FAKULTÄT FÜR INFORMATIK
TECHNISCHE UNIVERSITÄT MÜNCHEN

Complete Subgraphs of Random Graphs

Thomas Schickinger

FAKULTÄT FÜR INFORMATIK
TECHNISCHE UNIVERSITÄT MÜNCHEN

Lehrstuhl für Theoretische Informatik

Complete Subgraphs of Random Graphs

Thomas Schickinger

Vollständiger Abdruck der von der Fakultät für Informatik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. Dr. h.c. Wilfried Brauer
Prüfer der Dissertation:		
	1.	Univ.-Prof. Dr. Angelika Steger
	2.	Univ.-Prof. Dr. Hans-Jürgen Prömel, Humboldt-Universität zu Berlin

Die Dissertation wurde am 21.08.2002 bei der Technischen Universität München eingereicht und durch die Fakultät für Informatik am 19.12.2002 angenommen.

Abstract

A classical theorem by Erdős, Kleitman and Rothschild on the structure of triangle-free graphs states that with high probability such graphs are bipartite. Our first main result refines this theorem by investigating the structure of the 'few' triangle-free graphs which are not bipartite. We prove that with high probability these graphs are bipartite up to a few vertices. Similar results hold if we replace triangle-free by $K_{\ell+1}$-free and bipartite by ℓ-partite.

In our second main result we examine the class of ε-regular graphs in the context of the famous Regularity Lemma by Szemerédi. Whereas the case of dense graphs is well understood, the application of the Regularity Lemma for sparse random graphs still lacks an important keystone. This led to a conjecture by Kohayakawa, Łuczak and Rödl, which is considered one of the most important open problems in the theory of random graphs. The conjecture states that a fixed subgraph H occurs with extremely high probability in sufficiently dense ε-regular graphs. We prove this conjecture for the subgraphs $H = K_4$ and $H = K_5$.

Acknowledgments

First and foremost I would like to express my sincere and deep gratitude to my supervisor Angelika Steger. Her enthusiasm for theory aroused my own interest in the subject, her advice led me to the right problems and her continuous support finally enabled me to solve them.

The cooperative and even amicable atmosphere at the Chair for Efficient Algorithms helped me to enjoy my work and to alleviate my frustration when yet another proof had crashed. Many of my colleagues have become friends over time. I am indebted to Ernst W. Mayr for his unselfish support and generosity. Special thanks go to Stefanie Gerke for untiringly pushing on when all approaches had failed and for posing questions I had lost the drive to ask. In addition to Angelika and Steffi, I had the pleasure to work with Stefan Bischof and Mark Scharbrodt as coauthors. The cookies and the tea of my room mate Hanjo Täubig were an indispensable prerequisite of this work, and also the cheerful comments of my former room mate and good friend Alex Hall. Alex, Hanjo, Steffi and Uli Voll deserve an especially cordial 'thank you' for volunteering to proof-read parts of this thesis. Among my other colleagues I feel particularly obliged to Thomas Erlebach, Volker Heun and Klaus Holzapfel, who will hopefully all know why.

This thesis has strongly been influenced by our long standing excellent relationship to Humboldt-Universität zu Berlin. With best thanks to my Berlin coauthors Hans Jürgen Prömel and Anusch Taraz, I remember the inspiring visits to HU Berlin and many fruitful discussions. I additionally owe a special acknowledgment of my gratitude to Hans Jürgen Prömel for his invaluable support.

Finally, writing this page makes me think of the people I love and whose love has accompanied me up to here. I am happy and deeply grateful that you are there!

Contents

1

Introduction

The importance of graphs in theoretical computer science and discrete mathematics cannot probably be overrated. It is needless to count, for instance, the numerous algorithmic applications, where graphs play an essential rôle. Many real-world structures naturally map to graphs, which explains the outstanding practical impact of graph theoretical results. And, of course, besides their influence on many fields of computer science, graphs hold an eminent place in discrete mathematics. Their importance as fundamental combinatorial structure is highlighted by the beautiful and deep results which the research in this field has produced in great number.

An important branch of graph theory studies the structure of random graphs, e.g., the well-known models $G_{n,p}$ or $G_{n,m}$. Since the pioneering work by Erdős and Renyi around 1960 a rich theory on this subject has been developed (for an account on the history of random graph theory we refer the reader to the monographs [Bol85] and [JŁR00]). Random graph theory has become an invaluable tool, e.g., in the construction of graphs with specific properties, in average case analysis, and in the design of algorithms.

Already in the seminal paper [ER60] the occurrence of fixed subgraphs has been studied, and today this problem is well-understood for $G_{n,p}$ and $G_{n,m}$. However, for certain applications these models do not suffice. Whereas classical random graph theory deals with the problem

> *What is the structure of a typical, i.e., random graph (with a given density)?*

it might be necessary to focus on graphs with a given property $\mathcal{P}$. Given such a property $\mathcal{P}$ like, e.g., 'the graph is planar' or 'the graph does not contain a triangle', we would like to answer the question

> *What is the structure of a typical graph (with a given density) that satisfies property $\mathcal{P}$?*

Formally speaking, we draw a graph uniformly at random from the family of all graphs which satisfy $\mathcal{P}$. In the sequel we will refer to such random graphs as *structural random graphs*. In comparison to the extensive literature on $G_{n,p}$ and $G_{n,m}$, there are only few results on structural random graphs. This might be due to the fact that many tools from probability theory, which are so successful in $G_{n,p}$ and $G_{n,m}$, cannot *a priori* be applied to structural random graphs. Note that structural random graphs are not defined by a sequence of elementary random experiments like choosing the edges in $G_{n,p}$. Thus the underlying probability space tends to be rather difficult to capture.

In this thesis we will discuss problems concerning structural random graphs where the analysis is based on *direct counting* of the graphs under consideration. For one of our results this even leads to an order of magnitude of the probabilities which could not be achieved using classical probabilistic bounds. In an impressive way this shows the power of direct counting.

Hence, from a technical point of view, direct counting represents the thread which interweaves this thesis. Our specific results deal with the occurrence of complete subgraphs K_ℓ in random graphs.

For a graph $G = (V, E)$ the clique number $\omega(G) := \max\{\ell \in \mathbb{N} \mid K_\ell \subseteq G\}$, i.e., the size of the largest complete subgraph, is a frequently studied structural parameter. It is thus natural to ask for the typical structure of a graph, given a bound on $\omega(G)$. This subject will be treated in Chapter 3, where we investigate *random K_ℓ-free graphs*. A classical result by Erdős, Kleitman and Rothschild [EKR76] and its generalization by Kolaitis, Prömel and Rothschild [KPR87] state that random $K_{\ell+1}$-free graphs are almost always ℓ-partite. We further refine these results by looking at the family of $K_{\ell+1}$-free graphs which are not ℓ-partite. It turns out that this family of graphs exhibits a distinct hierarchical structure.

Via a conjecture by Kohayakawa, Łuczak and Rödl [KŁR97] the research on the evolution of random H-free graphs (i.e., the characterization of their structure depending on the number of edges in the graph) is closely interconnected to the structure of *random ε-regular graphs*. Consider a fixed graph H with chromatic number $\chi(H)$. If H can be shown to occur with extremely small (so-called subexponentially small) probability as a subgraph in a random ε-regular graphs, this implies that random H-free graphs are $(\chi(H) - 1)$-partite up to an arbitrarily small fraction of edges [Łuc00].

But the implications of the aforementioned conjecture extend far beyond the evolution of H-free graphs. Generally speaking, this conjecture represents the missing keystone for applying the famous *Regularity Lemma* by Szemerédi [Sze76] to sparse random graphs. In the monograph [JŁR00] this conjecture was therefore termed *one of the most important open questions in the theory of random graphs*. In Chapter 4 we prove the two 'smallest' open cases of this conjecture, namely, $H = K_4$ and $H = K_5$.

Outline of this thesis In Chapter 2 we clarify some notation. Since we mostly employ standard terminology, the reader may prefer to consult this part of the thesis only on demand.

In the subsequent two chapters we discuss the results sketched above. Chapter 3 deals with the structure of typical K_ℓ-free graphs. Then in Chapter 4 our results on the conjecture by Kohayakawa, Łuczak and Rödl are presented. Both chapters contain a section on the background of our results which is meant to provide the reader which is not familiar with the topic with a brief, intuitive overview.

2

Preliminaries

Graphs Let $G = (V, E)$ be a graph. We use the notation $v_G := |V|$ and $e_G := |E|$. A graph with $v_G = n$ and $e_G = m$ will be called an (n, m)-*graph*.

$\Gamma(v)$ denotes the *neighborhood* of vertex $v \in V$. For sets of vertices $S \subseteq V$ we let $\Gamma(S) := \bigcup_{v \in S} \Gamma(v) \setminus S$. In general, $\Gamma^i(S)$ denotes the *ith neighborhood* of a set $S \subseteq V$, i.e., $\Gamma^0(S) = S$ and

$$\Gamma^{i+1}(S) = \{v \in V \setminus [\Gamma^0(S) \cup \ldots \cup \Gamma^i(S)] \mid \exists x \in \Gamma^i(S), \{x, v\} \in E\}$$

for $i \geq 0$.

The *density* of the (bipartite) subgraph introduced between two disjoint vertex sets $A, B \subseteq V$ is given by

$$d(A, B) = \frac{e(A, B)}{|A| \cdot |B|},$$

where $e(A, B) := |\{\{u, v\} \in E \mid u \in A \ \wedge \ v \in B\}|$ denotes the number of edges between A and B. We will also use the abbreviation $E(A, B) := \{\{u, v\} \in E \mid u \in A \ \wedge \ v \in B\}$.

We introduce identifiers for certain types of graphs. The circle on n vertices is given by C_n. The complete graph on n vertices is denoted by K_n. The complete ℓ-partite graph K_ℓ^p consists of ℓ partitions of p vertices such that all edges between all pairs of partitions are present but there is no edge inside the partitions.

For $W \subseteq V$ let $G[W]$ denote the subgraph *induced* by W, i.e., $G[W]$ consists of the vertex set W and the edge set $\binom{W}{2} \cap E$. In contrast to that we say that

$H = (V', E')$ is a *weak subgraph* if $V' \subseteq V$ and $E' \subseteq E$. In this case write $H \subseteq G$. If $H \nsubseteq G$, we say that G is *H-free*.

For a graph G we let $\chi(G)$ denote the *chromatic number* of G.

By $G_{n,p}$ we denote, the binomial *random graph* on n vertices, where each edge occurs independently with probability p. Furthermore, let $G_{n,m}$ be a graph on n vertices and m edges drawn uniformly at random. We say that a random graph satisfies a certain property *asymptotically almost surely* if this property occurs with probability $1 - o(1)$.

Miscellaneous With $\log n = \log_2 n$ we denote the logarithm to base 2. For $i \geq 1$ the *iterated logarithm* $\log^{(i)}(n)$ is recursively defined by $\log^{(1)}(n) = \log(n)$ and $\log^{(i+1)}(n) = \log(\log^{(i)}(n))$.

Let $n^{\underline{k}} := n!/(n-k)! = n(n-1)\dots(n-k+1)$ denote the *k-th falling factorial*. A binomial coefficient $\binom{n}{k}$ may then be written as $\binom{n}{k} = n^{\underline{k}}/k!$.

For binomial coefficients we will often use the following inequality which follows immediately from *Vandermonde's convolution*:

$$\binom{a}{x}\binom{b}{y} \leq \binom{a+b}{x+y},$$

which is immediately clear due to combinatorial arguments (Instead of choosing x elements from a set of cardinality a and then y elements from a set of cardinality b, we pick in total $x + y$ from the union of both sets. This leads to over-counting, since we do not have to choose exactly x elements from a in this case.).

$H(x)$ denotes the *entropy function* and is defined as

$$H(x) := -x \log x - (1-x)\log(1-x).$$

This function is used in the well known bound

$$\binom{n}{\lambda n} \leq 2^{nH(\lambda)}.$$

3

Structure of K_ℓ-free graphs

3.1 Introduction

What is a random triangle-free graph? – In 1976 Erdős, Kleitman and Rothschild [EKR76] showed that if one conditions a random graph $G_{n,\frac{1}{2}}$ on being triangle-free, then such a graph is almost always bipartite, i.e.,

$$\Pr[\, G \text{ bipartite } \mid G = G_{n,\frac{1}{2}} \text{ is triangle-free} \,] = 1 - o(1). \tag{3.1}$$

Recall that the probability space of random graphs $G_{n,\frac{1}{2}}$ corresponds to the uniform distribution on the set of all graphs with n vertices.

While (3.1) surely answers the question of what the structure of a random triangle-free graph is, it is also in some sense unsatisfactory, as, clearly, a bipartite graph does not capture many interesting properties a triangle-free graph might have. Consider, for instance, the famous result of Erdős [Erd59] which shows that there exist graphs with high chromatic number and high girth (size of the smallest cycle). Hence, in particular, triangle-free graphs may have a high chromatic number instead of being bipartite.

In this chapter we thus investigate the structure of those triangle-free graphs which are not bipartite. As it turns out, this class is surprisingly well-structured. Let us call a graph $G = (V, E)$ *i-quasibipartite* if and only if there exist i vertices $v_1, \ldots, v_i \in V$ such that $G[V \setminus \{v_1, \ldots, v_i\}]$ is bipartite. Then we prove

- almost all triangle-free graphs which are not bipartite are 1-quasibipartite,

- almost all triangle-free graphs which are not bipartite or 1-quasibipartite are 2-quasibipartite,

and in general for all $i \geq 1$

- almost all triangle-free graphs which are not i'-quasibipartite for $i' \leq i$ are $i+1$-quasibipartite.

We also show that similar results hold for the class of all graphs which do not contain a clique $K_{\ell+1}$. For these graphs a hierarchy of i-quasi-ℓ-partite graphs can be found, extending a result of Kolaitis, Prömel and Rothschild [KPR87].

The results presented in this chapter have appeared in [PSS02] and [PSS01].

Outline of this chapter In Section 3.2 we briefly state the main result of this chapter. Section 3.3 introduces the Kleitman-Rothschild method which is essential for the proof of our results. Thereafter we discuss related results in Section 3.4. In particular, we review the current knowledge on the structure of typical H-free graphs (including their evolution) and point out algorithmic applications of these results. Section 3.5 presents a self-contained proof of our main result limited to triangle-free graphs. Due to its simplicity we believe that it has an instructive value on its own. The proof of the main result in full generality then follows in Section 3.6.

3.2 Main result

In order to state our result precisely we introduce some notation. Throughout the rest of the chapter we assume that ℓ is an arbitrary, but fixed integer with $\ell \geq 2$.

We consider graphs with labeled vertices. With $\mathcal{F}_0^\ell(n)$ we denote the set of all $K_{\ell+1}$-free graphs on n vertices, and the set $\mathcal{P}_0^\ell(n)$ contains all ℓ-partite graphs. Using these abbreviations the result from Kolaitis, Prömel and Rothschild [KPR87] can be stated as follows.

Theorem 3.1 ([KPR87]) *For $\ell \geq 2$ almost all $K_{\ell+1}$-free graphs are ℓ-partite, i.e.,*

$$|\mathcal{F}_0^\ell(n)| = (1+o(1))|\mathcal{P}_0^\ell(n)|.$$

Note that the special case $\ell = 2$ corresponds to the result of Erdős, Kleitman and Rothschild [EKR76].

We extend this result as follows: Let $\mathcal{P}_i^\ell(n)$ denote the set of all $K_{\ell+1}$-free graphs $G = (V, E)$ on n vertices which can be made ℓ-partite by removing i vertices but not by removing only $i - 1$ vertices, i.e., there exist vertices $v_1, \dots, v_i \in V$ such that $G[V \setminus \{v_1. \dots, v_i\}] \in \mathcal{P}_0^\ell(n-i)$, whereas for all choices of vertices $v'_1, \dots, v'_{i-1} \in V$ we have $G[V \setminus \{v'_1, \dots, v'_{i-1}\}] \notin \mathcal{P}_0^\ell(n - (i - 1))$. We call these $K_{\ell+1}$-free graphs *i-quasi-ℓ-partite*.

Additionally, we define the set $\mathcal{F}_i^\ell(n) := \mathcal{F}_0^\ell(n) \setminus [\mathcal{P}_0^\ell(n) \cup \ldots \cup \mathcal{P}_{i-1}^\ell(n)]$, i.e., $\mathcal{F}_i^\ell(n)$ contains all $K_{\ell+1}$-free graphs which cannot be made ℓ-partite by removing at most $i - 1$ vertices. For these graphs the following extension to Theorem 3.1 can be shown.

Theorem 3.2 *For all $i \geq 0$ and $\ell \geq 2$,*

$$|\mathcal{F}_i^\ell(n)| = (1 + o(1))|\mathcal{P}_i^\ell(n)|.$$

3.3 Background: Kleitman-Rothschild method

3.3.1 Motivation

For simplicity's sake let us concentrate for the moment on the case $i = 0$ and $\ell = 2$, i.e., we consider triangle-free and bipartite graphs. For brevity of notation we let $\mathcal{F}_i(n) := \mathcal{F}_i^2(n)$ and $\mathcal{B}_i(n) := \mathcal{P}_i^2(n)$ for $i \geq 0$. With this notation the famous result of [EKR76] reads

$$\Pr[G \in \mathcal{B}_0(n) \mid G = G_{n,\frac{1}{2}} \in \mathcal{F}_0(n)] = 1 - o(1), \tag{3.2}$$

where G is a random graph $G_{n,\frac{1}{2}}$.

Typical tools in probability theory are designed for probability spaces which are defined as the product of several independent random experiments, or, more generally, as random processes. Unfortunately, no step by step construction of graphs in $\mathcal{F}_0(n)$ is known, and apparently such a construction ought to have a rather intricate structure. This is due to the fact that the condition $G \in \mathcal{F}_0(n)$ introduces complicated dependencies between the occurrences of different edges. A process that produces a random graph $G \in \mathcal{F}_0(n)$ would have to keep track of these dependencies, which seems to be a quite difficult task.

Moreover, if we were able to devise such a random construction of graphs in $\mathcal{F}_0(n)$, this would presumably yield a proof for (3.2), too. Due to $\mathcal{B}_0(n) \subseteq \mathcal{F}_0(n)$ it suffices to show that

$$|\mathcal{F}_0(n)| \le (1 + o(1)) \cdot |\mathcal{B}_0(n)|. \tag{3.3}$$

The two abilities to randomly construct a combinatorial object and to count the number of such objects usually go hand in hand. It is typically possible to switch between these two sides of the medal by exchanging the random choices with bounds on the number of possibilities.

Note that $|\mathcal{B}_0(n)|$ is rather easy to determine. The upper bound $|\mathcal{B}_0(n)| \le 2^n \cdot 2^{n^2/4}$ immediately follows from the observation that there are 2^n possibilities to choose a 2-coloring of V. Furthermore, the number of possible choices for the edges between two color classes of size x and $n-x$ is bounded by $2^{x(n-x)} \le 2^{n^2/4}$. A rough lower bound on $|\mathcal{B}_0(n)|$ can also be obtained without too much technical effort (cf. e.g. [PS96b]). In [PS95] the following quite precise estimate for the number of arbitrary ℓ-partite graphs $|\mathcal{P}_0^\ell(n)|$ has been given (see also [Prö86]).

Theorem 3.3 *[PS95] For $\ell \ge 2$ we have*

$$|\mathcal{P}_0^\ell(n)| = \Theta(2^{\frac{\ell-1}{2\ell}n^2 + n\log \ell - \frac{\ell-1}{2}\log n}).$$

□

Clearly, for bipartite graphs we obtain $|\mathcal{B}_0(n)| = \Theta(2^{\frac{1}{4}n^2 + n - \frac{1}{2}\log n})$.

In brief, when we want to prove (3.3) and thus (3.2), we face the following situation. We have to compare the cardinalities of the graph classes $\mathcal{F}_0(n)$ and $\mathcal{B}_0(n)$, where we know precise bounds for $|\mathcal{B}_0(n)|$. However, there is not much hope to determine $|\mathcal{F}_0(n)|$.

In the sequel we will describe the Kleitman-Rothschild method, which uses an approach that involves neither probabilistic tools nor counting of the absolute value of $|\mathcal{F}_0(n)|$. Instead, this method uses induction to compare the rate of growth of $|\mathcal{F}_0(n) \setminus \mathcal{B}_0(n)|$ and $|\mathcal{B}_0(n)|$. If the former cardinality, interpreted as a function of n, can be shown to grow much slower than $|\mathcal{B}_0(n)|$, this will suffice to prove (3.3).

3.3.2 Technique

In a sequence of two papers ([KR70] and [KR75]) Kleitman and Rothschild succeeded in counting the number of different partial orders on a set of n

(labeled) elements. The method they invented for proving this result, subsequently also called the *Kleitman-Rothschild method*, is based on a clever inductive counting argument. Even though the Kleitman-Rothschild method was first used for partial orders, its highlights were within graph theory. Here the starting paper was by Erdős, Kleitman and Rothschild [EKR76] who proved that almost all triangle-free graphs are bipartite (cf. (3.2) and (3.3)). In the following years this result was generalized in many respects (cf. e.g. [LR84], [KPR87], [PS92c], [PS92a]). For further discussion of the method and its history, we refer the interested reader to [Ste93] or [PST01].

The Kleitman-Rothschild method is applicable in situations like the following. Let two families of sets $\mathcal{X}_n$ and $\mathcal{Y}_n$ be given such that $G \in \mathcal{Y}_n$ implies $G \in \mathcal{X}_n$. For the sake of concreteness we will think of these sets as families of graphs, where the parameter n denotes the number of vertices.

We intend to show that

$$|\mathcal{X}_n| = (1 + o(1)) \cdot |\mathcal{Y}_n|. \tag{3.4}$$

Assume that we have rather good estimates on $|\mathcal{Y}_n|$, but we do not know how to bound $|\mathcal{X}_n|$. In classical applications of the Kleitman-Rothschild method the actual goal is to estimate $|\mathcal{X}_n|$. To this aim one tries to guess the structure of most graphs in $\mathcal{X}_n$ and defines a corresponding family $\mathcal{Y}_n$ such that $|\mathcal{Y}_n|$, hopefully, can then be counted more easily. Together with (3.4) this yields the desired bound on $|\mathcal{X}_n|$.

For (3.4) it suffices to prove that there exist only very few graphs in $\mathcal{X}_n \setminus \mathcal{Y}_n$, or, more precisely, $|\mathcal{X}_n \setminus \mathcal{Y}_n|/|\mathcal{Y}_n| = o(1)$. This is achieved by the following two steps:

1. Define suitable *bad sets* $\mathcal{B}_n^1, \ldots, \mathcal{B}_n^k \subseteq \mathcal{X}_n$, for which it can be shown that
$$\mathcal{X}_n \setminus \mathcal{Y}_n \subseteq \mathcal{B}_n^1 \cup \ldots \cup \mathcal{B}_n^k.$$
2. Prove that
$$|\mathcal{B}_n^i|/|\mathcal{Y}_n| = o(1) \quad \text{for } i = 1, \ldots, k$$
using *inductive counting*.

The inductive counting approach used in the second step represents the core idea of the Kleitman-Rothschild method. In order to prove that $|\mathcal{B}_n^i|$ is much smaller than $|\mathcal{Y}_n|$ it is not necessary to estimate the absolute cardinalities of the sets. Instead it suffices to obtain bounds on the *growth rates*

$$\frac{|\mathcal{Y}_n|}{|\mathcal{Y}_{n-x}|} \quad \text{and} \quad \frac{|\mathcal{B}_n^i|}{|\mathcal{X}_{n-x}|} \quad \text{for suitable } x \in \mathbb{N}.$$

A bound of the form

$$\frac{|\mathcal{Y}_n|}{|\mathcal{Y}_{n-x}|} \geq g(n) \tag{3.5}$$

is usually straightforward, as we have assumed that $|\mathcal{Y}_n|$ is comparatively easy to count.

In classical applications of the Kleitman-Rothschild method the graph family $\mathcal{X}_n$ is *decreasing*, i.e., $G \in \mathcal{X}_n$ implies that $H \in \mathcal{X}_n$ for all subgraphs $H \subseteq G$. As a typical example for a decreasing family consider the set of triangle-free graphs $\mathcal{F}_0(n)$.

Let $G = (V, E) \in \mathcal{X}_n$. The definition of a bad set $\mathcal{B}_n^i$ typically involves properties of a small part of G, e.g., a single vertex or a (small) set $W \subseteq V$. As $\mathcal{X}_n$ is decreasing, it follows that $G' := G[V \setminus W] \in \mathcal{X}_{n-x}$ for $x := |W|$. Hence a bound of the form

$$\frac{|\mathcal{B}_n^i|}{|\mathcal{X}_{n-x}|} \leq f(n) \tag{3.6}$$

can be obtained by counting the number of possibilities to combine W and $G' \in \mathcal{X}_{n-x}$ such that the resulting graph belongs to $\mathcal{B}_n^i$. A clever definition of $\mathcal{B}_n^i$ implies severe restrictions on these possibilities and thus a small bound $f(n)$ can be shown. Combining (3.5) and (3.6) we conclude that

$$\frac{|\mathcal{B}_n^i|}{|\mathcal{Y}_n|} \leq \frac{|\mathcal{B}_n^i|}{|\mathcal{X}_{n-x}|} \cdot \frac{|\mathcal{X}_{n-x}|}{|\mathcal{Y}_{n-x}|} \cdot \frac{|\mathcal{Y}_{n-x}|}{|\mathcal{Y}_n|} \leq \frac{f(n)}{g(n)} \cdot (1 + o(1)),$$

where the bound on $|\mathcal{X}_{n-x}|/|\mathcal{Y}_{n-x}|$ follows by induction. If $f(n)/g(n) = o(1)$, the claim is proved.

Our application of the Kleitman-Rothschild method which will be presented in the remainder of the chapter differs in one aspect from the 'standard' approach described above. Observe that the set $\mathcal{F}_i(n)$ with $i \geq 1$ is, in contrast to $\mathcal{F}_0(n)$, not decreasing. We cope with this problem using two different techniques. Firstly, we derive a rough estimate on the ratio $\mathcal{F}_i(n)/\mathcal{F}_0(n)$. Thus we can compare the size of the bad sets to $|\mathcal{F}_0(n)|$ as in the standard approach, which will suffice for certain parts of the proof. Secondly, for other parts where higher accuracy is needed, we define the bad sets and the vertices W which are removed for the inductive counting in such a way that the remaining graph $G[V \setminus W]$ still belongs to $\mathcal{F}_i(n)$. This can be achieved although $\mathcal{F}_i(n)$ is not decreasing *in general*, i.e., for arbitrary subgraphs W.

3.4 Related work

Typical H-free graphs As we have already mentioned, the structure of triangle-free graphs was first investigated in the seminal paper [EKR76]. Later this result was generalized in [KPR87] to K_ℓ-free graphs for arbitrary

$\ell \geq 3$ (cf. Theorem 3.1). Written in a probabilistic style the main result of [KPR87] reads

$$\Pr[G \text{ is } \ell\text{-partite} \mid G = G_{n,\frac{1}{2}} \text{ is } K_{\ell+1}\text{-free}] = 1 - o(1).$$

In addition to that, similar results have been proved for certain non-complete subgraphs. In [LR84] almost all C_ℓ-free graphs are shown to be bipartite for every odd integer ℓ. Finally, in [PS92b] a characterization of graphs H is given such that almost all H-free graphs are ℓ-partite (see also [HPS93] for a further refinement of this result).

For arbitrary graphs H the number of H-free graphs has been bounded by $2^{(1-\frac{1}{\chi(H)-1})\frac{n^2}{2}+o(n^2)}$ in [EFR86]. However, this bound is weaker than the bounds for certain classes of H-free graphs which are implied by the results mentioned above in combination with Theorem 3.3. For instance, the number of K_ℓ-free graphs for $\ell \geq 3$ is thus of order $\Theta(2^{\frac{l-2}{2l-2}n^2+n\log(l-1)-\frac{l-2}{2}\log n})$, whereas [EFR86] only yields the estimate $2^{\frac{l-2}{2l-2}n^2+o(n^2)}$ in this case.

In addition to this rich theory on H-free graphs in the classical sense, where *weak subgraphs* H are forbidden, also graphs with forbidden *induced subgraphs* H have been studied. Interestingly, the results for such graphs closely parallel the results for forbidden weak subgraphs (cf. [PS91] [PS92a] [PS92d] [PS93b]). In [PS93a] a survey and comparison of the results for forbidden induced and weak subgraphs is given.

Algorithmic applications Techniques and results like the ones we have listed above also gain influence which extends well beyond the field of 'pure' graph theory, focusing on structural insight. Random graphs have had significant impact on the average case analysis of algorithms and on the quest for hard instances, e.g., of $\mathcal{NP}$-complete problems.

In [Wil84] it has been shown that graph-coloring, a classical $\mathcal{NP}$-complete problem, may be solved for random graphs in constant expected time. If we wish to decide whether a graph is ℓ-colorable, we simply search for cliques $K_{\ell+1}$. For a typical random graph $G_{n,\frac{1}{2}}$ such a clique can be found by looking at just a few vertices. Of course, this suffices to show that the graph is not ℓ-colorable. For graphs which pass this test without a $K_{\ell+1}$ being found the problem is solved by complete enumeration. Since there are very few such graphs, this still leads to constant execution time on the average.

Although elegant and simple, this 'solution' of the coloring problem is in some sense unsatisfactory and leads to a natural question. How crucial is the rôle of cliques, i.e., can we still achieve good performance on the average if the graph is $K_{\ell+1}$-free? In [PS92c] (cf. also [DF89]) this question has been answered in the affirmative by presenting an algorithm with expected running time $\mathcal{O}(n^2)$ on random K_ℓ-free graphs. The proof of this result makes substantial use of the Kleitman-Rothschild method.

Evolution of H-free graphs Up to now we have discussed results on the structure of a typical H-free graph, i.e., the structure of a graph randomly chosen from the set of *all* H-free graphs. Hence note that such results do not tell us much about the structure of *sparse* H-free graphs, i.e., graphs with $o(n^2)$ edges, since such graphs are usually outnumbered by *dense* graphs with $\Theta(n^2)$ edges.

Thus a natural direction of further research is to study the *evolution* of H-free graphs. Here the number of edges m is introduced as a new parameter (in addition to the number of vertices n), and the structure of H-free graphs on n vertices with m edges is investigated. This follows the usual approach in classical random graph theory, introduced in the seminal paper [ER60], where the structure of random graphs $G_{n,m}$ depending on m is studied.

A first result for this line of research was given in [PS96b], where the evolution of triangle-free graphs was examined. However, the case that the order of magnitude of m lies between $n^{3/2}$ and $n^{7/2}$ remained open. Finally, this problem was solved in [OPT].

Theorem 3.4 *[OPT]* (Evolution of K_3-free graphs) *Let* $t_3 = \frac{\sqrt{3}}{4} n^{3/2}\sqrt{\log n}$. *For* $\varepsilon > 0$ *we have*

$$\Pr[G_{n,m} \textit{ is bipartite} \mid G_{n,m} \textit{ is } K_3\textit{-free}] = \begin{cases} 1 - o(1) & \textit{if } m = o(n), \\ o(1) & \textit{if } \frac{n}{2} \le m \le (1-\varepsilon)t_3, \\ 1 - o(1) & \textit{if } (1+\varepsilon)t_3 \le m. \end{cases} \qquad \square$$

Remark 3.5 *The main result of [OPT] is actually stronger than the statement of Theorem 3.4. Similar thresholds t_ℓ are shown for arbitrary odd cycles C_ℓ.*

For the proof of Theorem 3.4 a result from [Łuc00] is employed, which states that all triangle-free graphs with $m \ge Cn^{3/2}$ edges are almost bipartite, i.e., they can be made bipartite by deleting $\lfloor \delta m \rfloor$ edges. Here for any $\delta > 0$ we can find a sufficiently large constant C such that this claim holds.

This result of [Łuc00] is actually much stronger, as it is formulated for arbitrary H-free graphs, given any fixed graph H. However, the proof is based on a conjecture by Kohayakawa, Łuczak and Rödl (cf. Conjecture 4.19 which will be studied in Chapter 4), which has only been verified for the case that H is a tree or a cycle (cf. Section 4.3.2).

For a formal statement of the result in [Łuc00] we call a graph (δ, ℓ)*-partite* if it can be made ℓ-partite by deleting at most δm edges. Furthermore, for an arbitrary graph H let

$$d_2(H) := \max\left\{\frac{e_F - 1}{v_F - 2} \mid F \subseteq H, v_F > 2\right\} \tag{3.7}$$

denote the 2-*density* of H. Observe that

$$d_2(K_\ell) = \frac{\binom{\ell}{2} - 1}{\ell - 2} = \frac{1}{2}\frac{\ell^2 - \ell - 2}{\ell - 2} = \frac{1}{2}(\ell + 1),$$

and thus $d_2(K_3) = 2$. Now the result in [Łuc00] reads as follows.

Theorem 3.6 *[Łuc00] Let H be a graph with $\chi(H) = \ell + 1 \geq 3$ for which Conjecture 4.19 holds. Then for every $\delta > 0$ there exists $C > 0$ such that*

$$\Pr[G \text{ is } (\delta, \ell)\text{-partite} \mid G = G_{n,m} \text{ is } H\text{-free}] = 1 - o(1),$$

provided that $m \geq Cn^{2-1/d_2(H)}$. □

Theorem 3.6 shows that there is a strong connection between the structure of H-free graphs treated in this chapter and the occurrence of subgraphs $H \subseteq G$ in so-called *ε-regular graphs* treated in Conjecture 4.19 which we will study in Chapter 4. Hence, progress in proving Conjecture 4.19 provides new tools for attacking the evolution of H-free graphs. In Chapter 4 we prove Conjecture 4.19 for $H = K_4$ and $H = K_5$.

There is also a second promising approach for studying the evolution of H-free graphs. In [Ste99] an alternative proof is given (up to constants) for the crucial part of Theorem 3.4, i.e, the second 1-statement (This statement represents the actual breakthrough achieved by [OPT], as the other parts of Theorem 3.4 were essentially proved before.). This alternative proof uses the Kleitman-Rothschild method, which will also represent our most important tool in this chapter.

The survey [PST01] summarizes the interrelations between counting combinatorial objects, analyzing their typical structure and studying their evolution.

Typical structure of untypical K_ℓ-free graphs Among the above mentioned previous research the main result presented in this chapter takes the following position. Primarily Theorem 3.2 generalizes Theorem 3.1 by characterizing the typical structure of the untypical, i.e., rare graphs in $\mathcal{F}_0^\ell(n) \setminus \mathcal{P}_0^\ell(n)$.

Furthermore, the notion 'i-quasi-ℓ-partite', which is defined in terms of deleted vertices, bears some similarity with the notion '(δ, ℓ)-partite' used in [Łuc00], which is based on the deletion of edges. However, there is no direct connection between [Łuc00] and Theorem 3.2, as [Łuc00] studies the evolution of H-free graphs, whereas in this chapter we will not bound the number of edges in the graph *a priori*.

3.5 Simple proof for bipartite graphs

In this section we give a proof of Theorem 3.2 for the case $\ell = 2$ and $i = 1$, i.e., we show that almost all triangle-free graphs which are not bipartite can be made bipartite by removing a single vertex. This corresponds to the following special case of Theorem 3.2.

Theorem 3.7 *Almost all triangle-free graphs which are not bipartite, are quasi-bipartite, i.e.,*

$$|\mathcal{F}_1(n)| = (1 + o(1)) \cdot |\mathcal{B}_1(n)|.$$

The following definition specifies some properties which we expect to hold for a 'typical' bipartite graph. Based on the negation of these properties we will later define sets of 'strange' graphs, i.e., graphs with unusual properties. These sets will be used as bad sets in the Kleitman-Rothschild method, i.e., we will show that the cardinality of these sets is negligible in comparison to $|\mathcal{B}_1(n)|$.

Definition 3.8 *We introduce the following abbreviations for properties of a graph $G = (V, E)$ with $|V| = n$:*

- *(P1) Minimum degree is not too small: $|\Gamma(v)| \geq 2 \log n$ for all $v \in V$.*
- *(P2) Large second neighborhood: $|\Gamma(Q)| \geq (\frac{1}{2} - \frac{2}{\log\log n})n$ for all $Q \subseteq V$ and $|Q| = \log n$.*

The following lemmas show some results on the structure of graphs for which the specified properties hold. The proofs of these results, i.e., of Lemma 3.9 and Lemma 3.11, are implicit in [Ste93] and [PST01] but we also briefly include them here for clarity and completeness.

Lemma 3.9 *Let $G = (V, E)$ be a graph on n vertices, where n is sufficiently large, and $C \subseteq V$ be a cycle with $|C| \leq 9$. If it holds that $|\Gamma^2(v)| \geq (\frac{1}{2} - \frac{2}{\log\log n})n$ for all $v \in C$ and that $|\Gamma^2(x) \cap \Gamma^2(y)| \leq \frac{1}{100}n$ for all edges $\{x, y\}$ of C, then the cycle cannot be odd.*

Proof For simplicity we will only show that there is no cycle C_3. The proof for C_5, C_7 and C_9 is similar.

Let $G = (V, E)$ denote a graph and let $C = (v_1, v_2, v_3)$ be a cycle in G which satisfies the conditions of the lemma. In the sequel we write $R_i := \Gamma^2(v_i)$ and $\alpha := \frac{2}{\log\log n}$, $\beta := \frac{1}{100}$ for short. We directly obtain the following estimates:

$$\begin{aligned} |R_1 \cup R_2| &= |R_1| + |R_2| - |R_1 \cap R_2| \geq (1 - 2\alpha - \beta)n \\ |\overline{R}_1 \cap \overline{R}_2| &= n - |R_1 \cup R_2| \leq (2\alpha + \beta)n. \end{aligned}$$

Hence,

$$|R_3| \leq |\overline{R}_1 \cap \overline{R}_2| + |R_3 \cap R_1| + |R_3 \cap R_2| \leq (2\alpha + 3\beta)n,$$

contradicting the assumption on the minimum size of R_3. $\square$

Now we define a property for sets of edges. If this property and also the properties from Definition 3.8 hold for a graph, then this graph must be bipartite.

Definition 3.10 *Given a graph $G = (V, E)$ with $|V| = n$ we define the following property for a set of edges $F \subseteq E$:*

(P3) *Few cycles C_5: $|\Gamma^2(x) \cap \Gamma^2(y)| \leq \frac{1}{100}n$ for all $\{x, y\} \in F$.*

Lemma 3.11 *If a graph $G = (V, E)$ satisfies the properties (P1), (P2) and (P3) for all edges and $|V| = n$ is sufficiently large, then G is bipartite.*

Proof We construct a 2-coloring of G as follows: By Lemma 3.9 G contains neither a C_3 nor a C_5 nor a C_7 nor a C_9.

Now choose an arbitrary edge $\{x, y\} \in E$. For brevity let $Q_x := \Gamma(x)$, $Q_y := \Gamma(y)$, $R_x := \Gamma^2(x)$ and $R_y := \Gamma^2(y)$. R_x and R_y are stable because otherwise we could find a C_5. Similarly we conclude that $R_x \cap R_y = \emptyset$.

We denote all other vertices by $S := V \setminus [Q_x \cup Q_y \cup R_x \cup R_y]$ and partition them in two classes:

$$S_x = \{v \in S \mid R_v \cap R_x \neq \emptyset\} \quad \text{and} \quad S_y = \{v \in S \mid R_v \cap R_y \neq \emptyset\}.$$

Observe that $S_x \cup S_y = S$, since R_x and R_y cover almost the whole graph and R_v is big for all v. Hence, for every vertex v the intersection $R_v \cap (R_x \cup R_y)$ cannot be empty. Moreover, $R_x \cup S_x$ and $R_y \cup S_y$ are stable and $S_x \cap S_y = \emptyset$ because there are no cycles C_7 and C_9 (see Figure 3.1).

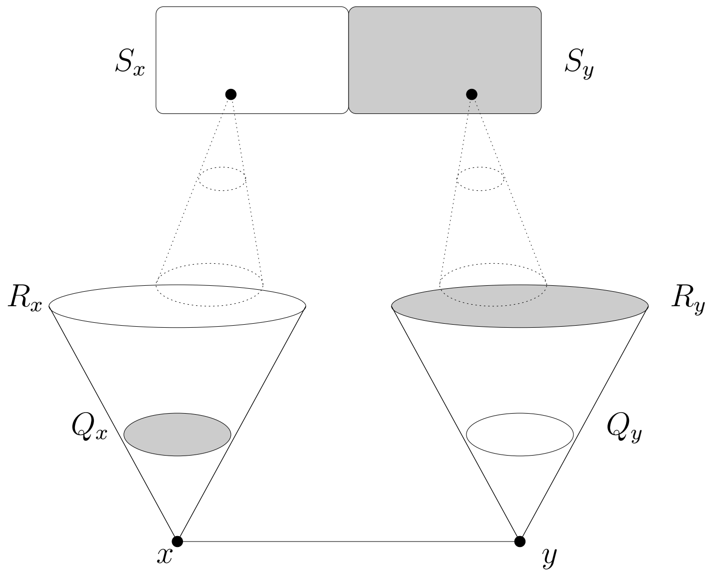

Figure 3.1: Finding color classes in Lemma 3.11

Hence, we have found a 2-coloring with the color classes $R_x \cup S_x \cup Q_y$ and $R_y \cup S_y \cup Q_x$. $\square$

Consider a graph G which satisfies properties (P1) and (P2). Furthermore, assume that the graph is composed of a bipartite subgraph on $n-2$ vertices and two additional vertices. The following lemma shows how to find two short vertex disjoint odd cycles in G. Later in the paper we will apply the lemma in order to exploit the fact that a graph with two such cycles remains non-bipartite even if an edge of a cycle is removed.

Lemma 3.12 *Let $G = (V, E)$ be a graph with two vertices $v_1, v_2 \in V$ such that $G' := G[V \setminus \{v_1, v_2\}] \in \mathcal{B}_0(n)$. Consider an arbitrary 2-coloring of G' with color classes S_1, S_2 and assume that for $j \in \{1, 2\}$ there are vertices $w_j^{(1)} \in (S_1 \cap \Gamma(v_j))$ and $w_j^{(2)} \in (S_2 \cap \Gamma(v_j))$ such that $w_1^{(1)}, w_1^{(2)}, w_2^{(1)}, w_2^{(2)}$ are pairwise different. If G' satisfies the properties (P1) and (P2) and n is sufficiently large we can find two vertex disjoint cycles C_7 in G.*

Proof We look for vertex disjoint paths P_j in G' of length five that connect $w_j^{(1)}$ and $w_j^{(2)}$ for $j \in \{1, 2\}$: By (P1) we can find sets $Q_j^{(1)} \subseteq \Gamma(w_j^{(1)}) \cap S_2$ and $Q_j^{(2)} \subseteq \Gamma(w_j^{(2)}) \cap S_1$ of size at least $\log n$. Since $\Gamma(Q_j^{(1)})$ and $\Gamma(Q_j^{(2)})$ are almost as large as one partition of V they are obviously connected by an edge and a path can be found.

Assume that the cycle at v_1 is fixed first. By (P1) the degrees of $w_j^{(1)}$ and $w_j^{(2)}$ are large and, hence, we are able to choose the set $Q_2^{(1)}$ and $Q_2^{(2)}$ in such a way that the resulting cycle at v_2 is vertex disjoint from the cycle at v_1. □

We also need a few results on the growth of $|\mathcal{B}_0(n)|$ and $|\mathcal{B}_1(n)|$. Later we will use them to show that the size of the 'bad sets' grows asymptotically slower than $|\mathcal{B}_1(n)|$ and, thus, these sets contain only a negligible number of graphs.

Lemma 3.13 *For $i \in \{0,1\}$ and all sufficiently large n*

$$\log \frac{|\mathcal{B}_i(n-1)|}{|\mathcal{B}_i(n)|} \leq -\frac{n-i-1}{2}.$$

Proof We construct pairwise different graphs in $\mathcal{B}_i(n)$ as follows. First we choose a graph $G \in \mathcal{B}_i(n-1)$ for the first $n-1$ vertices. By definition, this graph G contains a stable set of size at least $\frac{n-i-1}{2}$. The n-th vertex can be connected to this stable set in at least $2^{\frac{n-i-1}{2}}$ many ways. This shows that

$$|\mathcal{B}_i(n)| \geq 2^{\frac{n-i-1}{2}} \cdot |\mathcal{B}_i(n-1)|,$$

which is equivalent to the claimed inequality. □

Note that Theorem 3.3 implies the case $i = 0$ of Lemma 3.13, but the above rough estimate already suits our needs. However, we will use Theorem 3.3 to obtain a short proof for the following rather obvious technical lemma.

Lemma 3.14 *There is a constant $\rho > 0$ such that for all sufficiently large n*

$$\log \frac{|\mathcal{B}_0(n-1)|}{|\mathcal{B}_1(n)|} \leq \rho.$$

Proof The following procedure yields pairwise different graphs in $\mathcal{B}_1(n)$: We construct a cycle $C = (v_1, \ldots, v_5)$ on the first five vertices and choose a graph $G' \in \mathcal{B}_0(n-5)$ for the remaining $n-5$ vertices. Let S_1, S_2 denote an arbitrary 2-coloring of G'. We connect v_1 and v_3 to S_1 and v_2 and v_4 to S_2. There are $2^{2|S_1|+2|S_2|} = 2^{2(n-5)}$ possibilities for this. Note that the resulting graph contains no triangles because the neighbors in G' of adjacent vertices on the cycle are disjoint. Hence, we conclude that

$$|\mathcal{B}_1(n)| \geq |\mathcal{B}_0(n-5)| \cdot 2^{2(n-5)}.$$

Using Theorem 3.3 one easily checks that

$$\frac{|\mathcal{B}_0(n-1)|}{|\mathcal{B}_0(n-5)|} = \Theta\left(\frac{2^{\frac{1}{4}n^2-\frac{1}{2}n+\frac{1}{4}+n-1-\frac{1}{2}\log(n-1)}}{2^{\frac{1}{4}n^2-\frac{5}{2}n+\frac{25}{4}+n-5-\frac{1}{2}\log(n-5)}}\right) = \Theta(2^{2n}).$$

Then the lemma follows immediately. □

Remark 3.15 *Note that $|\mathcal{B}_0(n-1)|/|\mathcal{B}_1(n)|$ is actually much smaller than the bound in Lemma 3.14. Comparing Lemma 3.14 to Lemma 3.13 one would expect an exponentially small expression. But since a rough estimate suffices for our proofs we only state this almost trivial bound.*

For the application of the Kleitman-Rothschild method we partition $\mathcal{F}_1(n)$ into $\mathcal{B}_1(n)$ and several 'bad' sets of graphs with 'unlikely' properties based on (P1) up to (P3). Then we show that the cardinality of those bad sets is negligible.

$\mathcal{X}(n)$ The set of all graphs in $\mathcal{F}_1(n) \setminus \mathcal{B}_1(n)$ which contain a vertex v such that $|\Gamma(v)| \leq 3\log n$,

$\mathcal{Y}(n)$ The set of all graphs in $\mathcal{F}_1(n) \setminus [\mathcal{B}_1(n) \cup \mathcal{X}(n)]$ which contain a set Q of size $\log n$ such that $|\Gamma(Q)| \leq (\frac{1}{2} - \frac{1}{\log\log n})n$,

$\mathcal{Z}(n)$ The set of all graphs G in $\mathcal{F}_1(n) \setminus [\mathcal{B}_1(n) \cup \mathcal{X}(n) \cup \mathcal{Y}(n)]$ which contain an edge $\{x, y\}$ and sets $Q_x \subseteq \Gamma(x)$ and $Q_y \subseteq \Gamma(y)$ of size $|Q_x| = |Q_y| = \log n$ such that $G - \{x, y\} \in \mathcal{F}_1(n-2)$ and $|\Gamma(Q_x) \cap \Gamma(Q_y)| > \frac{1}{100}n$,

First we have to show that $\mathcal{B}_1(n)$ and the 'bad' sets cover $\mathcal{F}_1(n)$.

Lemma 3.16 *It holds that*

$$\mathcal{F}_1(n) \subseteq \mathcal{B}_1(n) \cup \mathcal{X}(n) \cup \mathcal{Y}(n) \cup \mathcal{Z}(n)$$

for all sufficiently large n.

Proof Consider a graph $G = (V, E)$ in $\mathcal{F}_1(n) \setminus [\mathcal{B}_1(n) \cup \mathcal{X}(n) \cup \mathcal{Y}(n) \cup \mathcal{Z}(n)]$. Then the properties (P1) and (P2) hold for G by the definitions of $\mathcal{X}(n)$ and $\mathcal{Y}(n)$, and are still satisfied if one or two vertices are deleted from G.

Assume that there is an edge $\{x, y\}$ such that $G' := G - \{x, y\} \notin \mathcal{F}_1(n-2)$. Since $G' \in \mathcal{F}_0(n-2) = \mathcal{F}_1(n-2) \cup \mathcal{B}_0(n-2)$ we may conclude that $G' \in \mathcal{B}_0(n-2)$.

Consider an arbitrary 2-coloring of G'. Note that $\Gamma(x) \cap \Gamma(y) = \emptyset$ because G is triangle-free. Furthermore, x and y must have neighbors in both color classes S_1 and S_2 since they are part of odd cycles in G. Otherwise we could deduce that $G \in \mathcal{B}_0(n) \cup \mathcal{B}_1(n)$ which contradicts our choice of G.

Assume that there is one color class, say S_1, and a vertex w such that $\Gamma(x) \cap \Gamma(y) \cap S_1 = \{w\}$. Then the vertices x, y, w would form a triangle and we get a contradiction. Thus, x and y have at least two disjoint neighbors in both color classes and we can find two vertex disjoint odd cycles C_1 and C_2 with $|C_1|, |C_2| \leq 7$ using Lemma 3.12. It follows by the definition of $\mathcal{Z}(n)$ that (P3) holds for all edges of C_1 and C_2. Lemma 3.9 then shows that C_1 and C_2 cannot be odd and we obtain a contradiction. Therefore, we may conclude that $G - \{x, y\} \in \mathcal{F}_1(n-2)$ for all edges $\{x, y\}$.

Hence, by the definition of $\mathcal{Z}(n)$ it holds for all $\{x, y\} \in E$ that $|\Gamma^2(x) \cap \Gamma^2(y)| \leq \frac{1}{100}n$ and Lemma 3.11 proves that G is bipartite, which once again yields a contradiction. □

The following lemmas help us to estimate the cardinality of the bad sets.

