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Valuation of mortgage products with stochastic prepayment-intensity models

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Abstract

This thesis is concerned with the valuation of mortgage products with uncertain time of termination. In particular, we develop new valuation models for agency mortgage-backed securities (MBS) as they are traded in the US market. Standard US mortgages feature a prepayment option which is often not exercised optimally. This causes uncertainty with respect to the time of termination of a mortgage contract and makes the valuation of mortgage-backed securities a mathematically challenging task. Building on recently introduced stochastic prepayment-intensity models for individual mortgage contracts, we develop new mathematically consistent valuation models for mortgage-backed securities. This modelling approach can also be considered as an extension of the more traditional, purely econometric MBS valuation models which are very popular in practice.

The intensity-based modelling framework also allows us to develop a closed-form approximation formula for the value of agency MBS. Compared to existing MBS valuation approaches in the academic and practitioner-oriented literature, which usually rely on Monte-Carlo simulations or costly numerical methods to solve multidimensional partial differential equations, our closed-form approximation approach offers a computationally highly efficient alternative. We apply this approach to some selected portfolio management applications with MBS, which require frequent product revaluations under different scenarios and thus computationally efficient valuation routines.

Furthermore, we consider the valuation of reverse mortgages in this thesis. Reverse mortgages also feature uncertainty with respect to the time of termination of the contract and their mathematical valuation is thus nontrivial. We develop a consistent valuation model, again based on a stochastic termination-intensity, and illustrate our approach with some examples directed towards the German market, where reverse mortgages are not yet available.

Zusammenfassung

Die im amerikanischen Markt üblichen Hypothekenkredite beinhalten eine Option, die es dem Kreditnehmer erlaubt, den Kredit jederzeit vorzeitig und ohne Vorfälligkeitsentschädigung zu tilgen (prepayment). Die Existenz der prepayment-Option und die Tatsache, dass viele Kreditnehmer die Option suboptimal ausüben, erzeugen Unsicherheit hinsichtlich des Terminierungszeitpunktes von Hypothekenkontrakten und machen die finanzmathematische Bewertung von Hypothekendarlehen (mortgages) und Mortgage-Backed Securities (MBS) zu einem anspruchsvollen Problem. Aufbauend auf intensitätsbasierten Modellen für individuelle Hypthekenkredite, werden in dieser Dissertation Bewertungsmodelle für MBS entwickelt, die auch als Erweiterung der in der Praxis gebräuchlichen, rein ökonometrischen Modelle interpretiert werden können.

Der intensitätsbasierte Ansatz ermöglicht es zudem, eine approximative, geschlossene Bewertungsformel für Mortgage-Backed Securities mit festem Zinssatz herzuleiten. Im Vergleich zu bestehenden MBS-Bewertungsroutinen, die üblicherweise eine Monte-Carlo Simulation oder aufwändige numerische Verfahren zur Lösung mehrdimensionaler partieller Differentialgleichungen erfordern, bietet die entwickelte geschlossene Approximationsformel eine numerisch sehr effiziente Bewertungsalternative. Diese ermöglicht es auch, MBS im Rahmen einiger ausgewählter Anwendungen im Portfoliomanagement zu betrachten, die eine wiederholte Produktbewertung unter verschiedenen Szenarien erfordern.

Abschließend werden in dieser Dissertation Reverse Mortgages betrachtet. Die mathematische Bewertung von Reverse Mortgages ist nicht-trivial, da deren Terminierungszeitpunkt ebenfalls zufällig ist. Der in dieser Arbeit entwickelte mathematisch konsistente Bewertungsansatz basiert, wie bereits die Bewertung von MBS, auf einer stochastischen Terminierungsintensität. Das Bewertungsmodell wird schließlich mit einigen Beispielen für den deutschen Markt illustriert, in dem Reverse Mortgages bisher nicht erhältlich sind.

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Contents

Chapter 1

Introduction

1.1 Motivation

Mortgage loans in general and mortgage-backed securities in particular constitute an important segment of any major debt market. The outstanding amount of all residential mortgage loans in the US market, on which we will primarily focus in this thesis, was USD 10.9 trillion at the end of 2006[1]. A total of USD 6.4 trillion had been securitised and sold to the secondary market in the form of mortgage-backed securities (MBS) and mortgage trusts (such as, e.g., real estate investment trusts). The most important segment of the secondary mortgage market are undoubtedly the so-called agency mortgage-backed securities, i.e. mortgage-backed securities issued and guaranteed by three agencies: The Government National Mortgage Association (Ginnie Mae, GNMA), the Federal National Mortgage Association (Fannie Mae, FNMA) and the Federal Home Loan Mortgage Corporation (Freddie Mac, FHLMC). The cumulative outstanding principal of all agency-MBS added up to USD 4.0 trillion at the end of 2006. The tremendous importance of mortgage products in the US fixed-income market becomes even clearer if one compares these amounts with the total amount of all (marketable, interest-bearing) outstanding US treasury debt, which equalled USD 4.6 trillion in 2006.

Standard residential mortgages in the US feature full prepayment flexibility, i.e. the mortgagors are allowed to prepay their mortgages at any time at a price of 100% of the outstanding notional. This prepayment option embedded in the mortgage contract causes uncertainty with respect to the time of termination of the mortgage contract and makes the valuation of mortgage

[1]Source: Statistical Supplement to the Federal Reserve Bulletin, June 2007. Available at www.federalreserve.gov

products a mathematically challenging task. This is particularly true for the valuation of MBS where pools of mortgages have to be evaluated. The first academic and practitioner-oriented articles which were concerned with the pricing of mortgages, explicitly taking into account the prepayment optionality, appeared in the early 1980s when mathematical finance and the pricing of financial derivatives had only just emerged as a field of research in its own right. Since then, a vast body of literature and models on the pricing of mortgage products has developed. These approaches can in general be classified as econometric, option-theoretic or, rather recently, intensity-based. Since there is no consensus, neither in academia nor in practice, which of these general approaches is the 'best' or most promising one, research in all directions remains active.

While we will give a brief overview of the existing literature concerned with each of the three approaches, we will focus on the intensity-based approach in this thesis, which has been applied to the pricing of individual mortgage contracts recently, but not yet explicitly to the pricing of MBS (to the author's best knowledge). The intensity-based approach will prove to be useful to tackle two major challenges regarding MBS valuation. The first challenge is the mathematical pricing routine which should be consistent with mathematical and financial theory and, at the same time, must be able to take into account that mortgagors behave notoriously sub-optimal. The second challenge is the computational burden associated with most existing MBS valuation techniques, which often causes problems in practice. This holds in particular for risk and portfolio management applications where possibly large portfolios of MBS have to be revaluated frequently under different scenarios.

In addition to MBS, we will consider reverse mortgages in detail in this thesis. Reverse mortgages are sold to older homeowners who receive either a lump sum or a fixed annuity from the mortgage lender, for which no interest payments have to be made during the lifetime of the contract. The reverse mortgage contract is terminated when the mortgagor dies or sells the house. At this point of time, all outstanding debt including all accrued interest has to be paid back, capped at the house sale proceeds. Since, of course, the time of termination of the contract is random, reverse mortgages also fall into the category of mortgage products with uncertain time of termination for which a mathematically consistent valuation model is non-trivial. Reverse mortgages are still a niche product in the US and are not yet available in most European countries, among them Germany. Yet, given the demographic development in these countries, the potential market for reverse mortgages in Europe is huge. Despite the well-acknowledged potential of the product, the academic literature on reverse mortgages, in particular concerning a mathematically consistent valuation, remains scarce.

1.2 Objectives and structure

The main objective of this thesis is to develop new valuation approaches for mortgage products with uncertain time of termination, based on stochastic intensity modelling. It is our aim to use this rather new concept in mathematical finance, which has become popular in the context of credit risk modelling recently, and to fine-tune it to the pricing of MBS and reverse mortgages. Concerning MBS, we want to improve on existing pricing models with respect to the challenges associated with MBS valuation as already stated in the previous section. Concerning reverse mortgages, it is our aim to develop a complete and consistent pricing model for different contract specifications, which has not been done before in the academic literature (to the author's best knowledge).

In addition to the theoretical development of the models, we will also apply the model to real market data where possible. We will thus discuss and take the reader through the whole model building process, from the theoretical formulation of the model to parameter estimation and calibration. We will also discuss the performance of our models where this is feasible and consider selected risk and portfolio optimisation topics. In the case of reverse mortgages, we will provide empirical results directed towards the German market.

The remainder of this thesis is organised as follows: In Chapter 2 we define how we understand the term 'prepayment' in this thesis. Moreover, we give a short overview of products with prepayment features in general and of mortgage products in particular. In order to familiarise the reader with the products considered in this thesis, we will then introduce the basic characteristics of MBS as they are traded in the US market. Finally, we will introduce reverse mortgages in more detail. Chapter 3 provides the reader with the mathematical concepts which we need later in our MBS valuation models. This chapter is also intended to familiarise the reader with the mathematical notation used in this thesis. While no substantial new contributions are contained in Chapter 3 some calculations related to interest-rate theory are carried out, which we will use in the subsequent chapters.

In Chapter 4 we provide an overview of the existing approaches for mortgage and MBS valuation and give a detailed motivation for the need of further research in this field. Chapters 5 and 6 can be considered as the innovative core of this thesis concerning MBS valuation. In Chapter 5 we develop a new MBS valuation model for fixed-rate MBS based on stochastic intensity modelling. We explicitly consider the relation between option-adjusted spreads (an excess return measure commonly used in practice) and real-world and market implied prepayment speed patterns. The theoretical foundation for this is Theorem 5.1 which adapts results from intensity models in other contexts to the modelling of MBS. It offers the necessary mathematical rigour to extend ideas from previous mortgage and MBS modelling approaches and embeds them into a mathematically well-defined model framework. We give empirical calibration examples, consider model sensitivities with respect to yield curve shifts and model parameters and discuss how the model can be used to price adjustable-rate mortgage-backed securities and collateralized mortgage obligations (CMOs). This model, however, has one inconvenience shared by many previous modelling approaches: the pricing requires a computationally expensive Monte-Carlo simulation. This problem is tackled in Chapter 6, where we propose a closed-form approximation formula, based on a slightly different model specification. This new closed-form approximation formula presented in Theorem 6.6 offers an easy-to-compute alternative to previous approaches in the literature concerned with MBS valuation in closed-form. Compared to existing models, our approach has the advantage that it does not require any numerically complex techniques. Again, we calibrate and validate the model empirically with historically observed MBS market prices. Chapter 7 is intended to embed our models into the existing literature and explicitly discusses the contribution of our MBS modelling approaches, which naturally completes the discussion in Chapter 4.

In Chapter 8 we present some selected portfolio optimisation problems, based on simulated scenarios, and include fixed-rate MBS into the universe of available assets. In an empirical study we show how an asset allocation strategy including MBS can outperform classical stock/bond portfolios. These empirical portfolio optimisation studies require a large number of MBS evaluations under different scenarios and have only become feasible due to our computationally highly efficient closed-form approximation approach.

Our pricing framework for reverse mortgages is presented in Chapter 9. We consider both, fixed-rate and adjustable-rate reverse mortgage contracts, explicitly take into account the possibility of losses for the lender (mainly resulting from longevity risk) and discuss the maximum payments a homeowner can receive from the mortgage lender under certain constraints. The results are illustrated with data from the German market. Finally, Chapter 10 concludes.

Chapter 2

Mortgage products and prepayment

In the first part of this chapter we will give an overview of the products with prepayment features, in particular with respect to the US market. Moreover, we will specify explicitly what we mean by 'prepayment' and 'prepayment risk'. These terms are not always unambiguously used, neither in the academic nor in the practitioner-oriented literature. The second part of this chapter is concerned with the most important asset class associated with prepayment risk: Mortgage-backed securities. While we can only provide an overview of the product characteristics, the most important subtypes and trading mechanics, this section is intended to familiarise the reader with the securities for which we will develop valuation models in the subsequent chapters. Prominent examples of textbooks covering mortgage-backed securities in a more detailed and extensive way include Fabozzi (2006) and Hu (1997), where legal, economic, structural, trading and pricing aspects of MBS are covered. Fabozzi (1998) features practitioner-oriented articles concerned with various aspects of MBS valuation, while Young et al. (1999) provide a detailed overview of MBS trading and settlement issues. Finally, reverse mortgages are introduced in the last section of this chapter.

2.1 Prepayment and prepayment risk: A definition

Prepayment is commonly understood as a borrower's decision to exercise an early repayment option in a financial contract. In order to price this optionality, the borrower's call policy must be anticipated correctly. This, however, is not always possible. The following definition formally establishes how we understand prepayment and prepayment risk in this thesis. It generalises the definition given by, e.g., François (2003) who exclusively considers callable debt.

Definition 2.1. *(Prepayments and prepayment risk)*
Prepayments are (contractually permitted) notional cash flows which occur earlier or later than expected, deviating from the anticipated call or put policy of the counterparty in a financial contract. Prepayment risk is the risk resulting from these cash flow deviations.

This definition is very broad and it is able to accommodate both the prepayment risk in a bank's assets and the prepayment risk in a bank's liabilities. In the case of a liability, prepayment risk stems from a lender's option to withdraw funds or to deposit money earlier or later than anticipated. Definition 2.1 also makes clear that we understand prepayment risk exclusively as a special kind of market price risk resulting from the uncertain time of termination (or partial termination) of the contract. Occasionally, the term 'extension risk' is used for the risk of cash flows which occur later than anticipated, increasing the duration of a financial product's cash flow stream. The term 'prepayment risk' is then used for the risk of cash flows which occur earlier than anticipated, decreasing the duration of the cash flow stream. In this thesis, however, we will not explicitly make this distinction and use the term 'prepayment risk' in its general form as defined in Definition 2.1. The fact that a counterparty's call or put policy can not be perfectly anticipated for some products may have various reasons. First, any prepayment model which tries to capture the prepayment behaviour may be misspecified due to, e.g., omission of factors or erroneous assumptions. Second, the counterparty may simply not behave optimally for lack of financial interest and/or sophistication.

The most important product class featuring prepayment risk are undoubtedly mortgage-backed securities. Mortgage-backed securities (MBS) can be considered as a particular subtype of asset-backed securities (ABS) where the assets backing the security's cash flows are mortgage loans. In general, ABS which feature call flexibility, e.g. ABS backed by Home-Equity or Retail Auto loans, also belong to the class of prepayment-sensitive assets. Of course, beside interest-rate and prepayment risk, an ABS investor may also be exposed to credit risk, which is in many cases the major source of risk and thus very often the primary focus of an ABS investor. ABS and MBS inherit their prepayment-sensitivity from the underlying loans. Prepayment risk of individual loans may in fact serve as the basis for assessing the prepayment risk of more complex products such as MBS. The prepayment risk in callable bonds is explicitly addressed in François (2003), who discusses the theoretical implications and provides empirical evidence. Yet, in most callable debt valuation models it is assumed that the borrower does make the optimal call decision which may be anticipated and priced by an adequate model (see, e.g., Artzner and Delbaen (1992) or Acharya and Carpenter (2002)).

In addition to the previously described prepayment-sensitive assets one may also want to consider liabilities with put features in a prepayment risk context, i.e. products where the depositor has the right to withdraw funds flexibly. An example are Municipal Guaranteed Investment Contracts (GICs). In the US, GICs are used by municipalities in conjunction with social or infrastructure projects. In order to finance these projects municipalities issue bonds whose proceeds are then transferred into a GIC agreement to be used for the project development or as a reserve account for the bond issues. Furthermore, special GIC accounts are usually created for the project's proceeds which are then used for interest and principal repayments to the bondholders. Fund withdrawals and future deposits are often flexible and may depend on various factors which are usually directly related to the project which is being financed and to the call features of the corresponding bonds. As a consequence, the timing and sometimes also the amount of cash flows is hard to anticipate, which results in prepayment risk.

2.2 Mortgage-backed securities (MBS)

In this section we will briefly present the major structures and features of mortgage-backed securities. Although MBS are one of the most important asset classes in the US, they are a unique instrument whose valuation remains highly complex. This is mainly caused by the prepayment feature inherent in the mortgage loans underlying a MBS. In the last subsection we will shortly summarise the basic loan and amortisation calculations for mortgages and MBS.

2.2.1 Subtypes of MBS and trading mechanics

Residential vs. Commercial MBS

The first criterion to classify the different subtypes of MBS in the US market is the nature of the underlying mortgage loans. While we focus on securities backed by residential mortgages in this thesis, securities backed by commercial mortgages (CMBS) also constitute an important part of the MBS market. However, the structure of a particular CMBS will largely be determined by the individual characteristics of the underlying commercial mortgage(s) and, as a consequence, the prepayment behaviour and risk of these securities can often be assessed only by taking into account these individual characteristics. Moreover, the primary concern of an investor in CMBS is usually the credit risk component in view of which the prepayment risk often plays only a minor or even negligible role.

Term and amortisation schedule

A second, natural classification criterion is the term of the underlying mortgages and their amortisation schedule. While 30 year, fully amortising mortgages are still the most common type of mortgage, mortgages and MBS with shorter maturities (e.g., 15 or 20 years) exist as well. More exotic amortisation structures include, for example, balloon mortgages and graduated-payment mortgages. Balloon mortgages have a 30 year amortisation schedule, but are due in just five or seven years, while the monthly payments of a graduated-payment mortgage are lower during the first year of the loan and then rise gradually, so that the loan is fully amortised after the 30 year term. Because of the low initial payments, a graduated-payment mortgage may feature negative amortisation in the early years.

Fixed-rate and adjustable-rate MBS

In the early 1980s adjustable-rate mortgages (ARMs) were introduced as an alternative to the traditional fixed-rate mortgages. Adjustable-rate mortgages and adjustable-rate MBS usually have a 6 month or 1 year floating money market or treasury rate as reference index rate, such as the 6 month LIBOR rate or the 1 year CMT (constant maturity treasury) rate. Yet, more exotic indices such as the COFI (cost of fund index) are also common. A great majority of ARMs have periodic reset Caps and Floors as well as life time Caps, reducing the impact of interest-rate changes for the borrower. A combination of fixed-rate and adjustable-rate mortgages are the so-called hybrids, which are adjustable-rate mortgages with an initial fixed-rate period of usually three, five, seven or ten years. For example, the notation 5/1/30 is commonly used for a hybrid mortgage with maturity 30 years and an annually fixed adjustable rate after an initial tenor of five years.

Pass-throughs, Pay-throughs and CMOs

In a pass-through security, the monthly mortgage payments, which contain interest, scheduled principal and prepayments, are directly 'passed' from mortgagors 'through' the issuer to the investor. Usually these payments to the investor are delayed, e.g., by 14 or 19 days in the case of GNMA securities (see Table 2.1). A delay of 14 means that the first payment to the investor is made at the 15th (instead of the first) of the month following the record date and every month thereafter. In a pay-through security, mortgage payments are transformed by the issuer before they are passed on to the investor. Pay-through structures have particularly gained popularity in the form of collateralized mortgage obligations (CMOs). In a CMO, payments and especially prepayments of the underlying mortgage pool are assigned to different tranches. It is thus possible to create tranches with different expected average lives, prepayment risk exposures and even interest rate agreements from one underlying pool. CMOs are thus able to satisfy the increasingly diversified risk appetite of investors. We will discuss some exemplary CMO structures in more detail in Chapter 5.4.

Agency and Private-label MBS

The Government National Mortgage Association (GNMA, Ginnie Mae) as well as the government-sponsored Federal National Mortgage Association (FNMA, Fannie Mae) and the Federal Home Loan Mortgage Corporation (FHLMC, Freddie Mac) play a crucial role in the US MBS market. These institutions act as guarantor for mortgage pools, guaranteeing full and timely payment of interest and principal to the investor. GNMA securities, which feature the full faith and credit of the US government, can thus be considered default-free from the investor's point of view since a possible default of any of the mortgages in the underlying pool simply results in prepayment of the outstanding notional of the respective loan by GNMA. GNMA, FNMA and FHLMC securities are usually called agency MBS, have highly standardised structures, trading and settlement mechanics and constitute the largest, most liquid and most important part of the MBS market. Private-label MBS have individual characteristics with respect to structure, credit quality and liquidity. In the following we will particularly focus on the GNMA I and GNMA II programs, whose securities we will use in the following chapters for the empirical validation of our modelling approaches. Table 2.1, which is adapted from Hu (1997), p. 17 and p. 18, summarises the major features of these securities.

Trading mechanics

Trading in the US agency pass-through market can be divided into to-be-announced (TBA) trading and pool-specific trading. While in a pool-specific trade both parties agree on the exact pool to be delivered, the seller in a

Feature	GNMA I	GNMA II
Issuer	Single lender	Multiple lenders
Type of loans	Newly originated, backed by federal agencies (e.g., FHA[2], VA[3])	Newly originated, backed by federal agencies (e.g., FHA, VA)
Minimum pool size	USD 1 mio	USD 0.25 mio per lender
Mortgage Rate Range	All must have the same rate	Must be within 100 bps of the lowest rate in the pool
Servicing/ Guarantee fee	50 bps	50-150 bps
Payment delay	14 days	19 days

Table 2.1: Major characteristics of GNMA I and GNMA II pass-through MBS

TBA trade has the right to choose the pool(s), which must satisfy some requirements of good delivery (see, e.g., Young et al. (1999) for details). The counterparties only agree on the crucial parameters, i.e. agency, term, coupon, par amount and price (for example, USD 20 mio of GNMA 30 year 7% pass-throughs at a price of USD 0.9875 per USD 1 face amount). In the so-called TBA vintage-market, the buyer can also specify the origination year of the pool(s). In 2005, USD 251 billion of agency MBS were traded on a daily average[4]. The market for TBA fixed-rate agency MBS remains the most liquid and most mature market segment.

2.2.2 Prepayment

As previously discussed, the mortgagors' right to prepay is a crucial feature of MBS. A MBS investor is always short the prepayment option which makes MBS highly complex instruments. The exercise of the prepayment option by a mortgagor may have several reasons. The first, and most important reason is usually called refinancing incentive. When mortgage refinancing rates drop, the mortgagor may have the possibility to refinance the mortgage at a lower rate. In this sense, the prepayment option can be compared

[2]Federal Housing Administration
[3]Department of Veterans Affairs
[4]Source: The Bond Market Association (www.bondmarkets.com)

to an American-style interest-rate option. However, different mortgagors in a mortgage pool may experience different constraints regarding the ability to refinance. These constraints may contain, for example, transaction costs or the opportunity costs that a mortgagor faces when he/she spends time renegotiating mortgage conditions with possible lenders. These costs are certainly a reason for the fact that refinancing related prepayment on pool level is rather heterogeneous. Yet, the relationship between falling interest rates and rising prepayments is well-established and can also be confirmed with the data which is available for this thesis. Figure 2.1 shows the development of the 10 year US treasury par yield together with prepayment speeds of some selected GNMA securities with different coupons. The prepayment speeds are expressed as single monthly mortalities (SMM).

Beside the refinancing incentive, prepayment may be caused by house sales due to relocation or death of the mortgagor. A mortgagor's default equally leads to prepayment for the investor in the case of agency MBS and, finally, a change in personal wealth (e.g., by an inheritance, an unexpected bonus payment, etc.) or simply new loan preferences of an individual mortgagor may prompt full or partial prepayment of a mortgage. These non-refinancing related prepayments are usually subsumed under the term 'turnover prepayment' or 'baseline prepayment'.

Prepayment speeds are usually expressed as single monthly mortality rates (SMM), as annualised constant prepayment rate (CPR) or as a percentage of the Public Securities Association standard assumption (PSA). The SMM in month t simply measures the percentage of the outstanding notional which is paid back to the investor after the interest and regular principal repayment of the corresponding month. Let $A(t)$ be the outstanding notional (after scheduled repayments) of a MBS at time t according to the original amortisation schedule without any prepayments and let $PF(t)$ denote the pool factor at time t, i.e. the actual notional amount outstanding at time t. Then, given the mortgages prepayment history of the MBS up to time $t-1$, we get[5]:

$$p_{\text{SMM}}(t) = \frac{\widehat{PF}(t) - PF(t)}{\widehat{PF}(t)}$$

where

$$\widehat{PF}(t) := PF(t-1) \cdot \frac{A(t)}{A(t-1)}.$$

Note that, given the prepayment history up to month $t-1$, $\widehat{PF}(t)$ would be the outstanding notional at time t if there were no further prepayments in

[5] We write p_{SMM} for the single monthly mortality (SMM) due to notational consistency with the subsequent chapters.

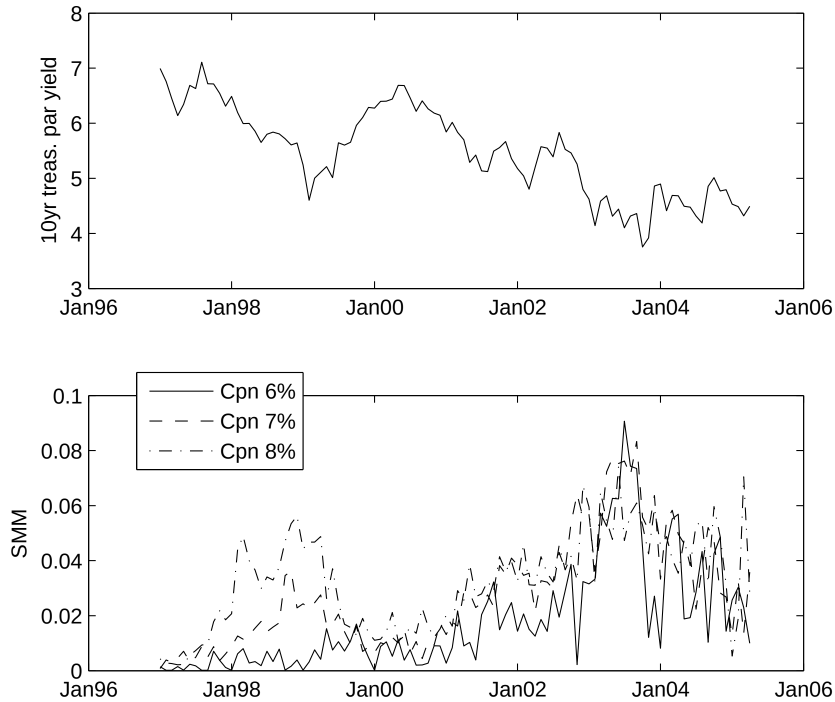

Figure 2.1: Historical 10yr treasury par yield (top) and prepayment speeds of some selected MBS of the GNMA II program with coupons of 6%, 7% and 8% respectively (bottom).

month t. The annualised counterpart of the SMM is the CPR, which can be obtained from the SMM by

$$\mathrm{CPR}(t) = 1 - (1 - p_{\mathrm{SMM}}(t))^{12}. \tag{2.1}$$

Finally, the PSA speed is given by

$$\mathrm{PSA}(t) = \begin{cases} 100 \cdot \frac{\mathrm{CPR}(t)}{0.2 \cdot t} & \text{for} \quad t < 30 \\ 100 \cdot \frac{\mathrm{CPR}(t)}{6} & \text{for} \quad t \geq 30 \end{cases}.$$

The PSA speed goes back to the standard assumption of the Public Securities Association, where prepayment speeds are modelled as a linear function of the security's age, rising from 0 to a CPR value of 6% during the first 30 months of the MBS. They are then assumed to remain constant at 6% (CPR).

Multiplied with some scalar, the PSA standard remains a simple, but popular tool in the markets. Figure 2.2 shows the 100% standard PSA curve and, for comparison purposes, the 50% standard PSA curve and the 200% standard PSA curve. Assuming that prepayment speeds are deterministic according to the PSA standard assumption, the value of any MBS is straightforward to calculate by simply discounting the deterministic future cash flows to the present day. These prices are often used for comparison purposes in the market.

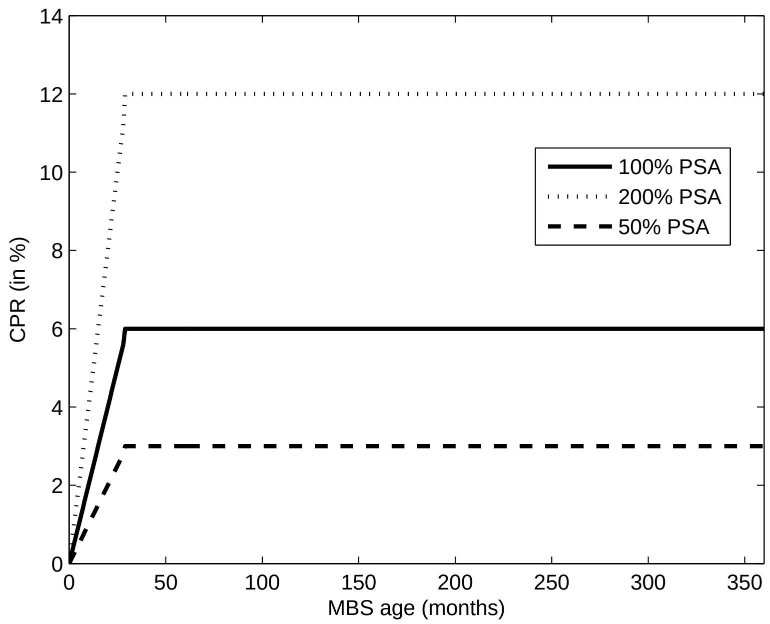

Figure 2.2: Standard PSA curves

2.2.3 Basic MBS cash flow conventions

In the following we will finally describe the complete cash flow structure of a mortgage pool with loans that are fully amortising with equal monthly payments. These cash flows include prepayments, the regular principal payments according to the amortisation schedule and the interest payments. The results are also summarised, for example, in the paper by Kariya et al. (2002). Let $i := \mathrm{WAC}/12$ be the monthly interest rate which corresponds to the mortgage pool's weighted average coupon (WAC) and let MP denote the monthly mortgage payment of the mortgagor, which contains the interest payment $I(t)$ and the regular principal repayment $RP(t)$, if we assume that prepayments are not allowed. For a mortgage with T months to maturity at time $t = 0$, we obtain the defining equation for MP by using i as internal rate of return:

$$\begin{aligned} A(0) &= \mathrm{MP}/(1+i)^1 + \mathrm{MP}/(1+i)^2 + ... + \mathrm{MP}/(1+i)^T \\ &= \mathrm{MP} \cdot \sum_{j=1}^{T} \frac{1}{(1+i)^j} \\ &= \mathrm{MP} \cdot \frac{\frac{1}{1+i} - \frac{1}{(1+i)^{T+1}}}{1 - \frac{1}{1+i}} \\ &= \mathrm{MP} \cdot \frac{1-(1+i)^{-T}}{i}. \end{aligned} \tag{2.2}$$

Thus,

$$\mathrm{MP} = A(0) \cdot \frac{i}{1-(1+i)^{-T}} \quad .$$

Equation (2.2) can of course be generalised, so that for any month t, $0 \leq t \leq T$, the outstanding notional according to the original amortisation schedule is given by

$$A(t) = \mathrm{MP} \cdot \frac{1-(1+i)^{-(T-t)}}{i}.$$

The scheduled interest payment according to the amortisation schedule without any prepayments, which has to be made by the mortgagors in month t, $1 \leq t \leq T$, is given by

$$I(t) = A(t-1) \cdot i = \mathrm{MP} \cdot (1-(1+i)^{-(T-t+1)}) = i \cdot A(0) \cdot \frac{1-(1+i)^{-(T-t+1)}}{1-(1+i)^{-T}} \quad .$$

Finally, the regular principal payment is given by

$$RP(t) = \mathrm{MP} - I(t) = i \cdot A(0) \cdot \frac{(1+i)^{-(T-t+1)}}{1-(1+i)^{-T}} \quad .$$

In a mortgage pool with prepayments, the difference between the outstanding notional according to the original amortisation schedule without prepayments and the actual pool factor, i.e. $A(t) - PF(t)$, can be considered as a quantity which reflects the magnitude of prepayments in the pool's history up to time t. This difference, or alternatively the ratio $PF(t)/A(t)$, is commonly referred to as the pool's burnout. We will also use the burnout as an explanatory variable in our hybrid-form model presented in Chapter 5.

Now, let C denote the monthly coupon of the MBS with a pool of similar mortgages and let S be the monthly servicing and guarantee spread. By market convention, the gross coupon i of the MBS is given by $i = C + S$. Defining by

$$\bar{I}(t) := PF(t-1) \cdot i$$

the actual interest paid by the mortgagors in month t, the actual cash flow $CF(t)$ paid to the investor in month t, $1 \leq t \leq T$, is given by

$$\begin{aligned}
CF(t) &= (PF(t-1) - PF(t)) + \frac{C}{C+S} \cdot \bar{I}(t) \\
&= \frac{PF(t-1)}{A(t-1)} \cdot \left(A(t-1) - \frac{PF(t)}{PF(t-1)} \cdot \frac{A(t-1)}{A(t)} \cdot A(t) \right) \\
&\quad + C \cdot PF(t-1) \\
&= \frac{PF(t-1)}{A(t-1)} \cdot \left(A(t-1) - \frac{PF(t)}{\widehat{PF}(t)} \cdot A(t) \right) + C \cdot PF(t-1) \\
&= \frac{PF(t-1)}{A(t-1)} \cdot (A(t-1) - A(t) + A(t) \cdot p_{\text{SMM}}(t)) + C \cdot PF(t-1).
\end{aligned}$$

I.e. the cash flow paid to the investor at time t is given by the sum of the regular principal payment, prepayment and interest payment (with the servicing fee deducted).

Remark 2.2. *(Monthly mortgage payment)*
Since in this thesis we are primarily interested in MBS from an investor's point of view, we will use the term 'monthly mortgage payment' for the quantity

$$M(t) \cdot \Delta t := A(t-1) - A(t) + \frac{C}{C+S} \cdot I(t) = RP(t) + \frac{C}{C+S} \cdot I(t), \quad 1 \leq t \leq T,$$

where $\Delta t = \frac{1}{12}$ unless explicitly specified otherwise. I.e., $M(t) \cdot \Delta t$ is the monthly payment received by the investor without any prepayments. The difference between $M(t) \cdot \Delta t$ and the earlier defined $MP = RP(t) + I(t)$ is the servicing spread which has to be paid by the mortgagors, but is not passed through to the investor. Note that, unlike MP, $M(t) \cdot \Delta t$ is not constant. However, since the servicing spread is usually small (50 basis points in the case of GNMA I securities), the changes of $M(t) \cdot \Delta t$ over time are also small.

2.3 Reverse mortgages

Reverse mortgages were first introduced in the US in the late 1980s. While for most of the time demand for reverse mortgages remained low, the US reverse mortgage market has experienced considerable growth in the last years and is now commonly viewed as a market with huge potential. In 2005, 43,131 reverse mortgage contracts were originated in the US, compared to 6,640 in 2001 (see Eschtruth et al. (2006)). A reverse mortgage allows home-rich, cash-poor older homeowners to access their housing wealth for consumption without selling the house and without having to take a conventional home equity mortgage which would require regular interest and loan amortisation payments. The most popular reverse mortgage program in the US is the Home Equity Conversion Mortgage (HECM), which is available to homeowners over the age of 62 who fulfil certain eligibility criteria. In a HECM loan, payments to the mortgagor are made as a lump sum at origination of the reverse mortgage contract, as a lifetime income or as a flexible line of credit. A reverse mortgage loan has to be paid back including all accrued interest when the mortgagor dies or sells the house or, depending on the contract, when the mortgagor moves out of the house. The amount which has to be paid back is, however, capped at the house sale proceeds.

