Maximum Ratio Combining of Correlated Rayleigh Fading Channels With Imperfect Channel Knowledge

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***Abstract*—The bit error probability for binary modulation and multiple correlated Rayleigh fading diversity branches is derived. The receiver performs maximum ratio combining of the diversity branches based on noisy channel estimates. Our results provide new analytical insights into performance, design, and optimization of some known communication receivers.**

***Index Terms*—Bit error probability (BEP), correlated channel, imperfect channel knowledge, maximum ratio combining (MRC), Rayleigh fading.**

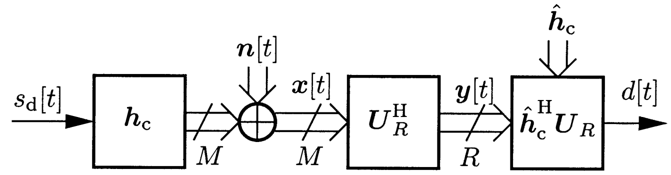

Fig. 1. MRC of R diversity branches with imperfect channel knowledge (ML channel estimates) after decorrelation and dimension reduction to the R dominant signal dimensions (cf. (2)).

I. Introduction

Very often it is assumed for performance analysis of communication systems, that channel coefficients are uncorrelated and identically distributed. However, in many practical cases communication channels have correlated coefficients or coefficients with distinct second order moments. Examples are the spatial channel at an antenna array with half-wavelength spacing or a channel impulse responses with exponentially decaying power delay spectrum. There exist receiver architectures for CDMA systems, which exploit correlations in the channel to reduce receiver complexity and to improve channel estimation, e.g., a temporal rake, space-time rake, space-time eigenrake [1]–[3], or multislot rank reduced channel estimation [4]. An analysis of their performance in terms of bit error probability (BEP) has to take into account channel estimation errors to capture significant performance issues. An approximation of the BEP using previous results for identical eigenvalues of the channel correlation matrix [5], i.e., uncorrelated diversity branches, is not possible and sufficient as distinct eigenvalues result in new performance issues [3].

In this contribution we restrict the analysis on binary signals (BPSK), which are transmitted over multiple correlated Rayleigh fading diversity paths. The receiver performs maximum-likelihood (ML) channel estimation to optimally combine the received signals using maximum ratio combining (MRC).

To obtain the BEP for this case we use Turin's result on the characteristic function of real Hermitian quadratic forms [6]. It was first used by [7]–[9] for MRC of BPSK signals and uncorrelated diversity branches with identical distribution. Proakis summarized the results in [5] and gave an alternative derivation for this case. In [10] he followed a different approach in order to describe the performance of coherent reception of M-PSK signals in the presence of channel estimation errors.

Based on the system model in Section II, we review the case of perfect channel knowledge (Section III). The derivation of the BEP with imperfect channel knowledge is given in Section IV. The significance of these results is illustrated in an example (see Section V).

Notation

$\mathbf{1}_M$ denotes the $M \times M$ identity matrix, $\hat{\boldsymbol{a}}$ an estimate of the random vector $\boldsymbol{a}$, and $\boldsymbol{e}_m$ the mth column of an identity matrix. $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{H}}$ are the transpose and complex-conjugate transpose of a matrix. $\mathrm{E}[\bullet]$ is the expectation operator and $\mathrm{diag}([\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M])$ a block-diagonal matrix with the matrices $\boldsymbol{A}_i$ on its diagonal.

II. Model

After a block of N pilot symbols $s_{\mathrm{p}}[t]$ with variance $\mathrm{E}[|s_{\mathrm{p}}[t]|^2] = P_{\mathrm{p}}$, BPSK data symbols $s_{\mathrm{d}}[t] \in \{-\sqrt{P_{\mathrm{d}}}, +\sqrt{P_{\mathrm{d}}}\}$ with $\mathrm{E}[|s_{\mathrm{d}}[t]|^2] = P_{\mathrm{d}}$ are transmitted over M correlated spatio-temporal diversity paths with $\boldsymbol{h}_{\mathrm{c}} \sim \mathcal{N}_{\mathrm{c}}(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{h}_{\mathrm{c}}})$, i.e., zero mean circularly symmetric Gaussian distribution with $\boldsymbol{R}_{\boldsymbol{h}_{\mathrm{c}}} = \mathrm{E}[\boldsymbol{h}_{\mathrm{c}}\boldsymbol{h}_{\mathrm{c}}^{\mathrm{H}}] = \boldsymbol{U}\boldsymbol{\Lambda}_M\boldsymbol{U}^{\mathrm{H}}$ (see Fig. 1). $\boldsymbol{h}_{\mathrm{c}}$ may consist of coefficients from the spatial as well as temporal channel domain. The eigenvalues λ_m, $m \in \{1, \ldots, M\}$, in $\boldsymbol{\Lambda}_M = \mathrm{diag}([\lambda_1, \ldots, \lambda_M])$ are distinct and sorted in decreasing order. The discrete-time received signal

