Minimax Mean-Square-Error Transmit Wiener Filter

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***Abstract*— A minimax solution of the *transmit Wiener filter* (TxWF) allocating identical *mean-square-errors* (MSEs) to *all* users is derived for *multi-user multiple-input single-output* (MU-MISO) systems with transmit processing. The conventional TxWF minimizes the *sum* MSE of all users and features the possibility to allocate different MSEs to the individual users depending on the given channel realization. In contrast, the modified *minimax* version is characterized by an equal user treatment resulting from a minimization of the maximum single-user MSE. Although this fairness can also be achieved by a *transmit zero-forcing filter* (TxZF), the minimax TxWF meets this equality with a reduced sum MSE since no other constraint is active than the fairness. Simulation results reveal, that our new approach features a decreased outage probability compared to the conventional TxWF and the TxZF, i.e., *bit-error-rates* (BERs) exceeding a certain threshold become less likely. Moreover, the new criterion (equal single-user MSEs) translates itself into nearly identical BERs to all users for *every* channel realization.**

I. Introduction

In order to combat the negative influence of the multi-user interference in the downlink of a multi-user system with multiple transmit antennas, several filter strategies have been transferred from the receiver to the transmitter. Especially for systems with decentralized receivers, where the individual users cannot combine their received signals, precoding can be applied when *channel state information* (CSI) is available at the transmitter side as it is in *time division duplex* (TDD) systems. Filter types that have been transferred are for example the *transmit matched filter* (TxMF) [1], [2] maximizing the similarity between the desired symbols and the processed received symbols, and the *transmit zero-forcing* filter (TxZF) [3], [4] which completely eliminates the multi-user interference and leads to an equal user treatment. The conventional *transmit Wiener filter* (TxWF) [5], [6] minimizes the *sum* MSE assigning different link qualities to the users.

We present a modified version of the conventional TxWF suspending this unequal treatment of the individual users and simultaneously assuring smaller MSE values than its *zero-forcing* counterpart. This is achieved by dropping the complete interference removal constraint and replacing it by the *identical MSEs* constraint.

This paper is organized as follows: In Section II, we discuss the system model underlying all derivations and simulations, Section III briefly reviews the conventional TxWF and the TxZF. The analytical derivation of the *minimax* mean-square-error TxWF is dealt with in Section IV. Having presented some simulation results in Section V, this paper concludes in Section VI.

II. System Model and Notation

The downlink of the broadcast channel is illustrated in Fig. 1. The transmitter is equipped with N_a antenna elements and serves K decentralized receivers. We assume a frequency flat channel, hence the propagation can be described by the complex-valued $K \times N_\mathrm{a}$ channel matrix $\boldsymbol{H} = [\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K]^\mathrm{T}$. The channel coefficients $\boldsymbol{h}_k = [h_{k,1}, \ldots, h_{k,N_\mathrm{a}}]^\mathrm{T} \in \mathbb{C}^{N_\mathrm{a}}$ of user k are the realizations of independent zero-mean Gaussian random variables $\mathsf{h}_{k,n}$, $k \in \{1, \ldots, K\}$, and $n \in \{1, \ldots, N_\mathrm{a}\}$ with variance

$$\mathrm{E}_{\mathsf{h}}[|\mathsf{h}_{k,n}|^2] = \sigma^2_{\mathsf{h}_k} \;\forall n,$$

so the channel weight from each antenna to user k is assumed to exhibit the same average path loss. The long-term average channel power of user k herewith reads as

$$\mathrm{E}_{\mathsf{h}}\left[\|\boldsymbol{\mathsf{h}}_k\|_2^2\right] = N_\mathrm{a}\sigma^2_{\mathsf{h}_k},$$

and different average path losses resulting from different distances of the users to the base station manifest in different values $\sigma^2_{\mathsf{h}_k}$. We define the *mean channel power ratio* $MCPR_{k,\ell}$ as the ratio of the average channel powers of users k and ℓ, i.e.,

$$MCPR_{k,\ell} = \frac{\mathrm{E}_{\mathsf{h}}\left[\|\boldsymbol{\mathsf{h}}_k\|^2\right]}{\mathrm{E}_{\mathsf{h}}\left[\|\boldsymbol{\mathsf{h}}_\ell\|^2\right]}. \tag{1}$$