Despite the recent success in the US, reverse mortgages are still not available in most European countries, among them Germany, on which we will focus in the empirical examples in Chapter 9. This is particularly surprising since in Germany the demographic development implies that there will be more and more elderly people in the near future without children. The access to home equity for consumption after retirement seems even more attractive without any direct heirs. The US experience and the demographic development in Europe suggest that the potential market for reverse mortgages in Germany and in other European countries will be huge. The most apparent reason for the reluctance of financial institutions to offer reverse mortgages may be the risk of longevity. A mortgage lender experiences losses if at termination of the contract the total outstanding loan amount exceeds the house value. This may obviously occur if the mortgagor attains a very high age. This risk must of course be taken into account for the pricing and subsequent valuation of the reverse mortgage contract. The question of how to price a reverse mortgage contract by adequately taking into account the risk that the total amount of the loan may exceed the house value at termination of the contract is not trivial. In a very recent paper, Wang et al. (2007) address this issue and consider survivor bonds and survivor swaps for reverse mortgages within an actuarial approach. Apart from this recent contribution the academic literature on the valuation of reverse mortgage contracts remains scarce.

A further reason for the reluctance to offer reverse mortgages may be the fear of adverse selection and moral hazard effects, which are discussed in detail in Davidoff and Welke (2005) and Shiller and Weiss (2000). Adverse selection means that mortgagors expecting an exceptionally long life, a particularly low mobility or with houses which appreciate at particularly low rates preferably enter into reverse mortgage contracts. Davidoff and Welke (2005) also give two dimensions of moral hazard. First, reverse mortgages may make it less attractive to sell the house. Second, a mortgagor with a reverse mortgage has less incentive to invest in property maintenance. While the latter issue is hard to measure empirically, Davidoff and Welke (2005) come to the conclusion that neither adverse selection nor moral hazard is guaranteed by the structure of the reverse mortgage industry in the US and even give empirical evidence for advantageous selection. Advantageous selection means that reverse mortgagors on average move out of their houses faster (by death or voluntarily) than older people without a reverse mortgage contract. We thus do not further take adverse selection and moral hazard effects into account.

Following Definition 2.1, the termination of the reverse mortgage contract by death or house sale can be considered as prepayment. We do not take into account the possibility of prepayment due to refinancing of the reverse mortgage, i.e. the possibility to prepay a reverse mortgage contract in order to get a new one with lower rates. There are two reasons why we do not consider refinancing prepayment for reverse mortgages. First, we concentrate on the German market. In Germany, it is still the market convention that refinancing-related prepayment of conventional mortgage loans is not permitted without penalty payments to compensate the mortgage lender. Thus, from a mortgage lender's point of view, it makes little sense to introduce reverse mortgages with refinancing-prepayment options as long as the standard mortgage products do not incorporate these options. Second, even in the US where mortgagors are used to having prepayment options in their mortgage contracts, refinancing related prepayment of reverse mortgages is very rare (see Davidoff and Welke (2005)). This may be explained by the high closing fees (6.8% on average), which makes refinancing expensive, and by the very nature of reverse mortgages. Reducing a monthly payment of a conventional mortgage by refinancing is certainly more attractive than reducing the accrued interest of a reverse mortgage which most of the mortgagors will never pay back during their lifetime anyway. All prepayment risk associated with the reverse mortgage contracts considered in this thesis therefore stems from the mortgagor's death and mobility. It is important to notice that these risks are unsystematic and may thus be considered diversifiable. We will, however, briefly comment on the modelling consequences which a systematic prepayment option would imply at the end of Chapter 9.

Chapter 3

Mathematical preliminaries

In this chapter we introduce the basic mathematical concepts which we need to develop valuation models for mortgage-backed securities and reverse mortgages. The first two sections of this chapter are also intended to familiarise the reader with the mathematical notation which will then be maintained throughout this thesis. While we cite original articles and further literature sources where appropriate, notation and presentation of the necessary preliminaries from interest-rate market theory are mainly based on Zagst (2002a). Beside Zagst (2002a), Bingham and Kiesel (2004) or Brigo and Mercurio (2006) are two further examples of textbooks covering stochastic processes, financial market theory and, in particular, interest-rate theory in a more detailed way. For the basics of point processes and, in particular, intensity-based financial modelling the books by Bielecki and Rutkowski (2002), Schönbucher (2003), Schmid (2004) or Brigo and Mercurio (2006) are good references where intensity-based models are applied in the context of credit risk. Schmid (2004) also treats the Kalman filtering method which we will present in the last section of this chapter.

3.1 The Cauchy problem

While we assume that the basic concepts of probability theory, stochastic processes and stochastic calculus are known to the reader, we would like to recall shortly the so-called Cauchy problem and the Feynman-Kac representation of the Cauchy problem since these concepts will be crucial in some proofs in the following parts of this thesis. For this purpose let us start with an n-dimensional Ito-process $X(t)$ on a complete filtered probability space

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ defined by

$$X(t) = x_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(t)dW(t) \tag{3.1}$$

for which we write in the usual way

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) = \mu(t)dt + \sum_{j=1}^{m} \sigma_j(t)dW_j(t).$$

$W(t) = (W_1(t), ..., W_m(t))'$ is an m-dimensional Wiener process, X(0) is $\mathcal{F}_0$-measurable and μ, σ are progressively measurable stochastic processes with

$$\int_0^t |\mu_i(s)|ds \quad < \quad \infty \tag{3.2}$$

$$E_Q\left[\int_0^t \sigma_{ij}^2(s)ds\right] \quad < \quad \infty \tag{3.3}$$

Q-almost surely for all $t \geq 0$, $i = 1, ..., n$, $j = 1, ..., m$.

If there exists an n-dimensional stochastic process X of the form (3.1) with $\mu(t) = \mu(X(t), t)$ and $\sigma(t) = \sigma(X(t), t)$ satisfying (3.2) and (3.3), the process $X(t)$ is called the strong solution of the following stochastic differential equation (see, e.g., Zagst (2002a), p. 36):

$$\begin{aligned} dX(t) &= \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \\ X(0) &= x_0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *(Existence and uniqueness)*
Let μ and σ in (3.4) be continuous functions such that for all $t \geq 0$, $x, y \in \mathbb{R}$ and for some constant $K > 0$ the following conditions hold:

(i) Lipschitz condition:

$$||\mu(x,t) - \mu(y,t)|| + ||\sigma(x,t) - \sigma(y,t)|| \leq K \cdot ||x - y||$$

(ii) Growth condition:

$$||\mu(x,t)||^2 + ||\sigma(x,t)||^2 \leq K^2 \cdot \left(1 + ||x||^2\right)$$

Then there exists a unique, continuous strong solution X of the stochastic differential equation (3.4) and a constant C, depending only on K and $T \geq 0$, such that

$$E_Q\left[||X(t)||^2\right] \leq C \cdot \left(1 + ||x||^2\right) \cdot e^{C \cdot t}$$

for all $t \in [0, T]$. *Moreover,*

$$E_Q\left[sup_{0\leq t\leq T}||X(t)||^2\right] < \infty.$$

In Zagst (2002a), p. 36f., some special cases of this theorem are discussed. A formal proof can be found, for example, in Korn and Korn (1999), p.127-133.

Definition 3.2. *(Cauchy Problem)*
Let $D : \mathbb{R}^n \to \mathbb{R}$, $r : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ *be continuous and* $T > 0$ *be arbitrary but fixed. The problem to find a function* $v : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ *which is continuously differentiable in* t *and twice continuously differentiable in* x *and solves the partial differential equation*

$$\begin{aligned} v_t(x,t) + \sum_{i=1}^{n} \mu_i(x,t) \cdot v_{x_i}(x,t) & \\ +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x,t) \cdot v_{x_i x_j}(x,t) &= r(x,t) \cdot v(x,t) \\ v(x,T) &= D(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$, *where* $a_{ij} := \sum_{k=1}^{m} \sigma_{ik}(x,t) \cdot \sigma_{jk}(x,t)$ *and* X *is the unique strong solution of the stochastic differential equation (3.4), is called the Cauchy problem.*

Now, define

$$P_0(t,s) := e^{\int_t^s r(X(u),u)du}.$$

Under sufficient regularity conditions for μ, σ, v, r, D (for details on regularity conditions see, e.g., Karatzas and Shreve (1991) or Korn and Korn (1999)), it can be shown that

$$\begin{aligned} v(x,t) &= E_Q[P_0^{-1}(t,T) \cdot D(X(T))|\mathcal{F}_t] \qquad (3.5) \\ &= E_Q[e^{-\int_t^T r(X(u),u)du} \cdot D(X(T))|\mathcal{F}_t] \end{aligned}$$

is the solution of the Cauchy problem (see, e.g., Zagst (2002a), p. 38ff.). The representation (3.5) is called the Feynman-Kac representation of the Cauchy problem.

We have introduced the Cauchy problem and the Feynman-Kac representation in its general form. Applied to interest-rate contingent claims this is a crucial result which we will frequently need and refer to in the rest of this thesis.

3.2 Interest-rate markets

3.2.1 General definitions

We start our overview of interest-rate market theory with the most important primary asset, the zero-coupon bond. The zero-coupon bond price $P(t,T)$ at the point of time t is the price one has to pay to get back 1 at maturity T. The zero-rate is defined in the usual way by

$$R(t,T) := -\frac{\ln P(t,T)}{T-t}$$

and its limit as T approaches t by

$$r(t) := R(t,t) := -\lim_{\Delta t \to 0} \frac{\ln P(t,t+\Delta t)}{\Delta t} = -\frac{\partial}{\partial T} \ln P(t,T)\bigg|_{T=t}.$$

The interest rate $r(t)$ is called the short rate. A contract in which two parties at time t agree to exchange at a future point of time T_1 a zero-coupon bond with maturity $T_2 - T_1$ is called a forward starting zero-coupon bond, denoted by $P(t,T_1,T_2)$. Buying a number of $P(t,T_1,T_2)$ zero-coupon bonds for a price of $P(t,T_1)$ at time t and an obligation to reinvest the amount one receives at T_1 into a zero-coupon bond with maturity $T_2 - T_1$ results in an identical portfolio as simply buying a zero-coupon bond $P(t,T_2)$ at time t. It is thus easy to see that the price of the forward starting zero-coupon bond is given by

$$P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}.$$

The forward zero-rate is given by

$$R(r,T_1,T_2) = -\frac{\ln P(t,T_1,T_2)}{T_2-T_1} = -\frac{\ln P(t,T_2) - \ln P(t,T_1)}{T_2-T_1}$$

and the forward short rate by

$$\begin{aligned} f(t,T) := R(t,T,T) &:= -\lim_{\Delta t \to 0} \frac{\ln P(t,T+\Delta t) - \ln P(t,T)}{\Delta t} \\ &= -\frac{\partial}{\partial T} \ln P(t,T). \end{aligned}$$

with $f(t,t) = r(t)$. The next instrument we would like to introduce is the cash account, which is defined in the usual way by

$$P_0(t) := e^{\int_0^t r(s)ds}.$$

I.e., the cash account describes a (random) payment of $P_0(t)$ which results from an investment of one dollar today (time 0) into infinitely many consecutive forward starting zero-coupon bonds with infinitesimal time to maturity when the investment is made successively in time.

Our interest-rate market is modelled by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ where the prices of the primary assets, the zero-coupon bonds, are driven by an m-dimensional Wiener process W. The zero-coupon bond prices are described by

$$d_t P(t,T) = \mu_P(t,T)dt + \sigma_P(t)dW(t) = \mu(t)dt + \sum_{j=1}^{m} \sigma_{P,j}(t)dW_j(t) \quad (3.6)$$

for all $t \in [0,T]$ with progressively measurable stochastic processes μ_p and σ_P such that for all T

$$\int_0^T |\mu_P(s,T)|ds \quad < \quad \infty \quad Q-\text{a.s.} \quad (3.7)$$

$$E_Q\left[\int_0^T \sigma^2_{P,j}(s,T)ds\right] \quad < \quad \infty \quad (3.8)$$

for all $j = 1, ..., m$. The discounted zero-coupon bond prices are given by

$$\widetilde{P}(t,T) := P_0^{-1}(t) \cdot P(t,T), \;\; 0 \leq t \leq T.$$

An important concept in interest-rate market theory and mathematical finance in general is the concept of an equivalent martingale measure, i.e. a probability measure $\widetilde{Q}$ on $(\Omega, \mathcal{F})$ equivalent to Q under which the discounted price processes $\widetilde{P}(t,T)$ are $\widetilde{Q}$-martingales. A major characteristic of interest-rate markets is the existence of infinitely many primary assets, since there are infinitely many maturities T with $T \leq T^*$, where T^* denotes the maximum time horizon of our interest-rate market. A probability measure $\widetilde{Q}$ is an equivalent martingale measure if it is an equivalent martingale measure for any finite interest-rate market, i.e. for any interest-rate market with a finite number of zero-coupon bonds. The following theorem, which is adapted from Zagst (2002a), p. 103, states the conditions under which such an equivalent martingale measure exists.

Theorem 3.3. *(Existence of equivalent martingale measure) Suppose that there exists an m-dimensional progressively measurable stochastic process γ such that:*

(i) The following Novikov condition holds for γ*:*

$$E_Q\left[e^{\frac{1}{2}\cdot\int_{t_0}^{T^*}||\gamma(s)||^2 ds}\right] < \infty.$$

(ii) The no-arbitrage condition

$$\mu_P(t,T) - \sigma_P(t,T)\gamma(t) = r(t)\cdot P(t,T)$$

holds for all $t_0 \le t \le T \le T^*$*.*

Furthermore let the probability measure $\widetilde{Q}$ *on* $(\Omega, \mathcal{F}_{T^*}) = (\Omega, \mathcal{F})$ *be defined by*

$$\frac{d\widetilde{Q}}{dQ} = L(\gamma, T). \tag{3.9}$$

with $L(\gamma,t) := e^{-\int_{t_0}^t \gamma(s)' dW(s) - \frac{1}{2}\cdot\int_{t_0}^t ||\gamma(s)||^2 ds}$*. Then, the stochastic process* $\widetilde{W}$ *defined by*

$$d\widetilde{W} := \gamma(t)dt + dW(t), \quad t \in [t_0, T^*] \tag{3.10}$$

is a $\widetilde{Q}$*-Wiener process and the discounted price processes* $\widetilde{P}(t,T)$ *have the following representation in terms of* $\widetilde{W}$*:*

$$\begin{aligned} d\widetilde{P}_0(t) &= 0 \\ d\widetilde{P}(t,T) &= \widetilde{\sigma}_P(t,T)d\widetilde{W}(t) \end{aligned}$$

for $t_0 \le t \le T \le T^*$*. Furthermore,*

$$dP(t,T) = r(t)\cdot P(t,T)dt + \sigma_P(t,T)d\widetilde{W}(t).$$

If the martingale condition

$$E_{\widetilde{Q}}\left[\int_{t_0}^{T^*} ||\widetilde{\sigma}_P(s,T)||^2 ds\right] < \infty$$

is satisfied for all $t_0 \le T \le T^*$*, then* $\widetilde{Q}$ *is an equivalent martingale measure.*

Proof. See Zagst (2002a), p. 104f. □

The existence of an equivalent martingale measure is important for the pricing of contingent claims. A (European) contingent claim (with maturity T) is a random variable $D(T)$, with $e^{-\int_t^T r(s)ds}\cdot D(T)$ lower bounded for all $t \in [0,T]$, on $(\Omega, \mathcal{F}_T)$.

In this thesis we will assume that the interest market is complete and that the equivalent martingale measure $\widetilde{Q}$ is unique. Thus, every contingent claim in our interest-rate market is attainable (i.e. for each contingent claim there exists a hedging strategy replicating the contingent claim) and the price $V_D(t)$ of the contingent claim D with maturity T_D is given by the risk-neutral valuation formula (see, e.g., Zagst (2002a), p. 107):

$$V_D(t) = P_0(t) \cdot E_{\widetilde{Q}} \left[P_0^{-1}(t) \cdot D(T_D) | \mathcal{F}_t \right].$$

3.2.2 The Vasicek and Hull-White Models

In interest-rate market theory, one of the major challenges is to find a model which is able to describe the price movements of the universe of zero-coupon bonds with different maturities, i.e. to find a model which adequately captures the dynamics of the term structure of interest rates. One-factor models like the Vasicek model and the Hull-White model, which we present in this section, or the Cox-Ingersoll-Ross (CIR) model, which will be the topic of the following section, still play a key role in interest-rate theory. A particularly appealing feature of these one-factor models is their analytical tractability which makes it possible to price interest-rate derivatives such as bond options, Caps and Floors in closed form. This is often not the case in more complex multi-factor models. The Vasicek model was originally developed in Vasicek (1977) and extended in Hull and White (1990) to the Hull-White model. The original paper concerned with the CIR model is Cox et al. (1985). For a more complete overview of one and multi-factor interest rate models, also with respect to tests and implementations, see, e.g., Rebonato (1998) or Brigo and Mercurio (2006).

In the Hull-White model the (risk-free) short rate is given by the dynamics (under the real-world measure Q):

$$dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r(t)$$

where a_r, σ_r are some positive constants, W_r is a 1-dimensional Wiener process and $\theta_r(t)$ is a deterministic function. If $\theta_r(t)$ is a constant, the Hull-White model reduces to the model considered by Vasicek (1977).

Now assume that there exists a progressively measurable stochastic process $\gamma(t)$ such that

$$\left. \frac{d\widetilde{Q}}{dQ} \right|_{\mathcal{F}_t} = e^{-\int_0^t \gamma(s)' dW(s) - \frac{1}{2} \cdot \int_0^t ||\gamma(s)||^2 ds}.$$

Further assume that $\gamma(t)$ satisfies the Novikov condition (i) in Theorem 3.3 and that there exists a constant $\lambda_r \in \mathbb{R}$ such that

$$\gamma(t) = \lambda_r \sigma_r r(t).$$

Then, according to Theorem 3.3,

$$d\widetilde{W}_r := \gamma(t)dt + dW_r(t), \quad t \in [t_0, T^*]$$

is a $\widetilde{Q}$-Wiener process. Defining $\hat{a}_r := a_r + \lambda_r \sigma_r^2$ the dynamics of the short rate under the equivalent martingale measure $\widetilde{Q}$, also called the risk-neutral measure, are given by:

$$dr(t) = (\theta_r(t) - \hat{a}_r r(t))dt + \sigma_r d\widetilde{W}_r.$$

The function $\theta_r(t)$ in the Hull-White model is given by

$$\theta_r(t) := f_T(0,T)|_{T=t} + \hat{a}_r \cdot f(0,t) + \frac{\sigma_r^2}{2\hat{a}_r}(1 - e^{-2\hat{a}_r t}). \tag{3.11}$$

This choice of $\theta_r(t)$ ensures that the Hull-White model is arbitrage-free, i.e. that the model prices of the zero-coupon bonds replicate the currently observed market prices. In fact, the initial yield curve is a model input for the Hull-White model via the market forward rates $f(0,t)$ and $\theta_r(t)$ is fitted to this input yield curve. For a constant θ_r, as in the Vasicek model, the yield curve is a model output.

In both the Vasicek model and the Hull-White model the price of a zero-coupon bond $P(t,T)$ is given by (see, e.g., Zagst (2002a), p. 136f.)

$$P(t,T) = e^{\hat{A}(t,T) - \hat{B}(t,T) \cdot r(t)} \tag{3.12}$$

with

$$\begin{aligned} \hat{A}(t,T) &= \int_t^T (\frac{1}{2}\sigma_r^2 \hat{B}(l,T) - \theta_r(l)\hat{B}(l,T))dl, \\ \hat{B}(t,T) &= \frac{1}{\hat{a}_r}(1 - e^{-\hat{a}_r(T-t)}), \end{aligned}$$

which yields in the Vasicek case with a constant θ_r (see, e.g., Zagst (2002a), p. 126 or Brigo and Mercurio (2006), p. 59)

$$\hat{A}(t,T) = \left(\frac{\theta_r}{\hat{a}_r} - \frac{\sigma^2}{2\hat{a}_r^2}\right)[\hat{B}(t,T) - T + t] - \frac{\sigma^2}{4\hat{a}_r} \cdot \hat{B}(t,T)^2 \tag{3.13}$$

and in the Hull-White case (see, e.g., Zagst (2002a), p. 139)

$$\begin{aligned}\hat{A}(t,T) &= \ln\left(\frac{P(0,T)}{P(0,t)}\right) + \hat{B}(t,T)\cdot f(0,t)\\ &\quad -\frac{1}{2}\cdot\hat{B}(t,T)\cdot\sqrt{\frac{\sigma_r^2}{2\hat{a}_r}\cdot(1-e^{-2\hat{a}_r\cdot t})}. \end{aligned} \quad (3.14)$$

Finally, we want to discuss some distributional properties of the short rate under the real-world measure Q, which we will need explicitly in Chapter 9. Since a linear stochastic differential equation (SDE)

$$dX(t) = (H\cdot X(t) + J(t))dt + VdW(t) \quad (3.15)$$

with an m-dimensional stochastic process X, $H\in\mathbb{R}^{m\times m}, V\in\mathbb{R}^{m\times m}$, $J: [0,\infty)^m\to\mathbb{R}^m$ continuous, has the unique strong solution

$$X(t) = e^{H\cdot t}X(0) + \int_0^t e^{H\cdot(t-l)}J(l)dl + \int_0^t e^{H\cdot(t-l)}VdW(l)$$

(see, e.g., Karatzas and Shreve (1991), [5.6]), we get by defining $m=1$, $X(t)=r(t)$, $H=-a_r$, $J(t)=\theta_r(t)$ and $V=\sigma_r$:

$$r(t) = e^{-a_r\cdot t}r(0) + \int_0^t e^{-a_r(t-l)}\theta_r(l)dl + \int_0^t e^{-a_r(t-l)}\sigma_r dW_r(l). \quad (3.16)$$

In the Vasicek case, (3.16) simplifies to

$$r(t) = e^{-a_r\cdot t}\left[r(0) + \frac{\theta_r}{a_r}\cdot(e^{a_r\cdot t}-1) + \int_0^t e^{a_r\cdot l}\sigma_r dW_r(l)\right].$$

Obviously, the distribution of both $r(t)$ and $\int_0^T r(t)dt$ is normal and a straightforward calculation yields the formulas for the expectation and variance of $\int_0^T r(t)dt$ in the Vasicek model, given $\mathcal{F}_0$ (see, e.g., Mamon (2004) for a detailed derivation):

$$\begin{aligned} E_Q\left[\int_0^T r(t)dt|\mathcal{F}_0\right] &= \left(r(0)-\frac{\theta_r}{a_r}\right)\cdot B(0,T) + \frac{\theta_r}{a_r}\cdot T,\\ Var_Q\left[\int_0^T r(t)dt|\mathcal{F}_0\right] &= V(0,T) \end{aligned}$$

with

$$B(0,T) := \frac{1}{a_r}(1-e^{-a_r\cdot T}) \quad (3.17)$$

$$V(0,T) := \frac{\sigma_r^2}{a_r^2}\left(T + \frac{2}{a_r}e^{-a_rT} - \frac{1}{2a_r}e^{-2a_rT)} - \frac{3}{2a_r}\right). \quad (3.18)$$

In the Hull-White model, (3.16) can be written in the form (see Brigo and Mercurio (2006), p. 73)

$$r(t) = \alpha(t) + \int_0^t e^{-a_r(t-l)}\sigma_r dW_r(l), \tag{3.19}$$

where

$$\alpha(t) := f(0,t) + \frac{\sigma_r^2}{2a_r^2} \cdot (1 - e^{-a_r \cdot t})^2.$$

As in the Vasicek model, the distribution of both $r(t)$ and $\int_0^T r(t)dt$ in the Hull-White model is obviously normal and from (3.19) we can calculate the expectation and variance of $\int_0^T r(t)dt$ given $\mathcal{F}_0$ under the real-world measure Q.

Lemma 3.4. *In the Hull-White model as previously introduced it holds that under the real-world measure Q*

$$\int_0^T r(t)dt \quad \sim \quad N(a_T; V(0,T)), \tag{3.20}$$

where

$$\begin{aligned} a_T \quad := \quad & -\ln P(0,T) + \frac{\sigma_r^2}{a_r^2} \cdot [T - 2B(0,T) \\ & + \frac{1}{2a_r} \cdot (1 - e^{-2a_r \cdot T})] \end{aligned} \tag{3.21}$$

and $B(0,T)$, $V(0,T)$ are as defined in (3.17) and (3.18), respectively.

Proof. The fact that $\int_0^T r(t)dt$ is normally distributed follows directly from (3.19), as previously stated. For the expectation, given $\mathcal{F}_0$, we obtain from (3.19):

$$\begin{aligned} E_Q\left[\int_0^T r(t)dt\right] &= -\ln P(0,T) + \frac{\sigma_r^2}{2a_r^2} \cdot \int_0^T (1 - e^{-a_r \cdot t})^2 dt \\ &= -\ln P(0,T) + \frac{\sigma_r^2}{2a_r^2} \cdot \left[T - 2 \cdot \int_0^T e^{-a_r \cdot t} dt \right. \\ & \left. + \int_0^T e^{-2 \cdot a_r \cdot t} dt \right] \end{aligned}$$

A straightforward calculation of the integrals yields

$$\begin{aligned} E_Q\left[\int_0^T r(t)dt\right] &= -\ln P(0,T) + \frac{\sigma_r^2}{2a_r^2} \cdot \left[T - \frac{2}{a_r} \cdot (1 - e^{-a_r \cdot T}) \right. \\ & \left. + \frac{1}{2a_r} \cdot (1 - e^{-2a_r \cdot T})\right], \end{aligned}$$

from which (3.21) follows with the definition of $B(0,T)$.

Moreover, we can calculate the variance of $\int_0^T r(t)dt$ from (3.19), using Fubini's theorem:

$$
\begin{aligned}
Var_Q\left[\int_0^T r(t)dt\right] &= Var_Q\left[\int_0^T \int_0^t e^{-a_r(t-l)}\sigma_r dW_r(l)dt\right] \\
&= Var_Q\left[\sigma_r \cdot \int_0^T e^{-a_r \cdot t} \cdot \int_0^t e^{a_r \cdot l} dW_r(l)dt\right] \\
&= Var_Q\left[\sigma_r \cdot \int_0^T e^{a_r \cdot l} \cdot \left(\int_l^T e^{-a_r \cdot t}dt\right) dW_r(l)\right] \\
&= Var_Q\left[\sigma_r \cdot \int_0^T \frac{1}{a_r} \cdot (1 - e^{-a_r \cdot (T-t)})dW_r(t)\right] \\
&= \frac{\sigma_r^2}{a_r^2} \cdot Var_Q\left[\int_0^T (1 - e^{-a_r \cdot (T-t)})dW_r(t)\right].
\end{aligned}
$$

Due to the Ito isometry (see, e.g., Zagst (2002a), p.24) we finally obtain

$$
\begin{aligned}
Var_Q\left[\int_0^T r(t)dt\right] &= \frac{\sigma_r^2}{a_r^2} \cdot \int_0^T (e^{-a_r \cdot (T-t)} - 1)^2 dt \\
&= \frac{\sigma_r^2}{a_r^2} \cdot \left[\int_0^T e^{-2a_r \cdot (T-t)}dt - 2\int_0^T e^{-a_r \cdot (T-t)}dt + T\right] \\
&= \frac{\sigma_r^2}{a_r^2} \cdot \left[T + \frac{1}{2a_r} \cdot \left(1 - e^{-2a_r \cdot T}\right) - \frac{2}{a_r} \cdot \left(1 - e^{-a_r \cdot T}\right)\right] \\
&= V(0,T)
\end{aligned}
$$

□

3.2.3 The Cox-Ingersoll-Ross Model

One major inconvenience of the Hull-White model, as introduced in the previous section, is the fact that interest rates may become negative, which is often considered unrealistic. This is not the case in the model developed by Cox et al. (1985), which is known as the CIR model. Yet, in its original version, this model is not able to provide an exact fit of the initially observed yield curve, similar to the Vasicek model. Arbitrage-free extensions of the CIR model have been proposed in the literature (see, e.g., Brigo and Mercurio (2001) or Schmid (2004)). These efforts, however, lead in general to a loss in analytical tractability, significantly complicate numerical calculations, and closed-form pricing of common interest-rate derivatives may become infeasible in arbitrage-free CIR extensions. We thus work with the original

CIR model in this thesis.

In the CIR model the (non-defaultable) short rate is given by the dynamics (under the real-world measure Q):

$$dr(t) = (\theta_r - a_r r(t))dt + \sigma_r \sqrt{r(t)} dW_r(t), \tag{3.22}$$

where θ_r, a_r, σ_r are some positive constants with $2\theta_r > \sigma_r^2$ and $W_r(t)$ is a 1-dimensional Wiener process. Assuming again that there exists a progressively measurable stochastic process $\gamma(t)$ such that

$$\left.\frac{d\widetilde{Q}}{dQ}\right|_{\mathcal{F}_t} = e^{-\int_0^t \gamma(s)' dW(s) - \frac{1}{2}\cdot\int_0^t ||\gamma(s)||^2 ds},$$

that $\gamma(t)$ satisfies the Novikov condition (i) in Theorem 3.3 and that there exists a constant $\lambda_r \in \mathbb{R}$ such that

$$\gamma(t) = \lambda_r \sigma_r \sqrt{r(t)},$$

then, according to Theorem 3.3,

$$d\widetilde{W}_r := \gamma(t)dt + dW_r(t), \quad t \in [0, T^*]$$

is a $\widetilde{Q}$-Wiener process and

$$dr(t) = (\theta_r - \hat{a}_r r(t))dt + \sigma_r \sqrt{r(t)} d\widetilde{W}_r(t)$$

are the dynamics of the short rate under the risk-neutral measure $\widetilde{Q}$ with $\hat{a}_r := a_r + \lambda_r \sigma_r^2$.

The CIR model is, as well as the Hull-White model, a short-rate model with affine term structure and the zero-coupon bond prices in the CIR model are given by (see Cox et al. (1985))

$$P(t,T) = e^{\hat{A}(t,T) - \hat{B}(t,T)\cdot r(t)} \tag{3.23}$$

with

$$\begin{aligned} \hat{A}(t,T) &= \frac{2\theta_r}{\sigma_r^2} \cdot \ln\left[\frac{\gamma \cdot e^{\kappa_2 \cdot (T-t)}}{\kappa_1 - e^{-\gamma\cdot(T-t)}}\right] \\ \hat{B}(t,T) &= \frac{1 - e^{-\gamma(T-t)}}{\kappa_1 - \kappa_2 e^{-\gamma(T-t)}} \end{aligned}$$

and $\gamma := \sqrt{\hat{a}_r^2 + 2\sigma_r^2}$, $\kappa_1 := \frac{\hat{a}_r}{2} + \frac{\gamma}{2}$, $\kappa_2 := \frac{\hat{a}_r}{2} - \frac{\gamma}{2}$. While in the Vasicek and Hull-White models, the distribution of the short rate is Gaussian, as discussed in the previous section, the distribution of the short rate in the CIR model is the non-central χ^2-distribution. More precisely, if we consider the distribution under the risk-neutral measure, it holds that given $\mathcal{F}_0$ (see Cox et al. (1985))

$$2 \cdot c \cdot r(t) \sim \chi^2(2q + 2, 2u),$$

where

$$c := \frac{2\hat{a}_r}{\sigma_r^2 \cdot (1 - e^{-\hat{a}_r \cdot t})}, \tag{3.24}$$

$$u := c \cdot r(0) \cdot e^{-\hat{a}_r \cdot t}, \tag{3.25}$$

$$q := \frac{2\theta_r}{\sigma_r^2} - 1$$

and $\chi^2(a, b)$ denotes the non-central χ^2-distribution with degrees of freedom parameter a and non-centrality parameter b. Of course, if we replace $\hat{a}_r$ by a_r in (3.24) and (3.25) we obtain the short-rate distribution under the real-world measure Q.

We conclude this section by remarking that, despite the analytical inconveniences of the non-central χ^2-distribution compared to the normal distribution, it is possible to derive closed-form formulas for options on zero-coupon bonds in the CIR model, as well as in the Vasicek and Hull-White models. (see, e.g., Brigo and Mercurio (2006)). Thus, many common interest-rate derivatives such as Caps and Floors can conveniently be priced in all short-rate models which we use in this thesis.

3.3 Point processes and intensities

Since we will need the concepts of point processes and intensities in our valuation models in the following chapters, we give a brief overview of the basic ideas and theorems in this section. Applied to financial modelling, intensity-based models are often labelled 'reduced-form' models and have become a popular tool, particularly in the context of credit risk modelling.

3.3.1 Theoretical overview

We start with a point or counting process $N(t)$ which we define on the probability space $(\Omega, \mathcal{G}, Q)$ by

$$N(t) = \sum_i 1_{\{\tau_i \leq t\}},$$

where $\{\tau_i, i \in \mathbb{N}\}$ is a collection of stopping times with respect to some filtration $\{\mathcal{F}_t^N\}_{t\geq 0}$, indexed in ascending order. Throughout this thesis, we will also assume that $\tau_i \neq \tau_j$ for $i \neq j$ (i.e. $\tau_i < \tau_{i+1}$ for all i) and that the point process is nonexplosive, i.e. $\lim_{n\to\infty} \tau_n = \infty$. The process $N(t)$ can thus be considered a stochastic process, counting the number of events associated with the stopping times τ_i. We assume that $(\Omega, \mathcal{G}, Q)$ is equipped with three filtrations $\{\mathcal{G}_t\}_{t\geq 0}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathcal{F}_t^N\}_{t\geq 0}$. Let $\{\mathcal{F}_t^N\}_{t\geq 0}$ be the filtration generated by the counting process $N(t)$ and let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by all other considered processes, excluding the counting process. Let furthermore

$$\{\mathcal{G}_t\}_{t\geq 0} = \{\mathcal{F}_t\}_{t\geq 0} \vee \{\mathcal{F}_t^N\}_{t\geq 0}.$$

The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is called 'background filtration' by Schönbucher (2003). We will assume throughout that for any $t \in (0, T^*]$ the σ-fields $\mathcal{F}_{T^*}$ and $\mathcal{F}_t^N$ are conditionally independent (under the martingale measure $\widetilde{Q}$) given $\mathcal{F}_t$. This is equivalent to the assumption that $\{\mathcal{F}_t\}_{t\geq 0}$ has the so-called martingale invariance property with respect to $\{\mathcal{G}_t\}_{t\geq 0}$ and for any $t \in (0, T^*]$ and any $\widetilde{Q}$-integrable $\mathcal{F}_{T^*}$-measurable random variable X we have $E_{\widetilde{Q}}[X|\mathcal{G}_t] = E_{\widetilde{Q}}[X|\mathcal{F}_t]$ (see, e.g., Bielecki and Rutkowski (2002), p. 242 for details). The following definition introduces the concept of intensity.

Definition 3.5. *(Intensity)*
Let $N(t)$ be a point process as previously introduced, adapted to the filtration $\{\mathcal{F}_t^N\}_{t\geq 0}$ and let $\gamma(t)$ be a nonnegative $\mathcal{F}_t$-progressively measurable process with

$$\int_0^t \gamma(s)ds < \infty$$

Q-a.s. for all t. If for all nonnegative $\mathcal{F}_t$-predictable processes $C(t)$ the equality

$$E_Q\left[\int_0^\infty C(s)dN(s)\right] = E_Q\left[\int_0^\infty C(s)\gamma(s)ds\right]$$

holds, the point process $N(t)$ is said to admit the $(Q, \mathcal{F}_t)$-intensity $\gamma(t)$.

