathcal{I}$, $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$ is always fulfilled. In that case, EASs and EFASs are equivalent.

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble $\mathscr{C}_n^{\mathbf{\Lambda},\mathbf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. We have 3 types of edges. Edges of the first type are connected to the VNs in $\mathcal{I}$ and don't carry flip messages, the edges of the second type are the ones connected to the VNs in $\mathcal{I}$ and carrying flip messages and the edges of the third type are connected to the VNs in $\mathcal{V} \setminus \mathcal{I}$ and carry flip messages. Compared to ASs, we differentiate between the edges connected to the VNs in $\mathcal{V} \setminus \mathcal{I}$ that carry flip messages and the ones not carrying flip messages. This is needed to include the additional constraint on the VNs in $\mathcal{V} \setminus \mathcal{I}$ imposed by the FAS. As for EASs, there are b type 2 edges. Let $\alpha(a, e, b, l)$ be the number of ways to choose a VNs such that exactly e type 1 edges, b type 2 edges emanate from them and l type 3 edges emanate from the other VNs and for each VNs $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$. Similar to the FAS case, we obtain

$$\alpha(a, e, b, l) = \operatorname{coeff}\left(f(t, s_1, s_2, s_3)^n, t^a s_1^e s_2^b s_3^l\right).$$

Let $\beta(e, b, l)$ be the number of ways to choose e type 1 edges, b type 2 edges and l type 3 edges such that there are exactly b unsatisfied CNs connected to exactly one VN from $\mathcal{I}$ and where all satisfied CNs of type τ are connected to $\mathsf{d}_{\min,\tau}$ VNs from $\mathcal{I}$. The corresponding generating function is $\sum_{e,b,l} \beta(e, b, l) y^b x_1^e x_2^b x_3^l$. We clarify briefly the derivation of $A_\tau(x_1, x_2, x_3)$. Each satisfied CN is connected to $\mathsf{d}_{\min,\tau}$ VN from $\mathcal{I}$, its generating function is then

$$g_{\mathsf{c}}(x_1) := 1 + W^{(\tau)}_{\mathsf{d}_{\min,\tau}} x_1^{\mathsf{d}_{\min,\tau}}.$$

For an unsatisfied super CN of type τ, the received vector $\boldsymbol{z}$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{z}) = 1$. Since $t_\tau \geq 1$, using BDD, $\hat{\boldsymbol{c}} = \mathbf{0}$. Thus, for EFASs, each super CN sends exactly one flip message to its neighboring VN from $\mathcal{I}$. Its generating function is $g_{\bar{\mathsf{c}}}(x_2) := n_\tau x_2$. Consider now an unsatisfied SPC node connected to exactly one VN from $\mathcal{I}$. It sends flip messages to all its neighboring VNs. Its generating function is $g_{\bar{\mathsf{c}}}(x_2, x_3) := n_\tau x_2 x_3^{n_\tau - 1}$. We have

$$\beta(e,b,l) = \mathrm{coeff}\left(g(x_1,x_2,x_3)^n, x_1^e x_2^b x_3^l\right).$$

Let Z_1, Z_2 and Z_3 be three RVs indicating, respectively, the number of type 1, type 2 and type 3 edges, where each VN in $\mathcal{I}$ is connected to strictly less type 2 edges than to type 1 edges and each VN in $\mathcal{V} \setminus \mathcal{I}$ of degree j is connected to strictly less than $j/2$ type 3 edges. Further, let Z_4 be a RV that is equal to 1 if there are exactly b unsatisfied CNs and each of the other CNs is satisfied, and to 0 otherwise. We have

$$\mathsf{E}^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{EFAS}}(a,b) = \binom{n}{a} \Pr\{Z_4 = 1\} \tag{8.101}$$

and

$$\begin{aligned}\Pr\{Z_4 = 1\} =& \sum_{e,l} \Pr\{Z_1 = e, Z_2 = b, Z_3 = l\} \Pr\{Z_4 = 1 | Z_1 = e, Z_2 = b, Z_3 = l\} \\ =& \sum_{e,b,l} \frac{\mathrm{coeff}\left(f(t,s_1,s_2,s_3)^n, t^a s_1^e s_2^b s_3^l\right)}{\binom{n}{a}} \frac{\mathrm{coeff}\left(g(x_1,x_2,x_3,y)^n, x_1^e x_2^b x_3^l\right)}{\binom{n\bar{\mathsf{d}}_\mathsf{v}}{e+b}\binom{e+b}{e}\binom{n\bar{\mathsf{d}}_\mathsf{v}-e-b}{l}}.\end{aligned} \tag{8.102}$$

■

We study next the normalized asymptotic distribution of $(\theta n, \gamma n)$ EFASs and present an efficient way to compute it.

Theorem 8.9. The normalized asymptotic distribution of $(\theta n, \gamma n)$ EFASs is

$$\begin{aligned}G^{\mathbf{\Lambda},\mathbf{P}}_{\mathrm{EFAS}}(\theta,\gamma) =& (\bar{\mathsf{d}}_\mathsf{v} - \gamma)\ln(\bar{\mathsf{d}}_\mathsf{v} - \gamma) - \bar{\mathsf{d}}_\mathsf{v}\ln(\bar{\mathsf{d}}_\mathsf{v}) - (\bar{\mathsf{d}}_\mathsf{v} - \gamma)\ln(1 + x_1 s_1 + x_3 s_3) \\ &+ \gamma\ln(\gamma) - \theta\ln(t) - \gamma\ln(x_2 s_2) + \ln\left(g(x_1,x_2,x_3)\right) \\ &+ \ln\left(f(t,s_1,s_2,s_3)\right)\end{aligned} \tag{8.103}$$

where $t, s_1, s_2, s_3, x_1, x_2, x_3$ are the unique positive solutions of

$$t\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial t} = \theta \tag{8.104}$$

$$s_1\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_1} = x_1\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_1} = \tilde{e}^\star \tag{8.105}$$

$$s_2\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_2} = x_2\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_2} = \gamma \tag{8.106}$$

$$s_3\frac{\partial \ln f(t,s_1,s_2,s_3)}{\partial s_3} = x_3\frac{\partial \ln g(x_1,x_2,x_3)}{\partial x_3} = \tilde{l}^\star \tag{8.107}$$

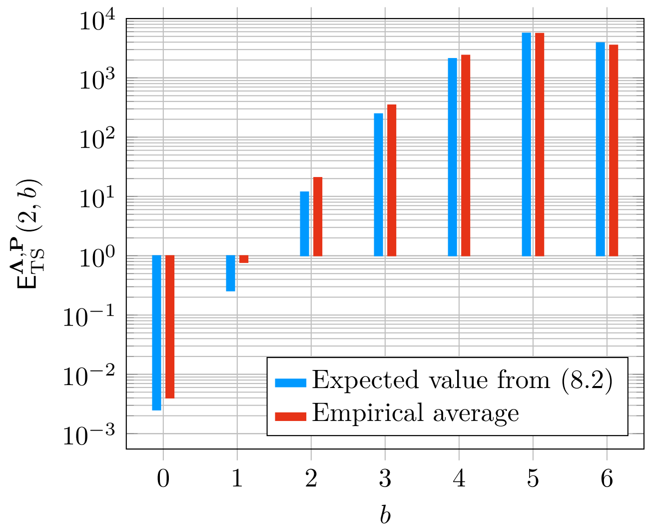

Figure 8.2: Expected and average distributions of $(2, b)$ TSs of the ensemble in Example 8.1 for $n = 155$.

Table 8.1: The error profiles for the PBF decoder.

SNR [dB]	(2,3)	(2,4)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(4,4)	(4,6)	(4,7)	(4,8)	(5,7)	(6,7)
8.6	17	797	9	57	2	32	68	0	2	7	7	1	1
9	16	820	12	73	4	18	48	1	0	7	1	0	0
9.4	17	872	5	63	0	15	27	1	0	0	0	0	0

where $f(t, s_1, s_2, s_3)$ and $g(x_1, x_2, x_3)$ are defined in (8.98) and (8.99), respectively, and

$$\tilde{e}^{\star} = (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\frac{x_1 s_1}{1 + x_1 s_1 + x_3 s_3} \tag{8.108}$$

$$\tilde{l}^{\star} = (\bar{\mathsf{d}}_{\mathsf{v}} - \gamma)\frac{x_3 s_3}{1 + x_1 s_1 + x_3 s_3}. \tag{8.109}$$

Consider now a regular GLDPC code ensemble, where all the CNs are associated with the same linear code $\mathscr{C}$. If $\mathscr{C}$ is an SPC, we obtain Lemma 6.7 and Theorem 6.12. If the CNs are super CNs, then an EFAS is equivalent to an EAS, and we obtain 8.7 and Theorem 8.6.

Example 8.1. Consider regular GLDPC code ensembles with VN degree 3, length $n \in \{155, 930\}$ and the $(31, 26)$ Hamming code as component code. We generate from the length 155 ensemble 1000 random codes without using any girth optimization techniques. We provide in Fig. 8.2 the average multiplicity of the $(2, b)$ TSs within these codes and compare them with the expected enumerator given in (8.88) to check the presented theoretical results. Remarkably, the average multiplicities of TSs are close to the ensemble averages.

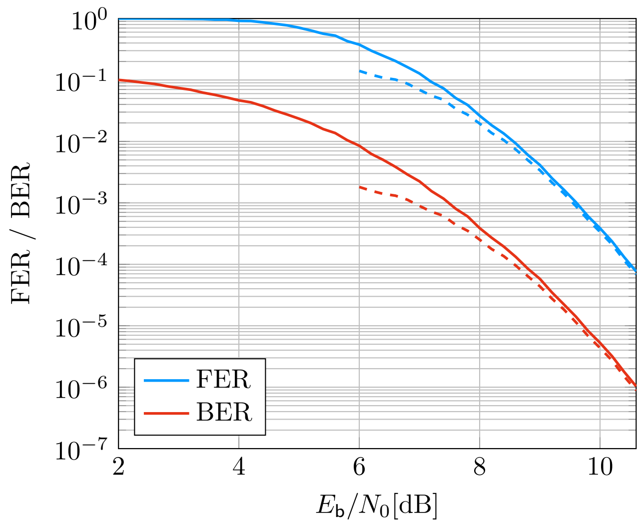

Figure 8.3: FER and BER versus E_b/N_0[dB] for the PBF decoder. The dashed lines represent the contribution of the dominant $(2, 4)$ FAS to the FER and BER.

In the following, we present experimental results to check the effect of the TSs and (fully) ASs on the performance of GLDPC codes. We consider a code picked randomly from the length 155 ensemble. We transmit the all-zero codeword over a biAWGN channel. We perform a hard decision on the received sequence, thereby the biAWGN channel is converted to a BSC, and apply the PBF algorithm. We set the maximum number of iterations $\ell_{\max} = 20$. Since the all-zero codeword is transmitted, we say that a VN is corrupt after decoding if its corresponding final estimate is one. For each channel realization leading to a decoding failure, we check if the subgraph containing the corrupted VNs and their neighboring CNs is a TS or (fully) AS. In this case, we determine its size. We collected 1000 error frames at SNRs $E_\mathsf{b}/N_0 \in \{8.6, 9, 9.4\}$. Table 8.1 shows the obtained error profiles, which are converging errors, i.e., the VN estimates remain the same in the last few iterations. All the errors provided in Table 8.1 are FASs and thus TSs and ASs. This confirms the stability of FAS under the parallel bit flipping decoder. In this example, all decoding failures are caused by FASs.