$$\boldsymbol{x}[t] = \boldsymbol{h}_{\mathrm{c}} s_{\mathrm{d}}[t] + \boldsymbol{n}[t] \quad \in \mathbb{C}^M \tag{1}$$

is disturbed by additive complex Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathrm{c}}(\mathbf{0}, \sigma_{\mathrm{n}}^2\mathbf{1}_M)$. The case of colored noise can be treated applying a noise-whitening filter at the receiver.

To focus on the signal subspace containing most of the signal power, i.e., the subspace spanned by the R eigenvectors $\boldsymbol{U}_R$ corresponding to the largest eigenvalues of $\boldsymbol{R}_{\boldsymbol{h}_{\mathrm{c}}}$, some reduced

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rank receivers [2], [11] project the received signal $\boldsymbol{x}[t]$ on its basis. This yields a reduced number of R diversity branches in

$$\boldsymbol{y}[t] = \boldsymbol{U}_R^{\mathrm{H}}\boldsymbol{x}[t] \text{ with } \boldsymbol{U}_R = \boldsymbol{U}[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_R] \in \mathbb{C}^{M\times R} \quad (2)$$

and correlation matrix $\boldsymbol{R}_{\boldsymbol{y}} = P_{\mathrm{d}}\boldsymbol{\Lambda}_R + \sigma_{\mathrm{n}}^2\boldsymbol{1}_R$, which are decorrelated as needed for the derivations in Section IV. Note, that $\boldsymbol{y}[t]$ represents a sufficient statistic, if $\lambda_m = 0\ \forall\, R < m \le M$, and that for $R = M$ no information is lost, but uncorrelated signals in $\boldsymbol{y}[t]$ are obtained. Now, the signal $d[t]$ used for decision is obtained by MRC [12] of the remaining R decorrelated diversity branches

$$d[t] = \mathrm{Re}\left\{(\boldsymbol{U}_R^{\mathrm{H}}\hat{\boldsymbol{h}}_{\mathrm{c}})^{\mathrm{H}}\boldsymbol{y}[t]\right\} \quad (3)$$

based on estimated channel coefficients $\hat{\boldsymbol{h}}_{\mathrm{c}}$ transformed by $\boldsymbol{U}_R$. The coefficients are obtained from the N pilot symbols $s_{\mathrm{p}}[t]$ by ML channel estimation [13]

$$\hat{\boldsymbol{h}}_{\mathrm{c}} = \boldsymbol{h}_{\mathrm{c}} + \frac{1}{NP_{\mathrm{p}}}\sum_{t=1}^{N}\boldsymbol{n}[t]s_{\mathrm{p}}^*[t]. \quad (4)$$

Equation (3) can be rewritten as

$$\begin{aligned} d[t] &= \frac{1}{2}\left(\hat{\boldsymbol{h}}_{\mathrm{c}}^{\mathrm{H}}\boldsymbol{U}_R\boldsymbol{y}[t] + \hat{\boldsymbol{h}}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{U}_R^*\boldsymbol{y}[t]^*\right) \\ &= \sum_{m=1}^{R}\boldsymbol{v}_m^{\mathrm{H}}\boldsymbol{Q}_2\boldsymbol{v}_m = \boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}_{2R}\boldsymbol{v} \end{aligned} \quad (5)$$