The following theorems are adapted from Schmid (2004), p.60, and are concerned with crucial properties, existence and uniqueness of intensities.

Theorem 3.6. *(Martingale Characterisation)*
If $N(t)$ admits the $(Q, \mathcal{F}_t)$-intensity $\gamma(t)$, then $N(t)$ is nonexplosive and

$$M(t) := N(t) - \int_0^t \gamma(s)ds \tag{3.26}$$

is a ($\mathcal{G}_t$-local) martingale. Conversely, let $N(t)$ be a nonexplosive point process adapted to $\mathcal{F}_t^N$, and suppose that for some nonnegative $\mathcal{F}_t$-progressively measurable process $\gamma(t)$ and for all $n \geq 1$,

$$N(t \wedge \tau_n) - \int_0^{t \wedge \tau_n} \gamma(s) ds$$

is a $(Q, \mathcal{G}_t)$-martingale. Then, $\gamma(t)$ is the $(Q, \mathcal{F}_t)$-intensity of $N(t)$.

Proof. See Schmid (2004), p. 60, and Brémaud (1981), p.27f. □

The integral

$$\Gamma(t) := \int_0^t \gamma(s) ds \tag{3.27}$$

is usually called the compensator of $N(t)$.

Theorem 3.7. *(Existence and Uniqueness of Predictable Intensity)*
Let $N(t)$ be a point process with a $(Q, \mathcal{F}_t)$-intensity $\gamma(t)$. Then one can find a $(Q, \mathcal{F}_t)$-intensity $\tilde{\gamma}(t)$ which is $\mathcal{F}_t$-predictable. Now, let $\tilde{\gamma}(t)$ and $\bar{\gamma}(t)$ be two $(Q, \mathcal{F}_t)$-intensities of $N(t)$ which are $\mathcal{F}_t$-predictable. Then $\tilde{\gamma}(t) = \bar{\gamma}(t)$ $Q(d\omega)dN(t, \omega)$ almost everywhere.

Proof. See Schmid (2004), p. 60, and Brémaud (1981), p.31. □

Let us assume for the moment that we have only one stopping time τ such that $N(t) = 1_{\{\tau \leq t\}}$. I.e. $N(t)$ is the indicator function associated with some event τ, for example the prepayment time of a mortgage. Let us further assume that $N(t)$ admits the $(Q, \mathcal{F}_t)$-intensity $\gamma(t)$. Then, recalling that $M(t)$ as defined in (3.26) is a martingale,

$$M_{t \wedge \tau} := N(t) - \int_0^{t \wedge \tau} \gamma(s) ds$$

is also a martingale and it is straightforward to see that

$$\begin{aligned} E_Q[N(t+\epsilon) - N(t) | \mathcal{G}_t] &= E_Q[M_{(t+\epsilon)\wedge\tau} - M_{t\wedge\tau} | \mathcal{G}_t] \\ &\quad + E_Q\left[\int_t^{t+\epsilon} \gamma(s) \cdot 1_{\{s<\tau\}} ds\right] \\ &= M_{t\wedge\tau}(t) - M_{t\wedge\tau}(t) \\ &\quad + E_Q\left[\int_t^{t+\epsilon} \gamma(s) \cdot 1_{\{s<\tau\}} ds\right] \\ &= E_Q\left[\int_t^{t+\epsilon} \gamma(s) \cdot 1_{\{s<\tau\}} ds\right]. \end{aligned} \tag{3.28}$$

Furthermore, it can be shown (see, e.g., Schmid (2004), p.61) that

$$\gamma(t) \cdot 1_{\{t \leq \tau\}} = \lim_{\epsilon \to 0+} \frac{Q^\tau(t, t+\epsilon)}{\epsilon}, \tag{3.29}$$

where

$$Q^\tau(t, t+\epsilon) := Q(\tau \in (t, t+\epsilon] | \mathcal{G}_t) = E_Q[N(t+\epsilon) - N(t) | \mathcal{G}_t]$$

is the probability that the event τ occurs in the time period from t to $t+\epsilon$. Thus, the intensity $\gamma(t)$ can be considered as the arrival rate of the event associated with τ, given all information at time t. If, for example, τ is associated with prepayment of a particular mortgage loan, we can conclude that the probability of prepayment over the next infinitesimal time interval of length ϵ is approximately given by $\gamma(t) \cdot \epsilon$. From (3.28) and (3.29) it follows that

$$Q(\tau \in (t, T] | \mathcal{G}_t) = E_Q\left[\int_t^T \gamma(s) \cdot 1_{\{s<\tau\}} ds | \mathcal{G}_t\right].$$

Lemma 3.8. *(Survival probability)*
Let τ be a stopping time with a bounded intensity γ or with an intensity satisfying the integrability conditions as stated in, e.g., Duffie (1998), p. 5. Fixing some time $T > 0$, let for $t < T$

$$Y_t := E_Q\left[e^{-\int_t^T \gamma(s) ds} | \mathcal{F}_t\right].$$

Then, if $Y_\tau - Y_{\tau-}$ is zero almost surely,

$$Q(\tau \in (t, T] | \mathcal{G}_t) = (1 - Y_t) \cdot 1_{\{\tau > t\}}.$$

Proof. The lemma is taken from Schmid (2004), p. 62. A proof can be found in Duffie (1998), p. 4f. □

Thus the 'survival' probability, i.e. the probability that the event associated with τ has not occurred until time T, is given by:

$$Q(\tau > T | \mathcal{G}_t) = E_Q\left[e^{-\int_t^T \gamma(s) ds} | \mathcal{F}_t\right] \cdot 1_{\{\tau > t\}}. \tag{3.30}$$

As a next step we generalise the previously introduced concept of point processes and attach a 'marker' to each event τ_i. We consider the double sequence $\{(\tau_i, Y_i), i \in \mathbb{N}\}$, where the stopping times τ_i are responsible for the timing of the event(s) and the marker variables Y_i, drawn from a measurable space $(E, \mathcal{E})$, determine the magnitude. The double sequence $\{(\tau_i, Y_i), i \in \mathbb{N}\}$ is called a marked point process. In order to formalise the concept of marked point processes we need to define jump measures, which are special cases of the more generally defined random measures.

Definition 3.9. *(Random measure)*
$\nu : \Omega \times \mathcal{E} \times \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}_+$ *is a random measure if for every* $\omega \in \Omega$, $\nu(\omega, \cdot, \cdot)$ *is a measure on* $((E \times \mathbb{R}_+), \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+))$ *and* $\nu(\omega, E, \{0\}) = 0$.

Definition 3.10. *(Jump measure)*
The jump measure of a marked point process $\{(\tau_i, Y_i), i \in \mathbb{N}\}$ *is a random measure on* $E \times \mathbb{R}_+$ *such that for all* $E' \in \mathcal{E}$*:*

$$\mu(\omega, E', [0,t]) = \int_0^t \int_{E'} \mu(\omega, de, ds) := \sum_{i=1}^{\infty} 1_{\{\tau_i(\omega) \le t\}} 1_{\{Y_i(\omega) \in E'\}}$$

for all $\omega \in \Omega$.

Note at this point that with the previous definitions the counting process associated with any marked point process, i.e. the number of jumps of a marked point process until a given time t, is given by:

$$N(t) = \int_0^t \int_E 1 \cdot \mu(de, ds).$$

Before we proceed with the definition of the compensator measure of a marked point process, we consider the special case more closely, where the marker space E contains only the element $\{1\}$. In this case the marked point process simply reduces to the earlier defined counting process $N(t)$. If the intensity γ is constant, $N(t)$ is usually called a Poisson process. If the intensity $\gamma(t)$ is a (non-constant) deterministic function of time, the process is usually called a time-inhomogeneous Poisson process. The next step is the incorporation of stochastic intensities. This yields a doubly stochastic Poisson process, also called Cox process, which can be defined as follows (see also Schönbucher (2003), p. 121):

Definition 3.11. *(Cox process)*
A point process $N(t)$ *with intensity process* $\gamma(t)$ *is a Cox process if, conditional on the background filtration* $\{\mathcal{F}_t\}_{t \ge 0}$, $N(t)$ *is a time-inhomogeneous Poisson process with intensity* $\gamma(t)$.

Note that this definition ensures that the Cox process can not be measurable with respect to $\{\mathcal{F}_t\}_{t \ge 0}$. Thus, knowledge of the intensity process does not reveal any information about the realisation of $N(t)$.

For the extension of the concept of the compensator (see (3.27)) from point processes to marked point processes, we need to define predictability in the context of random measures. The analogue to the $\mathcal{F}_t$-predictability of the process $C(s)$ in Definition 3.5 and of the intensity $\gamma(t)$ (see Theorem 3.7) is given in the following definition for stochastic functions and random measures.

Definition 3.12. *(Predictable stochastic function, random measure)*
A predictable stochastic function $f : (\Omega \times \mathbb{R}_+) \times E \to \mathbb{R}$ is a function which is measurable with respect to the σ-algebra $\mathcal{P} \otimes \mathcal{E}$, where $\mathcal{P}$ is the σ-algebra generated by the adapted left-continuous processes on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, Q)$. A random measure ν is called predictable if for every predictable stochastic function f, the integral process

$$X(\omega, t) := \int_0^t \int_E f(\omega, e, s) \nu(\omega, de, ds)$$

is again a predictable process.

The compensator measure of the jump measure of a marked point process $\mu(\omega, de, ds)$, whose existence and uniqueness has been proven by , e.g., Liptser and Shiryaev (2001), Chapter 18.3, is defined as follows:

Definition 3.13. *(Compensator measure)*
Let $f(\omega, e, s)$ be a predictable stochastic function and let $\mu(\omega, de, ds)$ be the jump measure of a marked point process. The compensator measure $\nu(\omega, de, dt)$ is the unique (a.s.) predictable random measure with the following property: $M(\omega, t)$, defined by

$$M(\omega, t) := \int_0^t \int_E f(\omega, e, s) \mu(\omega, de, ds) - \int_0^t \int_E f(\omega, e, s) \nu(\omega, de, ds),$$

is a martingale for all predictable stochastic functions f.

An easy, but at the same time the most important example for our purposes, is the Cox process which we will use in the following chapters for our MBS valuation models. If $\mu(\omega, de, ds)$ is the jump measure of a Cox process with (stochastic) intensity process $\gamma(t)$, it is straightforward to see that the compensator measure of the Cox process is given by

$$\nu(\omega, de, ds) = \delta_{Y=1}(de) \gamma(t) dt,$$

where Y denotes the marker variable and $\delta_{Y=1}$ is defined by

$$\delta_{Y=1} = \begin{cases} 1 & \text{for } Y = 1 \\ 0 & \text{otherwise} \end{cases} .$$

In Section 3.2 about interest-rate market theory we have already introduced the concept of an equivalent martingale measure which we need as a pricing measure for interest-rate derivatives. Theorem 3.3 states how the Radon-Nikodym derivative (3.9) determines which processes become Wiener processes after a transition from the probability measure Q to the equivalent martingale measure $\widetilde{Q}$ (see (3.10)). Considering marked point processes, it is evident that the compensator measure is, in general, affected by such a change of measure since the compensator measure describes the probabilities of the marked point process dynamics. So, the question arises which form the compensator measure takes after the measure change. The answer is given by the Girsanov theorem for marked point processes, which, in its general form as given below, is valid for probability spaces supporting marked point processes and diffusions.

Theorem 3.14. *(Girsanov theorem for marked point processes)*
Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, Q)$ be a filtered probability space which supports an n-dimensional Q-Wiener process $W(t)$ and a marked point process with jump measure $\eta(de, dt)$. The marker e of the marked point process is drawn from the mark space $(E, \mathcal{E})$. The compensator measure of $\eta(de, dt)$ is assumed to take the form $\nu^Q(de, dt) = K^Q(t, de)\gamma^Q(t)dt$ under Q. Here $\gamma^Q(t)$ is the Q-intensity of the arrivals of the point process and $K^Q(t, de)$ is the Q-conditional distribution of the marker on $(E, \mathcal{E})$.
Let ϕ be an n-dimensional predictable process and $\Phi(e, t)$ a non-negative predictable function with

$$\int_0^t |\phi_i(s)|^2 ds < \infty, \qquad \int_0^t \int_E |\Phi(e, s)| K^Q(t, de)\gamma^Q(s) ds < \infty$$

for any finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \phi(t)dW(t) + \int_E (\Phi(e, t) - 1)(\eta(de, dt) - \nu^Q(de, dt)).$$

Assume that $E_Q(L(t)) = 1$ for finite t. Define the probability measure $\widetilde{Q}$ with

$$\left.\frac{d\widetilde{Q}}{dQ}\right| \mathcal{F}_t = L(t), \qquad \forall t \geq 0.$$

Then:

(i) The process $\widetilde{W}$ with $\widetilde{W}(0) = 0$ and

$$d\widetilde{W}(t) := dW + \phi(t)dt$$

is a $\widetilde{Q}$-Wiener process.

(ii) The predictable compensator measure of η under $\widetilde{Q}$ is

$$\nu^{\widetilde{Q}}(de, dt) = \Phi(e, t)\nu^Q(de, dt) \tag{3.31}$$

(iii) Define $\mu(t) := \int_E \Phi(e,t)K^Q(t,de)$ *and* $L_E(e,t) = \frac{\Phi(e,t)}{\mu(t)}$ *for* $\mu(t) > 0$, $L_E(e,t) = 1$ *otherwise. The intensity of the counting process of the arrivals of the marked point process under* $\widetilde{Q}$ *is*

$$\gamma^{\widetilde{Q}}(t) = \mu(t)\gamma^Q(t). \tag{3.32}$$

(iv) The conditional distribution of the marker under $\widetilde{Q}$ *is*

$$K^{\widetilde{Q}}(t,de) = L_E(e,t)K^Q(t,de).$$

Proof. The theorem is taken from Schönbucher (2003), p. 108. A formal proof can be found in Jacod and Shiryaev (1987), Chapter III.3. □

3.3.2 Application to the pricing of contingent claims

The concepts introduced in the previous subsection are particularly useful for the pricing of contingent claims whose payoffs depend on a random, exogenously driven termination time associated with a particular event. As previously mentioned, intensity-based models have become very popular in the context of credit risk modelling. In particular, intensity-based models are often used for the pricing of defaultable bonds and credit derivatives. In this thesis, we will apply them to the modelling of prepayment. Both applications have in common that if a particular event occurs a financial contract is terminated prior to its final maturity. Let us assume that we have a stopping time τ associated with the termination event and with the first jump of a counting process $N(t)$ with (stochastic) intensity $\gamma(t)$. We denote by D the payoff received by the owner of the contingent claim at the final maturity T, if $\tau > t$. Moreover, we denote by $S(t)$ the stream of cash flows received by the owner of the claim until τ and by $Z(t)$ the recovery payoff at τ, if $\tau \leq T$. The following two definitions, adapted from Schmid (2004), p. 209f., formalise the concept of a contingent claim which can be terminated prior to its final maturity.

Definition 3.15. *(Non-terminable contingent claim)*
We call a triple (D,S,T) consisting of a cumulative dividend process S (any $\mathcal{F}_t$ *adapted process of integrable variation), the* $\mathcal{F}_T$*-measurable random variable D and the time* $T < T^*$ *at which D is paid a (European) non-terminable contingent claim.*

Definition 3.16. *(Terminable contingent claim)*
A (European) terminable contingent claim is a triple $[(D,S,T),Z,\tau]$ *consisting of*

- *a non-terminable (European) contingent claim (D,S,T) yielding payoffs* $\int_0^{T\wedge\tau} dS(u)$ *over the time interval* $[0, T\wedge\tau]$*, and a final payoff* D *at time* T*, provided the event associated with the stopping time* τ *has not occurred until time* T*.*
- *a* $\mathcal{F}_t$*-predictable process* Z *describing the payoff upon occurrence of the event which leads to termination of the contingent claim.*
- *a* $\{\mathcal{F}_t^N\}_{t\geq 0}$ *stopping time* τ *valued in* $[0,\infty)$*, describing the stochastic structure of the arrival time of the event.*

Theorem 3.17. *(Value of a terminable contingent claim)*
If the stopping time τ *admits the (stochastic) intensity* γ*, the value process* $V^{Cl}(t)$ *of a terminable contingent claim* $[(D,S,T),Z,\tau]$ *admits the following representation for* $t\in[0,T]$*:*

$$\begin{aligned} V^{Cl}(t) &= E_{\widetilde{Q}}\Bigg[\int_{(t,T]} e^{-\int_t^u r(s)ds}\cdot 1_{\{\tau>u\}}dS(u) + \int_{(t,T]} e^{-\int_t^u r(s)ds}\cdot Z(u)d1_{\{\tau\leq u\}} \\ &\quad + e^{-\int_t^T r(s)ds}\cdot D\cdot 1_{\{\tau>T\}}\Bigg|\mathcal{G}_t\Bigg] \\ &= 1_{\{\tau>t\}}\cdot E_{\widetilde{Q}}\Bigg[\int_{(t,T]} e^{-\int_t^u (r(s)+\gamma(s))ds}(dS(u)+\gamma(u)Z(u)du)\Bigg|\mathcal{F}_t\Bigg] \\ &\quad + 1_{\{\tau>t\}}\cdot E_{\widetilde{Q}}\Bigg[e^{-\int_t^T (r(s)+\gamma(s))ds}\cdot D\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{3.33}$$

Proof. The theorem is an adapted version of Theorem 8.2.1 in Bielecki and Rutkowski (2002), p. 230f. See also Schmid (2004), p. 210. □

Now, consider the example of a typical non-defaultable, fully amortising mortgage, which can be prepaid at any time. The stopping time τ is thus associated with the time of prepayment, the recovery process is equal to the (deterministic) outstanding notional $A(t)$ according to the amortisation schedule as defined in Chapter 2.2.3, and the payoff at the final maturity T of the contract, if $\tau > T$, is equal to 0. Moreover, if we assume for the moment that the mortgage payment $M(t)$, comprising interest and scheduled principal repayments, is made continuously, it follows that

$$dS(t) = M(t)dt.$$

According to (3.33), the value of the mortgage $V^{Mo}(t)$ is then given by:

$$\begin{aligned} V^{Mo}(t) &= 1_{\{\tau>t\}}\cdot E_{\widetilde{Q}}\Bigg[\int_t^T \Big(M(u)\cdot e^{-\int_t^u (r(s)+\gamma(s))ds} \\ &\quad + A(u)\cdot\gamma(u)\cdot e^{-\int_t^u (r(s)+\gamma(s))ds}\Big)du\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{3.34}$$

If the mortgage payments $M(t_k) \cdot \Delta t_k$ are made at discrete points of time t_1,...,$t_K = T$ with $\Delta t_k := t_k - t_{k-1}$, it follows that the value of the mortgage contract at time $t_0 \leq t \leq t_1$ is given by

$$\begin{aligned} V^{Mo}(t) &= 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K} e^{-\int_t^{t_k}(r(s)+\gamma(s))ds} \cdot M(t_k) \cdot \Delta t_k \\ &\quad + \int_t^T A(u) \cdot \gamma(u) \cdot e^{-\int_t^u (r(s)+\gamma(s))ds} du \,\Bigg|\mathcal{F}_t\Bigg] . \end{aligned} \tag{3.35}$$

If the mortgage payments are made at discrete points of time, the outstanding principal $A(t)$ remains constant between to payment dates. If we then approximate the integral in (3.35) by sums we obtain

$$\begin{aligned} V^{Mo}(t) &= 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K} \Bigg(e^{-\int_t^{t_k}(r(s)+\gamma(s))ds} \cdot (M(t_k) \cdot \Delta t_k \\ &\quad + A(t_k) \cdot \gamma(t_k) \cdot \Delta t_k) + R_{t_k}\Bigg) \Bigg|\mathcal{F}_t\Bigg] , \end{aligned} \tag{3.36}$$

where R_{t_k} are the error terms resulting from the approximation of the integral by sums. Note that R_{t_k} is small if Δt_k is small and that $R_{t_k} \to 0$ for $\Delta t_k \to 0$. We discuss the error term in more detail in the appendix. Neglecting the error terms R_{t_k}, we obtain the approximate value of a mortgage, which we denote by

$$\begin{aligned} V^{Mo}_{app}(t) &:= 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K} e^{-\int_t^{t_k}(r(s)+\gamma(s))ds} \cdot (M(t_k) \cdot \Delta t_k \\ &\quad + A(t_k) \cdot \gamma(t_k) \cdot \Delta t_k) \Bigg|\mathcal{F}_t\Bigg] . \end{aligned} \tag{3.37}$$

(3.34) and (3.37) are important and useful results which we will often refer to in the remainder of this thesis.

3.4 The Kalman filter

This section is concerned with the Kalman filter and with maximum likelihood estimation for state space models, which goes back to Kalman (1960). For a more extensive and detailed discussion of Kalman filtering techniques see, for example, Harvey (1989) or the paper by Koopman et al. (1999), where efficient algorithms for filtering, moment smoothing and simulation smoothing in state space models are presented. Kalman filtering is also discussed and applied to credit spread data in Schmid (2004).

In this thesis we only deal with linear Gaussian state space models, which consist of a transition equation and a measurement equation. The transition equation describes the dynamics of an unobservable state vector, while the measurement equation relates an observable variable to the state vector. The linear Gaussian state space model (in discrete time) is given by:

$$\alpha_t = c_t + T \cdot \alpha_{t-1} + H \cdot \epsilon_t, \quad t = 1, ..., T \quad \text{(transition equation)} \tag{3.38}$$

$$Y_t = d_t + Z \cdot \alpha_t + G \cdot \epsilon_t, \quad t = 1, ..., T \quad \text{(measurement equation)} \tag{3.39}$$

where

α_t	is the unobservable $m \times 1$ state vector at time t
Y_t	is the $N \times 1$ observation vector at time t
c_t, d_t	are unknown fixed effects at time t with dimension $N \times 1$ and $m \times 1$ respectively
ϵ_t	is the $r \times 1$ disturbance vector, where usually $r = m + N$
T, Z, G, H	are the deterministic system matrices with dimensions $m \times m$, $N \times m$, $m \times r$, $N \times r$.

Here, we only consider the case where the matrices T, Z, G, H are constant over time. Furthermore, we assume that the disturbance vectors $\{\epsilon_t\}_{t=1,...,T}$ are independent identically distributed (iid) multivariate-normal random vectors with expectation 0 and with the identity matrix I as covariance, i.e.

$$\epsilon_t \sim N_r(0, I)$$

and that the initial state vector is drawn from a normal distribution with expectation a_0 and covariance P_0, i.e.

$$\alpha_1 \sim N_m(a_0, P_0).$$

When the initial conditions are not explicitly defined, one can assume that the initial state vector is fully diffuse and choose $a_0 = 0$ and $P_0 = \kappa \cdot I$, where κ is some large scalar (see Koopman et al. (1999), p. 117). The maximum likelihood estimation of the parameters in the state space model is based on the Kalman filter which is given in the following:

Algorithm 1. *(Kalman Filter)*

(i) *Set* $t = 0$. *Specify* a_0, P_0

(ii) *Set* $t = t + 1$.
Evaluate the prediction equations:

$$\begin{aligned} a_{t|t-1} &= T \cdot a_{t-1} + c_t \\ P_{t|t-1} &= T \cdot P_{t-1} \cdot T' + GG' \end{aligned}$$

(iii) *Evaluate the update equations:*

$$\begin{aligned} a_t &= a_{t|t-1} + P_{t|t-1} \cdot Z' \cdot F_t^{1} \cdot (y_t - Z \cdot a_{t|t-1} - d_t) \\ P_t &= P_{t|t-1} - P_{t|t-1} \cdot Z' \cdot F_t^{-1} \cdot Z \cdot P_{t|t-1} \end{aligned}$$

with $F_t := Z \cdot P_{t|t-1} \cdot Z' + HH'$.

(iv) *If* $t = T$ *stop, else go back to (ii).*

The following theorem states the distributional properties of the quantities in the Kalman filter.

Theorem 3.18. *(Kalman Filter properties)*
For $t = 1, ..., T$ *it holds that*

$$\left(\begin{array}{c} \alpha_t \\ Y_t \end{array} \right) \Bigg| y_1, ..., y_{t-1} \sim$$

$$N_{m+N} \left(\left(\begin{array}{c} a_{t|t-1} \\ Z \cdot a_{t|t-1} + d_t \end{array} \right), \left(\begin{array}{cc} P_{t|t-1} & P_{t|t-1} \cdot Z' \\ Z \cdot P_{t|t-1} & F_t \end{array} \right) \right)$$

and that

$$\alpha_t | y_1, ..., y_t \sim N_m(a_t, P_t).$$

In particular, a_t *is the minimum mean square estimate of* α_t*, given the data* $y_1, ..., y_t$.

Proof. A proof can be found, e.g., in Harvey (1989), p. 109f. □

Theorem 3.18 delivers the necessary distributional properties for the calculation of the likelihood function of the model, which can be derived by prediction error decomposition (see, e.g., Harvey (1989), Chapter 3.4, for details). With the observations $y_1, y_2, ..., y_T$ and the model parameter vector ϕ, the log-likelihood, up to some constants, is given by

$$\log l(y_1, ..., y_T; \phi) = \sum_{t=1}^{T} \log p(y_t|y_1, ..., y_{t-1}; \phi) \propto -\sum_{t=1}^{T} (\log |F_t| + v_t' F_t^{-1} v_t). \tag{3.40}$$

Thus, maximum likelihood estimates of the parameters ϕ can be obtained by maximising the expression

$$f(\phi|y_1, ..., y_T) = -\sum_{t=1}^{T} (\log |F_t| + v_t' F_t^{-1} v_t).$$

Chapter 4

Mortgage and MBS valuation

Traditionally, the academic literature on the valuation of mortgage-backed securities could be divided into two general categories: The structural, option-based approach where prepayment is related to a mortgagor's rational decision to exercise the prepayment option inherent in the mortgage contract and the econometric approach where an empirically estimated prepayment function, often within a proportional hazard framework, is used to forecast prepayment cash flows. Recently, however, advances in the field of credit risk modelling have motivated a series of new research papers which are concerned with the valuation of mortgage loans and MBS using approaches borrowed from this field. While Nakagawa and Shouda (2005) define an unobservable prepayment cost process which they compare with the firm value process in the default risk literature, intensity-based modelling approaches seem particularly suitable for prepayment modelling and thus for the pricing of mortgages and MBS. While intensity-based prepayment and mortgage valuation models are closely related to the more traditional econometric models, they offer both mathematical rigour and the flexibility of econometric models with respect to explanatory factors and variables. Despite the fact that the purely econometric models are still widely preferred in practice, they have often been subject to criticism for their lack of mathematical rigour (see, e.g., Kagraoka (2002)).

In this chapter we will give an overview of existing mortgage and MBS valuation models and explain the basic ideas underlying the econometric, the option-based and the intensity-based modelling approaches. We will then comment on the shortcomings of the existing models and discuss the current frontiers and further challenges, in particular concerning MBS valuation, which motivate the subsequent MBS chapters in this thesis.

4.1 The different model classes

4.1.1 Econometric models

The traditional, econometric mortgage and MBS valuation models rely on a purely statistical modelling of prepayment rates. In this model class, the prepayment speed is usually considered as response variable in some regression model, where the most important explanatory variables are usually some function(s) of interest rates. Of course, the universe of further potential explanatory variables for prepayment is huge. The most common further explanatory variables include loan age, loan size, seasonal and geographic effects and, if prepayment is considered at pool level, the pool burnout effect. Spahr and Sunderman (1992) provide a good overview of the early econometric models which were first developed in the late 1980s. Schwartz and Torous (1989) and Richard and Roll (1989) remain two popular and frequently cited papers dating back to this period. The Schwartz/Torous model is based on a proportional hazard framework. In their model, the prepayment speed $p(t)$ is given by:

$$p(t; x(t), \theta) = p_0(t, \gamma, \lambda) \cdot e^{x_t' \beta}, \tag{4.1}$$

where t is the time from origination of the mortgage contract,

$$\theta = (\gamma, \lambda, \beta_1, \beta_2, \beta_3, \beta_4)'$$

is the parameter vector which has to be estimated statistically from historical prepayment data, $p_0(t, \gamma, \lambda)$ is the baseline hazard and $x_t = (x_1(t), ..., x_4(t))'$ contains the following explanatory variables:

$$\begin{aligned} x_1(t) &:= c - l(t-s) \\ x_2(t) &:= x_1(t)^3 \\ x_3(t) &:= \ln \frac{PF(t)}{A(t)} \end{aligned}$$

$$x_4(t) := \begin{cases} 1 & \text{if} \quad t = \text{May-August} \\ 0 & \text{if} \quad t = \text{September-April} \end{cases} .$$

Here, c is the mortgage contract rate, l is the default-free consol yield, s accounts for a time lag of three months and $PF(t)$, $A(t)$ are the pool factor and the outstanding loan amount according to the original amortisation schedule as already specified in Chapter 2.2.3. The baseline hazard function p_0 is given by the log-logistic hazard function, i.e.

$$p_0(t, \gamma, \lambda) = \frac{\gamma\lambda(\gamma t)^{\lambda-1}}{1 + (\gamma t)^{\lambda}},$$

In order to value mortgage-backed securities the prepayment speed function (4.1) is used to simulate cash flows. First, the short rate and the consol yield are simulated under the risk-neutral pricing measure according to the interest-rate model developed by Brennan and Schwartz (1979). Then, given the interest-rate scenario, the prepayment speed is calculated according to (4.1). Contractually obligatory and prepayment cash flows to the mortgage-backed security holder can then easily be determined (see Chapter 2.2.3). The present value of these cash flows gives a realisation of the security value. Finally, applying the Monte-Carlo principle, averaging over all simulated scenarios yields the theoretical price of the mortgage-backed security.

Since the previously introduced early models of the late 1980s many other statistical specifications of the prepayment speed function (4.1) have been proposed in the academic and practitioner-oriented literature. On the practitioner side, the available prepayment data and experience has increased considerably during the last years, in line with the substantial growth of the MBS market. Nowadays, highly specialised commercial consultancies offer econometric prepayment models which include an ever-increasing universe of explanatory variables, in particular loan- and pool-level variables[6]. Moreover, recent advances in statistics have motivated a series of papers with more sophisticated regression techniques for the prepayment speed function, particularly in the academic literature. Two examples are Maxam and LaCour-Little (2001), who use a nonparametric kernel regression, and Popova et al. (2007), who specify a Bayesian mixture of regression models. Parameter estimation in the latter model is carried out by Markov Chain Monte Carlo techniques.

Once the theoretical price of a MBS has been calculated (usually applying a Monte-Carlo simulation as previously described), this theoretical price can be compared to the market price of the respective security. Usually, it can be observed for any specification of the empirical prepayment model that MBS market prices are below the theoretical prices. This price difference can be expressed as a spread on the benchmark interest-rate curve used for discounting. The spread is called option-adjusted spread (OAS) and is commonly understood as a compensation for prepayment risk in practice. For a given OAS (continuously compounded), the theoretical price of a MBS in

[6]Examples of commercial vendors in this field are Andrew Davidson & Co., Inc. (www.ad-co.com), Interactive Data Corp. (www.bondegde.com) and Applied Financial Technology (www.aftgo.com). Andrew Kalotay Associates, Inc. (www.kalotay.com) is another consultancy offering an option-based prepayment model.

one particular interest-rate and prepayment scenario is given by

$$V_{MBS}(t_0) = \sum_{j=1}^{J} CF_{t_j} \cdot \prod_{l=1}^{j} e^{-(R(t_{l-1}, t_{l-1}+\Delta t_l)+OAS)\cdot \Delta t_l}, \tag{4.2}$$

where CF_{t_j} are the MBS cash flows corresponding to the remaining payment dates $t_1, ..., t_J$, $R(t_{l-1}, t_{l-1} + \Delta t_l)$ is the (continuously compounded) interest-rate from time t_{l-1} to $t_{l-1} + \Delta t_l$ as simulated at time t_{l-1} and $\Delta t_l := t_l - t_{l-1}$. The OAS is then adjusted iteratively until the theoretical price (after averaging over all scenarios) matches the market price of the security. We will discuss the theoretical justification and the implications of the existence of the OAS later in Chapter 5.

4.1.2 Option-theoretic models

Option-theoretic models for mortgages and MBS were first introduced in the early 1980s. The paper by Dunn and McConnell (1981) was the first publication which explicitly develops a pricing model for fixed-rate mortgages and GNMA pass-throughs based on option-pricing techniques. The basic principle underlying the option-based approach is the observation that a mortgage can be considered as a portfolio of a non-callable mortgage loan and an American-style call option on the underlying loan with a strike price equal to par, which the mortgage lender/investor is short. Thus, the value of a mortgage contract $V^{Mo}(t)$ is given by:

$$V^{Mo}(t) = V^{Mo}_{NC}(t) - V_{PrOp}(t), \tag{4.3}$$

where $V^{Mo}_{NC}(t)$ denotes the value of the non-callable mortgage and $V_{PrOp}(t)$ is the value of the prepayment option.

A first critical observation concerning (4.3) is the fact that if the prepayment option is exercised optimally, as assumed in the common mathematical models for American-style options, the option value will always be larger or equal to its intrinsic value. Thus, the mortgage value can never exceed par, assuming that there are no transaction costs. This fact was already noted in Dunn and McConnell (1981), who allow for sub-optimal prepayment and are therefore able to obtain mortgage values exceeding par. A further critical issue in option-theoretic models is the treatment of pool heterogeneity for the pricing of mortgage pools and MBS. Theoretically, the value of a mortgage pool (and thus of the corresponding MBS up to servicing fees) is given by the sum of the values of the individual mortgages in the pool. The early option-theoretic models commonly assumed that all homeowners behave identically. This assumption implies, however, that all prepayments of mortgages with similar characteristics occur simultaneously, which is, of course, far from reality (see Chapter 2.2.2). This problem was explicitly addressed in a couple of subsequent publications, e.g. in Stanton (1995) or Kau and Slawson (2002) which are two examples of frequently cited papers concerned with the option-based approach. In the Stanton (1995) model it is assumed that (1) mortgage holders face heterogeneous transaction costs and that (2) they make prepayment decisions only at (random) discrete intervals. Heterogeneous transaction costs alone are not able to explain the empirically observed prepayment behaviour as described in Chapter 2.2.2. Even with heterogeneous transaction costs, there would still be a critical level for each transaction cost at which all mortgagors with the corresponding transaction cost level would prepay immediately. If interest rates then rise and fall again to this level, there would be no further refinancing-prepayment since all mortgage holders who would optimally prepay would already have done so (see Stanton (1995)). Hence the need for the second assumption in order to obtain a more realistic prepayment model.

Most option-theoretic models derive the price of the mortgage $V^{Mo}(t)$ at time t by solving the partial differential equation (PDE) for the mortgage value with some finite difference, backward-induction method. In the Stanton-model, interest rates are assumed to follow a 1-factor CIR model as introduced in Chapter 3.2.3. It can be shown by standard argumentation (see, e.g., Dunn and McConnell (1981) or Stanton (1995) for details) that $V^{Mo}(t)$ satisfies the PDE

$$\frac{1}{2}\sigma_r^2 \cdot r \cdot \frac{\partial^2 V^{Mo}}{\partial r^2} + (\theta_r - \hat{a}_r r) \cdot \frac{\partial V^{Mo}}{\partial r} + \frac{\partial V^{Mo}}{\partial t} + C = r \cdot V^{Mo},$$

subject to appropriate boundary conditions, where $C(t)$ is the (continuously paid) coupon of the mortgage. The time boundary condition for a fully amortising mortgage is, obviously $V^{Mo}(r, T) = 0$, since all principal has been repaid when the mortgage matures at time T. Moreover, it is optimal to refinance the mortgage if

$$V^{Mo}(r, t) > A(t) \cdot (1 + X),$$

where $A(t)$ is the loan amount outstanding at time t and X are (proportional) transaction costs. This yields the typical boundary condition of a callable bond (see Dunn and McConnell (1981) and Stanton (1995) for further details on boundary conditions).