We can see from Table 8.1 that the $(2, 4)$ FAS is the dominant FAS. Simulation results of the considered GLDPC code are shown in Fig. 8.3 in terms of FER and BER versus E_b/N_0. The dashed lines represent the contribution of the dominant $(2, 4)$ FAS to the FER and BER. Note that at high SNR, the FER and BER are dominated by the $(2, 4)$ FAS.

Next, we use the derived theoretical results to estimate the error floor performance of the length 930 ensemble. We picked 50 random codes from this ensemble and simulated their performance under the parallel bit flipping algorithm over a BSC obtained by hard-

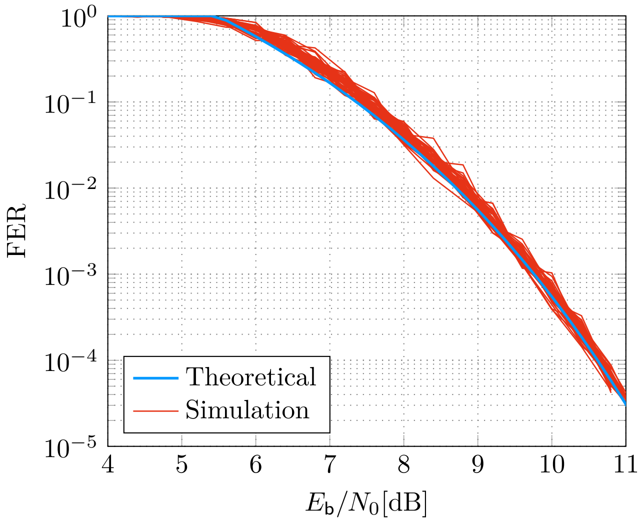

Figure 8.4: FER versus E_b/N_0[dB] for length $n = 930$ codes drawn from the GLDPC code ensemble in Example 8.1 under parallel bit flipping decoding with BDD at the CNs and the predicted average performance (error floor).

quantizing the output of a binary-input additive white Gaussian noise channel. The performance of the codes is provided in Fig. 8.4 in terms of FER versus E_b/N_0[dB]. In Fig. 8.4 an analytic estimate of the average error probability at large signal-to-noise ratios is given. The estimate is based on Eq. 1 in [43], where we considered the dominant $(2, 4)$ FAS. As multiplicity of $(2, 4)$ FASs, we employed the ensemble enumerator from (8.88). The codes have an error floor performance that is close to the estimated average error probability derived from the proposed analysis. Similar results were observed for other block lengths.

The normalized logarithmic asymptotic distribution of TSs and (fully) ASs of this ensemble are depicted in Fig. 8.5 for $\Delta \in \{0.005, 0.05, 0.1, 0.2, 0.3, 0.4\}$. We observe that the gap between the normalized logarithmic asymptotic distributions of TSs and ASs vanishes for small θ.

Example 8.2. Consider the rate 2/5 ensemble with $\Lambda_3 = 1$, $\mathrm{P}_1 = 0.8, \mathrm{P}_2 = 0.2$, $\mathscr{C}_1$ is the $(7, 6)$ SPC code and $\mathscr{C}_2$ is the $(7, 4)$ Hamming code. The normalized logarithmic asymptotic distributions of (elementary) TSs and (fully) ASs, of this ensemble for $\Delta \in \{0.005, 0.05, 0.1, 0.3, 0.5\}$ are depicted in Fig. 8.6 and Fig. 8.7, respectively. Fig. 8.8 compares the exact value of the normalized logarithmic asymptotic distribution of TSs, obtained from Theorem 8.1, and the approximation obtained from Corollary 8.1 for this ensemble. Observe that the approximations are accurate for small values of θ. A comparison of the exact value of the relative minimum Δ-TS size and its corresponding approximation obtained from (8.18) for this ensemble and for different values of Δ is shown in Table 8.2.

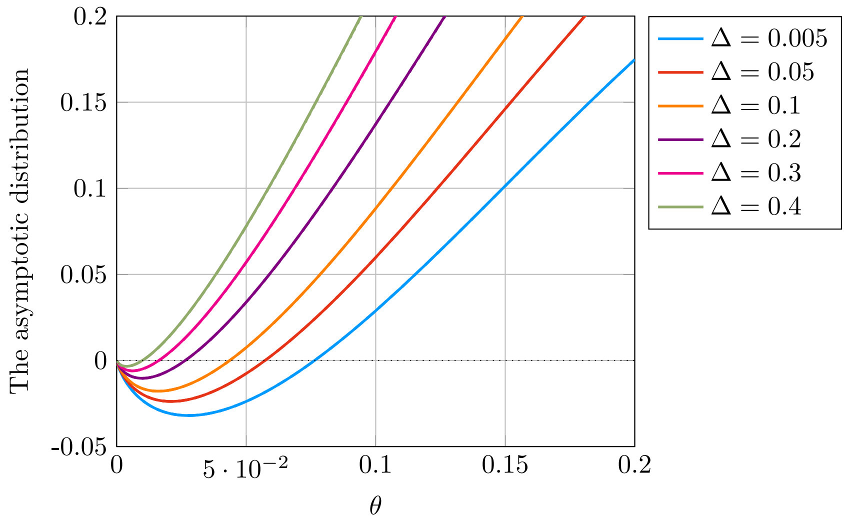

Figure 8.5: The normalized logarithmic asymptotic distributions of TSs (solid lines), ASs (dashed lines), and fully ASs (dotted lines) of the ensemble in Example 8.1.

The relative minimum Δ-AS sizes are also provided. It can be observed that the values obtained from (8.18), which we derived by analyzing the asymptotic distribution of TSs for the small θ case, are very good approximations of the relative minimum Δ-TS sizes. We can see that the values of $\theta^{\star}_{\mathrm{TS}}$ are very close to the ones of $\theta^{\star}_{\mathrm{AS}}$ especially for small θ. In fact, as we can see in Fig. 8.6 and Fig. 8.7, the gap between the normalized logarithmic asymptotic distributions of TSs and ASs vanishes for small values of θ.

We observed that this code ensemble has better TS and AS properties than the regular $(3, 7)$ LDPC ensemble. For instance, for $\Delta = 0.005$ we have $\theta^{\star}_{\mathrm{TS}} = 0.0241249$ while for the regular ensemble, we have $\theta^{\star}_{\mathrm{TS}} = 0.0118767$. Both code ensembles have the same VN degree distribution. For the regular $(3, 7)$ LDPC code ensemble, all CNs are SPC codes and for this ensemble some of the SPC CNs are replaced by the $(7, 4)$ Hamming code. This matches the approach in [133] to construct GLDPC codes by converting some of the SPC CNs involving a TS into super CN corresponding to a stronger linear block code. By converting some of the SPC CNs to super checks, the PBF decoder could correct the errors in the TS and thus eliminate it as shown in [133]. In fact, this method improves the TS properties of the code especially if the linear block code has good distance properties and thus a high error correcting capability under BDD.

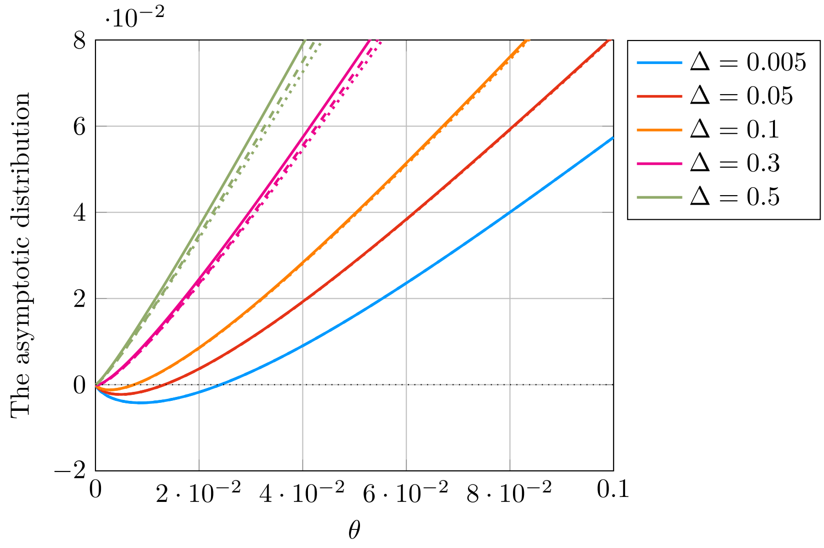

Figure 8.6: Normalized logarithmic asymptotic distributions of TSs (solid lines), ASs (dashed lines) and fully ASs (dotted lines) of the ensemble in Example 8.2.

8.3 Trapping and Absorbing Set Enumerators for Protograph-Based Ensembles

We derived in [66] the finite-length and asymptotic enumerators of (elementary) TSs and (fully) ASs for protograph-based GLDPC code ensembles.

8.3.1 Trapping and Elementary Trapping Set Distributions

In this section, we derive the finite-length and asymptotic TS enumerators for protograph-based GLDPC codes. Define the VN weight vector $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \dots, \epsilon_{n_\mathsf{P}}]$, where ϵ_j is the number of VNs of type $\mathtt{v}_j$ in $\mathcal{T}_{a,b}$. Clearly, we have $0 \le \epsilon_j \le Q \; \forall j \in \{1, 2, \dots, n_\mathsf{P}\}$ and

$$\sum_{j=1}^{n_\mathsf{P}} \epsilon_j = a. \tag{8.110}$$

Define the edge weight vector $\boldsymbol{w}(\boldsymbol{\epsilon}) = (w_g)_{g \in \mathscr{E}}$ where w_g is the the number of edges of type g in $\mathcal{T}_{a,b}$. For a given $\boldsymbol{\epsilon}$, we have $w_g = \epsilon_j$ if $g \in \mathscr{E}^\mathsf{P}_{\mathtt{v}_j}$. Define next the vector weight enumerating function (VWEF) of a binary linear code which we will use to derive the TS and (fully) AS enumerators.