Lemma 3.17 *For all sufficiently large n*

$$\log \frac{|\mathcal{X}(n)|}{|\mathcal{F}_1(n-1)|} \leq 4(\log n)^2.$$

Proof Consider a graph $G = (V, E) \in \mathcal{X}(n)$ and a vertex $x \in V$. By definition of $\mathcal{X}(n)$ we know that $G[V \setminus \{x\}] \notin \mathcal{B}_0(n-1)$. Hence, all graphs in $\mathcal{X}(n)$ can be constructed as follows. First choose the vertex v and a graph $G \in \mathcal{F}_1(n-1)$ on $V \setminus \{v\}$ (in at most $n \cdot |\mathcal{F}_1(n-1)|$ ways). Then choose the set $\Gamma(v)$. As there are at most

$$\sum_{j=0}^{3\log n} \binom{n-1}{j} \leq n^{3\log n}$$

ways to do this, it follows that

$$|\mathcal{X}(n)| \leq |\mathcal{F}_1(n-1)| \cdot n \cdot n^{3\log n}.$$

The lemma is an immediate consequence of this. □

Lemma 3.18 *For all sufficiently large n*

$$\log \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n-\log n)|} \leq \left(\frac{1}{2} - \frac{1}{2\log\log n}\right) n \log n.$$

Proof Construct all graphs in $\mathcal{Y}(n)$ as follows. First choose the set Q and a triangle-free graph G on $V \setminus Q$. This can be done in at most $\binom{n}{\log n} \cdot |\mathcal{F}_0(n - \log n)|$ ways. Then, we have less than 2^n possibilities to fix the set $R = \Gamma(Q)$. Additionally, there are at most $2^{|Q|\cdot|R|} \leq 2^{\log n \cdot (\frac{1}{2} - \frac{1}{\log\log n})n}$ possible choices for the edges between Q and R and less than $2^{(\log n)^2}$ possibilities for the edges inside Q. All in all we get

$$\begin{aligned}\log \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n-\log n)|} &\leq 2(\log n)^2 + n + \left(\frac{1}{2} - \frac{1}{\log\log n}\right) n \log n \\ &\leq \left(\frac{1}{2} - \frac{1}{2\log\log n}\right) n\log n\end{aligned}$$

for n sufficiently large. □

Lemma 3.19 *For all sufficiently large n*

$$\log \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_1(n-2)|} \leq \left(1 - \frac{1}{2000}\right) n.$$

Proof Construct all graphs in $\mathcal{Z}(n)$ as follows. First choose two vertices x and y, a triangle-free graph $G' \in \mathcal{F}_1(n-2)$ on $V \setminus \{x, y\}$, and appropriate sets Q_x, Q_y in less than $n^2 \cdot |\mathcal{F}_i(n-2)| \cdot n^{2\log n}$ ways. Let for conciseness of notation $R_x = \Gamma(Q_x)$ and $R_y = \Gamma(Q_y)$ and observe that R_x and R_y are determined by the choice of Q_x and Q_y. Finally, connect x and y to $V \setminus \{x, y\}$. As no vertex in R_x (R_y) may be connected to x (y) and no vertex in $V \setminus (R_x \cup R_y)$ may be connected to both x and y there are at most

$$\begin{aligned}&2^{|R_x\setminus R_y|+|R_y\setminus R_x|} \cdot 3^{n-|R_x\cup R_y|} \\ &\quad\leq 2^{|R_x|+|R_y|-2|R_x\cap R_y|+\frac{7}{4}(n-|R_x|-|R_y|+|R_x\cap R_y|)} \\ &\quad\leq 2^{\frac{7}{4}n-\frac{3}{4}(|R_x|+|R_y|)-\frac{1}{4}|R_x\cap R_y|} \leq 2^{n-\frac{1}{1000}n}\end{aligned}$$

ways to do this. Recall that by definition of $\mathcal{Y}(n)$ we have $|R_x|, |R_y| \geq (\frac{1}{2} - \frac{1}{\log\log n})n$ and that by assumption $|R_x \cap R_y| \geq \frac{1}{100}n$. Furthermore, one immediately checks that $\log 3 \leq \frac{7}{4}$. Putting everything together we obtain

$$\log \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_i(n-2)|} \leq 2\log n + 2(\log n)^2 + n - \frac{1}{1000}n \leq \left(1 - \frac{1}{2000}\right) n$$

for n sufficiently large. □

Now that we have estimated the cardinality of all bad sets, we are in a position to prove Theorem 3.7.

Proof of Theorem 3.7 Set $\gamma := 2^{\frac{1}{3000}}$. We will show that there exists a constant $c \geq 1$ such that

$$|\mathcal{F}_1(n)| \leq (1 + c\gamma^{-n})|\mathcal{B}_1(n)| \tag{3.8}$$

for all $n \in \mathbb{N}$. Choose n_0 large enough so that all lemmas above and all asymptotic estimates below hold for $n \geq n_0$. Subsequently, choose $c \geq 1$ such that (3.8) is satisfied for all $n \leq n_0$.

We conclude the proof by induction on n. So assume that (3.8) is satisfied for all graphs on n' vertices with $n' < n$. By Lemma 3.16 we deduce that

$$|\mathcal{F}_1(n)| \leq |\mathcal{B}_1(n)| + |\mathcal{X}(n)| + |\mathcal{Y}(n)| + |\mathcal{Z}(n)|.$$

Hence, it suffices to show that the ratio of $|\mathcal{X}(n)|$, $|\mathcal{Y}(n)|$ and $|\mathcal{Z}(n)|$ to $|\mathcal{B}_1(n)|$ is at most $\frac{c}{3}\gamma^{-n}$.

By Lemma 3.17, Lemma 3.13 and the induction hypothesis we conclude that

$$\begin{aligned}
\frac{|\mathcal{X}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{X}(n)|}{|\mathcal{F}_1(n-1)|} \cdot \frac{|\mathcal{F}_1(n-1)|}{|\mathcal{B}_1(n-1)|} \cdot \frac{|\mathcal{B}_1(n-1)|}{|\mathcal{B}_1(n)|} \\
&\leq 2^{4(\log n)^2} \cdot \underbrace{(1 + c\gamma^{-n+1})}_{\leq 2c} \cdot 2^{-\frac{1}{2}(n-2)} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

Similarly, using Lemma 3.18, Lemma 3.13, Lemma 3.14 and Theorem 3.1 (resp. the result from [EKR76] which suffices here) we obtain

$$\begin{aligned}
\frac{|\mathcal{Y}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{Y}(n)|}{|\mathcal{F}_0(n-\log n)|} \cdot \frac{|\mathcal{F}_0(n-\log n)|}{|\mathcal{B}_0(n-\log n)|} \cdot \\
&\qquad \frac{|\mathcal{B}_0(n-\log n)|}{|\mathcal{B}_1(n-\log n+1)|} \cdot \prod_{j=0}^{\log n-2} \frac{|\mathcal{B}_1(n-j-1)|}{|\mathcal{B}_1(n-j)|} \\
&\leq 2^{(\frac{1}{2}-\frac{1}{2\log\log n})n\log n} \cdot 2c \cdot \rho \cdot 2^{\sum_{j=0}^{\log n-2}[-\frac{1}{2}(n-j-2)]} \\
&\leq 2^{(\frac{1}{2}-\frac{1}{2\log\log n})n\log n} \cdot 2c \cdot \rho \cdot 2^{-\frac{1}{2}(\log n-1)(n-\log n-4)} \\
&\leq \frac{c}{3} \cdot 2^{-\frac{1}{2\log\log n}\cdot n\log n+n} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

Finally, we use Lemma 3.19, Lemma 3.13 and the induction hypothesis to show that

$$\begin{aligned}
\frac{|\mathcal{Z}(n)|}{|\mathcal{B}_1(n)|} &\leq \frac{|\mathcal{Z}(n)|}{|\mathcal{F}_1(n-2)|} \cdot \frac{|\mathcal{F}_1(n-2)|}{|\mathcal{B}_1(n-2)|} \cdot \frac{|\mathcal{B}_1(n-2)|}{|\mathcal{B}_1(n-1)|} \cdot \frac{|\mathcal{B}_1(n-1)|}{|\mathcal{B}_1(n)|} \\
&\leq 2^{(1-\frac{1}{2000})n} \cdot (1 + c\gamma^{-n+2}) \cdot 2^{-\frac{1}{2}(n-3)-\frac{1}{2}(n-2)} \leq \frac{c}{3}\gamma^{-n}.
\end{aligned}$$

This completes the proof. $\square$

3.6 Proof of the general case

This section contains the proof of Theorem 3.2 for arbitrary values of ℓ and i, and is organized as follows. In Section 3.6.1 we collect a few lemmas on the growth rate of $|\mathcal{P}_i^\ell(n)|$. In Section 3.6.2 the bad sets are defined and their sizes are examined. The results of these two sections are then combined in Section 3.6.3 and Theorem 3.2 is proved.

For our asymptotic estimates in this section we will often assume that n is sufficiently large. This will not always be explicitly mentioned.

3.6.1 Growth rates of ℓ-partite graphs

In this section we show some estimates on the growth rate of $|\mathcal{P}_i^\ell(n)|$ considered as a function of n and i. Later we will compare these growth rates to the growth rates of the bad sets and it will turn out that these sets grow asymptotically slower than $|\mathcal{P}_i^\ell(n)|$. Hence, for n sufficiently large the cardinality of the bad sets is tiny in comparison to $|\mathcal{P}_i^\ell(n)|$.

Lemma 3.20 *For all $i \geq 0$ and n sufficiently large,*

$$\log \frac{|\mathcal{P}_i^\ell(n-1)|}{|\mathcal{P}_i^\ell(n)|} \leq -\frac{\ell-1}{\ell}n + i + 1.$$

Proof We construct pairwise different graphs G in $\mathcal{P}_i^\ell(n)$ as follows: We choose a graph $G' = (V', E') \in \mathcal{P}_i^\ell(n-1)$ for the first $n-1$ vertices. Consider i vertices $v_1, \ldots, v_i \in V'$ such that $G'' := G'[V' \setminus \{v_1, \ldots, v_i\}]$ is ℓ-partite and fix an arbitrary ℓ-coloring of G''. Clearly, this coloring contains a color class S with at most $\frac{n-1-i}{\ell}$ vertices. If we don't connect the n-th vertex v to $S \cup \{v_1, \ldots, v_i\}$ one easily verifies that the resulting graph G cannot contain a $K_{\ell+1}$.

Hence, there are at least $2^{n-1-i-\frac{n-1-i}{\ell}}$ possibilities to connect v to G'. Therefore,

$$|\mathcal{P}_i^\ell(n)| \geq |\mathcal{P}_i^\ell(n-1)| \cdot 2^{(1-\frac{1}{\ell})n-i-1}$$

and the claim follows. □

From this lemma we easily deduce a corollary which will later come in handy.

Corollary 3.21 *For $i \geq 0$ and all integral positive functions $g(n) \in \mathcal{O}(\log n)$,*

$$\log \frac{|\mathcal{P}_i^\ell(n - g(n))|}{|\mathcal{P}_i^\ell(n)|} = -\frac{\ell - 1}{\ell} n \cdot g(n) + \mathcal{O}(g(n)^2).$$

Proof We repeatedly apply Lemma 3.20 and obtain

$$\begin{aligned}
\log \frac{|\mathcal{P}_i^\ell(n - g(n))|}{|\mathcal{P}_i^\ell(n)|} &= \sum_{j=0}^{g(n)-1} \log \frac{|\mathcal{P}_i^\ell(n - g(n) + j)|}{|\mathcal{P}_i^\ell(n - g(n) + j + 1)|} \\
&\leq \sum_{j=0}^{g(n)-1} \left(-\frac{\ell - 1}{\ell}(n - g(n) - j + 1) + i + 1 \right) \\
&= \left(-\frac{\ell - 1}{\ell} n + i + 1 \right) g(n) - \sum_{j=0}^{g(n)-1} j \\
&= -\frac{\ell - 1}{\ell} n \cdot g(n) + \mathcal{O}(g(n)^2).
\end{aligned}$$

□

Finally, we will need a lemma on the relation between the number of ℓ-partite graphs and the number of i-quasi-ℓ-partite graphs.

Lemma 3.22 *For all $i > 0$ and n sufficiently large,*

$$\frac{|\mathcal{P}_0^\ell(n - i(2\ell + 1))|}{|\mathcal{P}_i^\ell(n)|} \leq 1.$$

Proof For the proof of this lemma we need a small $K_{\ell+1}$-free graph H with chromatic number $\ell + 1$. H will serve as a witness that a graph G which contains H as a subgraph is not ℓ-colorable.

Let $H = (S, F)$ be a graph on $2\ell + 1$ vertices with

$$S = \{v, v_1, \ldots, v_\ell, v'_1, \ldots, v'_\ell\}.$$

F is assumed to be a minimal set of edges such that

1. v is connected to all vertices $v_1, \ldots, v_\ell$.
2. $G[\{v'_1, \ldots, v'_\ell\}]$ is a complete graph K_ℓ.

3. For every $i \in \{1, \ldots, \ell\}$: $G[\{v_i, v'_1, \ldots, v'_{i-1}, v'_{i+1}, \ldots, v'_\ell\}]$ is a complete graph K_ℓ.

It is easy to see that such a graph H is $K_{\ell+1}$-free (Observe that there is no edge between v_i and v'_i. Furthermore, v clearly does not belong to a subgraph $K_{\ell+1}$.). Moreover, at least $\ell + 1$ colors are necessary to color it: Assume that there is a legal ℓ-coloring. Then $v'_1, \ldots, v'_\ell$ must be colored with all ℓ different colors. Hence, v_i and v'_i must receive the same color for all $i \in \{1, \ldots, \ell\}$. Since v is connected to all vertices $v_1, \ldots, v_\ell$ we obtain a contradiction.

We construct pairwise different graphs $G = (V, E)$ in $\mathcal{P}^\ell_i(n)$ as follows: On the first $i(2\ell + 1)$ vertices i disjoint copies of H are fixed. Then we choose a graph $G' \in \mathcal{P}^\ell_0(n - i(2\ell + 1))$ for the remaining vertices. □

3.6.2 Bad sets

In order to exploit the fact that we consider only $K_{\ell+1}$-free graphs we want to identify dense subgraphs and parts of the graph with many edges incident to such a dense subgraph. If we have found a dense subgraph, we can conclude that it may be connected to the rest of the graph only in a quite restricted way, since otherwise a $K_{\ell+1}$ would be constructed.

We assume that

$$\varepsilon_1, \varepsilon_{\mathcal{A}}, \varepsilon_{\mathcal{D}}, \varepsilon'_{\mathcal{D}}, \varepsilon_{\mathcal{E}}, \text{ and } \varepsilon_{\mathcal{R}}$$

are positive constants. Later we will explain how these constants have to be chosen.

Definition 3.23 *A $q(k)$-set $Q = \bigcup_{j=1}^k Q_j$ in a graph $G = (V, E)$ on $n = |V|$ vertices consists of k pairwise disjoint vertex sets $Q_1, \ldots, Q_k \subseteq V$ such that $|Q_1| = \ldots = |Q_k| = f_k(n) := \lceil \log^{(k^2-k+2)}(n) \rceil$ and $\{x, y\} \in E$ for all $x \in Q_h, y \in Q_j$ with $1 \le h < j \le k$.*
Given a $q(k)$-set $Q = \bigcup_{j=1}^k Q_j$ we denote by $R(Q)$ the set

$$R(Q) := \{v \in V \setminus Q \mid |\Gamma(v) \cap Q_j| \ge \varepsilon_{\mathcal{R}} |Q_j| \ \text{ for all } j = 1, \ldots, k\}.$$

For ease of notation we assume that a $q(0)$-set denotes the empty set and that $R(\emptyset) = V$. We also use the term q-set as short hand for a $q(\ell - 1)$-set.

In the sequel we assume without loss of generality that the constants ε_1, $\varepsilon_{\mathcal{A}}$, $\varepsilon_{\mathcal{D}}$, $\varepsilon'_{\mathcal{D}}$, $\varepsilon_{\mathcal{E}}$, and $\varepsilon_{\mathcal{R}}$ are chosen such that they satisfy the following inequalities for all sufficiently large n and arbitrary k and w with $1 \le k < \ell$ and $\frac{n}{4\ell} \le w \le n$:

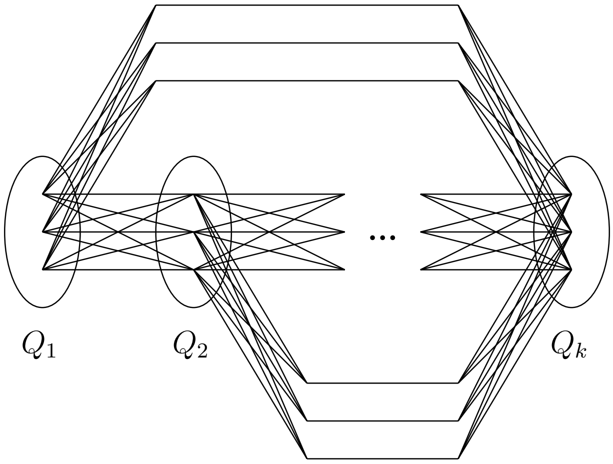

Figure 3.2: Structure of $q(k)$-set

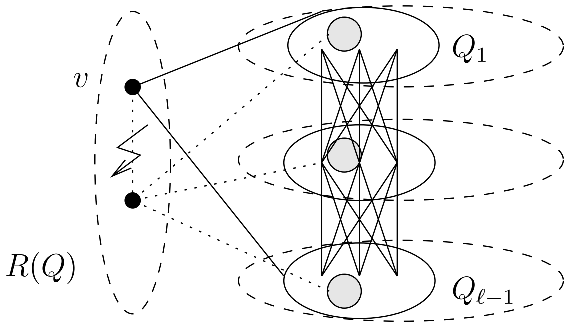

Figure 3.3: Structure of $R(Q)$ for q-set Q

1.	$k\varepsilon_{\mathcal{R}} f_k(n)\binom{f_k(n)}{\varepsilon_{\mathcal{R}} f_k(n)} \leq 2^{\frac{1}{4}\varepsilon_{\mathcal{A}} f_k(n)}$	**6.**	$\varepsilon'_{\mathcal{D}} \leq \frac{1}{32\ell}$
2.	$3\varepsilon_{\mathcal{A}} \leq \varepsilon_1 \frac{1}{16\ell}$	**7.**	$2\ell\varepsilon_{\mathcal{A}} \leq \frac{1}{64\ell}$
3.	$\binom{w}{\frac{w}{2}-\varepsilon_{\mathcal{D}} n} \leq 2^{w-\varepsilon_1 n}$	**8.**	$2\ell\varepsilon_{\mathcal{A}} \leq \frac{\varepsilon_{\mathcal{E}}}{2^{\ell+1}}$
4.	$\frac{1}{2}\varepsilon_{\mathcal{D}} \leq \varepsilon'_{\mathcal{D}}$	**9.**	$\varepsilon_{\mathcal{E}} \leq \frac{1}{\ell} - (\ell-1)\varepsilon_{\mathcal{A}}$
5.	$H\left(\frac{\frac{1}{2}-\varepsilon'_{\mathcal{D}}}{1-\varepsilon_{\mathcal{D}}}\right) \leq 1-\varepsilon_1$	**10.**	$2\varepsilon_{\mathcal{A}} + 4\ell\varepsilon_{\mathcal{E}} + \ell^2\varepsilon_{\mathcal{E}} \leq \frac{1}{10\ell}$

In order to see that this set of inequalities has a solution consider the ordering

$$\sigma := (\varepsilon'_{\mathcal{D}}, \varepsilon_{\mathcal{D}}, \varepsilon_1, \varepsilon_{\mathcal{E}}, \varepsilon_{\mathcal{A}}, \varepsilon_{\mathcal{R}}).$$

If the constants are fixed according to σ it is quite straightforward to check that the inequalities can be satisfied by choosing values that are sufficiently small.

How to construct bad sets Using these constants we will now define a family of bad sets. For every bad set two things will be done: First, the cardinality of the bad set is compared to the cardinality of (quasi-)ℓ-partite graphs. In Section 3.6.3 it will turn out that these bounds are strong enough. Additionally, structural results are shown for the graphs in $\mathcal{F}_i^\ell(n)\setminus\mathcal{P}_i^\ell(n)$ that don't belong to the bad sets defined so far. As we define more and more bad sets we obtain more and more knowledge about the structure of the remaining graphs. In the end we will derive contradictory results on this structure and, thus, no such graphs remain. This means that we have covered the set of graphs in $\mathcal{F}_i^\ell(n) \setminus \mathcal{P}_i^\ell(n)$ with the bad sets.

Before going into details we will explain the intuition behind the bad sets: Assume that Theorem 3.2 holds. Hence, most graphs in $\mathcal{F}_i^\ell(n)$ look like a typical graph in $\mathcal{P}_i^\ell(n)$. Without a formal proof we claim that the structure of a typical i-quasi-ℓ-partite graph is similar to the structure of a random ℓ-partite graph with i additional vertices (connected to the rest of the graphs in a suitable way). Note that the partitions of random ℓ-partite graphs have approximately equal cardinality, i.e., they contain about $\frac{n}{\ell}$ vertices, and edges between the partitions occur independently with probability about $\frac{1}{2}$.

The bad sets are characterized by properties which violate this intuition. Hence, we expect that these graphs occur very 'unlikely'. The lemmas on the cardinality of the bad sets verify this intuitive argument.

Now we define the first two types of bad sets: Consider a typical graph in $\mathcal{F}_i^\ell(n)$ and a set X of vertices that belongs to k color classes. Assume that $|X|$ is not too small and that there are many vertices of each of the k colors. We expect for every vertex $v \in V \setminus X$ which does not belong to one of the k color classes of X that $|\Gamma(v) \cap X| \approx \frac{1}{2}|X|$, because this is the typical situation we would encounter in a random graph. We call this intuitive argument the *neighborhood assumption*.

A $q(k)$-set Q belongs to k color classes. Hence, most of the about $\frac{\ell-k}{\ell}n$ vertices in the other $\ell-k$ color classes should have approximately $\frac{1}{2}|Q|$ neighbors in Q and, thus, $|R(Q)| \approx \frac{\ell-k}{\ell}n$, i.e., $R(Q)$ contains almost all the other $\ell - k$ color classes. This intuition is violated by the definition of the sets $\mathcal{A}_i^\ell(n,k)$. The intuition behind $\mathcal{B}_i^\ell(n,k)$ is similar.

Definition 3.24 *For $k \in \{1,\ldots,\ell-1\}$ and $i \geq 0$ we define the following sets of graphs*

$$\mathcal{A}_i^\ell(n,k) := \{G \in \mathcal{F}_i^\ell(n) \setminus \mathcal{P}_i^\ell(n) \mid \exists\, q(k)\text{-set } Q : |R(Q)| \leq (\tfrac{\ell-k}{\ell} - \varepsilon_{\mathcal{A}})n\},$$

and for $k \in \{0,\ldots,\ell-2\}$ let

$$\begin{aligned}\mathcal{B}_i^\ell(n,k) := \{G \in \mathcal{F}_i^\ell(n) \setminus \mathcal{P}_i^\ell(n) \mid \exists v \in V,\ \exists q(k)\text{-set } Q \ :\ Q \subseteq \Gamma(v),\\ |R(Q)| \geq (\tfrac{\ell-k}{\ell} - \varepsilon_{\mathcal{A}})n, |R(Q) \cap \Gamma(v)| \leq f_k(n)\}.\end{aligned}$$

Lemma 3.25 *For $k \in \{1, \ldots, \ell - 1\}$, $i \geq 0$ and n sufficiently large it holds that*

$$\log \frac{|\mathcal{A}_i^\ell(n,k)|}{|\mathcal{F}_0^\ell(n - kf_k(n))|} \leq \left(\frac{\ell-1}{\ell} - \frac{\varepsilon_\mathcal{A}}{2\ell}\right) \cdot kf_k(n) \cdot n.$$

Proof Figure 3.4 visualizes how the bad set $\mathcal{A}_i^\ell(n,k)$ is counted.

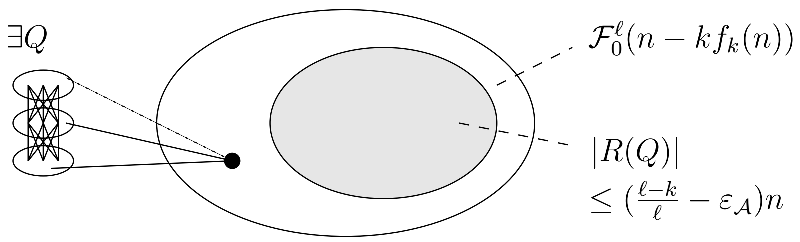

Figure 3.4: Bad set $\mathcal{A}_i^\ell(n,k)$

We construct all graphs $G = (V, E) \in \mathcal{A}_i^\ell(n,k)$ as follows: First we choose k sets $Q_1, ..., Q_k \subseteq V$ which form a $q(k)$-set $Q = Q_1 \cup \ldots \cup Q_k$ (at most $\binom{n}{f_k(n)}^k \leq n^{kf_k(n)}$ possibilities) and the edges inside Q (at most $2^{(kf_k(n))^2}$ possibilities). Then we fix a graph on the remaining $n - kf_k(n)$ vertices in $V \setminus Q$ (at most $|\mathcal{F}_0^\ell(n - kf_k(n))|$ possibilities) and a suitable set $R(Q) \subseteq V \setminus Q$ (at most 2^n possibilities). Finally, we connect Q to $R(Q)$ (at most $2^{kf_k(n)\cdot|R(Q)|}$ possibilities) and Q to $V \setminus [Q \cup R(Q)]$: To this aim we choose for every vertex $x \in V \setminus [Q \cup R(Q)]$ a set Q_{i_0} with $|Q_{i_0} \cap \Gamma(x)| \leq \varepsilon_\mathcal{R}|Q_{i_0}|$. Then we fix these neighbors and connect x in an arbitrary way to all other sets Q_j. Hence, for the edges from Q to $V \setminus [Q \cup R(Q)]$ there are at most

$$\left[k \cdot \sum_{j=0}^{\varepsilon_\mathcal{R} f_k(n)} \binom{f_k(n)}{j} \cdot 2^{(k-1)f_k(n))}\right]^{(n-|R(Q)|-kf_k(n))}.$$

possibilities. By Inequality 1 this expression is less than

$$2^{(\frac{1}{4}\varepsilon_\mathcal{A} f_k(n)+(k-1)f_k(n))(n-|R(Q)|)}.$$

Thus, we deduce that

$$\begin{aligned}
\log \frac{|\mathcal{A}_i^\ell(n,k)|}{|\mathcal{F}_0^\ell(n - kf_k(n))|} &\leq kf_k(n)\log n + (kf_k(n))^2 + n + kf_k(n) \cdot |R(Q)| \\
&\quad + \left(\frac{1}{4}\varepsilon_\mathcal{A} f_k(n) + (k-1)f_k(n)\right)(n - |R(Q)|) \\
&\leq \left((k-1)f_k(n) + \frac{1}{2}\varepsilon_\mathcal{A} f_k(n)\right) \cdot n + |R(Q)| \cdot f_k(n)
\end{aligned}$$

$$\leq \left(k-1+\frac{1}{2}\varepsilon_{\mathcal{A}}+\frac{\ell-k}{\ell}-\varepsilon_{\mathcal{A}}\right)\cdot f_k(n)\cdot n$$
$$\leq \left(k-\frac{k}{\ell}-\frac{1}{2}\varepsilon_{\mathcal{A}}\right)\cdot f_k(n)\cdot n.$$

□

In the next proofs we will repeatedly use the following observation: Let $G=(V,E)$ be a graph with $G\in\mathcal{F}_i^\ell(n)\setminus\mathcal{P}_i^\ell(n)$. Now delete an arbitrary vertex $v\in V$ and consider the graph $G':=G[V\setminus\{v\}]$. Then $G'\in\mathcal{F}_i^\ell(n-1)$. This follows directly from the fact that $G\notin\mathcal{P}_0^\ell(n)\cup\ldots\cup\mathcal{P}_{i-1}^\ell(n)\cup\mathcal{P}_i^\ell(n)$ by definition and, thus, $G'=G[V\setminus\{v\}]\notin\mathcal{P}_0^\ell(n-1)\cup\ldots\cup\mathcal{P}_{i-1}^\ell(n-1)$.

Lemma 3.26 *For $k\in\{0,\ldots,\ell-2\}$, $i\geq 0$ and n sufficiently large it holds that*

$$\log\frac{|\mathcal{B}_i^\ell(n,k)|}{|\mathcal{F}_i^\ell(n-1)|}\leq\left(\frac{\ell-1}{\ell}-\varepsilon_{\mathcal{A}}\right)\cdot n.$$

Proof Figure 3.5 visualizes how the bad set $\mathcal{B}_i^\ell(n,k)$ is counted.

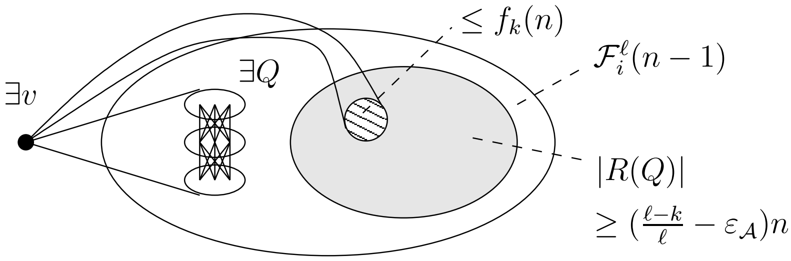

Figure 3.5: Bad set $\mathcal{B}_i^\ell(n,k)$

We construct all graphs $G=(V,E)\in\mathcal{B}_i^\ell(n,k)$ as follows: First we choose a vertex $v\in V$ and a graph $G'\in\mathcal{F}_i^\ell(n-1)$ on $V\setminus\{v\}$ (at most $n|\mathcal{F}_i^\ell(n-1)|$ possibilities). Then we fix a suitable $q(k)$-set $Q\subseteq V\setminus\{v\}$ (at most $n^{kf_k(n)}$ possibilities) and connect v to $R(Q)$ (at most $f_k(n)\cdot\binom{|R(Q)|}{f_k(n)}\leq n^{f_k(n)}$ possibilities). Finally, we connect v to $V\setminus[\{v\}\cup R(Q)]$ (at most $2^{n-|R(Q)|}$ possibilities).

All in all, we obtain

$$\log\frac{|\mathcal{B}_i^\ell(n,k)|}{|\mathcal{F}_i^\ell(n-1)|}\leq\log n+(k+1)f_k(n)\cdot\log n+n-|R(Q)|$$
$$\leq\left(\frac{k}{\ell}+2\varepsilon_{\mathcal{A}}\right)\cdot n\leq\left(\frac{\ell-1}{\ell}-\varepsilon_{\mathcal{A}}\right)\cdot n.$$

□

The exclusion of the sets $\mathcal{A}_i^\ell(n,k)$ and $\mathcal{B}_i^\ell(n,k)$ already enables us to obtain our first structural corollary.

Corollary 3.27 *Let $\mathcal{A}_i^\ell(n) = \bigcup_{k=1}^{\ell-1} \mathcal{A}_i^\ell(n,k)$ and $\mathcal{B}_i^\ell(n) = \bigcup_{k=0}^{\ell-2} \mathcal{B}_i^\ell(n,k)$. For n sufficiently large, all graphs $G = (V,E) \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n)]$ have the property that for every vertex $v \in V$ there exists a q-set Q_v with $Q_v \subseteq \Gamma(v)$.*

The proof of Corollary 3.27 is completely analogous to the corresponding result (Lemma 1.7) in [KPR87]. In particular, we also employ the following lemma from [KPR87] which generalizes a result of Bollobás and Erdős [BE73]:

Lemma 3.28 ([KPR87]) *Let $0 < \varepsilon < 1$ and $k \geq 0$. Then for all $N \geq N_0(\varepsilon,k)$ the following holds. If G is a graph with vertex set $A_0 \cup \ldots \cup A_k$, where A_j are pairwise disjoint sets of size $|A_j| = N$ and $|\Gamma(u) \cap A_j| \geq \varepsilon N$ for all $u \in A_0$ and all $j = 1, \ldots, k$, then there exist subsets $A_0' \subset A_0, \ldots, A_k' \subset A_k$ such that $|A_j'| = \lceil \log^{(2k)}(N) \rceil$ for all $j = 0, \ldots, k$ and such that A_0' is completely connected to all A_j', i.e., we have $\Gamma(u) \supseteq \bigcup_{j=1}^{k} A_j'$ for all $u \in A_0'$.*

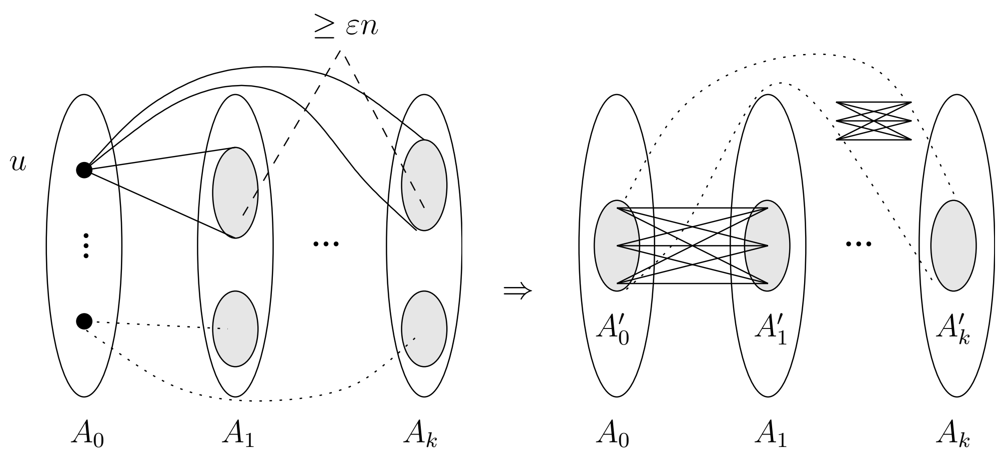

Figure 3.6: Construction of cliques using Lemma 3.28

We include a proof of Corollary 3.27 for sake of completeness and as we will also reuse the proof idea later on in the proof of Corollary 3.33.

Proof of Corollary 3.27 Consider an arbitrary graph $G = (V,E) \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n)]$ and an arbitrary vertex $v \in V$. We will show that $\Gamma(v)$ contains a q-set.

Because of $G \notin \mathcal{B}_i^\ell(n,0)$ we conclude that $|\Gamma(v)| \geq f_0(n) \geq f_1(n)$ and, thus, $\Gamma(v)$ contains a $q(1)$-set Q of size $f_1(n)$. We finish the proof by induction on k. Hence, assume that we have already found a $q(k)$-set $Q = \bigcup_{j=1}^k Q_j$ in $\Gamma(v)$ for a certain k with $1 \leq k \leq \ell - 2$.

Since $G \notin \mathcal{A}_i^\ell(n,k)$ it holds that $|R(Q)| \geq (\frac{\ell-k}{\ell} - \varepsilon_{\mathcal{A}})n$ and due to $G \notin \mathcal{B}_i^\ell(n,k)$ we deduce that $|\Gamma(v) \cap R(Q)| > f_k(n)$. Thus, let A_0 denote an arbitrary subset of $\Gamma(v) \cap R(Q)$ with $|A_0| = f_k(n)$. By applying Lemma 3.28 on A_0 and $A_j = Q_j, 1 \leq i \leq k$ we obtain sets $Q'_1, \ldots, Q'_{k+1} \subseteq \Gamma(v)$ which form a $q(k+1)$-set since

$$\begin{aligned} \lceil \log^{(2k)}(f_k(n)) \rceil &= \lceil \log^{(2k)}(\lceil \log^{(k^2-k+2)}(n) \rceil) \rceil \geq \lceil \log^{((k+1)^2-(k+1)+2)}(n) \rceil \\ &= f_{k+1}(n). \end{aligned}$$

□

Up to now we have only excluded graphs which contain a q-set Q for which $R(Q)$ is too small. The next definition also excludes graphs for which $R(Q)$ is too large.

Definition 3.29 *Let $\mathcal{C}_i^\ell(n)$ denote the set of all graphs $G = (V,E)$ in $\mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n)]$ that contain a vertex v and a q-set $Q = \bigcup_{j=1}^{\ell-1} Q_j \subseteq \Gamma(v)$, such that $|R(Q)| \geq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$.*

Lemma 3.30 *For all $i > 0$ and n sufficiently large,*

$$\log \frac{|\mathcal{C}_i^\ell(n)|}{|\mathcal{F}_i^\ell(n-1)|} \leq \left(\frac{\ell-1}{\ell} - \frac{1}{2}\varepsilon_{\mathcal{A}} \right) \cdot n.$$

Proof Figure 3.7 visualizes how the bad set $\mathcal{C}_i^\ell(n)$ is counted.

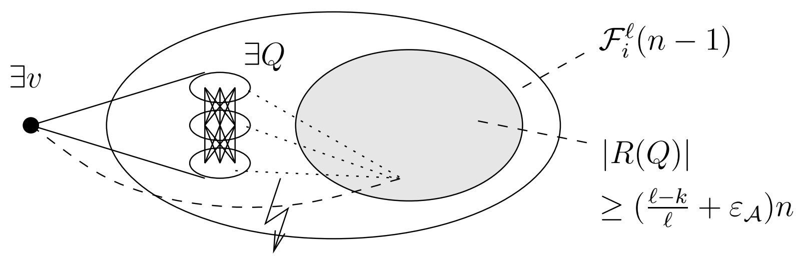

Figure 3.7: Bad set $\mathcal{C}_i^\ell(n)$

We construct all graphs $G = (V, E) \in \mathcal{C}_i^\ell(n)$ as follows: First we choose a vertex $v \in V$ and a graph $G' \in \mathcal{F}_i^\ell(n-1)$ on $V \setminus \{v\}$ (at most $n|\mathcal{F}_i^\ell(n-1)|$ possibilities). Then we choose an appropriate set Q in $V \setminus \{v\}$ (at most $n^{(\ell-1)f_{\ell-1}(n)}$ possibilities).

Note that v cannot be connected to any vertex in $R(Q)$, as otherwise G would contain a $K_{\ell+1}$. Therefore, if $|R(Q)| \geq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$ then there are at most $2^{(\frac{\ell-1}{\ell} - \varepsilon_{\mathcal{A}})n}$ possibilities to connect v to $V \setminus \{v\}$. It follows that

$$\begin{aligned}
\log \frac{|\mathcal{C}_i^\ell(n)|}{|\mathcal{F}_i^\ell(n-1)|} &\leq \log n + (\ell-1)f_{\ell-1}(n) \cdot \log n + \frac{\ell-1}{\ell}n - \varepsilon_{\mathcal{A}}n \\
&\leq \left(\frac{\ell-1}{\ell} - \frac{1}{2}\varepsilon_{\mathcal{A}}\right) \cdot n.
\end{aligned}$$

□

Before we derive our next structural corollary we first introduce another bad set. The bad set $\mathcal{D}_i^\ell(n)$ is once again based on the neighborhood assumption.

Definition 3.31 *Let $\mathcal{D}_i^\ell(n)$ denote the set of all graphs $G = (V, E)$ in $\mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n)]$ that contain three vertices u, v, w and q-sets $Q_u \subseteq \Gamma(u)$, $Q_v \subseteq \Gamma(v)$, $Q_w \subseteq \Gamma(w)$ such that the following conditions are satisfied for $W := R(Q_w) \setminus [R(Q_u) \cup R(Q_v)]$. $|W| \geq \frac{1}{4\ell}n$ and*

(i) $|\Gamma(u) \cap W| \leq \frac{|W|}{2} - \varepsilon_{\mathcal{D}}n$, or

(ii) $|\Gamma(u) \cap \Gamma(v) \cap W| \leq \frac{|W|}{4} - \varepsilon'_{\mathcal{D}}n$.

Lemma 3.32 *For all $i \geq 0$ and n sufficiently large,*

$$\log \frac{|\mathcal{D}_i^\ell(n)|}{|\mathcal{F}_i^\ell(n-1)|} \leq \left(\frac{\ell-1}{\ell} - \varepsilon_{\mathcal{A}}\right) \cdot n.$$

Proof Figure 3.8 visualizes how the bad set $\mathcal{D}_i^\ell(n)$ is counted.

We construct all graphs in $\mathcal{D}_i^\ell(n)$ as follows: First we choose the vertex u and a graph $G' \in \mathcal{F}_i^\ell(n-1)$ for the remaining vertices. Then we fix the vertices v and w and the q-sets Q_u, Q_v and Q_w. There are at most

$$n^3 \cdot \binom{n}{(\ell-1)f_{\ell-1}(n)}^3 \cdot |\mathcal{F}_i^\ell(n-1)| \leq n^{4\log n} \cdot |\mathcal{F}_i^\ell(n-1)|$$

possibilities for that.

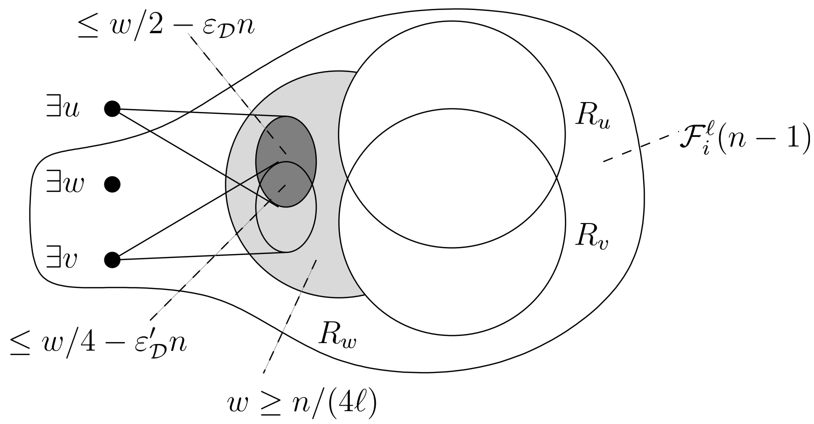

Figure 3.8: Bad set $\mathcal{D}_i(n)$

Assume that Condition (i) of Definition 3.31 holds. Note that $|R(Q_u)| \geq (\frac{1}{\ell} - \varepsilon_{\mathcal{A}})n$ since $G \notin \mathcal{A}_i^\ell(n)$. This implies that the number of choices for the edges from u to $V \setminus [R(Q_u) \cup W]$ is bounded from above by

$$2^{n-|R(Q_u)|-|W|} \leq 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}})n-|W|}.$$

Furthermore, there are at most

$$\sum_{j=0}^{\frac{1}{2}|W|-\varepsilon_{\mathcal{D}} n} \binom{|W|}{j} \leq |W| \cdot \binom{|W|}{\frac{1}{2}|W| - \varepsilon_{\mathcal{D}} n} \leq n \cdot 2^{|W|-\varepsilon_1 n}$$

possibilities for the edges from u to W, where the last estimate follows by Inequality 3. All in all, we obtain the following upper bound on the number of choices:

$$n^{4\log n} \cdot |\mathcal{F}_i^\ell(n-1)| \cdot 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}}-\varepsilon_1)n}.$$

If, on the other hand, Condition (ii) of Definition 3.31 holds we may safely assume that Condition (i) does not hold for any choice of the vertices u, v and w and their q-sets. Hence, it follows that $|\Gamma(v)\cap W| =: \xi \geq \frac{1}{2}|W| - \varepsilon_{\mathcal{D}} n \geq (\frac{1}{8\ell} - \varepsilon_{\mathcal{D}})n$. First we fix the edges from u to $V \setminus [R(Q_u) \cup (\Gamma(v) \cap W)]$. The number of choices is bounded from above by

$$2^{n-|R(Q_u)|-|\Gamma(v)\cap W|} \leq 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}})n-\xi}.$$

For the edges from u to $\Gamma(v) \cap W$ there are at most

$$\sum_{j=0}^{\frac{1}{4}|W|-\varepsilon'_{\mathcal{D}} n} \binom{\xi}{j} \leq n \cdot \binom{\xi}{\frac{1}{4}|W| - \varepsilon'_{\mathcal{D}} n} \leq n \cdot 2^{\xi \cdot H((\frac{1}{4}|W|-\varepsilon'_{\mathcal{D}} n)/\xi)}$$

possibilities. For $\varepsilon'_{\mathcal{D}} \geq \frac{1}{2}\varepsilon_{\mathcal{D}}$ (cf. Inequality 4) one easily checks that

$$\frac{\frac{1}{4}|W| - \varepsilon'_{\mathcal{D}} n}{\xi} \leq \frac{\frac{1}{4}|W| - \varepsilon'_{\mathcal{D}} n}{\frac{1}{2}|W| - \varepsilon_{\mathcal{D}} n} \leq \frac{(\frac{1}{2} - \varepsilon'_{\mathcal{D}})n}{(1 - \varepsilon_{\mathcal{D}})n} \leq \frac{1}{2}.$$

Thus,

$$H\left(\frac{\frac{1}{4}|W| - \varepsilon'_{\mathcal{D}} n}{\xi}\right) \leq H\left(\frac{(\frac{1}{2} - \varepsilon'_{\mathcal{D}})n}{(1 - \varepsilon_{\mathcal{D}})n}\right) \leq 1 - \varepsilon_1$$

by Inequality 5. The total number of choices is therefore bounded from above by

$$\begin{aligned} & n^{4\log n} \cdot |\mathcal{F}_i^\ell(n-1)| \cdot 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}})n - \varepsilon_1 \xi} \\ \leq \quad & n^{4\log n} \cdot |\mathcal{F}_i^\ell(n-1)| \cdot 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}} - \varepsilon_1(\frac{1}{8\ell} - \varepsilon_{\mathcal{D}}))n} \\ \leq \quad & n^{4\log n} \cdot |\mathcal{F}_i^\ell(n-1)| \cdot 2^{(1-\frac{1}{\ell}+\varepsilon_{\mathcal{A}} - \varepsilon_1 \frac{1}{16\ell})n}, \end{aligned}$$

applying Inequality 6. Using Inequality 2 we can combine the two cases and obtain

$$\begin{aligned} \log \frac{|\mathcal{D}_i^\ell(n)|}{|\mathcal{F}_i^\ell(n-1)|} \quad & \leq \quad 1 + 4(\log n)^2 + n \cdot \left(1 - \frac{1}{\ell} + \varepsilon_{\mathcal{A}} - \varepsilon_1 \frac{1}{16\ell}\right) \\ & \leq \quad 5(\log n)^2 + n \cdot \left(1 - \frac{1}{\ell} + \varepsilon_{\mathcal{A}} - 3\varepsilon_{\mathcal{A}}\right) \leq n \cdot \left(1 - \frac{1}{\ell} - \varepsilon_{\mathcal{A}}\right). \end{aligned}$$

□

Now we are in a position to prove our next structural corollary. It shows that all graphs which have not yet been excluded remain i-quasi-ℓ-partite after the deletion of at most ℓ vertices.

Corollary 3.33 *For all graphs $G \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n)]$ and arbitrary vertices $V' \subseteq V$ with $|V'| =: k \leq \ell$ it holds that $G' := G[V \setminus V'] \in \mathcal{F}_i^\ell(n-k)$.*

Proof Consider a graph $G \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n)]$. It suffices to show that $G \notin \bigcup_{j=i+1}^{\ell+i} \mathcal{P}_j^\ell(n)$ because if this condition holds at least $\ell + i + 1$ vertices must be deleted in order to make the graph ℓ-partite.

Assume that $G = (V, E) \in \mathcal{P}_j^\ell(n)$ for some $j \in \{i+1, \ldots, \ell+i\}$. Then we can find vertices $v_1, \ldots, v_j$ such that $G' = (V', E') := G[V \setminus \{v_1, \ldots, v_j\}] \in \mathcal{P}_0^\ell(n-j)$. Let $S_1, \ldots, S_\ell$ be a valid ℓ-coloring of G'. Figure 3.9 visualizes this situation.

Consider an arbitrary vertex $x \in S_1$ with q-set $Q_x \subseteq \Gamma(x)$. It is easy to see that $R(Q_x) \subseteq S_1 \cup \{v_1, \ldots, v_j\}$. Hence, $|S_1| \geq (\frac{1}{\ell} - 2\varepsilon_{\mathcal{A}})n$ since $G \notin \mathcal{A}_i^\ell(n)$. By symmetry this also holds for $S_2, \ldots, S_\ell$.

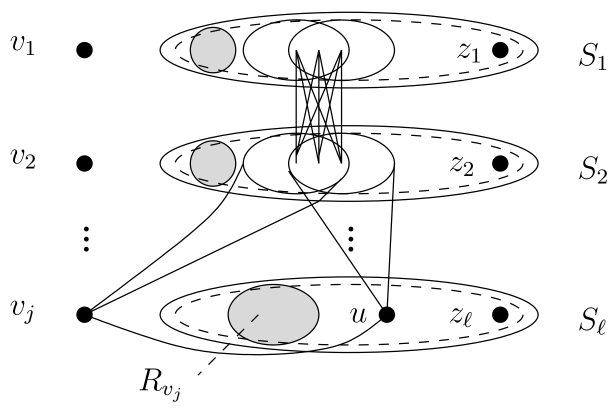

Figure 3.9: Situation in proof of Corollary 3.33

For brevity let $v_\ell =: v$. By Q_v we denote a q-set with $Q_v \subseteq \Gamma(v)$ which exists due to Corollary 3.27. Without loss of generality we assume that S_ℓ is a color class where the cardinality of the intersection with $R(Q_v)$ is maximum. By $G \notin \mathcal{C}_i^\ell(n)$ it follows that $|R(Q_v) \cap S_k| \leq \frac{1}{2}(\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$ for $k = 1, \ldots, \ell - 1$.

v must have a neighbor u in S_ℓ. Otherwise, G could be made ℓ-partite by removing just $v_2, \ldots, v_j$. Let $Q_u \subseteq \Gamma(u)$ be a q-set. As we have already seen, $R(Q_u) \subseteq S_\ell \cup \{v_1, \ldots, v_j\}$.

For the color classes $S_1, \ldots, S_{\ell-1}$ we fix vertices $z_1, \ldots, z_{\ell-1}$ with $z_k \in S_k$ and q-sets $Q_{z_k} \subseteq \Gamma(z_k)$ for $k = 1, \ldots, \ell - 1$. Note that $R(Q_{z_k}) \subseteq S_k \cup \{v_1, \ldots, v_j\}$. Furthermore, we define $W_k := R(Q_{z_k}) \setminus [R(Q_u) \cup R(Q_v)]$. Since $R(Q_{z_k}) \cap R(Q_u) \subseteq \{v_1, \ldots, v_j\}$ and $R(Q_v) \cap R(Q_{z_k}) \subseteq (R(Q_v) \cap S_k) \cup \{v_1, \ldots, v_j\}$, we have

$$\begin{aligned}
|W_k| &\geq |R(Q_{z_k})| - j - |R(Q_v) \cap S_k| \geq \left(\frac{1}{\ell} - \varepsilon_{\mathcal{A}}\right) n - j - \frac{1}{2}\left(\frac{1}{\ell} + \varepsilon_{\mathcal{A}}\right) n \\
&\geq \left(\frac{1}{2\ell} - 2\varepsilon_{\mathcal{A}}\right) n \geq \frac{1}{4\ell} \cdot n,
\end{aligned}$$

where the last inequality is a consequence of Inequality 7. Since $G \notin \mathcal{D}_i^\ell(n)$ and because of Inequality 6 it follows that

$$|\Gamma(u) \cap \Gamma(v) \cap W_k| \geq \left(\frac{1}{16\ell} - \varepsilon'_{\mathcal{D}}\right) n \geq \frac{1}{32\ell} \cdot n$$

Using the same approach as in the proof of Corollary 3.27 we construct a $K_{\ell-1}$ in $\Gamma(u) \cap \Gamma(v)$: By induction on k we show that $\Gamma(u) \cap \Gamma(v) \cap (S_1 \cup \ldots \cup S_k) =: N_k$ contains a $q(k)$-set. The base case $k = 0$ is trivial.