Stanton (1995) assumes that the transaction costs X of a mortgagor in a mortgage pool are random and follow a Beta-distribution. Furthermore, he assumes that each month there is a probability

$$p_{ex} = 1 - e^{-\lambda/12},$$

for some parameter $\lambda > 0$, that the mortgage is prepaid for exogenous reasons. This corresponds, in some sense, to the baseline prepayment in the empirical prepayment models. If it is optimal to prepay, the mortgagor does so only with a probability of

$$p_{ref} = 1 - e^{-(\lambda+\rho)/12},$$

for a further parameter $\rho > 0$, corresponding to the idea that mortgagors only decide to refinance at (random) points of time. Denoting by $V^{Mo,u}(t)$ the value of the mortgage conditional on the prepayment option remaining unexercised and by x a realisation of X, the mortgage value (to the mortgage lender/investor) at time t can be calculated by:

$$V^{Mo}(r,t) = \begin{cases} (1-p_{ex}) \cdot V^{Mo,u}(r,t) + p_{ex} \cdot A(t) & \text{if} \\ & V^{Mo,u}(r,t) \leq A(t) \cdot (1+x) \\ (1-p_{ref}) \cdot V^{Mo,u}(r,t) + p_{ref} \cdot A(t) & \text{otherwise} \end{cases} \tag{4.4}$$

The value $V^{Mo}(r,t)$ can be determined by (4.4) at any point of time t working backwards through the time grid from maturity $t = T$, once all parameters have been estimated from historical prepayment data. The value of a MBS on a pool of similar mortgages can be obtained from adding the values of the individual mortgages, which differ in their respective transaction cost realisation x.

Yet, the ability to explain market prices of most option-theoretic models and thus their success in practice has been limited so far. This fact is discussed in detail in Kalotay et al. (2004). In this recent paper, it is pointed out that most option-based models are only able to explain market prices of premium securities (which clearly exceed par), by 'assigning artificially high transaction costs to a fraction of the homeowners in the mortgage pool'. The authors suggest a new approach which, within the class of option-theoretic models, works with two different yield curves. One for discounting MBS cash flows and one to model the call strategy of homeowners. Pool heterogeneity is introduced by dividing the pool into financial engineers (who refinance optimally), leapers (who refinance too early) and laggards (who refinance too late).

4.1.3 Intensity-based models

Intensity-based prepayment and mortgage valuation models are closely related to the econometric models as introduced earlier in this chapter. In fact, they can be regarded as an extension of the hazard-based econometric models. Intensity-based prepayment models do not only consider the individual risk that a loan will terminate given a statistically determined hazard-rate, but allow for randomness of the hazard-rate itself (beyond the possibly stochastic explanatory variables, such as interest rates, in a statistical prepayment model). This general concept is well known and widely applied in credit risk modelling (for some references see Chapter 3.3), where these models are usually called reduced-form models. While they offer the necessary mathematical rigour, intensity-based models also offer flexibility with respect to the specification of the intensity process for applications in both credit risk and prepayment risk modelling and are, in general, analytically well tractable. Intensity-based prepayment modelling was first applied to the pricing of mortgage contracts by Kau et al. (2004) and Kau et al. (2006). They develop a pricing model for individual mortgage contracts taking into account both prepayment and the possibility of default. In Kau et al. (2004) the (stochastic) baseline prepayment and baseline default processes are explicitly specified as CIR processes as introduced in Chapter 3.2.3 in the context of interest-rate modelling.

Another interesting publication concerned with an intensity-based approach to the pricing of mortgage contracts is Goncharov (2005). Goncharov (2005) shows that the generic reduced-form pricing formula for a mortgage contract (3.34) can be written in the form

$$\begin{aligned} V^{Mo}(t) &= 1_{\{\tau>t\}} \cdot \Bigg(A(t) \\ &\quad + E_{\widetilde{Q}} \left[\int_t^T (m - r(u)) \cdot A(u) \cdot e^{-\int_t^u (r(s)+p(s))ds} du \right] \Bigg), \end{aligned} \tag{4.5}$$

where m is the (continuously compounded) mortgage rate and all other quantities are as previously defined. (4.5) can be used to derive the endogenous mortgage rate. The endogenous mortgage rate of a fixed-rate mortgage is the rate m^0 for which the mortgage value is equal to the loan amount at origination of the contract, i.e. for which the mortgage is priced at par. The endogenous mortgage rate is thus the mortgage rate implied by the current (at the time of origination) riskless yield curve and by the prepayment behaviour of a representative mortgagor. The superscript t in m^t denotes the point of time when the mortgage rate is fixed. From the par value condition

$V^{Mo}(0) = A(0)$ it follows immediately from (4.5) that

$$E_{\widetilde{Q}}\left[\int_0^T m^0 \cdot A(u) \cdot e^{-\int_0^u (r(s)+p(s))ds} du\right] = E_{\widetilde{Q}}\left[\int_0^T r(u) \cdot A(u) \cdot e^{-\int_0^u (r(s)+p(s))ds} du\right],$$

which yields

$$m^0 = \frac{E_{\widetilde{Q}}\left[\int_0^T r(u) \cdot A(u) \cdot e^{-\int_0^u (r(s)+p(s))ds} du\right]}{E_{\widetilde{Q}}\left[\int_0^T \cdot A(u) \cdot e^{-\int_0^u (r(s)+p(s))ds} du\right]}. \tag{4.6}$$

Note, however, that (4.6) is not a formula, but a nonlinear equation since the outstanding loan amount $A(s)$ (in any case) and $p(s)$ (for any serious prepayment model specification) depend on m^0. In the most general case, $p(s)$ may depend not only on the contract rate at origination m^0, but also on the future mortgage rates m^s. In this case, (4.6) is a functional equation. Goncharov (2005) also gives a theorem which guarantees the existence of a solution within his general modelling framework. The calculation of this solution is, however, numerically challenging. The question of how to calculate m^0 in a numerically efficient way is addressed in two subsequent papers (Goncharov et al. (2006), Goncharov (2007)).

Further contributions in the field of intensity-based prepayment and mortgage valuation were made recently (and independently of most of the research presented in this thesis) by Gorovoy and Linetsky (2007) and Rom-Poulsen (2007) who develop semi-analytical MBS pricing formulas which we will discuss in more detail later.

4.2 Current frontiers and further challenges

In the previous section, we have already mentioned that the econometric models remain highly popular in practice and that the OAS, as defined in (4.2), is a common and broadly accepted quantity in the MBS markets. Its interpretation, however, has become subject to discussion. It is a common view among practitioners that the OAS represents a risk premium for prepayment risk. Levin and Davidson (2005) point out, however, that random oscillations of actual prepayments around the model's predictions should be diversifiable and should not lead to any additional risk compensation premium. They thus interpret the OAS as a compensation for non-diversifiable uncertainty which is systematic in trend and unexplained by an otherwise best-guess prepayment model. Kupiec and Kah (1999) argue in a rather similar direction and attribute the existence of the OAS to the omission of important prepayment factors in the risk-neutral Monte-Carlo simulation process. Indeed, in the risk-neutral pricing framework there is no scope for economic risk premia since under a risk-neutral pricing measure all traded assets are expected to earn the risk-free rate. These recent considerations have directed researchers' attention to the probability measure associated with the prepayment process. Kagraoka (2002) points out that, 'surprisingly', this has not been an issue before despite the fact that practitioners have been employing the OAS procedure for decades. He emphasizes that it is, of course, not sure that the prepayment process under the pricing measure is similar to that under the real-world measure.

The intensity-based modelling approach delivers the necessary mathematical apparatus to deal with a change of measure for the pricing of mortgages. In fact, Kau et al. (2006) note that 'each source of randomness in the model has to be converted from real form to its risk-neutral form'. In their model specification in Kau et al. (2004), this leads to some additional model parameters which they calibrate to a data sample of individual mortgage contracts.

When making the transition from the valuation of individual mortgage contracts to MBS, however, some additional topics arise. The first aspect is credit quality. While for the valuation of individual mortgage contracts default risk is certainly an issue, agency-MBS are guaranteed by their respective issuer. GNMA securities, as previously discussed, have the full faith and credit of the US government so that they can be considered default-free for the investor and the US treasury curve can be used as benchmark curve. For FNMA and FHLMC securities a AAA corporate curve may the most appropriate benchmark curve. The second aspect is liquidity. Since agency MBS traded on a TBA basis are highly liquid securities, liquidity effects can be expected to be comparatively unimportant, while this may not be the case for the pricing of potentially illiquid individual mortgages. Moreover, the treatment of mortgagor heterogeneity is an important issue, as already discussed in the previous section. While this is a crucial point for any option-theoretic model, it is an intrinsic feature of econometric and intensity-based models that mortgagors with identical mortgages behave differently. This is caused by the fact that the specification of a prepayment probability/intensity automatically implies that, given a certain state of the economy, the prepayment of an individual mortgage remains random. Yet, it is also a well observed fact in the mortgage markets that past refinancing incentives due to low mortgage refinancing rates affect prepayment speeds at pool level in the present and future. This effect is commonly referred to as 'burnout' and called 'an essential phenomenon of mortgage behaviour' by Levin (2001). Levin's model, which belongs to the class of econometric models, explicitly separates the pool's mortgagors into an active (ready-to-refinance) and a passive, pure turnover part (including those mortgagors that are not able or not willing to refinance their loans due to, e.g., individual transaction costs or simply lack of financial interest). The incorporation of a burnout factor as an additional explanatory variable into the prepayment model as, e.g., in the model developed by Schwartz and Torous (1989) (the variable $x_3(t)$ in (4.1)), is the traditional way of accounting for burnout. This approach, which we will also take in our model in Chapter 5, has the advantage that a rather ad-hoc a priori assumption of mortgagor heterogeneity in a homogenous mortgage pool is not necessary.

In addition to the pool heterogeneity and burnout considerations, an MBS pricing model should be able to establish some relation between OAS levels and MBS market prices. In the option-theoretic model developed by Kalotay et al. (2004) the OAS of a MBS is a fixed input to the model, while it is an output in the common econometric models, calculated according to (4.2). However, following the argumentation outlined earlier in this section, the OAS itself is not a theoretically well justified quantity. To the author's best knowledge, the paper by Levin and Davidson (2005) is the only paper so far, where this argumentation is used to develop a prepayment-risk-neutral valuation model for agency MBS, which directly targets market prices without any need for an OAS. They do this by introducing two additional stochastic prepayment risk factors for pricing purposes (called prepayment multipliers), which scale the historically estimated refinancing-prepayment and the baseline prepayment functions respectively. Mean-reverting Vasicek-processes for these risk-factors are proposed (among other suggestions), with parameters which can then be calibrated to MBS market prices. While their model has the desirable feature that MBS market prices can be targeted directly, the introduction of the two prepayment-risk factors seems rather ad-hoc and in some sense artificial. In fact, no mathematical connection is made between the prepayment rates observed in real-world and the expected prepayment rates implied from market prices. However, as we will show in our own modelling approaches in the subsequent chapters, their basic idea can well be embedded into an intensity-based model framework which delivers the necessary mathematical apparatus while maintaining all desirable features of the Levin/Davidson approach.

A further challenge is the computational burden associated with MBS valuation. In general, this holds for option-based approaches as well as for intensity-based and traditional econometric approaches. In option-theoretic models most authors use backward induction valuation approaches on multidimensional grids to solve the partial differential equation (PDE) which the mortgage/MBS value must satisfy, as already discussed in the previous section. The grid dimension is determined by the number of factors which enter into the prepayment modelling. Thus, the grid points grow exponentially with the number of factors, making most numerical PDE methods computationally costly. The computational burden for the pricing of MBS is particularly high for the econometric models where a computationally expensive Monte-Carlo simulation is usually used for cash flow projection.

The computational burden of MBS valuation can constitute a serious problem, particularly when dealing with large portfolios of MBS which have to be revaluated frequently, e.g. in a risk or portfolio management context. Yet, in such an environment, a fast-to-compute closed-form approximation of a security's value would be sufficient for most purposes. This fact is also discussed in the papers by Collin-Dufresne and Harding (1999) and Sharp et al. (2006), which are concerned with closed-form formulas for mortgages and MBS. The previously mentioned papers by Gorovoy and Linetsky (2007) and Rom-Poulsen (2007) are two further recent contributions in this direction. The Sharp et al. (2006) paper, however, only addresses the valuation of a single fixed-rate mortgage contract by a purely option-theoretic approach for which a closed-form approximation is derived by the use of singular perturbation theory for PDEs. For a generalization of this approach to the valuation of MBS one would still have to deal with the non-optimal and heterogeneous prepayment behaviour of the different mortgagors in a (a priori homogeneous) MBS pool.

The model developed by Collin-Dufresne and Harding (1999) was originally set up as an option-theoretic model, too. Rom-Poulsen (2007) shows, however, that the Collin-Dufresne model can be embedded into an intensity framework. While with this model Collin-Dufresne and Harding are able to explain most of the historical price variation of an exemplarily chosen security, their model has a couple of shortcomings. First, their modelling framework is limited to one stochastic factor (the risk-free short rate). Second, the relation between interest rates and prepayments is strictly linear, which is not in line with the empirically well established S-curve shape of the refinancing incentive (see, e.g., Levin and Daras (1998)). Finally, their model does not allow for any path-dependent explanatory variables such as the previously explained burnout effect. The intensity-based Rom-Poulsen model can be considered as an extension of the Collin-Dufresne/Harding model, allowing for a quadratic interest rate/prepayment relationship which is somewhat more flexible than a purely linear relationship. Both the Collin-Dufresne/Harding and the Rom-Poulsen model offer a semi-analytic formula for the valuation of mortgages and MBS involving systems of partial differential equations which have to be solved numerically. Numerical complexity is also a critical issue in the approach by Gorovoy and Linetsky (2007). While the authors develop a closed-form formula for the valuation of mortgages based on eigenfunction expansion techniques, the computation of mortgage and MBS values requires numerically complex and parameter-sensitive techniques and should thus equally be considered as semi-analytic. We will address these issues explicitly in Chapter 6, where we develop a new, easy-to-compute closed-form approximation formula for the pricing of agency MBS within an intensity-based framework.

Chapter 5

A new hybrid-form MBS valuation model

In this chapter we present a new prepayment-risk-neutral valuation model for MBS which basically extends the proportional hazard model for individual mortgage contracts presented by Kau et al. (2004). Yet, we use different mean-reverting processes for the interest-rate and baseline prepayment factors and explicitly account for the dependence between baseline turnover prepayment and general economic conditions. This is done by adding a third factor which is fitted to the quarterly GDP growth in the US, making our model a hybrid-form model. We label our model 'prepayment-risk-neutral' since we directly target market prices in the spirit of Levin and Davidson (2005) without the need of any OAS input. The existence of the OAS is, as previously discussed, dubious from a theoretical point of view. Nevertheless, our model also allows for a traditional OAS analysis within the same modelling framework.

In the first section we present our model and provide the necessary mathematical background. Details of the parameter estimation and calibration process are discussed subsequently. We apply our model to data of GNMA 30yr fixed-rate MBS-pass-throughs and discuss the empirical results and their economic implications. The final two sections in this chapter are concerned with an extension of our model to the pricing of adjustable-rate MBS and CMOs.

5.1 The model set-up for a fixed-rate MBS

A crucial part of every valuation model for MBS is an adequate interest-rate model. We use a 1-factor Hull-White type model, as presented in Chapter

3.2.2, where the non-defaultable short rate r is defined by the dynamics (under the real-world measure Q)

$$dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r(t). \tag{5.1}$$

The time-dependent mean-reversion level $\theta_r(t)$ is fitted to the initial term-structure and its functional form is as given in (3.11). Then, a stochastic prepayment process $p(t)$ is considered in a proportional hazard framework. Corresponding to the model set-up of Kau et al. (2004), the basic idea behind our approach is to capture the turnover component of prepayment in a baseline hazard process, identical for all MBS of the same type, while the pool-specific refinancing components are captured through individual explanatory variables such as the contract rate spread to current mortgage benchmark rates, the pool burnout, etc. Since we find strong empirical evidence for the dependence of the turnover component of prepayment and the quarterly GDP growth in the US (which will be discussed later in this chapter) we use a 2-factor model for the baseline hazard process and fit the second factor to the GDP growth data. For both, the baseline hazard, which we incorporate into the overall prepayment process in an exponential way to ensure that prepayment speeds are non-negative, and the general economic conditions represented by the quarterly GDP growth, we assume a mean-reverting process with constant mean-reversion level following Vasicek (1977). Since we only consider GNMA securities in the empirical parts of this chapter we assume one common baseline hazard process for all MBS.

So, for an MBS with individual covariate vector $\mathbf{x}(t)$ the prepayment processes have the form:

$$p(t) = e^{f(\mathbf{x}(t),\boldsymbol{\beta})+p_0(t)}, \tag{5.2}$$

$$dp_0(t) = (\theta_p + b_{pw}w(t) - a_p p_0(t))dt + \sigma_p dW_p(t), \tag{5.3}$$

$$dw(t) = (\theta_w - a_w w(t))dt + \sigma_w dW_w(t), \tag{5.4}$$

where $f(\mathbf{x}(t),\boldsymbol{\beta})$ is some function of the time-dependent covariate vector of the MBS (containing, e.g., contract rate spread and pool burnout) and of the regression parameter vector $\boldsymbol{\beta}$, $p_0(t)$ is the common baseline hazard process, $w(t)$ represents the quarterly US GDP growth and W_p, W_w are independent Wiener processes with respect to Q.

To describe the hazard rate or (instantaneous) prepayment speed $p(t)$ of a mortgage pool we use the intensity framework as introduced in Chapter 3.3. Consider a complete filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, Q)$ which supports the Wiener processes W_r, W_p, W_w and a counting process $N(t)$, counting the number of mortgages in a pool that have already been prepaid at the point of time t. We define $N(t)$ as a doubly stochastic Poisson process, i.e. as a Cox process. Of course, in the Cox Process framework there is no maximum number of jumps, so that we have to assume at this point that there are infinitely many mortgages in a pool. We will come back to this issue later. In addition to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ we again consider the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by all of the previously considered processes except the counting process $N(t)$. We assume that $N(t)$ has a $\{\mathcal{F}_t\}$-measurable intensity $\gamma(t)$ with $\int_0^t \gamma(s)ds < \infty$ for all $t \geq 0$. Then,

$$M(t) := N(t) - \int_0^t \gamma(s)ds$$

is an ($\mathcal{G}_t$-local) martingale (see Theorem 3.6) and the existence of a unique $\mathcal{F}_t$-predictable version of the intensity $\gamma(t)$ is assured (see Theorem 3.7). The expected increment of the Cox process is given by (see (3.29))

$$E_Q\left(dN(t)|\mathcal{G}_t\right) = \gamma(t)dt.$$

As a next step, we account for the fact that there are only finitely many mortgages in a pool and approximate $dN(t)$ by $\sum_{k=1}^K dN_k(t)$ where $N_k(t)$ denotes the one-jump prepayment indicator process of the k-th mortgage in the pool which, at time t, has a value of 0 if the mortgage has not been prepaid previously, 1 otherwise, and K is the total number of mortgages in the pool. Assuming that the time of prepayment of one mortgage does not influence the probability of prepayment of other mortgages, but that prepayment probabilities of different mortgage are driven by the same background processes generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (an assumption that is maintained at all stages of our modelling approach), it holds that, as K goes to infinity,

$$\frac{1}{K}\sum_{k=1}^{K} dN_k(t) \xrightarrow{\mathcal{D}} \gamma(t)dt, \tag{5.5}$$

where '$\xrightarrow{\mathcal{D}}$' denotes convergence in distribution. A formal proof of this relation can be found in Kagraoka (2002) as a consequence of the Central Limit Theorem for Processes (see Jacod and Shiryaev (1987) [VIII 3.46]). Since in this thesis we are dealing with large mortgage pools with a large number of individual mortgages in each pool we can conclude that

$$p(t)dt \approx E_Q\left(\frac{1}{K}\sum_{k=1}^{K} dN_k(t)|\mathcal{G}_t\right) \approx \gamma(t)dt, \tag{5.6}$$

where the first part is a consequence of the law of large numbers and the second part follows from (5.5).

So far, we have only considered the dynamics and properties of the processes under the real-world measure. The key to the transition from the real-world measure Q to an equivalent martingale measure $\widetilde{Q}$ is the Girsanov theorem for marked point processes which we have already stated in its general version (Theorem 3.14). With a few structural assumptions we can derive the form of the processes (5.1) to (5.4) under the risk-neutral pricing measure (uniquely specified by our assumptions and calibration later), summarised in the following theorem.

Theorem 5.1. *Let $\phi' = (\phi_r, \phi_p, \phi_w)$ be a three-dimensional predictable process and $\Phi(t)$ a non-negative predictable function with*

$$\int_0^t |\phi_i(s)|^2 ds < \infty, \quad i = r, p, w, \quad \int_0^t |\Phi(s)| p(s) ds < \infty$$

for any finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \sum_{i=r,p,w} \phi_i(t) dW_i(t) + (\Phi(t) - 1)(dN(t) - p(t)dt).$$

Assume that $E_Q(L(t)) = 1$ for finite t. Define the probability measure $\widetilde{Q}$ with

$$\left.\frac{d\widetilde{Q}}{dQ}\right| \mathcal{F}_t = L(t), \qquad \forall t \geq 0.$$

Further assume that there are constants $\lambda_r, \lambda_p, \lambda_w$ such that

$$\begin{aligned} \phi_r(t) &= \lambda_r \sigma_r r(t) \\ \phi_p(t) &= \lambda_p \sigma_p p_0(t) \\ \phi_w(t) &= \lambda_w \sigma_w w(t) \end{aligned}$$

and assume that

$$\Phi(t) = (p(t))^{\mu - 1} \tag{5.7}$$

for some constant $\mu \in \mathbb{R}$. Then,

$$\tilde{p}(t) = e^{\mu \cdot (f(\mathbf{x}(t), \boldsymbol{\beta}) + p_0(t))}$$

is the intensity of the counting process $N(t)$ under $\widetilde{Q}$ and the processes (5.1), (5.3), (5.4) have the following dynamics under $\widetilde{Q}$:

$$\begin{aligned} dr(t) &= [\theta_r(t) - (a_r + \lambda_r \sigma_r^2) r(t)]dt + \sigma_r d\widetilde{W}_r(t) \\ dp_0(t) &= [\theta_p + b_{pw} w(t) - (a_p + \lambda_p \sigma_p^2) p_0(t)]dt + \\ &\quad + \sigma_p d\widetilde{W}_p(t) \\ dw(t) &= [\theta_w - (a_w + \lambda_w \sigma_w^2) w(t)]dt + \sigma_w d\widetilde{W}_w(t), \end{aligned}$$

where $\widetilde{W}_r, \widetilde{W}_p, \widetilde{W}_w$ are independent $\widetilde{Q}$-Wiener processes.

Proof. The proposition follows from the Girsanov theorem as stated in Theorem 3.14. Recall that our marked point process $N(t)$ is a Cox process with intensity $p(t)$. Thus, the marker space E contains only the element $\{1\}$ and, denoting the marker variable by Y, the compensator measure $\nu(de, dt)$ has the form:

$$\nu(de, dt) = \delta_{Y=1}(de)p(t)dt.$$

The Girsanov theorem now yields that (see (3.31) and (3.32))

$$\tilde{p}(t) = \Phi(t)p(t)$$

and by the structural assumption (5.7) we get

$$\tilde{p}(t) = e^{\mu \cdot (f(\mathbf{x}(t), \boldsymbol{\beta}) + p_0(t))}.$$

Furthermore, the Girsanov theorem ensures that, with $d\widetilde{W}_i(t) := dW_i(t) + \phi_i(t)dt$, $\widetilde{W}_i$ is a Wiener-process under $\widetilde{Q}$ for $i = r, p, w$ and we finally get the dynamics of the processes $r(t), p_0(t), w(t)$ under the (martingale) measure $\widetilde{Q}$ by standard argumentation (see, e.g., Zagst (2002a) [4.4-4.5] for details). □

Note at this point that the structural assumption (5.7) is a little unconventional. In most other applications the assumption that $\Phi(t) = \mu^*$ for some constant μ^* is the norm. In our case, however, it is more convenient to assume a structure as in (5.7), since this leads to the (multiplicative) risk-adjustment parameter μ in the overall prepayment process which is clearly identifiable against the risk-adjustment parameters λ_p and λ_w in the baseline prepayment process $p_0(t)$. The convenience of the previously described structural assumption will become clear in the subsequent section where we discuss the interaction of the risk-adjustment parameters in their economic context.

The value $V(0)$ of the MBS at time $t = 0$ can finally be calculated as the expectation of the security's discounted future cash flows under the risk-neutral pricing measure $\widetilde{Q}$. We denote by $A(t_k)$ the regular principal amount outstanding on payment date t_k according to the original amortisation schedule without any prepayments. Moreover, if we denote by $M(t_k) \cdot \Delta t_k$ the original monthly mortgage payment (i.e. the sum of interest and scheduled principal repayment) and by K the number of payment dates until final maturity of the MBS, we get the following cash flows at each payment date t_k:

- The monthly mortgage payment $M(t_k) \cdot \Delta t_k \cdot \prod_{j=1}^{k-1}(1 - p_{\text{SMM}}(t_j))$
- The prepaid principal $A(t_k) \cdot p_{\text{SMM}}(t_k) \cdot \prod_{j=1}^{k-1}(1 - p_{\text{SMM}}(t_j))$.

Here $p_{\text{SMM}}(t_k)$ denotes the prepayment speed expressed as single monthly mortality. The continuously compounded annual prepayment speed $p(t)$ can be converted to a (discrete) constant prepayment rate by

$$CPR(t_k) = e^{p(t)} - 1,$$

from which the single monthly mortality is obtained by the relation (2.1). We can conclude:

Theorem 5.2. *The value $V(0)$ of a fixed-rate MBS at time $t = 0$ is given by:*

$$\begin{aligned} V(0) &= E_{\tilde{Q}}\Bigg[\sum_{k=1}^{K} c_{t_k} \cdot \left(\prod_{j=1}^{k-1}(1 - \tilde{p}_{SMM}(t_j))\right) \\ &\qquad \cdot (\tilde{p}_{SMM}(t_k) \cdot A(t_k) + M(t_k) \cdot \Delta t)\Bigg], \end{aligned} \tag{5.8}$$

where $c_{t_k} = e^{-\int_0^{t_k} r(s)ds}$.

Due to the path dependence introduced through the explanatory variables we have no alternative to a computationally costly Monte-Carlo simulation to evaluate (5.8) at this point. Note that (5.8) is a version of (3.35) with discretised prepayment rates. In (5.8) the prepayment rates $\tilde{p}_{\text{SMM}}(t_k)$ are expressed as single monthly mortalities, which is convenient for the Monte-Carlo evaluation. If prepayment speeds were expressed as continuous annualised rates, the overall 'survival probability' up to time t_{k-1} inside the expectation would be given by $e^{-\int_0^{t_{k-1}} p(s)ds}$ instead of $\prod_{j=1}^{k-1}(1 - \tilde{p}_{\text{SMM}}(t_j))$ and (5.8) would take a form similar to (3.35). The Monte-Carlo algorithm used to evaluate (5.8) is given in the appendix.

5.2 Application to market data

5.2.1 Parameter estimation and model calibration

Interest rate and real-world prepayment model

The available data for this study consists of US treasury strip par rates and monthly historical prepayment data for large issues of 30yr fixed-rate mortgage-backed securities of the GNMA I and GNMA II programs. We use the historical pool data of a total of eight individual mortgage pools for the empirical prepayment model (see Table 5.2 for the pool numbers). The corresponding MBS were issued between 1993 and 1996 with more than USD 50m of residential mortgage loans in each of the eight pools and have coupons between 6% and 9%, so that both discounts and premiums are included in our sample. Discount MBS are securities with a low coupon which are traded below 100% while premiums feature high coupons and market prices above 100%. After the months of very high prepayment speeds in 2002-2004 (compare Figure 2.1) the mortgage pools considered for parameter estimation in this study were not large enough any more to maintain the assumptions based on large sample properties. We therefore discard the prepayment data of these pools in 2005 for parameter estimation in the prepayment model. Weekly US treasury strip zero rates, obtained from the par rates by standard bootstrapping, from 1993 to 2005 are used for the estimation of the parameters of the interest-rate process. Since the focus of our model is not on explanatory variables for prepayment, we restrict the set of covariates to those that are usually stated as the most important ones: the spread between the weighted-average coupon (WAC) of the mortgage pool and the 10yr treasury par yield which is commonly used as proxy for mortgage rates (see, e.g., Goncharov (2005) for some discussion concerning this choice of proxy) and the burnout which we define in line with the definition given in Schwartz and Torous (1989):

$$burnout(t) = \ln\left(\frac{PF(t)}{A(t)}\right),$$

where $PF(t)$ is the actual principal amount outstanding at time t and $A(t)$ is the remaining principal amount according to the amortisation schedule without any prepayments, as previously defined. In order to account for the usual S-curve shape of the influence of the refinancing incentive (see, e.g., Levin and Daras (1998) or Figure 5.1), expressed by the spread covariate, we choose the arc-tangent as functional form. The arc-tangent function was also used by Asay et al. (1987). Furthermore, our empirical results could be improved by incorporating the burnout covariate as cubic term in addition to the linear term. Finally, our covariate function f has the form

$$\begin{aligned} f(\mathbf{x}(t), \boldsymbol{\beta}) &= \beta_1 \cdot \arctan(\beta_2 \cdot (spread(t) + \beta_3)) + \\ &\quad + \beta_4 \cdot burnout(t) + \beta_5 \cdot burnout(t)^3. \end{aligned} \tag{5.9}$$

Parameter estimation for the short-rate process.
We estimate the parameters a_r, σ_r, λ_r of the interest rate model with a

Kalman filter for state space models as introduced in Chapter 3.4 with measurement and transition equations as given in the following.

Recall that the price of a zero-coupon bond with maturity T at the point of time t, denoted by $P(t,T)$, in the Hull-White type short-rate model (5.1) is given by (see Chapter 3.2.2):

$$\begin{aligned} P(t,T) &= e^{A(t,T)-B(t,T)r(t)}, \\ A(t,T) &= \int_t^T (\frac{1}{2}\sigma_r^2 B(l,T)^2 - \theta_r(l)B(l,T))dl, \\ B(t,T) &= \frac{1}{\hat{a}_r}(1-e^{-\hat{a}_r(T-t)}), \end{aligned}$$

where $\hat{a}_r = a_r + \lambda_r \cdot \sigma_r^2$. At the point of time t_k we observe for maturities τ_i, $i = 1, ..., n$, the treasury strip rates $R(t_k, t_k + \tau_i) = -\frac{lnP(t_k,t_k+\tau_i)}{(T-t)}$. With $a(t,T) = -\frac{A(t,T)}{T-t}$, $b(t,T) = \frac{B(t,T)}{T-t}$ the measurement equation of the state space model is then given by:

$$\begin{pmatrix} R(t_k, t_k+\tau_1) \\ \vdots \\ R(t_k, t_k+\tau_n) \end{pmatrix} = \begin{pmatrix} a(t_k, t_k+\tau_1) \\ \vdots \\ a(t_k, t_k+\tau_n) \end{pmatrix} + \begin{pmatrix} b(0,\tau_1) \\ \vdots \\ b(0,\tau_n) \end{pmatrix} \cdot r(t_k) + \epsilon_k, \quad (5.10)$$

where we assume that the measurement error follows an n-dimensional Normal distribution with expectation vector $\mathbf{0}$ and covariance matrix $h_r^2 \cdot \mathbf{I}_n$, i.e. $\epsilon_k \sim N_n(\mathbf{0}, h_r^2 \cdot \mathbf{I}_n)$. The transition equation can be derived by the Hull-White short-rate dynamics which yield (see (3.16)):

$$r_{k+1} = e^{-a_r \cdot \Delta t_{k+1}} r_k + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\theta_r(l)dl + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\sigma_r dW(l),$$

if we define $r_k := r(t_k)$ and $\Delta t_{k+1} := t_{k+1} - t_k$. By approximating $\theta_r(l)$ by $\theta_r(t_k)$ in the integral and defining

$$\eta_{k+1} := \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\sigma_r dW(l)$$

we finally get the transition equation of the state space model:

$$r_{k+1} = e^{-a_r \cdot \Delta t_{k+1}} r_k + \int_0^{\Delta t_{k+1}} e^{-a_r \cdot l}\theta_r(t_k)dl + \eta_{k+1} \quad (5.11)$$

with

$$\eta_{k+1} \sim N_1\left(0, \frac{\sigma_r^2}{2a_r}(1-e^{-2a_r \Delta t_{k+1}})\right).$$

The results of the parameter estimation for the interest-rate model (and for the real-world prepayment model described in the subsequent paragraph) are summarised in Table 5.1. The estimated standard errors of the parameter estimators are obtained by a moving block bootstrapping procedure as introduced in the appendix for which we choose a block length of 100 for the weekly interest-rate data and a block length of 20 for the monthly prepayment data. In the block bootstrapping procedure, the blocks are then randomly concatenated to obtain series with the same length as the respective original sample series. The empirical standard deviation of the respective estimator in a total of 50 bootstrap replications yields the standard error estimates as given in Table 5.1.

	Parameter	Estimate	(Std. error)
Short-rate process	a_r	0.11	(0.0044)
	σ_r	0.0088	$(8.6 \cdot 10^{-5})$
	λ_r	-1380.8	(48.16)
	h_r	0.0005	$(1.6 \cdot 10^{-5})$
GDP growth process	θ_w	0.019	(0.0012)
	a_w	1.43	(0.087)
	σ_w	0.002	$(1.3 \cdot 10^{-5})$
Baseline prepayment process	θ_p	-3.77	(0.30)
	a_p	1.20	(0.064)
	σ_p	0.88	(0.024)
	b_{pw}	-88.4	(22.93)
	h_p	0.70	(0.005)
Regression parameters	β_1	0.67	(0.10)
	β_2	0.92	(0.24)
	β_3	-1.55	(0.12)
	β_4	0.003	(0.013)
	β_5	0.007	(0.0015)

Table 5.1: Estimates of the interest-rate model and real-world prepayment model parameters where h_r and h_p are the measurement std. errors of the respective state space models.

The estimates of the interest-rate model parameters yield an average mean-reversion level of the short rate of 4.8% (i.e. $\frac{1}{K}\sum_{k=1}^{K}\theta_r(t_k)/a_r = 4.8\%$), which seems to be a fairly appropriate value given that the average observed

3-month rate was 5.1% during the time horizon used for parameter estimation. For the estimation of the prepayment parameters we use a two-stage procedure. We first estimate the parameters θ_w, a_w, σ_w of the GDP growth process by Maximum-Likelihood and the parameters $\theta_p, a_p, \sigma_p, b_{pw}$ again by a Kalman filter for state space models with the historical prepayment speeds as observables.