The channel coefficients in $\boldsymbol{H}$ are assumed to be perfectly known. Otherwise, a robust paradigm has to be applied [7]. The data symbols of the K users are stacked in $\boldsymbol{\mathsf{s}} = [\mathsf{s}_1, \ldots, \mathsf{s}_K]^\mathrm{T} \in \mathbb{C}^K$ and are precoded by the transmit filter $\boldsymbol{P} \in \mathbb{C}^{N_\mathrm{a} \times K}$. At the receiver side, Gaussian noise $\boldsymbol{\mathsf{n}} = [\mathsf{n}_1, \ldots, \mathsf{n}_K]^\mathrm{T} \in \mathbb{C}^K$ is added. The scalar weighting by $\beta^{-1} \in \mathbb{R}_+$ for all users allows for an amplitude correction necessary due to the limited transmit power at the sender.

Notation: Deterministic vectors and matrices are denoted by lower and upper case italic bold letters, whereas the respective random variables are written in sans serif font. The operators $\mathrm{E}[\cdot]$, $\mathrm{tr}(\cdot)$, $(\cdot)^\mathrm{H}$, $(\cdot)^\mathrm{T}$ stand for expectation with respect to symbols and noise, trace of a matrix, Hermitian transposition, and transposition, respectively. The scalar element in row b and column c of the matrix $\boldsymbol{A}$ is denoted by $[\boldsymbol{A}]_{b,c}$. The diag-operator stacks the diagonal elements $[\boldsymbol{A}]_{k,k}$ $(k \in \{1, \ldots, n\})$

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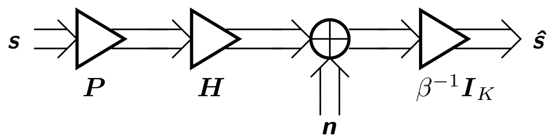

Figure 1. Downlink of multi-user *multiple-input single-output* (MU-MISO) system.

of an arbitrary $n \times n$ matrix $\boldsymbol{A}$ in a $n \times 1$ column vector, i.e., $\operatorname{diag}(\boldsymbol{A}) = [[\boldsymbol{A}]_{1,1}, \ldots, [\boldsymbol{A}]_{n,n}]^{\mathrm{T}}$. Expectation with respect to the channel is denoted by $\mathrm{E}_{\boldsymbol{h}}[\cdot]$, and the $K \times K$ identity matrix is denoted by $\boldsymbol{I}_K$ whose k-th column is $\boldsymbol{e}_k$.

III. Review of two common Transmit Filters

As depicted in Fig. 1, the estimate $\hat{\boldsymbol{s}}$ (at the input of the quantizer) for the transmitted symbols $\boldsymbol{s}$ can be written as

$$\hat{\boldsymbol{s}} = \beta^{-1}\boldsymbol{H}\boldsymbol{P}\boldsymbol{s} + \beta^{-1}\boldsymbol{n}. \tag{2}$$

The *sum* MSE ε including all K single-user MSEs ε_k herewith reads as

$$\begin{aligned}\varepsilon = \sum_{k=1}^{K} \varepsilon_k &= \operatorname{tr}\left(\mathrm{E}\left[(\boldsymbol{s}-\hat{\boldsymbol{s}})(\boldsymbol{s}-\hat{\boldsymbol{s}})^{\mathrm{H}}\right]\right) \\ &= \operatorname{tr}(\boldsymbol{R}_{\boldsymbol{s}}) - \beta^{-1}\operatorname{tr}(\boldsymbol{H}\boldsymbol{P}\boldsymbol{R}_{\boldsymbol{s}}) - \beta^{-1}\operatorname{tr}(\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{P}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}) \\ &\quad + \beta^{-2}\operatorname{tr}(\boldsymbol{H}\boldsymbol{P}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{P}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}) + \beta^{-2}\operatorname{tr}(\boldsymbol{R}_{\boldsymbol{n}}),\end{aligned} \tag{3}$$

where we make use of the Hermitian noise covariance matrix $\boldsymbol{R}_{\boldsymbol{n}} = \mathrm{E}[\boldsymbol{n}\boldsymbol{n}^{\mathrm{H}}] \in \mathbb{C}^{K\times K}$ and the Hermitian symbol covariance matrix $\boldsymbol{R}_{\boldsymbol{s}} = \mathrm{E}[\boldsymbol{s}\boldsymbol{s}^{\mathrm{H}}] \in \mathbb{C}^{K\times K}$. In order to be able to resolve the *single-user* MSEs ε_k, we define the MSE *vector*

$$\boldsymbol{\epsilon} = \operatorname{diag}\left(\mathrm{E}\left[(\boldsymbol{s}-\hat{\boldsymbol{s}})(\boldsymbol{s}-\hat{\boldsymbol{s}})^{\mathrm{H}}\right]\right) \in \mathbb{R}_{+}^{K}. \tag{4}$$

The MSE ε_k of user k corresponds to the k-th entry of $\boldsymbol{\epsilon}$, i.e., $\varepsilon_k = [\boldsymbol{\epsilon}]_{k,1}$.