Now assume that we have already found a $q(k)$-set Q with $Q \subseteq N_k$. Consider the set $L := [\Gamma(u) \cap \Gamma(v) \cap S_{k+1}] \setminus R(Q)$. $|L|$ is small because, clearly, $R(Q) \cap (S_1 \cup \ldots \cup S_k) = \emptyset$ and, therefore,

$$
\begin{aligned}
|V| = n & \geq |S_1| + \ldots + |S_k| + |R(Q)| + |L| \\
& \geq \left(\frac{1}{\ell} - 2\varepsilon_{\mathcal{A}}\right) \cdot k \cdot n + \left(\frac{\ell - k}{\ell} - \varepsilon_{\mathcal{A}}\right) n + |L| \\
& = (1 - 2k\varepsilon_{\mathcal{A}} - \varepsilon_{\mathcal{A}})n + |L|
\end{aligned}
$$

and, thus, $|L| \leq 2(k+1)\varepsilon_{\mathcal{A}} n \leq 2\ell\varepsilon_{\mathcal{A}} n \leq \frac{1}{64\ell} \cdot n$ by Inequality 7. This implies that $|\Gamma(u) \cap \Gamma(v) \cap S_{k+1} \cap R(Q)| = |\Gamma(u) \cap \Gamma(v) \cap S_{k+1}| - |L| \geq \frac{1}{64\ell} \cdot n$. Hence, as in the proof of Corollary 3.27 we can construct a $q(k)$-set within N_k.

This yields a contradiction because we can easily find a $K_{\ell+1}$ composed of the vertices u, v and an arbitrary $K_{\ell-1}$ taken from the $q(\ell-1)$-set within $N_{\ell-1} \subseteq \Gamma(u) \cap \Gamma(v)$. □

Finally, we define our last type of bad sets and derive a bound on their cardinality.

Definition 3.34 *For $k \in \{1, \ldots, l\}$ let $\mathcal{E}_i(n,k)$ denote the set of all graphs $G = (V,E) \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n)]$ that contain a k-clique $S_k = \{v_1, \ldots, v_k\}$ such that $G' := G[V \setminus S_k] \in \mathcal{F}_i^\ell(n-k)$ with*

$$S_k \text{ is not contained in a } k+1\text{-clique}, \tag{3.1}$$

and for each vertex v_j there exists a q-set $Q_j \subseteq \Gamma(v_j)$ such that $|R(Q_j)| \geq \frac{n}{l} - \varepsilon_{\mathcal{A}} n$ and

$$U := V \setminus \left(S_k \cup \bigcup_{j=1}^{k} R(Q_j)\right)$$

satisfies

$$|U| \geq \varepsilon_{\mathcal{E}} n. \tag{3.2}$$

Lemma 3.35 *For $k \in \{1, \ldots, l\}$, $i \geq 0$ and n sufficiently large,*

$$\log \frac{|\mathcal{E}_i(n,k)|}{|\mathcal{F}_i^\ell(n-k)|} \leq \frac{\ell - 1}{\ell} nk - \frac{\varepsilon_{\mathcal{E}}}{2^{\ell+1}} n.$$

Proof Figure 3.10 visualizes how the bad set $\mathcal{E}_i(n,k)$ is counted.

We construct all graphs $G = (V,E)$ in $\mathcal{E}_i(n,k)$ as follows: First we choose $S_k = \{v_1, \ldots, v_k\}$ and a suitable graph $G' \in \mathcal{F}_i^\ell(n-k)$ on the vertices $V \setminus S_k$ (at most $n^k \cdot |\mathcal{F}_i^\ell(n-k)|$ possibilities).

Now we choose the q-sets $Q_1, \ldots, Q_k \subseteq V$ such that the corresponding sets $R(Q_j)$ satisfy the conditions of Definition 3.34 (at most $n^{(\ell-1)kf_{\ell-1}(n)}$ possibilities).

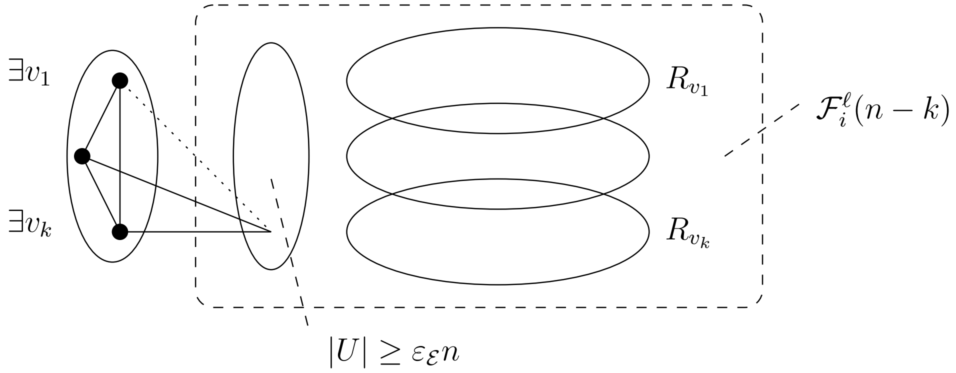

Figure 3.10: Bad set $\mathcal{E}_i(n,k)$

Finally, we connect S_k to $V \setminus S_k$. For every $1 \leq h \leq k$ and every h-tupel $1 \leq i_1 < \ldots < i_h \leq k$ we define the set $T_{i_1,\ldots,i_h} = (R(Q_{i_1}) \cap \ldots \cap R(Q_{i_h})) \setminus \bigcup_{j \notin \{i_1,\ldots,i_h\}} R(Q_j)$. Then the following holds for the set U by definition:

$$U = V \setminus \left(\bigcup_{h=1}^{k} \bigcup_{i_1,\ldots,i_h} T_{i_1,\ldots,i_h} \cup S_k \right),$$

The number of possibilities to connect S_k to $T_{i_1,\ldots,i_h}$ is at most $2^{(k-h)|T_{i_1,\ldots,i_h}|}$. To see this, observe that v_j and $R(Q_j)$ cannot be connected.

By assumption (3.1) of Definition 3.34 there are at most $(2^k-1)^{|U|} \leq 2^{(k-2^{-k})\cdot|U|}$ possibilities to connect S_k to U. Putting all this together we obtain at most

$$2^{(k-2^{-k})\cdot|U|} \cdot 2^{\sum_{h=1}^{k}\sum_{i_1,\ldots,i_h}|T_{i_1,\ldots,i_h}|(k-h)}$$

possible choices for the connections from S_k to $V \setminus S_k$. It holds that

$$\begin{aligned}
\sum_{h=1}^{k}\sum_{i_1,\ldots,i_h}|T_{i_1,\ldots,i_h}|(k-h) &= k\left(\sum_{h=1}^{k}\sum_{i_1,\ldots,i_h}|T_{i_1,\ldots,i_h}|\right) - \sum_{h=1}^{k}\sum_{i_1,\ldots,i_h}|T_{i_1,\ldots,i_h}|h \\
&\leq k|V \setminus U| - \sum_{j=1}^{k}|R(Q_j)| \\
&\leq kn - k|U| - k\left(\frac{n}{l} - \varepsilon_{\mathcal{A}} n\right) \\
&= \frac{l-1}{l}nk - k|U| + k\varepsilon_{\mathcal{A}} n.
\end{aligned}$$

Now the lemma follows by Inequality 8:

$$\log\frac{|\mathcal{E}_k(n)|}{|\mathcal{F}_i^\ell(n-k)|} \leq k\log n + (l-1)kf_{\ell-1}(n)\log n + (k-2^{-k})|U|$$

$$
\begin{aligned}
& +\frac{l-1}{l}nk - k|U| + k\varepsilon_{\mathcal{A}}n \\
\leq\ & \frac{l-1}{l}nk - 2^{-k}|U| + 2k\varepsilon_{\mathcal{A}}n \leq \frac{l-1}{l}nk - \frac{\varepsilon_{\mathcal{E}}}{2^k}n + 2k\varepsilon_{\mathcal{A}}n \\
\leq\ & \frac{l-1}{l}nk - \frac{1}{2^{l+1}}\varepsilon_{\mathcal{E}}n.
\end{aligned}
$$

□

Corollary 3.36 *Let $\mathcal{E}_i^\ell(n) := \bigcup_{k=1}^{l} \mathcal{E}_i^\ell(n,k)$. For all sufficiently large n, the graphs $G \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n) \cup \mathcal{E}_i^\ell(n)]$ have the following property:*

For $1 \leq k < \ell$, every clique of size k is contained in a clique of size ℓ.

Proof Consider an arbitrary graph $G \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n) \cup \mathcal{E}_i^\ell(n)]$. Assume that $S = \{v_1, \ldots, v_k\}$ is a maximal clique in G of size $1 \leq k < \ell$. For every vertex $v_j \in S$ there is a q-set $Q_j \subseteq \Gamma(v_j)$. Since $G \notin \mathcal{C}_i^\ell(n)$, it follows that $\sum_{j=1}^{k} |R(Q_j)| \leq k(\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$, i.e., we obtain for the set U of Lemma 3.35 that

$$
|U| \geq \left(1 - \frac{k}{\ell} - k\varepsilon_{\mathcal{A}}\right) n \geq \left(\frac{1}{\ell} - (\ell-1)\varepsilon_{\mathcal{A}}\right) n \geq \varepsilon_{\mathcal{E}} n,
$$

where the last inequality is a consequence of Inequality 9. This contradicts the assumption that $G \notin \mathcal{E}_k^\ell(n)$. □

Now we have defined enough bad sets in order to cover all graphs in $\mathcal{F}_i^\ell(n) \setminus \mathcal{P}_i^\ell(n)$. This is shown in the following lemma.

Lemma 3.37 *For all sufficiently large n, $\mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n) \cup \mathcal{E}_i^\ell(n)]$ is empty.*

Proof Assume there exists a graph $G = (V, E) \in \mathcal{F}_i^\ell(n) \setminus [\mathcal{P}_i^\ell(n) \cup \mathcal{A}_i^\ell(n) \cup \mathcal{B}_i^\ell(n) \cup \mathcal{C}_i^\ell(n) \cup \mathcal{D}_i^\ell(n) \cup \mathcal{E}_i^\ell(n)]$. We will show that G has to be ℓ-partite, which contradicts the definition of the set $\mathcal{F}_i^\ell(n)$.

Recall that due to Corollary 3.27 there exists a q-set $Q_v \subseteq \Gamma(v)$ for every vertex $v \in V$. The corresponding sets $R(Q_v)$ are denoted by $R(v)$ for brevity. In the sequel we collect some auxiliary results on the structure of the graph G:

(1) For all $v \in V$ it holds that $(\frac{1}{\ell} - \varepsilon_{\mathcal{A}}) \cdot n \leq |R(v)| \leq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}}) \cdot n$.

This is an immediate consequence of the fact that $G \notin \mathcal{A}_i^\ell(n, \ell-1) \cup \mathcal{C}_i^\ell(n)$.

(2) For $v, v' \in V$ with $\{v, v'\} \in E$ it holds that $|R(v) \cap R(v')| \leq 2\varepsilon_{\mathcal{E}} n$.

This can be seen as follows: By Corollary 3.36 we deduce that v and v' belong to an ℓ-clique $S = \{v_1, \ldots, v_\ell\}$. Since this clique cannot be contained in an $(\ell+1)$-clique it follows by Definition 3.34 that

$$\left| \bigcup_{j=1}^{l} R(v_j) \right| \geq n - \varepsilon_{\mathcal{E}} n.$$

As $|R(v_j)| \leq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$ for all $1 \leq i \leq \ell$ by (1), we conclude that (cf. Inequality 8)

$$|R(v) \cap R(v')| \leq (\ell \varepsilon_{\mathcal{A}} + \varepsilon_{\mathcal{E}})n \leq 2\varepsilon_{\mathcal{E}} n.$$

(3) For all $v \in V$ and $x \in R(v)$ it holds that $|R(v) \cap R(x)| \geq \frac{9}{10\ell} n$.

Since $x \in R(v)$ we conclude by the definition of $R(v)$ that there are vertices $v_1, \ldots, v_{\ell-1}$ in Q_v such that $v_1, \ldots, v_{\ell-1}, x$ as well as $v_1, \ldots, v_{\ell-1}, v$ form an ℓ-clique. Set $T := \bigcup_{j=1}^{\ell-1} R(v_j)$. By (1) and (2) we deduce using Inequality 8 that

$$\begin{aligned} |T| &\geq \left((\ell-1)\left(\frac{1}{\ell} - \varepsilon_{\mathcal{A}}\right) - \binom{\ell-1}{2} \cdot 2\varepsilon_{\mathcal{E}} \right) n \\ &\geq \left(\frac{\ell-1}{\ell} - (\ell-1)\varepsilon_{\mathcal{A}} - (\ell-1)\ell\varepsilon_{\mathcal{E}} \right) n \geq \left(\frac{\ell-1}{\ell} - \ell^2 \varepsilon_{\mathcal{E}} \right) n \end{aligned}$$

and that

$$|R(x) \cap T| \leq 2\ell\varepsilon_{\mathcal{E}} n, \qquad |R(v) \cap T| \leq 2\ell\varepsilon_{\mathcal{E}} n.$$

This implies that

$$|R(x) \cup R(v)| \leq |R(x) \cap T| + |R(v) \cap T| + (V \setminus T) \leq \left(4\ell\varepsilon_{\mathcal{E}} + \frac{1}{\ell} + \ell^2\varepsilon_{\mathcal{E}} \right) n.$$

Due to $|R(v)|, |R(x)| \geq (\frac{1}{l} - \varepsilon_{\mathcal{A}})n$ it follows that

$$\begin{aligned} |R(x) \cap R(v)| &= |R(x)| + |R(v)| - |R(x) \cup R(v)| \\ &\geq \left(\frac{2}{\ell} - 2\varepsilon_{\mathcal{A}} \right) n - \left(4\ell\varepsilon_{\mathcal{E}} + \frac{1}{\ell} + \ell^2\varepsilon_{\mathcal{E}} \right) n \\ &\geq \left(\frac{1}{\ell} - 2\varepsilon_{\mathcal{A}} - 4\ell\varepsilon_{\mathcal{E}} - \ell^2\varepsilon_{\mathcal{E}} \right) n \geq \frac{9}{10\ell} n \end{aligned}$$

by Inequality 10.

After these preliminary observations we are in a position to prove the lemma: Let $S = \{v_1, \ldots, v_\ell\}$ be an ℓ-clique which exists due to Corollary 3.36. For $j = 1, \ldots, \ell$ we set

$$S(v_j) := \left\{ v \in V \;\middle|\; |R(v) \cap R(v_j)| \geq \frac{2}{3}\frac{n}{\ell} \right\}.$$

Consider two vertices x, y that belong both to $S(v_j)$ and which are connected by an edge. As $|R(v_j)| \leq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$ by (1) it follows that $|R(x) \cap R(y)|$ is at least, say, $\frac{1}{4\ell}n$. This contradicts (2) (cf. Inequality 7) and we conclude that the sets $S(v_j)$ are stable.

On the other hand, observe that (1) together with $|\bigcup_{j=1}^{\ell} R(v_j)| \geq n - \varepsilon_{\mathcal{E}} n$ implies that for every vertex $x \in V$ there must be an index $j \in \{1, \ldots, \ell\}$ such that $R(x) \cap R(v_j) \neq \emptyset$. Consider an arbitrary vertex $y \in R(x) \cap R(v_j)$. By (3) it follows that $|R(x) \cap R(y)| \geq \frac{9}{10\ell}n$ and $|R(v_j) \cap R(y)| \geq \frac{9}{10\ell}n$. Since $|R(y)| \leq (\frac{1}{\ell} + \varepsilon_{\mathcal{A}})n$ we conclude that $|R(x) \cap R(v_j)| \geq \frac{2}{3}\frac{n}{\ell}$. From that we deduce that $\bigcup_{j=1}^{\ell} S(v_j) = V$, i.e., G is ℓ-colorable. $\square$

3.6.3 Inductive counting

Finally, we are in a position to show our main result of this chapter.

Proof of Theorem 3.2 We will show the slightly stronger statement that

$$|\mathcal{F}_i^{\ell}(n)| = (1 + c_i \cdot 2^{-\gamma n})|\mathcal{P}_i^{\ell}(n)|.$$

for appropriate constants $c_i \geq 1$ (where $c_0 \leq c_i$ for $i \geq 1$) and $\gamma > 0$.

By Lemma 3.37 it follows that

$$|\mathcal{F}_i^{\ell}(n)| \leq |\mathcal{P}_i^{\ell}(n)| + |\mathcal{A}_i^{\ell}(n)| + |\mathcal{B}_i^{\ell}(n)| + |\mathcal{C}_i^{\ell}(n)| + |\mathcal{D}_i^{\ell}(n)| + |\mathcal{E}_i^{\ell}(n)|.$$

Thus, it suffices to show that $|\mathcal{X}(n)|/|\mathcal{P}_i^{\ell}(n)| = \frac{1}{5\ell}2^{-\gamma n}$ for all bad sets $\mathcal{X}(n)$, i.e.,

$$\begin{aligned}\mathcal{X}(n) \;\in\; & \{\mathcal{A}_i^{\ell}(n,1), \ldots, \mathcal{A}_i^{\ell}(n,\ell-1), \mathcal{B}_i^{\ell}(n,0), \ldots, \mathcal{B}_i^{\ell}(n,\ell-2),\\ & \mathcal{C}_i^{\ell}(n), \mathcal{D}_i^{\ell}(n), \mathcal{E}_i^{\ell}(n,1), \ldots, \mathcal{E}_i^{\ell}(n,\ell)\}.\end{aligned}$$

We proceed by induction on n. Note that we may assume without loss of generality that the claim is true for all $n \leq n_0$ for an arbitrarily large constant n_0. This is due to the fact that we can simply choose $c_i \geq 1$ sufficiently large. For the induction step we note that for all bad sets $\mathcal{X}(n)$ apart from $\mathcal{A}_i^{\ell}(n,1), \ldots, \mathcal{A}_i^{\ell}(n,\ell-1)$ one easily checks by Lemma 3.26, Lemma 3.30, Lemma 3.32 and Lemma 3.35 that for $\gamma > 0$ sufficiently small we have

$$\log \frac{|\mathcal{X}(n)|}{|\mathcal{F}_i^{\ell}(n-k)|} \leq \left(\frac{\ell-1}{\ell} - 3\gamma\right) nk$$

for a suitable constant $k \geq 1$ and n sufficiently large.

Furthermore, it holds that

$$\frac{|\mathcal{X}(n)|}{|\mathcal{P}_i^\ell(n)|} = \frac{|\mathcal{X}(n)|}{|\mathcal{F}_i^\ell(n-k)|} \cdot \frac{|\mathcal{F}_i^\ell(n-k)|}{|\mathcal{P}_i^\ell(n-k)|} \cdot \frac{|\mathcal{P}_i^\ell(n-k)|}{|\mathcal{P}_i^\ell(n)|}.$$

By the induction hypothesis and Corollary 3.21 we deduce that

$$\begin{aligned}
\log \frac{|\mathcal{X}(n)|}{|\mathcal{P}_i^\ell(n)|} &\leq \left(\frac{\ell-1}{\ell} - 3\gamma\right) nk + \log(1+c_i) - \frac{\ell-1}{\ell} nk + \mathcal{O}(1) \\
&\leq \log(c_i) - 2\gamma n
\end{aligned}$$

and, consequently, $|\mathcal{X}(n)|/|\mathcal{P}_i^\ell(n)| \leq \frac{c_i}{5\ell} 2^{-\gamma n}$.

For the bad sets $\mathcal{X}(n) \in \{\mathcal{A}_i^\ell(n,1), \ldots, \mathcal{A}_i^\ell(n,\ell-1)\}$ we proceed similarly by observing that

$$\log \frac{|\mathcal{X}(n)|}{|\mathcal{F}_0^\ell(n-g(n))|} \leq \left(\frac{\ell-1}{\ell} - 3\gamma\right) ng(n)$$

for $g(n) = kf_k(n) = \mathcal{O}(\log n)$ due to Lemma 3.25. Moreover, it holds that

$$\begin{aligned}
\frac{|\mathcal{X}(n)|}{|\mathcal{P}_i^\ell(n)|} &= \frac{|\mathcal{X}(n)|}{|\mathcal{F}_0^\ell(n-g(n))|} \cdot \frac{|\mathcal{F}_0^\ell(n-g(n))|}{|\mathcal{P}_0^\ell(n-g(n))|} \cdot \frac{|\mathcal{P}_0^\ell(n-g(n))|}{|\mathcal{P}_i^\ell(n-g(n)+i(2\ell+1))|} \cdot \\
&\quad \frac{|\mathcal{P}_i^\ell(n-g(n)+i(2\ell+1))|}{|\mathcal{P}_i^\ell(n)|}.
\end{aligned}$$

Applying the induction hypothesis, Corollary 3.21 and Lemma 3.22 we conclude that

$$\begin{aligned}
\log \frac{|\mathcal{X}(n)|}{|\mathcal{P}_i^\ell(n)|} &\leq \left(\frac{\ell-1}{\ell} - 3\gamma\right) n\,g(n) + \log(1+c_0) + 0 \\
&\quad - \frac{\ell-1}{\ell} n\,g(n) + \mathcal{O}(g(n)^2) \\
&\leq \log(c_i) - 2\gamma n.
\end{aligned}$$

Once again, it follows that $|\mathcal{X}(n)|/|\mathcal{P}_i^\ell(n)| \leq \frac{c_i}{5\ell} 2^{-\gamma n}$ and the induction step is completed. $\square$

4

Complete subgraphs of ε-regular graphs

4.1 Introduction

Szemerédi's Regularity Lemma [Sze76] represents one of the most important tools in modern combinatorics. In graph theory this lemma and its variants have been applied to numerous problems but its application is mainly restricted to dense graphs, i.e., graphs with $\Theta(n^2)$ edges.

Independently, Kohayakawa and Rödl [Koh97] have devised versions of the Regularity Lemma which can be used for sparse graphs, too. However, the power of the sparse Regularity Lemma is yet limited by the fact that an additional lemma, which plays a crucial rôle in many applications and is rather easy to prove in the dense case, does not immediately carry over to sparse graphs.

It has been shown that this so-called embedding lemma does not hold for all sparse graphs, but there is strong evidence that it should hold in a probabilistic sense, i.e., for 'most' graphs. Kohayakawa, Łuczak and Rödl [KŁR97] have formulated a conjecture what such a sparse embedding lemma should look like. Due to the importance of this conjecture for the applicability of the Regularity Lemma to sparse random graphs, it is sometimes regarded to be *one of the most important open questions in the theory of random graphs* (cf. [JŁR00]). However, only a few special cases have been verified so far.

The main difficulty in proving the conjecture stems from the fact that the statement involves extremely small probabilities. These probabilities are too small to attack them with standard tools from probability theory. Thus custom-tailored counting methods must be devised.

In this chapter we prove the two 'smallest' open cases of the conjecture by Kohayakawa, Łuczak and Rödl. This also has immediate implications for a conjecture related to extremal H-free subgraphs of random graphs. It yields a conceptually simple proof of the case $H = K_4$, previously verified in [KŁR97], and solves the the case $H = K_5$, which has been open up to now.

The results discussed in this chapter have been presented in January 2002 at workshops in Oberwolfach and Zürich but have not yet been published in written form. A manuscript [GPS+02] for the case $H = K_4$ has been submitted for publication.

Outline of this chapter In Section 4.2 we briefly state our main result. Section 4.3 is meant as a gentle introduction to the conjecture by Kohayakawa, Łuczak and Rödl and its context. We briefly review Szemerédi's Regularity Lemma, its algorithmic variants, and its application in extremal graph theory (cf. Section 4.3.1). Then we introduce a sparse variant and the initially mentioned conjecture (cf. Section 4.3.2). Section 4.4 discusses related work.

The following sections contain the proof of our results and are organized as follows. In general, the proofs for the two cases $H = K_4$ and $H = K_5$ are intertwined as they are based on the same tools. Therefore, most parts of the proof are equally important for both cases. However, certain sections which refer only to one of the cases are marked with (K_4) resp. (K_5) in the heading. A reader who is only interested in one of the two cases may safely skip parts of the proof which deal with the other case. We prefer repeating certain parts of the proof for the two cases rather than introducing cross-references in order to render the sections for the different cases self-contained.

Section 4.5 contains an outline of the proof. Section 4.6 then introduces some notation and comparatively simple but necessary technical results. Section 4.7 presents a tool for proving that a certain set of graphs is small. Finally, the rather lengthy Section 4.8 contains the main part of the proof for both cases $H = K_4$ and $H = K_5$.

4.2 Main result

The conjecture by Kohayakawa et al. and thus also our main result deal with ℓ-partite graphs of the following structure.

Definition 4.1 (((ε,n,m)-regular) *A bipartite graph $B = (U \dot\cup W, E)$ with $|U| = |W| = n$ and $|E| = m$ is called* (ε,n,m)-regular *if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon n$ and $|W'| \geq \varepsilon n$,*

$$\left| |E(U',W')| - m \cdot \frac{|U'| \cdot |W'|}{n^2} \right| \leq \varepsilon m \cdot \frac{|U'| \cdot |W'|}{n^2}.$$

Definition 4.2 (Regular ℓ-partite graphs) *Consider a fixed graph H on ℓ vertices. An ℓ-partite graph $G = (V_1 \dot\cup \ldots \dot\cup V_\ell, E)$ is called* (H,n,m)-graph *if*

- $|V_1| = \ldots = |V_\ell| = n$, *and*
- $|E(V_i,V_j)| = m$ *for all $\{i,j\} \in E(H)$, and $E(V_i,V_j) = \emptyset$ otherwise.*

A (H,n,m)-graph is said to be $(H,n,m;\varepsilon)$-regular *if $G[V_i,V_j]$ is (ε,n,m)-regular for all $\{i,j\} \in E(H)$.*

Let $\mathcal{S}(H,n,m;\varepsilon)$ denote the set of all $(H,n,m;\varepsilon)$-regular graphs and let

$$\mathcal{F}(H,n,m;\varepsilon) := \{G \in \mathcal{S}(H,n,m;\varepsilon) \mid H \not\subseteq G\}.$$

Since in this chapter we concentrate on the case, when H is a complete graph, we set $\mathcal{S}_\ell(n,m;\varepsilon) := \mathcal{S}(K_\ell,n,m;\varepsilon)$ and $\mathcal{F}_\ell(n,m;\varepsilon) := \mathcal{F}(K_\ell,n,m;\varepsilon)$. Using this notation we now state the main result of this chapter.

Theorem 4.3 *For any $\beta > 0$ and $\ell \in \{4,5\}$ there exist constants $\varepsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}_\ell(n,m;\varepsilon)| \leq \beta^m \binom{n^2}{m}^{\binom{\ell}{2}}$$

for all $m \geq Cn^{2-2/(\ell+1)}$, $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_0$.

Theorem 4.3 directly corresponds to the cases $H = K_4$ and $H = K_5$ of the previously mentioned conjecture by Kohayakawa, Łuczak and Rödl. For a formal statement we refer the reader to Conjecture 4.19 on page 56.

4.3 Background: Regularity and sparse graphs

This section is intended to give a gentle introduction to the background of the conjecture which we are going to (partially) prove in the remainder of this chapter. However, this is not meant to attain the level neither of a comprehensive survey nor of a detailed text book. Instead, we will often appeal to the intuition of the reader and skip lengthy and rather technical proofs.

For more details and applications we refer the reader to the excellent survey articles on the Regularity Lemma [KS96] and [Koh97] which deal with dense resp. sparse graphs. Meanwhile the Regularity Lemma is also treated in many text books on graph theory and random graphs in particular (cf. e.g. [Die97] [Bol98] [Łuc00]).

4.3.1 Szemerédi's Regularity Lemma for dense graphs

When proving a conjecture of Erdős and Turán [ET36] on arithmetic progressions, Szemerédi showed an auxiliary lemma [Sze75] [Sze76] which since then has experienced a remarkable success story. This lemma, which is nowadays known as *Szemerédi's Regularity Lemma*, became a particularly important tool in modern combinatorics and especially in graph theory. For a comprehensive survey on applications of the Regularity Lemma we refer the reader to [KS96].

Informally speaking, the Regularity Lemma states that the vertex set V of any graph $G = (V, E)$ can be partitioned into vertex sets $V_0, V_1, \ldots, V_k$ for a (controllable) constant k such that most bipartite graphs induced by pairs (V_i, V_j) have a pseudorandom structure.

This pseudorandomness is expressed by the notion of *ε-regularity*, which means that the edges are very uniformly distributed in the graph, as it is the case, e.g., for random graphs $G_{n,p}$. This leads to the following definition.

Definition 4.4 (ε-regularity) *Let $\varepsilon > 0$ and consider a graph $G = (V, E)$ with two disjoint vertex sets $A, B \subseteq V$. The pair (A, B) is called* ε-regular *if*

$$|d(X, Y) - d(A, B)| \leq \varepsilon.$$

for all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$.

Recall that $d(X, Y) := e(X, Y)/(|X||Y|)$ denotes the density of the pair X, Y.

Definition 4.4 can be interpreted as follows. Let the pair (A, B) be ε-regular. Then in every pair of subsets (X, Y), where X and Y contain at least a linear number of vertices, we can find approximately as many edges $e(X, Y)$ as we expect by the density $d(A, B)$. More precisely we have

$$(d(A,B) - \varepsilon) \cdot (|X| \cdot |Y|) \leq e(X,Y) \leq (d(A,B) + \varepsilon) \cdot (|X| \cdot |Y|). \tag{4.1}$$

For a random graph $G_{n,p}$ we expect $\mathbb{E}[e(X,Y)] = p \cdot |X| \cdot |Y|$ and by standard concentration bounds from probability theory we easily obtain

$$|e(X,Y) - \mathbb{E}[e(X,Y)]| = o(1) \cdot \mathbb{E}[e(X,Y)] \;\Rightarrow\; |d(X,Y) - p| = o(1)$$

with high probability. Thus, as far as the number of edges between 'large', i.e., linear, subsets is concerned, an ε-regular pair behaves as a random graph $G_{n,p}$ for $p = d(A, B)$.

The Regularity Lemma shows that every graph may be decomposed into such pseudorandom ε-regular structures. It can be stated as follows.

Definition 4.5 (Regular partitions) *Let a graph $G = (V, E)$ be given. A partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k = V$ is called (ε, k)*-regular *if*

- $|V_1| = \cdots = |V_k|$,
- $|V_0| < \varepsilon n = \varepsilon|V|$ *(exceptional vertices),*
- *all but at most εk^2 of the pairs (V_i, V_j) are ε-regular (exceptional pairs).*

Lemma 4.6 *[Sze76]* (Regularity Lemma) *For every $\varepsilon > 0$ and $k_0 \geq 1$ there exist $N_0(\varepsilon, k_0) \geq 1$ and $K_0(\varepsilon, k_0) \geq k_0$ such that for every $G = (V, E)$ with $|V| \geq N_0$ there is a (ε, k)-regular partition with $k_0 \leq k \leq K_0$,.*

Obviously, such a decomposition can be extremely helpful for proving results on the structure of a graph. As we will see soon, it is quite straightforward to see that ε-regularity implies strong bounds, e.g., on the degree of individual vertices and the overlap of neighborhoods. Hence standard arguments often suffice to find certain structures first in the ε-regular pairs and then in the original graph. This sometimes turns the proof of rather deep results into a routine task. Due to its power the Regularity Lemma has become an extremely valuable and widely used tool.

Note that there exist quite a few versions of Regularity Lemmas. In particular there are important variants which deal with multiple graphs sharing the same set of vertices or which extend to hypergraphs (cf. [Ste90] [Chu91] [FR92]). However, for brevity we will refer to Lemma 4.6 as 'the' Regularity Lemma.

The original proof of the Regularity Lemma is non-constructive, but later algorithms for finding (ε, k)-regular partitions have been developed (cf. [ADL+92]). In particular, this led to new insight in the approximability of well-known optimization problems, like, e.g., MAXCUT or QUADRATICASSIGNMENT (cf. e.g. [FK96]). We refer the reader to [KR00] for a survey on algorithmic variants and applications of the Regularity Lemma.

Structure of ε-regular pairs

In order to understand the importance of the Regularity Lemma it is necessary to study the structure of ε-regular pairs. In the sequel we collect a few well-known facts about this structure. For a more complete exposition we refer the reader to the survey [KS96] or the books [Die97] [Bol98] [JŁR00].

Although the Regularity Lemma only concerns subsets of linear size, it is easy to obtain the following rather strong result on the degree of individual vertices.

Lemma 4.7 (Degrees in ε-regular pairs) *Consider an ε-regular pair (A, B). For any subset $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$ and $d_Y(v) := |\Gamma(v) \cap Y|$ we have*

$$|\{v \in A \mid d_Y(v) > (1+\varepsilon)d(A,B) \cdot |Y|\}| < \varepsilon|A|,$$

and

$$|\{v \in A \mid d_Y(v) < (1-\varepsilon)d(A,B) \cdot |Y|\}| < \varepsilon|A|.$$

Proof For the proof of the first claim let

$$X^> := \{v \in A \mid d_Y(v) > (1+\varepsilon)d(A,B) \cdot |Y|\}$$

and check Definition 4.4 for the sets $X^>$ and B. For $|X^>| \geq \varepsilon|A|$ this immediately yields a contradiction. The proof of the second claim is analogous. □

Lemma 4.7 shows that almost all vertices approximately have the 'right' degree, i.e., the degree which we would expect in a random graph, into a set Y of linear size. We can even strengthen this result without much effort.

Lemma 4.8 (Intersection property) *Consider an ε-regular pair (A, B) and let $\ell \geq 1$. For $d = d(A, B)$ and any subset $Y \subseteq B$ with $(d-\varepsilon)^{\ell-1}|Y| \geq \varepsilon|B|$ we have*

$$|\{\{v_1, \ldots, v_\ell\} \subseteq A \mid |Y \cap \Gamma(v_1) \cap \ldots \cap \Gamma(v_\ell)| \leq (d-\varepsilon)^\ell |Y|\}| < \ell\varepsilon|A|^\ell.$$

Proof Similar to the proof of Lemma 4.7. Use induction on ℓ. □

Applications of the Regularity Lemma usually aim at finding certain substructures, i.e., subgraphs with a certain property. By means of Lemma 4.8 this is rather easily achieved. We are going to demonstrate this by a small example (cf. Figure 4.1).

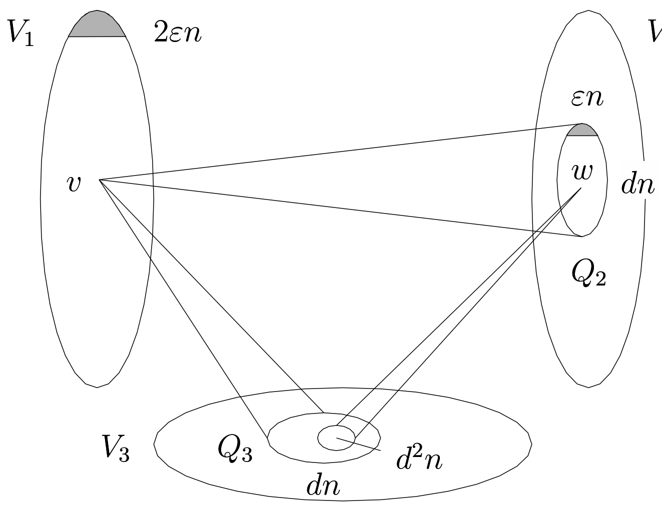

Figure 4.1: A simple example for an embedding lemma

Consider a tripartite graph $G = (V, E)$ with $V = V_1 \dot{\cup} V_2 \dot{\cup} V_3$ and $|V_1| = |V_2| = |V_3| = n$. For all pairs $\{i, j\} \subseteq \{1, 2, 3\}$ we assume that (V_i, V_j) is ε-regular and that $e(V_i, V_j) = m \geq dn^2$ resp. $d(V_i, V_j) \geq d$. We aim at finding a subgraph $K_3 \subseteq G$.

By Lemma 4.7 there exist at most $2\varepsilon n$ vertices $v \in V_1$ such that $d_j(v) < (1-\varepsilon)dn$ for $j \in \{2, 3\}$. Take one such vertex v and consider the neighborhoods $Q_j := \Gamma_j(v)$. Provided that, say, $d \geq 10\varepsilon$ we obtain $|Q_j| \geq (1-\varepsilon)dn \geq 5\varepsilon n$. Thus we can apply Lemma 4.7 one more time to show that there exist at most εn vertices $v \in Q_2$ such that $d_{Q_3}(v) < (1-\varepsilon)d|Q_3|$. Consequently, we can find a vertex $w \in Q_2$ with $d_{Q_3}(w) \geq (1-\varepsilon)d|Q_3| \geq (1-\varepsilon)^2 d^2 n$, and the vertices v, w clearly belong to many triangles $K_3 \subseteq G$.

Based on the ideas sketched in the preceding example the following lemma can be proved. Embedding lemmas of that kind are common in applications of the Regularity Lemma.

Lemma 4.9 (An embedding lemma) *For every $d > 0$, there exist $\varepsilon > 0$ and $N_0 \geq 1$ such that the following holds. Let $H = (V_H, E_H)$ be a graph with $V_H = \{1, \ldots, k\}$. Consider a graph $G = (V_G, E_G)$ and k disjoint subsets $V_1, \ldots, V_k \subseteq V_G$ with $|V_1| = \ldots = |V_k| = N \geq N_0$ such that all pairs (V_i, V_j) with $\{i, j\} \in E_H$ are ε-regular and have density $d(V_i, V_j) \geq d$. Then G contains H as a subgraph.* □

For a proof of Lemma 4.9 we refer the reader to the literature, e.g., [Die97].

Extremal graph theory and the Regularity Lemma

In extremal graph theory questions like 'What is the structure of a K_ℓ-free graph with a maximum number of edges?' are studied. Generally speaking, we investigate the structure of graphs which are extremal for the occurrence of certain substructures. If we put it the other way round, this means that we are interested in properties which imply the existence of such substructures.

A quantity which plays a central rôle in many classical results of extremal graph theory is given by

$$\mathrm{ex}(F, G) := \max\{e(H) \mid G \not\subseteq H \subseteq F\}$$

for graphs F and G, i.e., $\mathrm{ex}(F, G)$ denotes the maximum number of edges a G-free subgraph of F may have. Using this notation we state Turán's Theorem [Tur41], which is generally considered the starting point of extremal graph theory.

Theorem 4.10 *[Tur41]* (Turán's Theorem)

$$\mathrm{ex}(K_n, K_\ell) = \left(1 - \frac{1}{\ell - 1}\right)\frac{n^2}{2} =: t_{\ell-1}(n).$$ □

Theorem 4.10 states that every graph with more that $t_{\ell-1}(n)$ edges contains a subgraph $G = K_\ell$. The following famous theorem by Erdős, Stone and Simonovits generalizes this result to arbitrary subgraphs G. Theorem 4.11 basically says that the maximum number of edges which a G-free graph may have is a function of the chromatic number $\chi(G)$.

Theorem 4.11 *[ES46] [ES66]* (Erdős-Stone-Simonovits-Theorem) *For every graph G with $\chi(G) \geq 3$ we have*

$$\mathrm{ex}(K_n, G) = \left(1 - \frac{1}{\chi(G) - 1} + o(1)\right)\binom{n}{2}.$$

The Regularity Lemma can be used to obtain a rather simple proof of Theorem 4.11. We briefly sketch this proof here because it demonstrates the fundamental strategy which recurs in many applications of the Regularity Lemma.

Proof of Theorem 4.11 (Sketch) It is easy to show that it suffices to prove Theorem 4.11 for graphs K_ℓ^p (recall that K_ℓ^p denotes the complete ℓ-partite graph where every partition consists of p vertices) and $p \in \mathbb{N}$ sufficiently large. For a detailed discussion of this observation and of the remaining proof see, e.g., [Die97].

Let $G = (V, E)$ be a graph with more than $(1 - \frac{1}{\ell-1} + \beta)\binom{n}{2}$ edges for some $\beta > 0$. Observe that $\chi(K_\ell^p) = \ell$. Hence it remains to show that G contains a subgraph K_ℓ^p, provided that $|V|$ is sufficiently large.

Applications of the Regularity Lemma usually consist of the following three steps:

1. Obtain an (ε, k)-regular partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k = V$ by the Regularity Lemma (cf. Lemma 4.6), choosing ε sufficiently small and k_0 sufficiently large (We will see later what 'sufficiently' means.).

2. Consider a graph $R = (V_R, E_R)$ on the vertex set $V_R = \{1, \dots, k\}$, where $\{i, j\} \in E_R$ if and only if the pair (V_i, V_j) is ε-regular and contains a non-negligible number of edges. R is often referred to as the *reduced graph*.

   If the constants for this 'non-negligible' number and for the 'sufficiently' small resp. large values in the previous step are suitably chosen, the graph R must contain many edges. Note that most edges in G belong to pairs $E(V_i, V_j)$ such that $\{i, j\} \in R$, since only the following groups of edges are 'lost':

   (i) Edges inside a partition V_i. This can be controlled by choosing k_0 sufficiently large and thus $|V_i| \leq n/k_0$ sufficiently small.

   (ii) Edges incident to V_0. As $|V_0| \leq \varepsilon n$ only few edges can belong to this group if ε is small.

(iii) Edges between pairs (V_i, V_j) such that $e(V_i, V_j)$ is 'negligible'. Only few pairs (V_i, V_j) may belong to this group, since otherwise not enough space remains for the (many) edges in G which do not belong to the first two groups.

Based on these arguments and by choosing ε and k_0 in a suitable way, one can show that $|E_R| > (1 - \frac{1}{\ell-1})\frac{k^2}{2} = t_{\ell-1}$. By Turán's Theorem (cf. Theorem 4.10) it follows that R contains a subgraph K_ℓ.

3. By Lemma 4.9 it follows that G contains a subgraph K_ℓ. Using a strengthened version of Lemma 4.9 one can even show that $K_\ell^p \subseteq G$, and Theorem 4.11 is proved.

Since this sketch of the proof strategy suffices for our purposes, we conclude our exposition at this point and refer the reader to the literature for more details. □

The above steps (construct an (ε, k)-regular partition using the Regularity Lemma, find a suitable subgraph of the reduced graph R, go back to the graph G using an embedding lemma) occur in many proofs which apply the Regularity Lemma.

4.3.2 KŁR-Conjecture for sparse graphs

Regularity Lemmas for sparse graphs

Note that the statement of the Regularity Lemma is only useful for dense graphs. In the sequel we will call a graph $G = (V, E)$ *dense* if it has $\Theta(n^2)$ edges. Otherwise G is called *sparse*. For sparse pairs $A, B \subseteq V$ with $e(A, B) = o(n^2)$ it follows that $d(A, B) = o(1)$. In this case (4.1) obviously becomes meaningless.

However, by suitable modifications of the definition of regularity it is possible to derive Regularity Lemmas for sparse graphs. This was independently observed by Kohayakawa and Rödl (cf. [Koh97]). In the sequel we present a rather simple version of a sparse Regularity Lemma, which will suffice for our introductory exposition.

Let a graph $G = (V, E)$ be given. For a disjoint pair of sets $A, B \subseteq V$ the *normalized density* is defined by

$$\bar{d}(A, B) := \frac{e(A, B)}{|A||B|} \frac{\binom{V}{2}}{|E|} = d(A, B) \cdot \frac{\binom{V}{2}}{|E|}.$$

For the sake of completeness we set $\bar{d}(A, B) = 0$ if $|E| = 0$.

The scaling factor $\binom{V}{2}/|E|$ ensures that $\bar{d}(A, B)$ does not necessarily become small for sparse graphs, i.e., for $|E| = o(|V|^2)$. By replacing the density $d(A, B)$ with the normalized density $\bar{d}(A, B)$ we get a new notion of regularity which is useful also for sparse graphs.

Definition 4.12 (Strong ε-regularity) *Let $\varepsilon > 0$ and consider a graph $G = (V, E)$ with two disjoint sets $A, B \subseteq V$. The pair A, B is called* strongly ε-regular *if*

$$|\bar{d}(X, Y) - \bar{d}(A, B)| \leq \varepsilon$$

for all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$.

The goal of a sparse Regularity Lemma is to find strongly ε-regular partitions, which are identical to ε-regular partitions up to the fact that strong ε-regularity must hold for most pairs (V_i, V_j).

Definition 4.13 (Strongly regular partitions) *Let a graph $G = (V, E)$ be given. A partition $V_0 \dot{\cup} V_1 \dot{\cup} \cdots \dot{\cup} V_k = V$ is called* strongly (ε, k)-regular *if*

- $|V_1| = \cdots = |V_k|$,
- $|V_0| < \varepsilon n = \varepsilon|V|$ *(exceptional vertices),*
- *all but at most εk^2 of the pairs (V_i, V_j) are strongly ε-regular (exceptional pairs).*

For reusing the idea of the proof of the original Regularity Lemma an additional assumption has to be made. The graph G that shall be regularly partitioned must satisfy certain bounds which show that the density of linear subgraphs is not too big. The concept of γ-boundedness stated below represents one way to formalize that.

Definition 4.14 ((b, γ)-boundedness) *Let $b \geq 1$ and $\beta > 0$. A graph $G = (V, E)$ is* (b, γ)-bounded *if for every pair of disjoint subsets $A, B \subseteq V$ with $|A|, |B| \geq \gamma|V|$ we have $\bar{d}(A, B) \leq b$.*

Now we are in a position to state a sparse version of the Regularity Lemma. As in the dense case several versions of sparse Regularity Lemmas have been devised. Lemma 4.15 is a rather simple and weak variant, but it should suffice to point out the basic concepts. Our notation follows [JŁR00].

Lemma 4.15 *[Koh97]* (Sparse Regularity Lemma) *For every $\varepsilon > 0$ and $k_0, b \geq 1$ there exist $\gamma = \gamma(\varepsilon, b, k_0) > 0$, and $K_0(\varepsilon, k_0, b) \geq k_0$ such that for every (b, γ)-bounded graph $G = (V, E)$ with $|V| \geq k_0$ there is a strongly (ε, k)-regular partition.* □

Applying the sparse Regularity Lemma to $G_{n,m}$

The restriction to (b, γ)-bounded graphs does not severely limit the applicability of Lemma 4.15. In the sequel we will apply the Regularity Lemma to sparse random graphs $G_{n,m}$, where $m = o(n^2)$. To this aim we will firstly show that such random graphs are indeed (b, γ)-bounded.

Definition 4.16 (γ-uniformity) *A graph $G = (V, E)$ is* γ-uniform *if for every pair of disjoint subsets $A, B \subseteq V$ with $|A|, |B| \geq \gamma|V|$ we have*

$$1 - \gamma \leq \bar{d}(A, B) \leq 1 + \gamma.$$

Lemma 4.17 (Uniformity of random graphs) *For $m = \omega(n)$ and $\gamma > 0$ a random graph $G = G_{n,m}$ is asymptotically almost surely γ-uniform.*

Proof Follows easily by Chernoff/Höffding bounds. □

Lemma 4.17 implies in particular that G is, say, $(2, \gamma)$-bounded.

It is tempting to transfer well-known (deterministic) results for dense graphs which can be proved using the Regularity Lemma to sparse random graphs. In [KŁR97] the following conjecture was presented, which resembles the Erdös-Stone-Simonovits Theorem (cf. Theorem 4.11) in the deterministic setting (see also [HKŁ95] [HKŁ96], where the conjecture was already outlined).

Conjecture 4.18 *[KŁR97]* (Conjecture on $\mathrm{ex}(G_{n,m}, H)$) *Let H be a non-empty graph with $v_H \geq 3$. Then asymptotically almost surely*

$$\mathrm{ex}(G_{n,m}, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) m,$$

provided that $m = \omega(n^{2-1/d_2(H)})$.

Similar to the place of the Erdős-Stone-Simonovits Theorem in extremal graph theory, Conjecture 4.18 would be at the core of a *random extremal graph theory* — a theory which would study the structure of extremal subgraphs of random graphs and which is yet to be written.

A natural proof strategy for Conjecture 4.18 would proceed, analogously to the proof of Theorem 4.11, by using the sparse Regularity Lemma. And, indeed, the first part of the proof goes through quite easily. Given a fixed constant $\gamma > 0$, consider an arbitrary subgraph J of a random graph $G = G_{n,m}$ with at least $(1 - \frac{1}{\chi(H)-1} + \gamma)m$ edges. We intend to show that asymptotically almost surely $H \subseteq J$ for any choice of $J \subseteq G$.

For Step 1 we use the sparse Regularity Lemma. In Step 2 we can bound the edges outside the regular partition due to the uniformity of random graphs (cf. Lemma 4.17). Hence, as in the deterministic case, a subgraph K_p can be found in the reduced graph.

Unfortunately, Step 3 fails, since we do not have a counterpart of Lemma 4.9, the embedding lemma, for sparse graphs. Recall that the basic idea for the proof of Lemma 4.9 was to repeatedly exploit the fact that most vertices in ε-regular graphs have the right degree into subsets of linear size (cf. Lemma 4.7). Such a lemma still holds for sparse strongly ε-regular graphs. However, the expected degree of a vertex is equal to $\Theta(n \cdot \frac{m}{n^2}) = \Theta(\frac{m}{n}) = o(n)$ for $m = o(n^2)$. As the size of the neighborhood of a typical vertex is thus sublinear, the repeated application of Lemma 4.7 for estimating the size of the common neighborhood of several vertices fails. It even turns out that not only the proof technique for Lemma 4.9 does not carry over to the sparse case, but even that there are graphs which represent counterexamples for a direct analog of Lemma 4.9. In [KR01] such a counterexample (attributed to Łuczak) is cited and a result of a similar flavor is shown.

Conjecture by Kohayakawa, Łuczak and Rödl

Since there is no *deterministic* embedding lemma for sparse graphs, it is natural to ask, whether there is a *probabilistic* variant, which guarantees the existence of a specific subgraph with high probability. However, a rather simple heuristic argument reveals that such a lemma would presumably have to hold with extremely high probability in order to be useful in applications. More specifically, it will turn out that we require a probability for the occurrence of a subgraph H which is at least as large as $1 - \beta^m$ for a suitably small β.