Parameter estimation for the GDP growth process.
The dynamics of the GDP growth process in (5.4) are again given by a SDE of the form (3.15). Thus, we get

$$\begin{aligned} w(t_{k+1}) &= e^{-a_w \Delta t_{k+1}} w(t_k) + \int_{t_k}^{t_{k+1}} e^{-a_w(t_{k+1}-l)} \theta_w dl + \\ &\quad + \int_{t_k}^{t_{k+1}} e^{-a_w(t_{k+1}-l)} \sigma_w dW_w(l) \end{aligned}$$

and it follows that

$$\begin{aligned} w(t_{k+1})|w(t_k) &\sim N_1(c, d^2), \\ c &= e^{-a_w \Delta t_{k+1}} w(t_k) + \frac{\theta_w}{a_w}(1 - e^{-a_w \Delta t_{k+1}}), \\ d^2 &= \frac{\sigma_w^2}{2a_w}(1 - e^{-2a_w \Delta t_{k+1}}). \end{aligned}$$

We obtain Maximum-Likelihood estimates of the parameters θ_w, a_w, σ_w by maximising the likelihood function

$$L(\theta_w, a_w, \sigma_w) = \prod_{k=1}^{K} \varphi_{w(t_k)|w(t_{k+1})},$$

where $\varphi_{w(t_k)|w(t_{k+1})}$ denotes the p.d.f. of the Normal distribution with parameters c and d^2 as defined above.

Parameter estimation for the prepayment process.
The measurement equation of the state space model is given by (5.2) with the historically observed prepayment speeds (as SMM) and f as specified in (5.9):

$$\begin{pmatrix} \ln(p_{\mathrm{SMM},1}(t_k)) \\ \vdots \\ \ln(p_{\mathrm{SMM},N}(t_k)) \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}_1(t_k), \boldsymbol{\beta}) \\ \vdots \\ f(\mathbf{x}_N(t_k), \boldsymbol{\beta}) \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot p_0(t_k) + \boldsymbol{\epsilon}_k, \quad (5.12)$$

where we assume that $\boldsymbol{\epsilon}_k \sim N_N\left(\mathbf{0}, h_p^2 \cdot \mathbf{I}_N\right)$. The transition equation for the (unobservable) baseline prepayment hazard is given by (5.3). For stability reasons, we use $w(t)$ as an external input to the model and define $X(t) = p_0(t)$, $H = -a_p$, $J(t) = \theta_p + b_{pw}w(t)$, $V = \sigma_p$ to get a SDE of the form (3.15). Similar to the derivation of the transition equation (5.11) of the interest-rate model the transition equation of the prepayment state space model is

$$p_0(t_{k+1}) = e^{-a_p \Delta t_{k+1}} \cdot p_0(t_k) + \frac{\theta_p + b_{pw} w(t_k)}{a_p} \cdot (1 - e^{-a_p \Delta t_{k+1}}) + \eta_{k+1}$$

with

$$\eta_{k+1} \sim N_1 \left(0, \frac{\sigma_p^2}{2a_p}(1 - e^{-2a_p \Delta t_{k+1}}) \right).$$

The values for the estimates of the GDP growth process (see again Table 5.1) yield a mean-reversion level of $\theta_w/a_w = 1.3\%$ which is identical to the actually observed average. For all optimisation steps we use a combined Downhill-Simplex/Simulated Annealing algorithm as described in Press et al. (1992).

Before we discuss the statistical properties of our estimates in the real-world prepayment state space model and proceed to the calibration and interpretation of the prepayment-risk-adjustment parameters $\lambda_p, \lambda_w, \mu$ in the following subsection, we want to give some empirical justification for the incorporation of the GDP growth rate as a second factor of the prepayment model. While it is often recognised that the baseline component of prepayment is correlated to general economic conditions, nobody (to the author's best knowledge) has made the effort of explicitly modelling such a dependence structure by considering prepayment jointly with a factor such as the quarterly GDP growth. In order to investigate the value of such an economic factor, we consider the differences $d_i(t)$ between the actual, historically observed prepayment speeds $p_i(t)$ of the i-th MBS in the sample and those predicted by the covariates, i.e. not taking into account the baseline hazard in (5.2):

$$d_i(t) = \ln p_{\text{SMM},i}(t) - f(\mathbf{x}_i(t), \boldsymbol{\beta}), \quad i = 1, ..., M. \tag{5.13}$$

We consider the average difference $d(t) := \frac{1}{M}\sum_{i=1}^{M} d_i(t)$ of the M different MBS pools used for prepayment parameter estimation as an estimate for the baseline hazard prepayment process $p_0(t)$ and investigate the correlation between the estimated baseline hazard prepayment and the quarterly GDP growth process (monthly data for the GDP growth process was obtained by cubic spline interpolation). With the MatLab-function *corr*, the Pearson correlation coefficient for a 6-month lag between GDP growth rates and prepayments is estimated as -0.4 with a p-value of 0, clearly rejecting the hypothesis of no correlation. Since a lag of 6 months results in the highest significance level (compared to a lag of 3 and 9 months) we incorporate this time lag into our modelling. Note at this point that, of course, data on GDP growth are published with some delay. We account for this delay, so that, when speaking of a 6-month time lag between GDP growth rates and prepayments, we compare, e.g., prepayments in July with the quarterly GDP growth rate in January of the same year, published a few months later.

The negative sign of the correlation (and of the parameter b_{pw}) may be surprising at first sight. One possible explanation for this may be the fact that we have not separated prepayment from default. Default is certainly more likely in times of an adverse economic environment with sluggish growth. The time lag of 6 months suggests that it takes about half a year from a worsening of the general economic conditions to a rise in mortgagors' defaults or simply to a mortgagor's decision to 'downsize' a mortgage loan by selling the house and moving to a smaller one (which equally leads to higher prepayment rates). We leave the explicit modelling of default as a separate source of prepayment risk (from a GNMA-investor's point of view) for further research.

We finally want to test and verify the statistical assumptions of the prepayment state space model and of the Kalman filtering algorithm. It is

Pool	t-test	Box-Ljung-test	ARCH-test	Lilliefors-test
GN 354627	0	0	0	1
GN 351408	0	0	0	0
GN 352166	0	0	0	0
G2 2034	0	0	0	1
G2 2054	0	0	0	0
G2 2305	0	1	0	0
G2 2148	0	0	0	1
G2 1856	0	0	0	1

Table 5.2: Tests for the hypotheses 1. $E[u_t] = 0, \forall t$ (second column), 2. No serial correlation in (u_t) (third column), 3. No first-order heteroscedasticity in (u_t) (fourth column), 4. (u_t) are drawn from a normal distribution (fifth column). A value of 0 indicates that the respective hypothesis can not be rejected at the 5% level, a value of 1 indicates that the hypothesis can be rejected at the 5% level.

essential to assume that the Kalman filter innovations (i.e. the standardised residuals; see, e.g., Schmid (2002) [3.6] for further details) are iid random variables. Furthermore, the model specifications of the state space model require the residuals to be normally distributed with mean 0. To verify these assumptions we apply a couple of tests to the innovations

$$u_t := \frac{\ln p_{\mathrm{SMM}}(t) - \ln \hat{p}_{\mathrm{SMM}}(t)}{\sqrt{Var(\ln p_{\mathrm{SMM}}(t) - \ln \hat{p}_{\mathrm{SMM}}(t))}}, \quad t = 1, ..., T$$

of our model where $\hat{p}_{\mathrm{SMM}}(t)$ is the prepayment speed predicted by our Kalman filter. First of all, we test the hypothesis that $E[u_t] = 0$, $t = 1, ..., T$, with a simple t-test. We then test for serial correlation by applying the Box-Ljung test (see Ljung and Box (1978)). First-order heteroscedasticity is tested with the ARCH-test, which goes back to Engle (1982). We finally apply the Lilliefors-test to test the Normal distribution assumption (see Lilliefors (1967)). We use the Matlab implementation of these tests and apply them to each of the eight mortgage pools whose prepayment history we use for parameter estimation. Table 5.2 shows the results from which we can conclude that, altogether, the assumptions of the Kalman filter algorithm are sufficiently satisfied for our prepayment data.

In order to illustrate the regression parameter estimates, we show the historical prepayment rates available for this study (as SMM) and plot the estimated prepayment speed as a function of the spread covariate and of the burnout covariate when the baseline prepayment is set to its estimated mean-reversion level (Figure 5.1). While our observations are quite noisy, the general S-curve structure of the data can well be recognised. The noise in the data could of course be reduced, if aggregated data instead of pool level data are used. This, however, would imply that pool level covariates such as the burnout could not be incorporated into the prepayment model. While our estimated prepayment function captures the general structure of the data well, Figure 5.1 seems to indicate that our estimated S-curve slightly underestimates the steepness of the refinancing-incentive. A reason for this may be the fact that in our state-space model prepayment speeds enter as logarithms into the measurement equation (5.12) and prepayment observations close to 0 for high values of the spread covariate may thus become quite influential in the maximum likelihood estimation. The relation between the burnout and our expected prepayment speed is as expected. For a highly 'burnt-out' pool (i.e. a mortgage pool with a low value of the *burnout* variable), our expected prepayment speeds are lower than for a comparable fresh pool. This relation is reflected in the positive sign of the regression parameters β_4 and β_5.

While the statistical fine-tuning of any empirically estimated prepayment function may be an important issue for further research, this is not the primary focus of this thesis, as we have already pointed out earlier. In this thesis we are not primarily interested in explaining historically observed prepayment rates statistically, but in the pricing of MBS for which we obtain highly satisfactory results with our estimates, as we will discuss in the subsequent sections.

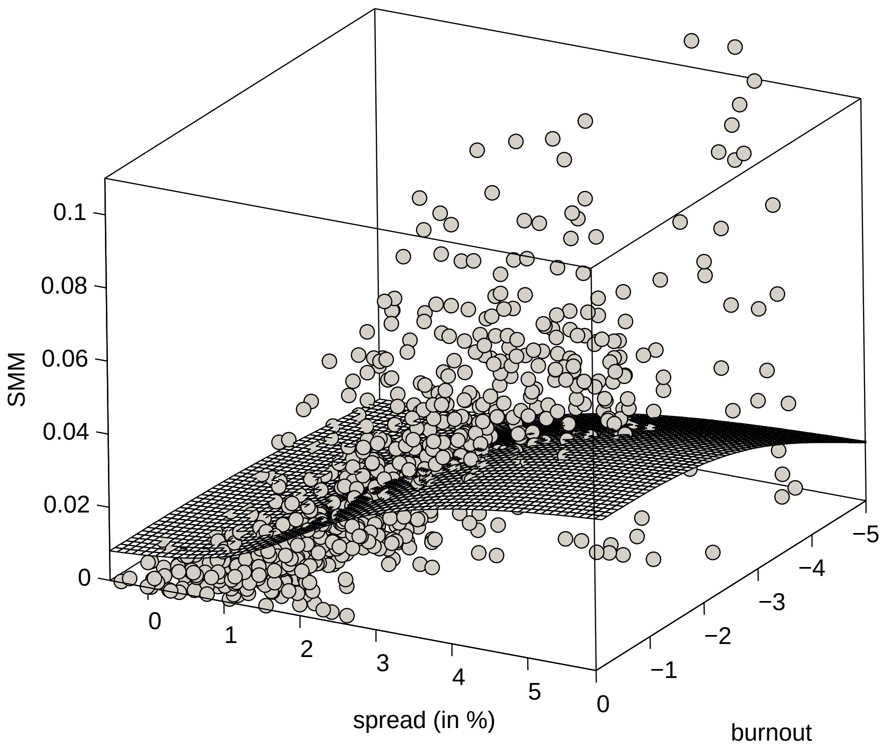

Figure 5.1: Historically observed SMM values and estimated prepayment speed (SMM) as a function of the covariates *spread* and *burnout* when the baseline hazard process is set equal to its mean-reversion level.

Prepayment-risk-neutral model

In general, there is no active and liquid market for an individual mortgage pool. One obvious reason for this is the limited size of individual pools. For the calibration of the prepayment-risk-adjustment parameters we therefore consider market prices of generic GNMA 30yr fixed-rate MBS as quoted in Bloomberg (Bloomberg ticker GNSF) for trading on a TBA (to-be-announced) basis. We consider coupons between 4.5% and 8%, so that both discounts and premiums are included. GNMA securities which are traded on a TBA basis are highly liquid securities, so that we do not have to worry about liquidity effects/premia. Even for those securities with a coupon well below or well above the current coupon, Bid-Ask spreads are usually not higher than 2 ticks with a tick size of USD $\frac{1}{32}$. When we speak of market prices we refer to the Ask-prices.

We estimate the prepayment-risk-adjustment parameters μ, λ_p, λ_w by minimising the Euclidean norm of the vector of differences between the market prices and model prices of the securities on each sample day. All calculations are carried out without accounting for any OAS, i.e. with an OAS equal to 0 for all securities. By setting the OAS-target equal to 0, we price with the treasury curve as benchmark curve, which seems to be the most appropriate curve for GNMA securities since these securities feature the full faith and credit of the US Government. Of course, any other curve could be used as benchmark curve if desired. Once the parameters have been calibrated, one can hardly expect all theoretical prices to match market prices exactly for all MBS securities. The OAS equivalent in a risk-neutral valuation framework (with a target of 0 in the calibration procedure) could be compared to the 'prOAS'-measure recently introduced by Levin and Davidson (2005). We will also use their 'prOAS'-term in the following and point out that the prOAS should not be regarded as any kind of risk premium, but simply as a measure of unsystematic residual pricing error.

Levin and Davidson (2005) emphasize the necessity of a two-risk factor model in order to account for the two distinct market fears in the MBS market: refinancing understatement and turnover overstatement. These two distinct market fears explain why, in the traditional OAS valuation approach, it is not uncommon to observe higher OAS levels for both discounts and premiums compared to MBS around the current-coupon level. On the one hand, an investor in discounts experiences losses if the turnover component is overestimated and pure turnover-related prepayment is slower than expected. In this case the average life of the security is extended, decreasing the cash flow stream's present value. On the other hand, the refinancing component is the major concern of an investor in premiums since the average life of premiums decreases if refinancing-related prepayment is faster than originally estimated, pulling the security's present value towards 100%. This would evidently result in a loss for the holder of a premium MBS.

These considerations are fully accounted for in our model since we have the (multiplicative) risk-adjustment parameter μ and the two (additive) risk-adjustment parameters λ_p and λ_w. For parameter values of μ larger than 1, the refinancing S-curve is stretched, i.e. the prepayment incentive induced by higher spreads between the WAC and the 10yr treasury par yield is accelerated. The parameters λ_p and λ_w only affect the baseline prepayment speed. Note that the mean-reversion level of the Vasicek process for $p_0(t)$ is negative in real-world when we set the GDP growth process $w(t)$ to its mean-reversion level. Thus, for values of μ larger than 1 (as in our estimates in Table 5.3), the process $p_0(t)$ will take much smaller values (larger in absolute terms), potentially reducing the overall prepayment speed for both discounts and premiums. Now, for positive values of λ_p and (with much less significant consequences) λ_w, the mean-reversion level of the process p_0 will be pulled back into the positive direction, bringing back the overall prepayment speed to sensible levels for discounts and premiums in the same (additive) way. With our structural assumption for the prepayment intensity under the risk-neutral pricing measure as given in (5.7) we can therefore accelerate prepayments for premiums while, at the same time, decelerate prepayments for discounts under the risk-neutral pricing measure. We can thus account for both, the market fear of turnover overstatement for discounts and the market fear of refinancing understatement for premiums, in our prepayment-risk-neutral pricing approach.

Figure 5.2 illustrates how the prepayment-risk-adjustment parameters

Parameter	18-Oct-2005	04-Nov-2005	12-Dec-2005
μ	2.2	2.4	2.0
λ_p	2.7	3.0	2.2
λ_w	-10.2	-8.1	-7.0

Table 5.3: Estimates of the prepayment-risk-adjustment parameters on three (arbitrarily chosen) dates.

account for the two types of prepayment risk as previously discussed. Under the risk-neutral pricing measure prepayment speeds are slower for low spreads which extends the average life of discounts. Contrarily, prepayment speeds for high spreads rise under the risk-neutral pricing measure, shortening the average life of premiums and thus clearly reflecting the market fear of refi-

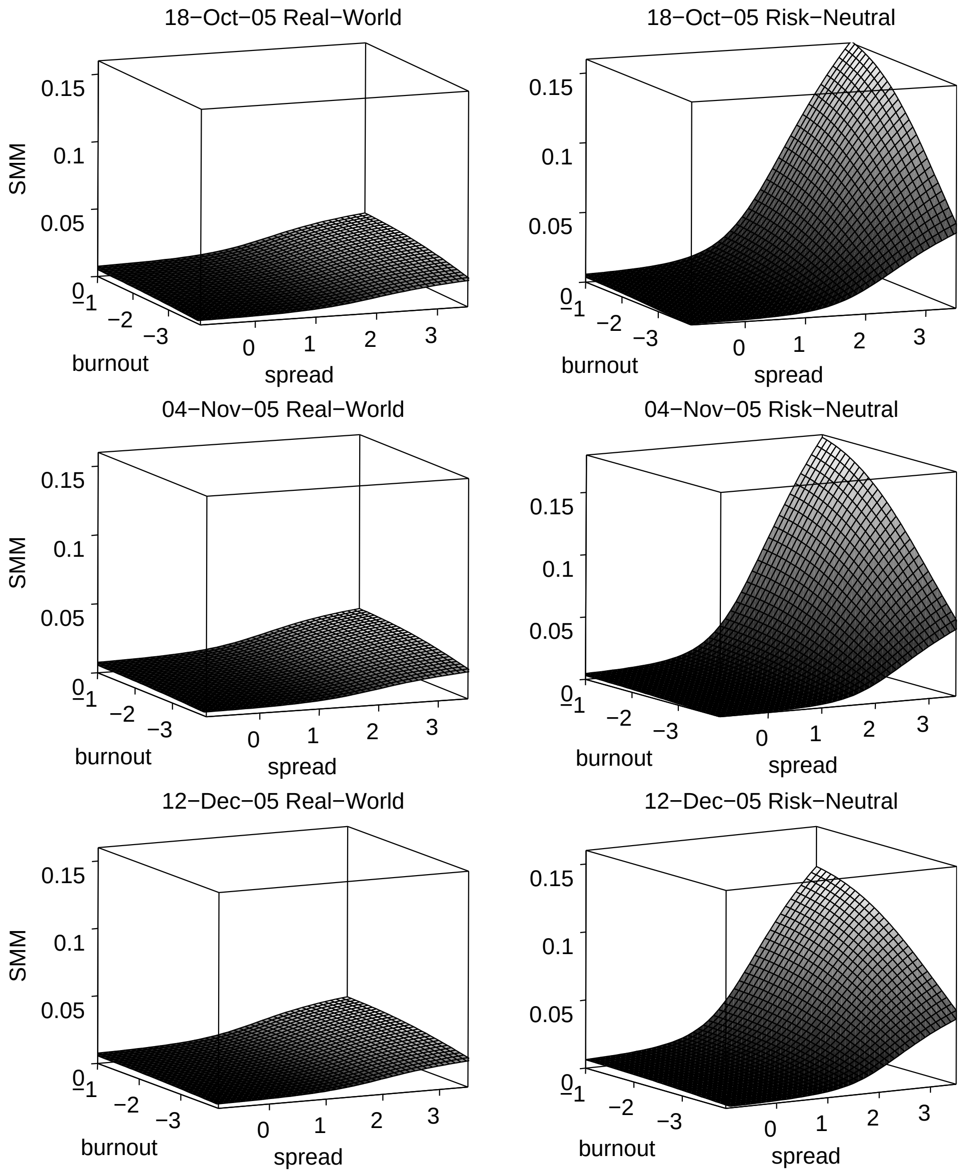

Figure 5.2: Expected prepayment speed as a function of the two covariates *spread* (in %) and *burnout* in real-world and under the risk-neutral pricing measure when the baseline hazard process is set equal to its mean-reversion level.

nancing understatement.

Note also at this point that a traditional OAS valuation is of course easy to perform within our modelling framework by simply setting $\mu = 1$ and $\lambda_p = \lambda_w = 0$. In this case prepayments would be forecast under the real-world measure and the OAS would be needed to equate the model prices to the observed market prices.

5.2.2 Prices and option-adjusted spreads

Figure 5.3 shows the traditional OAS and the prOAS values of our model and for a sample of GNMA securities on three arbitrarily chosen sample days. On each of the three days the current coupon was between 5.5% and 6%. For comparison purposes we also show the OAS levels as quoted in Bloomberg based on the Bloomberg prepayment model. Of course, it is hard

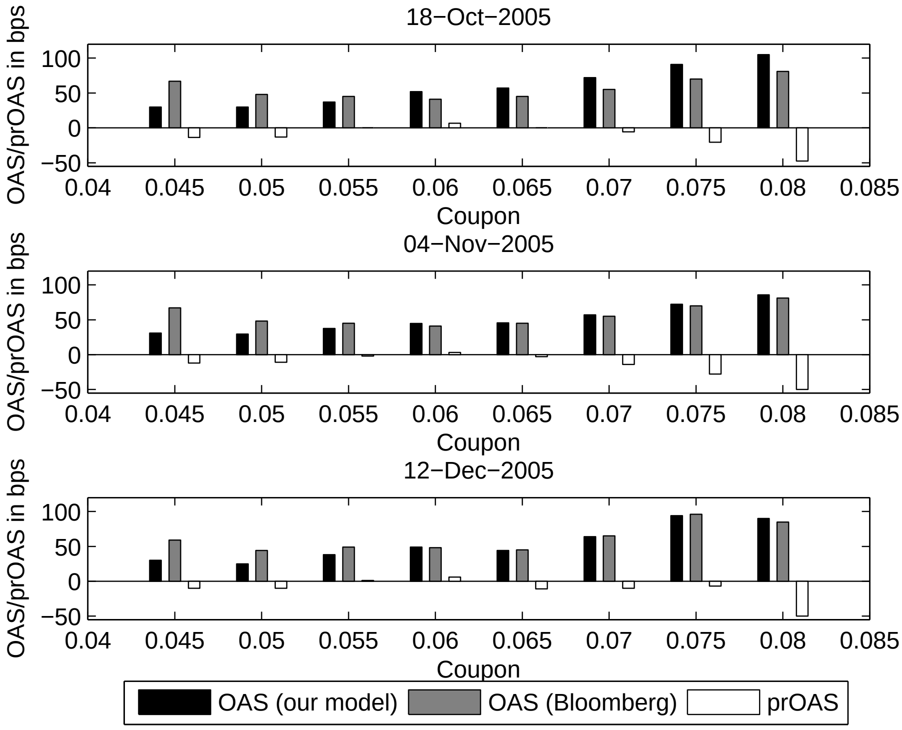

Figure 5.3: OAS according to our model, OAS as quoted in Bloomberg and 'prOAS' according to our model on 18-Oct-2005 (top), 04-Nov-2005 (centre) and 12-Dec-2005 (bottom) for a series of generic 30yr fixed-rate MBS of the GNMA I program with different coupons (Bloomberg ticker GNSF).

to compare OAS levels derived from different prepayment models. As already discussed in Kupiec and Kah (1999), it is very common in the MBS markets that OAS estimates of different brokers vary widely, attributable to different interest-rate and prepayment model assumptions. This fact may provide a further line of argumentation for prepayment-risk-neutral models like the one presented here. In addition to the prOAS levels in Figure 5.3 we also show the market prices of the GNMA securities in our sample directly compared to the risk-neutral model prices in Figure 5.4. These plots confirm that, generally, our model successfully explains market prices of generic fixed-rate GNMA pass-throughs (see also Chapter 6 and in particular Figures 6.5 and 6.6 for further empirical evidence on the performance of our model). Note that this is also true if we calibrate the risk-adjustment parameters only once to the data on 18-Oct-2005 and leave the parameters unchanged for our additional sample dates 04-Nov-2005 and 12-Dec-2005. As a quantitative measure of the accuracy of our pricing approach we consider the linear regression model

$$V_i^{market} = a + b \cdot V_i^{model} + \epsilon_i, \quad \epsilon_i \overset{iid}{\sim} N(0, \sigma_\epsilon^2), \quad i = 1, ...I \tag{5.14}$$

where V^{market} denotes the market prices of the MBS, V^{model} the prices of the MBS according to our prepayment-risk-neutral valuation model and $I = 24$ is the total number of observed market prices in our sample (we consider 8 securities on 3 different days). Obviously, the estimates of the regression parameters a and b should be close to 0 and 1 respectively. The actual estimates together with the R^2 value of the regression are reported in Table 5.4 in comparison to the values which we obtain with a 1-factor baseline model (i.e. without the GDP factor).

	Parameter	Estimate	95%-Conf.Int.
Regression (11) with 2-factor baseline prepayment model $R^2 = 98.5\%$	a	0.023	[-0.030;0.077]
	b	0.982%	[0.929;1.035]
Regression (11) with 1-factor baseline prepayment model $R^2 = 96.2\%$	a	0.0069	[-0.082;0.096]
	b	0.985	[0.898;1.072]

Table 5.4: Parameter estimates and R^2 of the regression model (5.14) when the model prices are calculated with the 2-factor baseline prepayment model and when the model prices are calculated with a 1-factor baseline prepayment model without the GDP growth process.

Since in the 2-factor baseline prepayment model the confidence intervals for a and b are narrower around 0 and 1 respectively and the R^2 value is higher, these results indicate that the GDP growth factor adds explanatory power to our prepayment-risk-neutral pricing model. The (in-sample) average absolute pricing error of our model, i.e. the mean of the absolute differences between the model prices and the market prices, is 59 basis points in our sample compared to 105 basis points for the 1-factor baseline prepayment model. When we consider out-of-sample prices, i.e. we use the risk-adjustment parameters calibrated to the data of 18-Oct-2005 for pricing on the two other sample days, we obtain an average absolute pricing error of 61 basis points for our model while the average absolute pricing error of the 1-factor baseline prepayment model is 76 basis points. These results provide further evidence for the usefulness of the GDP growth rate factor.

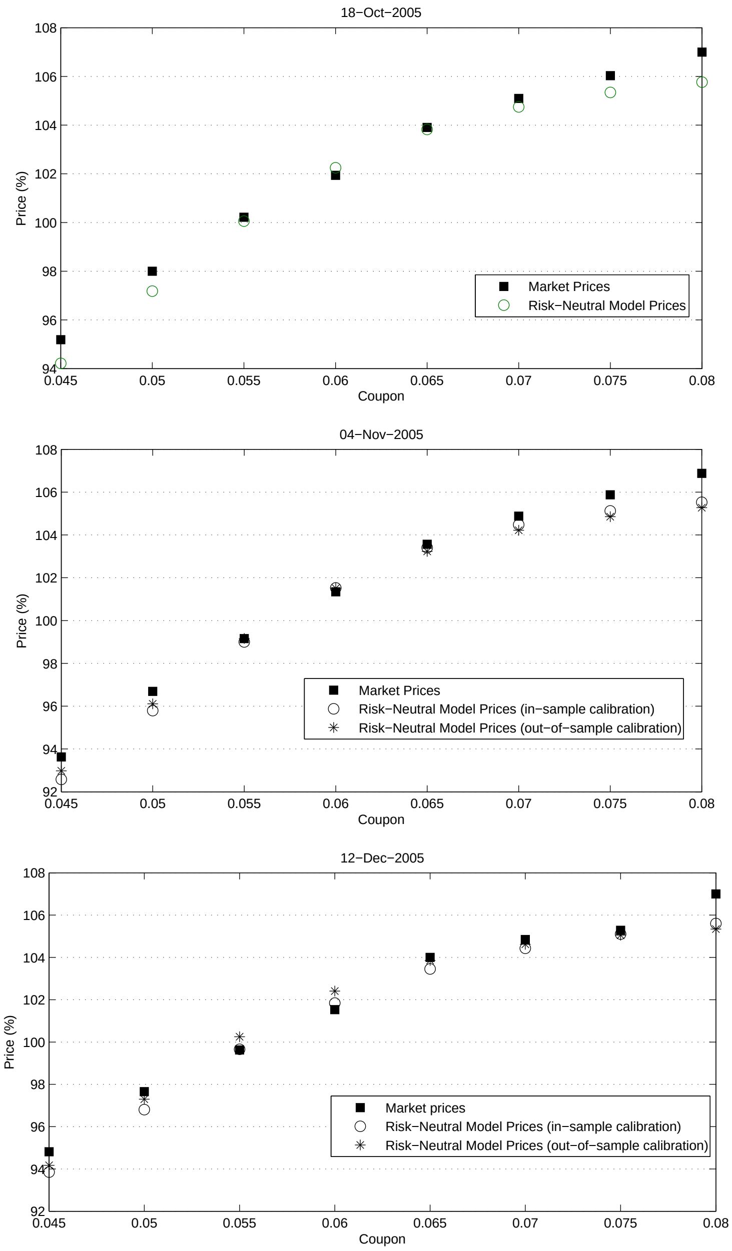

Figure 5.4: Market prices and model prices on 18-Oct-2005 (top), 04-Nov-2005 (centre) and 12-Dec-2005 (bottom) for a series of generic 30yr fixed-rate MBS of the GNMA I program with different coupons (Bloomberg ticker GNSF). Out-of-sample calibration means that we do not recalibrate the risk-adjustment parameters on the respective day, but use the parameter values of 18-Oct-2005 instead.

5.2.3 Effective duration, convexity and parameter sensitivities

Effective duration and convexity are two important quantities to measure the interest-rate risk exposure of a mortgage-backed security with respect to parallel shifts of the yield curve. Since these quantities are easy to determine they are very popular in practice, particularly in the context of MBS portfolio management. A regular bond's duration is defined as the price sensitivity of the bond with respect to parallel shifts of the yield curve and can easily be calculated as the weighted average of the times when payments are made with the weights being equal to the proportion of the bond's total present value provided by the payment of the respective payment time (see, e.g., Hull (2003), p. 112f. for details). For a mortgage-backed security, however, the basic duration concept can not be readily applied since a shift of the yield curve inevitably leads to changes in the prepayment behaviour and therefore changes the cash flows of the security. The modified cash flows resulting from changes in interest rates are accounted for in the effective duration D_{eff} which can be defined as follows (see, e.g. Hu (1997), p. 46):

$$D_{eff} = \frac{V_{-} - V_{+}}{2 \cdot V \cdot \Delta y}, \tag{5.15}$$

where V is the security's present value, V_{-} is the security's value after a parallel downward shift of the zero-rate curve of Δy and V_{+} is the security's value after a parallel upward shift of the yield curve of the same size. Analogously, the effective convexity C_{eff} , i.e. the second-order sensitivity of the value of a mortgage-backed security with respect to parallel shifts of the yield curve can be defined as (see again Hu (1997), p. 47):

$$C_{eff} = \frac{V_{-} - 2 \cdot V + V_{+}}{V \cdot (\Delta y)^2} \cdot 1\%, \tag{5.16}$$

where the scaling by 1% is done due to market convention. While the convexity of a regular bond is usually positive, most mortgage-backed securities (with the possible exception of very deep discounts and very high premium securities) feature negative convexities. Negative convexity means that, with respect to parallel shifts of the yield curve, an MBS has more downside risk in the case of rising interest rates than upside potential in the case of falling rates. This can easily be explained by the prepayment feature inherent in MBS. If interest rates rise, prepayments tend to slow down, which extends the cash flow stream's average life. Thus, the negative effect of the rising rates on the present value of the security's cash flow stream is intensified. Contrarily, prepayments accelerate if interest rates fall, counteracting the effect of falling rates on the present value of the security. In Figure 5.5 we show the effective durations and convexities across the whole coupon range according to our prepayment-risk-neutral valuation model for our sample date 12-Dec-2005. For comparison purposes we also show the values as provided by Bloomberg based on the Bloomberg prepayment model. Note that the effective durations and convexities were calculated with a parallel shift of the yield curve of 50 basis points, i.e. $\Delta y = 0.005$.

In addition to the sensitivities of the MBS prices to parallel shifts of

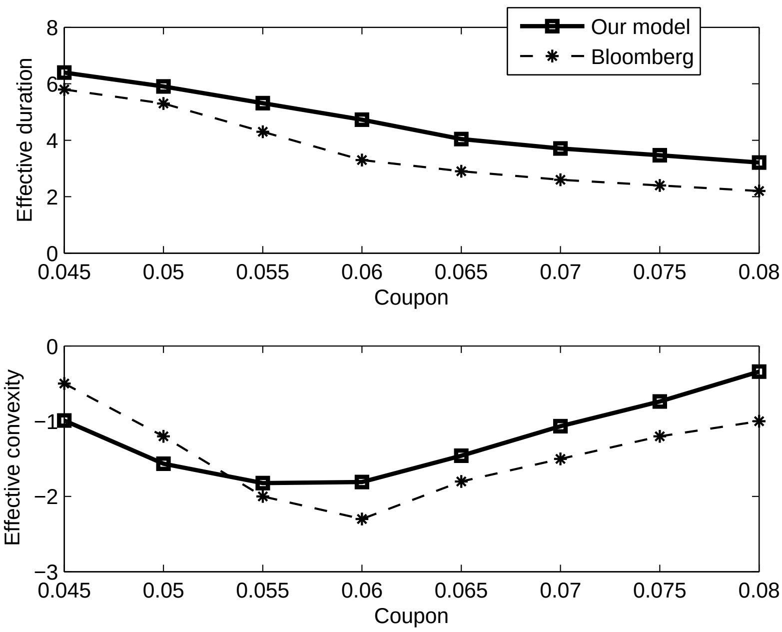

Figure 5.5: Effective durations and convexities of some GNMA MBS with different coupons on the sample day 12-Dec-2005 according to our model and according to Bloomberg.

the yield curve, we also investigate the price sensitivities with respect to changes in some of the model parameters on the same sample date 12-Dec-2005. For this purpose we gradually increase each model parameter from 60% of its value as stated in Tables 5.1 and 5.3 to 140%. Note that in the scenarios where we change the volatility parameters σ_r, σ_p and σ_w we also change λ_r, λ_p and λ_w so that the parameters $\hat{a}_r$, $\hat{a}_p$ and $\hat{a}_w$ remain unchanged.

The effect of the interest-rate volatility parameter σ_r. Figure 5.6 shows that

MBS prices across the whole coupon range decrease if σ_r increases. This can easily be explained by the prepayment option which an MBS investor is short. Consistent with basic option theory, the value of the mortgagors' prepayment option increases if the interest-rate volatility increases. Thus, MBS prices decrease.

The effect of the parameter $\hat{a}_r$. Figure 5.6 shows that changes of the mean reversion speed parameter $\hat{a}_r$ within the analysed scope do not significantly affect MBS prices. Recall that the interest-rate model we use is a Hull-White model, where the function $\theta_r(t)$ ensures that for any parameter value of $\hat{a}_r$ the current term structure is perfectly fit. Thus, if $\hat{a}_r$ changes, so does the function $\theta_r(t)$, so that the model remains arbitrage-free as already discussed in Chapter 3.2. Yet, if $\hat{a}_r$ increases, the average speed with which the short-rate reverts to its mean reversion level increases. However, within the analysed scope, the effect of an increased mean-reversion speed of the short rate seems negligible.