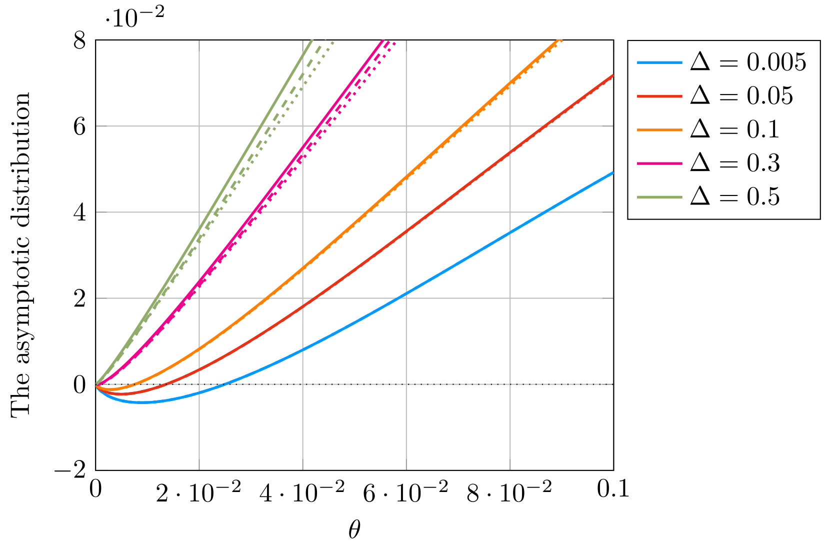

Figure 8.7: Normalized logarithmic asymptotic distributions of elementary TSs (solid lines), elementary ASs (dashed lines), and elementary fully ASs (dotted lines) of the ensemble in Example 8.2.

Definition 8.7. Let $\mathscr{C}$ be an (n, k) linear code. The VWEF of $\mathscr{C}$ is defined by

$$W_{\mathscr{C}}(\boldsymbol{x}) = \sum_{\boldsymbol{c}\in\mathscr{C}} \boldsymbol{x}^{\boldsymbol{c}} \tag{8.111}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_{n_i}]$, $x_i, i \in \{1, 2, \ldots, n\}$ are dummy variables.

Lemma 8.11. The average number of (a, b) TSs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\mathsf{E}_{\mathrm{TS}}^{\mathsf{P}}(a, b) = \sum_{\boldsymbol{\epsilon}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right)}{\prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}^{\mathsf{d}_{\mathsf{v}_j}-1}} \tag{8.112}$$

where

$$A_i(\boldsymbol{x}_i, y) = W_{\mathscr{C}_i}(\boldsymbol{x}_i) + y \left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 + x_g) - W_{\mathscr{C}_i}(\boldsymbol{x}_i)\right] \tag{8.113}$$

and $\mathscr{C}_i$ is the linear block code corresponding to c_i and $\boldsymbol{x} = (x_g)_{g\in\mathscr{E}}$, $\boldsymbol{x}_i = (x_g)_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}$, y and $x_g, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$ are dummy variables.

Proof. Consider the Tanner graph of a code drawn uniformly at random from the ensemble

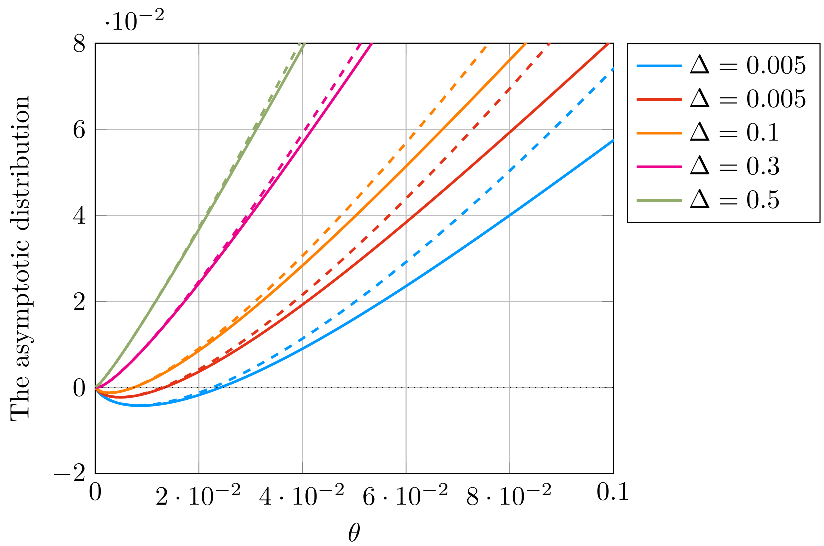

Figure 8.8: Normalized logarithmic asymptotic distribution of TSs of the ensemble in Example 8.2. The dashed lines denote the corresponding approximation obtained from Corollary 8.1.

$\mathscr{C}_n^{\mathsf{P}}$. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. The edges connected to a VN $\mathtt{v}$ are assigned the value chosen for $\mathtt{v}$. Given $\boldsymbol{\epsilon}$, each $\mathtt{v}_j \in \mathcal{V}$ has ϵ_j replicas in $\mathcal{T}_{a,b}$. Since there are Q copies of each VN type in the lifted graph, the number of VN sets with weight vector $\boldsymbol{\epsilon}$ is

$$N_{\mathtt{v}}(\boldsymbol{\epsilon}) = \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}. \tag{8.114}$$

Similarly, the number of edge sets with weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ is

$$N_e(\boldsymbol{w}(\boldsymbol{\epsilon})) = \prod_{g\in\mathscr{E}} \binom{Q}{w_g} = \prod_{j=1}^{n_{\mathsf{P}}} \binom{Q}{\epsilon_j}^{\mathsf{d}_{\mathtt{v}_j}}. \tag{8.115}$$

Let $\mathcal{I}$ be a set of VNs. We assign to each of these VNs the value one and the other VNs the value zero. Denote by $N_{\mathtt{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))$ the number of configurations with edge set weight vector $\boldsymbol{w}(\boldsymbol{\epsilon})$ that give exactly b unsatisfied CNs. Its generating function is

$$\sum_{b,\boldsymbol{w}(\boldsymbol{\epsilon})} N_{\mathtt{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) y^b \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})}.$$

A CN is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0 and it is unsatisfied otherwise.

Table 8.2: The exact values of the relative minimum Δ-TS sizes, their corresponding approximations obtained from (8.18) and the relative minimum Δ-AS sizes for the ensemble in Example 8.2.

Δ	$\theta^{\star}_{\mathrm{TS}}$	$\theta^{\star}_{\mathrm{TS}}$ from (8.18)	$\theta^{\star}_{\mathrm{AS}}$
0.005	0.02412490	0.02257472	0.02412533
0.05	0.01327771	0.01285350	0.01330189
0.1	0.00740617	0.00730286	0.00746339
0.15	0.00414895	0.00413289	0.00422695
0.2	0.00227137	0.00227527	0.00235458
0.25	0.00119249	0.00119784	0.00126824
0.3	5.89737e-04	5.92944e-04	6.50374e-04
0.35	2.69265e-04	2.70679e-04	3.12226e-04
0.4	1.10733e-04	1.11223e-04	1.37493e-04

Consider a CN of type c_i. If it is satisfied, its generating function is

$$g_{\mathsf{c}}(\boldsymbol{x}_i, y) := y^0 \sum_{\boldsymbol{c}\in\mathscr{C}_i} \boldsymbol{x}_i^{\boldsymbol{c}} = W_{\mathscr{C}_i}(\boldsymbol{x}_i). \tag{8.116}$$

If the CN is unsatisfied then its generating function is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y) := y^1 \sum_{\boldsymbol{c}\in\mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_i}}\setminus\mathscr{C}_i} \boldsymbol{x}_i^{\boldsymbol{c}} = y\left[\prod_{g\in\mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}(1+x_g) - W_{\mathscr{C}_i}(\boldsymbol{x}_i)\right]. \tag{8.117}$$

The sum $g_{\mathsf{c}}(\boldsymbol{x}_i, y) + g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i, y)$ yields $A_i(\boldsymbol{x}_i, y)$. Considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon})) = \mathrm{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)^Q, \boldsymbol{x}^{\boldsymbol{w}(\boldsymbol{\epsilon})} y^b\right). \tag{8.118}$$

Substituting these expressions in

$$\mathsf{E}^{\mathsf{P}}_{\mathrm{TS}}(a, b) = \sum_{\boldsymbol{\epsilon}} N_{\mathsf{v}}(\boldsymbol{\epsilon}) \frac{N_{\mathsf{c}}(b, \boldsymbol{w}(\boldsymbol{\epsilon}))}{N_e(\boldsymbol{w}(\boldsymbol{\epsilon}))} \tag{8.119}$$

completes the proof. ■

Remark 8.2. Evaluating $W_{\mathscr{C}_i}(\boldsymbol{x}_i)$ in (8.113) is complex for some (n_i, k_i) linear codes $\mathscr{C}_i$ if k_i is large. In that case, the following adapted version of the MacWilliams identity [131] might be useful, where we consider VWEF instead of WEF.

Theorem 8.10. Let $\mathscr{C}$ be an (n,k) binary linear code with VWEF $W_{\mathscr{C}}(\boldsymbol{x})$. The VWEF of its dual code $\mathscr{C}^{\perp}$ is

$$W_{\mathscr{C}^{\perp}}(\boldsymbol{x}) = \frac{1}{2^k} W_{\mathscr{C}}(\boldsymbol{x}') \prod_{i=1}^{n} (1 + x_i) \tag{8.120}$$

where $\boldsymbol{x}' = [\boldsymbol{x}'_1, \boldsymbol{x}'_2, \ldots, \boldsymbol{x}'_n]$ and

$$x'_i = \frac{1 - x_i}{1 + x_i} \quad i = 1, 2, \ldots, n. \tag{8.121}$$

The proof of Theorem 8.10 can be found in Appendix 8.4.2.

We next present a simple way to compute the normalized logarithmic asymptotic distribution of TSs for the ensemble $\mathscr{C}_n^{\mathsf{P}}$.

Theorem 8.11. The normalized asymptotic distribution of $(\theta n, \gamma n)$ TSs is

$$G_{\mathrm{TS}}^{\mathsf{P}}(\theta,\gamma) = \frac{1}{n_{\mathsf{P}}} \sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i, y) - \theta \ln t - \gamma \ln y - \sum_{j=1}^{n_{\mathsf{P}}} \left[\frac{\mathsf{d}_{\mathsf{v}_j} - 1}{n_{\mathsf{P}}} H(n_{\mathsf{P}} \tilde{\epsilon}_j^{\star}) + \sum_{g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \tilde{\epsilon}_j^{\star} \ln x_g \right]. \tag{8.122}$$

The values x_g for $g \in \mathscr{E}$, the value y and $\tilde{\epsilon}_j^{\star}$ for $j \in \{1, \ldots, n_{\mathsf{P}}\}$ are the unique positive solutions of

$$x_g \frac{\partial \ln A_i(\boldsymbol{x}_i, y)}{\partial x_g} = n_{\mathsf{P}} \tilde{w}_g^{\star} \quad \forall i \in \{1, \ldots, m_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}} \tag{8.123}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{8.124}$$

$$(\mathsf{d}_{\mathsf{v}_j} - 1) \ln \left(\frac{n_{\mathsf{P}} \tilde{\epsilon}_j^{\star}}{1 - n_{\mathsf{P}} \tilde{\epsilon}_j^{\star}} \right) = \sum_{g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}} \ln x_g + \mu \tag{8.125}$$

where (8.123) holds for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}$ and μ is chosen to satisfy $\sum_j \tilde{\epsilon}_j^{\star} = \theta$, $A_i(\boldsymbol{x}_i, y)$ is defined in (8.113) and $\tilde{w}_g^{\star} = \tilde{\epsilon}_j^{\star}$ if $g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}$. The proof is similar to the proof of Theorem 6.17.