A. Review of the Transmit Wiener Filter (TxWF)

The transmit Wiener filter minimizes the *sum* MSE ε subject to a power constraint at the transmitter side. Its optimization reads as [5], [6]

$$\{\boldsymbol{P}_{\mathrm{WF}}, \beta_{\mathrm{WF}}\} = \underset{\{\boldsymbol{P},\beta\}}{\operatorname{argmin}} \sum_{k=1}^{K} \varepsilon_k \quad \text{s.t.:} \quad \mathrm{E}\left[\|\boldsymbol{P}\boldsymbol{s}\|_2^2\right] \le E_{\mathrm{tr}}. \tag{5}$$

The constraint $\mathrm{E}\left[\|\boldsymbol{P}\boldsymbol{s}\|_2^2\right] \le E_{\mathrm{tr}}$ assures that the symbol-averaged transmit power does not exceed the maximum level E_{tr}. Inserting (3) into the optimization in (5) and solving for $\boldsymbol{P}$ and β by means of Lagrangian multipliers finally yields

$$\boldsymbol{P}_{\mathrm{WF}} = \beta_{\mathrm{WF}}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \frac{\operatorname{tr}(\boldsymbol{R}_{\boldsymbol{n}})}{E_{\mathrm{tr}}}\boldsymbol{I}_{N_{\mathrm{a}}}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}, \tag{6}$$

and β_{WF} is chosen such that the equality in the constraint in (5) holds. Note that different single-user MSEs $\varepsilon_{\mathrm{WF},k}$ are returned by the solution in (6) depending on the channel realization such that the *sum* MSE $\varepsilon_{\mathrm{WF}}$ is minimum. Moreover, (5) does not optimize the individual MSEs.

B. Review of the Transmit Zero-Forcing Filter (TxZF)

The transmit *zero-forcing* filter results from the TxWF by adding the unbiasedness and complete interference cancellation constraint, i.e. $\beta^{-1}\boldsymbol{H}\boldsymbol{P} = \boldsymbol{I}_K$. Thus, the *sum* MSE ε in (3) simplifies to $\varepsilon = \beta^{-2}\operatorname{tr}(\boldsymbol{R}_{\boldsymbol{n}})$. If degrees of freedom are available (i.e., when $N_{\mathrm{a}} > K$) having satisfied the unbiasedness constraint, this weighted trace is minimized:

$$\begin{aligned}\{\boldsymbol{P}_{\mathrm{ZF}}, \beta_{\mathrm{ZF}}\} &= \underset{\{\boldsymbol{P},\beta\}}{\operatorname{argmin}}\, \beta^{-2}\operatorname{tr}(\boldsymbol{R}_{\boldsymbol{n}}) \\ &\text{s.t.:}\;\; \beta^{-1}\boldsymbol{H}\boldsymbol{P} = \boldsymbol{I}_K \text{ and } \mathrm{E}\left[\|\boldsymbol{P}\boldsymbol{s}\|_2^2\right] \le E_{\mathrm{tr}}.\end{aligned} \tag{7}$$

The solution of (7) reads as

$$\begin{aligned}\boldsymbol{P}_{\mathrm{ZF}} &= \beta_{\mathrm{ZF}}\boldsymbol{H}^{\mathrm{H}}\left(\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}\right)^{-1}, \\ \beta_{\mathrm{ZF}} &= \sqrt{\frac{E_{\mathrm{tr}}}{\operatorname{tr}((\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}})^{-1}\boldsymbol{R}_{\boldsymbol{s}})}},\end{aligned} \tag{8}$$

leading to single-user MSEs

$$\varepsilon_{\mathrm{ZF},k} = \beta_{\mathrm{ZF}}^{-2}\sigma_{\boldsymbol{n}_k}^2, \tag{9}$$

with $\sigma_{\boldsymbol{n}_k}^2 = [\boldsymbol{R}_{\boldsymbol{n}}]_{k,k}$. If all users exhibit the same noise variance $\sigma_{\boldsymbol{n}_k}^2 = \sigma_{\boldsymbol{n}}^2$, identical single-user MSEs are allocated to them, i.e., $\varepsilon_{\mathrm{ZF},k} = \varepsilon_{\mathrm{ZF},\ell}\ \forall k,\ell$. However, the resulting *sum* MSE $\varepsilon_{\mathrm{ZF}}$ is lower bounded by that of the TxWF.