The effect of the parameter μ. As already discussed in the previous paragraphs, the prepayment-risk adjustment parameter μ inversely scales the overall prepayment intensity. Thus, the larger μ the lower the overall prepayment intensity and the longer the average lives of the MBS. Hence, in general, an increase in μ can be expected to increase the value of premiums and, at the same time, decrease the value of discounts. Very large values of μ, however, imply that there is virtually no prepayment, i.e. that the prepayment option is never exercised and thus worthless. For large values of μ, the MBS prices therefore equal the prices of a security without any prepayment feature. These prices are, obviously, higher than the regular MBS prices across the whole coupon range. This explains the sensitivity pattern as shown in Figure 5.6.

The effect of the parameter β_1. Figure 5.6 shows that MBS prices across the whole coupon range decrease if β_1 increases. An increase of β_1 means that the intensity of prepayment increases if rates rise. Thus, more people exercise their prepayment option when it is advantageous to do so, which reduces the value of both discount and premium MBS.

The effect of the parameter β_2. Figure 5.7 shows that MBS prices across the whole coupon range decrease if β_2 increases. The parameter β_2 determines the shape of the refinancing S-curve, modelled via the arctangent-function in our model. If β_2 increases, the S-curve is jolted. I.e., prepayment intensities react more sensitively to changes in interest-rates and refinancing incentives, reducing the value of MBS across the whole coupon range.

The effect of the parameters θ_p and $\hat{a}_p$. Both parameters θ_p and $\hat{a}_p$ determine the mean-reversion level of the baseline prepayment process $p_0(t)$. The larger the absolute value of θ_p (remember that θ_p is negative, see Table 5.1) and the smaller $\hat{a}_p$, the higher the average baseline prepayment. Then, the same reasoning as for the effects of changes in the overall prepayment speed by changes of the parameter μ applies. The previously discussed effects for the parameter μ explain the patterns in Figure 5.7 for the parameters θ_p and $\hat{a}_p$.

The effect of the parameter σ_p. Figure 5.7 shows that changes in the parameter value of σ_p do not have any major effects on the MBS prices within the analysed scope. This may be explained by the fact that the baseline prepayment is not a systematic prepayment component. E.g., a higher volatility of baseline prepayment may lead to some particularly high or particularly low prepayment rates when this is advantageous for the investor as well as in situations when this is not advantageous for the investor. I.e., a slightly higher or lower volatility of baseline prepayment rates does not systematically affect MBS prices.

The effect of the parameter b_{pw}. Figure 5.8 shows that if b_{pw} increases (in absolute terms; recall that b_{pw} is also negative) premiums appreciate while prices of discounts slightly decrease. This can again be explained by the fact that if b_{pw} increases (in absolute terms), baseline prepayment on average decreases, increasing the value of premiums and reducing the value of discounts as expected.

The effect of the parameters θ_w and $\hat{a}_w$. The parameters θ_w and $\hat{a}_w$ determine the mean-reversion level of the GDP growth process and, through the b_{pw} term the average mean-reversion level of the baseline prepayment. The larger the value of θ_w and the smaller the value of $\hat{a}_w$, the smaller the average baseline prepayment. While the effects on the MBS prices are generally small within the analysed scope, Figure 5.8 shows that the expected effects can at least be confirmed for premium securities.

The effect of the parameter σ_w. Figure 5.8 shows that changes in the parameter value of σ_w do not have any major effects on the MBS prices within the analysed scope. Since the GDP growth process only affects the baseline prepayment, which is an unsystematic source of prepayment, the same reasoning as previously discussed for the parameter σ_p applies.

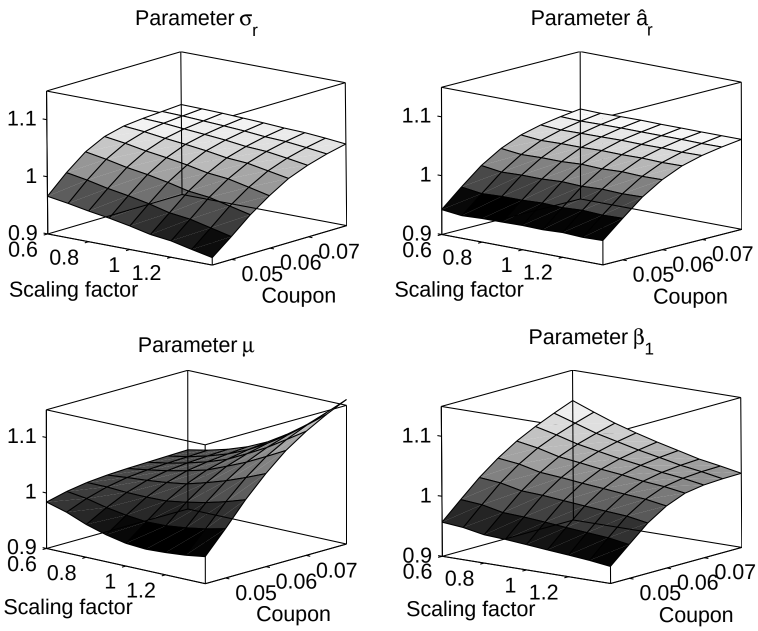

Figure 5.6: MBS price sensitivities with respect to changes in model parameters (I).

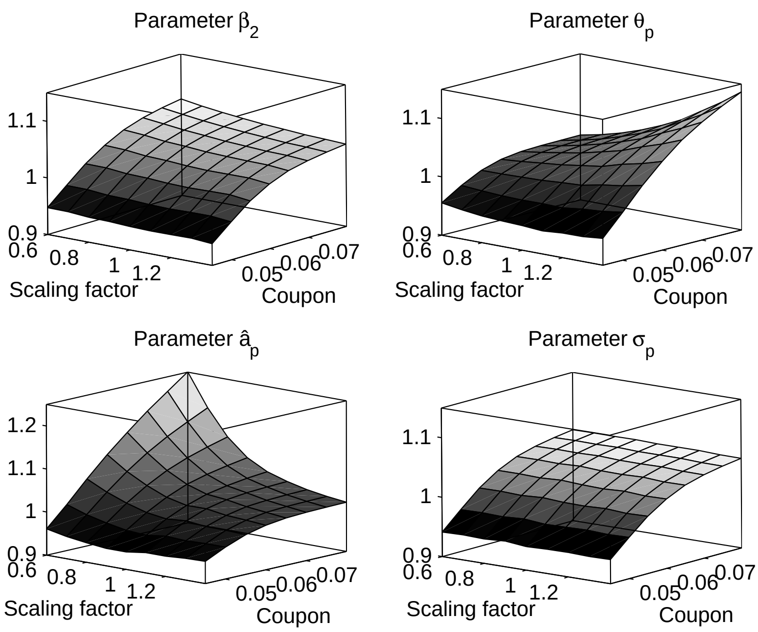

Figure 5.7: MBS price sensitivities with respect to changes in model parameters II).

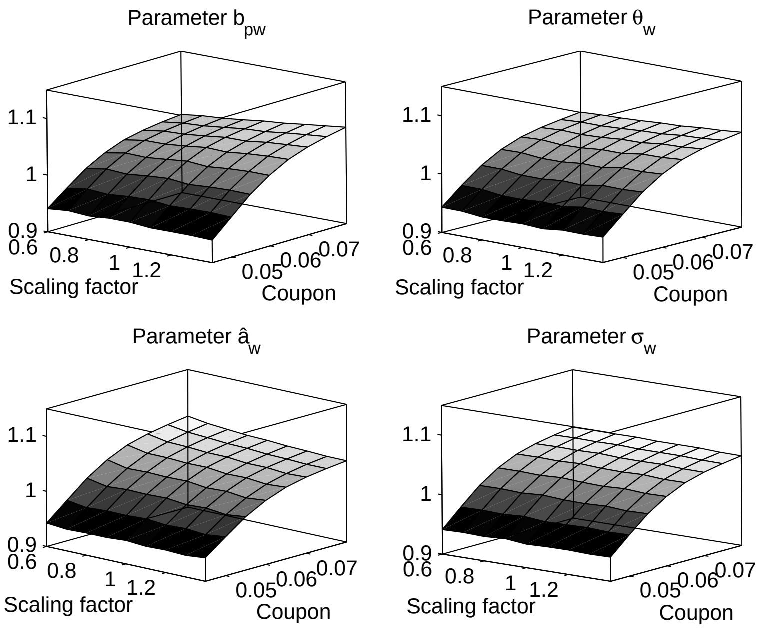

Figure 5.8: MBS price sensitivities with respect to changes in model parameters (III).

5.3 Adjustable-Rate MBS

Our model as introduced in the previous section for fixed-rate MBS can quite easily be extended to the pricing of adjustable-rate MBS. In Chapter 2.2.1 we have already mentioned that adjustable-rate MBS usually have a 6 month or 1 year floating money market or treasury rate as reference index and that they have periodic reset Caps and Floors as well as life time Caps, reducing the impact of interest-rate changes for the borrower. In the case of GNMA adjustable-rate MBS, which we will focus on in this study, the reference index is the 1 year constant maturity treasury (CMT) rate and the net margin (i.e. the spread above the index the adjustable-rate MBS adjusts to) is 150 basis points. Moreover, all GNMA adjustable-rate MBS (GNMA ARMs in the following) have a coupon reset frequency of 1 year, a lifetime Cap of 5% above the initial rate and an annual periodic reset Cap and Floor of 1%. Thus, the coupon adjustment at any fixing date can never exceed 1%, neither upward nor downward. Initial rates of adjustable-rate mortgages are often lower than the fully indexed rate. These low initial rates are usually labelled 'teaser-rates'.

Evidently, a pure floating-rate mortgage does not feature any prepayment risk since the coupon of the mortgage always reflects current market conditions and prepayment does not lead to any losses for the investor. Yet, GNMA ARMs are not pure floaters, as previously discussed. The embedded Caps and Floors, the long tenor and the existence of teaser rates are the reasons why an investor in GNMA ARMs is exposed to prepayment risk which can not be neglected entirely (see also, e.g., Ambrose and LaCour-Little (2001)).

For the valuation of GNMA ARMs we can straightforwardly extend our approach for fixed-rate MBS as described in the previous section. The only change we have to make is the refinancing-prepayment function, as given in (5.9) for fixed-rate MBS, since reasons for prepayments of ARMs differ from those of fixed-rate mortgages. Following Davis (2004) we consider pool age, ARM-to-ARM refinancing incentive and ARM-to-FRM refinancing incentive as possible explanatory variables. We do not consider seasonality effects, allow, however, for interaction effects between the ARM-to-ARM and the ARM-to-FRM refinancing incentives. The data sample for the statistical modelling in this section consists of historical prepayment and coupon data of a total of 18 GNMA ARMs during the time interval April 2001 to March 2006.

Ambrose and LaCour-Little (2001) report that ARM prepayments are low directly after origination, peak around the first rate reset date and remain constant afterwards (with some smaller peaks around subsequent rate reset dates in their sample). This behaviour can easily be explained by the teaser rates. A teaser rate terminates with the first rate reset (after 13-15 months for GNMA ARMs) and the incentive to refinance, possibly to a new adjustable-rate contract with a new teaser rate, peaks. This general pattern can also be observed in our data (see Figure 5.9). We therefore define the first explanatory variable in our GNMA ARM refinancing-prepayment model by

$$z_1(t) = \begin{cases} \text{pool age} & \text{if} \quad \text{pool age} \leq 14 \\ 14 & \text{if} \quad \text{pool age} > 14 \end{cases} .$$

Similar to Davis (2004) we consider the spread between the current weighted

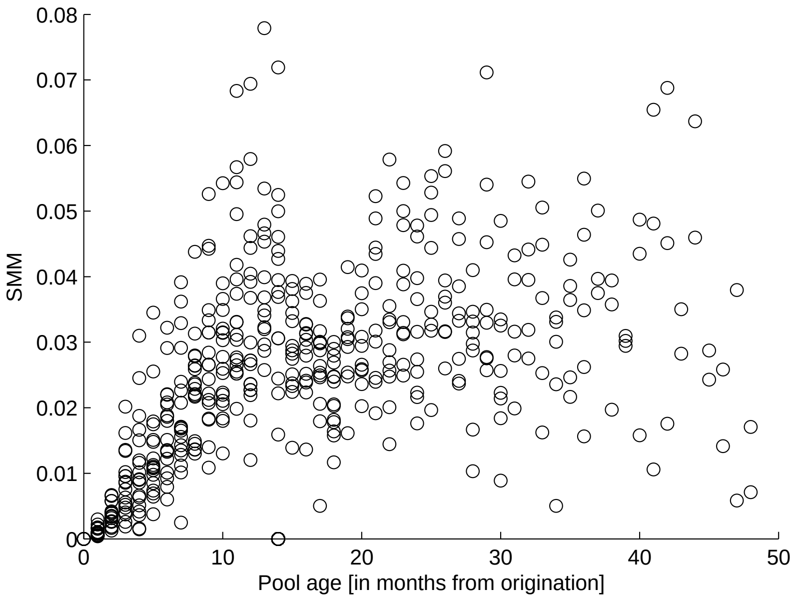

Figure 5.9: Observed GNMA ARM prepayment speeds (as SMM) vs. pool age.

average coupon (CWAC) and the current 1 year CMT rate as a proxy for the ARM-to-ARM refinancing incentive. The ARM-to-FRM incentive is modelled via the spread between the 10 year CMT rate (which we again consider to be a proxy for long-term mortgage refinancing rates) and the current 1 year CMT rate. We thus have the explanatory variables

$$\begin{aligned} z_2(t) &= CWAC(t) - \text{1-year CMT (t)} \qquad [\text{ in } \%], \\ z_3(t) &= \text{10-year CMT (t)} - \text{1-year CMT (t)} \qquad [\text{ in } \%]. \end{aligned}$$

We also include the explanatory variable

$$z_4(t) = z_2(t) \cdot z_3(t)$$

which allows us to account for interaction effects between $z_2(t)$ and $z_3(t)$. Such an interaction effect is useful since ARM-to-ARM refinancing and ARM-to-FRM refinancing influence each other. In fact, the regression surface in Figure 5.10 shows that the highest prepayment rates can be expected if either the spread between the CWAC and the 1-year CMT rate is high (indicating a clear ARM-to-ARM refinancing incentive) or if the slope of the yield curve is low (indicating an ARM-to-FRM refinancing incentive). With the covariate

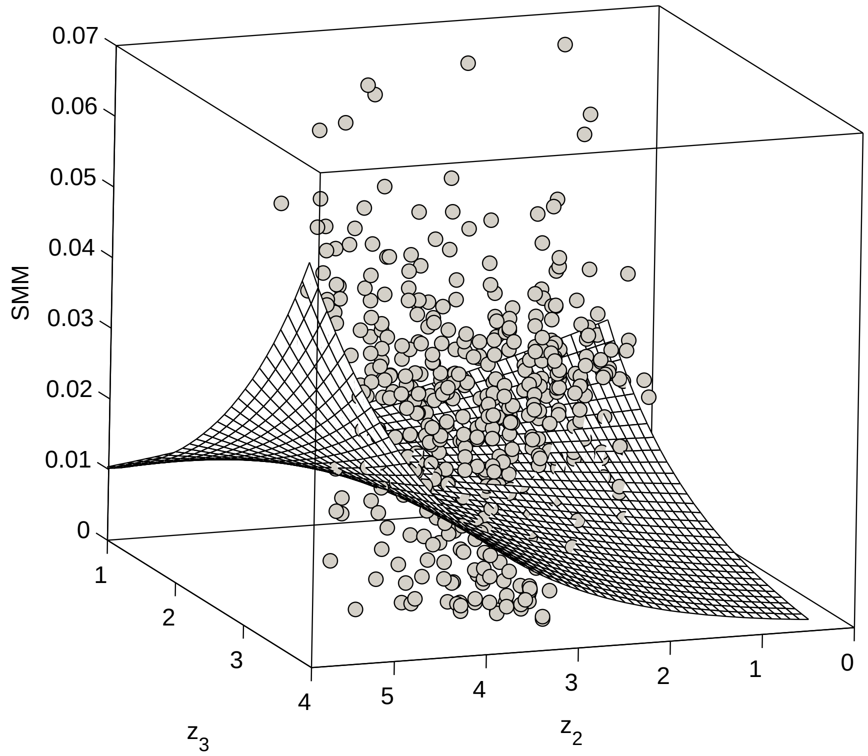

Figure 5.10: Observed GNMA ARM prepayment speeds (as SMM) vs. fitted regression surface if the baseline prepayment process is set to its mean-reversion level and $z_1 = 14$.

vector $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ and the regression parameter vector $\alpha = (\alpha_1, \alpha_3, \alpha_3, \alpha_4)$, our refinancing-prepayment function for GNMA ARMs has the form

$$f_{ARM}(z(t), \alpha) = \alpha_1 z_1(t) + \alpha_2 z_2(t) + \alpha_3 z_3(t) + \alpha_4 z_4(t). \tag{5.17}$$

The overall prepayment speed is thus given by

$$p_{ARM}(t) = e^{f_{ARM}(z(t),\alpha)+p_0(t)},$$

where $p_0(t)$ is the baseline prepayment process as defined in (5.3). We assume for simplicity that the baseline prepayment process is the same for fixed-rate and adjustable-rate mortgages. The components of the regression parameter vector α can again be estimated by Kalman filtering techniques, as already discussed for our fixed-rate MBS model in the previous section. The results of the parameter estimation are shown in Table 5.5. Since regression pa-

Parameter	Estimate
α_1	0.1124
α_2	-0.4377
α_3	-1.0515
α_4	0.2697

Table 5.5: Estimates of the regression parameter vector in our GNMA ARM model.

rameters in a model with interaction effects between explanatory variables (reflected in our variable $z_4(t)$) can not be interpreted directly, we illustrate them with respect to the spread covariates in Figure 5.10. For any given value of the prepayment-risk-adjustment parameters μ, λ_p and λ_w, we can now price GNMA ARMs by Monte-Carlo simulation. The Monte-Carlo estimation procedure, as described in the appendix for fixed-rate MBS, has to be adapted only slightly. Instead of the fixed-rate MBS refinancing-prepayment function $f(x(t), t)$ we have to use the function $f_{ARM}(z(t), t)$ as given in (5.17). Furthermore, the weighted average coupon does of course not remain constant but has to be adjusted according to the simulated index rate, taking into account the Caps and Floors embedded in the contract.

The prepayment-risk-adjustment parameters μ, λ_p and λ_w can again be calibrated to market data. In order to give an empirical example, we consider market data of 20-Aug-2006. On this particular day, prices for GNMA ARMs with current coupons of 4%, 4.5% and 5% were actively quoted on a TBA basis. Calibrating the prepayment-risk-adjustment parameters to the prices of these securities (by minimising the Euclidean norm of differences between model and market prices) yields the estimates as given in Table 5.6. These results are line with the estimates obtained in our fixed-rate MBS model and the same interpretations apply. In Table 5.7 we show the market

Parameter	Estimate
μ	2.7
λ_p	3.1
λ_w	-14.4

Table 5.6: Prepayment-risk-adjustment parameters calibrated to GNMA ARM prices on the sample day 20-Aug-2006.

prices and our model prices. These results indicate that our model is able to explain market prices of GNMA ARMs successfully. Yet, further empirical evidence, in particular with respect to parameter stability over time, would be interesting. This is however, beyond the scope of this thesis and remains a topic for further research.

	Coupon 4%	Coupon 4.5%	Coupon 5%
Market price	0.9800	0.9841	1.0059
Model price	0.9847	0.9912	0.9995
Rel. pricing error	0.5%	0.7%	0.6%

Table 5.7: GNMA ARM market and model prices according to our prepayment-risk-neutral GNMA ARM pricing model on the sample day 20-Aug-2006.

5.4 Collateralized Mortgage Obligations

In this section we will apply our model to the pricing of Collateralized Mortgage Obligations (CMOs). We have already introduced CMOs briefly in Chapter 2 and have mentioned that the cash flows of the underlying mortgage pool are assigned to different tranches. Depending on the individual securitisation scheme, these tranches may have very different characteristics. The most basic type of CMO is the sequential-pay CMO, sometimes also called plain-vanilla or clean CMO. In a sequential-pay CMO each tranche receives regular interest payments. Principal payments (scheduled payments and prepayments) are, however, assigned sequentially to the different tranches, usually labelled A-Tranche, B-Tranche, etc. I.e., principal payments are assigned to the Tranche A alone until it is completely retired. Once this is the case, all principal payments are assigned to Tranche B, etc. Figure 5.12 illustrates the cash flow pattern of a sequential-pay CMO with three tranches.

More complex CMO structure may include Planned Amortization Class (PAC) tranches and Targeted Amortization Class (TAC) tranches. PAC tranches guarantee a fixed principal repayment schedule as long as prepayment rates remain in a certain corridor, e.g. 75% PSA to 300% PSA. TAC tranches have a designated target speed. If prepayments are equal or above the target speed, e.g. 100% PSA, the principal allocation to the TAC tranche follows a prespecified schedule. Thus, PAC tranches offer protection against prepayment risk up to a certain degree. Unlike the PAC tranches, a TAC tranche is not protected from extension if prepayments are slower than expected and therefore offer only one-sided prepayment variability protection. Excess prepayments which are above the PAC and TAC tranche schedules are absorbed by so-called companion or support tranches. These tranches feature a very high uncertainty with respect to cash flow timing and therefore bear a large part of the prepayment risk associated with the underlying mortgage pool. In addition to PAC, TAC and support tranches, complex CMOs may also have accrual tranches (usually labelled Z-tranche), which do not receive interest payments during a certain lockout period, Interest-Only tranches, Principal-Only tranches, Floating-Rate Tranches and Inverse Floating-Rate tranches. Finally, residual tranches collect remaining cash flows from the underlying pool after the obligations to the other tranches have been met.

The pricing of these tranches within our modelling framework is perfectly possible. The Monte-Carlo simulation used to price the underlying pass-through security according to (5.8) simply has to be changed to accommodate the cash flow structuring rules of the CMO. Since virtually every CMO features individual characteristics, liquid market prices of CMO tranches are not readily available. Since we do not have liquid market data, it is impossible at this stage to validate our model empirically using CMO market prices. We will however, briefly illustrate how our model works for an easy sequential-pay CMO structure with three tranches as illustrated in Figure 5.12. We assume that all tranches are equally large (i.e. 1/3 of the underlying pool's total notional amount) and that the underlying pool is the GNMA 5.5% generic pool on the sample day 12-Dec-2005 as already used in Chapter 5.2.2. Furthermore, we assume that the coupon for each of the three tranches is equally 5.5%. Our prepayment-risk neutral model prices of the tranches (with respect to a notional of 1), as well as the respective expected average lives, are given in Table 5.8.

In addition to the different expected average lives, CMO tranches also differ with respect to their model risk exposure. The different model risk

Tranche	A	B	C
Model Price	1.0081	0.9866	0.9866
Exp. Average Life (yrs.)	1.98	7.72	17.02

Table 5.8: Model prices and expected average lives of the three tranches in the sequential-pay CMO example.

exposure of the three tranches in our CMO example is illustrated exemplarily in Figure 5.11. Analogously to Chapter 5.2.3, we gradually scale the interest-rate volatility parameter σ_r. The price sensitivity with respect to changes of the parameter σ_r evidently increases from Tranche A to Tranche C. This can again be explained by the value of the prepayment option which increases when the interest-rate volatility increases. Since the tranches with lower principal repayment priority have longer average lives, the value of the prepayment option inherent in these tranches is evidently more sensitive with respect to changes in interest-rate volatility than the value of the prepayment option inherent in tranches with a higher principal repayment priority. This should be taken into account when investing in CMO tranches.

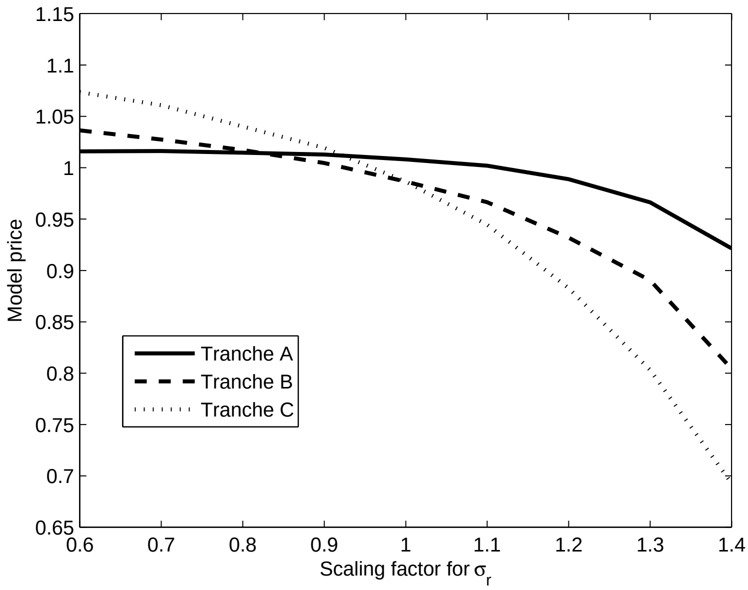

Figure 5.11: Price sensitivities of the three tranches in the sequential-pay CMO example with respect to changes in the interest-rate model volatility parameter σ_r.

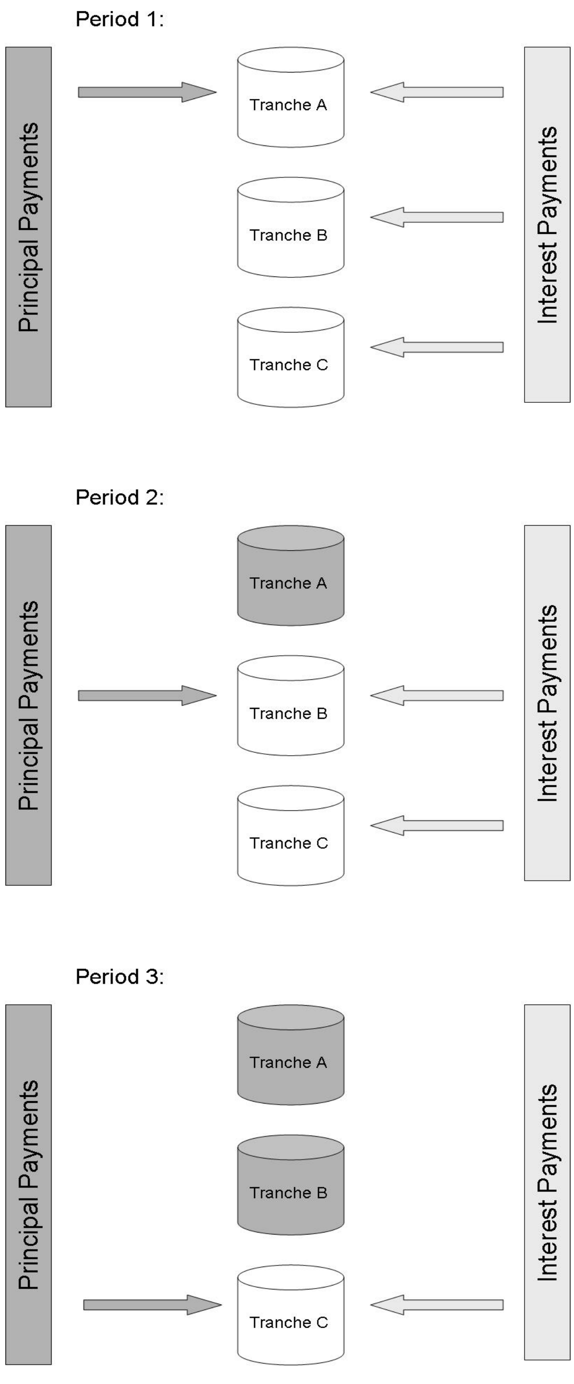

Figure 5.12: Cash flow structure of a sequential-pay CMO with three tranches.

Chapter 6

A closed-form approximation for fixed-rate MBS

The valuation of mortgage-backed securities is usually considered a computationally expensive problem. This holds for both the option-theoretic approaches and the traditional econometric approaches, as already discussed in Chapter 4. In this chapter we develop an approximate closed-form formula for the value of fixed-rate (agency) MBS and, as corollaries, for Interest-Only and Principal-Only securities. The valuation is again based on a stochastic prepayment intensity approach similar in spirit to the approach presented in Chapter 5.

In many intensity-based modelling approaches for the valuation of defaultable bonds and credit derivatives closed-form pricing formulas are available. It is our aim in this chapter to apply these techniques to prepayment-sensitive MBS. Interestingly, in our intensity-based model we find that a closed-form solution of the MBS valuation problem leads to rather similar challenging calculations as in the approach developed by Collin-Dufresne and Harding (1999). As already discussed in Chapter 4.2, the Collin-Dufresne/Harding model, as well as its extension developed by Rom-Poulsen (2007), has a couple of shortcomings. In our framework, however, we are able to address some of the previously mentioned shortcomings which may be problematic in certain situations and for certain types of MBS. It is straightforward to incorporate additional stochastic factors into our model. Similar to the approach taken in Chapter 5, we do this by modelling the (non refinancing-related) baseline prepayment process via two stochastic factors, where the second factor is fit to the GDP growth in the US. We thus account for the dependence between general economic conditions and turnover-related prepayment in our model. The baseline prepayment is also supposed to capture defaults which, in the case of agency MBS, simply result in prepayment for an investor. In addition to this, we account for the usual S-curve shape of the refinancing-incentive/prepayment relation by a sectionwise linear approximation. This is quite similar to the approach presented in Gorovoy and Linetsky (2007) (which was developed independently from the research presented here). We find that this approximation does have an important effect across the whole coupon range.

While in our modelling framework it is straightforward to conduct a classical OAS valuation, we are again primarily interested in a prepayment-risk-neutral valuation. This also allows us to assess the performance of our model quantitatively by directly comparing market to model prices.

6.1 The model set-up

Our starting point here is the valuation of a single mortgage contract. We assume that the time of prepayment of one mortgage does not influence the probability of prepayment of other mortgages and that the pool is homogeneous (w.r.t. mortgage maturity, coupon, etc. and thus w.r.t. individual prepayment probabilities). Thus, the value of the MBS can be calculated as the value of an individual mortgage multiplied with the number of mortgages in the pool. While this assumption is problematic in option-based models where one would have to establish some additional features accounting for heterogeneous prepayment-option exercise behaviour, this is not the case in the reduced-form framework. We further assume that partial prepayment is not possible.

Consider a mortgage contract with payment dates $t_1, ..., t_K$, define $\Delta t_k := t_k - t_{k-1}$ (years) and set $t_0 = 0$. On each payment date t_k, $k = 1, ..., K$, the mortgage payment $M(t_k) \cdot \Delta t_k$, containing both interest and regular repayments, has to be made until the time of prepayment. At the time of prepayment t_τ (or at the final maturity of the mortgage), the remaining principal balance according to the amortization schedule $A(t_\tau)$ is paid back in a lump sum. Thus, all cash flows depend on the time of prepayment. Following (3.35) and (3.37), the approximate value $V_{app}^{Mo}(0)$ of the mortgage contract at time 0 admits the representation

$$V_{app}^{Mo}(0) = E_{\widetilde{Q}}\left[\sum_{k=1}^{K}(M(t_k) \cdot \Delta t_k + A(t_k) \cdot \pi(t_k) \cdot \Delta t_k) \cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right], \tag{6.1}$$

where $r(t)$ is the (risk-free) short-rate process, $\pi(t)$ is the (continuous, annualised) prepayment intensity/prepayment speed process and $\widetilde{Q}$ is the risk-neutral pricing measure. Formula (6.1) can also be considered a discretised version of the continuous time valuation formula in Goncharov (2005). Note that (6.1) implies that $A(t_k)$ is to be understood here as the outstanding balance according to the amortisation schedule before the regular principal repayment has been made on the payment date t_k. In the credit risk literature, where formulas such as (6.1) are common, the process $\pi(t)$ is usually the default intensity. If the default intensity process can be set up within a Gaussian framework, a closed-form representation of formula (6.1) is, in general, possible. In many models and applications in practice the default intensity is modelled independently from $r(t)$. In the case of prepayment modelling, however, the prepayment intensity process $\pi(t)$ can not be assumed to be independent of the interest-rate process $r(t)$ since it is a well known fact that mortgage borrowers are more likely to refinance their loans and thus prepay their mortgages when interest rates decline, as previously discussed. In the following we will again decompose the overall prepayment intensity into the two independent components refinancing-related prepayment $\pi_{\text{refi}}(t)$ and baseline prepayment $\pi_0(t)$, i.e. we get for the continuous, annualised prepayment speed

$$\pi(t) = \pi_{\text{refi}}(t) + \pi_0(t). \tag{6.2}$$

Note that we have specified the prepayment intensity here in a slightly different way than in the model presented in the previous chapter. The prepayment intensity $\pi(t)$ in our closed-form model presented here is the analogue of $\ln p(t)$ in the previous chapter.

Following the argumentation in Chapter 5 based on the Girsanov theorem for marked point processes, we again introduce a multiplicative prepayment-risk adjustment parameter μ so that, under the risk-neutral pricing measure $\widetilde{Q}$, the prepayment process has the dynamics

$$d\tilde{\pi}(t) = \mu \cdot (d\pi_{\text{refi}}(t) + d\pi_0(t)). \tag{6.3}$$

Note that (6.3) implies that we assume that the intensity under the risk-neutral pricing measure $\widetilde{Q}$ is given by the intensity under the real-world measure, multiplied with some constant μ. I.e., we assume that

$$\Phi(t) = \mu$$

instead of $\Phi(t) = (p(t))^{\mu-1}$ in Theorem 5.1. We will discuss the refinancing component and the turnover component of prepayment separately in the following before we finally put all components together for our closed-form formula.

The short-rate model and the refinancing component

A crucial component of every MBS valuation model is an adequate model for the interest-rate term structure. For our closed-form formula we use a 1-factor CIR model as introduced in Chapter 3.2.3, which has proven to be better suited for our closed-form approximation approach than the Hull-White model used in Chapter 5. While an extensions to a two-factor CIR model is possible in our modelling framework, we leave this for further research. Recall that in the basic CIR model, the risk-free short-rate dynamics under the risk-neutral measure $\widetilde{Q}$ are given by

$$dr(t) = (\theta_r - \hat{a}_r r(t))dt + \sigma_r \sqrt{r(t)} d\widetilde{W}_r(t), \tag{6.4}$$

where $\widetilde{W}_r$ is a $\widetilde{Q}$-Wiener process, $\hat{a}_r := a_r + \lambda_r \sigma_r^2$ with the market price of risk parameter λ_r and some positive constants θ_r, a_r, σ_r with $2\theta_r > \sigma_r^2$. Recall also that the zero-coupon bond prices in the CIR model can be calculated analytically. They are comprised in the following, more general, lemma:

Lemma 6.1. *In the CIR short-rate model and with $r^c(t) := c \cdot r(t)$ for some constant $c \geq -\frac{\hat{a}_r^2}{2\sigma_r^2}$, it holds that*

$$P^c(t,T) := E_{\widetilde{Q}}[e^{-\int_t^T r^c(s)ds} | \mathcal{F}_t] = e^{A^c(t,T) - B^c(t,T) r(t)} \tag{6.5}$$

where

$$\begin{aligned} B^c(t,T) &= c \cdot \frac{1 - e^{-\gamma^c (T-t)}}{\kappa_1 - \kappa_2 e^{-\gamma^c (T-t)}}, \\ A^c(t,T) &= \frac{2\theta_r}{\sigma_r^2} \log \left[\frac{\gamma^c e^{\kappa_2 \cdot (T-t)}}{\kappa_1 - \kappa_2 \cdot e^{-\gamma^c \cdot (T-t)}} \right] \end{aligned}$$

with $\gamma^c := \sqrt{\hat{a}_r^2 + 2\sigma_r^2 c}$, $\kappa_1 := \frac{\hat{a}_r}{2} + \frac{\gamma^c}{2}$ and $\kappa_2 := \frac{\hat{a}_r}{2} - \frac{\gamma^c}{2}$.