Note that to compute the normalized asymptotic distribution of ETSs, we need to replace $A_i(\boldsymbol{x}_i, y)$ given in (8.113) with

$$A_i(\boldsymbol{x}_i, y) = 1 + \sum_{\boldsymbol{c} \in \mathscr{C}_i : \mathrm{w_H}(\boldsymbol{c}) = \mathsf{d}_{\min, \mathsf{c}_i}} \boldsymbol{x}_i^{\boldsymbol{c}} + y \sum_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} x_g \tag{8.126}$$

where $\mathsf{w}_{\mathrm{H}}(\boldsymbol{c})$ is the Hamming weight of $\boldsymbol{c}$.

8.3.2 Absorbing and Elementary Absorbing Set Distributions

The paper [47] gives an elegant approach to compute the TS enumerator of protograph-based LDPC codes. The approach connects a flag VN to each CN, and each flag VN is assigned a bit to satisfy its CN equation. This way a new protograph is obtained. A (a, b) TS in the original protograph can be interpreted as a codeword of VN weight a and flag VN weight b in the new protograph. This method can be used to determine the TS enumerator of protograph-based GLDPC codes. However, for GLDPC codes, we cannot identify an AS solely from its underlying topological structure since, e.g., if the constraints imposed by a CN are not satisfied, then the node will not necessarily send flip messages in the bit flipping algorithm. Therefore, generating functions are needed to obtain the AS enumerator since they capture the behavior of the bit flipping decoder with BDD at the CNs.

In this section, we derive the AS and EAS enumerators for GLDPC codes and we develop an analytical method for evaluating the normalized logarithmic asymptotic distributions of ASs and EASs.

Lemma 8.12. The average number of (a, b) ASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\mathsf{E}_{\mathrm{AS}}^{\mathsf{P}}(a,b) = \sum_{\boldsymbol{e},\boldsymbol{w}} \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b\right)}{\prod_{g\in\mathscr{E}} \binom{Q}{e_g+w_g}\binom{e_g+w_g}{e_g}} \times \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}}\right) \tag{8.127}$$

where

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = \begin{cases} \frac{1}{2}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1+x_g^{(1)}) + \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g^{(1)})\right] & \\ \quad + y\frac{1}{2}\left[\prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1+x_g^{(2)}) - \prod_{g\in\mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}}(1-x_g^{(2)})\right] & \mathscr{C}_i \text{ is SPC} \\ W_{\mathscr{C}_i}(\boldsymbol{x}_i^{(1)}) + y\left[\sum_{\boldsymbol{z}\in\mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_i}}\setminus\mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{z}\cdot\hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{z}+\boldsymbol{z}\cdot\hat{\boldsymbol{c}}}\right] & \text{otherwise} \end{cases} \tag{8.128}$$

$$B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) = 1 + t \sum_{\boldsymbol{r}_j \in \mathcal{R}_j} (\boldsymbol{s}_j^{(1)})^{\mathbf{1}_j - \boldsymbol{r}_j} (\boldsymbol{s}_j^{(2)})^{\boldsymbol{r}_j} \tag{8.129}$$

and $\mathbf{1}_j$ is the length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ all-ones vector, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v}_\mathsf{j}}$ and Hamming weight $\leq \lfloor (\mathsf{d}_{\mathsf{v}_\mathsf{j}} - 1)/2 \rfloor$, and $\boldsymbol{s}^{(\ell)} = (s_g^{(\ell)})_{g \in \mathscr{E}}$, $\boldsymbol{s}_j^{(\ell)} = (s_g^{(\ell)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$, $\boldsymbol{x}^{(\ell)} = (x_g^{(\ell)})_{g \in \mathscr{E}}$, $\boldsymbol{x}_i^{(\ell)} = (x_g^{(\ell)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}$, y, t and $s_g^{(\ell)}, x_g^{(\ell)}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}, \ell = 1, 2$ are dummy variables, $\hat{\boldsymbol{c}}$ in (8.128) is the decoded vector with BDD and $\boldsymbol{z}$ is the received sequence at the CN. The multiplication in $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is an element-wise multiplication and the addition in $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is an element-wise modulo 2 addition. Note that the i-th entry of $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = \hat{c}_i = 1$ and zero otherwise and the i-th entry of $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = 1$ and $\hat{c}_i = 0$ and zero otherwise.

Proof. We randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. The edges connected to a VN v are assigned the value chosen for v. We define the edge weight vectors $\boldsymbol{e} = (e_g)_{g \in \mathscr{E}}$ and $\boldsymbol{w} = (w_g)_{g \in \mathscr{E}}$ where e_g represents the number of edges of type g in $\mathcal{A}_{a,b}$ not carrying flip messages and w_g is the number of edges of type g in $\mathcal{A}_{a,b}$ carrying flip messages.

Denote by $N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w})$ the number of configurations with edge set weight vectors $\boldsymbol{e}, \boldsymbol{w}$ that give exactly b unsatisfied CNs. Its generating function is

$$\sum_{b, \boldsymbol{e}, \boldsymbol{w}} N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}) y^b (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}}.$$

A CN is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0. We clarify briefly the derivation of $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ for super CNs. Consider a super CN of type c_i. If it is satisfied, then it doesn't send flip messages to its neighboring VNs in $\mathcal{I}$, i.e., all the edges emanating from $\mathcal{I}$ and connected to that CN don't carry flip messages. Thus, the generating function of a satisfied super CN is

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := y^0 \sum_{\boldsymbol{c} \in \mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} = W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(1)}).$$

For an unsatisfied CN, the received vector $\boldsymbol{z}$ is not a valid codeword. Given $\boldsymbol{z} \in \mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_\mathsf{i}}} \setminus \mathscr{C}_i$, we denote by $\hat{\boldsymbol{c}}$ the decoded vector with BDD. The i-th entry of $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = \hat{c}_i = 1$ and zero otherwise and the i-th entry of $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = 1$ and $\hat{c}_i = 0$ and zero otherwise. If $z_i = \hat{c}_i = 1$ (i-th entry of $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1), then the corresponding edge does not carry a flip message and if $z_i = 1$ and $\hat{c}_i = 0$ (the i-th entry of $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1), then the corresponding edge carries a flip message. The generating function of an unsatisfied super CN is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) := y \sum_{\boldsymbol{z} \in \mathbb{F}_2^{\mathsf{d_{c_i}}} \setminus \mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{z} \cdot \hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}}.$$

The SPC node is unsatisfied if it is connected an odd number of times to $\mathcal{I}$. In this case, it sends flip messages to all its neighboring VNs. The generating function of an unsatisfied SPC node is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, y) := \frac{1}{2} y \left[\prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 + x_g^{(2)}) - \prod_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 - x_g^{(2)}) \right].$$

Considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff} \left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} y^b \right) \tag{8.130}$$

where $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ is defined in (8.128).

Denote by $N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w})$ the number of configurations with a VNs and edge set weight vectors $\boldsymbol{e}, \boldsymbol{w}$ such that for each of these VNs $\mathsf{n}_{\mathsf{v}}^{(\mathsf{f})} \leq \lfloor (\mathsf{d_v} - 1)/2 \rfloor$. Its generating function is

$$\sum_{a, \boldsymbol{e}, \boldsymbol{w}} N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}}.$$

Consider a VN of type v_j. Let $\boldsymbol{r}^{(j)} = (r_g^{(j)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$ be a length $\mathsf{d_{v_j}}$ binary vector with $r_g^{(j)} = 1$ if the type g edge carries a flip message and $r_g^{(j)} = 0$ otherwise. Note that if the VN of type v_j belongs to $\mathcal{I}$, the Hamming weight of $\boldsymbol{r}^{(j)}$ should satisfy $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) = \sum_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}} r_g^{(j)} \leq \lfloor (\mathsf{d_{v_j}} - 1)/2 \rfloor$. We can either include this VN type in $\mathcal{I}$ or not. If we skip it we obtain the term 1 in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ corresponding to zero VNs and zero edges. If we include it in the set, we will have 1 node, $\mathsf{d_{v_j}} - \mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges emanating not carrying flip messages and $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges carrying flip messages with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \leq \lfloor (\mathsf{d_{v_j}} - 1)/2 \rfloor$. Considering all possible binary vectors $\boldsymbol{r}^{(j)}$, we obtain the second term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$. Taking into account all possible VN types and that there are Q VNs of each type, we obtain

$$N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) = \operatorname{coeff} \left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} \right) \tag{8.131}$$

where $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ is defined in (8.129).

The number of edge sets with weight vectors $\boldsymbol{e}$ and $\boldsymbol{w}$ is

$$N_e(\boldsymbol{e}, \boldsymbol{w}) = \prod_{g \in \mathscr{E}} \binom{Q}{e_g + w_g} \binom{e_g + w_g}{e_g}. \tag{8.132}$$

Substituting these expressions in

$$\mathsf{E}^{\mathsf{P}}_{\mathrm{AS}}(a, b) = \sum_{\boldsymbol{e}, \boldsymbol{w}} \frac{N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}) N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w})}{N_e(\boldsymbol{e}, \boldsymbol{w})} \tag{8.133}$$

completes the proof. ■

Theorem 8.12. The normalized asymptotic distribution of $(\theta n, \gamma n)$ ASs is

$$\begin{aligned} G^{\mathsf{P}}_{\mathrm{AS}}(\theta, \gamma) =& \frac{1}{n_{\mathsf{P}}} \sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) - \gamma \ln y + \frac{1}{n_{\mathsf{P}}} \sum_{j=1}^{n_{\mathsf{P}}} \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}) \\ & - \theta \ln t - \frac{1}{n_{\mathsf{P}}} \sum_{g \in \mathscr{E}} \ln \left(1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)} \right) \end{aligned} \tag{8.134}$$

The values t, $s_g^{(1)}$, $s_g^{(2)}$, $x_g^{(1)}$, $x_g^{(2)}$, for $g \in \mathscr{E}$ and y are the unique positive solutions of

$$t \frac{\partial \ln \prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial t} = n_{\mathsf{P}} \theta \tag{8.135}$$

$$s_g^{(1)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(1)}} = x_g^{(1)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(1)}} = n_{\mathsf{P}} \tilde{e}_g^{\star} \tag{8.136}$$

$$s_g^{(2)} \frac{\partial \ln B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})}{\partial s_g^{(2)}} = x_g^{(2)} \frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial x_g^{(2)}} = n_{\mathsf{P}} \tilde{w}_g^{\star} \tag{8.137}$$

$$y \frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)}{\partial y} = n_{\mathsf{P}} \gamma \tag{8.138}$$

where (8.136) and (8.137) are for all $i \in \{1, \ldots, m_{\mathsf{P}}\}, j \in \{1, \ldots, n_{\mathsf{P}}\}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j} \cap \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}$, $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)})$ and $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ are defined in (8.129) and (8.128), respectively, and

$$\tilde{e}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(1)} s_g^{(1)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)}} \tag{8.139}$$

$$\tilde{w}_g^{\star} = \frac{1}{n_{\mathsf{P}}} \frac{x_g^{(2)} s_g^{(2)}}{1 + x_g^{(1)} s_g^{(1)} + x_g^{(2)} s_g^{(2)}}. \tag{8.140}$$

The proof is similar to the one of Theorem 6.17. Note that to compute the normalized asymptotic distribution of EASs, we must replace $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y)$ given in (8.128) with

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, y) = \begin{cases} 1 + \sum\limits_{\substack{g_1, g_2 \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i} \\ g_1 \neq g_2}} x_{g_1}^{(1)} x_{g_2}^{(1)} + y \sum\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g^{(2)} & \text{if } \mathscr{C}_i \text{ is a SPC code} \\ 1 + \sum\limits_{\boldsymbol{c} \in \mathscr{C}_i : \mathrm{w_H}(\boldsymbol{c}) = \mathsf{d}_{\min, \mathsf{c}_i}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} + y \sum\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g^{(2)} & \text{otherwise.} \end{cases} \tag{8.141}$$

8.3.3 Fully and Elementary Fully Absorbing Set Distributions

In this section, we derive the FAS and EFAS enumerators for GLDPC codes and we develop an analytical method for evaluating the normalized logarithmic asymptotic distributions of FASs and EFASs.