IV. Derivation of the Equal Mean-Square-Error Transmit Wiener Filter

Motivated by the advantages of the transmit filters presented in the previous two subsections, we are seeking for a precoding filter which combines the merits of both worlds, i.e., identical single-user MSEs ε_k *and* a small *sum* MSE ε. If all receiving antennas were located at the same position and could cooperatively process the received signals, identical MSEs for all substreams can be obtained in a joint fashion as shown in [8]. But since the users are decentralized, the assignment of identical MSEs has to be done by the precoder.

A. The Equal Mean-Square-Error Transmit Wiener Filter

We set up the following optimization for the *equal MSE* (eMSE) transmit Wiener filter:

$$\begin{aligned}\{\boldsymbol{P}_{\mathrm{eMSE}}, \beta_{\mathrm{eMSE}}\} &= \underset{\{\boldsymbol{P},\beta\}}{\operatorname{argmin}} \sum_{k=1}^{K} \varepsilon_k \\ &\text{s.t.:}\;\; \varepsilon_1 = \varepsilon_\ell, \quad \forall \ell \in \{2,\ldots,K\}, \\ &\text{and } \mathrm{E}\left[\|\boldsymbol{P}\boldsymbol{s}\|_2^2\right] \le E_{\mathrm{tr}}.\end{aligned} \tag{10}$$

The first constraint in (10) ensures that all users feature the same MSE. For the solution of above optimization, we set up the Lagrangian function $L(\boldsymbol{P},\beta,\lambda_0,\lambda_2,\ldots,\lambda_K)$:

$$\begin{aligned}L(\boldsymbol{P},\beta,\lambda_0,\lambda_2,\ldots,\lambda_K) &= \sum_{k=1}^{K}\varepsilon_k + \sum_{k=2}^{K}\lambda_k(\varepsilon_1 - \varepsilon_k) \\ &\quad + \lambda_0\Big(\operatorname{tr}(\boldsymbol{P}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{P}^{\mathrm{H}}) - E_{\mathrm{tr}}\Big),\end{aligned} \tag{11}$$

which can be transformed into

$$\begin{aligned}L(\cdot)=&\sum_{k=1}^{K}\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{\epsilon}+\sum_{k=2}^{K}\lambda_k(\boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_k^{\mathrm{T}})\boldsymbol{\epsilon}+\lambda_0\Big(\mathrm{tr}(\boldsymbol{P}\boldsymbol{R_s}\boldsymbol{P}^{\mathrm{H}})-E_{\mathrm{tr}}\Big)\\ =&\,\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{\epsilon}\left(1+\sum_{k=2}^{K}\lambda_k\right)+\sum_{k=2}^{K}(1-\lambda_k)\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{\epsilon}\\ &+\lambda_0\Big(\mathrm{tr}(\boldsymbol{P}\boldsymbol{R_s}\boldsymbol{P}^{\mathrm{H}})-E_{\mathrm{tr}}\Big)\,.\end{aligned} \tag{12}$$

From the derivatives of $L(\cdot)$ with respect to $\boldsymbol{P}$ and β, we obtain the precoding matrix depending on the Lagrangian factors $\lambda_2,\ldots,\lambda_K$:

$$\boldsymbol{P}_{\mathrm{eMSE}}=\beta_{\mathrm{eMSE}}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{T}\boldsymbol{H}+\frac{\mathrm{tr}(\boldsymbol{T}\boldsymbol{R_n})}{E_{\mathrm{tr}}}\boldsymbol{I}_{N_{\mathrm{a}}}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{T}, \tag{13}$$

where we utilized the $K\times K$ diagonal matrix $\boldsymbol{T}$, whose main diagonal elements read as