with $\boldsymbol{v}_m = [\boldsymbol{u}_m^{\mathrm{H}}\hat{\boldsymbol{h}}_{\mathrm{c}} \quad y_m[t]]^{\mathrm{T}} = [\hat{h}_m \quad y_m[t]]^{\mathrm{T}}$, $\hat{h}_m \hat{=} \boldsymbol{u}_m^{\mathrm{H}}\hat{\boldsymbol{h}}_{\mathrm{c}}$, $\boldsymbol{v} = [\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_R^{\mathrm{T}}]^{\mathrm{T}}$, and $\boldsymbol{Q}_{2R} = 1/2\ (\boldsymbol{1}_R \otimes [\boldsymbol{e}_2, \boldsymbol{e}_1])$. $\boldsymbol{u}_m$ is the m-th column of $\boldsymbol{U}_R$. $\boldsymbol{v}$ is distributed as $\mathcal{N}_{\mathrm{c}}(\boldsymbol{0}, \boldsymbol{L})$ with correlation matrix $\boldsymbol{L} = \mathrm{E}[\boldsymbol{v}\boldsymbol{v}^{\mathrm{H}}] = \mathrm{diag}([\boldsymbol{L}_1, \ldots, \boldsymbol{L}_R])$ and

$$\begin{aligned} \boldsymbol{L}_m &= \begin{bmatrix} \mathrm{E}[|\hat{h}_m|^2] & \mathrm{E}[\hat{h}_m y_m^*[t]] \\ \mathrm{E}[\hat{h}_m^* y_m[t]] & \mathrm{E}[|y_m[t]|^2] \end{bmatrix} \\ &= \begin{bmatrix} \lambda_m + \frac{\sigma_{\mathrm{n}}^2}{NP_{\mathrm{p}}} & \sqrt{P_{\mathrm{d}}}\,\lambda_m \\ \sqrt{P_{\mathrm{d}}}\,\lambda_m & P_{\mathrm{d}}\,\lambda_m + \sigma_{\mathrm{n}}^2 \end{bmatrix} \end{aligned} \quad (6)$$

if $s_{\mathrm{d}}[t] = +\sqrt{P_{\mathrm{d}}}$ is the transmitted symbol.

III. BEP for Perfect Channel Knowledge

The bit error probability P_{b} for BPSK after maximum ratio combining of R correlated Rayleigh fading diversity paths, when the channel is known perfectly at the receiver, i.e., $N \to \infty$ or $P_{\mathrm{p}} \to \infty$, is (cf. [12], [14])

$$P_{\mathrm{b}} = \frac{1}{2}\sum_{m=1}^{R}\left(\prod_{\substack{k=1\\k\neq m}}^{R}\left(1 - \frac{\lambda_k}{\lambda_m}\right)\right)^{-1}\left[1 - \left(\sqrt{1 + \frac{1}{\lambda_m\gamma_{\mathrm{d}}}}\right)^{-1}\right]. \quad (7)$$

$\gamma_{\mathrm{d}} = P_{\mathrm{d}}/\sigma_{\mathrm{n}}^2$ is the ratio of transmit symbol power and noise variance. For simplicity, R distinct eigenvalues λ_m are assumed. The general case, where some eigenvalues are equal, can be treated equivalently. The case of R identical eigenvalues corresponds to a probability density function of $d[t]$, which is the Nakagami-m distribution more commonly known as χ_{2R}^2-distribution ($2R$ degrees of freedom). The solution is treated in every standard communication text book, e.g., [12], or in a unified way in [15].

IV. BEP for Imperfect Channel Knowledge

The characteristic function of the real quadratic form (5) is given in [6]. As $\boldsymbol{v}$ is zero mean it simplifies considerably. The characteristic function of $d[t]$ reads as[1]

$$\begin{aligned} \psi(\omega) &= \mathrm{E}[\mathrm{e}^{\mathrm{j}\omega d}] = \int_{\boldsymbol{v}} p(\boldsymbol{v})\exp(\mathrm{j}\omega\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}_{2R}\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \quad (8) \\ &= \prod_{m=1}^{R}\frac{1}{(1 - \mathrm{j}\omega\xi_m^+)(1 - \mathrm{j}\omega\xi_m^-)}, \quad (9) \end{aligned}$$

where $\xi_m^+ \ge 0$ and $\xi_m^- \le 0$ are the eigenvalues of $\boldsymbol{L}_m\boldsymbol{Q}_2$. For $s_{\mathrm{d}}[t] = +\sqrt{P_{\mathrm{d}}}$ they are given as

$$\xi_m^{\pm} = \sqrt{P_{\mathrm{d}}}\frac{\lambda_m}{2} \pm \frac{1}{2}\sqrt{\left(\lambda_m + \frac{\sigma_{\mathrm{n}}^2}{NP_{\mathrm{p}}}\right)(P_{\mathrm{d}}\lambda_m + \sigma_{\mathrm{n}}^2)}. \quad (10)$$