Lemma 8.13. The average number of (a, b) FASs in the Tanner graph of a code drawn randomly from the ensemble $\mathscr{C}_n^{\mathsf{P}}$ is

$$\begin{aligned} \mathsf{E}^{\mathsf{P}}_{\mathrm{FAS}}(a,b) = \sum_{\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}} & \frac{\operatorname{coeff}\left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} (\boldsymbol{x}^{(3)})^{\boldsymbol{l}} y^b \right)}{\prod_{g \in \mathscr{E}} \binom{Q}{e_g + w_g} \binom{e_g + w_g}{e_g} \binom{Q - e_g - w_g}{l_g}} \times \\ & \operatorname{coeff}\left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} (\boldsymbol{s}^{(3)})^{\boldsymbol{l}} \right) \end{aligned} \tag{8.142}$$

where

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) = \begin{cases} \frac{1}{2} \left[\prod\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 + x_g^{(1)}) + \prod\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (1 - x_g^{(1)}) \right] & \\ + y \frac{1}{2} \left[\prod\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (x_g^{(3)} + x_g^{(2)}) - \prod\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} (x_g^{(3)} - x_g^{(2)}) \right] & \mathscr{C}_i \text{ is SPC} \\ W_{\mathscr{C}_i}(\boldsymbol{x}_i^{(1)}) + y \sum\limits_{\boldsymbol{z} \in \mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_i}} \setminus \mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{z} \cdot \hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(3)})^{\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}} & \text{otherwise.} \end{cases} \tag{8.143}$$

$$B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)}) = \sum_{\boldsymbol{r}_j \in \mathcal{R}_j} (\boldsymbol{s}_j^{(3)})^{\boldsymbol{r}_j} + t \sum_{\boldsymbol{r}_j \in \mathcal{R}_j} (\boldsymbol{s}_j^{(1)})^{\boldsymbol{1}_j - \boldsymbol{r}_j} (\boldsymbol{s}_j^{(2)})^{\boldsymbol{r}_j} \tag{8.144}$$

and $\boldsymbol{1}_j$ is the length $\mathsf{d}_{\mathsf{v_j}}$ all-ones vector, $\mathcal{R}_j$ is the set of binary vectors of length $\mathsf{d}_{\mathsf{v_j}}$ and Hamming weight $\leq \lfloor (\mathsf{d}_{\mathsf{v_j}} - 1)/2 \rfloor$, and $\boldsymbol{s}^{(\ell)} = (s_g^{(\ell)})_{g \in \mathscr{E}}$, $\boldsymbol{s}_j^{(\ell)} = (s_g^{(\ell)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j}}$, $\boldsymbol{x}^{(\ell)} = (x_g^{(\ell)})_{g \in \mathscr{E}}$, $\boldsymbol{x}_i^{(\ell)} = (x_g^{(\ell)})_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}}$, y, t and $s_g^{(\ell)}, x_g^{(\ell)}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}, \ell = 1, 2, 3$ are dummy variables, $\hat{\boldsymbol{c}}$ in (8.143) is the decoded vector with BDD and $\boldsymbol{z}$ is the received sequence at the CN. The product $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is an element-wise multiplication and the sum $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is an element-wise modulo 2 addition. Note that the i-th entry of $\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = 0$ and $\hat{c}_i = 1$.

Proof. Randomly choose a set $\mathcal{I}$ of a VNs and assign the value 1 to each VN in the set with a uniform distribution over all possibilities. The edges connected to a VN $\mathtt{v}$ are assigned the value chosen for $\mathtt{v}$. Define the edge weight vectors $\boldsymbol{e} = (e_g)_{g\in\mathscr{E}}$, $\boldsymbol{w} = (w_g)_{g\in\mathscr{E}}$ and $\boldsymbol{l} = (l_g)_{g\in\mathscr{E}}$ where e_g represents the number of edges of type g in $\mathcal{F}_{a,b}$ not carrying flip messages and w_g is the number of edges of type g in $\mathcal{F}_{a,b}$ carrying flip messages and l_g is the number of edges of type g outside $\mathcal{F}_{a,b}$ carrying flip messages. We denote by $N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l})$ the number of configurations with edge set weight vectors $\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}$ that give exactly b unsatisfied CNs. Its generating function is

$$\sum_{b,\boldsymbol{e},\boldsymbol{w},\boldsymbol{l}} N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) y^b (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} (\boldsymbol{x}^{(3)})^{\boldsymbol{l}}.$$

A CN is satisfied if it recognizes a valid local codeword when the edges connected to $\mathcal{I}$ are assigned the value 1 and the other edges the value 0. We clarify briefly the derivation of $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)$ for a super CNs. Consider a super CN of type c_i. If it is satisfied, then it doesn't send flip messages to its neighboring VNs in $\mathcal{I}$, i.e., all the edges connected to that CN don't carry flip messages. Thus the generating function of a satisfied super CN is

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := y^0 \sum_{\boldsymbol{c}\in\mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} = W_{\mathcal{P}_i}(\boldsymbol{x}_i^{(1)}). \tag{8.145}$$

For an unsatisfied CN, the received vector $\boldsymbol{z}$ is not a valid codeword. Given $\boldsymbol{z} \in \mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_i}} \setminus \mathscr{C}_i$, we denote by $\hat{\boldsymbol{c}}$ the decoded vector with BDD. The i-th entry of $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = \hat{c}_i = 1$ and zero otherwise and the i-th entry of $\boldsymbol{z} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = 1$ and $\hat{c}_i = 0$ and zero otherwise and the i-th entry of $\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1 if $z_i = 0$ and $\hat{c}_i = 1$ and zero otherwise. If $z_i = \hat{c}_i = 1$ (i-th entry of $\boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1), then the corresponding edge connected to $\mathcal{I}$ does not carry a flip message and if $z_i = 1$ and $\hat{c}_i = 0$ (the i-th entry of $\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1), then the corresponding edge not connected to $\mathcal{I}$ carries a flip message and if $z_i = 0$ and $\hat{c}_i = 1$ (the i-th entry of $\hat{\boldsymbol{c}} + \boldsymbol{z} \cdot \hat{\boldsymbol{c}}$ is 1), then the corresponding edge not connected to $\mathcal{I}$ carries a flip message. The generating function of an unsatisfied super CN is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) := y \sum_{\boldsymbol{z}\in\mathbb{F}_2^{\mathsf{d}_{\mathsf{c}_i}} \setminus \mathscr{C}_i} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{z}\cdot\hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(2)})^{\boldsymbol{z}+\boldsymbol{z}\cdot\hat{\boldsymbol{c}}} (\boldsymbol{x}_i^{(3)})^{\hat{\boldsymbol{c}}+\boldsymbol{z}\cdot\hat{\boldsymbol{c}}}.$$

The SPC node is unsatisfied if it is connected an odd number of times to $\mathcal{I}$. In this case, it sends flip messages to all its neighboring VNs. The generating function of an unsatisfied

SPC node is

$$g_{\bar{\mathsf{c}}}(\boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) := \frac{1}{2} y \left[\prod_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (x_g^{(3)} + x_g^{(2)}) - \prod_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (x_g^{(3)} - x_g^{(2)}) \right]$$

and the generating function of a satisfied SPC node is

$$g_{\mathsf{c}}(\boldsymbol{x}_i^{(1)}, y) := \frac{1}{2} \left[\prod_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 + x_g^{(1)}) + \prod_{g \in \mathscr{E}_{\mathsf{c}_i}^{\mathsf{P}}} (1 - x_g^{(1)}) \right].$$

Considering all CN types and that there are Q CNs of each type c_i, we obtain

$$N_{\mathsf{c}}(b, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) = \operatorname{coeff} \left(\prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)^Q, (\boldsymbol{x}^{(1)})^{\boldsymbol{e}} (\boldsymbol{x}^{(2)})^{\boldsymbol{w}} (\boldsymbol{x}^{(3)})^{\boldsymbol{l}} y^b \right) \tag{8.146}$$

where $A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)$ is defined in (8.143).