$$\begin{aligned}[\boldsymbol{T}]_{1,1}&=1+\sum_{k=2}^{K}\lambda_k\\ [\boldsymbol{T}]_{k,k}&=1-\lambda_k,\ k\in\{2,\ldots,K\},\end{aligned} \tag{14}$$

and $\mathrm{tr}(\boldsymbol{T})=K$, cf. (12). The scalar weight β_{eMSE} is chosen such that $\mathrm{tr}(\boldsymbol{P}_{\mathrm{eMSE}}\boldsymbol{R_s}\boldsymbol{P}_{\mathrm{eMSE}}^{\mathrm{H}})=E_{\mathrm{tr}}$ holds. For $k=2$ users, a closed form solution for the Lagrangian multiplier λ_2 can be obtained by finding the real-valued roots of a fourth-order polynomial, whose coefficients depend on channel matrix $\boldsymbol{H}$, the symbol and noise covariance matrices $\boldsymbol{R_s}$ and $\boldsymbol{R_n}$, and the available maximum transmit power E_{tr}. For more than two users, numerical methods have to be applied in order to find the $K-1$ Lagrangian factors.

B. Relation to the Weighted Minimum Mean-Square-Error Wiener Filter

The Lagrangian function for the optimization of the conventional TxWF in (5) can be expressed by

$$L(\boldsymbol{P},\beta,\lambda_0)=\sum_{k=1}^{K}\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{\epsilon}+\lambda_0\Big(\mathrm{tr}(\boldsymbol{P}\boldsymbol{R_s}\boldsymbol{P}^{\mathrm{H}})-E_{\mathrm{tr}}\Big)\,. \tag{15}$$

Taking a closer look at the Lagrangian function of the *equal MSE* filter in (12), we observe that the equal MSE filter has the same structure as the conventional TxWF, but a *weighted MSE sum* $\sum_{k=1}^{K}[\boldsymbol{T}]_{k,k}\varepsilon_k$ is minimized instead of the ordinary MSE sum $\sum_{k=1}^{K}\varepsilon_k$:

$$\begin{aligned}\boldsymbol{\epsilon}&\to\boldsymbol{T}\boldsymbol{\epsilon}\\ \varepsilon_k&\to[\boldsymbol{T}]_{k,k}\varepsilon_k.\end{aligned} \tag{16}$$

The weights $[\boldsymbol{T}]_{k,k}$ are chosen such that the resulting single-user MSEs ε_k are identical. Due to this similarity of the equal MSE filter to the weighted minimum MSE filter, the Lagrangian factors $\lambda_2,\ldots,\lambda_K$ have to be chosen from the open set

$$\lambda_k\in(1-K,1)\ \forall k\in\{2,\ldots,K\}, \tag{17}$$

such that $\sum_{k=2}^{K}\lambda_K>-1$, since a negative weighting of a single-user MSE does not make sense. Consequently, the computational load of the numerical determination of the optimum Lagrangian factors is drastically reduced since only points in the interior of the $K-1$ dimensional hyper-cube with edge-length K have to be examined, see (17). Moreover, when $K=2$ and both users share the same noise and symbol variances, i.e., $[\boldsymbol{R_n}]_{1,1}=[\boldsymbol{R_n}]_{2,2}$ and $[\boldsymbol{R_s}]_{1,1}=[\boldsymbol{R_s}]_{2,2}$, the only thing which distinguishes the two users are their instantaneous channel powers $\|\boldsymbol{h}_1\|_2^2$ and $\|\boldsymbol{h}_2\|_2^2$, respectively. For this setup, the optimum Lagrangian factor $\lambda_{2,\mathrm{opt}}$ follows from the root of a second-order polynomial instead of a fourth-order one. The *instantaneous channel power ratio* shall be defined as

$$ICPR_{1,2}=\frac{\|\boldsymbol{h}_1\|_2^2}{\|\boldsymbol{h}_2\|_2^2}. \tag{18}$$

Imagine that both users reverse roles, and let the new variables *after* this reversal be denoted by a prime-superscript $'$. Then, we have

$$\begin{aligned}\boldsymbol{h}_1'&=\boldsymbol{h}_2,\\ \boldsymbol{h}_2'&=\boldsymbol{h}_1,\\ \varepsilon_1'&=\varepsilon_2,\\ \varepsilon_2'&=\varepsilon_1,\\ ICPR_{1,2}'&=\frac{1}{ICPR_{1,2}}.\end{aligned} \tag{19}$$