For the proof of Conjecture 4.18 we must show that $H \subseteq J$ for *every* subgraph $J \subseteq G = G_{n,m}$ with $e(J) \geq (1 - \frac{1}{\chi(H)-1} + \gamma)m$. Note that there are roughly 2^m choices for the subgraph J. Furthermore, it appears to be intractable to control the dependencies of the events '$H \subseteq J_1$' and '$H \subseteq J_2$'

for two different subgraphs $J_1, J_2 \subseteq G$. Thus there seems to be no better approach than to estimate $\Pr[H \not\subseteq J_0]$ for an arbitrary but fixed subgraph $J_0 \subseteq H$ and to use the trivial union bound

$$\Pr[\exists J \not\subseteq G = G_{n,m}: \ H \subseteq J] \leq 2^m \cdot \Pr[H \not\subseteq J_0].$$

Hence, the probabilistic embedding lemma should provide a bound on the probability $\Pr[H \not\subseteq J]$ which is much smaller than 2^{-m}.

A rather simple lower bound shows that such small probabilities are out of reach for random graphs $G_{n,m}$. To see this, fix a partition of the n vertices into $k := \chi(H) - 1$ partitions of size n/k (for simplicity's sake assume that k divides n). Clearly, if none of the m edges of $G = G_{n,m}$ lies within one of the k partitions, it follows that $H \not\subseteq G$. For $m \leq c\binom{n}{2}$ we deduce that

$$\Pr[H \not\subseteq G] \geq \left\{ \left(\frac{k-1}{k} \binom{n}{2} - m \right) \cdot \binom{n}{2}^{-1} \right\}^m \geq \left(1 - \frac{1}{k} - c \right)^m.$$

This implies that a probabilistic embedding lemma must exploit the fact that a fixed graph H shall be found in an ε-regular partition. From an intuitive point of view it seems possible that the ε-regularity might lead to smaller probabilities for the occurrence of a given subgraph H than in a classical random graph $G_{n,m}$. This is due to the fact that in ε-regular structures the edges are nicely distributed. More precisely, the bound on the number of edges between linear sets of vertices should imply that in given parts of the graph the local number of edges closely follows its expectation. For example the degree of most vertices should be proportional to the density and also the overlap of neighborhoods should occur as expected. This intuition turns out to be justified, as the analysis of our main result in this chapter will show.

In [KŁR97] Kohayakawa, Łuczak and Rödl conjectured that a probabilistic embedding lemma indeed exists. We cite a simplified version from [KR01].

Conjecture 4.19 *[KR01]* (KŁR-Conjecture) *Let H denote an arbitrary graph. For any $\beta > 0$ there exist constants $\varepsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}(H, n, m; \varepsilon)| \leq \beta^m \binom{n^2}{m}^{e(H)} \tag{4.2}$$

for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$ and $0 < \varepsilon \leq \varepsilon_0$.

Recall (3.7), where we have defined that

$$d_2(H) := \max\left\{\frac{e_F - 1}{v_F - 2} \mid F \subseteq H, v_F > 2\right\}.$$

Observe that $(H, n, m; \varepsilon)$-regular graphs directly correspond to the ε-regular structures which we encounter when we apply the (sparse) Regularity Lemma and identify a subgraph H in the reduced graph. There are only two minor technical differences. Firstly, the Regularity Lemma does not guarantee that $|E(V_i, V_j)|$ is identical for all $\{i, j\} \in E(H)$. However, it easy to see that a seemingly more general version of Conjecture 4.19 which holds for non-identical values of $|E(V_i, V_j)|$ can be deduced from the simpler version stated above (cf. [Łuc00]). Secondly, the sparse Regularity Lemma was formulated in terms of the normalized density $\bar{d}$ which uses the normalization factor $|E|/\binom{|V|}{2}$, whereas in the definition of (ε, n, m)-regularity we have the factor m/n^2. But clearly these two quantities only differ by a constant.

Conjecture 4.19 can directly be rephrased in a probabilistic style. To this aim let $\mathbb{G}(H, n, m)$ denote an (H, n, m)-graph drawn uniformly at random. Using this notation (4.2) reads

$$\Pr[G = \mathbb{G}(H, n, m) \in \mathcal{S}(H, n, m; \varepsilon) \ \wedge \ H \not\subseteq G] \leq \beta^m.$$

Using Chernoff/Höffding-bounds it is easy to see that

$$\Pr[G = \mathbb{G}(H, n, m) \in \mathcal{S}(H, n, m; \varepsilon)] = 1 - o(1) \tag{4.3}$$

for, say, $m \geq n \log n$, i.e., almost all (H, n, m)-graphs are $(H, n, m; \varepsilon)$-regular. Due to (4.3) the statement (4.2) is equivalent to

$$\Pr[H \not\subseteq G \mid G = \mathbb{G}(H, n, m) \in \mathcal{S}(H, n, m; \varepsilon)] \leq \beta^m \tag{4.4}$$

for suitably chosen constants ε and β. Thus Conjecture 4.19 indeed represents the probabilistic embedding lemma that we need for applications of the sparse Regularity Lemma, e.g. for the proof of Conjecture 4.18.

Interpretation of the threshold $n^{2-1/d_2(H)}$ The threshold $n^{2-1/d_2(H)}$ has the following intuitive interpretation. Assume that we generate a graph $G \in \mathcal{S}(H, n, m; \varepsilon)$ edge by edge, and consider the situation when almost all edges have been chosen. Only a linear fraction, say, κm, of the edges $E(V_i, V_j)$ between two arbitrary partitions V_i and V_j are about to be selected at random. It should be the case that the choices for these edges dominate the probability that G is H-free, since we need first the previously chosen edges to get many subgraphs which are easy to complete to a subgraph H. Then these edges impose strong restrictions on the choices for the remaining edges if the graph shall remain H-free.

In order to achieve an overall probability of β^m for the event '$H \subseteq G$', every single edge must complete a subgraph H with probability at least $1 - \gamma$ for a sufficiently small γ. Then we would obtain an overall probability of about $\gamma^{\kappa m}$, boldly assuming that the probabilities for the edges are independent. Such a small probability for a single edge can only be achieved if almost all of the n^2 positions for edges $E(V_i, V_j)$ are incident to subgraphs which are identical to H up to one missing edge. The expected number of such 'almost-H' subgraphs should be approximately equal to

$$\Theta\left(n^{v_H} \cdot \left(\frac{m}{n^2}\right)^{e_H - 1}\right). \tag{4.5}$$

In order to find out how big m has to be (restricting our attention merely to the order of magnitude) when 'almost-H' subgraphs occupy all but a tiny fraction of the n^2 positions for edges, we set

$$n^{v_H} \cdot \left(\frac{m}{n^2}\right)^{e_H-1} \overset{!}{=} n^2 \iff m^{e_H-1} = n^{2e_H - v_H} \iff m = n^{2-\frac{v_H-2}{e_H-1}}. \tag{4.6}$$

Observe that the number of occurrences of a subgraph H depends on the subgraph $F \subseteq H$ with the smallest expected number of occurrences (analogously to the well-known result on the occurrence of subgraphs in random graphs $G_{n,p}$, cf. e.g. [JŁR00]). Clearly, H cannot occur more often than F. In order to avoid over-counting we should therefore consider the following value of m as a suitable candidate for the searched threshold (cf. (3.7))

$$m = \max_{F \subseteq H} n^{2-\frac{v_F-2}{e_F-1}} = n^{2-1/d_2(H)}. \tag{4.7}$$

A second, more formal argument to obtain a lower bound for the threshold would be as follows. In [JŁR90] a quite precise bound on the probability of the event '$H \not\subseteq G_{n,p}$' has been given. This bound has been transferred to random graphs $G_{n,m}$ in [PS96a]. Roughly speaking, for non-bipartite graphs H it can be shown that

$$\Pr[H \not\subseteq G_{n,m}] = e^{-\Theta(\Phi_H)}, \tag{4.8}$$

where $\Phi_H := \min_{F \subseteq H}\{n^{v_F} \cdot (m/n^2)^{e_F}\}$ denotes the expected number of occurrences of the least-frequent subgraph $F \subseteq H$. Observe that Φ_H is increasing in m and that Φ_H can be bounded by $\Phi_H \leq m$ (consider the subgraph F which consists of a single edge).

We want to find out when we have $\Pr[H \not\subseteq G_{n,m}] = e^{-\Theta(m)}$. Because of (4.8) we set

$$\Phi_H \geq m \iff \min_{F \subseteq H} n^{v_F} \cdot \left(\frac{m}{n^2}\right)^{e_F} \geq m \iff \min_{F \subseteq H} n^{v_F} \cdot \left(\frac{m}{n^2}\right)^{e_F - 1} \geq n^2.$$

Comparing this with (4.6) and (4.7), we see that we obtain $\Phi_H = m$ for $m \geq n^{2-1/d_2(H)}$.

Analogously, it follows that $\Phi_H = o(m)$ and thus $\Pr[H \not\subseteq G_{n,m}] \geq e^{-o(m)}$ for $m = o(n^{2-1/d_2(H)})$. Let 'REG' denote the event that the random graph is strongly ε-regular. A standard application of Chernoff/Höffding-bounds shows that $\Pr[REG] \geq 1 - e^{-\Theta(m)}$ (similar to (4.3). We obtain

$$\begin{aligned} \Pr[H \not\subseteq G_{n,m} \mid REG] &\geq \Pr[H \not\subseteq G_{n,m} \cap REG] \\ &\geq \Pr[H \not\subseteq G_{n,m}] - \Pr[\neg REG] \\ &\geq e^{-o(m)} - e^{-\Theta(m)} = e^{-o(m)}. \end{aligned}$$

Thus a result like (4.4) cannot be true for $m = o(n^{2-1/d_2(H)})$.

Applications of the KŁR-conjecture Now let us return to applications of Conjecture 4.19. We have introduced Conjecture 4.19 as a tool needed for the proof of Conjecture 4.18 on $\mathrm{ex}(G_{n,m}, H)$. For a more detailed discussion of this subject we refer the reader to [JŁR00] where also a proof of the case $H = K_3$ can be found.

As we have already seen in Chapter 3, [Łuc00] contains the following theorem, which is based on Conjecture 4.19. We restate this theorem here for easier reference.

Theorem 3.6 *[Łuc00] Let H be a graph with $\chi(H) = \ell + 1 \geq 3$ for which Conjecture 4.19 holds. Then for every $\delta > 0$ there exists $C > 0$ such that*

$$\Pr[G \text{ is } (\delta, \ell)\text{-partite} \mid G = G_{n,m} \text{ is } H\text{-free}] = 1 - o(1),$$

provided that $m \geq Cn^{2-1/d_2(H)}$. □

The proof of Theorem 3.6 uses a sparse Regularity Lemma which slightly differs from Lemma 4.15, but the modifications are rather technical. The basic approach for applying the Regularity Lemma stays the same and is based on Conjecture 4.19 as probabilistic embedding lemma.

The following conjecture was formulated in [KŁR97] together with Conjecture 4.18. The statement refers to arbitrary subgraphs of random graphs $G_{n,m}$ and bears some similarity to Theorem 3.6. Therefore it comes as no surprise that Conjecture 4.19 can be used to prove it (cf. [JŁR00] [Beh02]).

Conjecture 4.20 *[KŁR97] Let H be a graph with $\chi(H) = \ell + 1 \geq 3$. Then for every $\delta > 0$ there exists $\gamma > 0$ such that asymptotically almost surely every H-free subgraph $F \subseteq G$ of a random graph $G = G_{n,m}$ with at least $e(F) \geq (1-\gamma)\mathrm{ex}(G_{n,m}, H)$ edges is (δ, ℓ)-partite, provided that $m = \omega(n^{2-1/d_2(H)})$.* □

It is worth mentioning that there exists an intimate connection between Conjecture 4.20 and Conjecture 4.18. For simplicity let us consider the case $\ell = 2$, and let H be a triangle. Observe that $n^{2-1/d_2(K_3)} = n^{2-1/2} = n^{3/2}$ and that $\mathrm{ex}(G_{n,m}, K_3) \approx (1 - 1/2)m = m/2$, provided that Conjecture 4.18 holds. Now assume the following (analogous to the statement of Conjecture 4.20):

> Every triangle-free subgraph $F \subseteq G$ of a random graph $G = G_{n,m}$ with at least $e(F) \geq m/2$ edges is asymptotically almost surely $(\delta, 2)$-partite, provided that $m \geq Cn^{3/2}$ and $C = C(\delta)$ is sufficiently large.

By Chernoff/Höffding-bounds it is rather easy to show that a random graph $G = G_{n,m}$ does not contain a bipartite subgraph with more than, say, $m/2 + \sqrt{mn}\log n = (1/2 + o(1))m$ edges for $m = \Omega(n^{3/2})$.

Now we will argue that the above assumption implies Conjecture 4.18. Consider a subgraph $J \subseteq G$ with $e(J) \geq (1/2 + \gamma)m$ edges. We intend to show that asymptotically almost surely all such subgraphs contain a triangle. Assume the contrary and let $J \subseteq G$ be a counterexample. Then, by our previous assumption, J is almost surely $(\delta, 2)$-partite. Thus there exists a bipartite subgraph $J' \subseteq G$ with $e(J') \geq (1/2 + \gamma - \delta)m$. By choosing δ smaller than, say, $\gamma/2$ we get the desired contradiction.

In addition to Conjecture 4.18 and Conjecture 4.20 which deal with results on extremal graphs, i.e., of the so-called *Turán-type*, there are also results of the *Ramsey-type*, where the sparse Regularity Lemma and Conjecture 4.19 are applicable. Again we refer the reader to the literature (cf. e.g. [JŁR00] [Beh02]) for more detailed accounts on this subject.

Given the powerful rôle which the Regularity Lemma plays for dense graphs, one can hope that, in addition to the applications sketched above, a proof of Conjecture 4.19 may lead to important insight and other new results on random graphs.

4.4 Related work

Extremal subgraphs of random graphs Only a few special cases of Conjecture 4.18 have been proved up to now. In fact, these cases are restricted to trees and cycles. For trees, Conjecture 4.18 follows from a simple application of the sparse Regularity Lemma. The first non-trivial cases considered in the literature are $H = K_3$ in [FR86] and $H = C_4$ in [Für94] (based on a idea from [KW82]), where results are presented from which Conjecture 4.18 essentially follows. Later Conjecture 4.18 has been fully verified for arbitrary cycles $H = C_\ell$ (cf. [HKŁ95] [HKŁ96] [KKS98]). Apart from that only the case $H = K_4$ has been solved (cf. [KŁR97]).

Recently, significant progress has been made towards a proof of Conjecture 4.18 for complete graphs $H = K_\ell$. In [KRS02] a corresponding result for arbitrary ℓ is shown, based on a technique introduced in [KR01]. However, this result, stated for random graphs $G_{n,p}$, assumes that $p = \omega(n^{-1/(\ell-1)})$, which is larger than the conjectured threshold $n^{-1/d_2(K_\ell)} = n^{-2/(\ell+1)}$.

Embedding lemma for sparse graphs As far as Conjecture 4.19 is concerned, the situation is very similar. The case that H is a tree can be shown without any effort. For cycles the papers [Für94] [HKŁ95] [KKS98], which deal with Conjecture 4.18, contain auxiliary results which come very close to Conjecture 4.19. Related results can also be found in [KK97] [Kre97]. In [Beh02] it is shown how these results can be turned into a formal proof of Conjecture 4.19 for arbitrary cycles.

In [KR01] a variant of Conjecture 4.19 for $H = K_\ell$ and arbitrary ℓ is announced. However, similar to the above mentioned result in [KRS02], this result only holds for random graphs $G_{n,p}$ with $p = \omega(n^{-1/(\ell-1)})$ (This formulation for subgraphs of random graphs $G_{n,p}$ essentially corresponds to the formulation of Conjecture 4.19 given in this chapter.). Thus, it still remains open to prove the conjecture for the threshold $n^{-1/d_2(K_\ell)} = n^{-2/(\ell+1)}$.

Our results verify the smallest open cases $H = K_4$ and $H = K_5$ of Conjecture 4.18, achieving exactly the presumed threshold. This yields a conceptually simple proof of the case $H = K_4$ of Conjecture 4.18, previously settled in [KŁR97], and a proof of the yet completely unsolved case $H = K_5$.

4.5 Outline of the proof

Due to (4.3) we know that random graphs $\mathbb{G}(H, n, m)$ are typically ε-regular. Hence we will base our following intuitive arguments on the assumption that the structure of ε-regular graphs is similar to random graphs.

Note that here and in the sequel we use the terms $(H, n, m; \varepsilon)$-regular and ε-regular interchangeably. Since we will never be dealing with ε-regular graphs in the sense of Definition 4.4 or Definition 4.12 this should not cause any confusion.

The case $H = K_4$

Let $G = G(n, m, 4) := \mathbb{G}(K_4, n, m)$ be a random subgraph of the complete 4-partite graph $K_{n,n,n,n}$ with n vertices in each partition and with m edges between each pair of partitions. As an alternative we may consider a binomial random subgraph $G = G(n, p, 4)$ of $K_{n,n,n,n}$ in which all edges are present independently with probability $p := m/n^2$.

Let $q := m/n$. Note that in a random graph $G(n, p, 4)$, the neighborhood of a vertex v in one of the partitions has expected size q, since $pn = (m/n^2)n = q$. Intuitively, conditioning on the ε-regularity of G should even increase the probability that the size of a neighborhood is close to its expectation because the edges in ε-regular graphs are very evenly spread, i.e., the ε-regularity helps to avoid untypical cases which may occur in random graphs (although with small probability). Based on this intuition, which can clearly be transferred to other sets than neighborhoods (cf. Lemma 4.8), we will now introduce the crucial ideas behind our proof strategy.

Finding K_4-candidates For every vertex $v \in V_1$ we try to find subgraphs which can be turned into a K_4 by adding a single edge from v to some vertex $v' \in V_4$. We call such a vertex v' a *clique candidate* of v. Note that we do not insist on a clique candidate v' being adjacent to v. We show that there are $\Omega(n)$ vertices $v \in V_1$ each of which has almost n clique candidates in V_4. Since it is very unlikely that the neighborhood of v does not intersect the big set of clique candidates, this will suffice to prove Theorem 4.3.

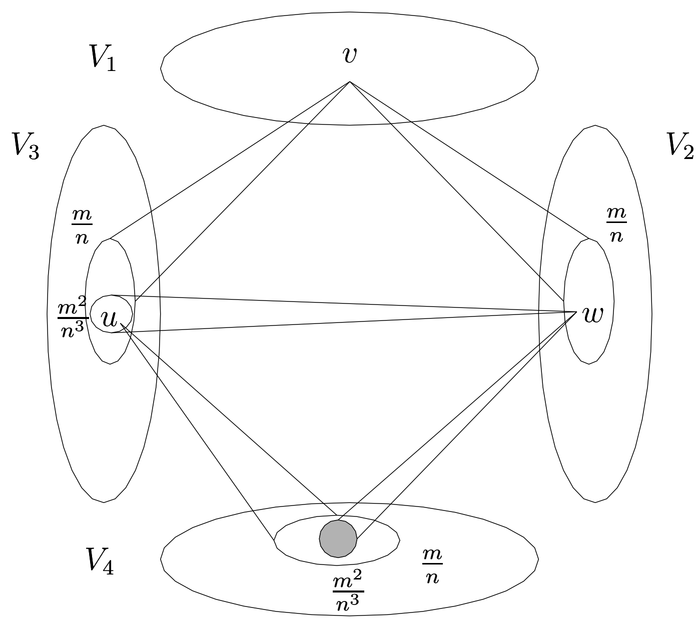

Figure 4.2: Idea for construction of K_4-candidates.

Figure 4.2 illustrates how the clique candidates in V_4 are found. As discussed earlier, a vertex $v \in V_1$ should have approximately q neighbors in V_2 and V_3. We denote the neighborhood of v in V_i by $\Gamma_i(v)$. Consider a vertex $w \in \Gamma_2(v)$.

We expect that $|\Gamma_3(v)\cap\Gamma_3(w)| \approx pq = (m/n^2)(m/n) = m^2/n^3$. The same argument shows that for every vertex $u \in \Gamma_3(v) \cap \Gamma_3(w)$ there are approximately m^2/n^3 vertices in $\Gamma_4(u) \cap \Gamma_4(w)$. These m^2/n^3 vertices are clique candidates of v. Note that the same arguments still hold, as far as the order of magnitude of the results is concerned, if we forbid at most, say, $(1-\xi)n$ vertices in V_4, provided that ξ is not too tiny. Hence, we may find m^2/n^3 clique candidates for every vertex in $\Gamma_3(v) \cap \Gamma_3(w)$, yielding a total of $(m^2/n^3)^2$ clique candidates. Finally, we repeat this process for the q vertices in $\Gamma_2(v)$, and get $(m^2/n^3)^2 q = (m^4/n^6)(m/n) = m^5/n^7$ clique candidates in V_4. If $m \geq Cn^{8/5}$ for C sufficiently large, we may thus expect that almost all vertices in V_4 are clique candidates of v.

Introducing bad sets From a more technical point of view our proof strategy proceeds as follows. We define families of 'bad' graphs, i.e., graphs with 'unusual' properties. By estimating the number of such graphs we show that the overwhelming majority of graphs does not possess these properties. Hence we collect more and more information on structural properties of typical ε-regular graphs. Based on this knowledge we identify certain substructures in the graph which will, in the end, lead to K_4-candidates. As a red thread which may guide the reader through the proof we briefly sketch the structural properties that we use.

Firstly, as an immediate (and well-known) consequence of ε-regularity, the *degrees* of single vertices are approximately equal to their expectation q.

Secondly, the neighborhood of sets with cardinality $\Theta(n^2/m)$ and $\Theta(m/n) = \Theta(q)$ is examined by introducing the notions of *covers* and *multicovers*. In particular, it turns out the the neighborhood of a single vertex in a set of cardinality $\Theta(q)$ has size $\Theta(q \cdot m/n^2) = \Theta(m^2/n^3)$, as one may expect due to the affinity with random graphs $G(n, m, 4)$.

Thirdly, we prove that a set of size $\Theta(m^2/n^3)$ and a single vertex have a common neighborhood of size $t = \Omega(n^2/m)$. Note that the expectation in a random graph $G(n, m, 4)$ amounts to $\Theta(m^2/n^3 \cdot n \cdot (m/n^2)^2) = \Theta(m^4/n^6)$ and for $m = \omega(n^{8/5})$ this is larger than t. The results of this step correspond to the auxiliary notion of a *triangle candidate cover*.

In a fourth step, we then combine multicovers and triangle candidate covers. Consider again a vertex $v \in V_1$ and its neighborhood $\Gamma_3(v)$ of size $\Theta(q)$. We prove that a single vertex $w \in V_2$ indeed has a neighborhood of size $\Theta(m^2/n^3)$ inside $\Gamma_3(v)$. This follows from the properties of a multicover. Moreover, these $\Theta(m^2/n^3)$ vertices close $\Omega(n^2/m)$ triangles (between the partitions V_2, V_3, and V_4), as implied by the properties of a triangle candidate cover. Together, this will enable us to introduce *triangle covers*, which show that almost all vertices in V_2 close many triangles with $\Gamma_3(v)$.

Now the fifth step is straightforward. We prove that the neighborhood $\Gamma_2(v)$ intersects with the many vertices in the triangle cover. This finally yields the desired K_4-*candidates*, since only the edge between V_1 and V_4 is missing from the structure. For one such vertex $w \in \Gamma_2(v)$ we obtain $t = \Omega(n^2/m)$ K_4-candidates. By iterating the process for $\Theta(q)$ vertices in $\Gamma_2(v)$ we get $\Omega(q \cdot t) = \Omega(m/n \cdot n^2/m) = \Omega(n)$ K_4-candidates.

As indicated above, a simple sixth step of the proof shows that this implies the existence of subgraphs K_4, since $\Gamma_4(v)$ is likely to overlap with the large set of K_4-candidates..

Due to technical reasons, i.e., as to obtain a sufficiently small probability, one major deviation from the scheme sketched above is necessary. When showing the existence of triangle covers we have to examine the neighborhoods of many vertices in V_1 at once. This makes the proof a bit more hairy than the rather clear strategy might suggest.

The case $H = K_5$

Our intuitive arguments are based on a random subgraph $G = G(n, m, 5)$ of the complete 5-partite graph $K_{n,n,n,n,n}$ with n vertices in each partition and m edges between each pair of partitions, resp. the corresponding binomial random graph $G = G(n, p, 5)$ in which all edges occur independently with probability $p = m/n^2$.

Finding K_5-candidates Our aim is to find $\Theta(n)$ K_5-candidates for every vertex $v \in V_1$, i.e., vertices $x \in V_5$ which are part of a subgraph that can be turned into a K_5 by adding a single edge from v to x. Figure 4.3 shows how this goal is achieved.

Consider three fixed vertices, say, $v \in V_1$, $w \in V_2$, and $u \in V_3$. The expected size of their common neighborhood $\Gamma_4(v) \cap \Gamma_4(w) \cap \Gamma_4(u)$ is equal to $\Theta(n \cdot (m/n^2)^3) = \Theta(m^3/n^5)$. For $m \geq Cn^{5/3}$ and a sufficiently large constant C we may thus assume that there exists a vertex $y \in \Gamma_4(v) \cap \Gamma_4(w) \cap \Gamma_4(u)$. For this vertex we again obtain that $\Gamma_5(y) \cap \Gamma_5(w) \cap \Gamma_5(u) \neq \emptyset$. Thus, for every triple (v, w, u) of vertices we obtain one K_5-candidate, provided that the vertices v, w, and u form a triangle.

For a fixed vertex $w \in \Gamma_2(v)$, we expect $\Theta(n \cdot (m/n^2)^2) = \Theta(m^2/n^3)$ vertices $u \in \Gamma_3(v) \cap \Gamma_3(w)$. Observe that $n^2/m = \mathcal{O}(m^2/n^3)$ for $m = \Omega(n^{5/3})$. Hence, we may assume that at least $\Theta(n^2/m)$ suitable vertices $u \in V_3$ exist. Note that the neighborhood of $\Theta(n^2/m)$ vertices has size $\Theta(n^2/m \cdot n \cdot m/n^2) = \Theta(n)$. Thus, taking more vertices u into consideration does not yield more disjoint K_5-candidates, since the neighborhoods $\Gamma_5(u)$ of these vertices are likely to overlap.

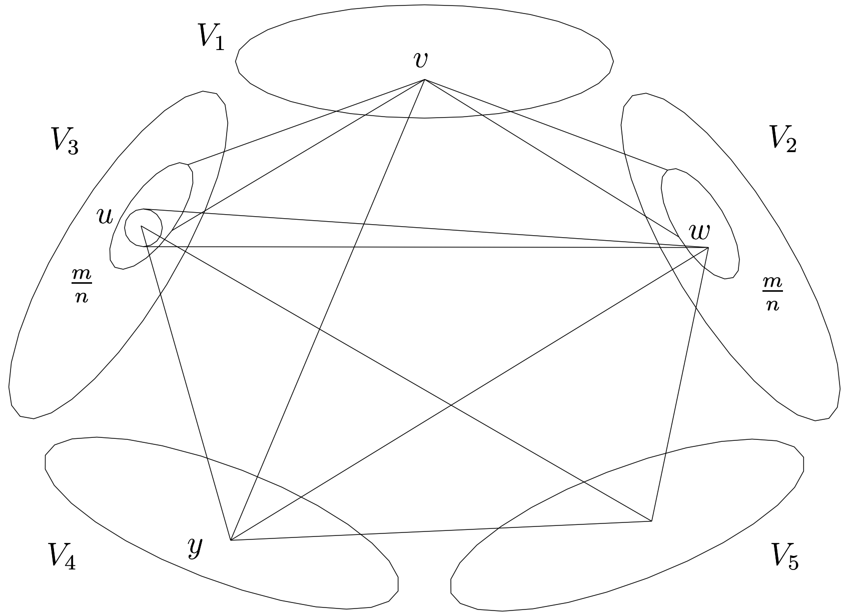

Figure 4.3: Idea for construction of K_5-candidates.

Finally, observe that there are $\Theta(m/n)$ vertices $w \in \Gamma_2(v)$. Hence, iterating over all vertices w and u we obtain $\Theta(m/n \cdot n^2/m \cdot 1) = \Theta(n)$ K_5-candidates.

Introducing bad sets For the case $H = K_5$ we use a similar proof technique as for the case $H = K_4$, namely we investigate the structure of typical ε-regular graphs by introducing suitable families of bad, i.e., untypical graphs.

Initially, we again rely on the previously mentioned results concerning the degrees of single vertices, as well as covers and multicovers. Additionally, we derive a result on the *overlapping neighborhood*, i.e., the common neighborhood of two vertices. Similar to multicovers we will show that two vertices share $\Theta(m^2/n^3)$ common neighbors.

Our next aim is to prove the existence of *square candidate covers*, which exhibit an analogous structure to the above mentioned triangle candidates covers. Basically, we want to show that a single vertex $w \in V_2$ and a set of size $\Theta(m^2/n^3)$ in V_3 typically belong to many subgraphs K_4 which extend over $t = \Omega(n^2/m)$ vertices in V_5. More precisely, we will also have an additional fixed vertex v and an edge from v to the subgraph K_4.

After having proved the existence of square candidate covers, we can finish the proof as in the case $H = K_4$. Consider a vertex $v \in V_1$ and its neighborhood $\Gamma_3(v)$ of size $\Theta(q)$. We prove that a single vertex $w \in V_2$ indeed has a neighborhood of size $\Theta(q \cdot m/n^2) = \Theta(m^2/n^3)$ inside $\Gamma_3(v)$. This follows from the properties of a multicover. Moreover, these $\Theta(m^2/n^3)$ vertices close

$t = \Omega(n^2/m)$ subgraphs K_4 (between the partitions V_2, V_3, V_4, and V_5, with additional edges from v to the vertices in V_4), as implied by the properties of a square candidate cover. Together, this will enable us to introduce *square covers*, which show that almost all vertices in V_2 close many squares with $\Gamma_j(v)$ for $v \in V_1$ (with an additional edge from v to V_4). Then K_5-candidates can be constructed analogously to the K_4-candidates in the case $H = K_4$ by combining the squares for all $\Theta(q)$ vertices in $\Gamma_2(v)$. This yields in total $\Omega(q \cdot t) = \Omega(n)$ K_5-candidates.

For the construction of square candidate covers we introduce yet another auxiliary structure, the so-called *cocovers*. A cocover consists of a set in V_3 of size $\Theta(n^2/m)$ and sets V_4 of size $\Theta(m^2/n^3)$. Together these sets should have

$$\Theta(n^2/m \cdot m^2/n^3 \cdot n \cdot (m/n^2)^2) = \Theta(m^3/n^4)$$

neighbors in V_5. Note that $m^3/n^4 = \Theta(n)$ for $m = \Theta(n^{5/3})$. Hence, we show that neighborhoods of the sets in the cocover 'tile' the partition V_5. For larger m, i.e., for $m = \omega(n^{5/3})$, we control the overlap between these neighborhoods by introducing the notion of *quasidisjointness*. Based on this quasidisjointness, we will show that cocovers almost directly correspond to the square candidate covers we have been looking for.

Technical difficulties

Of course, this outline of the proof is very rough and neglects several important aspects. Firstly, we must deal with random *ε-regular* graphs. This rules out many tools from probability theory which are formulated for product spaces. Hence, most of the time we will directly count the number of graphs with given properties. Secondly, a little deviation from the expectation has to be tolerated and corresponding error terms must be taken into account. Thirdly, so far we have only discussed the order of magnitude of the results, whereas the choice of suitable constants represents a delicate part of the proof. Finally, the order in which the neighborhoods and intersections of neighborhoods are considered is crucial because dependencies would spoil our proof strategy.

4.6 Preliminaries

4.6.1 Conventions and notation

In order to increase the clarity of the presentation we make use of the following conventions:

We will not introduce floors and ceilings when we are talking about integral terms, e.g., cardinalities of sets. Since we are only interested in the asymptotic behavior of those quantities, this would merely introduce lower order error terms which complicate the exposition. However, it would be a standard but laborious task to modify the proofs such that the integrality of all terms is respected. Similarly we will ignore whether certain quantities are integrally divisible, e.g., we will assume that we can partition a set of q vertices into exactly q/p sets of size p.

For neighborhoods inside specific partitions we will use the abbreviations $\Gamma_i(v) := \Gamma(v) \cap V_i$ and $d_i(v) = |\Gamma_i(v)|$. For $F \subseteq E$ we let $\Gamma_i^F(v) = \{u \in V_i \mid \{v, u\} \in F\}$ and define $d_i^F(v)$ accordingly.

To gain control over the size of certain sets, we will often need to deterministically fix a subset of a given cardinality in a larger set of vertices. In order to do so we assume that the vertices in V_i are ordered in an arbitrary but unique way, say $V_i = \{1, \ldots, n\}$. By $[A]_x$ we denote the set $B \subseteq A$ of size $|B| = x$ that contains the x smallest elements in A. If $|A| < x$, we define $[A]_x := A$.

All constants which are denoted by Greek letters are tacitly assumed to be smaller than, say, 10^{-3}. When we intend to show Theorem 4.3 for some fixed $\beta > 0$ all other auxiliary constants will depend on β. More specifically we assume that the auxiliary constants decrease as β decreases. In fact we will even require that all auxiliary constants approach zero as $\beta \to 0$. Formally speaking, an auxiliary constant ν is a function $\nu = \nu(\beta)$ depending on β. Note that if suitable constants ε_0, $C > 0$ and $n_0 > 0$ exist for a specific value of β, then these constants are also suitable for $\beta' \geq \beta$. Hence if the statement of Theorem 4.3 is proved for β it also holds for β'. We may therefore assume that certain constants are sufficiently small by choosing β sufficiently small.

For an auxiliary constant $\nu = \nu(\beta)$ we will call quantities of the type ν^x *super-exponentially small* in ν, as the basis ν can be made arbitrarily small by choosing a sufficiently small value for β.

We will not be interested in the actual values of the auxiliary constants for a given value of β. Instead we will be content to show that suitable auxiliary constants exist. Since the dependencies between β and the auxiliary constants are rather complicated to state explicitly, we introduce the relation '$\lll$'. Let $\nu = \nu(\beta)$ and $\mu = \mu(\beta)$ denote two auxiliary constants. Essentially $\nu \lll \mu$ means that $\nu^{\mathrm{poly}(\mu)} \leq \mu$ for any polynomial $\mathrm{poly}(x)$ provided that β and thus ν and μ are sufficiently small. This notion will suffice for all occasions when we need that a specific constant is chosen 'much' smaller than a certain other constant.

Definition 4.21 (Relation $\lll$) *Consider two functions $\nu, \mu : (0,1] \to (0,1]$ such that $\lim_{\beta\to 0} \nu(\beta) = \lim_{\beta\to 0} \mu(\beta) = 0$. We write $\nu \lll \mu$ if*

$$\forall C \geq 1 \;\exists \beta_0 > 0 \;\forall 0 < \beta \leq \beta_0 : \; C \cdot [\nu(\beta)]^{\mu(\beta)^C/C} \leq \mu(\beta)^C.$$

The following example shows that auxiliary constants ν and μ with $\nu \lll \mu$ indeed exist.

Example 4.22 *We have $e^{-e^{1/x}} \lll x$, since for all $C \geq 1$*

$$C \cdot \left(e^{-e^{1/x}}\right)^{x^C/C} = C \cdot \exp\left(-\frac{e^{1/x}}{C \cdot (1/x)^C}\right) \leq C \cdot e^{-1/x^C} \leq x^C$$

for x sufficiently small, with lots of room to spare.

The following proposition collects a few simple properties of $\lll$.

Proposition 4.23 (Simple properties of $\lll$)

(i) If $\nu \lll \mu$, then $\nu \leq \mu$ for sufficiently small β.

(ii) If $\rho \lll \nu$ and $\nu \leq \mu$, then $\rho \lll \mu$.

(iii) If $\nu \lll \mu$, then $\nu \lll \mu/k$ and even $\nu \lll \mu^k$ for all $k > 0$.

(iv) If $\nu \lll \mu$, then $k\nu \lll \mu$ and even $\nu^{1/k} \lll \mu$ for all $k > 0$.

(v) If $\nu \lll \mu$, then $\nu/\mu \leq \nu^{1/2}$ for sufficiently small β.

Proof (i) and (ii) follow directly from the definition. The proof of (iii) and (iv) is similar and we just show one case. Assume that $\nu \lll \mu$ and that $k \geq 1$ (otherwise the claim is trivial). In order to show that $\nu \lll \mu^k$ observe that

$$C\nu^{(\mu^k)^C/C} \leq (kC)\nu^{\mu^{kC}/(kC)} \leq \mu^{kC} \leq (\mu^k)^C$$

for β sufficiently small. Finally, (v) easily follows from $\nu^{1/2} \lll \mu$. $\square$

In the sequel, if we assume that ν is chosen such that $\nu \lll \mu$ for a given $\mu \geq 0$, we may assume for concreteness sake that $\mu = e^{-e^{1/\nu}}$. This is summarized in the following definition.

Definition 4.24 ($(\mu)^{\oplus}$ and $(\mu)^{\ominus}$) *Let $f(x) := e^{-e^{1/x}}$. For a function $\nu : (0,1] \to (0,1]$ we define $(\nu)^{\ominus} := f \circ \nu$. Accordingly we set $(\nu)^{\oplus} := f^{-1} \circ \nu$. Furthermore, $(\nu)^{\oplus}$ and $(\nu)^{\ominus}$ may be iterated. We let $(\nu)^{\oplus 1} := (\nu)^{\oplus}$ and $(\nu)^{\oplus i+1} := ((\nu)^{\oplus i})^{\oplus}$ for $i \geq 1$. The notation $(\nu)^{\ominus i}$ is defined accordingly.*

Recall Example 4.22 and observe that $\nu \lll (\nu)^{\oplus}$ and $(\nu)^{\ominus} \lll \nu$.

Theorem 4.3 only holds asymptotically, i.e., we shall only prove it for sufficiently large n. For the asymptotic estimates the number of edges m must be interpreted as a function $m = m(n)$ of the number of vertices. Furthermore, we can assume that $m \geq t(n)$ with a threshold $t(n) = \Theta(n^{5/3})$ for the case $H = K_5$ and $t(n) = \Theta(n^{8/5})$ for $H = K_4$. As we intend to prove a probability of β^m in Theorem 4.3, only terms which are exponential in m will really be important for our subsequent counting arguments. In particular, polynomial terms in n may safely be ignored. But observe that terms like 2^m are also negligible in comparison to β^m if β is sufficiently small. In order to neglect such terms at an early stage of the proof we introduce the relation $\lesssim^x$. Let $f(n), g(n)$ denote two functions in n. Intuitively, $f \lesssim^x g$ means that f is at most a large as g up to a multiplicative term C^x for a suitably chosen constant C.

Definition 4.25 (Relation $\lesssim$) *Consider two functions $f, g : \mathbb{N} \to \mathbb{R}^+$. For $x \geq 0$ we write $f \lesssim^x g$ if*

$$\exists C \geq 1 \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 : \;\; f(n) \leq C^x \cdot g(n).$$

Instead of $\lesssim^0$ we simply write $\lesssim$.

$f(n) \lesssim g(n)$ means that $f(n) \leq g(n)$ for n sufficiently large. Observe that $f(n) \lesssim^1 g(n)$ is equivalent to $f(n) = \mathcal{O}(g(n))$.

The following lemma shows that the relations $\lll$ and $\lesssim$ work well together. If $\nu \lll \mu$ and $f \lesssim^x g$ as well as $g \leq \nu^{\mu x} h$, then $f \leq \mu^x h$. Thus $\lesssim^x$ can be replaced by $\leq$, which is of course our goal when proving bounds on the number of graphs with a certain property. The only price that we have to pay for this is to replace $\nu^{\mu x}$ by μ^x, which is however still superexponentially small.

Lemma 4.26 (Connection between $\lll$ and $\lesssim$) *Consider $x > 0$, $\nu = \nu(\beta) > 0$, $\mu = \mu(\beta) > 0$ with $\nu \lll \mu$. Let $f(n) \lesssim^x g(n)$ and $g(n) \le \nu^{\mu x} \cdot h(n)$. Then there exists $\beta_0 > 0$ such that for all $0 < \beta \le \beta_0$,*

$$f(n) \le \mu^x \cdot h(n)$$

for n sufficiently large.

Proof Let $C \ge 1$ be a constant as in Definition 4.25 of $f(n) \lesssim^x g(n)$. Now we conclude by Definition 4.21 that there exists $\beta_0 > 0$ such that

$$C\nu^\mu \le C\nu^{\mu/C} \le \mu^C \le \mu$$

for $0 < \beta \le \beta_0$. For sufficiently large n we deduce that

$$f(n) \le C^x \cdot g(n) \le (C \cdot \nu^\mu)^x \cdot h(n) \le \mu^x \cdot h(n).$$

$\square$

4.6.2 Auxiliary technical results

Binomial coefficients

For future reference we collect a few bounds on binomial coefficients, which will be used repeatedly throughout the proof of Theorem 4.3.

Lemma 4.27 (Technical inequalities for binomial coefficients)

(i) If $0 \le b + x \le 0.9a$, then

$$\binom{a}{b} \le \binom{a-x}{b} e^{10b}.$$

(ii) If $0 \le x \le 1$, then

$$\binom{xa}{b} \le \binom{a}{b} x^b.$$

(iii) If $b := \sum_{i=1}^k b_i \le a$, then

$$\prod_{i=1}^k \binom{a}{b_i} \le \binom{a}{b} (2^k)^b.$$

Proof Using $1 - z \geq e^{-10z}$ for $z \leq 0.9$ we conclude that

$$\frac{\binom{a}{b}}{\binom{a-x}{b}} = \frac{a^{\underline{b}}}{(a-x)^{\underline{b}}} \leq \frac{a^b}{(a-x-b)^b} = \left(1 - \frac{x+b}{a}\right)^{-b} \leq e^{10(x+b)b/a} \leq e^{10b},$$

proving (i) (recall that $n^{\underline{z}}$ denotes the z-th falling factorial of n).

For the proof of (ii) observe that

$$\binom{xa}{b} = x^b \cdot \frac{a(a-\frac{1}{x})\ldots(a-\frac{b-1}{x})}{b!} \leq \binom{a}{b} \cdot x^b$$

Inequality (iii) follows by combinatorial arguments. Instead of choosing individual sets with cardinalities $b_1, \ldots, b_k$ from a set with a elements we pick $\sum_{i=1}^{k} b_i$ elements at once. Then we fix for each of the chosen elements to which of the k subsets it shall belong. Since there are at most 2^k possibilities for each chosen element, the claim follows. □

Basic counting arguments

When counting graphs with certain properties some arguments will recur. In the sequel we list these arguments in a form which abstracts from their actual application.

The following lemma is inspired by a very simple observation. If in a bipartite graph $G = (U \dot\cup W, E)$ all vertices in U have large degree, then there must also be many vertices in W with large degree. This is proved by an easy counting argument.

Lemma 4.28 (Overlap lemma) *Let $\alpha > 0$, and let $G = (U \dot\cup W, E)$ be a bipartite graph with $d(u) \geq \alpha|W|$ for all $u \in U$. Then for all $\beta > 0$,*

$$|\{w \in W \mid d(w) \geq \beta|U|\}| \geq \frac{\alpha - \beta}{1 - \beta}|W|.$$

Proof Assume for a contradiction that there are less than $\frac{\alpha-\beta}{1-\beta}|W|$ vertices in W with degree at least $\beta|U|$. It follows that

$$|E| < \frac{\alpha - \beta}{1 - \beta}|W| \cdot |U| + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right)|W| \cdot \beta|U| = \alpha|W| \cdot |U|.$$

On the other hand it is clear by the lower bound on the degree of the vertices in U that $|E| \geq \alpha|W| \cdot |U|$ and we get a contradiction. □

Although the idea behind Lemma 4.28 is quite simple, the following corollary will become a key stone in the proof of our main result.

Corollary 4.29 *Let $\varepsilon > 0$. If a bipartite graph $G = (U \dot\cup W, E)$ satisfies $d(u) \geq (1-\varepsilon)|W|$ for all $u \in U$, then*

$$|\{w \in W \mid d(w) \geq (1-\sqrt{\varepsilon})|U|\}| \geq (1-\sqrt{\varepsilon})|W|.$$

Proof Set $\alpha = 1-\varepsilon$ and $\beta = 1-\sqrt{\varepsilon}$ in Lemma 4.28. □

The union of k disjoint sets $A_1, \ldots, A_k$ with $|A_1| = \ldots = |A_n| = a$ obviously has cardinality $k \cdot a$. The following definition and lemma generalize this simple argument. Instead of assuming that $X_1, \ldots, X_k$ are disjoint, we will merely require that they overlap in a controlled manner.

Definition 4.30 (Quasidisjoint sets) *A family of sets $A_1, \ldots, A_b \subseteq V$ is called* s-quasidisjoint *if*

$$|\{i \in \{1, \ldots, b\} \mid x \in A_i\}| \leq s \quad \textit{for all } x \in V.$$

Lemma 4.31 (Sequential selection lemma) *Consider two separate sets V and B with $|B| = b$. The elements $x \in B$ are assigned corresponding sets $A_{x,1}, \ldots, A_{x,h} \subseteq V$. Assume that there is a subset $B^* \subseteq B$ with $|B^*| = (1-\nu)b$ such that $|A_{x,1}| = \ldots = |A_{x,h}| = a$ for $x \in B^*$ and the sets $(A_{x,l})_{x \in B^*, l \in \{1,\ldots,h\}}$ are s-quasidisjoint.*
A set $C \subseteq B$ with $|C| = c$ is called t-spreading *if $|\bigcup_{z \in C}(A_{z,1} \cup \ldots \cup A_{z,h})| \geq t$. If*

(i) $h \leq s$,

(ii) $4ts \leq \nu abh$,

(iii) $c \geq 4t/a$,

then

$$|\{C \in \binom{V}{c} \mid C \textit{ is not } t\textit{-spreading}\}| \leq (8\nu)^{c/2}\binom{n}{c}.$$

Proof Assume that the elements of $C \in \binom{V}{c}$ are chosen one-by-one uniformly at random. Observe that the ordering of C which is introduced by this random process does not influence whether C is t-spreading or not. We construct a set $X \subseteq \bigcup_{z \in C} A_z$ using the following algorithm:

- X is initialized with $X = \emptyset$.

- Assume that the elements $\{z_1, \ldots, z_{k-1}\}$ have already been chosen and now z_k is fixed uniformly at random in $B \setminus \{z_1, \ldots, z_{k-1}\}$. The element $z_k \in B$ is called 'good' if the following conditions are satisfied. Firstly, it must belong to B^*. Secondly, it must not yet have been added to C. And, thirdly, there exists a set $A_{z_k,i}$ such that $|A_{z_k,i} \setminus X| \geq a/2$, i.e., at least half of the elements in $A_{z_k,i}$ do not belong to an A-set of an already chosen element of C. For a good element z_k we add $[A_{z_k,i} \setminus X]_{a/2}$ to X (But note that the other sets $A_{z_k,i'}$ for $i' \neq i$ are not taken into account.). Then the next iteration of the algorithm (for z_{k+1}) begins.

Assume that at least $k_0 := 2t/a$ chosen elements are good. Then we have at the end of the algorithm $|X| \geq k_0 \cdot a/2 = t$. Thus C is t-spreading.

On the other hand we have $|X| \leq t$ as long as at most k_0 good elements have been chosen and added to C. Every single element of X belongs to at most s sets $A_{x,l}$ and thus the t elements in X correspond to at most ts occurrences of good elements in sets $A_{x,l}$. Consequently, at most $\frac{ts}{a/2}$ sets $A_{x,l}$ contain less than $a/2$ elements which do not belong to X. We call these sets $A_{x,l}$ destructed.

For an yet unchosen element $x \in B^*$ to become bad, all h sets $A_{x,1}, \ldots, A_{x,h}$ must be destructed. Consequently, at most

$$k_0 + \frac{2ts}{ah} = \frac{2t}{a} + \frac{2ts}{ah} \overset{(i)}{\leq} \frac{4ts}{ah} \overset{(ii)}{\leq} \nu b$$

elements of B^* are bad. Hence if we choose a vertex in B uniformly at random, we get a good vertex in B^* with probability at least $(1 - 2\nu)$.

Recall that we have $k_0 = 2t/a \overset{(iii)}{\leq} c/2$, provided that at most k_0 good vertices have been chosen. Thus we can bound the probability

$$\begin{aligned} &\Pr['\text{at most } k_0 \text{ good vertices are chosen}'] \\ \leq\ &\Pr['\text{at least } c/2 \text{ bad vertices are chosen}'] \leq 2^c \cdot (2\nu)^{c/2} \leq (8\nu)^{c/2}. \end{aligned}$$

This suffices to prove the claim. □

Technical counting lemmas

In our subsequent counting arguments we will show that certain substructures of the objects to be counted occur very rarely. This will then imply the desired bound on the number of the actual objects. Since the corresponding estimates are very much alike we prefer to state them in an abstract and unified way.

The objects which shall be counted will be referred to as 'bad', because later we will identify them with substructures of graphs for which the occurrence of a complete subgraph cannot be guaranteed. These structures must be shown to occur very rarely in order to prove Theorem 4.3. Hence, for our purposes they are 'bad'.

Let a ground set Z be given. Assume that we choose elements of a subset $X \subseteq Z$ one by one, and a single element is bad with probability at most ν. Obviously, for ν sufficiently small, X will contain only few bad elements with high probability (Such an argument has already been used in the proof of Lemma 4.31.). Observe that choosing the elements of X one by one introduces an ordering of X. This is important if the fact that a chosen vertex is bad depends on the previously chosen vertices. However, if we define the set X to be bad if and only if *all* orderings of the elements in X contain, say, at least, say, $\delta|X|$ bad elements, this approach will suffice to show that bad sets X occur very rarely.

The following lemma generalizes these arguments to choosing not only single elements but subsets $X_1, \ldots, X_r$ such that $X = X_1 \dot\cup \ldots \dot\cup X_r$. If the probability that an arbitrary subset X_i is bad, is sufficiently small, and the set X is bad if and only if *every* partition of X into subsets $X_1, \ldots, X_r$ contains many bad sets X_i, then this implies that bad sets X occur only with very small probability.

Lemma 4.32 (Bad steps lemma) *Let $\varepsilon > 0$, $\delta > 0$, and consider a set Z with $|Z| = n$. A set $X \in \binom{Z}{t}$ is called* (p, δ)-$\forall$-bad *if choosing disjoint (ordered) subsets $X_1, \ldots, X_r \subseteq X$ with $|X_1| = \ldots = |X_r| = p$ and $r = t/p$ always yields at least a δ-fraction of bad sets X_i.*
A subset X_i is bad if it satisfies a certain property $\Pi(X_i; X_1, \ldots, X_{i-1})$. Note that Π may depend on the previously chosen subsets $X_1, \ldots, X_{i-1}$. For any choice of $X_1, \ldots, X_{i-1}$ containing at most $(1-\delta)r$ good subsets X_i we assume that

$$|\{Y \in \binom{Z \setminus \bigcup_{l=1}^{i-1} X_l}{p} \mid Y \text{ satisfies } \Pi(Y; X_1, \ldots, X_{i-1})\}| \leq \varepsilon^p \binom{n}{p},$$

where ε^p is called exception probability.
For bad sets X with exception probability ε^p we have for $t \leq n/4$ and $\sqrt{\varepsilon} < e^{-11/\delta}$ that

$$|\{X \subseteq \binom{Z}{t} \mid X \text{ is } (p,\delta)\text{-}\forall\text{-bad (wrt. } \Pi) \}| \leq \varepsilon^{\delta t/2} \binom{n}{t}.$$

Proof It suffices to show that

$$\Pr['X \text{ is } (p,\delta)\text{-}\forall\text{-bad}'] \leq \varepsilon^{\delta t/2}$$

for a randomly chosen set $X \subseteq Z$ with $|X| = t$.