Consider a VN of type v_j. Let $\boldsymbol{r}^{(j)} = (r_g^{(j)})_{g \in \mathscr{E}_{\mathsf{v}_j}^{\mathsf{P}}}$ be a length $\mathsf{d}_{\mathsf{v}_j}$ binary vector with $r_g^{(j)} = 1$ if the type g edge carries a flip message and $r_g^{(j)} = 0$ otherwise. Note that for each VN of type v_j, the Hamming weight of $\boldsymbol{r}^{(j)}$ should satisfy $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \le \lfloor (\mathsf{d}_{\mathsf{v}_j} - 1)/2 \rfloor$. We can either include this VN type in $\mathcal{I}$ or not. If we skip it we obtain the first term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$ corresponding to zero VNs and $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \le \lfloor (\mathsf{d}_{\mathsf{v}_j} - 1)/2 \rfloor$ edges carrying flip messages and VNs outside $\mathcal{I}$. If we include it in $\mathcal{I}$, we will have 1 node, $\mathsf{d}_{\mathsf{v}_j} - \mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges not carrying flip messages and $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)})$ edges carrying flip messages with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{r}^{(j)}) \le \lfloor (\mathsf{d}_{\mathsf{v}_j} - 1)/2 \rfloor$. Considering all possible binary vectors $\boldsymbol{r}^{(j)}$, we obtain the second term in $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$. Taking into account all possible VN types and that there are Q VNs of each type, we obtain

$$N_{\mathsf{v}}(a, \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) = \operatorname{coeff} \left(\prod_{j=1}^{n_{\mathsf{P}}} B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})^Q, t^a (\boldsymbol{s}^{(1)})^{\boldsymbol{e}} (\boldsymbol{s}^{(2)})^{\boldsymbol{w}} (\boldsymbol{s}^{(3)})^{\boldsymbol{l}} \right) \tag{8.147}$$

where $B_j(t, \boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}, \boldsymbol{s}_j^{(3)})$ is defined in (8.144). The number of edge sets with weight vectors $\boldsymbol{e}$, $\boldsymbol{w}$ and $\boldsymbol{l}$ is

$$N_e(\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{l}) = \prod_{g \in \mathscr{E}} \binom{Q}{e_g + w_g} \binom{e_g + w_g}{e_g} \binom{Q - e_g - w_g}{l_g}. \tag{8.148}$$

Summing over all possible weight vectors $\boldsymbol{e}$, $\boldsymbol{w}$ and $\boldsymbol{l}$ completes the proof. ■

Theorem 8.13. The normalized asymptotic distribution of $(\theta n, \gamma n)$ FASs is

$$\begin{aligned} G^{\mathsf{P}}_{\mathrm{FAS}}(\theta,\gamma) =& \frac{1}{n_{\mathsf{P}}}\sum_{i=1}^{m_{\mathsf{P}}} \ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y) + \frac{1}{n_{\mathsf{P}}}\sum_{j=1}^{n_{\mathsf{P}}} \ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)}) \\ & - \theta \ln t - \gamma \ln y - \frac{1}{n_{\mathsf{P}}}\sum_{g\in\mathscr{E}} \ln\left(1 + x_g^{(1)}s_g^{(1)} + x_g^{(2)}s_g^{(2)} + x_g^{(3)}s_g^{(3)}\right) \end{aligned} \tag{8.149}$$

where t, $s_g^{(1)}$, $s_g^{(2)}$,$s_g^{(3)}$, $x_g^{(1)}$, $x_g^{(2)}$, $x_g^{(3)}$ for $g \in \mathscr{E}$ and y are the unique positive solutions of

$$t\frac{\partial \ln \prod_{j=1}^{n_{\mathsf{P}}} B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)})}{\partial t} = n_{\mathsf{P}}\theta \tag{8.150}$$

$$s_g^{(1)}\frac{\partial \ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)})}{\partial s_g^{(1)}} = x_g^{(1)}\frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y)}{\partial x_g^{(1)}} = n_{\mathsf{P}}\tilde{e}_g^{\star} \tag{8.151}$$

$$s_g^{(2)}\frac{\partial \ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)})}{\partial s_g^{(2)}} = x_g^{(2)}\frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y)}{\partial x_g^{(2)}} = n_{\mathsf{P}}\tilde{w}_g^{\star} \tag{8.152}$$

$$s_g^{(3)}\frac{\partial \ln B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)})}{\partial s_g^{(3)}} = x_g^{(3)}\frac{\partial \ln A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y)}{\partial x_g^{(3)}} = n_{\mathsf{P}}\tilde{l}_g^{\star} \tag{8.153}$$

$$y\frac{\partial \ln \prod_{i=1}^{m_{\mathsf{P}}} A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y)}{\partial y} = n_{\mathsf{P}}\gamma \tag{8.154}$$

where (8.151)-(8.153) are for all $i \in \{1,\dots,m_{\mathsf{P}}\}, j \in \{1,\dots,n_{\mathsf{P}}\}, g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{v}_j} \cap \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i},$, $B_j(t,\boldsymbol{s}_j^{(1)},\boldsymbol{s}_j^{(2)},\boldsymbol{s}_j^{(3)})$ and $A_i(\boldsymbol{x}_i^{(1)},\boldsymbol{x}_i^{(2)},\boldsymbol{x}_i^{(3)},y)$ are defined in (8.144) and (8.143), respectively, and

$$\tilde{e}_g^{\star} = \frac{1}{n_{\mathsf{P}}}\frac{x_g^{(1)}s_g^{(1)}}{1 + x_g^{(1)}s_g^{(1)} + x_g^{(2)}s_g^{(2)} + x_g^{(3)}s_g^{(3)}} \tag{8.155}$$

$$\tilde{w}_g^{\star} = \frac{1}{n_{\mathsf{P}}}\frac{x_g^{(2)}s_g^{(2)}}{1 + x_g^{(1)}s_g^{(1)} + x_g^{(2)}s_g^{(2)} + x_g^{(3)}s_g^{(3)}} \tag{8.156}$$

$$\tilde{l}_g^{\star} = \frac{1}{n_{\mathsf{P}}}\frac{x_g^{(3)}s_g^{(3)}}{1 + x_g^{(1)}s_g^{(1)} + x_g^{(2)}s_g^{(2)} + x_g^{(3)}s_g^{(3)}}. \tag{8.157}$$

The proof is omitted since it is similar to the one of Theorem 6.17.

To compute the normalized asymptotic distribution of EFASs, we must replace

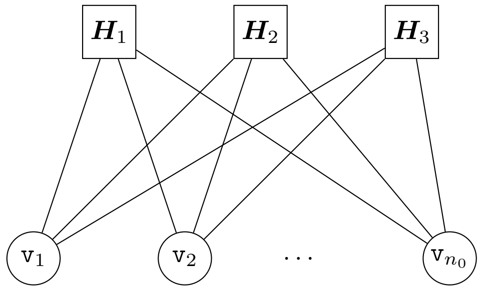

Figure 8.9: Protograph of the GLDPC code ensemble of the ensemble in Example 8.3.

$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y)$ in (8.143) with

$$A_i(\boldsymbol{x}_i^{(1)}, \boldsymbol{x}_i^{(2)}, \boldsymbol{x}_i^{(3)}, y) = \begin{cases} 1 + \sum\limits_{g,g' \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}: g \neq g'} x_g^{(1)} x_{g'}^{(1)} + y \sum\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g^{(2)} \prod\limits_{g' \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}: g' \neq g} x_{g'}^{(3)} & \mathscr{C}_i \text{ is SPC} \\ 1 + \sum\limits_{\boldsymbol{c} \in \mathscr{C}_i : \mathsf{w}_{\mathrm{H}}(\boldsymbol{c}) = \mathsf{d}_{\min,\mathsf{c}_i}} (\boldsymbol{x}_i^{(1)})^{\boldsymbol{c}} + y \sum\limits_{g \in \mathscr{E}^{\mathsf{P}}_{\mathsf{c}_i}} x_g^{(2)} & \text{else.} \end{cases} \tag{8.158}$$

where $\mathbf{1}_i$ is the length $\mathsf{d}_{\mathsf{c}_i}$ all-ones vector.

Note that, using BDD, for a received sequence $\boldsymbol{z}$ with $\mathsf{w}_{\mathrm{H}}(\boldsymbol{z}) = 1$ and $t_i \geq 1$, we have $\hat{\boldsymbol{c}} = \mathbf{0}$. That means, super check nodes never send flip messages to VNs outside $\mathcal{I}$. Thus, if all CNs are super check nodes, the condition that for each VN outside $\mathcal{I}$, $\mathsf{n}_{\mathsf{v}}^{(\mathrm{f})} < \mathsf{d}_{\mathsf{v}}/2$ is always fulfilled. In that case, EASs and EFASs are equivalent.

Example 8.3. Consider the rate 1/6 GLDPC code ensemble which has the protograph shown in Fig. 8.9 where $n_0 = 6$, $\mathcal{P}_1$ is a $(6, 3)$ shortened Hamming code with the parity-check matrix

$$\boldsymbol{H}_1 = \begin{bmatrix} 1\ 1\ 0 & \\ 0\ 1\ 1 & \boldsymbol{I}_3 \\ 1\ 0\ 1 & \end{bmatrix}$$

and $\mathcal{P}_2$ and $\mathcal{P}_3$ are $(6, 5)$ SPC codes. The normalized logarithmic asymptotic distribution of (elementary) TSs and (fully) ASs of this ensemble are depicted in Fig. 8.10 and Fig. 8.11 for $\Delta \in \{0.005, 0.05, 0.1, 0.3\}$. We observe that the gap between the normalized logarithmic asymptotic distributions of TSs and ASs is small and vanishes for small θ. Observe that this code ensemble has better TS and AS properties than the regular $(3, 6)$ LDPC ensemble. For instance, for $\Delta = 0.005$, we have $\theta^{\star}_{\mathrm{TS}} = 0.172580$ while for the regular ensemble, we have $\theta^{\star}_{\mathrm{TS}} = 0.0207989$. As mentioned previously, replacing some of the SPC CNs by a more powerful code improves the TS properties of the code especially if the linear block code has

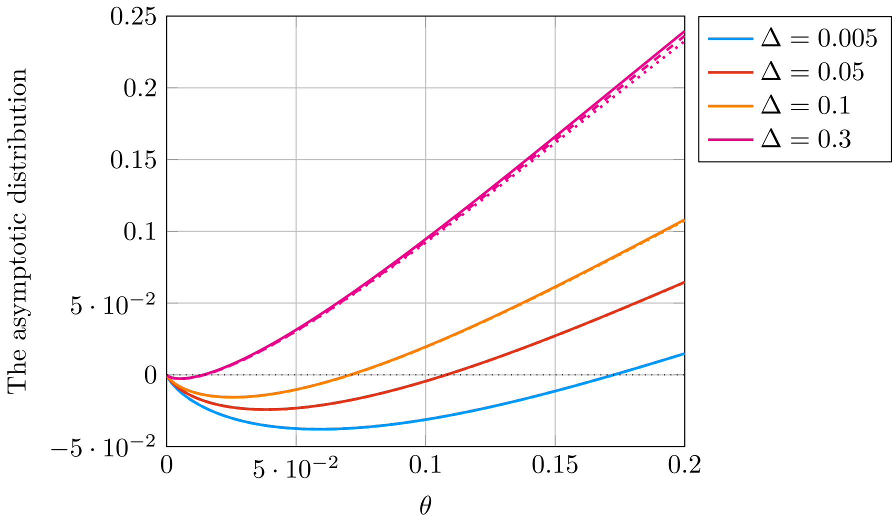

Figure 8.10: Normalized logarithmic asymptotic distributions of TSs (solid lines), ASs (dashed lines), and fully ASs (dotted lines) of the ensemble in Example 8.3.

good distance properties and thus a high error correcting capability under BDD.