Of course, the optimum MSE weighting coefficients $[\boldsymbol{T}_{\mathrm{opt}}]_{1,1}$ and $[\boldsymbol{T}_{\mathrm{opt}}]_{2,2}$ are also interchanged:

$$\begin{aligned}[\boldsymbol{T}_{\mathrm{opt}}]_{1,1}'=[\boldsymbol{T}_{\mathrm{opt}}]_{2,2}\Leftrightarrow 1+\lambda_{2,\mathrm{opt}}'=1-\lambda_{2,\mathrm{opt}}\\ [\boldsymbol{T}_{\mathrm{opt}}]_{2,2}'=[\boldsymbol{T}_{\mathrm{opt}}]_{1,1}\Leftrightarrow 1-\lambda_{2,\mathrm{opt}}'=1+\lambda_{2,\mathrm{opt}}.\end{aligned} \tag{20}$$

Summing up, inverting the instantaneous channel power ratio brings about a factor -1 of the optimum Lagrangian factor $\lambda_{2,\mathrm{opt}}$:

$$ICPR_{1,2}\to\frac{1}{ICPR_{1,2}}\quad\Rightarrow\quad\lambda_2\to-\lambda_2. \tag{21}$$

For the given case, this translates itself into the nice property that [1]

$$\begin{aligned}\|\boldsymbol{h}_1\|_2>\|\boldsymbol{h}_2\|_2\Leftrightarrow\lambda_{2,\mathrm{opt}}\in(-1,0)\\ \|\boldsymbol{h}_1\|_2<\|\boldsymbol{h}_2\|_2\Leftrightarrow\lambda_{2,\mathrm{opt}}\in(0,1),\end{aligned} \tag{22}$$

so only half of the open interval $(-1,1)$ actually has to be checked. In particular, when $ICPR_{1,2}=1$ then $\lambda_{2,\mathrm{opt}}=0$, so the conventional TxWF assigns identical single-user MSEs, i.e., $\varepsilon_{\mathrm{WF},1}=\varepsilon_{\mathrm{WF},2}$ in this case. If the mean channel power ratio $MCPR_{1,2}$ is much larger or much smaller than one, the instantaneous ratio $ICPR_{1,2}$ is also very likely to be far away from one, and consequently, $\lambda_{2,\mathrm{opt}}$ will strongly differ from zero. Thus, the difference between the conventional TxWF and the equal MSE TxWF becomes evident.

[1]The second-order polynomial, whose root is $\lambda_{2,\mathrm{opt}}$, has the form $P(\lambda_2)=a\lambda_2^2+b\lambda_2+c$ with $b<0$, $ac\geq 0$, and $a=\|\boldsymbol{h}_2\|_2^4-\|\boldsymbol{h}_1\|_2^4$. We can conclude that the roots of $P(\lambda_2)$ are positive for $\|\boldsymbol{h}_2\|_2\geq\|\boldsymbol{h}_1\|_2$, whereas they are negative otherwise.

C. Relation to the Minimax Mean-Square-Error Solution

A possible alternative fair transmit filter approach results from minimizing the maximum MSE among the users:

$$\begin{aligned}\{\boldsymbol{P}_{\text{minmax}}, \beta_{\text{minmax}}\} &= \underset{\{\boldsymbol{P},\beta\}}{\operatorname{argmin}} \|\boldsymbol{\epsilon}\|_\infty \\ \text{s.t.: } &\mathrm{E}\left[\|\boldsymbol{P}\boldsymbol{s}\|_2^2\right] \leq E_{\text{tr}}.\end{aligned} \tag{23}$$

In most cases, the minimax approach leads to identical MSEs for all users. However, this is not true for minimax optimizations in general. For example, if the MSE of one user, say $k_{\text{decoupled}}$, could be reduced without deteriorating the MSEs of the other users in the neighborhood of the equal MSE solution, the possible optimizers for the minimax approach would lead to equal MSEs for the users $k \neq k_{\text{decoupled}}$, whereas the MSE of user $k_{\text{decoupled}}$ would be smaller than or equal to the MSE of the other users. We see that the equal MSE solution is also part of the set of possible solutions for the minimax approach in this special case. It's straightforward to see that this statement also holds for the case of more than one decoupled user, i.e., the equal MSE transmit Wiener filter is always part of the solution set for the minimax optimization in (23).