The inverse Fourier transform can be computed with the help of the partial fraction expansion of (9). If the eigenvalues λ_m are distinct and $NP_{\mathrm{p}} \neq P_{\mathrm{d}}$, which is the case in practice, the denominator polynomial in (9) has distinct zeros. The partial fraction expansion in this case yields

$$\psi(\omega) = \sum_{m=1}^{R}\left(\frac{A_m}{1 - \mathrm{j}\omega\xi_m^+} + \frac{B_m}{1 - \mathrm{j}\omega\xi_m^-}\right) \quad (11)$$

with coefficients

$$\begin{aligned} A_m &= \prod_{\substack{i=1\\i\neq m}}^{R}\frac{1}{1 - \frac{\xi_i^+}{\xi_m^+}}\prod_{i=1}^{R}\frac{1}{1 - \frac{\xi_i^-}{\xi_m^+}}, \\ B_m &= \prod_{i=1}^{R}\frac{1}{1 - \frac{\xi_i^+}{\xi_m^-}}\prod_{i=1\, i\neq m}^{R}\frac{1}{1 - \frac{\xi_i^-}{\xi_m^-}}. \end{aligned} \quad (12)$$

The probability density function of the decision variable d conditioned on a transmitted data symbol $s_{\mathrm{d}} = +\sqrt{P_{\mathrm{d}}}$ is given by the inverse Fourier transform of the characteristic function

$$\mathrm{p}(d|s_{\mathrm{d}} = +\sqrt{P_{\mathrm{d}}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\omega)\exp(-\mathrm{j}\,\omega\,d)\,\mathrm{d}\omega, \quad (13)$$

which is

$$\begin{aligned} &\mathrm{p}(d|s_{\mathrm{d}} = +\sqrt{P_{\mathrm{d}}}) \\ &\quad = \sum_{m=1}^{R}\frac{A_m}{\xi_m^+}\mathrm{e}^{-d/\xi_m^+}\,\Theta(d) + \frac{B_m}{-\xi_m^-}\mathrm{e}^{-d/\xi_m^-}\,\Theta(-d) \end{aligned}$$

with $\Theta(d) = 1$, if $d > 0$, and $\Theta(d) = 0$, elsewhere. The probability of a bit error $\mathrm{P_b}$ for equally probable transmitted symbols, i.e., $\mathrm{p}(s_{\mathrm{d}} = -\sqrt{P_{\mathrm{d}}}) = \mathrm{p}(s_{\mathrm{d}} = +\sqrt{P_{\mathrm{d}}}) = 0.5$, and a maximum *a posteriori* decision device is (compare e.g., [12])

$$\begin{aligned} P_{\mathrm{b}} &= \int_{-\infty}^{0}\mathrm{p}(\delta|s_{\mathrm{d}} = +\sqrt{P_{\mathrm{d}}})\,\mathrm{d}\delta \quad (14) \\ &= \sum_{m=1}^{R}\frac{B_m}{-\xi_m^-}\left[-\xi_m^-\mathrm{e}^{-\delta/\xi_m^-}\right]_{\delta=-\infty}^{0} = \sum_{m=1}^{R}B_m. \quad (15) \end{aligned}$$

Using (12) and (10) the BEP can be nicely written in terms of key system parameters which are the eigenvalues $\{\lambda_m\}$ of the

[1] In the sequel we omit the time indices in our notation.

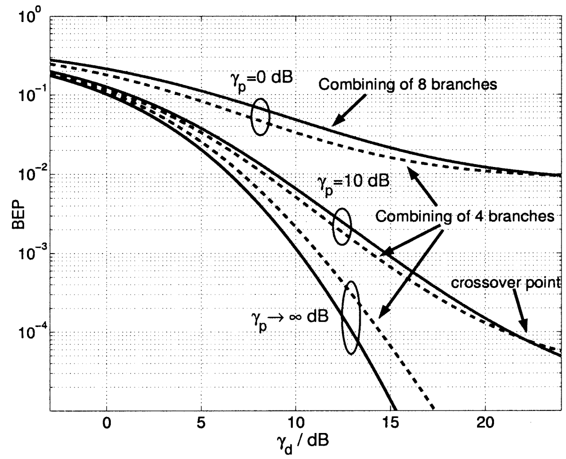

Fig. 2. MRC of $M=8$ temporal diversity branches using estimated channel coefficients ($N=6$). We compare the selection of all eight branches ($R=8$, solid line) with the selection of only $R=4$ branches (dashed line) for different SNR $\gamma_{\mathrm{p}} \in \{0\,\mathrm{dB}, 10\,\mathrm{dB}, \infty\,\mathrm{dB}\}$ in the pilot sequence.