As far as the underlying probability space is concerned, we do not consider the 'natural' definition, where each set in $\binom{Z}{t}$ is chosen with equal probability. Instead we assume that sets $X_1, \ldots, X_r$ of size p are chosen one by one and let $X = X_1 \cup \ldots \cup X_r$. Observe that the event 'X is bad' does not depend on this partial ordering of the elements in X, and thus we get the same probability.

If $X = X_1 \cup \ldots \cup X_r$ is bad, then at least δr sets X_i must be bad, as the bound on the number of bad sets X_i holds for any ordered choice of subsets $X_1, \ldots, X_r \subseteq S$. Applying Lemma 4.27 (i) and using $p + t \leq 2t \leq 0.9n$, we obtain

$$\Pr['X \text{ is } (p,\delta)\text{-}\forall\text{-bad}'] \leq 2^r \left(\frac{\varepsilon^p \binom{n}{p}}{\binom{n-t}{p}} \right)^{\delta r} \leq 2^r \cdot (\varepsilon^p e^{10p})^{\delta r} \leq (\varepsilon e^{11/\delta})^{\delta t} \leq \varepsilon^{\delta t/2}.$$

$\square$

Remark 4.33 *An important special case of Lemma 4.32 arises for $p = 1$. Here individual elements are judged to be bad and this is assumed to be the case for at most εn elements of Z for any choice of $x_1, \ldots, x_{i-1} \in X$. The set X is bad if at least $\delta|X|$ of its elements are bad for any ordering of X.*

The arguments leading to Lemma 4.32 essentially also hold for the case that X contains a subset which is (p, δ)-$\forall$-bad. Only the constants have to be adapted in a suitable way depending on the size of the bad subset. This implies the following corollary.

Corollary 4.34 *A set $X \subseteq \binom{Z}{t}$ is called α-*partially (p,δ)-$\forall$-bad *if there exists a subset $X' \subseteq X$ with $|X'| \geq \alpha t$ which is (p,δ)-$\forall$-bad (with respect to a certain property Π).*
If X' is bad with exception probability ε^p, and $\varepsilon^{\alpha\delta/4} \leq 1/5$, then

$$|\{X \subseteq \binom{Z}{t} \mid X \text{ is } \alpha\text{-partially } (p,\delta)\text{-}\forall\text{-bad (wrt. } \Pi)\}| \leq \varepsilon^{\alpha\delta t/4} \binom{n}{t},$$

provided that the preconditions of Lemma 4.32 hold.

Proof By Lemma 4.32 the number of possibilities to construct a suitable set X is bounded from above by

$$
\begin{aligned}
t \cdot \max_{\alpha t \le t' \le t} \varepsilon^{\delta t'/2} \binom{n}{t'} \binom{n}{t-t'} &\le t \cdot \varepsilon^{\alpha\delta t/2} \max_{\alpha t \le t' \le t} \binom{n}{\alpha t} \binom{n}{t - \alpha t} \\
&\overset{\text{L4.27 (iii)}}{\le} \left(5\varepsilon^{\alpha\delta/2}\right)^t \binom{n}{t} \le \varepsilon^{\alpha\delta t/4} \binom{n}{t}.
\end{aligned}
$$

□

The following lemma deals with the case that a partition $Z_1 \ldots, Z_p$ of the ground set Z for which certain sets $X \subseteq Z$ shall be counted is already given. X is bad if there is a non-negligible number of 'bad' sets $X_i \subseteq Z_i \cap X$, i.e., there are very few possibilities to choose such sets. Note that X is bad if bad subsets $X_1, \ldots, X_p \subseteq X$ with $X_i \subseteq Z_i$ for $i = 1, \ldots, p$ *exist*. This contrasts to Lemma 4.32, where *any* partition of X into subsets $X_1, \ldots, X_p$ must contain many bad subsets.

Lemma 4.35 (Bad choices lemma) *Let $\varepsilon > 0$, $\delta > 0$, and consider a set Z with $|Z| = n$. Furthermore, pairwise disjoint subsets $Z_1, \ldots, Z_p \subseteq Z$ of equal size $|Z_1| = \ldots = |Z_p| =: z$ are given. A set $X \in \binom{Z}{t}$ is called (p, r, δ)-*∃-bad *if there exist subsets $X_l \subseteq \binom{Z_l}{r}$ for δp indices $l \in \{1, \ldots, p\}$ such that all these subsets are bad, i.e., they satisfy a certain property $\Pi(X_l)$. If we have*

$$
\left|\left\{Y \in \binom{Z_l}{r} \mid Y \text{ satisfies } \Pi(Y)\right\}\right| \le \varepsilon^r \binom{z}{r}
$$

for all $l = 1, \ldots, p$, then ε^r is called exception probability.
For bad sets X with exception probability ε^r and $pr \ge \alpha t$ we have for ε sufficiently small that

$$
\left|\left\{X \subseteq \binom{Z}{t} \mid X \text{ is } (p, r, \delta)\text{-}\exists\text{-bad (wrt. } \Pi \text{ and } Z_1, \ldots, Z_p)\right\}\right| \le (8\varepsilon^{\alpha\delta})^t \binom{n}{t}.
$$

Proof We directly estimate the number of bad subsets by

$$
2^p \left(\varepsilon^r \binom{z}{r}\right)^{\delta p} \binom{n}{t - \delta r p} \le 2^p \varepsilon^{\alpha\delta t} \binom{n}{\delta r p} \binom{n}{t - \delta r p} \overset{\text{L4.27 (iii)}}{\le} (8\varepsilon^{\alpha\delta})^t \binom{n}{t}.
$$

□

4.7 General counting lemma

We want to investigate some properties of a typical regular graph and then show that a graph satisfying these properties contains a K_ℓ. More precisely, we shall see that a graph is untypical or 'bad' if it contains *linearly* many vertices which have a somewhat 'irregular' neighborhood.

The aim of this section is to provide a rather general counting lemma which can be applied to various definitions of what we mean by an 'irregular' neighborhood.

We will construct and thus count the number of untypical graphs by first fixing their edges up to $E[B, V_j]$, where $B \subseteq V_i$ for $i \neq j$ denotes a set of 'bad' vertices with $|B| = \Theta(n)$. The key idea then is that an appropriate definition of 'irregular' implies that, once all edges except those between B and V_j have been chosen, the number of 'irregular' neighborhoods is very small compared to the total number of possible neighborhoods (more precisely, $\pi^{d_v}\binom{n}{d_v}$ instead of $\binom{n}{d_v}$ for some suitably small $0 < \pi$, where $d_v = |\Gamma_j(v)|$.)

Definition 4.36 (Neighborhood function) *A neighborhood function $\mathcal{N}$ is given an ℓ-partite graph $G = (V_1 \cup \ldots \cup V_\ell, E)$, an set $B \subseteq V_i$, and a value d_v for each $v \in B$ and it computes sets $\mathcal{N}(v) \subseteq \binom{V_j}{d_v}$ for each $v \in B$.*

Definition 4.37 (Bad neighborhood function) *Let $\delta, \pi > 0$ and let $\mathcal{G} \subseteq \mathcal{S}_\ell(n, m; \varepsilon)$. A neighborhood function $\mathcal{N}$ is called a* bad neighborhood function *for the graph family $\mathcal{G}$ and the parameters δ, π if the following condition holds: For each $G = (V, E) \in \mathcal{G}$ there exist $1 \leq i, j \leq \ell$ and an set $B \subseteq V_i$ with $|B| \geq \delta n$ and $d_j(v) \geq q/2$ (recall that $q = m/n$) for all $v \in B$ such that for each $v \in B$*

$$\Gamma_j(v) \in \mathcal{N}(v) \quad \textit{and} \quad |\mathcal{N}(v)| \lesssim^q \pi^{d_v}\binom{n}{d_v},$$

where $\mathcal{N}(v)$ is applied to the graph $G' = (V, E')$ with edge set $E' := E \setminus E(B, V_j)$, the set B, and $d_v := d_j(v)$.

Figure 4.4 illustrates the concept of a bad neighborhood function. The following lemma shows how bad neighborhood functions are used to prove that the cardinality of a certain set of graphs is small.

Lemma 4.38 (Counting bad graphs) *Let $\pi, \delta > 0$ with $\pi \lll \delta$. If $\mathcal{N}$ is a bad neighborhood function for $\mathcal{G} \subseteq \mathcal{S}_\ell(n, m; \varepsilon)$ and parameters δ, π, then*

$$|\mathcal{G}| \leq \delta^m \binom{n^2}{m}^{\binom{\ell}{2}}$$

for $m = \omega(n \log n)$ and n sufficiently large.

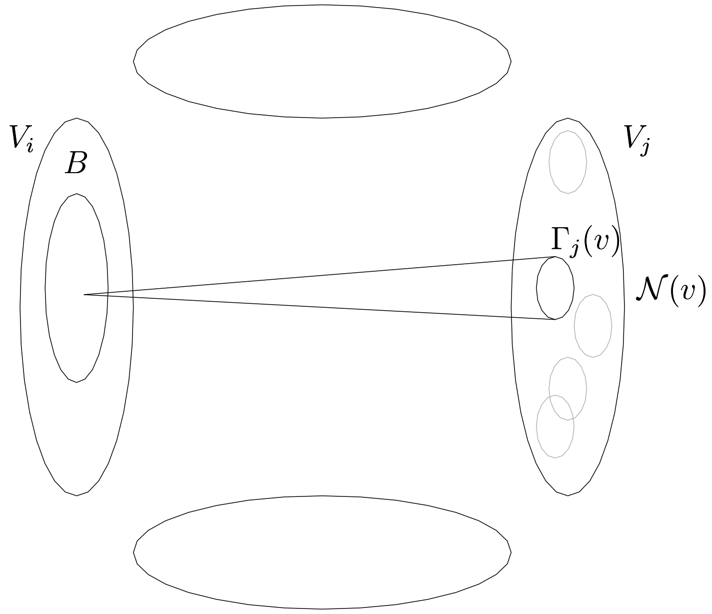

Figure 4.4: Bad neighborhood function

Proof We construct all graphs $G \in \mathcal{G}$ as follows: We start by choosing i, j with $i \neq j$, the set $B \subseteq V_i$ and the degree values $d_v := |\Gamma_j(v)|$ for $v \in V_i$. Observe that there are at most $\ell^2 \cdot n! \cdot n^n \leq 2^m$ possibilities for that, assuming that n is sufficiently large. Then we fix the edges in $E \setminus E(V_i, V_j)$ (at most $\binom{n^2}{m}^{\binom{\ell}{2}-1}$ possibilities). For the vertices $v \in V_i \backslash B$ we have at most $\binom{n}{d_v}$ choices to fix their remaining neighborhood $\Gamma_j(v)$. For each vertex $v \in B$ we choose a set from $\mathcal{N}(v)$ as its neighborhood $\Gamma_j(v)$. Observe that $\mathcal{N}(v)$ is completely determined by the part of G which has already been constructed.

From the assumption that $\mathcal{N}$ is a bad neighborhood function it follows that we may assume that there are at most $\pi^{d_v}\binom{n}{d_v}$ possibilities for that. Hence, we obtain that the total number of possibilities for choosing the edges between V_i and V_j is bounded by (with respect to $\lesssim^m$)

$$\begin{aligned}\left(\prod_{v \in V_i \backslash B} \binom{n}{d_v}\right) \cdot \left(\prod_{v \in B} \pi^{d_v} \binom{n}{d_v}\right) &\leq \pi^{|B| \cdot \min_{v \in B}\{d_v\}} \prod_{v \in V_i} \binom{n}{d_v} \\ &\leq \pi^{\delta n \cdot q/2} \binom{n^2}{m} = \pi^{\delta m/2} \binom{n^2}{m}.\end{aligned}$$

Thus it follows that

$$|\mathcal{G}| \lesssim^m \binom{n^2}{m}^{\binom{\ell}{2}-1} \cdot \pi^{\delta m/2} \binom{n^2}{m} \text{ and, consequently, } |\mathcal{G}| \leq \delta^m \cdot \binom{n^2}{m}^{\binom{\ell}{2}},$$

which proves the claim. $\square$

4.8 Constructing complete subgraphs

This section contains the proof of Theorem 4.3. We will collect several properties that most graphs in $\mathcal{S}_\ell(n, m; \varepsilon)$ satisfy. Finally, all these properties will then be combined to the actual proof of our main theorem.

4.8.1 Constants

For the proof of Theorem 4.3 we will need the following constants which depend on β:

$$\xi = (\beta)^{\ominus 2}, \mu = (\xi)^{\ominus 2}, \sigma = (\mu)^{\ominus 5}, \nu = (\sigma)^{\ominus}, \rho = (\nu^2)^{\ominus}.$$

The constant $\varepsilon > 0$ which specifies the regularity will be assumed to be sufficiently small, in particular much smaller than all other constants.

We will fix the constants only at the very end of the proof. All auxiliary results are formulated for arbitrary values of the constants and the conditions which must be satisfied are given explicitly. Nevertheless, the above list may serve as a quick reference for the relations between the constants, e.g., ρ will always be much smaller than all other constants (except ε).

In the sequel we will assume that $C_0 n^{5/3} \leq m \leq n^2/4$, where $C_0 := \rho^{-6}$, for the case $H = K_5$ and $C_0 n^{8/5} \leq m \leq n^2/4$ for the case $H = K_4$. Note that we have to proof Theorem 4.3 only up to $m \leq \alpha n^2$ for an arbitrary constant $\alpha > 0$, since for $m > \alpha n^2$ the deterministic embedding lemma, Lemma 4.9, takes over.

4.8.2 Further notation and conventions

By a *property* Π for some element x (e.g. a vertex or a set of vertices) we mean a condition which may be true or false for x. For a property Π let $V_i[\Pi]$ denote the vertices in V_i which satisfy Π. The contraposition of a property is given by $\neg\Pi$. The trivial property which is always satisfied is denoted by Ω. If the definition of a property Π incorporates another property Π', we will write $(\Pi \mid \Pi')$. For simplicity $(\Pi \mid \Omega)$ will be abbreviated by (Π). For a family of sets $\mathcal{X} \subseteq \binom{X}{t}$ (satisfying a certain property) we let $\bar{\mathcal{X}} := \binom{X}{t} \setminus \mathcal{X}$.

Throughout this section all results refer to a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$, and V_i, V_j and V_k denote different partitions of this graph. When counting the graphs which satisfy a certain property Π it makes no difference whether the values of i, j and k are fixed or variable. Since the number of graphs will usually be just a superexponentially small fraction of $|\mathcal{S}_\ell(n, m; \varepsilon)|$, the number of choices for i, j and k, which is clearly less than ℓ^3, does not carry weight.

However, it does not suffice to define the properties to hold for arbitrary values of i, j and k. Recall our basic counting strategy introduced in Section 4.7. We construct the graph by sequentially choosing the edges. The crucial point is that there exist partitions V_i and V_j for which we can significantly reduce the number of possible choices for the neighborhoods of certain 'bad' vertices $B \subseteq V_i$ by introducing a bad neighborhood function. Clearly, the definition of the bad neighborhood function must not depend on the edges between B and V_j. Thus we must be able to control the set of edges on which a property depends. We prefer to do this implicitly, as an explicit statement of these dependencies would result in very clumsy notation. However, in Section 4.8.16 we will further elaborate on that and discuss on which edges the various properties shall depend. For now we ignore this issue and assume that the properties hold for arbitrary values of i, j and k, whatever we need.

Furthermore, throughout the following counting arguments we first fix values of certain parameters, e.g., degrees of vertices or cardinalities of sets, which will then be used in the bounds on the number of graphs to be counted. By convention such parameters are assumed to take their worst-case value, i.e., the value which maximizes the number to be estimated. Formally speaking, we do not explicitly write the maximum $\max_t$ for every free parameter t in the formula.

The following cardinalities of sets of vertices will be used in the proof, where λ will be replaced by one of various constants defined above:

$$
\begin{aligned}
q &:= \tfrac{m}{n}, \\
q_\lambda &:= (1-\lambda)\lambda q, \\
p_\lambda &:= 2\lambda^{-1}\tfrac{n^2}{m}, \\
r_\lambda &:= (1-\lambda)\lambda^2\tfrac{m^2}{n^3}, \\
t_\lambda &:= \lambda^{-1}\tfrac{n^2}{m}, \\
h_\lambda &:= \lambda^{-1}\tfrac{m^3}{n^5}, \\
r_\lambda(x) &:= \tfrac{x}{2p_\lambda} = \tfrac{1}{4}\lambda\tfrac{m}{n^2}x.
\end{aligned}
$$

The following quantities depend on specific constants, which we have introduced in Section 4.8.1:

$$
\begin{aligned}
o_\mu &:= \sigma^{-5/2}\tfrac{n^2}{m}, \\
s_\mu &:= r_\sigma(r_\mu) = \tfrac{1}{4}(1-\mu)\mu^2\sigma\tfrac{m^3}{n^5}, \\
d_\mu &:= 2s_\mu \cdot \tfrac{q_\mu}{2o_\mu} = \tfrac{1}{4}(1-\mu)^2\mu^3\sigma^{7/2}\tfrac{m^5}{n^8}, \\
\tilde{p} &:= p_\mu/2 = \mu^{-1}\tfrac{n^2}{m}, \\
\tilde{r} &:= r_\mu(\sigma^2 q_\mu)/2 = \tfrac{1}{8}(1-\mu)\mu^2\sigma^2 q, \\
\hat{q} &:= (1-(\mu)^{\oplus 2})n/\tilde{p},
\end{aligned}
$$

$$\begin{aligned}\tilde{r}_1 &:= r_\mu(\mu q_\mu)/2 = \tfrac{1}{8}(1-\mu)\mu^3\tfrac{m^2}{n^3},\\ \tilde{r}_2 &:= r_\rho(\mu q_\mu)/2 = \tfrac{1}{8}(1-\mu)\rho\mu^2\tfrac{m^2}{n^3}.\end{aligned}$$

The above definitions will be restated later but we collect them here for better reference.

Obviously some of the above quantities are very similar, e.g., $\tilde{p} = t_\mu$. However, these quantities will have a different meaning in the proof and we hope that this redundancy will increase the readability of the exposition.

The constant ε will always refer to the regularity constant, i.e., to the constant used to define the graph class $\mathcal{S}_\ell(n, m; \varepsilon)$. We assume that ε is arbitrarily small, i.e., as small as we need it in all subsequent inequalities. In particular, ε is much smaller than all other constants.

4.8.3 Homogeneous sets

Unfortunately, the ε-regularity of a graph does not help much when considering subgraphs with $o(n)$ vertices. However, for our proofs we will mostly be content with a weaker property, which we call *homogeneity*. This property can be shown to be hereditary, i.e., it also holds for most subgraphs of a homogeneous graph.

The basic idea behind homogeneity consists in a characterization of the neighborhood of sets (not only of single vertices!).

Definition 4.39 (Vertices with a bad degree) *For a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ and a set $X \subseteq V_j$ we define*

$$\bar{D}_i(X) := \{v \in V_i \mid \ |d_X(v) - |X| \cdot \tfrac{m}{n^2}| \geq \varepsilon \cdot |X|\tfrac{m}{n^2}\}.$$

It is easy to see that in an ε-regular graph almost all vertices have the 'right' degree into any other partition.

Lemma 4.40 ($\bar{D}$ is small) *Let $X \subseteq V_j$. If $|X| \geq \varepsilon n$ then $|\bar{D}_i(X)| \leq 2\varepsilon n$.*

Proof Follows directly from the definition of ε-regularity (similar to the proof of Lemma 4.7). □

Since almost all vertices in an ε-regular graph have $\Theta(m/n)$ neighbors, one can deduce that the neighborhood of $\Theta(n^2/m)$ vertices typically has size $\Theta(m/n \cdot n^2/m) = \Theta(n)$. Furthermore, the neighborhoods of the single vertices have disjoint parts of size $\Theta(m/n)$, i.e., every vertex contributes an equal part to the combined neighborhood of all vertices. In the sequel we give a formal definition of this structure.

Definition 4.41 (Covering vertices) *Let $P \subseteq V_i$. $I_j(P;\nu)$ is defined as an (ordered) subset of $\tilde{P} \subseteq P$ with maximum cardinality such that the following property is satisfied: Let $\tilde{P} = \{v_1, \ldots, v_k\}$, then there exist pairwise disjoint sets $W_1 \subseteq \Gamma_j(v_1), \ldots, W_k \subseteq \Gamma_j(v_k)$ with $|W_1| = \ldots = |W_k| = q_\nu := (1-\nu)\nu q$. If there are several (ordered) sets $\tilde{P}$ with this property then $I_j(P;\nu)$ is assigned one of these sets in an arbitrary but unique way.*
Furthermore, we may restrict the vertices from which the vertices $\tilde{P}$ are chosen to all vertices in V_i which satisfy a certain property Π. Then we write $I_j(P;\nu \mid \Pi)$ and require that $I_j(P;\nu \mid \Pi) \subseteq V_i[\Pi]$.

Definition 4.42 (Covered neighborhood) *For $P \subseteq V_i$ the set $W_j(P;\nu)$ is defined as follows. Let $I_j(P;\nu) = \{v_1, \ldots, v_k\}$ then*

$$W_j(P;\nu) := \bigcup_{i=1}^{k} W_i.$$

In [KŁR96] (cf. Lemma 11) an argument has been presented which implies that in an ε-regular graph almost all sets P with $|P| = p_\nu = \nu^{-1}\frac{n^2}{m}$ have a large neighborhood, i.e., $|W_j(P;\nu)| \geq (1-\nu)n$. Intuitively, a set is homogeneous, if exactly this property holds. A weaker property than ε-regularity suffices to show the desired bound on $|W_j(P;\nu)|$ and this property forms the basis of the definition of homogeneity. In Section 4.8.4 we will then see how this property is used to prove the existence of sets P with large neighborhoods.

Definition 4.43 (Homogeneous sets) *A set $X \subseteq V_i$ is called ρ-homogeneous for V_j if for all $\nu \geq \varepsilon$ and all $P \subseteq X$ with $|P| \leq p_\varepsilon$ one of the following two conditions is satisfied:*

(i) $|W_j(P;\nu)| \geq (1-\nu)n$, *or*

(ii) $|(X \setminus P) \cap \bar{D}_i(V_j \setminus W_j(P;\nu))| \leq \rho|X|$.

The set $\mathcal{H}_j(Z,t;\rho) \subseteq \binom{Z}{t}$ contains all ρ-homogeneous sets in $\binom{Z}{t}$.

The intuition behind this definition is as follows. Assume that we want to assure that a typical set P with $\Theta(n^2/m)$ vertices has a covered neighborhood $W_j(P;\nu)$ of size $(1-\nu)n$. We choose P vertex by vertex uniformly at random and intend to show that with very high probability this leads to a set P with the desired size of the covered neighborhood. Definition 4.43 guarantees that in every step of this random process the probability for doing something 'wrong' is small.

Condition (i) can be interpreted as the termination case. If the covered neighborhood is already large enough, we are done. Condition (ii) assures that there are only very few vertices in X (which have not yet been added to P) with an untypical degree into the part of V_j which is not yet covered by W_j. This implies that by choosing a vertex in $X \setminus P$ uniformly at random, we are likely to get a vertex v which contributes a new set W_v to the covered neighborhood. Thus the covered neighborhood is guaranteed to grow quickly. This intuition will later lead to the proof of Lemma 4.49.

Now let us collect a few simple properties of homogeneity.

Proposition 4.44 (Simple properties of homogeneity)

(i) *ε-regularity implies 2ε-homogeneity of V_i for V_j.*

(ii) *For $\rho \leq \rho'$, ρ-homogeneous sets are also ρ'-homogeneous.*

(iii) *If a set X is ρ-homogeneous, then any subset $X' \subseteq X$ with $|X'| \geq \alpha|X|$ is ρ/α-homogeneous.*

Proof (i) is a simple consequence of Lemma 4.40. The other claims follow directly from the Definition 4.43. □

In a ρ-homogeneous set X we know that almost all vertices are 'good' for all choices of $P \subseteq X$. Here a vertex is 'good' if it has the right number of neighbors in the part of V_j which is not in the neighborhood W_j of P. When we gain more and more knowledge about typical properties of most vertices in the graph we have to strengthen the definition of homogeneity. For Π-enhanced ρ-homogeneous sets we additionally require that 'good' vertices satisfy a given property Π.

Definition 4.45 (Enhanced homogeneous sets) *A ρ-homogeneous set $X \subseteq V_i$ is called* Π-enhanced *for a property Π, if $|X[\neg\Pi]| \leq \rho|X|$. The set of Π-enhanced ρ-homogeneous sets in $\binom{Z}{t}$ is denoted by $\mathcal{H}_j(Z,t;\rho \mid \Pi) \subseteq \binom{Z}{t}$.*

The following lemma shows the crucial property of homogeneity, namely that this property is hereditary, i.e., it typically also holds for subsets.

Lemma 4.46 (Hereditary nature of homogeneity) *Let $\varepsilon^{-1}\frac{n^2}{m}\log n \leq t_2 \leq t_1 \leq n$ and $\rho_1 \lll \rho_2$. Consider a set $X \in \mathcal{H}_j(Z, t_1; \rho_1 \mid \Pi)$. Then*

$$|\overline{\mathcal{H}}_j(X; t_2, \rho_2 \mid \Pi)| \leq \rho_2^{t_2}\binom{t_1}{t_2}.$$

Proof Let us firstly consider the case $\Pi = \Omega$. There are most $n^{p_\varepsilon+1}$ possibilities to choose a set $P \subseteq X$ with $|P| \leq p_\varepsilon$. Furthermore, the number of choices for $\nu \geq \varepsilon$ which yield really different conditions in Definition 4.43 can be bounded from above by, say, n^2. This is due to the fact that ν only influences quantities which must (or can be assumed to) be integer, namely, q_ν and the bound $(1-\nu)n$ in Definition 4.43 (i) (see also Remark 4.55 where we encounter a similar problem). All together we obtain

$$n^2 \cdot n^{p_\varepsilon+1} = 2^{2\varepsilon^{-1}\frac{n^2}{m}\log n+3} \leq 2^{2t_2+3}.$$

Since this bound is small enough compared to the superexponentially small term $\rho_2^{t_2}$, it suffices to consider a fixed set $P \subseteq X$ and calculate the number of sets $Y \in \binom{X}{t}$ with $P \subseteq Y$ for which the conditions given in Definition 4.43 and Definition 4.45 are violated. For such a set Y we must either have that

$$|(Y \setminus P) \cap \bar{D}_i(V_j \setminus W_j(P;\nu))| > \rho_2|Y| \text{ and } |W_j(P;\nu)| < (1-\nu)n,$$

or $|Y[\neg\Pi]| > \rho_2|Y|$. Note that

$$|(X \setminus P) \cap \bar{D}_i(V_j \setminus W_j(P;\nu))| \leq \rho_1|X| \text{ if } |W_j(P;\nu)| < (1-\nu)n$$

and $|X[\neg\Pi]| \leq \rho_1|X|$.

Consider a set $Y' \subseteq Y \setminus P$ with $|Y'| = |Y|/2$ which contains all of the at least $\rho_2|Y| \geq \rho_2|Y'|$ bad vertices. Since such a set Y' exists, it follows that Y is $1/2$-partially $(1,\rho_2)$-$\forall$-bad with error probability $2\rho_1$. Applying Corollary 4.34 proves the claim. $\square$

4.8.4 Simple covers

In the previous section we have already encountered the following intuitive argument: In a random graph with edge probability m/n^2 a set of size n^2/m has $(n^2/m)\cdot n\cdot(m/n^2) = n$ incident edges on the average. Hence, we may expect that a set P of size $\Theta(n^2/m)$ in partition V_i covers a class V_j for $j \neq i$, i.e., $\Gamma_j(P) \approx V_j$. Such covering sets or *covers* will play a crucial rôle in our proof. Figure 4.5 illustrates the structure of a cover.

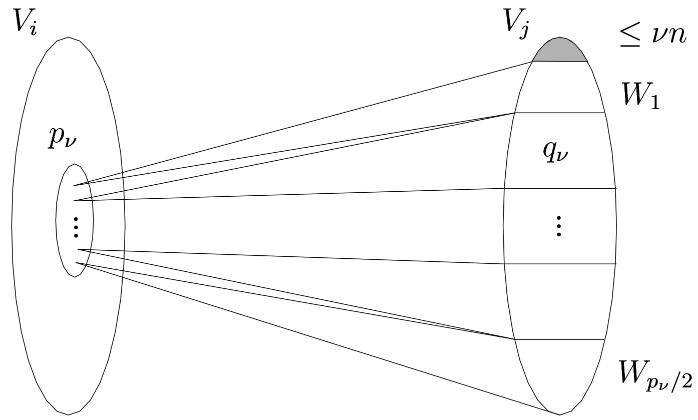

Figure 4.5: A ν-cover.

Definition 4.47 (Covers) *A set $P \subseteq V_i$ with $|P| = p_\nu := 2\nu^{-1}\frac{n^2}{m}$ is called a* ν-cover *of V_j if $|I_j(P;\nu)| \geq p_\nu/2$. For a ν-cover P we let $P^* := [I_j(P;\nu)]_{p_\nu/2}$.*

Note that for a ν-cover P we have

$$|\Gamma_j(P)| \geq |W_j(P;\nu)| \geq (p_\nu/2) \cdot q_\nu = (1-\nu)n. \tag{4.9}$$

The following definition parallels the definition of Π-enhanced ρ-homogeneous sets. A cover is called Π-qualified if the covering vertices can be chosen to satisfy Π.

Definition 4.48 (Qualified covers) *A ν-cover P of V_j is* Π-qualified *for a given property Π if $|I_j(P;\nu \mid \Pi)| \geq p_\nu/2$. Then we set $P^* := [I_j(P;\nu \mid \Pi)]_{p_\nu/2}$.*
For the set of Π-qualified covers in X we introduce the notation $\mathcal{P}_j(X;\nu \mid \Pi) \subseteq \binom{X}{p_\nu}$. Instead of $\mathcal{P}_j(X;\nu \mid \Omega)$ we simply write $\mathcal{P}_j(X;\nu)$.

The following lemma establishes the connection between homogeneous sets and covers. It shows that almost all sets of size p_ν (up to a superexponentially small fraction) in ρ-homogeneous sets are ν-covers.

Lemma 4.49 ($\bar{\mathcal{P}}$ is small) *If X is Π-enhanced ρ-homogeneous for V_j and $\nu \geq \varepsilon$ then*

$$|\bar{\mathcal{P}}_j(X;\nu \mid \Pi)| \leq (2\rho)^{p_\nu/4}\binom{|X|}{p_\nu}.$$

Proof Choose the vertices $v_1, \ldots, v_{p_\nu} \in P$ one by one. Let $U_k := \{v_1, \ldots, v_k\}$. If v_k satisfies Π and has at least q_ν neighbors in $V_j \setminus W_j(U_{k-1}; \nu)$, then clearly $|I_j(U_k; \nu) \setminus I_j(U_{k-1}; \nu)| \geq 1$. On the other hand we have $|I_j(U_k; \nu) \setminus I_j(U_{k-1}; \nu)| \leq 1$ by the definition of I_j. If $|I_j(U_k; \nu) \setminus I_j(U_{k-1}; \nu)| = 1$, we call v_k a good vertex.

Provided that $|I_j(U_k; \nu)| \geq p_\nu/2$, the set P is a ν-cover. Otherwise, we have $|W_j(U_k; \nu)| < (p_\nu/2) \cdot q_\nu = (1-\nu)n$. Note that the vertices in $D_i(V_j \setminus W_j(U_k; \nu))$ which satisfy Π are good. Hence, by Definition 4.43 and Definition 4.45 there are at most $2\rho|X|$ bad vertices in $X \setminus P$. Since $P \in \bar{\mathcal{P}}_j(X; \nu \mid \Pi)$ contains at least $p_\nu/2$ bad vertices, the set $\bar{\mathcal{P}}_j(X; \nu \mid \Pi)$ can be interpreted to be $(1, 1/2)$-$\forall$-bad with exception probability 2ρ, and the claim follows by Lemma 4.32. $\square$

4.8.5 Multicovers

In the previous section we have introduced covers of size $\Theta(n^2/m)$, whereas in our proof we will often be concerned with sets of size $\Theta(m/n) = \Theta(q)$, i.e., neighborhoods of vertices. The following definition therefore transfers the notion of a cover to such (larger) sets.

Definition 4.50 (Multicovers) *Let $\nu > 0$. We call a set $Q \subseteq V_i$ a ν-*multicover *of V_j if there exist pairwise disjoint subsets $P_1, \ldots, P_r \subseteq Q$ with $r = r_\nu(|Q|) := \frac{|Q|}{2p_\nu}$ and $|P_1| = \ldots = |P_r| = p_\nu$ such that P_i is a ν-cover of V_j for all $i = 1, \ldots, r$. Furthermore, we let $Q^*(\nu) := \bigcup_{k=1}^{r} P_k^*$, where the parameter ν indicates that Q shall be decomposed into ν-covers P_k in an arbitrary but unique way.*

Figure 4.6 shows the structure of a ν-multicover. Note that the P-sets actually occupy only one half of Q.

Observe that a multicover Q with $|Q| = \Theta(m/n)$ can be decomposed into $\Theta(m/n \cdot m/n^2) = \Theta(m^2/n^3)$ covers $P_1, \ldots, P_r$. Since $\Gamma_j(P_l) \approx V_j$ for all $l = 1, \ldots, r$, we expect that a typical vertex has degree r into Q (up to some constant factor). Observe that the average number of neighbors is $\Theta((m/n) \cdot (m/n^2)) = \Theta(m^2/n^3) = \Theta(r)$. This intuition is confirmed by the following definition and lemma.

Definition 4.51 (Covered neighborhood of multicovers) *Let $\nu > 0$. For a ν-multicover $Q \subseteq V_i$ we define*

$$C_j^F(Q; \nu) := \{w \in V_j \mid |\Gamma_i^F(w) \cap Q| \geq r_\nu(|Q|)/2\}.$$

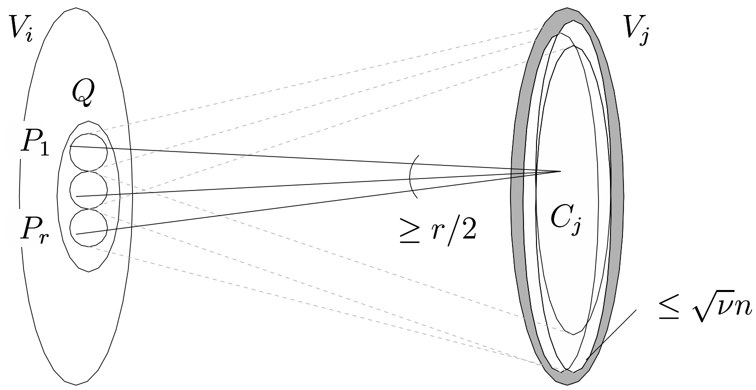

Figure 4.6: ν-multicover

Lemma 4.52 (Large covered neighborhoods) *Let $Q^*(\nu) = \bigcup_k P_k^* \subseteq V_i$ be a ν-multicover of V_j for some $\nu > 0$ with $\sqrt{\nu} \leq \frac{1}{2}$. Then*

$$|C_j(Q;\nu)| \geq (1-\sqrt{\nu})n.$$

For $F \subseteq E$ with $d_j(v) - d_j^F(v) \leq \nu^2 q$ we have

$$|C_j^F(Q;\nu)| \geq (1-3\sqrt{\nu})n.$$

Proof Let $P_1, \ldots, P_r$ with $r := r_\nu(|Q|)$ denote pairwise disjoint sets of size p_ν in Q which are ν-covers of V_j. Consider the auxiliary bipartite graph $B = (\{P_1, \ldots, P_r\} \dot\cup V_j, E_B)$, where $\{P_i, w\} \in E_B$ if and only if $w \in \Gamma_j(P_i)$ in G. $P_1, \ldots P_r$ can be interpreted as 'super-vertices' in G. From $|\Gamma_j(P_i)| \geq (1-\nu)n$ we deduce that all super-vertices P_i have degree at least $(1-\nu)n$ in B. Hence, we can apply Corollary 4.29 and obtain that

$$Z := \{w \in V_j \mid |\{i = 1, \ldots, r \mid w \in \Gamma_j(P_i)\}| \geq (1-\sqrt{\nu})r\}$$

satisfies $|Z| \geq (1-\sqrt{\nu})n$. By

$$(1-\sqrt{\nu})r \geq r/2 = r_\nu(|Q|)/2$$

we conclude that $Z \subseteq C_j(Q;\nu)$.

The second claim follows from the observation that

$$|\Gamma_j^F(P_i)| \geq (1-\nu)n - p_\nu \cdot \nu^2 q = (1-\nu)n - 2\nu\frac{n^2}{m} \cdot \frac{m}{n} = (1-3\nu)n.$$

The rest of the proof is completely analogous to the case $F = \emptyset$. □

For a ν-multicover Q the constant ν can be interpreted as the *accuracy* of the cover. The smaller ν is the more vertices in V_j belong to C_j. However, we have to pay a price for better accuracy. As ν decreases the size $p_\nu = 2\nu^{-1}\frac{n^2}{m}$ of the covers inside the multicover increases. This leads to a decrease of $r_\nu(|Q|)$. Thus vertices in C_j see less vertices inside Q as ν gets smaller. Due to this tradeoff we will need multicovers of different accuracy in our proof. Although it would be possible to list the different constants ν that we use, we prefer to replace the notion of a multicover by the more powerful notion of a supercover. Supercovers are ν-multicovers for a variety of constants ν.

Definition 4.53 (Supercovers) *A set Q is called a* τ-supercover *if every subset $Q' \subseteq Q$ with $|Q'| \geq \tau|Q|$ is a ν-multicover for all $\nu \geq \varepsilon$.*

At later stages of the proof we will not be content with finding multicovers or supercovers. Additionally, we will need that the vertices which make up the structure of the multicover, i.e., which possess the necessary covering neighborhoods, satisfy certain properties. To this aim we introduce the following definition.

Definition 4.54 (Qualified multi- and supercovers) *A ν-multicover Q of V_j is* Π-qualified *if the ν-covers that make up Q may be chosen such that they are Π-qualified. A τ-supercover Q is Π-qualified if every subset Q' with $|Q'| \geq \tau|Q|$ is a Π-qualified ν-multicover for all $\nu \geq \varepsilon$.*
Let $X \subseteq V_i$. $\mathcal{MQ}_j(X; t, \nu \mid \Pi)$ denotes the set of Π-qualified ν-multicovers with t vertices. For the set of Π-qualified τ-supercovers we write

$$\mathcal{Q}_j(X; t, \tau \mid \Pi) := \{Z \in \binom{X}{t} \mid \forall Z' \subseteq Z, |Z'| \geq \tau t : \\ Z' \in \bigcap_{\nu \geq \varepsilon} \mathcal{MQ}_j(X; |Z'|, \nu \mid \Pi)\}.$$

Remark 4.55 *The notation $\bigcap_{\nu\geq\varepsilon}$ in Definition 4.54 is justified by the fact that there is only a polynomial number of distinct choices for $\nu \geq \varepsilon$ for which the sets $\mathcal{MQ}_j(X; |Z'|, \nu \mid \Pi)$ are really different. Note that the parameter ν merely influences cardinalities of sets, namely p_ν and the bound on the size of $W_j(P; \nu)$. These quantities must be integers and thus there are at most, say, n^2 possible values for them.*

Analogously to Lemma 4.49 for covers, we now show that almost all subsets of a ρ-homogeneous set are ν-multicovers.

Lemma 4.56 ($\bar{\mathcal{Q}}$ is small) *Let $t \geq p_\rho$. If X is Π-enhanced ρ-homogeneous, then*

$$|\bar{\mathcal{Q}}_j(X; t, (\rho)^\oplus \mid \Pi)| \leq ((\rho)^\oplus)^t \binom{|X|}{t}.$$

Proof Assume that

$$|\overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)| \leq (2\rho)^{t'/16} \binom{|X|}{t'}, \tag{4.10}$$

where $(\rho)^\oplus t \leq t' \leq t$. By Remark 4.55 we conclude that (maximizing over t')

$$\begin{aligned} |\bar{\mathcal{Q}}_j(X; t, (\rho)^\oplus \mid \Pi)| &\lesssim^t (2\rho)^{t'/16} \binom{|X|}{t'} \binom{|X|}{t-t'} \overset{\text{L4.27 (iii)}}{\leq} 4^t \cdot (2\rho)^{t'/16} \binom{|X|}{t} \\ &\leq (8\rho^{(\rho)^\oplus/16})^t \binom{|X|}{t}, \end{aligned}$$

and the lemma directly follows. Hence it suffices to show (4.10).

Due to Lemma 4.49 the sets in $\overline{\mathcal{MQ}}_j(X; t', \nu \mid \Pi)$ are $(p_\nu, 1/2)$-$\forall$-bad with exception probability $(2\rho)^{p_\nu/4}$. By Lemma 4.32, (4.10) follows. □

Remark 4.57 *Observe that for the proof of Lemma 4.56 it would suffice to assume that X is $\rho^{1/k}$-homogeneous for some constant $k \geq 1$.*

The following definition and lemma show that the basic structure of a multicover remains intact if we forbid a set $X \subseteq V_j$. Only the number of vertices with covering neighborhoods and the size of the covering neighborhoods must be scaled down in a suitable way.

Definition 4.58 (Resistant multicovers) *Let $Q \subseteq V_i$ be a ν-multicover of V_j. Q is called X-*resistant *for $X \subseteq V_j$ if*

$$\begin{aligned} &\exists Q^{**} \subseteq Q^*, |Q^{**}| = \nu|Q| \\ &\forall u \in Q^{**}\ \exists W'_u \subseteq \Gamma_j(u) \setminus X, |W'_u| = \nu q_\nu : \\ &\qquad \text{the sets } W'_u \text{ are } r_\nu(|Q|)\text{-quasidisjoint}. \end{aligned}$$

Lemma 4.59 (All multicovers are resistant) *Every ν-multicover $Q \subseteq V_i$ of V_j is X-resistant for $X \subseteq V_j$, provided that $|X| \leq (1 - 2\sqrt{\nu})n$.*

Proof We count the number of occurrences of vertices $x \in V_4$ in sets W_u of the ν-multicover Q. Every vertex in $C_j(Q;\nu)$ corresponds to at least $r_\nu(|Q|)/2$ such occurrences.

Since $|C_j(Q;\nu)| \geq (1-\sqrt{\nu})n$ by Lemma 4.52, it follows that $|C_j(Q;\nu)\setminus X| \geq \sqrt{\nu}n$. We conclude that at least

$$\sqrt{\nu}n \cdot r_\nu(|Q|)/2 \geq \frac{1}{8}\nu^{3/2}\frac{m}{n}|Q| \tag{4.11}$$

occurrences remain if we restrict the sets W_u to vertices in $V_j \setminus X$.

Assume that there are not enough vertices left from the original sets W_u (i.e. in the ν-multicover) for the sets W_u' in the X-resistant ν-multicover. If the number of suitable sets is too small to meet the requirements of Definition 4.58, we can bound the number of remaining occurrences in sets W_u as follows. Firstly, for every vertex in Q there may exist νq_ν such occurrences. Secondly, for at most $\nu|Q|$ vertices $u \in Q$ we may have as many as q_ν occurrences, i.e., the whole covering neighborhood W_u belongs to $V_j \setminus X$. Combining these choices (and counting over) we obtain

$$\#\text{occurrences} \quad \leq \quad |Q|\cdot \nu q_\nu + \nu|Q|\cdot q_\nu = 2\nu q_\nu|Q| \leq 2\nu^2\frac{m}{n}|Q|.$$

This obviously contradicts (4.11) for ν sufficiently small. □

4.8.6 Structure of simple neighborhoods

Now it is time to introduce the first few of several properties by which we intend to characterize 'good' graphs, i.e., graphs for which the occurrence of a complete subgraphs can be guaranteed. These properties will refer to the size and the structure of the neighborhood of a single vertex.

We say that a vertex $v \in V_i$ satisfies the *degree property* if the following condition (D) is met:

$$(D) \qquad (1-\varepsilon)q \leq d_j(v) \leq (1+\varepsilon)q. \tag{4.12}$$

The next lemma shows that most vertices in ε-regular graphs satisfy (D).

Lemma 4.60 (Most vertices satisfy (D)) *For a graph $G \in \mathcal{S}_\ell(n,m;\varepsilon)$ we have*

$$|V_i[(\neg D)]| \leq \ell^2\varepsilon n.$$

Proof Follows directly from Lemma 4.40. □

A vertex $v \in V_i$ satisfies the *homogeneity property* (H_ρ) if the following condition is met:

$$(H_\rho \mid \Pi) \quad \Gamma_j(v) \text{ is } (\Pi\text{-enhanced}) \ \rho\text{-homogeneous for } V_k. \tag{4.13}$$

For the conjunction of properties $(H_\rho) \wedge (D)$ we introduce the shortcut (H_ρ, D). The following simple fact is stated here for future reference.

Proposition 4.61 *For a vertex v which satisfies (H_ρ, D) we conclude that any subset $Q' \subseteq Q := \Gamma_j(v)$ with $|Q'| \geq \tau q$ is $2\rho/\tau$-homogeneous.*

Proof To see this note that $|Q| \leq (1+\varepsilon)q \leq 2q$ due to property (D). Then Proposition 4.61 follows directly from Proposition 4.44. □

We say that a vertex $v \in V_i$ satisfies the *cover property* (C_ρ) if the following condition is met:

$$(C_\rho \mid \Pi) \quad \Gamma_j(v) \text{ is a } (\Pi\text{-qualified}) \ \rho\text{-supercover of } V_k. \tag{4.14}$$

Observe that (C_ρ) is monotonous, i.e., (C_ρ) implies $(C_{\rho'})$ for $\rho' \geq \rho$. This is due to the fact that a ρ-supercover is also a ρ'-supercover for $\rho' \geq \rho$. For (H_ρ) an analogous statement holds due to Proposition 4.44 (ii). This will facilitate the handling of the constants later in the proof because we do not have to pay attention that the constants do not get 'too small'.

Since the homogeneity property and the cover property bear some similarity (namely both deal with the existence of covers and multicovers), we combine them to the property

$$(Q_\rho \mid \Pi) := (H_\rho \mid \Pi) \wedge (C_\rho \mid \Pi).$$

As for all subsequently defined properties we introduce a set of bad graphs with respect to (Q_ρ), namely

$$\mathcal{B}_\ell^Q(n, m; \varepsilon, \rho) := \{G \in \mathcal{S}_\ell(n, m; \varepsilon) \mid |V_i[(\neg Q_\rho, D)]| \geq (\rho)^\oplus n\}, \tag{4.15}$$

and show that only very few such bad graphs exist.

Lemma 4.62 ($\mathcal{B}_\ell^Q$ is small)

$$|\mathcal{B}_\ell^Q(n, m; \varepsilon, \rho)| \leq ((\rho)^\oplus)^m \binom{n^2}{m}^{\binom{\ell}{2}}.$$

Proof Consider the set of bad vertices $B \subseteq V_i[(\neg Q_\rho, D)]$ with $|B| \geq (\rho)^\oplus n$ implied by (4.15).

Case 1: There are at least $(\rho)^{\oplus}n/2$ bad vertices for which $\Gamma_j(v)$ is not ρ-homogeneous, i.e., (H_ρ) is not satisfied. We define the neighborhood function

$$\mathcal{N}(v) := \{X \in \binom{V_j}{d_v} \quad | \quad X \in \overline{\mathcal{H}}_k(V_j; d_v, \rho)\}$$

Observe that $d_v \geq q/2$, and thus $d_v \geq \varepsilon^{-1}\frac{n^2}{m}\log n$. Since V_i is 2ε-homogeneous due to the ε-regularity of the graph and $2\varepsilon \lll \rho$ by the assumption that ε is sufficiently small, we deduce by Lemma 4.46 that $|\mathcal{H}_k(V_j; d_v, \rho)| \leq \rho^{d_v}\binom{n}{d_v}$. We obtain $|\mathcal{N}(v)| \leq \rho^{d_v}\binom{n}{d_v}$. Now the claim follows for this case by Lemma 4.38.

Case 2: There are at least $(\rho)^{\oplus}n/2$ bad vertices for which $\Gamma_j(v)$ is no ρ-supercover, i.e. (C_ρ) is not satisfied. The corresponding neighborhood function is given by

$$\mathcal{N}(v) := \{X \in \binom{V_j}{d_v} \quad | \quad X \in \bar{\mathcal{Q}}_k(V_j; d_v, \rho)\}$$

Recall that V_i is 2ε-homogeneous. By Lemma 4.56 we have $|\bar{\mathcal{Q}}_k(V_j; d_v, \rho)| \leq |\bar{\mathcal{Q}}_k(V_j; d_v, (\varepsilon)^{\oplus})| \leq ((\varepsilon)^{\oplus})^{d_v}\binom{n}{d_v}$ and we can thus derive a similar bound on $|\mathcal{N}(v)|$ as in the previous case. This completes the proof. □

Remark 4.63 *The proof of Lemma 4.62 merely relies on the fact that V_i is 2ε-homogeneous and that $(\varepsilon)^{\oplus} \leq \rho$. The proof could remain unchanged if V_i were just α-homogeneous for some α with $(\alpha)^{\oplus} \leq \rho$.*

Assume that vertex $v \in V_i$ has a ν-multicover Q in its neighborhood $\Gamma_k(v)$. Since $C_j(Q;\nu)$ contains $(1-\nu)n$ vertices in V_j and ν is assumed to be small, we expect that the neighborhood $\Gamma_j(v)$ and $C_j(Q;\nu)$ overlap significantly. This leads to the following property, which we will only need for the proof of the case $H = K_5$.