V. Simulation Results

Fig. 2 shows the individual MSEs ε_1 (triangle down marker) and ε_2 (triangle up marker) of the equal MSE Wiener filter depending on the Lagrangian factor λ_2 for a specific channel realization with an instantaneous channel power ratio $ICPR_{1,2} \approx 1/4$ (cf. Eq. 18). The noise and symbol covariance matrices are scaled identity matrices, and the power limitation E_{tr} is chosen such that $E_{\text{tr}}/\operatorname{tr}(\boldsymbol{R_n}) = 10$. First of all, we observe that for $\lambda_2 = 0$, i.e., for the *conventional* TxWF, different MSEs $\varepsilon_{\text{WF},1}$ and $\varepsilon_{\text{WF},2}$ are achieved. Since user 2 has the stronger channel, $\varepsilon_{\text{WF},1} > \varepsilon_{\text{WF},2}$ holds in this case, so the MSE *ratio* $\varepsilon_1/\varepsilon_2$ (square marker) is larger than one (approx. 1.7). Obviously, the *sum* MSE (star marker) has its minimum for $\lambda_2 = 0$. For $ICPR_{1,2} < 1$, the optimum Lagrangian factor $\lambda_{2,\text{opt}}$ turns out to be larger than zero, $\lambda_{2,\text{opt}} \approx 0.37$. Given $\lambda_2 = \lambda_{2,\text{opt}}$, the *sum* MSE $\varepsilon_{\text{eMSE},1} + \varepsilon_{\text{eMSE},2} \approx 0.465$ of the equal MSE Wiener filter is only slightly increased compared to the sum MSE $\varepsilon_{\text{WF},1} + \varepsilon_{\text{WF},2} \approx 0.44$ of the *conventional* TxWF, but still, it offers a much smaller sum MSE than the *zero-forcing* filter TxZF (dashed line with diamond marker) with $\varepsilon_{\text{ZF},1} + \varepsilon_{\text{ZF},2} \approx 0.73$.

Fig. 3 shows the relative frequency of a system outage depending on the transmit SNR $10\log_{10}(E_{\text{tr}}/\operatorname{tr}(\boldsymbol{R_n}))$, meaning that the *uncoded bit-error-rate* of at least one user rises above the outage threshold of 0.15, since a reliable communication link cannot be established with an appropriate channel code for example. The results are based on 72000 independent channel realizations, where $K = 2$ users are served by $N_{\text{a}} = 2$ antenna elements. The mean channel power ratio is $MCPR_{1,2} = 1$ with $\mathrm{E}_{\boldsymbol{h}}[\|\boldsymbol{h}_1\|_2^2] + \mathrm{E}_{\boldsymbol{h}}[\|\boldsymbol{h}_2\|_2^2] = 4$. The modulation alphabet is QPSK. The transmit matched filter TxMF (star marker) [2], [1], saturates at a relative frequency of about 0.7, i.e. in 70 percent of the cases the BER of at least one user is above 0.15. This results from the fact that the TxMF does not

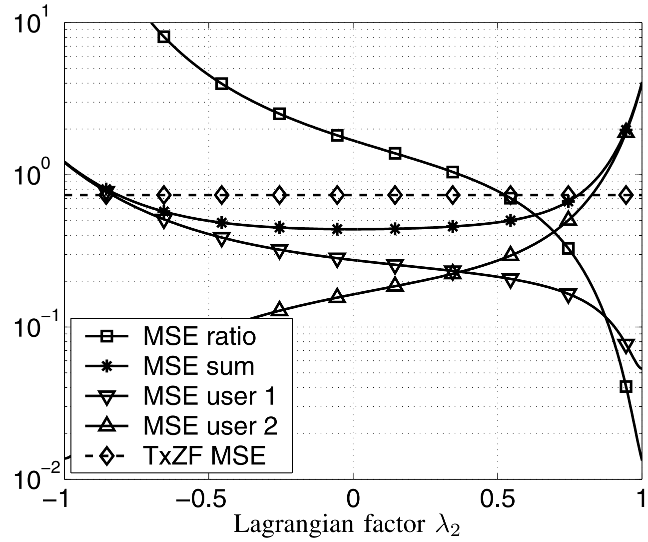

Figure 2. MSE characteristics of the equal MSE Wiener filter for different Lagrangian factors λ_2.