channel correlation matrix $\boldsymbol{R}_{\boldsymbol{h}_{\mathrm{c}}}$, the ratio of transmit power and noise variance of the data channel $\gamma_{\mathrm{d}} = P_{\mathrm{d}}/\sigma_{\mathrm{n}}^2$ and of the pilot channel $\gamma_{\mathrm{p}} = NP_{\mathrm{p}}/\sigma_{\mathrm{n}}^2$, respectively. Insights on their influence on the system can be obtained from this result. See

$$\begin{aligned} &\mathrm{P_b}\left(\{\lambda_m\}, \gamma_{\mathrm{p}}, \gamma_{\mathrm{d}}\right) \\ &= \sum_{m=1}^{R} \left[\prod_{i=1}^{R} \left(1 - \frac{\lambda_i \left(1 + \sqrt{\left(1+\frac{1}{\lambda_i \gamma_{\mathrm{p}}}\right)\left(1+\frac{1}{\lambda_i \gamma_{\mathrm{d}}}\right)}\right)}{\lambda_m \left(1 - \sqrt{\left(1+\frac{1}{\lambda_m \gamma_{\mathrm{p}}}\right)\left(1+\frac{1}{\lambda_m \gamma_{\mathrm{d}}}\right)}\right)} \right) \right. \\ &\quad \left. \times \prod_{\substack{i=1 \\ i\neq m}}^{R} \left(1 - \frac{\lambda_i \left(1 - \sqrt{\left(1+\frac{1}{\lambda_i \gamma_{\mathrm{p}}}\right)\left(1+\frac{1}{\lambda_i \gamma_{\mathrm{d}}}\right)}\right)}{\lambda_m \left(1 - \sqrt{\left(1+\frac{1}{\lambda_m \gamma_{\mathrm{p}}}\right)\left(1+\frac{1}{\lambda_m \gamma_{\mathrm{d}}}\right)}\right)} \right) \right]^{-1}. \end{aligned} \tag{16}$$

V. Applications

The performance of a temporal rake [16], which performs maximum ratio combining of the delayed signals after despreading, is evaluated. We assume perfect autocorrelation properties of the spreading sequence. Thus, the temporal diversity branches of the channel can be resolved perfectly. The taps are uncorrelated (uncorrelated scattering) and their power distribution decays exponentially, i.e., $\boldsymbol{U}_M = \mathbf{1}_M$ and

$$\boldsymbol{\Lambda}_M = \mathrm{diag}([0.5, 0.25, 0.13, 0.06, 0.03, 0.016, 0.008, 0.004]).$$

Using (7) and (16) we can compute the BEP for this case (see Fig. 2). Due to hardware constraints in practice a rake selects (temporal rank reduction) and combines the – on average – strongest taps only as described in (2). In Fig. 2 the influence on performance is compared, when all 8 diversity branches ($R = M = 8$) or just the strongest four ($R=4$) are selected and combined. In case of perfect channel knowledge ($\gamma_{\mathrm{p}} \to \infty$ dB) BEP increases because of the neglected signal power in the remaining four eigenvalues. However, when only noisy channel estimates are available ($\gamma_{\mathrm{p}} \in \{0\,\mathrm{dB}, 10\,\mathrm{dB}\}$), we observe, that the BEP is smaller for 4 selected branches due to the smaller number of channel coefficients, which have to be estimated. This tradeoff [11] between channel estimation quality, i.e., smaller variance of the channel estimate, and neglected signal power results in a crossover point for a certain SNR. For example, selecting the 4 most significant diversity branches is better than combining all 8 branches for an SNR $\gamma_{\mathrm{d}} \le 21$ dB in case of $\gamma_{\mathrm{p}} = 10$ dB, which can be described and optimized with (16).

Another application is the spatial eigenrake [1], [2]. For example, the signals received with an antenna array with half wavelength equidistant element spacing are often correlated, i.e., $\boldsymbol{h}_{\mathrm{c}}$ now models the spatial channel. Selecting the R eigenvectors $\boldsymbol{U}_R$ as in (2), which is beamforming with R beams, results in the same tradeoffs as for the temporal rake, which can now be described using the expressions from above ((16) and (7)).

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