Proof. For $c = 1$ we have the well-known formulas for zero-coupon bond prices in the CIR model. For $c \geq 0$ in general, we get the dynamics of $r^c(t)$ under $\widetilde{Q}$ by a simple application of the Ito-formula and obtain:

$$r^c(t) = (\theta_r^c - \hat{a}_r r^c(t))dt + \sigma_r^c \sqrt{r^c(t)} d\widetilde{W}_r(t)$$

with

$$\begin{aligned} \theta_r^c &= c \cdot \theta_r, \\ \sigma_r^c &= \sqrt{c} \cdot \sigma_r \end{aligned}$$

and the statement follows directly from, e.g., Zagst (2002a), p.126/127. For $-\frac{\hat{a}_r^2}{2\sigma_r^2} \le c < 0$, however, the result is less straightforward. We therefore explicitly give the detailed proof in the following.

From the Feynman-Kac representation of the Cauchy-Problem (see (3.5)) we know that $P^c(t,T)$ must satisfy:

$$P_t^c + (\theta_r - \hat{a}_r r) P_r^c + \frac{1}{2} \cdot \sigma_r^2 \cdot r \cdot P_{rr}^c = c \cdot r \cdot P^c \tag{6.6}$$

with boundary condition $P^c(T,T) = 1$. Since

$$\begin{aligned} P_r^c &= -B^c \cdot P^c, \\ P_t^c &= P^c \cdot (A_t^c - r \cdot B_t^c), \\ P_{rr}^c &= (B^c)^2 \cdot P^c, \end{aligned}$$

it follows from (6.6) that

$$\begin{aligned} A_t^c(t,T) - \theta_r B^c(t,T) - r \cdot (c - \frac{1}{2} \cdot \sigma_r^2 \cdot (B^c(t,T))^2 & \\ + B_t^c(t,T) - \hat{a}_r \cdot B^c(t,T)) &= 0 \end{aligned}$$

with $A^c(T,T) = B^c(T,T) = 0$. This leads to the Riccati-style equations

$$c - \frac{1}{2} \cdot \sigma_r^2 \cdot (B^c(t,T))^2 + B_t^c(t,T) - \hat{a}_r B^c(t,T) = 0$$

with $B^c(T,T) = 0$ and

$$A_t^c(t,T) = \theta_r B^c(t,T)$$

with $A^c(T,T) = 0$. Thus, it remains to show that

$$\begin{aligned} B^c(t,T) &= c \cdot \frac{1 - e^{-\gamma^c(T-t)}}{\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)}}, \\ A^c(t,T) &= \frac{2\theta_r}{\sigma_r^2} \log \left[\frac{\gamma^c e^{\kappa_2 \cdot (T-t)}}{\kappa_1 - \kappa_2 \cdot e^{-\gamma^c \cdot (T-t)}} \right] \end{aligned}$$

with $\gamma^c := \sqrt{\hat{a}_r^2 + 2\sigma_r^2 c}$, $\kappa_1 := \frac{\hat{a}_r}{2} + \frac{\gamma^c}{2}$ and $\kappa_2 := \frac{\hat{a}_r}{2} - \frac{\gamma^c}{2}$ solve the Riccati equations for $c \ge -\frac{\hat{a}_r^2}{2\sigma_r^2}$. Since (for $c \neq 0$)

$$\begin{aligned} \frac{\frac{1}{2} \cdot \sigma_r^2 \cdot (B^c(t,T))^2}{c} - \frac{B_t^c(t,T)}{c} + \frac{\hat{a}_r B^c(t,T)}{c} &= \frac{\frac{1}{2}\sigma_r^2 c \cdot (1 - e^{-\gamma^c(T-t)})^2}{(\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)})^2} \\ -\frac{(\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)})(-\gamma^c e^{-\gamma^c(T-t)}) - (1 - e^{-\gamma^c(T-t)})(-\kappa_2 \gamma^c e^{-\gamma^c(T-t)})}{(\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)})^2} & \\ +\frac{\hat{a}_r (1 - e^{-\gamma^c(T-t)})(\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)})}{(\kappa_1 - \kappa_2 e^{-\gamma^c(T-t)})^2} := \frac{Z}{N} & \end{aligned}$$

it remains to show that $Z = N$.

$$\begin{aligned} Z &= \frac{1}{2}\sigma_r^2 c - \sigma_r^2 c e^{-\gamma^c(T-t)} + \frac{1}{2}\sigma_r^2 c e^{-2\gamma^c(T-t)} + \kappa_1\gamma^c e^{-\gamma^c(T-t)} - \\ &\quad \kappa_2\gamma^c e^{-2\gamma^c(T-t)} - \kappa_2\gamma^c e^{-\gamma^c(T-t)} + \kappa_2\gamma^c e^{-2\gamma^c(T-t)} + \hat{a}_r\kappa_1 - \\ &\quad \hat{a}_r\kappa_1 e^{-\gamma^c(T-t)} - \hat{a}_r\kappa_2 e^{-\gamma^c(T-t)} + \hat{a}_r\kappa_2 e^{-2\gamma^c(T-t)} \\ &= \frac{1}{2}\sigma_r^2 c - \sigma_r^2 c e^{-\gamma^c(T-t)} + \frac{1}{2}\sigma_r^2 c e^{-2\gamma^c(T-t)} + \frac{\hat{a}_r}{2}\gamma^c e^{-\gamma^c(T-t)} + \\ &\quad \frac{1}{2}\gamma^{c2} e^{-\gamma^c(T-t)} - \frac{\hat{a}_r}{2}\gamma^c e^{-\gamma^c(T-t)} + \frac{1}{2}\gamma^{c2} e^{-\gamma^c(T-t)} + \frac{\hat{a}_r^2}{2} + \\ &\quad \frac{\hat{a}_r}{2}\gamma^c - \frac{\hat{a}_r^2}{2} e^{-\gamma^c(T-t)} - \frac{\hat{a}_r}{2}\gamma^c e^{-\gamma^c(T-t)} - \frac{\hat{a}_r^2}{2} e^{-\gamma^c(T-t)} + \\ &\quad \frac{\hat{a}_r}{2}\gamma^c e^{-\gamma^c(T-t)} + \frac{\hat{a}_r^2}{2} e^{-2\gamma^c(T-t)} - \frac{\hat{a}_r}{2}\gamma^c e^{-2\gamma^c(T-t)}. \end{aligned}$$

Simplifying further, we obtain:

$$\begin{aligned} Z &= \frac{\hat{a}_r^2}{2} + \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{2}\sigma_r^2 c + \\ &\quad e^{-\gamma^c(T-t)}(-\sigma_r^2 c + (\hat{a}_r^2 + 2\sigma_r^2 c) - \hat{a}_r^2 - \frac{\hat{a}_r}{2}\gamma^c + \frac{\hat{a}_r}{2}\gamma^c) \\ &\quad e^{-2\gamma^c(T-t)}(\frac{1}{2}\sigma_r^2 c + \frac{\hat{a}_r^2}{2} - \frac{\hat{a}_r}{2}\gamma^c) \\ &= \frac{\hat{a}_r^2}{2} + \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{2}\sigma_r^2 c + \\ &\quad e^{-\gamma^c(T-t)} \cdot \sigma_r^2 c + e^{-2\gamma^c(T-t)}(\frac{1}{2}\sigma_r^2 c + \frac{\hat{a}_r^2}{2} - \frac{\hat{a}_r}{2}\gamma^c). \end{aligned}$$

Now, since

$$\begin{aligned} \kappa_1^2 &= \frac{\hat{a}_r^2}{4} + \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{4}(\hat{a}_r^2 + 2\sigma_r^2 c) = \frac{\hat{a}_r^2}{2} + \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{2}\sigma_r^2 c, \\ -2\kappa_1\kappa_2 &= -2(\frac{\hat{a}_r^2}{4} - \frac{1}{4}\gamma^{c2}) = -\frac{\hat{a}_r^2}{2} + \frac{\hat{a}_r^2}{2} + \sigma_r^2 c = \sigma_r^2 c; \\ \kappa_2^2 &= \frac{\hat{a}_r^2}{4} - \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{4}(\hat{a}_r^2 + 2\sigma_r^2 c) = \frac{\hat{a}_r^2}{2} - \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{2}\sigma_r^2 c, \end{aligned}$$

we get:

$$\begin{aligned} N &= \frac{\hat{a}_r^2}{2} + \frac{1}{2}\hat{a}_r\gamma^c + \frac{1}{2}\sigma_r^2 c \\ &\quad + e^{-\gamma^c(T-t)} \cdot \sigma_r^2 c + e^{-2\gamma^c(T-t)}(\frac{1}{2}\sigma_r^2 c + \frac{\hat{a}_r^2}{2} - \frac{\hat{a}_r}{2}\gamma^c) \end{aligned}$$

and thus $Z = N$.

For $A^c(t,T)$ we get by comparing the original form of $A(t,T)$ in the CIR model

$$\begin{aligned} A^c(t,T) &= -\int_t^T \theta_r B^c(s,T)ds \\ &= c \cdot \frac{2\theta_r}{\sigma_r^2 c} \log\left[\frac{2\gamma^c e^{\frac{1}{2}(\hat{a}_r+\gamma^c)(T-t)}}{(\hat{a}_r+\gamma^c)(e^{\gamma^c(T-t)}-1)+2\gamma^c}\right] \\ &= \frac{2\theta_r}{\sigma_r^2} \log\left[\frac{2\gamma^c e^{\frac{1}{2}(\hat{a}_r+\gamma^c)(T-t)}}{(\hat{a}_r+\gamma^c)(e^{\gamma^c(T-t)}-1)+2\gamma^c}\right], \end{aligned}$$

which completes the proof. □

If $c = 1$ in Lemma 6.1, $P^c(t,T)$ is the price of a zero-coupon bond in the CIR model and we will write $P(t,T)$, γ, $B(t,T)$ and $A(t,T)$ instead of $P^c(t,T)$, γ^c, $B^c(t,T)$ and $A^c(t,T)$ in this case.

As already mentioned earlier, the refinancing incentive is usually modelled as a function of the spread between a security's weighted-average coupon (WAC) and current long-term interest rates which serve as a proxy for mortgage refinancing rates. While in some models (e.g., Levin and Daras (1998) or our approach in Chapter 5) the 10yr par yield is used, we use the 10yr zero yield here since this is a more convenient choice for our closed-form formula. Note that within the CIR framework the 10yr zero yield R_{10} is given by

$$R_{10}(t) = -a_{10} + b_{10} \cdot r(t), \tag{6.7}$$

where $a_{10} := \frac{A(t,t+10)}{10}$ and $b_{10} := \frac{B(t,t+10)}{10}$. Contrarily to Collin-Dufresne and Harding (1999) we do not use a purely linear functional form, but approximate an S-curve shape by defining

$$\pi_{\text{refi}}(t) = \beta \cdot \max(\min(\text{WAC} - R_{10}(t), \alpha), 0), \tag{6.8}$$

for some constant $\alpha > 0$, which results in a spread-refinancing prepayment relationship as shown in Figure 6.1. This functional form offers two major advantages compared to a purely linear functional form:

- The S-like relationship between the spread and the refinancing-driven prepayment, which has been confirmed empirically by, e.g., Levin and Daras (1998), is accounted for.

- Refinancing-driven prepayment can never become negative.

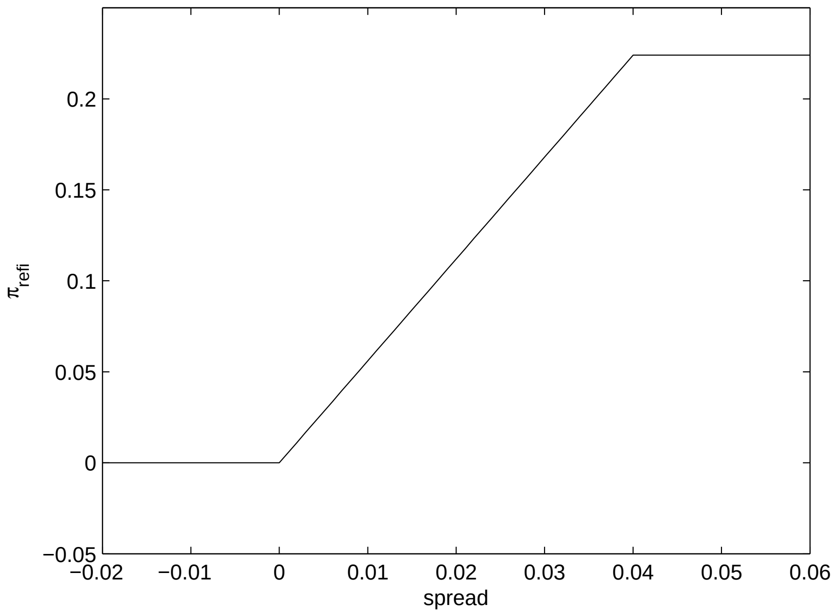

Figure 6.1: Assumed functional form of the relationship between the contract rate spread (i.e. the spread between the WAC and the current 10yr treasury zero rate) and the refinancing-related (annualised) prepayment speed. The parameter β in (6.8) is set to 5.6 as estimated later and α is set to 0.04.

These advantages also hold if we compare our approach to the model developed by Rom-Poulsen (2007). The quadratic interest-rate/refinancing relationship in the Rom-Poulsen model offers more flexibility than a purely linear functional form. Yet, it is not guaranteed that refinancing prepayment is always positive. Moreover, a quadratic relationship may yield non-sensical prepayment patterns in extreme situations (i.e. for very low or very high spread values).

Using (6.7) and noting that for some constants $a, b, c \in \mathbb{R}$, $b > c$, we have

$$\max(\min(a - x, b), c) = a - x + \max(x - (a - c), 0) - \max(a - b - x, 0),$$

formula (6.8) gets:

$$\begin{aligned} \pi_{\text{refi}}(t) &= \beta \cdot \text{WAC} + \beta a_{10} - \beta b_{10} r(t) \\ &\quad + \beta b_{10} \cdot \max\left(r(t) - \frac{\text{WAC} + a_{10}}{b_{10}}, 0\right) \\ &\quad - \beta b_{10} \cdot \max\left(\frac{\text{WAC} + a_{10} - \alpha}{b_{10}} - r(t), 0\right). \end{aligned} \tag{6.9}$$

Now, consider the term $E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right]$. Defining

$$\widetilde{r}(t) := r^{(1-\beta b_{10})}(t) = (1-\beta b_{10}) \cdot r(t)$$

we get by using (6.9):

$$\begin{aligned} E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right] &= E_{\widetilde{Q}}\Big[e^{-\int_0^{t_k}\beta\cdot\text{WAC}+\beta a_{10}+\widetilde{r}(s)ds}\cdot \\ &\quad e^{-\int_0^{t_k}\beta b_{10}\cdot\max\left(r(s)-\frac{\text{WAC}+a_{10}}{b_{10}},0\right)ds}\cdot \\ &\quad e^{\int_0^{t_k}\beta b_{10}\cdot\max\left(\frac{\text{WAC}+a_{10}-\alpha}{b_{10}}-r(s),0\right)ds}\Big]. \end{aligned} \tag{6.10}$$

The following theorem shows how we can calculate this quantity, up to an error term which will be discussed in more detail in the appendix.

Theorem 6.2. *Defining*

$$C(t_k) := e^{-t_k\cdot(\beta\cdot WAC+\beta a_{10})},$$

the expression

$$P^{refi}(0,t_k) := E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right]$$

in the previously introduced model setting can be written in the following way:

$$\begin{aligned} P^{refi}(0,t_k) &= C(t_k)\cdot\widetilde{P}(0,t_k) - C(t_k)\cdot\beta b_{10}\cdot\widetilde{\text{Cap}}(r,0,t_k,r_{Cap},\Delta t) \\ &\quad +C(t_k)\cdot\beta b_{10}\cdot\widetilde{\text{Floor}}(r,0,t_k,r_{Floor},\Delta t) \\ &\quad +C(t_k)\cdot E_{\widetilde{Q}}\left[\widetilde{R}\left(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k\right)\right], \end{aligned} \tag{6.11}$$

where, corresponding to Lemma 6.1,

$$
\begin{aligned}
\widetilde{P}(0,t_k) &= P^{(1-\beta b_{10})}(0,t_k)\\
r_{Cap} &:= \frac{WAC + a_{10}}{b_{10}}\\
r_{Floor} &:= \frac{WAC + a_{10} - \alpha}{b_{10}}\\
\widetilde{\text{Cap}}(r,0,T,r_X,\Delta t) &:= \sum_{k=1}^{T/\Delta t} \Delta t \cdot \Bigg[\frac{q+1+u_k}{c_k} - \frac{u_k}{c_k}\cdot \chi^2(2c_k r_X, 2q+6, 2u_k)\\
&\quad -\frac{q+1}{c_k}\cdot \chi^2(2c_k r_x, 2q+4, 2u_k) - r_X + r_X \cdot \chi^2(2c_k r_X, 2q+2, 2u_k)\Bigg]\\
\widetilde{\text{Floor}}(r,0,T,r_X,\Delta t) &:= \sum_{k=1}^{T/\Delta t} \Delta t \cdot \Bigg[r_X \cdot \chi^2(2c_k r_X, 2q+2, 2u_k)\\
&\quad -\frac{u_k}{c_k}\cdot \chi^2(2c_k r_X, 2q+6, 2u_k) - \frac{q+1}{c_k}\cdot \chi^2(2c_k r_X, 2q+4, 2u_k)\Bigg]\\
c_k &:= \frac{2\hat{a}_r}{\sigma_r^2\cdot(1-e^{-\hat{a}_r\cdot k\cdot\Delta t})}\\
u_k &:= c_k \cdot r(0)\cdot e^{-\hat{a}_r\cdot k\cdot \Delta t}\\
q &:= \frac{2\theta_r}{\sigma_r^2} - 1,
\end{aligned}
$$

and $\chi^2(\cdot;a,b)$ denotes the cdf of the non-central Chi-square distribution with degrees of freedom parameter a and non-centrality parameter b.
$\widetilde{R}\left(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k\right)$ is a term containing residual terms of the order $O(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k)$. v_k, w_k, z_k are defined as:

$$
\begin{aligned}
v_k &:= -\int_0^{t_k} \beta b_{10}\cdot max(r(s) - r_{Cap}, 0)\ ds\\
w_k &:= \int_0^{t_k} \beta b_{10}\cdot max(r_{Floor} - r(s), 0)\ ds\\
z_k &:= \int_0^{t_k} \widetilde{r}(s)\ ds
\end{aligned}
$$

Proof. After factoring out $C(t_k)$ in (6.10) we apply the approximation

$$e^{z_k+v_k+w_k} = e^{z_k} + v_k + w_k + \bar{R}, \tag{6.12}$$

where the term $\bar{R}$ contains residual terms of the order $O(v_k^2, w_k^2, z_k \cdot v_k, z_k \cdot w_k, v_k \cdot w_k)$. If we then approximate the integrals by sums, we obtain:

$$\begin{aligned}
E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right] &= C(t_k)\cdot\widetilde{P}(0,t_k) \\
-\beta b_{10}\cdot C(t_k)\cdot\Delta t\cdot&\sum_{k=1}^{\lceil\frac{t_k}{\Delta t}\rceil} E_{\widetilde{Q}}\left[\max\left(r(k\cdot\Delta t)-\frac{\text{WAC}+a_{10}}{b_{10}},0\right)\right] \\
+\beta b_{10}\cdot C(t_k)\cdot\Delta t\cdot&\sum_{k=1}^{\lceil\frac{t_k}{\Delta t}\rceil} E_{\widetilde{Q}}\left[\max\left(\frac{\text{WAC}+a_{10}-\alpha}{b_{10}}-r(k\cdot\Delta t),0\right)\right] \\
+C(t_k)\cdot E_{\widetilde{Q}}&\left[\widetilde{R}(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k)\right] \\
&= C(t_k)\cdot\widetilde{P}(0,t_k) \\
-\beta b_{10}\cdot C(t_k)\cdot\Delta t\cdot&\sum_{k=1}^{\lceil\frac{t_k}{\Delta t}\rceil}\int_{r_{Cap}}^{\infty}(r(k\cdot\Delta t)-r_{Cap})\,f(r(k\cdot\Delta t))dr(k\cdot\Delta t) \\
+\beta b_{10}\cdot C(t_k)\cdot\Delta t\cdot&\sum_{k=1}^{\lceil\frac{t_k}{\Delta t}\rceil}\int_{0}^{r_{Floor}}(r_{Floor}-r(k\cdot\Delta t))\,f(r(k\cdot\Delta t))dr(k\cdot\Delta t) \\
+C(t_k)\cdot E_{\widetilde{Q}}&\left[\widetilde{R}(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k)\right] \qquad (6.13)
\end{aligned}$$

where $f(\cdot)$ denotes the pdf of the short rate and where the residual term $\widetilde{R}$ also contains the error terms of order $O(\Delta t)$ resulting from the approximation of integrals by sums (see Appendix for a general discussion). Since we work with a CIR model here, we know from Cox et al. (1985) that the distribution of $2\cdot c_k\cdot r(k\cdot\Delta t)$ is the non-central χ^2-distribution with parameters $2q+2$ and $2u_k$, with c_k, u_k and q as previously defined (see also Chapter 3.2.3). From the recurrence relation (see Johnson et al. (1995), p. 442)

$$\begin{aligned}
\lambda\cdot\chi^2(x;\mu+4,\lambda) &= (\lambda-\mu)\cdot\chi^2(x;\mu+2,\lambda) \\
&\quad +(x+\mu)\cdot\chi^2(x;\mu,\lambda)-x\cdot\chi^2(x;\mu-2,\lambda)
\end{aligned} \qquad (6.14)$$

(for $\mu>2$) and from the relation (see Johnson et al. (1995), p. 443)

$$\frac{\partial\chi^2(x;\mu,\lambda)}{\partial x}=f(x;\mu,\lambda)=\frac{1}{2}(\chi^2(x;\mu-2,\lambda)-\chi^2(x;\mu,\lambda)) \qquad (6.15)$$

it follows with some easy calculations that

$$\int_0^b xf(x;\mu,\lambda)dx=\mu\cdot\chi^2(b;\mu+2,\lambda)+\lambda\cdot\chi^2(b;\mu+4,\lambda). \qquad (6.16)$$

Applying (6.16) to the first integral in (6.13), we obtain:

$$\int_{r_{Cap}}^{\infty} (r(k\cdot\Delta t) - r_{Cap})\, f(r(k\cdot\Delta t))dr(k\cdot\Delta t) \;=\; \frac{1}{2c_k}\Big[E_{\widetilde{Q}}[2c_k r(k\cdot\Delta t)] -$$

$$-(2q+2)\cdot\chi^2(2c_k r_{Cap}; 2q+4, 2u_k) - 2u_k\cdot\chi^2(2c_k r_{Cap}; 2q+6, 2u_k)\Big]$$

$$-r_{Cap}\cdot(1-\chi^2(2c_k r_{Cap}; 2q+2, 2u_k)).$$

Similarly,

$$\int_{0}^{r_{Floor}} (r_{Floor} - r(k\cdot\Delta t))\, f(r(k\cdot\Delta t))dr(k\cdot\Delta t) \;=$$

$$r_{Floor}\cdot\chi^2(2c_k r_{Floor}; 2q+2, 2u_k) - \frac{1}{2c_k}\Big[(2q+2)$$

$$\cdot\chi^2(2c_k r_{Floor}; 2q+4, 2u_k) + 2u_k\cdot\chi^2(2c_k r_{Floor}; 2q+6, 2u_k)\Big].$$

Noting that

$$E_{\widetilde{Q}}[2c_k r(k\cdot\Delta t)] = 2q + 2 + 2u_k,$$

formula (6.11) follows directly after rearranging of terms.

□

Note that the error term in Theorem 6.2 can be expected to be reasonably small, in particular for small values of t_k. The error term will be discussed in more detail in the appendix. Note also that the notation 'Cap' has not been chosen without motive. If one equates the linear interest rate at time t for the period from t to $t+\Delta t$ with the short rate $r(t)$, the expression

$$\max\left(r(k\cdot\Delta t) - \frac{(\mathrm{WAC}+a_{10})}{b_{10}}, 0\right)\cdot\Delta t$$

in (6.10) is simply the payoff of a standard caplet from $k\cdot\Delta t$ to $(k+1)\cdot\Delta t$ with cap rate $r_{Cap} := (\mathrm{WAC}+a_{10})/b_{10}$. A similar consideration applies for the notation 'Floor'. We typically have $\Delta t_k = 1/12$ (i.e. 1 month) for all $k = 1, ..., K$. Hence, $\Delta t = 1/12$ is a natural choice for the interval length of the discretisation in (6.11).

The baseline prepayment

We model the baseline or turnover component of prepayment within a two-factor Gaussian process framework where both factors follow Vasicek processes, similar to the specification of the baseline prepayment process in

Chapter 5. The second factor is fit to the GDP growth in the US, accounting for the dependence between general economic conditions and turnover prepayment. Of course, any other observable factor, e.g. a suitable house price index, could be used instead of or in addition to the GDP growth factor. While our empirical results have turned out to be satisfactory with the GDP growth as second factor in the baseline prepayment model, house prices have been used for example by Kariya et al. (2002), Sharp et al. (2006) or Downing et al. (2005). Our baseline prepayment processes are thus again given by their $\widetilde{Q}$-dynamics

$$\begin{aligned} d\pi_0(t) &= (\theta_p + b_{pw}w(t) - \hat{a}_p\pi_0(t))dt + \sigma_p d\widetilde{W}_p(t), \qquad (6.17) \\ dw(t) &= (\theta_w - \hat{a}_w w(t))dt + \sigma_w d\widetilde{W}_w(t), \end{aligned}$$

where $\widetilde{W}_p, \widetilde{W}_w$ are independent $\widetilde{Q}$-Wiener processes (independent of the previously defined $\widetilde{W}_r$) and $\hat{a}_i := a_i + \lambda_i\sigma_i^2$, $i = p, w$, for the two prepayment-risk-adjustment parameters λ_p, λ_w.

In order to be able to calculate (6.1) we have to evaluate the expression

$$\begin{aligned} \widetilde{P}^d(t,T) &:= E_{\widetilde{Q}}[e^{-\int_t^T(\widetilde{r}(s)+\pi_0(s))ds}|\mathcal{F}_t] \\ &= E_{\widetilde{Q}}[e^{-\int_t^T\widetilde{r}(s)ds}|\mathcal{F}_t] \cdot E_{\widetilde{Q}}[e^{-\int_t^T\pi_0(s)ds}|\mathcal{F}_t] \\ &=: \widetilde{P}(t,T) \cdot P^{base}(t,T), \end{aligned}$$

where $\widetilde{r}(t) := r^{(1-\beta b_{10})}(t)$, $\widetilde{P}(t,T) := P^{(1-\beta b_{10})}(t,T)$. The letter d in the superscript of $\widetilde{P}^d(t,T)$ is used in analogy to the reduced-form credit risk literature.

Theorem 6.3. *In the model set-up as previously introduced it holds that*

$$P^{base}(t,T) = e^{A^d(t,T)-C^d(t,T)\pi_0(t)-D^d(t,T)w(t)}$$

with

$$\begin{aligned} C^d(t,T) &= \frac{1}{\hat{a}_p}\left(1 - e^{-\hat{a}_p(T-t)}\right), \\ D^d(t,T) &= \frac{b_{pw}}{\hat{a}_p}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w} + \frac{e^{-\hat{a}_w(T-t)} - e^{-\hat{a}_p(T-t)}}{\hat{a}_w - \hat{a}_p}\right), \\ A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_p^2 C^d(l,T)^2 + \sigma_w^2 D^d(l,T)^2\right) \\ &\quad -\theta_p C^d(l,T) - \theta_w D^d(l,T)dl. \end{aligned}$$

Proof. From the Feynman-Kac representation of the Cauchy-Problem (see (3.5)) we know that $P^{base}(t,T)$ must satisfy:

$$\begin{aligned} P_t^{base} + (\theta_w - \hat{a}_w w)P_w^{base} + (\theta_p + b_{pw}\cdot w - \hat{a}_p\pi_0)P_{\pi_0}^{base} \\ +\frac{1}{2}\cdot(\sigma_p^2\cdot P_{\pi_0\pi_0}^{base} + \sigma_w^2\cdot P_{ww}^{base}) = \pi_0\cdot P^{base} \end{aligned}$$

Calculating the derivatives of P^{base} it follows that

$$\begin{aligned} A_t^d(t,T) - \pi_0(1-\hat{a}_pC^d(t,T) + C_t^d(t,T)) \\ -w(D_t^d(t,T) - \hat{a}_wD^d(t,T) + b_{pw}C^d(t,T)) \\ +\tfrac{1}{2}\cdot(\sigma_p^2C^d(t,T)^2 + \sigma_w^2D^d(t,T)^2) - \theta_pC^d(t,T) - \theta_wD^d(t,T) = 0. \end{aligned}$$

Thus, we obtain the system of linear differential equations

$$\begin{aligned} 1-\hat{a}_pC^d(t,T) + C_t^d(t,T) &= 0 \\ b_{pw}C^d(t,T) - \hat{a}_wD^d(t,T) + D_t^d(t,T) &= 0 \\ A_t^d(t,T) + \frac{1}{2}\cdot(\sigma_p^2C^d(s,T)^2 + \sigma_w^2D^d(s,T)^2) \\ -\theta_pC^d(s,T) - \theta_wD^d(s,T) &= 0 \end{aligned}$$

with $A^d(T,T)=0$, $C^d(T,T)=D^d(T,T)=0$. With some easy calculations it is straightforward to verify that the formulas as stated in Theorem 6.3 are the solutions of the linear differential equations above. □

Note, that we have associated the prepayment speed $\pi(t)$ with the intensity of prepayment. In our model specification in this chapter, however, $\pi(t)$ can have negative values, albeit, in general, with only small probabilities. Prepayments for ordinary fixed-rate MBS can, of course, never be negative. Furthermore the association of the process $\pi(t)$ with the prepayment intensity (and likewise the association of the baseline prepayment process $\pi_0(t)$ with the corresponding default intensity process in the credit risk literature) is not unproblematic from a technical point of view for the same reason: intensities can never be negative. We thus consider the processes $\pi(t)$ and $\pi_0(t)$ as proxies for the respective intensity processes. The fact that both $\pi(t)$ and $\pi_0(t)$ are negative only with small probabilities justifies this approach. With the parameter values as estimated in the following section (see Table 6.1), the probability that after one year the baseline prepayment is negative is just 2%, if we set the GDP growth constant and equal to its mean-reversion level and the initial baseline prepayment level to its mean-reversion level of approx. 17% (see also Schönbucher (2003), p. 167, for a further discussion of this topic in the context of credit risk modelling).

The closed-form approximation

With the ingredients developed in the previous subsections the expressions

- $E_{\widetilde{Q}}\left[M(t_k)\cdot e^{-\int_0^{t_1} r(s)ds}\right] = M(t_k)\cdot P(0,t_1)$
- $E_{\widetilde{Q}}\left[M(t_k)\cdot e^{-\int_0^{t_k} (r(s)+\pi(s))ds}\right] = M(t_k)\cdot P^{refi}(0,t_k)\cdot P^{base}(0,t_k)$

in (6.1) can readily be evaluated for all k, if the error term $\widetilde{R}$ is neglected. This is not yet the case for the terms involving $\pi(t_k)$ as a factor.

Lemma 6.4. *It holds that in the previously introduced model set-up*

$$\begin{aligned} E_{\widetilde{Q}}\left[\pi_0(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right] &= C(t_k)\cdot \widetilde{P}^d(0,t_k)\cdot \tilde{f}^d(0,t_k) \\ &-C(t_k)\cdot\beta b_{10}\cdot P^{base}(0,t_k)\cdot \tilde{f}^d(0,t_k)\cdot \widetilde{\text{Cap}}(r,0,t_k,r_{Cap},\Delta t_k) \\ &+C(t_k)\cdot\beta b_{10}\cdot P^{base}(0,t_k)\cdot \tilde{f}^d(0,t_k)\cdot \widetilde{\text{Floor}}(r,0,t_k,r_{Floor},\Delta t_k) \\ &+C(t_k)\cdot\beta b_{10}\cdot P^{base}(0,t_k)\cdot \tilde{f}^d(0,t_k) \\ &\cdot E_{\widetilde{Q}}\left[\widetilde{R}(\Delta t, v_k^2, w_k^2, z_k\cdot v_k, z_k\cdot w_k, v_k\cdot w_k)\right] \end{aligned}$$

where $\tilde{f}^d(0,t_k)$ is the "baseline spread forward rate", i.e.

$$\tilde{f}^d(0,t_k) = -\frac{\partial}{\partial t_k}\ln P^{base}(0,t_k)$$

and all other quantities are as previously defined.

Proof. As a first step, recall the well-known result (see, e.g., Schmid (2004), p. 243) saying that

$$E_{\widetilde{Q}}\left[e^{-\int_0^T r(l)dl} r(T)|\mathcal{F}_0\right] = -E_{\widetilde{Q}}\left[e^{-\int_0^T r(l)dl}|\mathcal{F}_0\right]\cdot\frac{\partial}{\partial T}\ln P(0,T). \qquad (6.18)$$

Now, if we use the independence between $(r(t),\pi_{\text{refi}}(t))$ and $\pi_0(t)$, apply (6.18) to

$$\begin{aligned} E_{\widetilde{Q}}\left[\pi_0(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right] &= E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right]\cdot \\ & \quad E_{\widetilde{Q}}\left[\pi_0(t_k)\cdot e^{-\int_0^{t_k}\pi_0(s)ds}\right], \end{aligned}$$

the lemma follows directly if we recall (6.11). □

This leaves us with the term $E_{\widetilde{Q}}\left[\pi_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right]$.

Lemma 6.5. *Within the previously introduced model set-up it holds that:*

$$E_{\widetilde{Q}}\left[\pi_{refi}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right] = -P^{base}(0,t_k)\cdot P^{refi}(0,t_k) \cdot\frac{\partial}{\partial t_k}\ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]. \tag{6.19}$$

Proof. If we define the t_k-forward measure Q^{t_k} in the usual way via its Radon-Nikodym derivative $L(T)$ with respect to $\widetilde{Q}$ by

$$L(t) = \frac{dQ^{t_k}}{d\widetilde{Q}}\bigg|\mathcal{F}_t = \frac{P(t,t_k)}{P(0,t_k)\cdot e^{\int_0^{t_k} r(s)ds}}$$

for $t \in [0, t_k]$ and use (6.18) we obtain:

$$\begin{aligned}
E_{\widetilde{Q}}\left[\pi_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi(s))ds}\right] &= E_{\widetilde{Q}}\left[\pi_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+\pi_{\text{refi}}(s))ds}\right]\cdot \\
&\quad P^{base}(0,t_k)\\
&= P(0,t_k)\cdot E_{Q^{t_k}}\left[\pi_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}\pi_{\text{refi}}(s)ds}\right]\cdot P^{base}(0,t_k)\\
&= -P(0,t_k)\cdot E_{Q^{t_k}}\left[e^{-\int_0^{t_k}\pi_{\text{refi}}(s)ds}\right]\cdot\\
&\quad \frac{\partial}{\partial t_k}\ln E_{Q^{t_k}}\left[e^{-\int_0^{t_k}\pi_{\text{refi}}(s)ds}\right]\cdot P^{base}(0,t_k)\\
&= -P^{refi}(0,t_k)\cdot\frac{\partial}{\partial t_k}\ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]\cdot P^{base}(0,t_k)
\end{aligned}$$

□

Note that by using Theorem 6.3 and Theorem 6.2, it is straightforward to evaluate the terms in (6.19), if the error term $\widetilde{R}$ is neglected.