We say that a vertex $v \in V_i$ satisfies the *triangle property* if the following condition (T_ρ) is met:

$$(T_\rho) \quad \forall Q \subseteq \Gamma_k(v),\ Q \text{ is a } \rho\text{-multicover}: \ |\Gamma_j(v) \setminus C_j(Q;\rho)| \leq (\rho)^{\oplus}q. \quad (4.16)$$

This leads to the bad set

$$\mathcal{B}_\ell^T(n,m;\varepsilon,\rho) \ := \ \{G \in \mathcal{S}_\ell(n,m;\varepsilon) \mid \ |V_i[(\neg T_\rho, D)]| \geq (\rho)^{\oplus 2}n\}, \qquad (4.17)$$

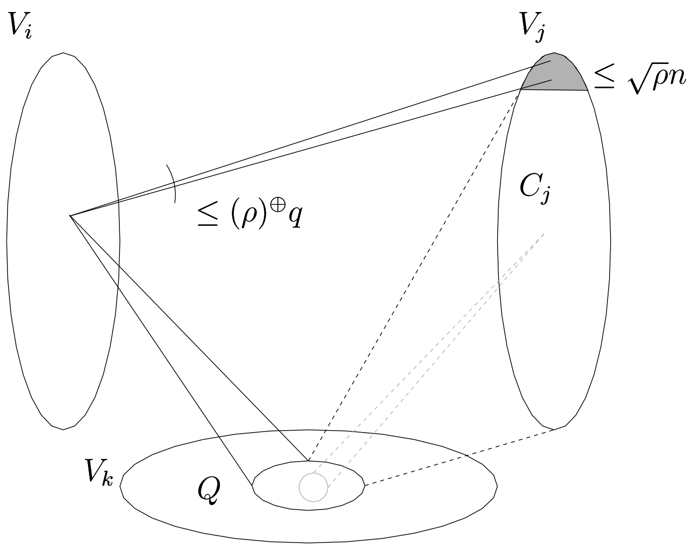

Figure 4.7: Property (T_ρ)

which can be shown to be small. Figure 4.7 illustrates property (T_ρ). The picture also makes clear how $\mathcal{B}_\ell^T$ is counted. Since C_j comprises almost the whole partition V_j, the number of possibilities to choose the neighbors of v in $V_j \setminus C_j$ is tiny. This immediately implies that $\mathcal{B}_\ell^T$ is small, too.

Lemma 4.64 ($\mathcal{B}_\ell^T$ is small)

$$|\mathcal{B}_\ell^T(n, m; \varepsilon, \rho)| \leq ((\rho)^{\oplus 2})^m \binom{n^2}{m}^{\binom{\ell}{2}}.$$

Proof We again use Lemma 4.38, i.e., we have to show that there exists an appropriate definition of a bad neighborhood function $\mathcal{N}$ for $(\rho)^{\oplus 2}n$ bad vertices $B \subseteq V_i[(\neg T_\rho, D)]$. We let

$$\mathcal{N}(v) := \{X \in \binom{V_j}{d_v} \quad | \quad \exists Q \subseteq \Gamma_k(v),\ Q \text{ is a } \rho\text{-multicover}: \\ |X \setminus C_j(Q_k; \rho)| \geq (\rho)^{\oplus} q\}.$$

When counting $|\mathcal{N}(v)|$ we first choose the set Q. Since $Q \subseteq \Gamma_k(v)$ and $d_k(v) \leq (1+\varepsilon)q$ we can bound the number of choices for Q by $2^{(1+\varepsilon)q}$. As $|C_j(Q;\rho)| \geq (1-\sqrt{\rho})n$ by Lemma 4.52, we deduce that

$$|\mathcal{N}(v)| \lesssim^q \binom{\sqrt{\rho}n}{(\rho)^{\oplus}q}\binom{n}{d_v - (\rho)^{\oplus}q} \overset{\text{L4.27 (ii, iii)}}{\leq} \rho^{(\rho)^{\oplus}q/2} \cdot 4^{d_v}\binom{n}{d_v} \leq ((\rho)^{\oplus})^{d_v}\binom{n}{d_v},$$

proving the claim. $\square$

4.8.7 Overlapping neighborhoods (K_5)

After we have collected several properties in the preceding section concerning the neighborhood of single vertices we will now examine the common neighborhood of two vertices.

Definition 4.65 (Overlapping neighborhoods) *For constants $\rho, \sigma > 0$ let $P \subseteq V_k$ denote an (H_ρ)-qualified σ-cover of V_j. A vertex $v \in V_i$ has an X-*resistant (ρ, σ)-overlapping neighborhood with P in V_j *for $X \subseteq V_j$ if*

$$\begin{aligned}
&\exists \tilde{P} \subseteq P^*, |\tilde{P}| = 2\sqrt{\sigma} p_\sigma \\
&\forall u \in \tilde{P}\ \exists R_u \subseteq (\Gamma_j(v) \cap \Gamma_j(u)) \setminus X, |R_u| = r_\sigma := (1-\sigma)\sigma^2 \tfrac{m^2}{n^3} : \\
&\qquad R_u \text{ is a } \rho'\text{-supercover, where } \rho' := (\rho)^\oplus \text{, and} \\
&\qquad \text{the sets } R_u \text{ are pairwise disjoint for all } u.
\end{aligned}$$

The overlapping neighborhood is called Π-qualified *if the sets R_u are Π-qualified ρ'-supercovers.*

A vertex $v \in V_i$ satisfies the *overlapping neighborhood property* if the following condition $(N_{\rho,\sigma})$ is met:

$$(N_{\rho,\sigma}[X]) \qquad \begin{aligned}&\forall P \subseteq V_k,\ P \text{ is a } (H_\rho)\text{-qualified } \sigma\text{-cover of } V_j : \\ &\quad v \text{ has an } X\text{-resistant } (\rho,\sigma)\text{-overlapping neighborhood} \\ &\quad \text{with } P \text{ in } V_j\end{aligned} \tag{4.18}$$

Figure 4.8 illustrates the structure of an overlapping neighborhood.
Let

$$\begin{aligned}
\mathcal{B}_\ell^N(n, m; \varepsilon, \rho, \sigma) \quad := \quad & \{G \in \mathcal{S}_\ell(n, m; \varepsilon) \mid \\
& \exists X \subseteq V_j, |X| \le (1 - (\sigma)^\oplus)n : \\
& |V_i[(\neg N_{\rho,\sigma}[X], D)]| \ge (\sigma)^{\oplus 2} n\}
\end{aligned} \tag{4.19}$$

denote the set of graphs which are bad with respect to property $(N_{\rho,\sigma})$.

There are two possibilities why a vertex $v \in V_i$ might fail to satisfy $(N_{\rho,\sigma})$. Firstly, there could be less than $2\sqrt{\sigma} p_\sigma$ vertices $u \in P$ such that v and u have a common neighborhood of size at least r_σ. In order to gain control on the location of the common neighborhood, we concentrate on vertices $u \in P^*$ and on the overlap of the covering neighborhood W_u with $\Gamma_j(v)$. Recall that the covering neighborhoods W_u are disjoint. Assume that for almost all vertices in P^* this overlap is tiny, i.e., there exist less than $2\sqrt{\sigma} p_\sigma$ vertices with

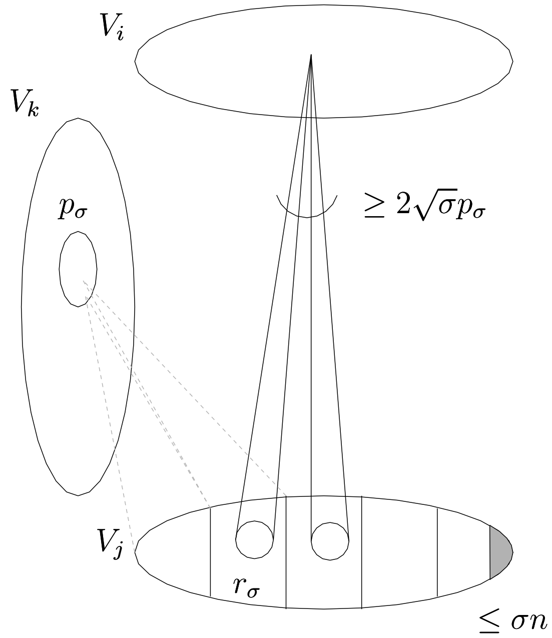

Figure 4.8: Property $(N_{\rho,\sigma})$

a common neighborhood of size at least r_σ. It will turn out that this is very improbable. To see this, note that the overwhelming majority of the neighbors of v must lie within a very small number of covering neighborhoods W_u. Hence, v cannot properly be attached to V_j, and, in turn, this implies that only very few graphs may have this property.

But there is a second way a vertex v might fail to satisfy $(N_{\rho,\sigma})$. Up to now we know that we can find a set $\tilde{P}$ and pairwise disjoint sets R_u of the right cardinality (as required by Definition 4.65). However, we have not yet guaranteed that the sets R_u are supercovers. To see why this is the case, note that the covering neighborhoods W_u are homogeneous, as we have assumed that the cover P is (H_ρ)-qualified. Recall that homogeneous sets contain very few sets which are not supercovers. Hence, we obtain a sufficiently small bound on the number of graphs for which too many vertices have too many non-supercovers in their neighborhood.

The following lemma formalizes these arguments.

Lemma 4.66 *Let $\rho' := (\rho)^\oplus \lll \sigma$. Then*

$$|\mathcal{B}_\ell^N(n, m; \varepsilon, \rho, \sigma)| \leq ((\sigma)^{\oplus 2})^m \binom{n^2}{m}^{\binom{\ell}{2}}.$$

Proof Since there are at most $2^n \le 2^m$ possibilities to choose X it suffices to count the graphs in $\mathcal{B}_\ell^N$ for fixed X. Let

$$|X| =: (1-\lambda)n \le (1-(\sigma)^{\oplus})n. \tag{4.20}$$

As sketched above the proof is divided into two cases. Later we will give a formal proof that these cases are really sufficient, i.e., they comprise all graphs in $\mathcal{B}_\ell^N$.

Case 1: This case shows that we can find a set $\tilde{P}$ and sets R_u of the right cardinality as required by Definition 4.65.

Assume that for at least $(\sigma)^{\oplus 2}n/2$ vertices in $V_i[(D)]$ there exists an (H_ρ)-qualified σ-cover $P \subseteq V_k$ of V_j such that the following condition is satisfied:

$$|\{u \in P^* \mid |(W_u \cap \Gamma_j(v)) \setminus X| \ge r_\sigma\}| \le 3\sqrt{\sigma}p_\sigma. \tag{4.21}$$

Due to (4.20) we conclude by Lemma 4.40 that $|\bar{D}_j(V_j \setminus X)| \le 2\varepsilon n$. Hence we can find $(\sigma)^{\oplus 2}n/3$ vertices $v \in V_i[(D)]$ which satisfy (4.21) and with

$$|\Gamma_j(v) \setminus X| \ge (1-\varepsilon)\frac{m}{n^2}|V \setminus X| \ge (1-\varepsilon)\lambda q.$$

We intend to apply Lemma 4.38 and define a suitable neighborhood function for these bad vertices by

$$\begin{aligned}\mathcal{N}(v) := \{Y \in \binom{V_j}{d_v} \quad | \quad & |Y \setminus X| \ge (1-\varepsilon)\lambda q \ \wedge \\ & \exists P \subseteq V_k,\ P \text{ is an } (H_\rho)\text{-qualified } \sigma\text{-cover of } V_j \\ & \exists \bar{P} \subseteq P^*, |\bar{P}| = |P^*| - 3\sqrt{\sigma}p_\sigma = (1-6\sqrt{\sigma})p_\sigma/2 \\ & \forall u \in \bar{P}:\ |(W_u \cap Y) \setminus X| \le r_\sigma\}.\end{aligned}$$

In order to count $|\mathcal{N}(v)|$ we first choose P and $\bar{P}$ (at most $n^{p_\sigma} \cdot 2^{p_\sigma} \lesssim 2^q$ possibilities). Let $F := \bigcup_{u \in \bar{P}} W_u$ and observe that

$$|F| = (1-6\sqrt{\sigma})p_\sigma/2 \cdot q_\sigma = (1-6\sqrt{\sigma})(1-\sigma)n \ge (1-7\sqrt{\sigma})n.$$

Now we fix

$$y_v := |Y \setminus X| \ge (1-\varepsilon)\lambda q \ge (1-\varepsilon)(\sigma)^{\oplus}q \quad \text{and} \quad f_v := |(Y \setminus X) \cap F|$$

(at most $n \cdot n \lesssim 2^q$ possibilities; recall that d_v is part of the input to $\mathcal{N}(v)$ and thus does not have to be counted.). We choose $d_v - y_v$ neighbors in X and y_v neighbors in $V_j \setminus X$. These y_v vertices are further subdivided. f_v neighbors are selected from $F \setminus X$ and $y_v - f_v$ neighbors come from $V_j \setminus (F \cup X)$. Since

$$f_v = |(Y \setminus X) \cap F| \le |\bar{P}| \cdot r_\sigma \le |P^*| \cdot r_\sigma = p_\sigma/2 \cdot r_\sigma \le \sigma q$$

and

$$y_v - f_v \geq ((1-\varepsilon)(\sigma)^{\oplus} - \sigma)q \geq (\sigma)^{\oplus} d_v/2,$$

we obtain (maximizing over y_v and f_v)

$$\begin{aligned} |\mathcal{N}(v)| &\lesssim^q \binom{n}{d_v - y_v}\binom{n}{f_v}\binom{7\sqrt{\sigma}n}{y_v - f_v} \overset{\text{L4.27 (ii, iii)}}{\leq} (7\sqrt{\sigma})^{y_v - f_v} \cdot 8^{d_v} \cdot \binom{n}{d_v} \\ &\leq \left(56\sqrt{\sigma}^{(\sigma)^{\oplus}/2}\right)^{d_v} \cdot \binom{n}{d_v} \leq ((\sigma)^{\oplus})^{d_v}\binom{n}{d_v}. \end{aligned}$$

This completes the proof of the first case.

Case 2: This case shall show that the sets R_u (which exist due to the previous case) are indeed supercovers.

Assume that for at least $(\sigma)^{\oplus 2} n/2$ vertices there exists a (H_ρ)-qualified σ-cover $P \subseteq V_k$ such that

$$\begin{aligned} &\exists \bar{P} \subseteq P^*, |\bar{P}| = \sigma p_\sigma \\ &\forall u \in \bar{P}\ \exists R_u \subseteq (W_u \cap \Gamma_j(v)) \setminus X, |R_u| = r_\sigma :\ R_u \text{ is } \underline{\text{no}}\ \rho'\text{-supercover}\} \end{aligned}$$

Again we will apply Lemma 4.38 and define the neighborhood function

$$\begin{aligned} \mathcal{N}(v) := \{Y \in \binom{V_j}{d_v} \quad | \quad &\exists P \subseteq V_k,\ P \text{ is an } (H_\rho)\text{-qualified } \sigma\text{-cover} \\ &\exists \bar{P} \subseteq P^*, |\bar{P}| = \sigma p_\sigma \\ &\forall u \in \bar{P}\ \exists R_u \subseteq (W_u \cap Y) \setminus X, |R_u| = r_\sigma : \\ &\quad R_u \text{ is } \underline{\text{no}}\ \rho'\text{-supercover}\}. \end{aligned}$$

As in the previous case the number of possible choices for P and $\bar{P}$ is of lower order and can be neglected. Since P is assumed to be (H_ρ)-qualified, $\Gamma_j(u)$ is ρ-homogeneous for all $u \in P^*$. By Proposition 4.44, W_u with

$$|W_u| = q_\sigma \geq \tfrac{3}{4}\sigma q \geq \tfrac{\sigma}{2} d_j(u)$$

is thus $2\rho/\sigma \leq \rho^{1/2}$-homogeneous and the number of sets $R \in \binom{W_u}{r_\sigma}$ which are not ρ'-supercovers is bounded by $\rho'^{r_\sigma}\binom{q_\sigma}{r_\sigma}$ due to Lemma 4.56 and Remark 4.57. Now note that a set $Y \in \mathcal{N}(v)$ is $(p_\sigma/2, r_\sigma, \sigma)$-$\exists$-bad with exception probability ρ'^{r_σ}. We have

$$p_\sigma/2 \cdot r_\sigma = \sigma^{-1}\frac{n^2}{m} \cdot (1-\sigma)\sigma^2\frac{m^2}{n^3} = (1-\sigma)\sigma q \geq \sigma d_v/2.$$

Hence, it follows by Lemma 4.35 that

$$|\mathcal{N}(v)| \lesssim^q (8\rho'^{\sigma^2/2})^{d_v}\binom{n}{d_v} \leq \sigma^{d_v}\binom{n}{d_v},$$

as $\rho' \lll \sigma$. This proves the second case.

Cases are sufficient It remains to show that both cases comprise all graphs in $\mathcal{B}_\ell^N$. For graphs which are not contained in one of the cases, at least $(1-(\sigma)^{\oplus 2})n$ vertices in V_1 remain which satisfy the following properties for all (H_ρ)-qualified σ-covers $P \subseteq V_k$. Due to case 1 we can find $3\sqrt{\sigma}p_\sigma$ vertices $u \in P^*$ for which a suitable set $R_u \subseteq (W_u \cap Y) \setminus X$ exists. By case 2 at most σp_σ of them do not satisfy the requirements of Definition 4.65. Hence, enough vertices remain for the set $\tilde{P}$ and the proof is complete. □

4.8.8 Qualified vertices and covers

Graph classes for K_5-proof

For finding subgraphs K_5 we will consider vertices which satisfy certain combinations of the properties (Q_ρ), (T_ρ) and $(N_{\rho,\sigma})$. Therefore we introduce some abbreviations.

We let $(HT_\rho) := (H_\rho, T_\rho)$ and $(QT_\rho) := (Q_\rho, T_\rho)$. For the graphs in

$$\mathcal{S}_\ell^{QT}(n,m;\varepsilon,\rho) := \mathcal{S}_\ell(n,m;\varepsilon) \setminus (\mathcal{B}_\ell^{Q}(n,m;\varepsilon,\rho) \cup \mathcal{B}_\ell^{T}(n,m;\varepsilon,\rho))$$

we may assume that

$$|V_i[\neg(QT_\rho, D)]| \le |V_i[(\neg D)]| + |V_i[(\neg Q_\rho, D)]| + |V_i[(\neg T_\rho, D)]| \le 3(\rho)^{\oplus 2} n \quad (4.22)$$

by Lemma 4.60, (4.15) and (4.17).

Observe that the notation for the good sets $\mathcal{S}_\ell^X$ obeys the following convention. The superscript X indicates which bad sets $\mathcal{B}_\ell^X$ have already been excluded.

We define (HT_ρ^+) as $(H_\rho \mid HT_\rho)$ and let

$$\mathcal{B}_\ell^{HT+}(n,m;\varepsilon,\rho) := \{G \in \mathcal{S}_\ell^{QT}(n,m;\varepsilon,(\rho)^{\ominus 4}) \mid |V_i[(\neg HT_\rho^+, D)]| \ge (\rho)^{\oplus} n\}. \quad (4.23)$$

The notation $(H_\rho \mid HT_\rho)$ probably looks a bit contrived at first sight. However, note that the statement of the property (H_ρ) and thus also of $(H_\rho \mid HT_\rho)$ refers to partitions V_i, V_j and V_k. Hence by restricting the possibilities for i, j and k differently for (H_ρ) and (HT_ρ) the property $(H_\rho \mid HT_\rho)$ is well-defined. In Section 4.8.16 the reader will find a more detailed account on this. But for the time being we will ignore this technical difficulty in order to render the exposition simpler.

By (4.22) we have that

$$|V_i[(\neg HT_\rho)]| \le |V_i[\neg(QT_\rho, D)]| \le 3((\rho)^{\ominus 4})^{\oplus 2} n = 3(\rho)^{\ominus 2} n \le (\rho)^{\ominus} n$$

for graphs in $\mathcal{S}_\ell^{QT}(n,m;\varepsilon,(\rho)^{\ominus 4})$, since . Thus V_i is (HT_ρ)-enhanced $(\rho)^{\ominus}$-homogeneous. By Remark 4.63 we can proceed analogously to the proof of Lemma 4.62 and obtain that

$$|\mathcal{B}_\ell^{HT+}(n,m;\varepsilon,\rho)| \leq ((\rho)^{\oplus})^m \binom{n^2}{m}^{\binom{\ell}{2}}. \tag{4.24}$$

Accordingly, we define

$$\mathcal{S}_\ell^{QT+}(n,m;\varepsilon,\rho) := \mathcal{S}_\ell^{QT}(n,m;\varepsilon,(\rho)^{\ominus 4}) \setminus \mathcal{B}_\ell^{HT+}(n,m;\varepsilon,\rho),$$

For brevity we let $(QT_\rho^+) := (HT_\rho^+) \wedge (QT_\rho)$ and note that for graphs $G \in \mathcal{S}_\ell^{QT+}(n,m;\varepsilon,\rho)$,

$$|V_i[\neg(QT_\rho^+, D)] \leq (3(\rho)^{\ominus 2} + (\rho)^{\oplus})n \leq 4(\rho)^{\oplus} n \leq (\rho)^{\oplus 2} \tag{4.25}$$

due to (4.22) and (4.23).

Furthermore, we let $\sigma = \sigma(\mu) := (\mu)^{\ominus 5}$ and introduce the family of graphs

$$\mathcal{S}_\ell^{QT+N}(n,m;\varepsilon,\rho,\mu) := \mathcal{S}_\ell^{QT+}(n,m;\varepsilon,\rho) \setminus \mathcal{B}_\ell^{N}(n,m;\varepsilon,\rho,\sigma).$$

Recall that for $\rho' = (\rho)^{\oplus} \lll \sigma$, we have

$$|\mathcal{B}_\ell^{N}(n,m;\varepsilon,\rho,\sigma)| \leq ((\sigma)^{\oplus 2})^m \binom{n^2}{m}^{\binom{\ell}{2}} \tag{4.26}$$

by Lemma 4.66. For graphs $G \in \mathcal{S}_\ell^{QT+N}(n,m;\varepsilon,\rho,\mu)$ and all $X \subseteq V_j$ with $|X| \leq (1-(\sigma)^{\oplus})n$, we get

$$|V_i[\neg(QT_\rho^+, N_{\rho,\sigma}[X], D)]| \leq ((\rho)^{\oplus 2} + (\sigma)^{\oplus 2})n \leq 2(\sigma)^{\oplus 2} n \leq (\mu)^{\ominus 2} n \tag{4.27}$$

by (4.25) and (4.19).

Finally we deduce the property $(N_{\rho,\mu}^+[X])$ from $(N_{\rho,\mu}[X])$ by introducing the additional condition that P is (HT_ρ^+)-qualified instead of just (H_ρ)-qualified and that the overlapping neighborhoods R_u are also (HT_ρ)-qualified (cf. (4.18)). We let

$$\begin{aligned}\mathcal{B}_\ell^{N+}(n,m;\varepsilon,\rho,\mu) \quad := \quad & \{G \in \mathcal{S}_\ell(n,m;\varepsilon) \mid \\ & \exists X \subseteq V_j, |X| \leq (1-(\mu)^{\oplus})n : \\ & |V_i[(\neg N_{\rho,\mu}^+[X], D)]| \geq (\mu)^{\oplus 2} n\}.\end{aligned} \tag{4.28}$$

Proceeding along the lines of the proof of Lemma 4.66 we can show that

$$|\mathcal{B}_\ell^{N+}(n,m;\varepsilon,\rho,\mu)| \leq ((\mu)^{\oplus 2})^m \binom{n^2}{m}^{\binom{\ell}{2}}, \tag{4.29}$$

provided that $\rho' = (\rho)^{\oplus} \lll \mu$.

Graph classes for K_4-proof

For finding subgraphs K_4 we will not need the property (T_ρ) of graphs in $\mathcal{S}_\ell^{QT}(n,m;\varepsilon,\rho)$. Therefore, we define

$$\mathcal{S}_\ell^{Q}(n,m;\varepsilon,\rho) := \mathcal{S}_\ell(n,m;\varepsilon) \setminus \mathcal{B}_\ell^{Q}(n,m;\varepsilon,\rho).$$

Additionally, we will need the property $(C_\rho^+) := (C_\rho \mid C_\rho)$. Again the two occurrences of (C_ρ) refer to different partitions V_i, V_j and V_k. We let

$$\mathcal{B}_\ell^{C+}(n,m;\varepsilon,\rho) := \{G \in \mathcal{S}_\ell^{Q}(n,m;\varepsilon,(\rho)^{\ominus 2}) \mid |V_i[(\neg C_\rho^+, D)]| \geq (\rho)^{\oplus 2} n\}. \tag{4.30}$$

From Lemma 4.60 and (4.15) it follows immediately that for graphs $G \in \mathcal{S}_\ell^{Q}(n,m;\varepsilon,(\rho)^{\ominus 2})$,

$$|V_i[\neg(Q_\rho, D)]| \leq |V_i[(\neg Q_{(\rho)^{\ominus 2}}, D)]| + V_i[(\neg D)] \leq 2(\rho)^{\ominus} n, \tag{4.31}$$

and, consequently, V_i is (C_ρ)-enhanced $2(\rho)^{\ominus}n$-homogeneous. Thus we obtain

$$|\mathcal{B}_\ell^{C+}(n,m;\varepsilon,\rho)| \leq ((\rho)^{\oplus})^m \binom{n^2}{m}^{\binom{\ell}{2}} \tag{4.32}$$

due to Lemma 4.62 and Remark 4.63. Accordingly, we define

$$\mathcal{S}_\ell^{Q+}(n,m;\varepsilon,\rho) := \mathcal{S}_\ell^{Q}(n,m;\varepsilon,(\rho)^{\ominus 2}) \setminus \mathcal{B}_\ell^{C+}(n,m;\varepsilon,\rho).$$

For brevity we let $(Q_\rho^+) := (C_\rho^+) \wedge (Q_\rho)$. For graphs in $\mathcal{S}_\ell^{Q+}(n,m;\varepsilon,\rho)$ we get

$$|V_i[\neg(Q_\rho^+, D)]| \leq (2(\rho)^{\ominus} + (\rho)^{\oplus 2})n \leq 2(\rho)^{\oplus 2} n \tag{4.33}$$

due to (4.30) and (4.31).

4.8.9 Triangle candidate covers (K_4)

In the sequel four or more partitions V_i, $i \in \{1,\dots,4\}$ will be involved in our arguments for finding subgraphs K_4. However, the following results will just be used for a specific choice of these partitions. Therefore, we prefer to formulate the definitions and lemmas with direct references to the partitions under consideration.

Consider a vertex $v \in V_1$ and a set $Q \subseteq \Gamma_3(v)$. Assume that we are able to find a structure as the one shown in Figure 4.9, i.e., almost all vertices in V_2, namely the vertices in T, complete many triangles in V_4. Then it would be easy to show that there exists a vertex $w \in \Gamma_2(v) \cap T$, completing many K_4-candidates.

This leads to the following definition.

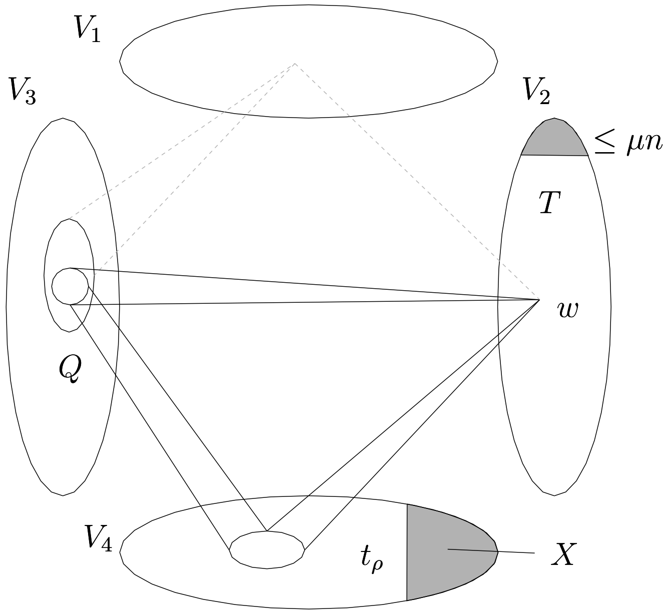

Figure 4.9: Triangle cover

Definition 4.67 (Triangle covers) *A set $Q \subseteq V_3$ is called a (ρ, μ)-triangle cover if there exists a set $T = T(Q) \subseteq V_2$ with $|T| \geq (1-\mu)n$ such that for all $w \in T$,*

$$|\Gamma_4(R) \cap \Gamma_4(w)| \geq t_\rho \quad \textit{for } R := Q \cap \Gamma_3(w),$$

where $t_\rho := \rho^{-1}\frac{n^2}{m}$.

Unfortunately, we are not able to show directly that most sets Q are triangle covers. Instead we take a little detour and introduce triangle candidate covers.

Definition 4.68 (Triangle candidate covers) *Let $G \in \mathcal{S}_4(n, m; \varepsilon)$ and $X \subseteq V_4$. A set $Q \subseteq V_3$ with $|Q| = q_\mu$ is called an X-*resistant *(ρ, μ)-*triangle candidate cover *if there exist sets $\tilde{Q} \subseteq Q$ with $|\tilde{Q}| = \mu q_\mu$ and $\tilde{T} = \tilde{T}(Q) \subseteq V_2$ with $|\tilde{T}| \geq (1-\rho^{1/4})n$ such that the following condition is satisfied:*

$$\begin{aligned} &\forall w \in \tilde{T}\ \exists \tilde{Q}^* \subseteq \tilde{Q}, |\tilde{Q}^*| \geq (1-\rho^{1/4})\mu q_\mu \\ &\forall u \in \tilde{Q}^*\ \exists R_u \subseteq \Gamma_4(u) \cap \Gamma_4(w) \setminus X, |R_u| = \tilde{r}_2 : \\ &\qquad \textit{the sets } R_u \textit{ are } r_\mu(q_\mu)\textit{-quasidisjoint for } u \in \tilde{Q}^*, \end{aligned}$$

where $\tilde{r}_2 := r_\rho(\mu q_\mu)/2 = \frac{1}{8}(1-\mu)\rho\mu^2\frac{m^2}{n^3}$.

A triangle candidate cover does not yet consider the edges between V_2 and V_3. But its structure is chosen in such a way that the set $R = Q \cap \Gamma_3(w)$ for a typical vertex $w \in V_2$ will have a sufficiently large neighborhood $\Gamma_4(R) \cap \Gamma_4(w)$. Note that most vertices in $\tilde{Q}$ have many common neighbors with w in V_4. Figure 4.10 illustrates the structure of a triangle candidate cover.

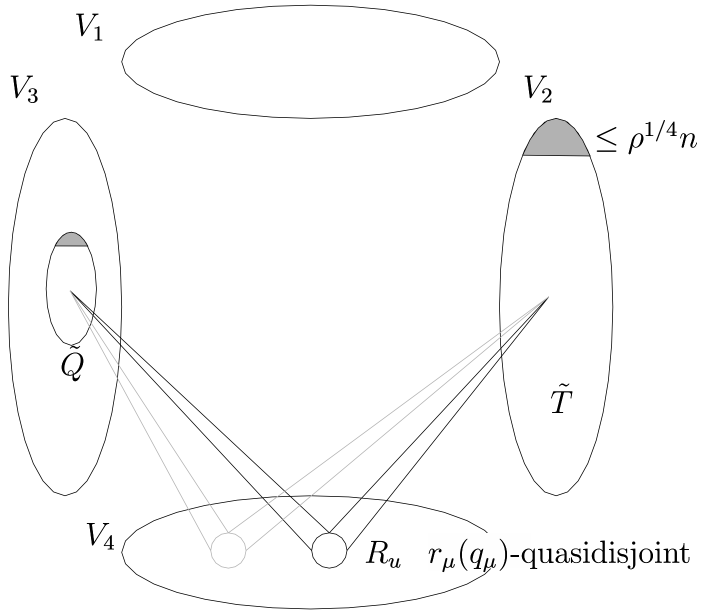

Figure 4.10: (ρ, μ)-triangle candidate cover

The following lemma shows that we have already encountered triangle covers. A close examination of (C_ρ)-qualified μ-multicovers Q will reveal that such sets Q already satisfy Definition 4.68.

Lemma 4.69 (Qualified multicovers are triangle candidate covers) *Let $Q \subseteq V_3$ with $|Q| = q_\mu$ be a (C_ρ)-qualified μ-multicover of V_4. Then Q is an X-resistant (ρ, μ)-triangle candidate cover for every $X \subseteq V_4$ with $|X| \leq (1-2\sqrt{\mu})n$, provided that $\mu^3 \geq \rho$.*

Proof Let an arbitrary set $X \subseteq V_4$ with $|X| \leq (1 - 2\sqrt{\mu})n$ be given. Due to Lemma 4.59 we know that Q is an X-resistant μ-multicover. We will show that Q^{**} satisfies the properties of $\tilde{Q}$ in Definition 4.68. Note that $|Q^{**}| = \mu q_\mu$ by Definition 4.58, as required in Definition 4.68.

Since Q is a μ-multicover of V_4, there exist $r_\mu(q_\mu)$-quasidisjoint covering neighborhoods $W'_u \subseteq \Gamma_4(u) \setminus X$ with $|W'_u| = \mu q_\mu$ for $u \in Q^{**}$. Note that $\mu q_\mu \geq \mu^2 q/2 \geq \mu^3 d_4(u)$ and $\mu^3 \geq \rho$. Hence, as Q is (C_ρ)-qualified and the vertices $u \in Q^{**} \subseteq Q^*$ thus satisfy (C_ρ), the sets W'_u are ρ-multicovers of V_2

and by Lemma 4.52 we have $|C_j(W'_u;\rho)| \geq (1-\sqrt{\rho})n$ for all $u \in Q^{**}$. By Corollary 4.29 we conclude that there is a set $\tilde{T} \subseteq V_2$ with $|\tilde{T}| \geq (1-\rho^{1/4})n$ such that every vertex $w \in \tilde{T}$ belongs to at least $(1-\rho^{1/4})\mu q_\mu$ sets $C_j(W'_u;\rho)$ for $u \in Q^{**}$.

In other words, for every $w \in \tilde{T}$ the set $\tilde{Q}^* := \{u \in Q^{**} \mid w \in C_j(W'_u;\rho)\} \subseteq Q^{**} \stackrel{!}{=} \tilde{Q}$ satisfies $|\tilde{Q}^*| \geq (1-\rho^{1/4})\mu q_\mu$.

Now for every $u \in \tilde{Q}^*$ define $R_u := \Gamma_4(w) \cap W'_u$ for a fixed vertex $w \in \tilde{T}$. Observe that by the definition of $\tilde{Q}^*$ we have $w \in C_j(W'_u;\rho)$, hence, by definition of the latter, $|R_u| = |\Gamma_4(w) \cap W'_u| \geq r_\rho(\mu q_\mu)/2 = \tilde{r}_2$. On the other hand, we have $W'_u \subseteq \Gamma_4(u) \setminus X$ by the definition of W'_u, and therefore $R_u \subseteq (\Gamma_4(u)\cap\Gamma_4(w))\setminus X$. Finally, note that the sets R_u are $r_\mu(q_\mu)$-quasidisjoint since they are subsets of the $r_\mu(q_\mu)$-quasidisjoint sets W'_u. □

As a first step from triangle candidate covers to triangle covers, we now show that for a typical vertex $w \in V_2$ the set $R := Q \cap \Gamma_3(w)$ completes many triangles with w. For $|Q| = \Theta(m/n)$ note that we may expect that $|R| = \Theta(m/n \cdot m/n^2) = \Theta(m^2/n^3)$. The following definition introduces 'bad' R-sets, i.e., sets $R \subseteq Q$ which have less than $t_\rho := \rho^{-1}\frac{n^2}{m}$ common neighbors $\Gamma_4(R) \cap \Gamma_4(w)$. The subsequent lemma then shows that for every vertex $w \in \tilde{T}(Q)$ there exist only very few bad R-sets. Since $\tilde{T}$ is large (cf. Definition 4.68), this will later suffice to obtain triangle covers. Note that we expect $\Theta(m^2/n^3 \cdot n \cdot (m/n^2)^2) = \Theta(m^4/n^6)$ common neighbors $\Gamma_4(R) \cap \Gamma_4(w)$, and that $n^2/m = o(m^4/n^6)$ for $m = \omega(n^{8/5})$. Thus, at least in expectation, R-sets should have t_ρ common neighbors.

Definition 4.70 (Bad R-sets) *Consider a graph $G \in \mathcal{S}_4(n,m;\varepsilon)$, $X \subseteq V_4$, $w \in V_2$ and a set $Q \subseteq V_3$ with $|Q| = q_\mu$ which is an X-resistant (ρ,μ)-triangle candidate cover. We define*

$$\bar{\mathcal{R}}^4(w,Q;X,\rho) := \{R \in \binom{\tilde{Q}}{\tilde{r}_1} \mid |(\Gamma_4(R) \cap \Gamma_4(w)) \setminus X| < t_\rho\},$$

where $t_\rho := \rho^{-1} \cdot \frac{n^2}{m}$ and $\tilde{r}_1 := r_\mu(\mu q_\mu)/2 = \frac{1}{8}(1-\mu)\mu^3\frac{m^2}{n^3}$.

Lemma 4.71 (Few bad R-sets) *Let Q be an X-resistant (ρ,μ)-triangle candidate cover and consider a vertex $w \in \tilde{T}(Q) \subseteq V_2$. Then*

$$|\bar{\mathcal{R}}^4(w,Q;X,\rho)| \leq \rho^{\tilde{r}_1/20}\binom{\mu q_\mu}{\tilde{r}_1}.$$

for $m \geq Cn^{8/5}$ and C sufficiently large.

Proof We intend to apply Lemma 4.31. To this aim we identify $B \equiv \tilde{Q}$, $B^* \equiv \tilde{Q}^*$, with $|B^*| \geq (1-\rho^{1/4})|B|$ due to Definition 4.68, and $A_{u,1} \equiv R_u$ for $u \in \tilde{Q}^*$. Since $A_{u,1} \subseteq \Gamma_4(u) \cap \Gamma_4(w) \setminus X$, it suffices to bound the number of non-t_ρ-spreading sets $R \in \binom{\tilde{Q}}{\tilde{r}_1}$.

Let us check the preconditions of Lemma 4.31. Condition (i) trivially holds since we have $h = 1$.

For Condition (ii) the left-hand side evaluates to $4\rho^{-1}\frac{n^2}{m}r_\mu(q_\mu) = \Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$, whereas the right-hand side amounts to $\rho^{1/4} \cdot \tilde{r}_2 \cdot \mu q_\mu = \Theta(m^2/n^3 \cdot m/n) = \Theta(m^3/n^4)$. Observe that $m/n = o(m^3/n^4)$ for $m = \omega(n^{3/2})$. Thus for sufficiently large n Condition (ii) is obviously satisfied.

For Condition (iii) we obtain

$$\frac{1}{8}(1-\mu)\mu^3\frac{m^2}{n^3} = \tilde{r}_1 \leq 4\rho^{-1}\frac{n^2}{m} \cdot \frac{1}{\tilde{r}_2} = \frac{32}{\rho^2(1-\mu)\mu^2}\frac{n^2n^3}{mm^2}.$$

This condition is satisfied for $m^5 \geq \rho^{-2}\mu^{-5}n^8$.

Now the claim follows immediately by Lemma 4.31. $\square$

4.8.10 Cocovers (K_5)

In the sequel four or more partitions V_i for $i \in \{1, \dots, 5\}$ will be involved in our arguments for finding subgraphs K_5. However, as in the case $H = K_4$, the following results will just be used for a specific choice of these partitions. Therefore, we prefer to formulate the definitions and lemmas with direct references to the partitions under consideration.

In Section 4.8.7 we have seen that a constant fraction of a typical set $P \subseteq V_4$ with $\Theta(n^2/m)$ vertices has a common neighborhood of size $\Theta(m^2/n^3)$ with a given vertex $u \in V_3$, yielding $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ common neighbors. If these common neighborhoods were (rather) disjoint for $\Theta(n^2/m)$ vertices $u_1, u_2, \dots$, we could expect that all common neighborhoods together 'tile' the n vertices in V_5, since $\Theta(m/n \cdot n^2/m) = \Theta(n)$. The aim of this section is to show that this is indeed the case.

More specifically, we will fix a vertex $v \in V_1$ and a vertex $u \in V_3$ with common neighborhood $R \in V_4$ of size $\Theta(m^2/n^3)$. Note that $m^2/n^3 \geq n^2/m$ for $m \geq n^{5/3}$. Hence, we can divide R into several subsets $P_1, P_2, \dots$ of size $\Theta(n^2/m)$ for which the above arguments hold, i.e., the common neighborhoods with u contain $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ vertices. If we take $\Theta(n^2/m)$ vertices $u_1, u_2, \dots \in V_3$, we expect their neighborhoods to tile V_5 (cf. Figure 4.11).

Sets $\{u_1, u_2, \dots\} \subseteq V_3$ for which such a structure as described above can be found will be called cocovers. Before turning this into a formal definition, we will introduce a new auxiliary notion, which is in a certain sense complementary to quasidisjointness (cf. Definition 4.30).

Definition 4.72 *Let $A_1, \ldots, A_b \subseteq V$. Then the k-*fold overlap *$OV_k((A_i)_{i=1,\ldots,b})$ is defined by*

$$OV_k((A_i)_{i=1,\ldots,b}) := \{x \in V \mid \ |\{i \in \{1, \ldots, b\} \mid x \in A_i\}| \geq k\}.$$

Definition 4.73 (Cocovers) *Let $v \in V_1$. A set $O \subseteq V_3$ with*

$$|O| = o_\mu := \sigma^{-5/2}\frac{n^2}{m}, \text{ where } \sigma := (\mu)^{\ominus 5},$$

*is called a (ρ, μ)-*cocover of V_5 via v and V_4 *if*

$$\begin{array}{l} \exists O^+ \subseteq O \\ \forall u \in O^+ \ \exists R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v), |R_u| = \sqrt{\sigma} r_\mu \\ \qquad R_u \text{ is a } (T_\rho)\text{-qualified } \sqrt{(\rho)^\oplus}\text{-supercover of } V_2 \\ \forall y \in R_u \ \exists W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u), |W_{u,y}| = r_\sigma \\ \qquad W_{u,y} \text{ is a } (\rho)^\oplus\text{-supercover of } V_2, \\ \qquad \text{the sets } W_{u,y} \text{ are } 2s_\mu\text{-quasidisjoint for all } u, y \\ \qquad \text{and } |OV_5| \geq (1-\mu)n \text{ for } OV_5 := OV_{s_\mu}((W_{u,y})_{u \in O^+, y \in R_u}), \end{array}$$

where $s_\mu := r_\sigma(r_\mu) = r_\mu/2p_\sigma = \frac{1}{4}(1-\mu)\mu^2\sigma\frac{m^3}{n^5}$.
Let $\mathcal{CO}_{3,5}(v; \rho, \mu) := \{O \in \binom{V_3}{o_\mu} \mid O \text{ is a } (\rho, \mu)\text{-cocover via } v\}$.

Figure 4.11 illustrates the structure of a (ρ, μ)-cocover.

Since the definition of a cocover is rather lengthy, a few comments are in order. Basically, a cocover results from a multiple application of property $(N_{\rho,\sigma})$. Note that the factor $\sqrt{\sigma}$ for the size of the part of O resp. $\tilde{P}$ in a cocover resp. an overlapping neighborhood is identical. This is no coincidence, as in the proof we will apply property $(N_{\rho,\sigma})$ (cf. Definition 4.65) in order to obtain the sets $W_{u,y}$. The forbidden set X in the definition of $(N_{\rho,\sigma}[X])$ can be used to control the overlap between neighborhoods which stem from several sequential applications of $(N_{\rho,\sigma})$. The control on the overlap between different sets $W_{u,y}$ is crucial for the remainder of the proof, as we want these sets to be as disjoint as possible (We will see later why this is so important.).

For future reference we note that

$$\frac{o_\mu r_\mu r_\sigma}{n s_\mu} = \frac{4(1-\mu)\mu^2(1-\sigma)\sigma^2}{\sigma^{5/2}(1-\mu)\mu^2\sigma} \cdot \frac{n^2 m^4 n^5}{nmn^6nm^3} = \frac{4(1-\sigma)}{\sigma^{3/2}} \tag{4.34}$$

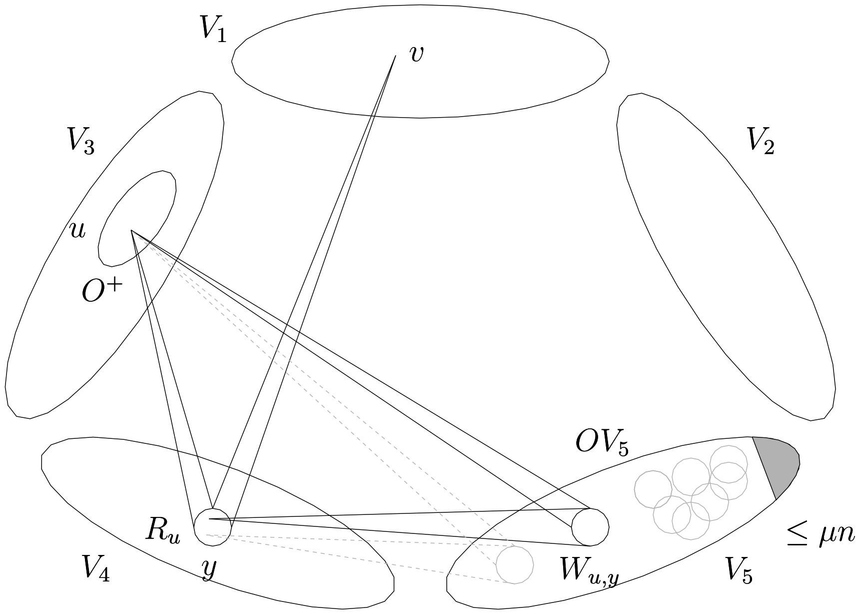

Figure 4.11: (ρ, μ)-cocover

Observe that by the definition of OV_5 there must be at least $(1-\mu)n \cdot s_\mu$ occurrences of vertices in sets $W_{u,y}$ if $|OV_5| \geq (1-\mu)n$. Hence we have for $|O^+| =: \alpha o_\mu$,

$$(1-\mu)n \cdot s_\mu \leq \alpha o_\mu \cdot \sqrt{\sigma} r_\mu \cdot r_\sigma \;\Rightarrow\; \alpha \geq \frac{(1-\mu)ns_\mu}{\sqrt{\sigma} o_\mu r_\mu r_\sigma} \overset{(4.34)}{=} \frac{(1-\mu)\sigma}{4(1-\sigma)} \geq \sigma/5$$

and thus

$$|O^+| \geq \sigma o_\mu/5. \tag{4.35}$$

As the sets $W_{u,y}$ are $2s_\mu$-quasidisjoint, we can derive a similar upper bound on the number of occurrences of vertices in sets $W_{u,y}$, and analogously we conclude that $(1-\mu)n \cdot 2s_\mu \geq \alpha o_\mu \cdot \sqrt{\sigma} r_\mu \cdot r_\sigma$. Hence we get

$$|O^+| \leq \sigma o_\mu/2. \tag{4.36}$$

The following lemma shows that almost all sets (of suitable size) are cocovers. The proof is split into two cases which correspond to the following approach.

Firstly, assume that sufficiently large sets $\tilde{R}_u \subseteq \Gamma_4(u) \cap \Gamma_4(v)$ exist for many vertices $u \in O^+$. Then we apply $(N_{\rho,\sigma})$ for a suitably chosen constant σ (which will be much smaller than μ) in order to obtain sets $W_{u,y}$ as required by Definition 4.73. Note that this should be possible if we consider graphs in $\mathcal{S}_\ell^{QT+N}(n, m; \varepsilon, \rho, \mu)$, since these graphs contain only very few vertices for which $(N_{\rho,\sigma})$ is not satisfied.

The second case shows that we may indeed assume that suitable sets $\tilde{R}_u$ can be found. To this aim we require that the given vertex $v \in V_1$ satisfies $(N^+_{\rho,\mu})$. It will turn out that sets $\tilde{R}_u$ can be found, provided that O is a σ-multicover. This is achieved by decomposing O into μ-covers for which $(N^+_{\rho,\mu})$ is applied. Then the existence of the sets $\tilde{R}_u$ follows directly from the definition of $(N^+_{\rho,\mu})$.

Lemma 4.74 ($\overline{\mathcal{CO}}$ is small) *Let $\rho' = (\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$. Consider $G \in \mathcal{S}_\ell^{QT+N}(n, m; \varepsilon, \rho, \mu)$ and a vertex $v \in V_1$ which satisfies $(N^+_{\rho,\mu}[\emptyset])$. Then*

$$|\overline{\mathcal{CO}}_{3,5}(v; \rho, \mu)| \leq ((\mu)^{\ominus})^{o_\mu} \binom{n}{o_\mu}.$$

Proof We distinguish two kinds of sets P which may belong to $\overline{\mathcal{CO}}_{3,5}$, as discussed above. Later we will show that these two cases indeed comprise all graphs in $\overline{\mathcal{CO}}_{3,5}$.

Case 1: Firstly we count sets $O \in \binom{V_3}{o_\mu}$ which contain μo_μ 'bad' vertices $u \in O$ with the property $(\neg CO)$. This property depends on an ordering of O, where O is bad if *all* orderings result in μo_μ bad vertices. Let $O = \{u_1, \ldots, u_{o_\mu}\}$. The vertex $u \equiv u_l$ is bad if it satisfies

$$\begin{aligned}(\neg CO) \qquad & (\exists \tilde{R}_u \subseteq \Gamma_4(u) \cap \Gamma_4(v), |\tilde{R}_u| = r_\mu : \\ & \qquad \tilde{R}_u \text{ is an } (HT_\rho)\text{-qualified } \rho'\text{-supercover}) \wedge \\ & u \text{ does not satisfy } (N_{\rho,\sigma}[Y_l]) \text{ for } Y_l \text{ as specified below.}\end{aligned}$$

Note the two different constants μ and σ which are used for the property (N) in this lemma. We require that $v \in V_1$ satisfies $(N^+_{\rho,\mu}[\emptyset])$ but we look for vertices $u \in O$ with property $(N_{\rho,\sigma}[Y_l])$. It will be very important for our subsequent estimates that σ has been chosen to be significantly smaller than μ.

Determine vertex by vertex whether u_l is bad for $l = 1, \ldots, o_\mu$. Initially we start with an empty set Y_1 and an empty family $\mathcal{Y}_1 \subseteq \binom{V_5}{r_\sigma}$ of vertex sets. If u_l is a good vertex which satisfies (CO) and for which a suitable set $\tilde{R}_u$ exists (as required by the definition of $(\neg CO)$), then u_l must satisfy $(N_{\rho,\sigma}[Y_l])$. Assume that $|Y_l| \leq (1-\mu)n$. Since $\tilde{R}_u$ is an (HT_ρ)-qualified ρ'-supercover, it is also an (HT_ρ)-qualified σ-multicover. Hence we can find $s_\mu = r_\sigma(r_\mu)$ σ-covers in $\tilde{R}_u$ which are (H_ρ)-qualified. We apply (4.18) and Definition 4.65 to every σ-cover and conclude that R_u satisfies

$\exists R_u \subseteq \tilde{R}_u, |R_u| = \sqrt{\sigma} r_\mu$
$\forall y \in R_u \; \exists W_{u,y} \subseteq (\Gamma_5(v) \cap \Gamma_5(u)) \setminus Y_l, |W_{u,y}| = r_\sigma$
$W_{u,y}$ is a ρ'-supercover of V_2, and the sets $W_{u,y}$ are s_μ-quasidisjoint.