8.4 Appendices

8.4.1 Proof of Corollary 8.1

We obtain expressions for t, s, x, y in terms of $\tilde{w}^\star$ and for $\tilde{w}^\star$ in terms of θ. We should analyze the equations (8.9)-(8.11) for the case where $\theta \to 0$ and $\gamma = \Delta\theta$. These equations can be rewritten as

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{ts^j}{1+ts^j} = \theta \tag{8.159}$$

$$\sum_{j=1}^{\mathsf{d}_\mathsf{v}^{\max}} \Lambda_j \frac{jts^j}{1+ts^j} = \tilde{w}^\star \tag{8.160}$$

$$\xi \sum_{\tau=1}^{n_c} \mathrm{P}_\tau \frac{\sum_{h=\mathsf{d}_{\min,\tau}}^{n_\tau} W_h^{(\tau)} h x^h + \sum_{h=1}^{n_\tau} \left(\binom{n_\tau}{h} - W_h^{(\tau)} \right) h x^h y}{1 + \sum_{h=\mathsf{d}_{\min,\tau}}^{n_\tau} W_h^{(\tau)} x^h + \sum_{h=1}^{n_\tau} \left(\binom{n_\tau}{h} - W_h^{(\tau)} \right) x^h y} = \tilde{w}^\star \tag{8.161}$$

$$\xi \sum_{\tau=1}^{n_c} \mathrm{P}_\tau \frac{\sum_{h=1}^{n_\tau} \left(\binom{n_\tau}{h} - W_h^{(\tau)} \right) x^h y}{1 + \sum_{h=\mathsf{d}_{\min,\tau}}^{n_\tau} W_h^{(\tau)} x^h + \sum_{h=1}^{n_\tau} \left(\binom{n_\tau}{h} - W_h^{(\tau)} \right) x^h y} = \Delta\theta \tag{8.162}$$

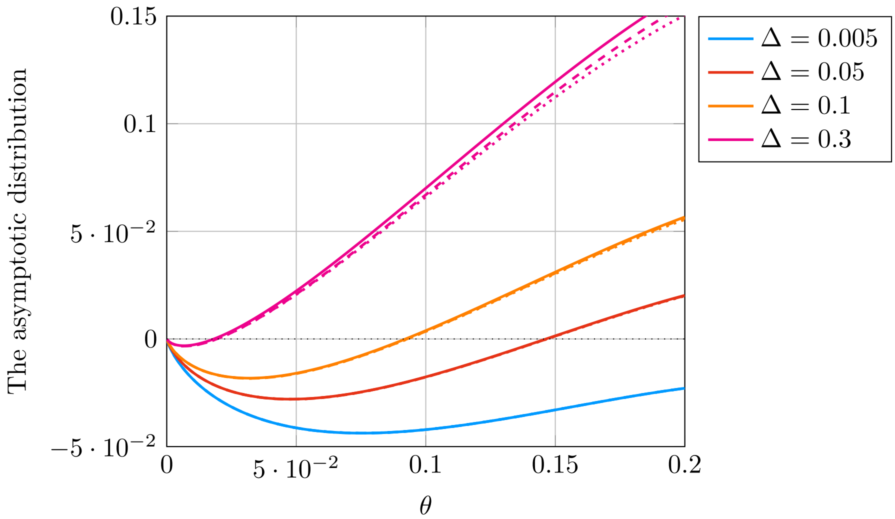

Figure 8.11: Normalized logarithmic asymptotic distributions of elementary TSs (solid lines), elementary ASs (dashed lines), and elementary fully ASs (dotted lines) of the ensemble in Example 8.3.

where $W_h^{(\tau)}$ is the number of codewords of Hamming weight h in $\mathscr{C}_\tau$. From (8.159) and (8.160), since for $\mathsf{d}_\mathsf{v}^{\min} \leq j \leq \mathsf{d}_\mathsf{v}^{\max}$ we have

$$\mathsf{d}_\mathsf{v}^{\min} \Lambda_j \frac{ts^j}{1+ts^j} \leq j\Lambda_j \frac{ts^j}{1+ts^j} \leq \mathsf{d}_\mathsf{v}^{\max} \Lambda_j \frac{ts^j}{1+ts^j}$$

by summing over $j = \mathsf{d}_\mathsf{v}^{\min}, \ldots, \mathsf{d}_\mathsf{v}^{\max}$ we obtain $\mathsf{d}_\mathsf{v}^{\min}\theta \leq \tilde{w}^\star \leq \mathsf{d}_\mathsf{v}^{\max}\theta$.

Thus, we have

$$\lim_{\theta \to 0} \tilde{w}^\star = 0 \tag{8.163}$$

and the notations $o(\theta)$ and $o(\tilde{w}^\star)$ are equivalent, i.e., for any function f, $f = o(\theta) \iff f = o(\tilde{w}^\star)$. Therefore, we will use $o(\theta)$ and $o(\tilde{w}^\star)$ interchangeably.

The left hand side of (8.162) should also be $o(1)$, that means for some h, we have $x^h y = o(1)$ and for all other h, $x^h y = o(\theta)$. Thus, we have

$$\Delta\theta(1 + o(1)) = \xi \sum_{\tau=1}^{n_c} \mathrm{P}_\tau n_\tau xy. \tag{8.164}$$

Because of (8.163), the left hand side of (8.161) must be $o(1)$, i.e., $x^h = o(1)$ for some h, $x^h = o(\tilde{w}^\star)$ for the other h, $x^h y = o(1)$ for some h and $x^h y = o(\tilde{w}^\star)$ for the other h. The left hand side of (8.161) is dominated by the terms corresponding to $h = 1, r$ where r is the smallest minimum distance over all CN types. We have

$$\tilde{w}^\star(1+o(1)) = \xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)} x^r + \xi \sum_{\tau=1}^{n_c} \mathrm{P}_\tau n_\tau xy. \tag{8.165}$$

From (8.164) and (8.165), we have

$$x = \left(\frac{\tilde{w}^\star - \Delta\theta}{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}} \right)^{1/r} (1+o(1)). \tag{8.166}$$

Substituting (8.164) into (8.12), we obtain

$$s = \frac{\tilde{w}^\star}{\bar{\mathsf{d}}_\mathsf{v}} \left(\frac{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}}{\tilde{w}^\star - \Delta\theta} \right)^{1/r} (1+o(1)). \tag{8.167}$$

Thus, (8.160) can be written as

$$\tilde{w}^\star(1+o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}} \mathsf{d}_\mathsf{v}^{\min} t s^{\mathsf{d}_\mathsf{v}^{\min}} \tag{8.168}$$

and thus

$$t = \frac{(\bar{\mathsf{d}}_\mathsf{v})^{\mathsf{d}_\mathsf{v}^{\min}}}{\Lambda_{\mathsf{d}_\mathsf{v}^{\min}} \mathsf{d}_\mathsf{v}^{\min} (\tilde{w}^\star)^{\mathsf{d}_\mathsf{v}^{\min}-1}} \left(\frac{\tilde{w}^\star - \Delta\theta}{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}} \right)^{\mathsf{d}_\mathsf{v}^{\min}/r} (1+o(1)). \tag{8.169}$$

Similarly, from (8.159), we have

$$\theta(1+o(1)) = \Lambda_{\mathsf{d}_\mathsf{v}^{\min}} t s^{\mathsf{d}_\mathsf{v}^{\min}}. \tag{8.170}$$

Thus,

$$\tilde{w}^\star = \mathsf{d}_\mathsf{v}^{\min} \theta (1+o(1)). \tag{8.171}$$

Substituting (8.171) into (8.166), (8.167) and (8.168), we obtain

$$x = \left(\frac{(\mathsf{d}_\mathsf{v}^{\min} - \Delta)\theta}{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}} \right)^{1/r} (1+o(1)) \tag{8.172}$$

$$s = \frac{\mathsf{d}_{\mathsf{v}}^{\min}}{\bar{\mathsf{d}}_{\mathsf{v}}} \left(\frac{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}}{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta} \right)^{1/r} \theta^{\frac{r-1}{r}} (1 + o(1)) \tag{8.173}$$

$$t = \frac{(\bar{\mathsf{d}}_{\mathsf{v}})^{\mathsf{d}_{\mathsf{v}}^{\min}}}{\Lambda_{\mathsf{d}_{\mathsf{v}}^{\min}} (\mathsf{d}_{\mathsf{v}}^{\min})^{\mathsf{d}_{\mathsf{v}}^{\min}}} \left(\frac{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta}{\xi r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}} \right)^{\mathsf{d}_{\mathsf{v}}^{\min}/r} \theta^{\frac{r-(r-1)\mathsf{d}_{\mathsf{v}}^{\min}}{r}} (1 + o(1)) \tag{8.174}$$

$$y = \frac{\Delta \xi^{\frac{1-r}{r}}}{\bar{\mathsf{d}}_{\mathsf{c}}} \left(\frac{r \sum_{\tau:\mathsf{d}_{\min,\tau}=r} \mathrm{P}_\tau W_r^{(\tau)}}{\mathsf{d}_{\mathsf{v}}^{\min} - \Delta} \right)^{1/r} \theta^{\frac{r-1}{r}} (1 + o(1)). \tag{8.175}$$

By substituting (8.172)-(8.175) into (8.8) and by using the Taylor series of $\ln(1+x)$ around $x = 0$, we obtain (8.16). Note that we obtain exactly the same result for ETSs.

8.4.2 Proof of Theorem 8.10

The following Lemma is useful to derive the Theorem.

Definition 8.8. Let f be a function defined over $\mathbb{F}_2^n$. The Hadamard transform $\hat{f}$ of f is defined as

$$\hat{f}(\boldsymbol{u}) = \sum_{\boldsymbol{v} \in \mathbb{F}_2^n} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} f(\boldsymbol{v}) \tag{8.176}$$

where $\boldsymbol{u} \cdot \boldsymbol{v}$ is the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$.

The following Lemma is a property of the Hadamard transform.

Lemma 8.14. Let f be a function defined over $\mathbb{F}_2^n$. We have

$$\sum_{\boldsymbol{u} \in \mathscr{C}^\perp} f(\boldsymbol{u}) = \frac{1}{|\mathscr{C}|} \sum_{\boldsymbol{u} \in \mathscr{C}} \hat{f}(\boldsymbol{u}). \tag{8.177}$$

Proof.

$$\begin{aligned} \sum_{\boldsymbol{u} \in \mathscr{C}} \hat{f}(\boldsymbol{u}) &= \sum_{\boldsymbol{u} \in \mathscr{C}} \sum_{\boldsymbol{v} \in \mathbb{F}_2^n} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} f(\boldsymbol{v}) \\ &= \sum_{\boldsymbol{v} \in \mathbb{F}_2^n} f(\boldsymbol{v}) \sum_{\boldsymbol{u} \in \mathscr{C}} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} \\ &= \sum_{\boldsymbol{v} \in \mathscr{C}^\perp} f(\boldsymbol{v}) \sum_{\boldsymbol{u} \in \mathscr{C}} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} + \sum_{\boldsymbol{v} \notin \mathscr{C}^\perp} f(\boldsymbol{v}) \sum_{\boldsymbol{u} \in \mathscr{C}} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} \end{aligned} \tag{8.178}$$

▷ For $\boldsymbol{v} \in \mathscr{C}^{\perp}$, $\boldsymbol{u} \cdot \boldsymbol{v} = 0 \ \forall \boldsymbol{u} \in \mathscr{C}$. Thus, we have

$$\sum_{\boldsymbol{v} \in \mathscr{C}^{\perp}} f(\boldsymbol{v}) \sum_{\boldsymbol{u} \in \mathscr{C}} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} = |\mathscr{C}| \sum_{\boldsymbol{v} \in \mathscr{C}^{\perp}} f(\boldsymbol{v}). \tag{8.179}$$