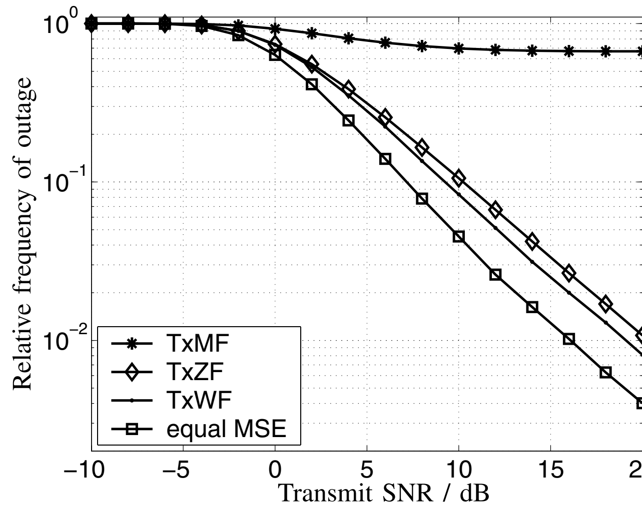

Figure 3. Relative frequency of system outage vs. transmit SNR for an *bit-error-ratio* threshold of 0.15, $MCPR_{1,2} = 1$.

account for the multi-user interference. To achieve a relative frequency of 0.1, the TxZF (diamond marker [3], [4]) requires a transmit SNR of 10.25 dB, the TxWF [5], [6] needs 9.25 dB. Surveying the performance of the equal MSE Wiener filter (square marker), we observe that approx. 3 dB less transmit SNR are sufficient to achieve the same relative frequency of 0.1 compared to the TxZF and about 2 dB compared to the TxWF (point marker). Note that the TxZF [3], [4] also assigns identical MSEs for *every* channel realization in case of identical noise variances of all users, whereas the TxWF assigns MSEs according to the channel and noise conditions.

In Fig. 4, the mean channel power ratio is raised to 10, so user 1 has on average a ten times stronger channel than user 2, but $\mathrm{E}_{\boldsymbol{h}}[\|\boldsymbol{h}_1\|_2^2] + \mathrm{E}_{\boldsymbol{h}}[\|\boldsymbol{h}_2\|_2^2] = 4$ still holds. All other parameters are also left unchanged. The uncoded *bit-error-rate* (BER) is

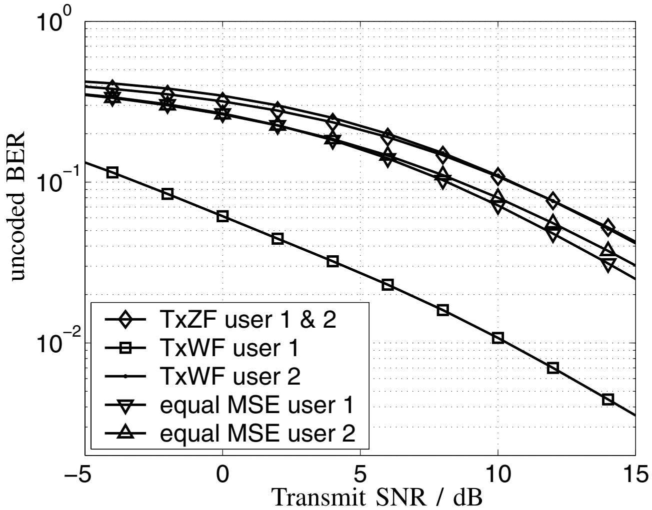

Figure 4. Uncoded BER vs. transmit SNR; *average* channel power of user 1 is ten times that of user 2, $MCPR_{1,2} = 10$.

plotted versus the transmit SNR for different filter types. For a BER of 0.1, the TxZF (diamond marker) needs about 10.5 dB, the equal MSE filter (tringle down marker for user 1, triangle up marker for user 2) needs only 8.6 dB. Although the MSEs of the two users are the same, the mapping towards BERs leads to slightly different BER values in this case. The unequal treatment of the two users for the TxWF (square marker for user 1, point marker for user 2) not only holds for the average, but also for every individual channel realization.

VI. Conclusion

We presented a solution of the minimax mean-square-error transmit filter which assigns identical MSEs to the individual users in a multi-user scenario, so equal link qualities can be allocated even under unfavorable channel conditions. In contrast to the transmit zero-forcing filter, this aim can be achieved with much smaller mean-square-errors for the links to the users. Moreover, the minimax MSE solution and the equal MSE solution were proven to be identical.

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