For a good vertex $u_l \equiv u$ (for which such a structure exists) we set $\mathcal{Y}_{l+1} = \mathcal{Y}_l \cup \{W_{u,y} \mid y \in R_u\}$. Then we proceed to choose u_{l+1}. Otherwise, for a bad vertex u, $\mathcal{Y}_l$ remains unchanged, i.e., we set $\mathcal{Y}_{l+1} = \mathcal{Y}_l$. In any case, we let $Y_{l+1} := OV_{s_\mu}(\mathcal{Y}_{l+1})$. If Y_l has more than $(1-\mu)n$ vertices, we stop adding new vertices to Y and vertex sets to $\mathcal{Y}$, setting $\mathcal{Y}_{l+1} = \mathcal{Y}_l$ and $Y_{l+1} = Y_l$ for all future steps.

The sets R_u and $W_{u,y}$ satisfy the requirements of Definition 4.73, which can be seen as follows. Since $\tilde{R}_u$ is a (HT_ρ)-qualified ρ'-supercover and $|R_u| = \sqrt{\sigma}|\tilde{R}_u|$, we conclude that R_u is a (T_ρ)-qualified $\rho'/\sqrt{\sigma}$-supercover. Now observe that $\rho'/\sqrt{\sigma} \le \sqrt{\rho'}$. Furthermore, the sets $W_{u,y}$ are $2s_\mu$-quasidisjoint, as the sets $W_{u,y}$ which are added to Y_l for one specific vertex $u \equiv u_l$ are s_μ-quasidisjoint and also the vertices in $V_5 \setminus Y_l$ are s_μ-quasidisjoint by the definition of Y_l.

Now let us return to counting the number of bad graphs. By the definition of $(\neg CO)$, a bad vertex u_l must satisfy $(\neg N_{\rho,\sigma}[Y_l])$. By (4.27) we conclude that sets O with the above mentioned property are $(1,\mu)$-$\forall$-bad with exception probability $(\mu)^{\ominus 2}$. Note that $|Y_l| \le (1-\mu/2)n \le (1-(\sigma)^{\oplus})n$ by construction. Due to Lemma 4.32 we can bound the number of such sets by $((\mu)^{\ominus 2})^{\mu o_\mu/2}\binom{n}{o_\mu} \le \frac{1}{2}((\mu)^{\ominus})^{o_\mu}\binom{n}{o_\mu}$.

Case 2: Secondly we consider sets $O \in \binom{V_3}{o_\mu}$ which are not (HT_ρ^+)-qualified μ-multicovers of V_4. Note that this implies $O \in \bar{\mathcal{Q}}_4(V_3; o_\mu, 1 \mid (HT_\rho^+))$.

By (4.25) we conclude that V_3 is (HT_ρ^+)-enhanced $(\rho)^{\oplus 2}$- and thus $(\mu)^{\ominus 3}$-homogeneous. Hence, by Lemma 4.56 it follows that the number of such sets O can be bounded by

$$((\mu)^{\ominus 2})^{o_\mu}\binom{n}{o_\mu} \le \frac{1}{2}((\mu)^{\ominus})^{o_\mu}\binom{n}{o_\mu}.$$

Cases are sufficient: To complete the proof it suffices to show that an (HT_ρ^+)-qualified μ-multicover $O \in \binom{V_3}{o_\mu}$, provided that an ordering exists such that there at most μo_μ vertices satisfying $(\neg CO)$ is a (ρ,μ)-cocover via v.

Assume that we iterate through the vertices according to the given ordering and finally obtain a set $Y := Y_{\mu o_\mu} \subseteq V_5$ with $|Y| \ge (1-\mu)n$. As discussed above the vertices for which new vertices have been added to Y form a set O^+ which satisfies the requirement of Definition 4.73. Thus the set O is in fact a cocover.

It remains to show that the case $|Y| < (1-\mu)n$ cannot occur. Let $O^* = \bigcup_{l=1}^{r'} P_l^*$, where P_l^* is an (HT_ρ^+)-qualified μ-cover and $r' = r_\mu(o_\mu)$. Since v satisfies $(N_{\rho,\mu}^+[\emptyset])$, it follows that v has an (HT_ρ)-qualified (ρ,μ)-overlapping neighborhood with P_l^* in V_4 for $l = 1,\ldots,r'$. Hence there are at least $r' \cdot 2\sqrt{\mu}p_\mu = \sqrt{\mu}o_\mu$ vertices $u \in O$ for which $\tilde{R}_u \subseteq (\Gamma_4(v) \cap \Gamma_4(u))$ with $|\tilde{R}_u| = r_\mu$ exists such that $\tilde{R}_u$ is an (HT_ρ)-qualified ρ'-supercover of V_5. Let O^* denote the set of these vertices.

If we remove all vertices from O^* which satisfy $(\neg CO)$, still at least μo_μ vertices remain, with lots of room to spare. Call the set of these vertices O^{**}. Since we iterate through the vertices in O according to the ordering implied by Case 1, the following holds. By the definition of property $(\neg CO)$ the vertices $u_l \in O^{**}$ satisfy $(N_{\rho,\sigma}[Y_l])$ where Y_l is constructed as defined in Case 1. Hence for each $u \equiv u_l \in O^{**}$ the vertices in $\bigcup_{y\in\tilde{R}_u} W_{u,y}$ are added to Y_l, as we have assumed that $|Y_l| \le |Y| < (1-\mu)n$. Note that the vertices in O^{**} satisfy all properties of vertices in O^+ given in Definition 4.73. Only the size of OV_5 is not yet clear. But since $|O^{**}| \ge \mu o_\mu$ exceeds the upper bound on O^+ given in (4.36), we obtain a contradiction. Hence, O^{**} cannot grow that large and it follows that $|Y| \ge (1-\mu)n$. Thus we have constructed a (ρ,μ)-cocover. □

The strength of the cocovers relies on the fact that their structure is difficult to destroy, since they incorporate many (quasi-)disjoint sets. Indeed we will show that in every subset of V_5 which is not too small a resistant cocover can be found. A resistant cocover has the same basic structure as a cocover, only the sizes of the various sets are suitably scaled down.

Definition 4.75 (Resistant Cocovers) *Let $v \in V_1$ and $X \subseteq V_5$. A set $O \subseteq V_3$ with $|O| = o_\mu$ is called X-*resistant (ρ,μ)-cocover of V_5 via v *if*

$$\begin{aligned}
&\exists O^* \subseteq O, |O^*| = 2\sigma^2 o_\mu \\
&\forall u \in O^*\ \exists R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v), |R_u| = \sigma r_\mu \\
&\qquad R_u \text{ is a } (T_\rho)\text{-qualified } \nu\text{-multicover of } V_2, \text{ where } \nu := (\rho)^\oplus \\
&\forall y \in R_u\ \exists W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u) \setminus X, |W_{u,y}| = \sigma r_\sigma \\
&\qquad W_{u,y} \text{ is a } \rho\text{-multicover of } V_2, \text{ and the sets } W_{u,y} \text{ are } 2s_\mu\text{-quasidisjoint.}
\end{aligned}$$

The following lemma shows that every cocover is X-resistant provided that X is not too big.

Lemma 4.76 (All cocovers are resistant) *Assume that $\rho' := (\rho)^\oplus \lll \sigma$, where $\sigma := (\mu)^{\ominus 5}$. Let $O \subseteq V_3$ with $|O| = o_\mu$ be a (ν,μ)-cocover and consider a set $X \subseteq V_5$ with $|X| \le (1-2\mu)n$. Then O is also an X-resistant (ν,μ)-cocover.*

Proof We count the number of occurrences of vertices $x \in V_5$ in sets $W_{u,y}$ for the (ν, μ)-cocover O. By counting occurrences we mean that a single vertex $x \in V_5$ is counted as many times as it occurs in a set $W_{u,y}$ for different vertices u, y.

Since $|OV_5 \cap V_5 \setminus X| \geq \mu n$ and every vertex $x \in OV_5$ corresponds to at least s_μ such occurrences, we conclude that at least $s_\mu \cdot \mu n$ occurrences remain if we restrict the sets $W_{u,y}$ to vertices in $V_5 \setminus X$.

Assume that there are not enough vertices left from the original sets R_u and $W_{u,y}$ (i.e. in the (ν, μ)-cocover) for the X-resistant (ν, μ)-cocover. Let us denote these (original) sets by $\hat{R}_u$ and $\hat{W}_{u,y}$ in order to distinguish them from the sets to be found for the X-resistant cocover. Recall that $|\hat{R}_u| = \sqrt{\sigma} r_\mu$ and $|\hat{W}_{u,y}| = r_\sigma$.

If the number of suitable sets R_u and $W_{u,y}$ is too small, we obtain the following bound on the number of remaining occurrences in sets $W_{u,y} \subseteq \hat{W}_{u,y} \setminus X$. Note that for every vertex $u \in O^+$ and all $\sqrt{\sigma} r_\mu$ neighbors $y \in \hat{R}_u$ there may remain σr_σ neighbors in $\hat{W}_{u,y}$. Secondly there may be σr_μ neighbors $y \in \hat{R}_u$ such that all r_σ neighbors in $\hat{W}_{u,y}$ remain. Finally, for at most $2\sigma^2 o_\mu$ vertices in O^+ and all $\sqrt{\sigma} r_\mu$ neighbors $y \in \hat{R}_u$ all r_σ neighbors in $\hat{W}_{u,y}$ may remain. Combining these numbers of occurrences of vertices $x \in V_5$ in W-sets leads to

$$\begin{aligned} \#\text{occurrences} &\leq |O^+| \cdot \sqrt{\sigma} r_\mu \cdot \sigma r_\sigma + |O^+| \cdot \sigma r_\mu \cdot r_\sigma + 2\sigma^2 o_\mu \cdot \sqrt{\sigma} r_\mu \cdot r_\sigma \\ &\overset{(4.36)}{\leq} \sigma^2 o_\mu r_\mu r_\sigma, \end{aligned}$$

using (4.36). Combining this with the lower bound $s_\mu \cdot \mu n$ we obtain using (4.34),

$$\mu s_\mu n \leq \sigma^2 o_\mu r_\mu r_\sigma \;\Rightarrow\; \mu \leq \sigma^2 \cdot \frac{4(1-\sigma)}{\sigma^{3/2}} \leq 4\sqrt{\sigma}.$$

This is obviously a contradiction.

It remains to show that the sets R_u and $W_{u,y}$ are indeed multicovers. This easily follows from the fact that the corresponding sets in the original (ρ, μ)-cocover were ρ'- or at least $\sqrt{\rho'}$-supercovers. Observe that the remaining sets R_u and $W_{u,y}$ are at least a σ-fraction of the original sets. As $\sqrt{\rho'} \leq \sigma$ this completes the proof. □

4.8.11 Square (candidate) covers (K_5)

The aim of this section is to find (many) squares K_4 in the graph. If many such subgraphs exist and their occurrences are scattered throughout the graph it seems plausible that at least some of them can be enlarged to a subgraph K_5. To this aim we introduce the set $SQU_5(w, Q, v)$:

$$SQU_5(w,Q,v) := \{x \in V_5 \quad | \quad \exists u \in \Gamma_3(w) \cap Q \; \exists y \in \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w) : \\ x \in \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w)\}.$$

We will use this definition for $Q \subseteq \Gamma_3(v)$ and $w \in \Gamma_2(v)$. Observe that $SQU_5(w,Q,v)$ then directly corresponds to the K_5-candidates which we want to construct in the end (cf. Figure 4.12).

The following definition introduces a structure which immediately leads to many K_5-candidates.

Definition 4.77 (Square covers) *Let*

$$S_2(Q,v;\rho) := \{w \in V_2 \mid |SQU_5(w,Q,v) \setminus X| \geq t_\rho\},$$

where $t_\rho := \rho^{-1}\frac{n^2}{m}$. *We call a set* $Q \subseteq V_3$ *a* (ρ,μ)-square cover *of* V_2 *via* v *if*

$$|S_2(Q,v;\rho)| \geq (1-\mu)n.$$

See Figure 4.12 for an illustration of the structure of a square cover.

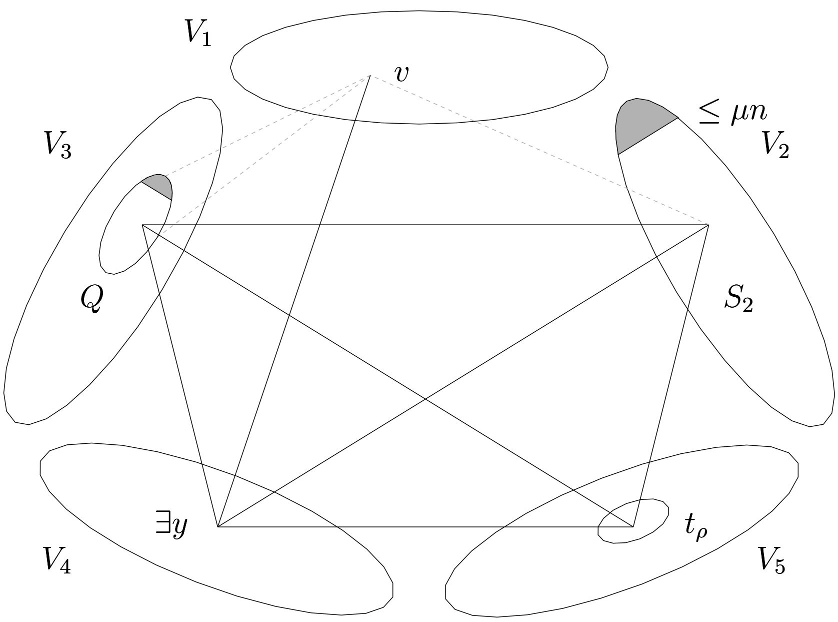

Figure 4.12: (ρ,μ)-square cover

Intuitively, for a square cover Q almost all vertices in V_2 belong to squares. Consider a vertex $v \in V_1$ and a square cover $Q \subseteq \Gamma_3(v)$. It would be easy to show that with very high probability $\Gamma_2(v)$ and $S_2(Q, v; \rho)$ intersect, yielding t_ρ K_5-candidates.

Unfortunately, we are not able to give a direct proof that almost all sets Q are square covers. Instead we will show a somewhat weaker statement which is based on so-called square candidate covers. Such a square candidate cover can be interpreted as a square cover where the edges from V_2 to V_3 are ignored, i.e., their presence is not yet guaranteed.

Definition 4.78 (Square candidate covers) *Let $v \in V_1$ and $X \subseteq V_5$. A set $Q \subseteq V_3$ with $|Q| = q_\mu$ is called an X-*resistant (ρ, μ)-square candidate cover *if there exist sets $\tilde{Q} \subseteq Q$ and $\tilde{T} = \tilde{T}(Q) \subseteq V_2$ with $|\tilde{Q}| = \sigma^2 q_\mu$, $|\tilde{T}| = (1 - 6\nu^{1/8})n$ and $\nu := \sqrt{(\rho)^\oplus}$ such that*

$$\begin{aligned}
&\forall w \in \tilde{T}\ \exists \tilde{Q}^* \subseteq \tilde{Q}, |\tilde{Q}^*| \geq (1 - 6\nu^{1/8})\sigma^2 q_\mu \\
&\forall u \in \tilde{Q}^*\ \exists Y_{u,w} \subseteq \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w), |Y_{u,w}| = \nu^3 \tfrac{m^3}{n^5} =: h_\nu \\
&\forall y \in Y_{u,w}\ \exists S_{u,w,y} \subseteq \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w) \setminus X, |S_{u,w,y}| \geq h_\rho \\
&\qquad \text{the sets } S_{u,w,y} \text{ are } d_\mu\text{-quasidisjoint for fixed } w,
\end{aligned}$$

where $d_\mu := 2s_\mu \cdot \frac{q_\mu}{2o_\mu}$.
Let $\mathcal{SC}_{3,2}(v, X; \rho, \mu)$ denote the set of X-resistant square candidates covers in V_3.

Note that

$$d_\mu = 2s_\mu \cdot \frac{q_\mu}{2o_\mu} = \frac{1}{4}(1-\mu)\mu^2\sigma\frac{m^3}{n^5} \cdot (1-\mu)\mu\frac{m}{n} \cdot \sigma^{5/2}\frac{m}{n^2} \geq \sigma^4\frac{m^5}{n^8}, \tag{4.37}$$

and, accordingly,

$$d_\mu \leq \sigma\frac{m^5}{n^8}. \tag{4.38}$$

Even if Definition 4.78 looks perhaps somewhat scary, the structure of square candidate covers is actually not too difficult, as Figure 4.13 illustrates.

In fact, it turns out that square candidate covers are not much more than cocovers. By comparing the structure of (X-resistant) cocovers (cf. Figure 4.11) and that of square candidate covers (cf. Figure 4.13) the astute reader will immediately notice that the differences are only moderate. Firstly, there are new edges between V_2 and V_4 resp. V_2 and V_5. However, these edges do not come as a surprise. Observe that the sets R_u and $W_{u,y}$ in a cocover are multicovers. Hence, we should expect the edges in the square candidate cover to be indeed present.

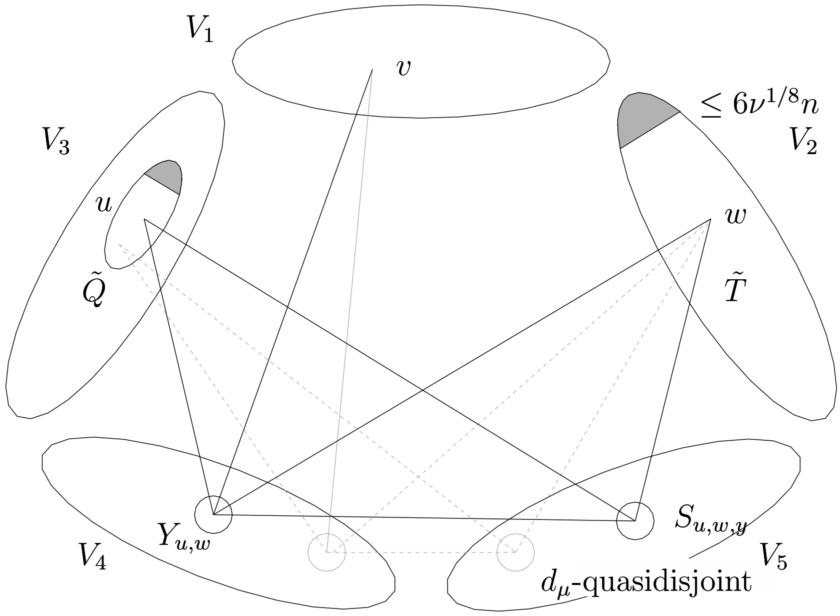

Figure 4.13: (ρ, μ)-square candidate cover

The second difference is even smaller. Square candidate covers have a larger cardinality than cocovers. Thus we will construct the former by combining sufficiently many cocovers.

The following lemma shows that a sufficiently large set Q which contains many cocovers is indeed a square candidate cover. This implies that the number of sets which are no square candidate cover is small. Let $\mathcal{SC}_{3,2}(v;\rho,\mu) \subseteq \binom{V_3}{q_\mu}$ denote all sets which can be decomposed into sets $O_1, \ldots, O_{q_\mu/o_\mu}$ with $|O_1| = \ldots = |O_{q_\mu/o_\mu}| = o_\mu$ such that at least half of these sets are (ρ,μ)-cocovers. In the sequel we will see that almost all sets in $\binom{V_3}{q_\mu}$ belong to $\mathcal{SC}_{3,2}(v;\rho,\mu)$ and that the latter set contains the X-resistant square candidate covers we have been looking for.

Lemma 4.79 (Many square candidate covers) *Let* $(\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$. *Consider* $G \in \mathcal{S}_\ell^{QT+N}(n,m;\varepsilon,\rho,\mu)$ *and a vertex* $v \in V_1$ *which satisfies* $(N^+_{\rho,\mu}[\emptyset])$. *Then*

$$|\overline{\mathcal{SC}}_{3,2}(v;\rho,\mu)| \leq \mu^{2q}\binom{n}{q_\mu}$$

and

$$\mathcal{SC}_{3,2}(v;\rho,\mu) \subseteq \mathcal{SC}_{3,2}(v,X;\rho,\mu) \quad \textit{for all } X \subseteq V_5 \textit{ with } |X| \leq (1-2\mu)n,$$

i.e., every set in $\mathcal{SC}_{3,2}(v;\rho,\mu)$ *is an* X*-resistant* (ρ,μ)*-square candidate cover.*

Proof We prove the two claims of the lemma separately.

$\overline{\mathcal{SC}}$ is small To prove the first claim it suffices to see that due to Lemma 4.74 every set in $\overline{\mathcal{SC}}_{3,2}(v;\rho,\mu)$ is $(o_\mu, 1/2)$-$\forall$-bad with an exception probability of $((\mu)^{\ominus})^{o_\mu}$. Hence by Lemma 4.32 we can deduce that there are at most $((\mu)^{\ominus})^{q_\mu/4}\binom{n}{q_\mu} \leq \mu^{2q}\binom{n}{q_\mu}$ such sets.

$\mathcal{SC}_{3,2}(v;\rho,\mu)$ contains X-resistant square candidate covers For the proof of the second claim let us consider arbitrary sets $Q \in \mathcal{SC}_{3,2}(v;\rho,\mu)$ and $X \subseteq V_5$ with $|X| \leq (1-2\mu)n$. By assumption, Q contains at least $q_\mu/(2o_\mu)$ (ρ,μ)-cocovers, which must be X-resistant due to Lemma 4.76.

Properties of single cocovers Consider an X-resistant (ρ,μ)-cocover O and a vertex $u \in O^*$. We let

$$F_u := \{\{y,w\} \in E(V_4,V_2) \mid y \in R_u, w \in \Gamma_2(y) \cap C_2(W_{u,y},\rho)\}.$$

We intend to show the following auxiliary claim:

(i) For at least $(1-3\sqrt{\nu})n$ vertices $w \in C_2^{F_u}(R_u;\nu)$ suitable sets $Y_{u,w}$ and $S_{u,w,y}$ for $y \in Y_{u,w}$ exist which satisfy the requirements of Definition 4.78.

Note that $d_2(y) - d_2^{F_u}(y) \leq (\rho)^{\oplus}q \leq \nu^2 q$ for $y \in R_u$ (recall that $\nu := \sqrt{(\rho)^{\oplus}}$, cf. Definition 4.78), since $R_u \subseteq \Gamma_4(u) \cap \Gamma_4(v)$ is (T_ρ)-qualified and $W_{u,y} \subseteq \Gamma_5(y) \cap \Gamma_5(u) \setminus X$ is a ρ-multicover. Hence by Lemma 4.52 we deduce that

$$|C_2^{F_u}(R_u;\nu)| \geq (1-3\sqrt{\nu})n.$$

For every vertex $w \in C_2^{F_u}(R_u;\nu)$ we have that

$$|\Gamma_4^{F_u}(w) \cap R_u| \geq r_\nu(\sigma r_\mu)/2 = \frac{1}{8}\nu\sigma r_\mu \cdot \frac{m}{n^2} \geq \frac{1}{9}\nu\sigma\mu^2\frac{m^3}{n^5} \geq \nu^3\frac{m^3}{n^5} = h_\nu,$$

as R_u is a ν-multicover (of V_2). Hence, we can find a set $Y_{u,w} \subseteq \Gamma_4(v)\cap\Gamma_4(u)\cap\Gamma_4(w)$ with $|Y_{u,w}| = h_\nu$ for which the following property is satisfied. Every vertex $y \in Y_{u,w}$ is incident to an edge $\{y,w\} \in F_u$. Due to the definition of F_u we deduce that $w \in C_2(W_{u,y},\rho)$. This leads to

$$|\Gamma_5(w) \cap W_{u,y}| \geq r_\rho(\sigma r_\sigma)/2 = \frac{1}{8}\rho\sigma r_\sigma \cdot \frac{m}{n^2} \geq \frac{1}{9}\rho\sigma^3\frac{m^3}{n^5} \geq \rho^3\frac{m^3}{n^5} \geq h_\rho.$$

Thus we can find a set $S_{u,w,y} \subseteq \Gamma_5(u)\cap\Gamma_5(y)\cap\Gamma_5(w)\setminus X$ with $|S_{u,w,y}| = h_\rho$ as required by Definition 4.78 for at least $(1-3\sqrt{\nu})n$ vertices $w \in C_2^{F_u}(R_u;\nu)$. This shows claim (i).

Now we combine claim (i) for a single vertex $u \in O^*$ to the subsequent claim (ii) for many vertices in O^*. By Corollary 4.29 it follows from (i) that at least $(1 - 3\nu^{1/4})n$ vertices in V_2 belong to $C_2^{F_u}(R_u; \nu)$ for at least $(1 - 3\nu^{1/4})2\sigma^2 o_\mu$ vertices $u \in O^*$. This holds for every X-resistant (ρ, μ)-cocover O.

The sets $(S_{u,w,y})_{u \in O^*, y \in Y_{u,w}}$ are $2s_\mu$-quasidisjoint because they are subsets of $2s_\mu$-quasidisjoint sets $W_{u,y}$. We obtain the following auxiliary claim.

(ii) There exists a set $GV_2(O) \subseteq V_2$ with $|GV_2(O)| = (1 - 3\nu^{1/4})n$ such that for every vertex $w \in GV_2(O)$ there exist $(1-3\nu^{1/4})2\sigma^2 o_\mu$ vertices $u \in O^*$ such that suitable sets $Y_{u,w}$ and $S_{u,w,y}$ satisfying the requirements of Definition 4.78 can be found. The sets $S_{u,w,y}$ are $2s_\mu$-quasidisjoint for fixed w.

Combining many cocovers Now assume that Q contains at least $z := q_\mu/(2o_\mu)$ X-resistant (ρ, μ)-cocovers $O_1, \ldots, O_z$. By Corollary 4.29 and (ii) we conclude that there are at least $(1 - 3\nu^{1/8})n$ vertices $w \in V_2$ such that $w \in GV_2(O_l)$ for at least $(1 - 3\nu^{1/8})z$ indices $l \in \{1, \ldots, z\}$. Thus for such a vertex w there are at least

$$(1 - 3\nu^{1/8})z \cdot (1 - 3\nu^{1/4})2\sigma^2 o_\mu \geq (1 - 6\nu^{1/8})\sigma^2 q_\mu$$

vertices $u \in \bigcup_{l=1}^{z} O_l^*$ such that suitable sets $Y_{u,w}$ and $S_{u,w,y}$ exist. These sets $S_{u,w,y}$ are $2s_\mu z$-quasidisjoint by construction.

In order to show that such a set Q satisfies Definition 4.78 it suffices to identify $\tilde{Q}$ with $\bigcup_{l=1}^{z} O_l^*$, since $|\bigcup_{l=1}^{z} O_l^*| = z \cdot 2\sigma^2 o_\mu = \sigma^2 q_\mu$. □

The following definition introduces square candidates of a set $R \subseteq V_3$ and a vertex $w \in V_2$, i.e., vertices $x \in V_5$ which would complete a square with w and a vertex $u \in R$ (with an additional edge to a given vertex $v \in V_1$) if the edge $\{u, w\}$ were indeed present. However, note that these edges, i.e., the edges between V_2 and V_3 are not yet taken into account.

Definition 4.80 (Square candidates) *For $v \in V_1$, $w \in V_2$ and $R \subseteq V_3$ the set of square candidates is given by*

$$\begin{aligned} SQC(w, R, v) := \{x \in V_5 \quad | \quad & \exists u \in R \; \exists y \in \Gamma_4(v) \cap \Gamma_4(u) \cap \Gamma_4(w) : \\ & x \in \Gamma_5(u) \cap \Gamma_5(y) \cap \Gamma_5(w)\} \end{aligned}$$

Square candidate covers Q were constructed in such a way that for a given vertex $w \in V_2$ almost all vertices in $\tilde{Q}$ complete $h_\nu \cdot h_\rho = \Theta(m^6/n^{10})$ square candidates with w.

Now assume that we choose several vertices $R \subseteq Q$ and require that the square candidates which belong to the vertices in R are 'rather' disjoint. If $|R|$ is sufficiently small, the quasidisjointness of the square candidates will suffice to show that this is indeed the case. Recall that the square candidates are $\Theta(m^5/n^8)$-quasidisjoint and at the threshold $m = \Theta(n^{8/5})$ it follows that they are $\Theta(1)$-quasidisjoint. Thus, the overlap should not cause real problems. Our aim is to choose $|R| = \Theta(m^2/n^3)$ vertices which complete $\Theta(n^2/m)$ square candidates. Note that this should be possible since $\Theta(m^2/n^3 \cdot m^6/n^{10}) = \Theta(m^8/n^{13})$ and $n^2/m = o(m^8/n^{13})$ for $m = \omega(n^{5/3})$.

The following definition introduces good R-sets, i.e., sets $R \subseteq \tilde{Q}$ with many square candidates, and the subsequent lemma shows that bad R-sets occur indeed very rarely. Here it will turn out to be essential that the sets $S_{u,w,y}$ in the square candidate cover have been shown to be quasidisjoint. Otherwise, neighborhoods of the vertices in a set R could cluster and we would not obtain the desired cardinality.

Definition 4.81 (Non-spreading R-sets) *Consider $v \in V_1$ and $X \subseteq V_5$. Let $Q \subseteq V_3$ be an X-resistant (ρ, μ)-square candidate cover and consider a vertex $w \in \tilde{T}(Q) \subseteq V_2$. We define*

$$\bar{\mathcal{R}}^5(w, Q, v; X, \rho) := \{R \in \binom{\tilde{Q}}{\tilde{r}} \mid |SQC(w, R, v) \setminus X| < t_\rho\},$$

where $t_\rho := \rho^{-1}\frac{n^2}{m}$ and $\tilde{r} := r_\mu(\sigma^2 q_\mu)/2 = \frac{1}{8}(1-\mu)\mu^2\sigma^2\frac{m^2}{n^3}$.

Lemma 4.82 (Few non-spreading R-sets) *Let Q be an X-resistant (ρ, μ)-square candidate cover and consider a vertex $w \in \tilde{T}(Q) \subseteq V_2$. Let $\nu := \sqrt{(\rho)^{\oplus}}$. Then*

$$|\bar{\mathcal{R}}^5(w, Q, v; X, \rho)| \le \nu^{\tilde{r}/20}\binom{\sigma^2 q}{\tilde{r}}$$

for $m \ge Cn^{5/3}$ and C sufficiently large.

Proof We intend to apply Lemma 4.31. To this aim we identify $B \equiv \tilde{Q}$, $B^* \equiv \tilde{Q}^*$, with $|B^*| \ge (1 - 6\nu^{1/8})|B|$, and $A_{u,1}, \dots, A_{u,h_\nu} \equiv (S_{u,w,y})_{y \in Y_{u,w}}$ for $u \in \tilde{Q}^*$ (cf. Definition 4.78). Note that $S_{u,w,y} \subseteq SQC(w, \{u\}, v) \setminus X$. Hence it suffices to bound the number of non-t_ρ-spreading sets $R \in \binom{\tilde{Q}}{\tilde{r}}$.

Let us check the preconditions of Lemma 4.31. For condition (i) observe that

$$h_\nu \le d_\mu \overset{(4.37)}{\Leftarrow} \nu^3\frac{m^3}{n^5} \le \sigma^4\frac{m^5}{n^8},$$

which clearly holds for $m = \omega(n^{3/2})$.

For condition (ii) we get

$$\begin{aligned} & 4t_\rho \cdot d_\mu \leq 6\nu^{1/8} \cdot h_\rho \cdot \sigma^2 q_\mu \cdot h_\nu \\ \overset{(4.38)}{\Leftarrow}\quad & \rho^{-1}\frac{n^2}{m}\sigma\frac{m^5}{n^8} \leq \nu\rho^3\frac{m^3}{n^5}\sigma^2\mu\frac{m}{n}\nu^3\frac{m^3}{n^5} \;\Leftarrow\; \frac{m^4}{n^6} \leq \mu\sigma\rho^4\nu^4\frac{m^7}{n^{11}}. \end{aligned}$$

The last inequality is satisfied for, say, $m^3 \geq \rho^{-6}n^5$.

Condition (iii) is satisfied due to

$$\tilde{r} \geq 4\frac{t_\rho}{h_\rho} \;\Leftarrow\; \frac{1}{9}\mu^2\sigma^2\frac{m^2}{n^3} \geq 4\rho^{-1}\frac{n^2}{m}\rho^{-3}\frac{n^5}{m^3} \;\Leftarrow\; \mu^2\sigma^2\frac{m^2}{n^3} \geq 40\rho^{-4}\frac{n^7}{m^4}$$

which is true for, say, $m^3 \geq \rho^{-6}n^5$.

Since all preconditions are satisfied, Lemma 4.31 gives

$$|\bar{\mathcal{R}}^5(w, Q, v; X, \mu)| \leq (48\nu^{1/8})^{\tilde{r}/2}\binom{\sigma^2 q}{q_\mu} \leq \nu^{\tilde{r}/20}\binom{\sigma^2 q}{q_\mu},$$

and the proof is complete. □

As we want to find many square candidate covers we are interested in vertices which satisfy the following property:

$$(S_{\rho,\mu}) \quad \forall Q \subseteq \Gamma_3(v), |Q| = q_\mu : \; Q \in \mathcal{SC}_{3,2}(v; \rho, \mu).$$

For this property we again define a corresponding set of bad graphs

$$\begin{aligned} \mathcal{B}_5^S(n, m; \varepsilon, \rho, \mu) \;&:=\; \{G \in \mathcal{S}_5^{QT+N}(n, m; \varepsilon, \rho, \mu) \mid \\ & \quad |V_1[(\neg S_{\rho,\mu}, N^+_{\rho,\mu}[\emptyset], D)]| \geq (\mu)^{\oplus} n\}. \end{aligned} \tag{4.39}$$

The following lemma shows that there are very few such bad graphs. A routine application of Lemma 4.38 will suffice for that.

Lemma 4.83 ($\mathcal{B}^S$ is small) *Let* $(\rho)^{\oplus} \lll \sigma = (\mu)^{\ominus 5}$. *Then*

$$|\mathcal{B}_5^S(n, m; \varepsilon, \rho, \mu)| \leq ((\mu)^{\oplus})^m \binom{n^2}{m}^{10}.$$

Proof We define the neighborhood function

$$\mathcal{N}(v) := \{X \in \binom{V_3}{d_v} \mid \exists Q \subseteq X, |Q| = q_\mu : \; Q \in \overline{\mathcal{SC}}_{3,2}(v, \rho, \mu)\}$$

By Lemma 4.79 we know that $|\mathcal{SC}_{3,2}(v;\rho,\mu)| \leq \mu^{2q}\binom{n}{q_\mu}$ for $v \in V_1[(N^+_{\rho,\mu}[\emptyset])]$. Letting $t := |Q|$ we deduce that

$$|\mathcal{N}(B)| \lesssim^q \mu^{2q}\binom{n}{q_\mu}\binom{n}{d_v - q_\mu} \overset{\text{L4.27 (iii)}}{\leq} \mu^{d_v}\binom{n}{d_v},$$

and the claim follows by Lemma 4.38. $\square$

For future reference we introduce the abbreviation

$$\begin{aligned}\mathcal{S}_5^{QT+N+S}(n,m;\varepsilon,\rho,\mu) \quad := \quad & \mathcal{S}_5^{QT+N}(n,m;\varepsilon,\rho,\mu) \setminus \\ & (\mathcal{B}_5^{N+}(n,m;\varepsilon,\rho,\mu) \cup \mathcal{B}_5^{S}(n,m;\varepsilon,\rho,\mu)).\end{aligned}$$

Note that for graphs $G \in \mathcal{S}_5^{QT+N+S}(n,m;\varepsilon,\rho,\mu)$ we have that

$$|V_1[(N^+_{\rho,\mu}[\emptyset], S_{\rho,\mu}, QT^+_\rho, D)]| \geq (1-(\mu)^{\ominus 2} - (n)^{\oplus 2} - (\mu)^{\oplus})n \geq (1 - 2(\mu)^{\oplus 2})n \tag{4.40}$$

by (4.27) on page 99, (4.28) and (4.39).

4.8.12 Cover families

Motivation for $H = K_5$ Although we have already defined square covers we have not yet proved their occurrence in typical graphs. Instead we have shown that there are many square candidate covers Q and that sets of size $\Theta(m^2/n^3)$ inside Q close $\Theta(n^2/m)$ square candidates. Observe that a typical vertex $w \in V_2$ has $\Theta(m^2/n^3)$ neighbors in Q for $|Q| = \Theta(m/n)$. If these neighbors indeed close $\Theta(n^2/m)$ square candidates and this is true for most vertices in V_2, this suffices to show that Q is a square cover.

Unfortunately, by considering one square candidate cover alone we do not achieve a sufficiently small probability for bad graphs. This is due to the fact that by Lemma 4.82 we only gain a factor $\tilde{r} = \Theta(m^2/n^3)$ in the exponent for a single bad neighborhood of size $\Theta(m^2/n^3)$ inside a specific square candidate cover Q. Observe that in the previous proofs we have needed $\Theta(m/n)$ in the exponent to obtain sufficiently small probabilities. This technical difficulty is overcome by considering a partition of V_3 into $\Theta(n^2/m)$ square candidate covers Q. By applying the above arguments to all these sets at once, we gain a factor of $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$ in the exponent, which will suffice to complete the proof.

Motivation for $H = K_4$ For the case $H = K_4$ we face a similar situation. Actually we want prove that many triangle covers exist. However, up to now we have only succeeded in proving that there are many triangle candidate covers Q and that most sets $R \subseteq Q$ with $|R| = \Theta(m^2/n^3)$ close a sufficient number of triangles (cf. Lemma 4.71). Again we get only a factor of order $\Theta(m^2/n^3)$ in the exponent for one bad R-set. Thus we want to combine $\Theta(n^2/m)$ such R-sets in order to cut down the probability to a term with exponent $\Theta(n^2/m \cdot m^2/n^3) = \Theta(m/n)$.

In the sequel we introduce such partitions of V_3. When applying these definitions we will need to work with a set $X_v \subseteq V_\ell$ of forbidden vertices for each vertex $v \in V_1$. These sets X_v will depend on sets A_v, the so-called processed neighborhoods, and functions f_v, named exclusion functions, which map A_v to X_v. The meaning of these names will later become clear, when we use cover families to iteratively construct subgraphs K_4 resp. K_5.

Anyway, the reader may prefer to ignore these rather technical aspects for the moment, pretending that $A_v = \emptyset$ and $f_v(A_v) = \emptyset$. Just remember that the cover families, which we will define now, may be restricted by choosing certain sets $X_v = f_v(A_v)$, but this feature is not essential to understand their basic structure.

Definition 4.84 (Independent vertex functions) *Let $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ and $\{i, j, k\} \subseteq \{1, \ldots, \ell\}$/ A function $f : 2^{V_i} \to 2^{V_k}$ is called* (i, j)-independent *if $f(Y)$ does not depend on $E[V_i \setminus Y, V_j]$ for all $Y \subseteq V_i$.*

Definition 4.85 (Cover families) *A* μ-cover family *consists of pairwise disjoint sets $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_1$ with $\tilde{p} := p_\mu/2$, $|Q_1| = \ldots = |Q_{\tilde{p}}| = q_\mu$. We consider a cover family in conjunction with*

- *an* inducing set $P^* = \{v_1, \ldots, v_{\tilde{p}}\}$ *such that $Q_l \subseteq \Gamma_3(v_l)$ for $l = 1, \ldots, \tilde{p}$,*
- processed neighborhoods $A_1, \ldots, A_{\tilde{p}} \subseteq V_2$ *with $|A_1|, \ldots, |A_{\tilde{p}}| \leq \mu^2 q$, and*
- $(2, 3)$*-independent* exclusion functions $\vec{f} = (f_1, \ldots, f_{\tilde{p}})$ *with $f_l : 2^{V_2} \to 2^{V_\ell}$ for $l = 1, \ldots, \tilde{p}$ (where f_l may also depend on v_l).*

In the application of Definition 4.85, P^* will belong to a μ-cover P and we will set $Q_1 = W_1, \ldots, Q_{\tilde{p}} = W_{\tilde{p}}$.

A cover family is enhanced by a property Π if most sets Z of size $\Theta(m^2/n^3)$ inside the sets $Q_1, \ldots, Q_{\tilde{p}}$ of size $\Theta(m/n)$ satisfy Π. The sets Z can be interpreted as potential neighborhoods of single vertices $w \in V_2$ inside the Q-sets. Our aim is to show that under this condition the real neighborhoods of most vertices $w \in V_2$ satisfy Π, too. If this is indeed the case, we call the cover family qualified for Π. The following definitions formally introduce enhanced and qualified cover families. Afterwards we will study the question under which conditions an enhanced cover family is qualified.

Definition 4.86 (Enhanced cover families) *A μ-cover family (with given inducing set, forbidden neighborhoods and exclusion functions) is called* Π-(ν, τ)-enhanced *for a family of properties $(\Pi_w)_{w \in V_2}$ if for $l = 1, \ldots, \tilde{p}$ there exists a set $\tilde{T}_l = \tilde{T}_l(v_l, A_l) \subseteq V_2$ with $|\tilde{T}_l| \geq (1-\mu)n$, and a μ-multicover (of V_2) $\tilde{Q}_l \subseteq Q_l \subseteq V_3$ with $|\tilde{Q}_l| = \tau q_\mu$ such that for all $w \in \tilde{T}_l$,*

$$|\{Z \in \binom{\tilde{Q}_l}{\tilde{r}} \mid Z \text{ does not satisfy } \Pi_w\}| \leq (\nu)^{\tilde{r}} \binom{\tau q_\mu}{\tilde{r}}, \tag{4.41}$$

where $\tilde{r} = \tilde{r}(\tau, \mu) := r_\mu(\tau q_\mu)/2 = \frac{1}{8}(1-\mu)\mu^2 \tau \frac{m^2}{n^3}$. The properties $\Pi_w = \Pi_w(v_l, A_l)$ may depend on w and v_l, as well as on A_l and $f_l(A_l)$.

Note that Definition 4.86 redefines $\tilde{r}$. When using enhanced cover families for the case $H = K_5$, $\tilde{r}$ will receive exactly the value given in Section 4.8.1. For the case $H = K_4$, however, we will set $\tilde{r} = \tilde{r}_1$. But reusing the identifier $\tilde{r}$ may remind the reader for which values of $\tilde{r}$ we will apply Definition 4.86.

Figure 4.14 illustrates the structure of an enhanced cover family. Later in the proof the sets Z with $|Z| = \tilde{r}$ will be neighborhoods of vertices $w \in V_2$. Note that the picture abstracts from the fact that the sets Z only reside in a small subset $\tilde{Q}_l \subseteq Q_l$.

Definition 4.87 (Qualified cover families) *A μ-cover family (with given inducing set and exclusion functions) is called* Π-τ-qualified *for a family of properties $(\Pi_w)_{w \in V_2}$ if for all processed neighborhoods $A_1, \ldots, A_{\tilde{p}} \subseteq V_2$ with $|A_1|, \ldots, |A_{\tilde{p}}| \leq \mu^2 q$ there exist $(1-\mu^{1/10})\tilde{p}$ indices $l \in \{1, \ldots, \tilde{p}\}$ such that*

$$(TC) \quad \begin{array}{l} \exists T = T(v_l, A_l) \subseteq V_2, |T| = (1 - 3(\mu)^{\oplus})n \\ \forall w \in T\ \exists R \subseteq \Gamma_3(w) \cap Q_l, |R| = \tilde{r} = \tilde{r}(\tau, \nu) : \ R \text{ satisfies } \Pi_w(v_l, A_l). \end{array}$$

For the sake of completeness let $T(v_l, A_l) = \emptyset$ for all indices l such that (TC) is not satisfied.

Note that T depends on v_l and A_l via $\Pi_w(v_l, A_l)$ but not on $v_{l'}$ and $A_{l'}$ for $l' \neq l$.

Figure 4.15 shows that the structure of enhanced and of qualified cover families is rather similar. Hence, it will not cause surprise that most enhanced

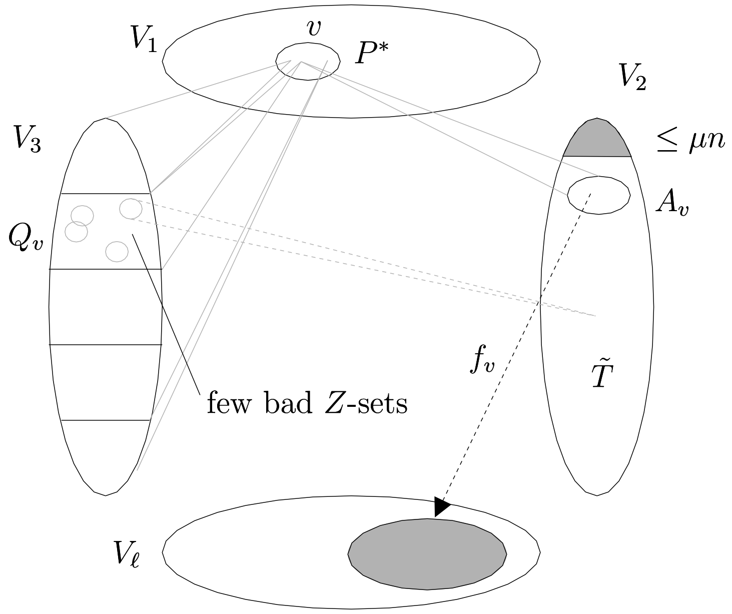

Figure 4.14: Π-(ν, τ)-enhanced cover family

cover families are also qualified. Assume that an enhanced cover family is given. Note that the Q-sets in the cover family are multicovers. Hence most vertices have $\Theta(m^2/n^3)$ neighborhoods inside the Q-sets. Due to (4.41) we conclude that these neighborhoods will typically satisfy Π_w. The following lemma shows how these two properties (existence of neighborhood and neighborhood satisfies Π_w) can be combined formally. A condition is given which implies that a Π-(ν, τ)-enhanced cover family is also Π-qualified. Afterwards we will prove a lemma which shows that this condition is almost always satisfied.

Lemma 4.88 (Condition for qualified cover families) *Consider a Π-(ν, τ)-enhanced μ-cover family with inducing set and exclusion functions. For given processed neighborhoods $A_1, \ldots, A_{\tilde{p}} \subseteq V_2$ with $|A_1|, \ldots, |A_{\tilde{p}}| \leq \mu^2 q$ we call a vertex $w \in V_2[(D)] \setminus \bigcup_{l=1}^{\tilde{p}} A_l$* non-spreading *if there exist at least $\mu\tilde{p}$ sets Q_l such that*

$$w \in \tilde{T}_l(v_l, A_l) \ \wedge \ \exists R \subseteq \Gamma_3(w) \cap \tilde{Q}_l, |R| = \tilde{r} : \ R \text{ does not satisfy } \Pi_w(v_l, A_l).$$

If there are at most $(\mu)^{\oplus} n$ non-spreading vertices for any choice of the processed neighborhoods $A_1, \ldots, A_{\tilde{p}}$, then the cover family is also Π_w-τ-qualified.

Proof Let arbitrary processed neighborhoods $A_1, \ldots, A_{\tilde{p}}$ be given and define

$$\tilde{C}_l := \tilde{T}_l(v_l, A_l) \cap C_2(\tilde{Q}_l; \mu)$$

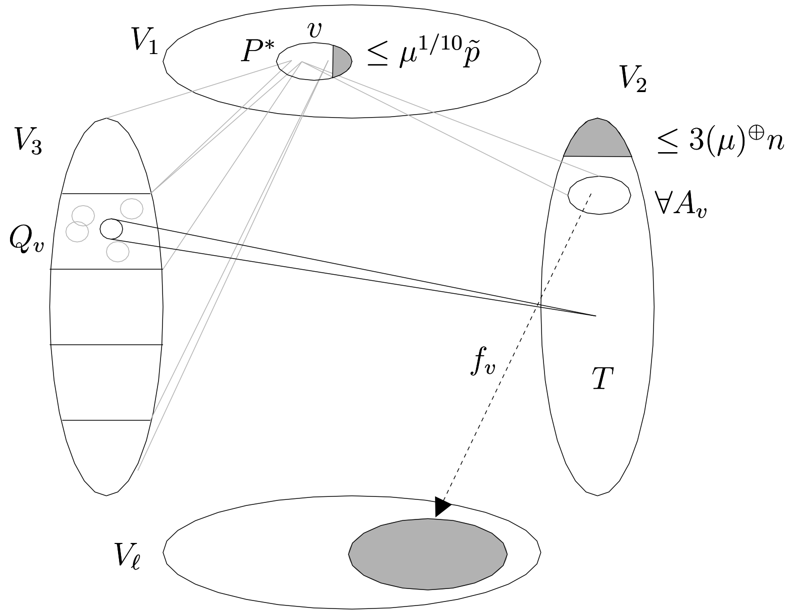

Figure 4.15: Π-τ-qualified cover family

for all indices $l \in \{1, \ldots, \tilde{p}\}$. By Definition 4.86 and Lemma 4.52 we have

$$|\tilde{C}_l| \geq (1 - \mu - \sqrt{\mu})n \geq (1 - 2\sqrt{\mu})n. \tag{4.42}$$

By Definition 4.51 such a vertex $w \in \tilde{C}_l \subseteq C_2(\tilde{Q}_l; \mu)$ has many neighbors in the corresponding sets $\tilde{Q}_l$, i.e., $|\tilde{Q}_l \cap \Gamma_3(w)| \geq r_\mu(\tau q_\mu)/2 = \tilde{r}$.