▷ For $\boldsymbol{v} \notin \mathscr{C}^{\perp}$:

Define the following sets: $\mathcal{I}_0(\boldsymbol{v}) = \{\boldsymbol{u} \in \mathscr{C} : \boldsymbol{u} \cdot \boldsymbol{v} = 0\}$ and $\mathcal{I}_1(\boldsymbol{v}) = \{\boldsymbol{u} \in \mathscr{C} : \boldsymbol{u} \cdot \boldsymbol{v} = 1\}$. Note that $\mathcal{I}_0(\boldsymbol{v}) \cup \mathcal{I}_1(\boldsymbol{v}) = \mathscr{C}$ and $\mathcal{I}_0(\boldsymbol{v}) \cap \mathcal{I}_1(\boldsymbol{v}) = \varnothing$. Let $\boldsymbol{u}^{\star} \in \mathscr{C}$ with $\boldsymbol{u}^{\star} \cdot \boldsymbol{v} = 1$ (such a vector exists since $\boldsymbol{v} \notin \mathscr{C}^{\perp}$). Define the set $\mathcal{I}^{\star} = \{\boldsymbol{u}^{\star} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{I}_0(\boldsymbol{v})\}$. $\mathcal{I}^{\star}$ is a coset of $\mathcal{I}_0(\boldsymbol{v})$ and as a result

$$|\mathcal{I}_0(\boldsymbol{v})| = |\mathcal{I}^{\star}|. \tag{8.180}$$

Let $\boldsymbol{w}^{\star} \in \mathcal{I}^{\star}$, i.e., $\boldsymbol{w}^{\star} = \boldsymbol{u}^{\star} + \boldsymbol{u}$, where $\boldsymbol{u} \in \mathcal{I}_0(\boldsymbol{v})$. We have

$$\boldsymbol{w}^{\star} \cdot \boldsymbol{v} = \boldsymbol{u}^{\star} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v} = 1$$

and

$$\mathcal{I}^{\star} \subseteq \mathcal{I}_1(\boldsymbol{v}). \tag{8.181}$$

Let $\boldsymbol{w}_1 \in \mathcal{I}_1(\boldsymbol{v})$ so $\boldsymbol{w}_1$ can be rewritten as $\boldsymbol{w}_1 = \boldsymbol{u}^{\star} + (\boldsymbol{u}^{\star} + \boldsymbol{w}_1)$. Note that $\boldsymbol{u}^{\star} + (\boldsymbol{u}^{\star} + \boldsymbol{w}_1) \in \mathcal{I}_0(\boldsymbol{v})$, i.e., $\boldsymbol{w}_1 \in \mathcal{I}^{\star}$. Hence, we have

$$\mathcal{I}_1(\boldsymbol{v}) \subseteq \mathcal{I}^{\star}. \tag{8.182}$$

From (8.181) and (8.182), we obtain $\mathcal{I}_1(\boldsymbol{v}) = \mathcal{I}^{\star}$ and as a result $|\mathcal{I}_1(\boldsymbol{v})| = |\mathcal{I}^{\star}|$. Thus from (8.180), $|\mathcal{I}_1(\boldsymbol{v})| = |\mathcal{I}_0(\boldsymbol{v})|$.

We can now calculate the second term in (8.178):

$$\begin{aligned} \sum_{\boldsymbol{v} \notin \mathscr{C}^{\perp}} f(\boldsymbol{v}) \sum_{\boldsymbol{u} \in \mathscr{C}} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} &= \sum_{\boldsymbol{v} \notin \mathscr{C}^{\perp}} f(\boldsymbol{v}) \left(\sum_{\boldsymbol{u} \in \mathcal{I}_0(\boldsymbol{v})} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} + \sum_{\boldsymbol{u} \in \mathcal{I}_1(\boldsymbol{v})} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} \right) \\ &= \sum_{\boldsymbol{v} \notin \mathscr{C}^{\perp}} f(\boldsymbol{v}) (|\mathcal{I}_0(\boldsymbol{v})| - |\mathcal{I}_1(\boldsymbol{v})|) = 0. \end{aligned} \tag{8.183}$$

From (8.178), (8.179) and (8.183) we obtain the result in (8.177).

■

Consider the function

$$f(\boldsymbol{u}) = \boldsymbol{x}^{\boldsymbol{u}} \tag{8.184}$$

and note that

$$W_{\mathscr{C}}(\boldsymbol{x}) = \sum_{\boldsymbol{u}\in\mathscr{C}} f(\boldsymbol{u}). \tag{8.185}$$

The Hadamard transform of f is

$$\begin{aligned}\hat{f}(\boldsymbol{u}) &= \sum_{\boldsymbol{v}\in\mathbb{F}_2^n} (-1)^{\boldsymbol{u}\cdot\boldsymbol{v}} f(\boldsymbol{v}) = \sum_{v_1\in\{0,1\}} \sum_{v_2\in\{0,1\}} \cdots \sum_{v_n\in\{0,1\}} \prod_{i=1}^{n} (-1)^{u_i v_i} x_i^{v_i} \\ &= \prod_{i=1}^{n} \sum_{v_i\in\{0,1\}} (-1)^{u_i v_i} x_i^{v_i} = \prod_{i=1}^{n} (1 + (-1)^{u_i} x_i) \\ &= \prod_{i=1}^{n} (1+x_i) \left(\frac{1-x_i}{1+x_i}\right)^{u_i}.\end{aligned} \tag{8.186}$$

We can now use Lemma in (8.14) to determine the VWEF of the dual code:

$$\begin{aligned}W_{\mathscr{C}^\perp}(\boldsymbol{x}) &= \sum_{\boldsymbol{u}\in\mathscr{C}^\perp} f(\boldsymbol{u}) = \frac{1}{|\mathscr{C}|} \sum_{\boldsymbol{u}\in\mathscr{C}} \prod_{i=1}^{n} \left(\frac{1-x_i}{1+x_i}\right)^{u_i} (1+x_i) \\ &= \frac{1}{|\mathscr{C}|} \prod_{j=1}^{n} (1+x_j) \sum_{\boldsymbol{u}\in\mathscr{C}} \prod_{i=1}^{n} \left(\frac{1-x_i}{1+x_i}\right)^{u_i} = \frac{1}{|\mathscr{C}|} \prod_{j=1}^{n} (1+x_j) W_{\mathscr{C}}(\boldsymbol{x}')\end{aligned} \tag{8.187}$$

where $\boldsymbol{x}' = [x'_1, x'_2, \ldots, x'_n]$ with $x'_i = \frac{1-x_i}{1+x_i}$.

9

Conclusions and Outlook

This thesis investigated low-complexity decoding algorithms for binary, non-binary, and generalized LDPC codes.

For binary LDPC codes and quantized message passing, optimized code ensembles were obtained that perform close to theoretical Shannon limits at high code rates. Our decoders provide a trade-off between decoding complexity and coding gain.

For non-binary LDPC codes, we considered different channel models: the QSC, QEC, the AWGN and Poisson channels with PPM modulation where the modulation order matches the field size. We introduced several decoding algorithms where the exchanged messages are: symbols from $\mathbb{F}_q$, lists of symbols from $\mathbb{F}_q$, or symbols from $\mathbb{F}_q$ together with their reliability scores. A DE analysis shows how our decoding algorithms improve decoding thresholds, closing the gap to the performance achieved by the SPA.

For GLDPC codes, we studied quantized decoding algorithms where the exchanged messages are binary or ternary, and the VN decoder can exploit soft information from the channel. At the CNs, the binary and ternary messages are obtained either by using BDD or by using optimum APP SISO decoding. In the latter case, the component decoder soft-output (i.e., the extrinsic likelihood ratios) is mapped to messages from the desired binary/ternary alphabet.

For future work, it would be interesting to derive a theoretical limit of quantized decoders (other than the Shannon limit) that takes the quantization of the exchanged messages into account. Further, it is well known that the thresholds of spatially coupled codes under SPA saturate at the maximum a-priori threshold. We observed that spatial coupling increases the iterative decoding thresholds (for instance, under TMP, QMP, and SMP) to specific values. The threshold saturation effect was observed numerically. A promising direction is determining what these values represent and proving the saturation theoretically.

We have also studied the error floor performance of binary, non-binary, and generalized LDPC codes. We reviewed the matrix enumeration method used in previous works to obtain the TS, AS, and FAS enumerators for binary regular LDPC code ensembles. We extended the method to irregular binary ensembles, and we explained that the technique is limited to unstructured binary ensembles, i.e., it cannot be applied to non-binary codes, structured ensembles, and GLDPC code ensembles. Therefore, we proposed generating functions to obtain the TS, AS, and FAS enumerators. We derived the TS and (elementary) AS enumerators for unstructured and (constrained and unconstrained) protograph-based non-binary LDPC code ensembles. Further, we proposed new definitions of the (elementary) TSs and (fully) ASs for GLDPC codes. Experimental results show that the proposed definitions yield graph structures that are harmful for bit flipping decoders. Future works can use the derived analysis to design LDPC and GLDPC codes free of harmful configurations.

10 Acronyms

AWGN additive white Gaussian noise

AR4JA accumulate-repeat-4-jagged-accumulate

BP belief propagation

MAP maximum a posteriori

SISO soft-input soft-output

SPA sum product algorithm

PMF probability mass function

CDF cumulative distribution function

PDF probability density function

LDPC low-density parity-check

FER frame error rate

SER symbol error rate

BICM bit-interleaved coded modulation

SNR signal-to-noise ratio

DE density evolution

LLR log-likelihood ratio

BER bit error rate

SER symbol error rate

biAWGN binary-input additive white Gaussian noise

PEG progressive edge-growth

LLR log-likelihood ratio

RV random variable

SC-LDPC spatially coupled low-density parity-check code

BSC binary symmetric channel

BEEC binary error and erasure channel

CN check node

VN variable node

TMP ternary message passing

QMP quaternary message passing

BMP binary message passing

QSC q-ary symmetric channel

QEEC q-ary error and erasure channel

SMP symbol message passing

AS absorbing set

EAS elementary absorbing set

TS trapping set

FAS fully absorbing set

EFAS elementary fully absorbing set

ETS elementary trapping set

DMC discrete memoryless channel

QMS Quantized Min-Sum

GLDPC generalized low-density parity-check

SPC single parity-check

APP a posteriori probability

SRLMP scaled reliability list message passing

U-NBPB unconstrained non-binary protograph-based

C-NBPB constrained non-binary protograph-based

CVWEF composition vector weight enumerator function

WEF weight enumerator function

VWEF vector weight enumerating function

SS stopping set

BDD bounded distance decoding

PBF parallel bit flipping

BCH Bose-Chaudhuri-Hocquengham

BSC binary symmetric channel

llv $\boldsymbol{L}$-vector

DMC discrete memoryless channel

PPM pulse position modulation

MQMS *matched* quantized min-sum

QMS quantized min-sum

RSMP reliability-based symbol message passing

QEC q-ary erasure channel

SEMP symbol and erasure message passing

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