For a vertex $w \in V_2$ let

$$L(w) := \{l \in \{1, \ldots, \tilde{p}\} \mid w \in \tilde{C}_l\}.$$

We are interested in vertices $w \in V_2$ such that $L(w)$ is big. Thus we define

$$S := \{w \in V_2 \mid |L(w)| \geq (1 - 2\mu^{1/4})\tilde{p}\}.$$

Using Corollary 4.29 we deduce from (4.42) that $|S| \geq (1 - 2\mu^{1/4})n$. We remove all non-spreading vertices and the vertices in $\bigcup_{l=1}^{\tilde{p}} A_l$ from S and call the remaining set S'. Observe that $|\bigcup_{l=1}^{\tilde{p}} A_l| \leq \mu^2 q \cdot \tilde{p} \leq \mu n$ and thus

$$|S'| \geq (1 - 2\mu^{1/4} - (\mu)^{\oplus} - \mu)n \geq (1 - 2(\mu)^{\oplus})n.$$

Let $R_l := [\tilde{Q}_l \cap \Gamma_3(w)]_{\tilde{r}} \subseteq Q_l \cap \Gamma_3(w)$ and note that $|R_l| = \tilde{r}$ for $l \in L(w)$. We define

$$L'(w) := \{l \in L(w) \mid R_l \text{ satisfies } \Pi_w(v_l, A_l)\}.$$

Since a vertex $w \in S'$ is not 'non-spreading' there may exist at most $\mu\tilde{p}$ sets Q_l such that $w \in \tilde{T}_l(v_l, A_l)$ but R_l does not satisfy $\Pi_w(v_l, A_l)$. As $w \in \tilde{T}_l(v_l, A_l)$ for $l \in L(w)$ we deduce that

$$|L'(w)| \geq |L(w)| - \mu\tilde{p} \geq (1 - 2\mu^{1/4} - \mu)\tilde{p} \geq (1 - 3\mu^{1/4})\tilde{p}.$$

We apply Corollary 4.29 one more time and obtain that there are at least

$$(1 - 3\mu^{1/8}) \cdot |S'| \geq (1 - 3\mu^{1/8}) \cdot (1 - 2(\mu)^{\oplus})n \geq (1 - 3(\mu)^{\oplus})n$$

vertices $w \in S'$ such that we can find $(1 - 3\mu^{1/8})\tilde{p} \geq (1 - \mu^{1/10})\tilde{p}$ indices $l \in \{1, \ldots, \tilde{p}\}$ such that $l \in L'(w)$. Observe that for these indices l there exists a suitable set $R \subseteq \Gamma_3(w) \cap Q_l$ which satisfies $\Pi_w(v_l, A_l)$ as required by Definition 4.87. This completes the proof. □

As announced previously, the following lemma achieves the desired probability of β^m for cover families to be Π-qualified by considering many cover families at once. These cover families will be induced by disjoint sets $P_1^*, \ldots, P_{\hat{q}}^* \subseteq V_1$ for suitably chosen $\hat{q}$. The forbidden neighborhoods for all cover families will be denoted by $(A_v)_{v \in V_1}$. By arbitrarily numbering the vertices in P_h^* for $h = 1, \ldots, \hat{q}$ we get a one to one correspondence between A_l and A_{v_l} for any vertex $v_l \in P_h^*$. If the sets $P_1^*, \ldots, P_{\hat{q}}^*$ are given, we use a vertex v and its index l inside 'its' P-set interchangeably, even without explicitly numbering the vertices in the P^*-sets since every numbering suits our needs.

Lemma 4.89 (Most graphs contain qualified cover families) *Let $\nu \lll \tau \leq \mu$. Consider a fixed tuple of $(2,3)$-independent exclusion functions $\vec{f} = (f_1, \ldots, f_{\tilde{p}})$. A graph $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ is* $\vec{f}$-non-spreading *for a given property Π if one of the following two conditions is satisfied:*

(i) There exists a Π-(ν, τ)-enhanced μ-cover family $Q_1, \ldots, Q_{\tilde{p}} \subseteq V_3$ induced by a set $P^* \subseteq V_1$, $|P| = \tilde{p}$ with processed neighborhoods $A_1, \ldots, A_{\tilde{p}} \subseteq V_2$ such that there are at least $(\mu)^{\oplus}n$ non-spreading vertices in V_2.

(ii) (i) is not satisfied and we can find disjoint sets $P_1^*, \ldots, P_{\hat{q}}^* \subseteq V_1[(D)]$ which induce Π-(ν, τ)-enhanced μ-cover families (which are also Π-τ-qualified due to Lemma 4.88) with $\hat{q} := (1 - (\mu)^{\oplus 3})n/\tilde{p}$ and $|P_1^*| = \ldots = |P_{\hat{q}}^*| = \tilde{p}$ such that there exist

- *processed neighborhoods $(A_v)_{v \in V_1} \in (2^{V_2})^n$ with $A_v \subseteq \Gamma_2(v)$ and $|A_v| \leq \mu^2 q$ for all $v \in V_1$,*
- *$(\mu)^{\oplus 2}\hat{q}$ indices $l \in \{1, \ldots, \hat{q}\}$ such that there exist $(\mu)^{\oplus 2}\tilde{p}$ vertices $v \in P_l^*$ with*

$$(\Gamma_2(v) \setminus A_v) \cap T(v, A_v) = \emptyset. \tag{4.43}$$

The set of $\vec{f}$-non-spreading graphs $\mathcal{B}_\ell^{\vec{f}}(n, m; \varepsilon, \nu, \mu, \tau \mid \Pi)$ satisfies

$$|\mathcal{B}_\ell^{\vec{f}}(n, m; \varepsilon, \nu, \mu, \tau \mid \Pi)| \leq ((\mu)^{\oplus 2})^m \binom{n^2}{m}^{\binom{\ell}{2}}.$$

Proof We distinguish two cases corresponding to the conditions (i) and (ii).

Case 1: We consider the case that condition (i) is met.

For choosing the sets $Q_1, \ldots, Q_{\tilde{p}}$ we have at most $n^{q_\mu \cdot \tilde{p}}$ possibilities. Note that $n^{q_\mu \cdot \tilde{p}} \leq n^{q\tilde{p}} \leq 2^m$ for $m = \omega(n \log n)$. For $A_1, \ldots, A_{\tilde{p}}$ the same bound $n^{q\tilde{p}} \leq 2^m$ applies. Furthermore, we have $n^{\tilde{p}}$ possibilities to choose P. Hence there are at most, say, 5^m possibilities to fix the cover family. Consequently, it suffices to count the number of graphs which are bad with respect to a specific cover family.

To this aim we plan to apply Lemma 4.38, where the bad vertices $B \subseteq V_2$ directly correspond to the $(\mu)^{\oplus} n$ non-spreading vertices in V_2. We define the neighborhood function

$$\begin{aligned} \mathcal{N}(w) \quad := \quad & \{X \in \binom{V_3}{d_w} \mid \exists L \subseteq \{1, \ldots, \tilde{p}\}, |L| = \mu\tilde{p} \;\; \forall l \in L : \\ & \exists R \subseteq X \cap \tilde{Q}_l, |R| = \tilde{r} : \;\; R \text{ does not satisfy } \Pi_w(l, A_l)\}. \end{aligned}$$

Now let us estimate $\mathcal{N}(w)$ for $w \in B$. Observe that we can determine the sets which satisfy Π_w independently of the edges $E(V_2, V_3)$ (For that we need in particular that the exclusion functions are $(2,3)$-independent.). By Definition 4.86 the bad vertices are $(\tilde{p}, \tilde{r}, \mu)$-$\exists$-bad with error probability $(\nu)^{\tilde{r}}$, as $w \in \tilde{T}_l(v_l, A_l)$ for $w \in B$. Note that

$$\tilde{r} \cdot \tilde{p} = \frac{1}{8}(1-\mu)\mu^2 \tau \frac{m^2}{n^3} \cdot \mu^{-1} \frac{n^2}{m} \geq \frac{1}{9}\mu\tau q \geq \frac{1}{10}\tau^2 d_w.$$

By Lemma 4.35 we obtain

$$|\mathcal{N}(w)| \leq (8\nu)^{\mu\tau^2 d_w/10} \binom{n}{d_w} \leq \tau^{d_w} \binom{n}{d_w},$$

and the proof of the first case is complete.

Case 2: Now we consider the case that condition (ii) is met.

We will apply Lemma 4.35 to restrict the number of neighbors of bad vertices $B \subseteq V_1$ (which we will define later) in V_2.

Let us first choose the sets $P_1^*, \ldots, P_{\hat{q}}^* \subseteq V_1$ (at most $n^n \leq 2^m$ possibilities) and the Q-sets of the induced cover families (at most $2^{2q \cdot n} = 4^m$ possibilities since we know the edges $E(B, V_3)$). Again this number of possible choices is small enough that it does not carry weight when proving the desired bound on the number of graphs in $\mathcal{B}_\ell^{\vec{f}}$. Note that by Lemma 4.88 and condition (i) the chosen Π-ν-enhanced μ-cover families are also Π-qualified.

Again we will use Lemma 4.38 to prove that the number of 'bad' graphs is small. Assume for the moment that the sets $(A_v)_{v\in V_1}$ are known and consider the $(\mu)^{\oplus 2}\hat{q}$ 'bad' indices $l \in \{1,\dots,\hat{q}\}$. For every 'bad' index there exist $(\mu)^{\oplus 2}\tilde{p}$ vertices in P_l^* which satisfy (4.43). By Definition 4.87 at most $\mu^{1/10}\tilde{p} \leq (\mu)^{\oplus 2}\tilde{p}/2$ vertices $v \in P_l^*$ do not satisfy (TC). Hence we can find $(\mu)^{\oplus 2}\tilde{p}/2$ bad vertices in every bad set P_l^* for which (TC) holds. These vertices form the bad set $B \subseteq V_1$. All in all we get

$$|B| \geq (\mu)^{\oplus 2}\hat{q}\cdot(\mu)^{\oplus 2}\tilde{p}/2 \geq ((\mu)^{\oplus 2})^2 n/4$$

bad vertices. For $v \in B$ we let

$$\mathcal{N}(v) \;:=\; \{X \in \binom{V_2}{d_v} \mid \exists A \subseteq X_v, |A| \leq \mu^2 q:\; (X\setminus A)\cap T(v,A) = \emptyset\}$$

Let $a_v := |A_v|$ for $v \in B$. There are at most $n^n \lesssim 2^m$ choices for the values $(a_v)_{v\in B}$. Hence we get (maximizing over a_v)

$$\begin{aligned} |\mathcal{N}(v)| &\lesssim^q \binom{n}{a_v}\cdot\binom{3(\mu)^{\oplus}n}{d_v - a_v} \overset{\text{L4.27 (ii, iii)}}{\leq} (3(\mu)^{\oplus})^{d_v - a_v}4^{d_v}\cdot\binom{n}{d_v} \\ &\leq (200(\mu)^{\oplus})^{d_v/2}\cdot\binom{n}{d_v}. \end{aligned}$$

This suffices to complete the proof using Lemma 4.38. □

The following corollary shows that Lemma 4.89 achieves the goal that for a typical vertex $v \in V_1$ we can find a neighbor $w \in \Gamma_2(v)$ such that v and w have a common neighborhood R of size $\Theta(m^2/n^3)$ such that R satisfies a given property Π_w.

Corollary 4.90 (Almost all vertices have good neighborhoods R) *Consider a graph $G \in \mathcal{S}_\ell(n,m;\varepsilon)\setminus\mathcal{B}_\ell^{\vec{f}}(n,m;\varepsilon,\nu,\mu,\tau \mid \Pi)$ and arbitrary processed neighborhoods $(A_v)_{v\in V_1}$ with $A_v \subseteq \Gamma_2(v)$ and $|A_v| \leq \mu^2 q$ for $v \in V_1$. Assume that there exist $P_1^*,\dots,P_{\hat{q}}^* \subseteq V_1$ inducing Π-(ν,τ)-enhanced cover families (as in (ii) of Lemma 4.89). Then we can find $TV_1 \subseteq V_1$ with $|TV_1| \geq (1-10(\mu)^{\oplus 2})n$ such that all vertices $v \in TV_1$ satisfy the property*

$$\exists w \in \Gamma_2(v)\setminus A_v\; \exists R \subseteq \Gamma_3(v)\cap\Gamma_3(w), |R| = \tilde{r}:\; R \text{ satisfies } \Pi_w(v,A_v).$$

Proof By Lemma 4.89 (ii) there are at least $(1-(\mu)^{\oplus 2})\hat{q}$ indices l such that P_l^* contains at least $(1-(\mu)^{\oplus 2})\tilde{p}$ vertices $v \in P_l^*$ for which (4.43) is not satisfied. For such a vertex v we conclude that there exists a vertex $w \in \Gamma_2(v)\setminus A_v$ such that $w \in T(v,A_v)$. By Definition 4.87 it follows that there exists a set $R \subseteq \Gamma_3(w)\cap Q_v \subseteq \Gamma_3(w)\cap\Gamma_3(v)$ with $|R| = \tilde{r}$ which satisfies $\Pi_w(v,A_v)$. Note that we can find at least

$$(1-(\mu)^{\oplus 2})\hat{q}\tilde{p} - \hat{q}(\mu)^{\oplus 2}\tilde{p} = (1-2(\mu)^{\oplus 2})\hat{q}\tilde{p} \geq (1-10(\mu)^{\oplus 3})n$$

such vertices $v \in V_1$. This completes the proof of the corollary. □

The statement of Corollary 4.90 can be simplified further. To this aim we introduce the following property of a vertex $v \in V_1$:

$$\begin{array}{ll}(R \mid \Pi) & \forall A \subseteq \Gamma_2(v), |A| \leq \mu^2 q \\ & \exists w \in \Gamma_2(v) \setminus A\ \exists R \subseteq \Gamma_3(v) \cap \Gamma_3(w), |R| = \tilde{r} : \\ & \qquad R \text{ satisfies } \Pi_w(v, A).\end{array}$$

Note the similarity between the property $(R \mid \Pi)$ and the property of vertices in TV_1, introduced in Corollary 4.90. The only difference consists in the fact that the ordering of the quantifiers has been swapped. In Corollary 4.90 we have seen that for all processed neighborhoods there exist many 'good' vertices TV_1. In contrast to that the following corollary shows that there exist many vertices in V_1 satisfying $(R \mid \Pi)$, i.e., they are 'good' for all processed neighborhoods.

Corollary 4.91 (Almost all vertices satisfy $(R \mid \Pi)$) *For a graph $G \in \mathcal{S}_\ell(n, m; \varepsilon) \backslash \mathcal{B}_\ell^{\vec{f}}(n, m; \varepsilon, \nu, \mu, \tau \mid \Pi)$. Assume that there exist sets $P_1^*, \ldots, P_{\hat{q}}^* \subseteq V_1$ inducing Π-(ν, τ)-enhanced cover families. Then we have $|GV_1| \geq (1-10(\mu)^{\oplus 2})n$, where $GV_1 := V_1[(R \mid \Pi)]$.*

Proof Assume that $|GV_1| < (1-10(\mu)^{\oplus 3})n$ and consider corresponding processed neighborhoods A_v for $v \in V_1$, i.e., for every vertex $v \in V_1 \setminus GV_1$ we define A_v in such a way that $(R \mid \Pi)$ is violated. Thus there is no R-set satisfying $\Pi_w(v, A_v)$ for these vertices.

Now observe that on the one hand the fact that a set R satisfies $\Pi_w(v, A_v)$ does not depend on the the choices for $A_{v'}$ with $v' \neq v$. Thus we have $TV_1 \subseteq GV_1$ for the given processed neighborhoods. On the other hand, Corollary 4.90 tells us that also for the given choice of the A-sets there must exist a large set TV_1, yielding a contradiction. □

When we want to apply Corollary 4.91 we will construct many μ-covers $P_1, P_2, \ldots \subseteq V_1$ such that the sets $P_1^*, P_2^*, \ldots \subseteq V_1$ are disjoint and induce enhanced cover families. The following lemma shows how these μ-covers can be obtained.

Lemma 4.92 (Find many disjoint covers) *Let $\rho, \mu > 0$ and assume that V_1 is (Π-enhanced) ρ-homogeneous. Then we can find (Π-qualified) μ-covers $P_1, \ldots, P_{\hat{q}}$, where $\bar{q} := (1-\sqrt{\rho})n/\tilde{p}$, such that the sets $P_1^*, \ldots, P_{\bar{q}}^*$ are disjoint.*

Proof We construct the sets $P_1, \ldots, P_{\hat{q}}$ one by one. For $l \in \{1, \ldots, \bar{q}\}$ note that

$$|V_1 \setminus (P_1^* \cup \ldots \cup P_{l-1}^*)| \leq n - (l-1)\tilde{p} \geq \sqrt{\rho}n/2.$$

Hence, the remaining part of V_1 is still $2\sqrt{\rho}$-homogeneous and by means of Lemma 4.49 we conclude that there exists a suitable set P_l. □

4.8.13 Finding cover families

Finding triangle cover families (K_4)

For $v \in V_1$ and $w \in V_2$ let

$$TRI(v, w) := \Gamma_4(\Gamma_3(v) \cap \Gamma_3(w)) \cap \Gamma_4(w).$$

For the case $H = K_4$ we will consider cover families with the 2-3-independent exclusion functions

$$f_l^{TRI}(A_l) = [\bigcup_{w \in A} TRI(v_l, w)]_{(1-2\sqrt{\mu})n}$$

and the family of properties

$$(R_\rho^4)_w \quad |\Gamma_4(R) \cap \Gamma_4(w) \setminus f_l^{TRI}(A_l)| \geq t_\rho$$

for $w \in V_2$ and $R \subseteq Q_l \cap \Gamma_3(w)$ with $|R| = \tilde{r}_1$.

We define

$$\mathcal{B}_4^R(n, m; \varepsilon, \rho, \mu) := \mathcal{B}_4^{f^{TRI}}(n, m, \varepsilon, \rho^{1/20}, \mu, \mu \mid R_\rho^4), \tag{4.44}$$

and accordingly we let

$$\mathcal{S}_4^{C+R}(n, m; \varepsilon, \rho, \mu) := \mathcal{S}_4^{C+}(n, m; \varepsilon, \rho, \mu) \setminus \mathcal{B}_4^R(n, m; \varepsilon, \rho, \mu).$$

By Lemma 4.89 we conclude that,

$$|\mathcal{B}_4^R(n, m; \varepsilon, \rho, \mu)| \leq ((\mu)^{\oplus 2})^m \binom{n^2}{m}^{10}, \tag{4.45}$$

since $\rho^{1/20} \lll \mu$.

We intend to apply Corollary 4.90 to graphs in $\mathcal{S}_5^{C+R}(n, m; \varepsilon, \rho, \mu)$. To this aim we first have to find suitable cover families. This is done as follows.

Lemma 4.93 (Covers induce triangle cover families) *Let $\rho \lll \mu$. For a (C_ρ^+, C_ρ, D)-qualified μ-cover $P \subseteq V_1$ the set P^* induces a (R_ρ^4)-$(\rho^{1/20}, \mu)$-enhanced μ-cover family for arbitrary processed neighborhoods $(A_v)_{v \in V_1}$.*

Proof Let $P^* = \{v_1, \dots, v_{\tilde{p}}\}$ and $Q_1 = W_1, \dots, Q_{\tilde{p}} = W_{\tilde{p}}$ with $|Q_1| = \ldots = |Q_{\tilde{p}}| = q_\mu$. Due to $v_l \in V_1[(C_\rho)]$ for $l = 1, \dots, \tilde{p}$, and since $|Q_l| = q_\mu \geq \rho \cdot d_3(v_l)$, it follows that Q_l is a μ-multicover of V_2.

Furthermore, by $v_l \in V_1[(C_\rho^+)]$ we conclude that Q_l is a (C_ρ)-qualified μ-multicover of V_4. Recall that $|f_l^{TRI}(A_l)| \leq (1 - 2\sqrt{\mu})n$. Thus Q_l is also an $f_l^{TRI}(A_l)$-resistant (ρ, μ)-triangle candidate cover (cf. Lemma 4.69).

Note that $|\tilde{T}(Q_l)| \geq (1 - \rho^{1/4})n \geq (1 - \mu)n$ due to Definition 4.68. Furthermore, we have $\tilde{r}(\mu, \mu) = r_\mu(\mu q_\mu)/2 = \tilde{r}_1$. By Lemma 4.71 it follows that the cover family $Q_1, \dots, Q_{\tilde{p}}$ is indeed (R_μ^4)-$(\rho^{1/20}, \mu)$-enhanced (cf. Definition 4.86). □

A cover family which is constructed using Lemma 4.93 yields the desired triangle covers. Observe that property (R_ρ^4) guarantees that the number of triangle candidates in $V_4 \setminus f_l^{TRI}(A_l)$ for a single vertex w is at least t_ρ. If there are only few non-spreading vertices for the cover family, then the cover family is also qualified (cf. Lemma 4.88). Observe that the large set T which exists for most Q-sets in a qualified cover family (cf. Definition 4.87) thus directly corresponds to the (large) set T in Definition 4.67 of triangle covers.

However, we do not have to construct the triangle covers explicitly. Recall that Lemma 4.89 serves a two-fold purpose, expressed by the two properties (i) and (ii) of $\vec{f}$-non-spreading graphs. Property (i) shows that most graphs, i.e., the 'good' graphs which are not $\vec{f}$-non-spreading, contain few non-spreading vertices and thus a cover family can be assumed to be qualified. Hence, as discussed above, triangle covers exist. Additionally, the neighborhood of most vertices in good graphs overlaps with the (large) set T due to property (ii). In our application of Lemma 4.89 this means that most vertices in V_1 have neighbors inside the triangle cover, and thus K_4-candidates exist (cf. Figure 4.9 on page 101).

Finding square cover families (K_5)

For the construction of square covers we will consider cover families with the 2-3-independent exclusion functions

$$f_l^{SQU}(A_l) = [\bigcup_{w \in A} SQC(w, \Gamma_3(w) \cap \Gamma_3(v_l), v_l)]_{(1-2\mu)n}$$

and the family of properties

$$(R_\rho^5)_w \quad |SQC(w, R, v_l) \setminus f_l^{SQU}(A_l)| \geq t_\rho \tag{4.46}$$

for $w \in V_2$ and $R \subseteq Q_l \cap \Gamma_3(w)$ with $|R| = \tilde{r}$.

We define

$$\mathcal{B}_5^R(n,m;\varepsilon,\rho,\mu) := \mathcal{B}_5^{f^{SQU}}(n,m,\varepsilon,\nu^{1/20},\mu,\sigma^2 \mid R_\mu^5), \tag{4.47}$$

where $\nu = \sqrt{(\rho)^{\oplus}}$ and $\sigma = (\mu)^{\ominus 5}$, and accordingly we let

$$\mathcal{S}_5^{QT+N+SR}(n,m;\varepsilon,\rho,\mu) := \mathcal{S}_5^{QT+N+S}(n,m;\varepsilon,\rho,\mu) \setminus \mathcal{B}_5^R(n,m;\varepsilon,\rho,\mu).$$

By Lemma 4.89 we conclude that

$$|\mathcal{B}_5^R(n,m;\varepsilon,\rho,\mu)| \le ((\mu)^{\oplus 2})^m \binom{n^2}{m}^{10}, \tag{4.48}$$

since $\nu^{1/20} \lll \sigma^2 \le \mu$.

We intend to apply Corollary 4.90 to graphs in $\mathcal{S}_5^{QT+N+SR}(n,m;\varepsilon,\rho,\mu)$. To this aim we first have to find suitable cover families. This is done as follows.

Lemma 4.94 (Covers induce square cover families) *Assume that $(\rho)^{\oplus} \le \sigma = (\mu)^{\ominus 5}$ and $6\nu^{1/8} \le \mu$ for $\nu = \sqrt{(\rho)^{\oplus}}$. For a $(N^+_{\rho,\mu}[\emptyset], S_{\rho,\mu}, C_\rho, D)$-qualified μ-cover $P \subseteq V_1$ the set P^* induces a (R_ρ^5)-$(\nu^{1/20},\sigma^2)$-enhanced μ-cover family for arbitrary processed neighborhoods $(A_v)_{v\in V_1}$.*

Proof Let $Q_1 = W_1, \dots, Q_{\tilde p} = W_{\tilde p}$. Due to the definition of $(S_{\rho,\mu})$ we conclude that $Q_1,\dots,Q_{\tilde p} \in \mathcal{S}_{3,2}(v;\rho,\mu)$ and by $|f_l^{SQU}(A_l)| \le (1-2\mu)n$ we deduce that

$$Q_l \in \mathcal{S}_{3,2}(v, f_l^{SQU}(A_l);\rho,\mu) \text{ for all } l = 1,\dots,\tilde p$$

due to Lemma 4.79, i.e., Q_l is a $f_l^{SQC}(A_l)$-resistant (ρ,μ)-square candidate cover.

By the definition of (C_ρ) it follows that the set $\tilde Q_l$ is a μ-multicover of V_2, as $|\tilde Q_l| = \sigma^2|Q|$ and $\sigma^2 \ge \rho$. Note that $|\tilde T(Q_l)| \ge (1-6\nu^{1/8})n \ge (1-\mu)n$ due to Definition 4.78. Furthermore, we have $\tilde r(\sigma^2,\mu) = r_\mu(\sigma^2 q_\mu)/2 = \tilde r$. By Lemma 4.82 it follows that the cover family $Q_1,\dots,Q_{\tilde p}$ is indeed (R_ρ^5)-$(\nu^{1/20},\sigma^2)$-enhanced (cf. Definition 4.86). □

Analogously to Lemma 4.93, a cover family which is constructed using Lemma 4.94 yields the desired square covers. The property (R_ρ^5) guarantees that the number of square candidates for a single vertex w is large. If there are only few non-spreading vertices for the cover family, then the cover family is also qualified (cf. Lemma 4.88), and the large set T which exists for most Q-sets in a qualified cover family (cf. Definition 4.87) thus directly corresponds to the set S_2 in Definition 4.77 of square covers. However, as we already saw in the case $H = K_4$, we do not have to construct the square covers explicitly, since Lemma 4.89 also shows that in most graphs and for most vertices $v \in V_1$ the neighborhood $\Gamma_2(v)$ and the set T will overlap. This implies the existence of K_5-candidates (cf. Figure 4.12 on page 111).

4.8.14 Clique candidates

Now we are finally in a position to construct clique candidates, i.e., subgraphs which are complete up to at most one edge.

Definition 4.95 (Clique candidates, owners and covers) *An i-*clique candidate *is a subgraph K_ℓ up to the edge between V_i and V_ℓ, i.e., this edge does not have to be present.*
*For two vertices $v \in V_i$ and $w \in V_j$ the set $CC_\ell(v, w)$ of clique candidates is given by all vertices $x \in V_\ell$ such that there exist an i-clique candidate which contains v and w. Accordingly, we define $CC_\ell(v, A) := \bigcup_{w \in A} CC_\ell(v, w)$ for $A \subseteq V_j$. The vertex v is a ξ-*clique covering vertex *if $|CC_\ell(v, \Gamma_j(v))| \geq (1 - \xi)n$.*

The following lemma shows that only very few clique free graphs with many clique covering vertices exist.

Lemma 4.96 (Only few clique covering vertices exist) *The number of graphs $G \in \mathcal{S}_\ell(n, m; \varepsilon)$ for which V_i contains $(\xi)^{\oplus} n$ ξ-clique covering vertices which satisfy (D) and*

$$(\neg K) \quad \Gamma_\ell(v) \cap CC_\ell(v, \Gamma_j(v)) = \emptyset$$

is bounded by

$$((\xi)^{\oplus})^m \binom{n^2}{m}^{\binom{\ell}{2}}.$$

Proof A simple application of Lemma 4.38 proves the claim. The bad set B consists of the $(\xi)^{\oplus} n$ ξ-clique covering vertices $v \in V_i$. Let

$$\mathcal{N}(v) := \{X \in \binom{V_\ell}{d_v} \mid X \cap CC_\ell(v, \Gamma_2(v)) = \emptyset\}.$$

It follows that $|\mathcal{N}(v)| \leq \binom{\xi n}{d_v} \leq \xi^{d_v} \binom{n}{d_v}$, which completes the proof. □

If there exist many clique covering vertices, this implies for most graphs that we can indeed find a clique due to Lemma 4.96. The following definition and lemma show how we can find many clique covering vertices.

Definition 4.97 (Clique candidate rich vertices) *A vertex $v \in V_1$ is ξ-*resistant μ-clique candidate rich *if*

$$\begin{aligned} &\forall A \subseteq \Gamma_j(v), |A| \leq \mu q : \\ &\quad (\exists w \in \Gamma_j(v) \setminus A : \ |CC_\ell(v, w) \setminus CC_\ell(v, A)| \geq t_\mu) \vee \\ &\quad |CC_\ell(v, A)| \geq (1 - \xi)n \end{aligned}$$

Lemma 4.98 (Clique candidate rich implies clique covering) *A ξ-resistant μ-clique candidate rich vertex $v \in V_1$ is also ξ-clique covering.*

Proof We will prove the claim by iteratively constructing a set $A \subseteq \Gamma_j(v)$ with $|A| \leq \mu q$ such that $|CC_\ell(v, A)| \geq (1-\xi)n$. Initially, A is empty. In every step of the iteration we add a vertex $w \in \Gamma_j(v) \setminus A$ to A such that $|CC_\ell(v, w) \setminus CC_\ell(v, A)| \geq t_\mu$. Definition 4.97 ensures that this can be done as long as $|CC_\ell(v, A)| < (1-\xi)n$. Since $CC_\ell(v, A)$ grows by at least t_μ vertices in every iteration and $t_\mu \cdot \mu q = n \geq (1-\xi)n$, this suffices to complete the proof. □

4.8.15 Proof of the main theorem

Case $H = K_4$

In the sequel we show using Lemma 4.98 that graphs in $\mathcal{S}_4^{C+R}(n, m; \varepsilon, \rho, \mu)$ contain many clique covering vertices, which implies that they contain a K_4 due to Lemma 4.96.

Lemma 4.99 (Find clique covering vertices for $H = K_4$) *For a graph $G \in \mathcal{S}_4^{C+R}(n, m; \varepsilon, \rho, \mu)$ there exist at least $(1-\gamma)n$ vertices in V_1 which are ξ-clique covering for $\xi \geq 2\sqrt{\mu}$ and $\gamma \geq 10(\mu)^{\oplus 3}$.*

Proof By Lemma 4.98 it suffices to show that V_1 contains $(1-\gamma)n$ vertices which are ξ-resistant μ^2-clique candidate rich (we have to use μ^2 instead of μ here for technical reasons, cf. the bound on the size of the processed neighborhoods in Definition 4.85).

Due to (4.33) on page 100 we conclude that V_1 is (Q_ρ^+, D)-enhanced $2(\rho)^{\oplus 2}$-homogeneous. Observe that

$$\bar{q} = (1 - 2\sqrt{(\rho)^{\oplus 2}})n/\tilde{p} \geq (1 - (\mu)^{\oplus 3})n/\tilde{p} = \hat{q}.$$

Thus, applying Lemma 4.92 we may construct (Q_ρ^+, D)-qualified μ-covers $P_1, \ldots, P_{\hat{q}} \subseteq V_1$ such that the sets $P_1^*, \ldots, P_{\hat{q}}^* \subseteq V_1[(Q_\rho^+, D)] \subseteq V_1[(C_\rho^+, C_\rho, D)]$ are disjoint. By Lemma 4.93 is follows that $P_1, \ldots, P_{\hat{q}}$ induce (R_ρ^4)-$(\rho^{1/20}, \mu)$-enhanced cover families. Hence a set GV_1 with $|GV_1| \geq (1 - 10(\mu)^{\oplus 2})n \geq (1-\gamma)n$ exists as indicated in Corollary 4.91. For a vertex $v \in GV_1$ and an arbitrary set $A \subseteq \Gamma_j(v)$ with $|A| \leq \mu^2 q$ we can find $w \in \Gamma_2(v) \setminus A$ and $R \subseteq \Gamma_3(v) \cap \Gamma_3(w)$ such that

$$|\Gamma_4(R) \cap \Gamma_4(w) \setminus f_v^{TRI}(A)| \geq t_\rho \geq t_{\mu^2}.$$

Note that $\Gamma_4(R) \cap \Gamma_4(w) \subseteq CC_4(v, w)$ and $CC_4(v, A) = f_v^{TRI}(A)$, provided that $|CC_4(v, A)| \leq (1 - 2\sqrt{\mu})n$. Thus, for every vertex $v \in GV_1$ we either have $|CC_4(v, A)| \geq (1 - 2\sqrt{\mu})n \geq (1 - \xi)n$ or

$$|CC_4(v, w) \setminus CC_4(v, A)| \geq |\Gamma_4(R) \cap \Gamma_4(w) \setminus f_v^{TRI}(A)| \geq t_{\mu^2}.$$

Consequently, the vertices in GV_1 satisfy the conditions given in Definition 4.97, and by Lemma 4.98 the claim follows. □

Now the proof of Theorem 4.3 for $H = K_4$ is essentially complete. By Lemma 4.62, (4.32), (4.45) and Lemma 4.96 we obtain

$$|\mathcal{S}_4^{C+R}(n, m; \varepsilon, \rho, \mu)| \geq (1 - (10(\xi)^\oplus)^m) \binom{n^2}{m}^6 \tag{4.49}$$

By Lemma 4.99 and Lemma 4.96 at most $(11(\xi)^\oplus)^m \binom{n^2}{m} \leq \beta^m \binom{n^2}{m}$ graphs in $\mathcal{S}_4(n, m; \varepsilon)$ exist which do not contain a subgraph K_4.

Case $H = K_5$

Using Lemma 4.98 we will now show that graphs in $\mathcal{S}_5^{QT+N+SR}(n, m; \varepsilon, \rho, \mu)$ contain many clique covering vertices, which implies that they contain a K_5 due to Lemma 4.96.

Lemma 4.100 (Find clique covering vertices) *Assume that $\gamma \geq 10(\mu)^{\oplus 3}$ and $\xi \geq 2\mu$. For a graph $G \in \mathcal{S}_5^{QT+N+SR}(n, m; \varepsilon, \rho, \mu)$ there exist at least $(1 - \gamma)n$ vertices in V_1 which are ξ-clique covering.*

Proof By Lemma 4.98 it suffices to show that V_1 contains $(1 - \gamma)n$ vertices which are ξ-resistant μ^2-clique candidate rich.

Due to (4.40) on page 118 we conclude that V_1 is $(N_{\rho,\mu}^+[\emptyset], S_{\rho,\mu}, QT_\rho^+, D)$-enhanced $2(\mu)^{\oplus 2}$-homogeneous. Using Lemma 4.92 we construct $\bar{q} = (1 - \sqrt{2(\mu)^{\oplus 2}})n/\tilde{p} \geq (1 - (\mu)^{\oplus 3})n/\tilde{p} = \hat{q}$ μ-covers $P_1, \dots, P_{\hat{q}}$ such that the sets

$$P_1^*, \dots, P_{\hat{q}}^* \subseteq V_1[(N_{\rho,\mu}^+[\emptyset], S_{\rho,\mu}, QT_\rho^+, D)] \subseteq V_1[(N_{\rho,\mu}^+[\emptyset], S_{\rho,\mu}, C_\rho, D)]$$

are disjoint. By Lemma 4.94 it follows that $P_1^*, \dots, P_{\hat{q}}^*$ induce (R_ρ^5)-$(\nu^{1/20}, \sigma^2)$-enhanced μ-cover families. Hence a set GV_1 with $|GV_1| \geq (1 - 10(\mu)^{\oplus 3})n \geq (1 - \gamma)n$ exists as indicated in Corollary 4.91. For a vertex $v \in GV_1$ and an arbitrary set $A \subseteq \Gamma_j(v)$ with $|A| \leq \mu^2 q$ we can find $w \in \Gamma_j(v) \setminus A$ and $R \subseteq \Gamma_k(v) \cap \Gamma_k(w)$ such that

$$|SQC(w, R, v) \setminus f_v^{SQU}(A)| \geq t_\rho \geq t_{\mu^2}$$

due to (4.46). Note that $SQC(w, R, v) \subseteq CC_5(v, w)$ and $CC_5(v, A) = f_v^{SQU}(A)$, provided that $|CC_5(v, A)| \leq (1 - 2\mu)n$. Thus, for every vertex $v \in TV_1$ we either have $|CC_5(v, A)| \geq (1 - 2\mu)n \geq (1 - \xi)n$ or

$$|CC_5(v, w) \setminus CC_5(v, A)| \geq |SQC(w, R, v) \setminus f_v^{SQU}(A)| \geq t_{\mu^2}.$$

Consequently, the vertices in GV_1 satisfy the conditions given in Definition 4.97, and V_1 is (γ, μ^2)-clique candidate rich. By Lemma 4.98 the claim follows. $\square$

Now the proof of Theorem 4.3 is essentially complete for $H = K_5$. Note that by Lemma 4.62, Lemma 4.64, (4.26), (4.29), Lemma 4.83, (4.48) and Lemma 4.96,

$$|\mathcal{S}_5^{QT+N+SR}(n, m; \varepsilon, \rho, \mu)| \geq (1 - (10(\xi)^{\oplus})^m) \binom{n^2}{m}^{10} \tag{4.50}$$

By Lemma 4.100 and Lemma 4.96 at most $(11(\xi)^{\oplus})^m \binom{n^2}{m} \leq \beta^m \binom{n^2}{m}$ graphs in $\mathcal{S}_5(n, m; \varepsilon)$ exist which do not contain a subgraph K_5.

Note that our proof actually yields a somewhat stronger result than Theorem 4.3. The condition (K) could be strengthened to

$$(\neg K') \quad |\Gamma_\ell(v) \cap CC_\ell(v, \Gamma_j(v))| \leq (1 - \tau)q$$

for a suitable constant $\tau > 0$ which is large in comparison to ξ. Using this we obtain that almost all neighbors $x \in \Gamma_\ell(v)$ of almost all clique covering vertices are part of a subgraph K_ℓ together with v.

4.8.16 Handling the dependencies

Up to now we have neglected for which partitions V_i, V_j, V_k the above defined properties shall hold. Instead we have simply assumed that the properties were applicable whenever we needed them. This section is devoted to an explicit treatment of this issue.

Note that we must not assume that the parameter i, j and k are all-quantified, even if this introduced only a tiny number of choices which would be negligible in the above counting arguments. However, this would cause certain properties to depend on edges which we have assumed to be still unfixed in our proofs, in particular in applications of Lemma 4.38.

In the sequel we will go through all properties which occur inside the definition of other properties and we will state for which choices of i, j and k we assume the properties to hold.

Case $H = K_4$

In the proof of the case $H = K_4$ only the property $(C_\rho^+) = (C_\rho \mid C_\rho)$ is based on another property. Here the property (C_ρ) on the righthand side will be used for $i = 3$, $j = 4$ and $k = 2$, whereas (C_ρ) on the lefthand side is applied for $i = 1$, $j = 3$ and $k = 4$.

Case $H = K_5$

Firstly, we consider the overlapping neighborhoods. Inside $(N_{\rho,\sigma})$, (H_ρ)-qualified covers occur. Here (H_ρ) is assumed to be satisfied for $i = 4$, $j = 5$, $k = 2$. Later we derive $(N_{\rho,\mu}^+)$ from $(N_{\rho,\mu})$. Here (HT_ρ^+)-qualified covers occur. Recall that $(HT_\rho^+) = (H_\rho \mid HT_\rho)$. Let us first examine the property $(HT_\rho) = (H_\rho, T_\rho)$. Here (H_ρ) is assumed to be satisfied for $i = 4$, $j = 5$, $k = 2$ (this must be identical to the values of (H_ρ) used in $(N_{\rho,\sigma})$). The property (T_ρ) shall hold for $i = 4$, $j = 2$ and $k = 5$.

We apply $(N_{\rho,\sigma})$ for $i = 3$, $j = 5$, $k = 4$, and $(N_{\rho,\mu}^+)$ is used with $i = 1$, $j = 4$, $k = 3$. Note that (H_ρ) used in $(N_{\rho,\sigma})$ does not depend on the edges $E(V_3, V_5)$, and (HT_ρ^+) used in $(N_{\rho,\mu}^+)$ does not depend on $E(V_1, V_4)$. Thus in both cases the edges $E(V_i, V_j)$ for $(i, j) = (3, 5)$ resp. $(i, j) = (1, 4)$ are *not* necessary to determine which vertices satisfy these auxiliary properties. Consequently, the application of Lemma 4.38 in the proof of Lemma 4.66 (and its modified variant for $(N_{\rho,\mu}^+)$ which we have not stated explicitly) goes through.

In the application of the properties $(N_{\rho,\sigma})$ and $(N_{\rho,\mu}^+)$, and thus of Definition 4.65, we must specify for which partitions the sets R_u are supercovers. In the case of $(N_{\rho,\sigma})$ we assume that the sets R_u are supercovers of V_2. For $(N_{\rho,\mu})^+$ we need that the sets R_u are supercovers of V_5 and V_2. The important observation is that in neither case the edges $E(V_1, V_3)$ are involved. Consequently, the definition of cocovers (and the auxiliary property $(\neg CO)$) does not depend on the edges $E(V_1, V_3)$. Hence, the proof of Lemma 4.79 goes through as intended.

Bibliography

[ADL+92] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma (extended abstract). In *Proc. 33rd Ann. IEEE Symp. on Foundations of Computer Science*, pages 473–481. IEEE, 1992. 48

[BE73] B. Bollobás and P. Erdős. On the structure of edge graphs. *Bull. London Math. Soc.*, 5:317–321, 1973. 31

[Beh02] M. Behrisch. Random graphs without a short cycle. Master's thesis, Humboldt-Universität zu Berlin, 2002. (in German). 59, 60, 61

[Bol78] B. Bollobás. *Extremal Graph Theory*. Academic Press, London, 1978.

[Bol85] B. Bollobás. *Random Graphs*. Academic Press, London, 1985. 1

[Bol98] B. Bollobás. *Modern Graph Theory*. Springer, 1998. 46, 48

[Chu91] F. R. K. Chung. Improved separators for planar graphs. *Graph Theory, Combinatorics and Applications*, 1:265–270, 1991. 48

[DF89] M. Dyer and A. Frieze. The solution of some random $\mathcal{NP}$-hard problems in polynomial expected time. *Journal of Algorithms*, 10:451–489, 1989. 13

[Die97] R. Diestel. *Graph Theory*. Springer, 1997. 46, 48, 50, 51

[EFR86] P. Erdős, P. Frankl, and V. Rödl. The asymptotic enumeration of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. *Graphs and Combinatorics*, 2:113–121, 1986. 13

[EKR76] P. Erdős, D. J. Kleitman, and B. L. Rothschild. Asymptotic enumeration of K_n-free graphs. In *International Colloquium on Combinatorial Theory*, volume 2 of *Atti dei Convegni Lincei*, pages 19–27, Roma, 1976. 2, 7, 9, 11, 12, 23

[ER60] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci.*, 5:17–61, 1960. 1, 14

[Erd59] P. Erdős. Graphs theory and probability. *Canad. J. Math.*, 11:34–38, 1959. 7

[ES46] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1089–1091, 1946. 51

[ES66] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966. 51

[ET36] P. Erdős and P. Turán. On some sequences of integers. *Journal of the London Mathematical Society*, 11(2):261–264, 1936. 46

[FK96] A. Frieze and R. Kannan. The regularity lemma and approximation schemes for dense problems. In *Proc. 37th Ann. IEEE Symp. on Foundations of Computer Science*. IEEE, 1996. 48

[FR86] P. Frankl and V. Rödl. Large triangle-free subgraphs in graphs without K_4. *Graphs and Combinatorics*, 2:135–144, 1986. 60

[FR92] P. Frankl and V. Rödl. The uniformity lemma for hypergraphs. *Graphs and Combininatorics*, 8:309–312, 1992. 48

[Für94] Z. Füredi. Random Ramsey graphs for the four-cycle. *Discrete Mathematics*, 126:407–410, 1994. 60, 61

[GPS$^+$02] S. Gerke, H. J. Prömel, T. Schickinger, A. Steger, and A. Taraz. K_4-free subgraphs of random graphs revisited. submitted for publication, 2002. 44

[HKŁ95] P. E. Haxell, Y. Kohayakawa, and T. Łuczak. Turán's extremal problem in random graphs: forbidding even cycles. *Journal of Combinatorial Theory B*, 64:273–287, 1995. 54, 61

[HKŁ96] P. E. Haxell, Y. Kohayakawa, and T. Łuczak. Turán's extremal problem in random graphs: forbidding odd cycles. *Combinatorica*, 16(1):107–122, 1996. 54, 61

[HPS93] C. Hundack, H. J. Prömel, and A. Steger. Extremal graph problems for graphs with a color-critical vertex. *Combinatorics, Probability, and Computing*, 2:465–477, 1993. 13

[JŁR90] S. Janson, T. Łuczak, and A. Ruciński. An exponential bound for the probability of nonexistence of a specified subgraph in a random graph. In M. Karoński, J. Jaworski, and A. Ruciński, editors, *Random Graphs '87*, pages 73–87, Chichester, 1990. John Wiley & Sons. 58

[JŁR00] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs*. John Wiley & Sons, 2000. 1, 3, 43, 48, 53, 58, 59, 60

[KK97] Y. Kohayakawa and B. Kreuter. Threshold functions for asymmetric Ramsey properties involving cycles. *Random Structures & Algorithms*, 11:245–276, 1997. 61

[KKS98] Y. Kohayakawa, B. Kreuter, and A. Steger. An extremal problem for random graphs and the number of graphs with large even girth. *Combinatorica*, 18:101–120, 1998. 61

[KŁR96] Y. Kohayakawa, T. Łuczak, and V. Rödl. Arithmetic progressions of length three in subsets of a random set. *Acta Arithmetica*, LXXV.2:133–163, 1996. 82

[KŁR97] Y. Kohayakawa, T. Łuczak, and V. Rödl. On K^4-free subgraphs of random graphs. *Combinatorica*, 17(2):173–213, 1997. 2, 43, 44, 54, 56, 59, 61

[Koh97] Y. Kohayakawa. Szemerédi's regularity lemma for sparse graphs. In F. Cucker and M. Shub, editors, *Foundations of Computational Mathematics*, pages 216–230, Berlin, Heidelberg, 1997. Springer-Verlag. 43, 46, 52, 54

[KPR87] Ph. G. Kolaitis, H. J. Prömel, and B. L. Rothschild. K_{l+1}-free graphs: Asymptotic structure and a 0–1 law. *Trans. Amer. Math. Soc.*, 303:637–671, 1987. 2, 8, 11, 12, 13, 31

[KR70] D. J. Kleitman and B. L. Rothschild. The number of finite topologies. *Proc. Amer. Math. Soc.*, 25:276–282, 1970. 10

[KR75] D. J. Kleitman and B. L. Rothschild. Asymptotic enumeration of partial orders on a finite set. *Trans. Amer. Math. Soc.*, 205:205–220, 1975. 10

[KR00] Y. Kohayakawa and V. Rödl. Algorithmic aspects of regularity. In G. Gonnet, D. Panario, and A. Viola, editors, *LATIN'2000*, volume 1776 of *Lecture Notes in Computer Science*, pages 1–17, Berlin, 2000. Springer. 48

[KR01] Y. Kohayakawa and V. Rödl. Regular pairs in sparse random graphs I. Submitted for publication, 2001. 55, 56, 61

[Kre97] B. Kreuter. *Probabilistic versions of Ramsey's and Turán's theorems*. PhD thesis, Humboldt-Universität zu Berlin, 1997. 61

[KRS02] Y. Kohayakawa, V. Rödl, and M. Schacht. The Turán Theorem for random graphs. Submitted for publication, 2002. 61

[KS96] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In D. Miklós, V. T. Sós, and T. Szőnyi, editors, *Combinatorics – Paul Erdős is eighty*, volume 2 of *Bolyai Society Mathematical Studies*, pages 295–352, Budapest, 1996. János Bolyai Mathematical Society. 46, 48

[KW82] D. J. Kleitman and K. J. Winston. On the number of graphs without 4-cycles. *Discrete Mathematics*, 41:167–172, 1982. 60

[LR84] E. Lamken and B. L. Rothschild. The number of odd–cycle–free graphs. In A. Hajnal, L. Lovász, and V.T. Sós, editors, *Colloq. Math. Soc. János Bolyai*, volume 37, pages 547–553, North–Holland, Amsterdam, 1984. 11, 13

[Łuc00] T. Łuczak. On triangle-free random graphs. *Random Structures & Algorithms*, 16:260–276, 2000. 2, 14, 15, 16, 46, 57, 59

[OPT] D. Osthus, H. J. Prömel, and A. Taraz. For which densities are random triangle-free graphs almost surely bipartite. to appear in Combinatorica. 14, 15

[Prö86] H. J. Prömel. Asymptotic enumeration of ℓ-colorable graphs. Technical Report 85405-OR, Universität Bonn, 1986. 10

[PS91] H. J. Prömel and A. Steger. Excluding induced subgraphs I: quadrilaterals. *Random Structures & Algorithms*, 2:55–71, 1991. 13

[PS92a] H. J. Prömel and A. Steger. Almost all Berge graphs are perfect. *Combinatorics, Probability and Computing*, 1:53–79, 1992. 11, 13

[PS92b] H. J. Prömel and A. Steger. The asymptotic number of graphs not containing a fixed color-critical subgraph. *Combinatorica*, 12:463–473, 1992. 13

[PS92c] H. J. Prömel and A. Steger. Coloring K_{l+1}-free graphs in linear expected time. *Random Structures & Algorithms*, 3:374–402, 1992. 11, 13

[PS92d] H. J. Prömel and A. Steger. Excluding induced subgraphs III: a general asymptotic. *Random Structures & Algorithms*, 3:19–31, 1992. 13

[PS93a] H. J. Prömel and A. Steger. Asymptotic structure of H-free graphs. In P.Seymour N.Robertson, editor, *Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference*, volume 147 of *Contemporary Mathematics*, pages 167–178. American Mathematical Society, 1993. 13

[PS93b] H. J. Prömel and A. Steger. Excluding induced subgraphs II: extremal graphs. *Discrete Applied Mathematics*, 44:283–294, 1993. 13

[PS95] H. J. Prömel and A. Steger. Random l-colorable graphs. *Random Structures & Algorithms*, 6(1):21–37, 1995. 10

[PS96a] H. J. Prömel and A. Steger. Counting H-free graphs. *Discrete Mathematics*, 154:311–315, 1996. 58

[PS96b] H. J. Prömel and A. Steger. On the asymptotic structure of sparse triangle-free graphs. *Journal of Graph Theory*, 21:137–151, 1996. 10, 14

[PSS01] H. J. Prömel, T. Schickinger, and A. Steger. On the structure of clique-free graphs. *Random Structures & Algorithms*, 19(1):37–53, 2001. 8

[PSS02] H. J. Prömel, T. Schickinger, and A. Steger. A note on triangle-free and bipartite graphs. *Discrete Mathematics*, 2002. Special issue in honor of the 65th birthday of Daniel J. Kleitman, to appear. 8

[PST01] H. J. Prömel, A. Steger, and A. Taraz. Asymptotic enumeration, global structure and constrained evolution. *Discrete Mathematics*, 229:213–233, 2001. 11, 15, 16

[Ste90] A. Steger. *The Kleitman-Rothschild method*. PhD thesis, Universität Bonn, 1990. (in German). 48

[Ste93] A. Steger. Asymptotic properties of H-free graphs, 1993. Habilitationsschrift, Universität Bonn. 11, 16

[Ste99] A. Steger. unpublished manuscript. Technische Universität München, 1999. 15

[Sze75] E. Szemerédi. On sets of integer sets containing no k elements in arithmetic progression. *Acta Arithmetica*, 27:199–245, 1975. 46

[Sze76] E. Szemerédi. Regular partitions of graphs. In *Problèmes Combinatoires et Théorie des Graphes*, volume 260 of *Colloques Internationaux CNRS*, pages 399–401, 1976. 3, 43, 46, 47

[Tur41] P. Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436–452, 1941. (in Hungarian). 50

[Wil84] H. S. Wilf. Backtrack: an $\mathcal{O}(1)$ expected time algorithm for the graph coloring problem. *Inform. Process. Lett.*, 18(119-121), 1984. 13

Index