We can finally summarise our results in the following theorem, where we use $\widetilde{V}_{app}$ as the value of a mortgage-backed security, if the error term $\widetilde{R}$ as previously defined is neglected.

Theorem 6.6. *For a fixed-rate mortgage-backed security with K outstanding payment dates at time 0 within the model specification as previously introduced it holds that:*

$$\widetilde{V}_{app}(0) = S_1 + S_2 + S_3 - \Delta_1 + \Delta_2 \tag{6.20}$$

with

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{K} M(t_k) \cdot \Delta t_k \cdot C(t_k) \cdot \widetilde{P}^d(0,t_k) \\
S_2 &= \sum_{k=1}^{K} C(t_k) \cdot \widetilde{P}^d(0,t_k) \cdot A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) \\
S_3 &= -\sum_{k=1}^{K} C(t_k) \cdot \widetilde{P}^d(0,t_k) \cdot A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln \left[\frac{P^{refi}(0,t_k)}{P(0,t_k)} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_1 &= \sum_{k=1}^{K} \widetilde{\mathrm{Cap}}(r,0,t_k,r_{Cap},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \\
&\quad \beta b_{10} \cdot \Bigg[M(t_k) \cdot \Delta t_k + A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) - A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln \left[\frac{P^{refi}(0,t_k)}{P(0,t_k)} \right] \Bigg] \\
\Delta_2 &= \sum_{k=1}^{K} \widetilde{\mathrm{Floor}}(r,0,t_k,r_{Floor},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \\
&\quad \beta b_{10} \cdot \Bigg[M(t_k) \cdot \Delta t_k + A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) - A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln \left[\frac{P^{refi}(0,t_k)}{P(0,t_k)} \right] \Bigg].
\end{aligned}
$$

Formula (6.20) can readily be evaluated once the model parameters have been estimated and calibrated. From Theorem 6.6 it is also easy to see how the most common mortgage derivatives, i.e. Interest-Only (IO) and Principal-Only (PO) securities, can be priced within our modelling framework. If we split up the mortgage payment $M(t_k)$ into the interest payment $M^I(t_k)$ and regular principal repayment $M^P(t_k)$, so that $M(t_k)$=$M^I(t_k)$ + $M^P(t_k)$, and denote

$$\begin{aligned}
S_1^I &:= M^I(t_k)\cdot\Delta t_1\cdot P(0,t_1)+\sum_{k=2}^{K} M^I(t_k)\cdot\Delta t_k\cdot C(t_k)\cdot\widetilde{P}^d(0,t_k)\\
\Delta_1^I &:= \sum_{k=1}^{K}\widetilde{\text{Cap}}(r,0,t_k,r_{Cap},\Delta t_k)\cdot C(t_k)\cdot P^{base}(0,t_k)\cdot\beta b_{10}\cdot M^I\cdot\Delta t_k\\
\Delta_2^I &:= \sum_{k=1}^{K}\widetilde{\text{Floor}}(r,0,t_k,r_{Floor},\Delta t_k)\cdot C(t_k)\cdot P^{base}(0,t_k)\cdot\beta b_{10}\cdot M^I\cdot\Delta t_k\\
S_1^P &:= S_1-S_1^I\\
\Delta_1^P &:= \Delta_1-\Delta_1^I\\
\Delta_2^P &:= \Delta_2-\Delta_2^I
\end{aligned}$$

we obtain the following two corollaries, where the notation $\widetilde{V}_{app}^{IO}(0)$ and $\widetilde{V}_{app}^{PO}(0)$ is again used for the calculation of values neglecting the error term $\widetilde{R}$. The two corollaries conclude this section.

Corollary 6.7. *The value $\widetilde{V}_{app}^{IO}(0)$ of an Interest-Only security with K outstanding payment dates at time 0 is given by:*

$$\widetilde{V}_{app}^{IO}(0)=S_1^I-\Delta_1^I+\Delta_2^I$$

Corollary 6.8. *The value $\widetilde{V}_{app}^{PO}(0)$ of a Principal-Only security with K outstanding payment dates at time 0 is given by:*

$$\widetilde{V}_{app}^{PO}(0)=S_1^P+S_2+S_3-\Delta_1^P+\Delta_2^P$$

6.2 Application to market data

6.2.1 Parameter estimation and model calibration

For the empirical evaluation of our closed-form approximation approach we use again the monthly historical prepayment data of the 30yr fixed-rate mortgage-backed securities of the GNMA I and GNMA II program which we have already used for parameter estimation in our model presented in Chapter 5. In addition to this we now use monthly historical prices of generic GNMA 30yr pass-through MBS with different coupons as traded on a to-be-announced (TBA) basis from 1996 to 2006 in order to assess the performance of our model. All data were obtained from Bloomberg.

Weekly US treasury strip zero rates, obtained from the par rates by standard bootstrapping, from 1993 to 2005 are used for the estimation of the parameters of the CIR interest-rate model. We estimate the CIR interest-model parameters with a state-space approach which integrates time-series information of different maturities, similar to the approach described in Geyer and Pichler (1999). Estimation of the unobservable state variables (i.e. of the short rate) is done with an approximative Kalman filter where the transition densities are supposed to be normal. For the maximisation of the log-likelihood we use again the combined Downhill Simplex/Simulated Annealing algorithm as described in Press et al. (1992). The parameters $\theta_p, a_p, \sigma_p, b_{pw}$ can again be estimated by Kalman filtering techniques, similar in spirit to the approach in Chapter 5. The measurement equation of the prepayment state space model is given by (6.2) with the historically observed annualised (continuous) prepayment rates $\pi(t_k)$ and π_{refi} as specified in (6.8). We obtain:

$$\begin{pmatrix} \pi_1(t_k) \\ \vdots \\ \pi_N(t_k) \end{pmatrix} = \begin{pmatrix} \pi_{1,\text{refi}}(t_k) \\ \vdots \\ \pi_{N,\text{refi}}(t_k) \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \pi_0(t_k) + \boldsymbol{\epsilon}_k, \tag{6.21}$$

where we assume that $\boldsymbol{\epsilon}_k \sim N_N\left(\mathbf{0}, h_p^2 \cdot \mathbf{I}_N\right)$. Since in 2002-2004 prepayment speeds were very high, we only use the data until 2004 for parameter estimation in the prepayment model in order to avoid noise in our observations caused by small pool sizes after 2004. The transition equation for the (unobservable) baseline prepayment process is again given by

$$\pi_0(t_{k+1}) = e^{-a_p \Delta t_{k+1}} \cdot \pi_0(t_k) + \frac{\theta_p + b_{pw} w(t_k)}{a_p} \cdot (1 - e^{-a_p \Delta t_{k+1}}) + \eta_{k+1}$$

with

$$\eta_{k+1} \sim N_1\left(0, \frac{\sigma_p^2}{2a_p}(1 - e^{-2a_p \Delta t_{k+1}})\right).$$

The estimates of the interest-rate model parameters and of the (real-world) prepayment model parameters for our closed-form approximation model are given in Table 6.1. The standard errors are again estimates obtained from a moving block bootstrap procedure.

In order to illustrate the parameter estimates in our closed-form approximation prepayment model, we plot again the historical prepayment rates (as SMM) in Figure 6.2, similar to the presentation in Figure 5.1 for our full Monte-Carlo valuation model. We also show the estimated single monthly mortalities according to our closed-form approximation model when the baseline prepayment process is set equal to its mean-reversion level. Note that

	Parameter	Estimate	(Std. error)
Short-rate process	θ_r	0.014	(0.0056)
	a_r	0.41	(0.12)
	σ_r	0.059	(0.0073)
	$\hat{a}_r$	0.20	(0.10)
	h_r	0.0044	$(4.8 \cdot 10^{-4})$
GDP growth process	θ_w	0.019	(0.0099)
	a_w	1.43	(0.79)
	σ_w	0.002	$(4.3 \cdot 10^{-4})$
Baseline prepayment process	θ_p	0.43	(0.20)
	a_p	0.75	(0.56)
	σ_p	0.12	(0.057)
	b_{pw}	-22.6	(5.03)
	h_p	0.085	(0.012)
Regression parameter	β	5.6	(0.90)

Table 6.1: Estimates of the interest-rate model and real-world prepayment model parameters in our closed-form approximation approach where h_r and h_p are the measurement std. errors of the respective state space models.

the burnout is not an explanatory variable in this model and the expected prepayment rates are thus independent of the burnout. Moreover, note that the spread is defined as the difference between the WAC of the mortgage pool and the 10yr CMT zero-rate here, while the 10yr CMT par rate was used in Figure 5.1. For the illustration in Figure 6.2 the continuously compounded annual prepayment speed $\pi(t)$ as given by (6.2) in our closed-form approximation model had to be converted to a single monthly mortality. This can easily be done by calculating the (discrete) constant prepayment rate

$$\mathrm{CPR}(t) = e^{\pi(t)} - 1,$$

from which the single monthly mortality is obtained by the relation (2.1).

In the next step, we turn our attention to the prepayment-risk adjustment parameters $\mu, \lambda_p, \lambda_w$. By simply setting $\mu = 1$ and $\lambda_p = \lambda_w = 0$ we can conduct a classical OAS analysis, similar to Chapter 5, since in this case the prepayment speed enters with its real-world dynamics into the overall model and the OAS is needed to equate model prices to actually observed market prices. Yet, we are again primarily interested in a prepayment-risk-neutral valuation following the argumentation in Levin and Davidson (2005) and our own discussion in Chapter 5. Using price data of different coupon levels of

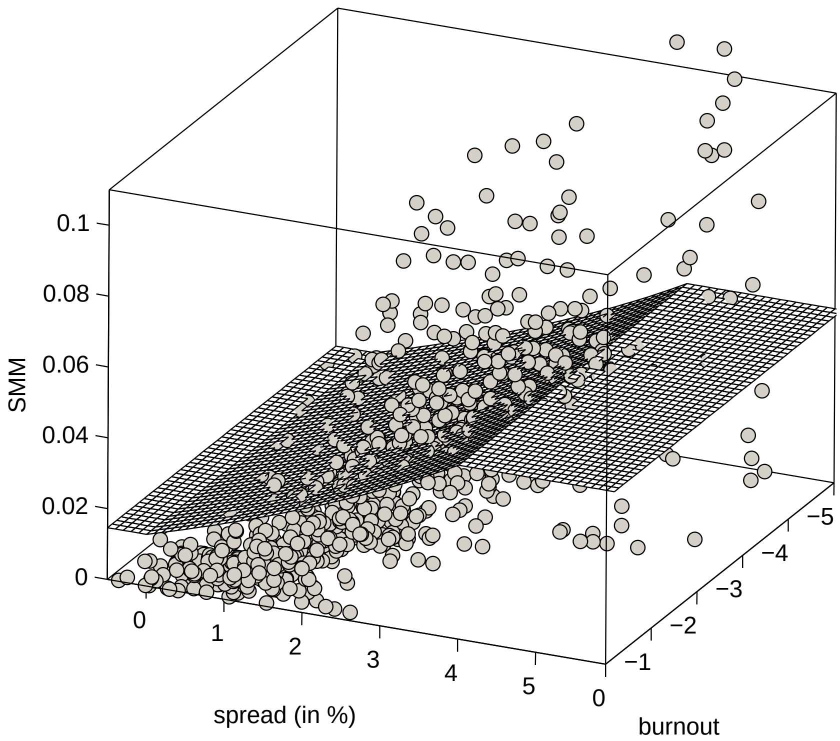

Figure 6.2: Historically observed SMM values and estimated prepayment speed (SMM) in our closed-form approximation model as a function of the covariates *spread* and *burnout* when the baseline hazard process is set equal to its mean-reversion level.

GNMA TBA pass-through securities we calibrate the prepayment-risk adjustment parameters in such a way that the Euclidean norm of the vector of differences between the market prices and model prices of the securities on a particular sample day is minimised. In this study we consider monthly price data of five generic GNMA TBA pass-throughs with coupons between 6% and 8% from 1996 to 2006. We recalibrate the risk-adjustment parameters once a year in October in order to account for changing perceptions of prepayment risk over time.

At this point we would like to recall briefly how the prepayment-risk adjustment parameters are able to account for the two distinct types of prepayment risk. The fact that there are two distinct types of prepayment risk, refinancing understatement and turnover overstatement, was already mentioned in Levin and Davidson (2005) and discussed in detail in Chapter 5.

On the one hand, an investor in discounts experiences losses if the turnover component is overestimated and pure turnover-related prepayment is slower than expected. In this case the average life of the security is extended, decreasing the cash flow stream's present value. On the other hand, the refinancing component is the major concern of an investor in premiums since the average life of premiums decreases if refinancing-related prepayment is faster than originally estimated. This would evidently result in a loss for the holder of a premium MBS. For $\mu > 1$ both, refinancing and baseline prepayment, is accelerated under the risk-neutral pricing measure, compared to the real-world measure. The parameters λ_p and λ_w, however, only affect the baseline prepayment. The higher λ_p the slower the expected prepayment rates under the risk-neutral pricing measure. For the estimates calibrated to the data of Oct-1996 we obtain $\mu = 1.28$, $\lambda_p = 10.0$ and $\lambda_w = -165$. For these estimates Figure 6.3 shows the expected prepayment rates under the risk-neutral pricing measure as a function of the spread variable. Compared to the real-world measure, higher prepayment rates are expected under the risk-neutral pricing measure in the premium area (i.e. for high spread values), while slower prepayment rates are expected in the discount area (i.e. for low values of the spread variable). In some sense, the expected prepayment speeds under the risk-neutral pricing measure could be considered as 'implied expected prepayment rates', implied by MBS market prices. Figure 6.4 shows how these 'implied expected prepayment rates' evolve over time when we re-estimate the prepayment-risk adjustment parameters once a year. The parameter μ varies around its mean 1.24 (with a standard deviation of 0.21), the parameter λ_p around 16.9 (std. dev. 18.0) and the parameter λ_w around -84 (std. dev. 68).

6.2.2 Model performance, prices & sensitivities

The main contribution of the modelling approach presented in this chapter is, as previously mentioned, to provide a closed-form (and thus computationally very efficient) approximation of the value of fixed-rate mortgage-backed securities. This is particularly useful for risk and portfolio management purposes where other valuation methods may not be feasible due to their computational burden. Yet, a closed-form approximation of the securities' values is only useful if the model is able to track major price movements of actually traded securities. In order to assess our model's performance and adequacy empirically, we use the price data of the GNMA TBA pass-throughs from 1996 to 2006. In order to simplify the analysis we assume that each MBS was issued 6 months before the valuation month. Since in our model we do not account for loan age effects anyway, this is not a major restriction. In

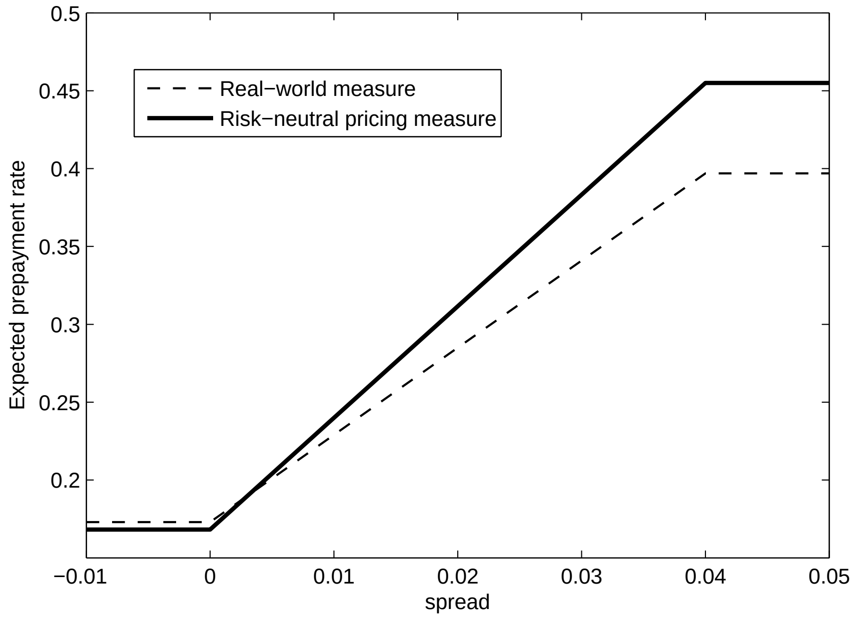

Figure 6.3: Expected prepayment rates under the real-world measure and under the risk-neutral pricing measure as a function of the spread variable with the prepayment-risk adjustment parameters calibrated to GNMA market prices of Oct-1996.

a first step, we price the securities with the risk-adjustment parameters recalibrated once a year, as described in the previous section. The results are shown in Figures 6.5 and 6.6 (together with the market prices for comparison purposes and the model prices of our full Monte-Carlo simulation model as presented in Chapter 5, obtained by Monte-Carlo simulation). The average absolute pricing error for each coupon is shown in Table 6.2. The overall average absolute pricing error over the entire sample is 159 basis points, i.e. just above 1.5%. In general our approach seems to work slightly better for premium securities than for discounts. The average absolute pricing error for all discount observations (i.e. observations with market prices below 100%) is 169 basis points in our sample, compared to 153 basis points for all premium observations (observations with market prices above 100%). If we only consider those observations with market prices between 98% and 102% (i.e. observations of MBS around the respective current coupon), we get an average absolute pricing error of 153 basis points.

In the same empirical setting, we want to compare the results of our

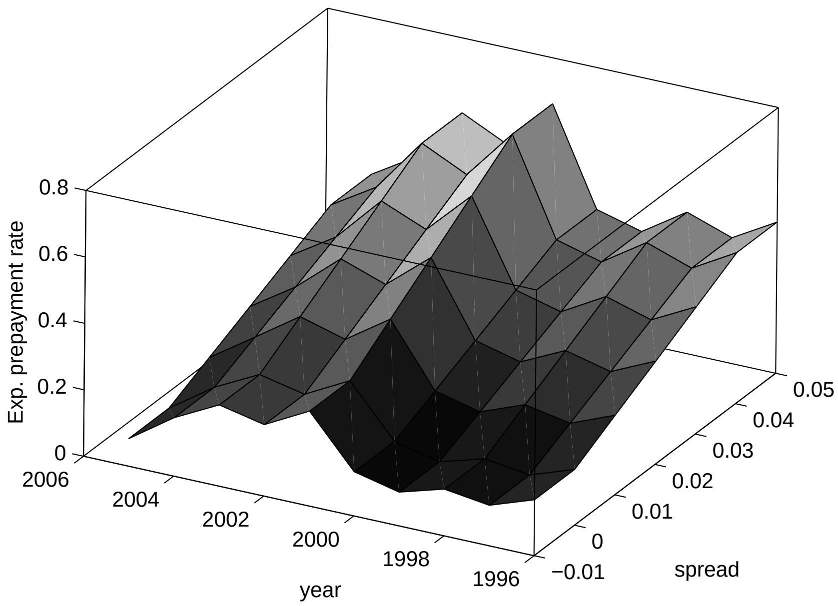

Figure 6.4: Expected prepayment speeds under the risk-neutral pricing measure from 1996 to 2006 as a function of the spread variable when the prepayment-risk adjustment parameters are recalibrated once a year.

modelling approach with some selected alternatives. First of all, we test the usefulness of the stochastic baseline prepayment factors compared to a constant prepayment parameter assumption as in the model developed by Gorovoy and Linetsky (2007). For this purpose we reformulate our model so that instead of (6.2) the overall prepayment intensity is given by

$$\pi(t) = c_0 + \pi_{\text{refi}}(t),$$

where c_0 is a constant and $\pi_{\text{refi}}(t)$ is as given in (6.8). Similar to the approach in Gorovoy and Linetsky (2007) we do not estimate c_0 historically but leave it free for calibration. The empirical results are shown in the third column of Table 6.2. The overall average absolute pricing error of 166 basis points (with a yearly recalibration) indicates that the two stochastic factors only add little to the pricing accuracy. However, a stochastic baseline prepayment specification has another advantage. In our model it is theoretically well justified to consider the baseline prepayment process in real-world, e.g. for real-world prepayment scenario generation for example in an asset-liability management study, and prepayment-risk-adjusted for MBS pricing purposes.

It is hard to justify theoretically that a constant baseline prepayment has a different value for real-world and pricing considerations. It is, however, highly likely that a historically estimated baseline prepayment constant would have a different value than the one calibrated to MBS market data.

In a second experiment we want to discuss the value of our piecewise linear approximation of the refinancing S-curve compared to a purely linear functional form like in the model developed by Collin-Dufresne and Harding (1999). In the Collin-Dufresne/Harding model, the refinancing (annualised) prepayment speed for a fixed-rate MBS with maturity T is given by

$$\pi_{\text{refi}}(t) = a_0 + a_1 \cdot \ln \frac{P(0,T)}{P(t,T)}$$

for some constants a_0, a_1. Thus, the spread explanatory variable is defined in a slightly different way compared to our model. This difference, as well as the fact that Collin-Dufresne and Harding (1999) use a Vasicek process for the short-rate, can be considered as minor differences between the models. Apart from the restriction to one stochastic factor, the major restriction in the Collin-Dufresne/Harding model is the purely linear form for the approximation of the refinancing S-curve. Within our model framework, we want to test empirically whether the piecewise linear approximation presented here does add explanatory power to the pricing model. For this purpose we re-estimate our model with a purely linear functional form. I.e., instead of (6.8) we set:

$$\pi_{\text{refi}}(t) = \beta \cdot (\text{WAC} - R_{10}(t)).$$

Note that there is no need for an intercept here since we still incorporate the baseline prepayment process $\pi_0(t)$. This, of course, makes the formula much easier since we do not have to deal with the rather complex formulas of Theorem 6.2. We also re-calibrate the risk-adjustment parameters once a year and price the five different coupon securities with this model from 1996 to 2006. The results as shown in the fourth column of Table 6.2 indicate that the piecewise linear approximation yields indeed better results than the purely linear functional form, almost across the whole coupon range. The overall average absolute pricing error is 266 basis points in the model with a purely linear functional form, compared to 159 basis points in our full model.

In a third step we compare our closed-form model with our full Monte-Carlo valuation model. Pricing with the model as presented in Chapter 5 requires a full Monte-Carlo procedure such that parameter (re-)calibration becomes computationally very expensive. The results are shown in the fifth column of Table 6.2. While, as expected, the full Monte-Carlo model is able to reduce the average absolute pricing error significantly, this improvement will have to be traded off against the elevated computational burden in practice. While on our regular personal computer one Monte-Carlo simulation for pricing the five MBS with different coupons simultaneously (using antithetic paths for variance reduction) takes approximately 15 minutes, all of the previously presented versions of our closed-form approximation approach only require a couple of seconds. Note also at this point that our closed-form approximation approach does not require any numerically complex procedures, in contrast to the models presented by Collin-Dufresne and Harding (1999), Rom-Poulsen (2007) and Gorovoy and Linetsky (2007).

The general idea of approximating the usual S-curve shape of the refinancing incentive by a piecewise linear function was also used by Gorovoy and Linetsky (2007), as previously mentioned (developed independently of the approach presented in this thesis). Their model, however, only introduces a floor to the refinancing incentive equal to 0 (similar to our approach) and does not cap refinancing prepayment for high spread values. In addition to this, the spread variable involves the short-rate, instead of the 10yr rate used here. The 10yr rate is certainly a more realistic proxy for mortgage rates which refinancing decisions are usually based on. A comparison of the accuracy of the Gorovoy/Linetsky approach and of our approach for the real-life TBA prices in this study would be highly interesting. Unfortunately, we were not able to produce comparable results with the Gorovoy/Linetsky model for our data. Taking the WAC minus some constant as parameter k in their definition of the spread variable, we encountered numerical problems in the eigenfunction expansion leading to unstable and, in some cases, non-sensical results when applied to our data. The numerical complexity of the Gorovoy/Linetsky model and the problems resulting from this have unfortunately made a consistent comparison based on our data infeasible.

	Average absolute pricing error			
Coupon	Closed-form approx. model	Constant baseline	Linear refi-incentive	Full MC val. model
6%	226	223	380	112
6.5%	187	167	142	115
7%	141	150	165	104
7.5%	116	126	230	98
8%	121	139	402	97
Overall	159	166	266	105

Table 6.2: Average absolute pricing errors of our closed-form approximation model, of reduced versions of our closed-form approximation model and of our full Monte-Carlo valuation model for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year.

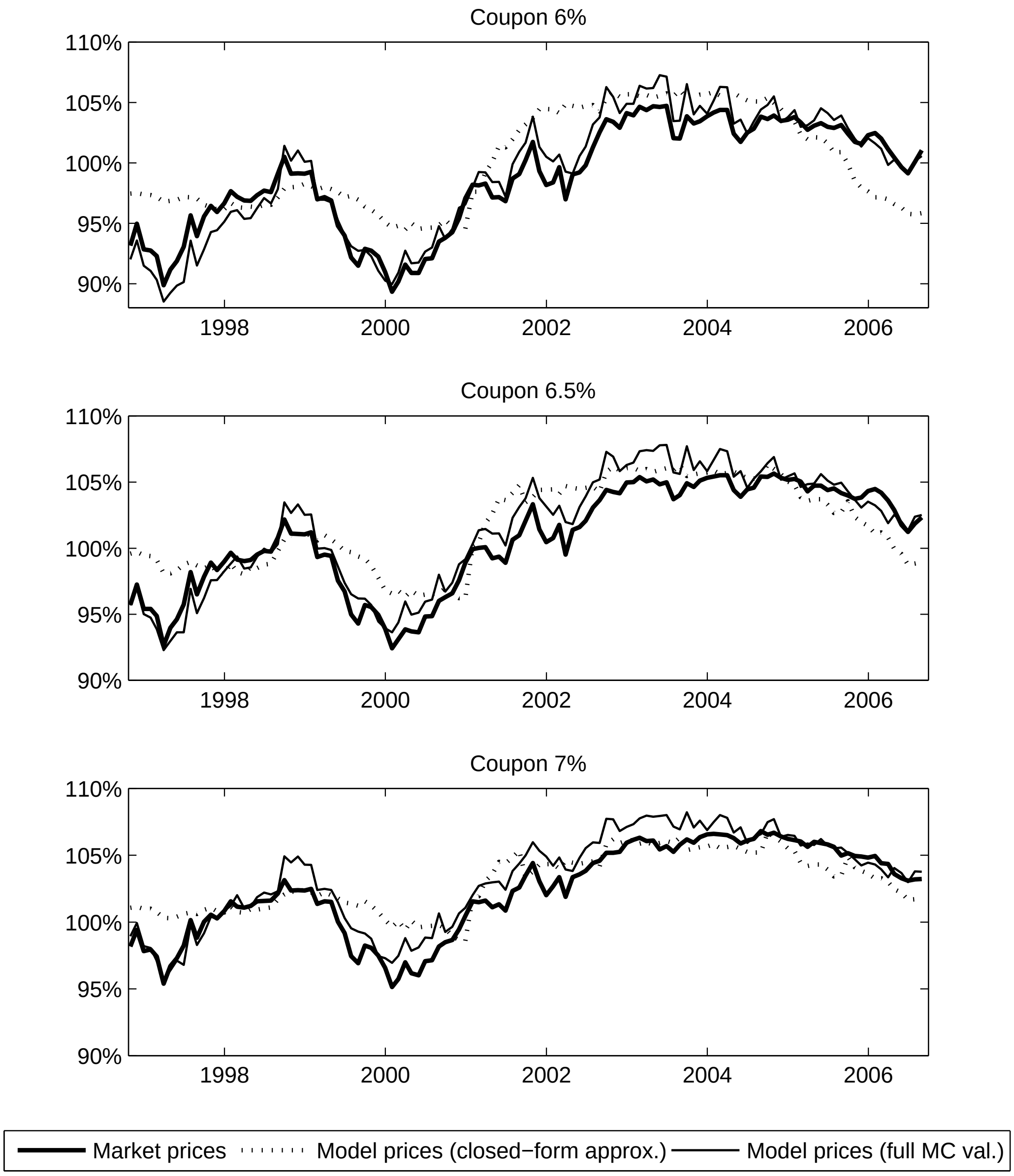

Figure 6.5: Market and model prices for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year (I).

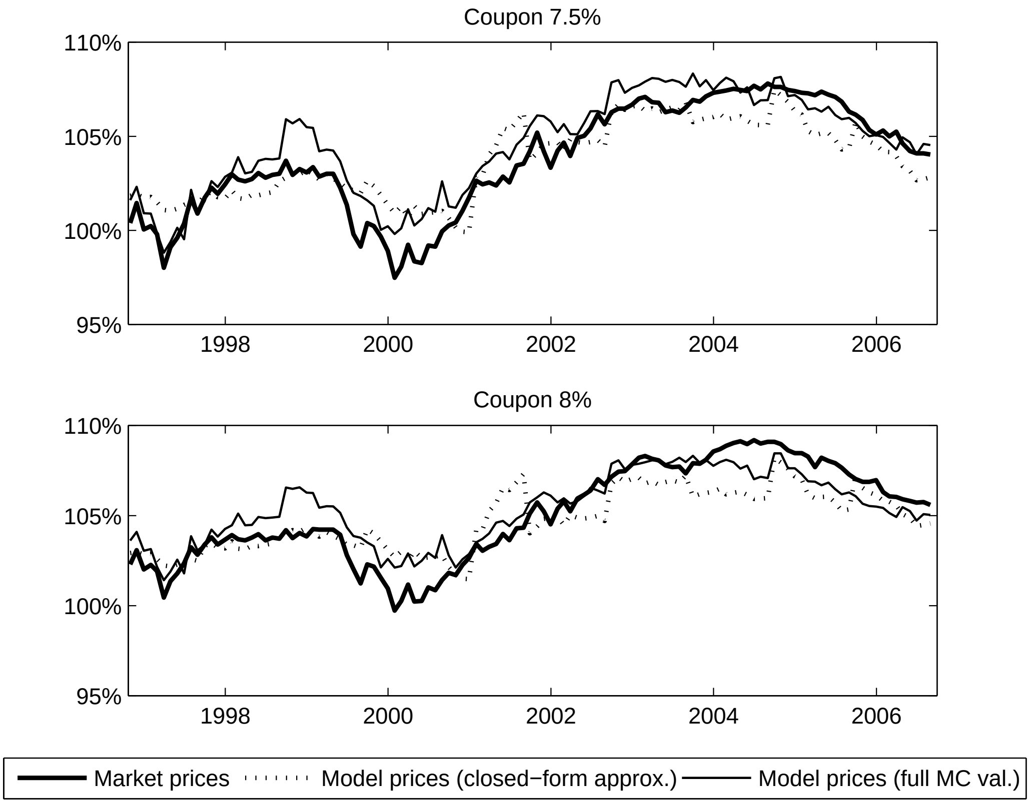

Figure 6.6: Market and model prices for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year (II).

Finally we give an empirical example of MBS effective durations and effective convexities calculated with our closed-form approximation model. In Chapter 5.2.3 we have already calculated effective durations and convexities for a sample of GNMA MBS with different coupons on the sample day 12-Dec-2005, based on our full Monte-Carlo valuation model (see Figure 5.5). Figure 6.7 shows the effective durations and convexities calculated with our closed-form approximation model for the same securities. Note, however, that the effective durations and convexities in Figure 6.7 are short-rate sensitivities since we work with a CIR short-rate model in our closed-form approximation approach. The effective durations in Figure 5.5 are durations/convexities in the proper sense (i.e. sensitivities w.r.t. parallel shifts of the whole yield curve) since in the Hull-White interest-rate model as used in Chapter 5 the whole yield curve is used as a model input via the function $\theta_r(t)$ (see (3.11)). This data example provides further evidence that our closed-form approximation yields consistent results and is indeed able to capture the basic characteristics of MBS, such as negative convexities.

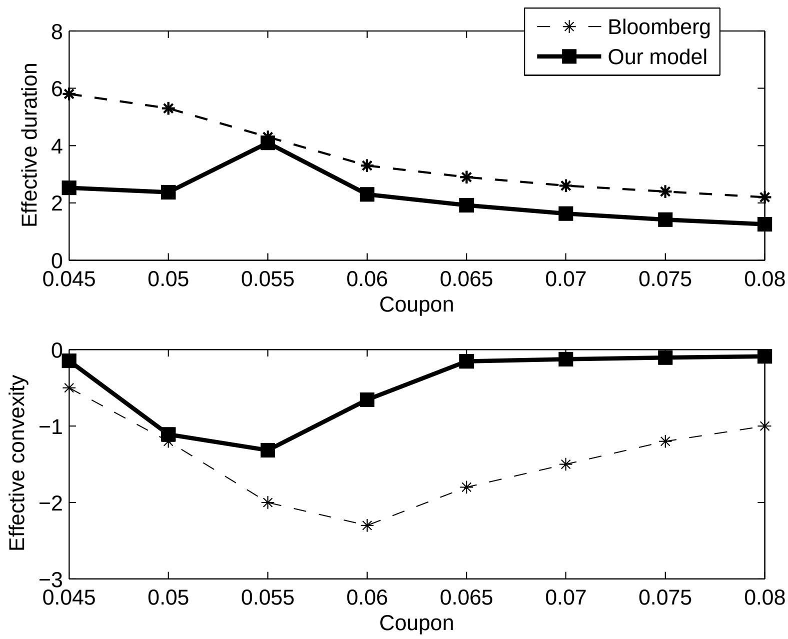

Figure 6.7: Effective (short-rate) durations and convexities of some GNMA MBS with different coupons on the sample day 12-Dec-2005 according to our closed-form approximation. Bloomberg effective durations and convexities are shown for comparison purposes.

Chapter 7

The contribution of our MBS pricing models

'All models are wrong, but some are useful.' (Box (1979))

In Chapter 4 we have motivated the need for further research concerning MBS valuation. In this chapter we will briefly summarise what has been achieved in this thesis. In particular, we will comparatively assess the contribution of our MBS pricing models and discuss the implications for their use in practice.

7.1 A comparative assessment

In the previous two chapters we have presented pricing models for agency MBS within an intensity-based modelling framework. In particular, we have extended the general model framework presented by Kau et al. (2004) for individual mortgage contracts to the pricing of MBS. Moreover, our model explicitly accounts for the general economic environment by the incorporation of a factor into the baseline hazard process which is fitted to the GDP growth rate. We have called our model 'prepayment-risk-neutral' since we are able to directly target market prices and thus do not need the theoretically dubious OAS in the pricing procedure. This is achieved by the introduction of prepayment-risk adjustment parameters which stem from the change of measure from the real-world measure to a risk-neutral pricing measure. We have shown that these prepayment-risk adjustment parameters can be calibrated to MBS market prices to obtain 'implied' prepayment patterns, in the sense of Levin and Davidson (2005). Our model therefore has the desirable features of the Levin/Davidson model combined with the mathematical rigour of the intensity-based modelling framework.

Furthermore, we have derived a closed-form approximation formula for the pricing of fixed-rate agency MBS with a slightly adjusted model specification. Our closed-form approximation is computationally highly efficient and reduces the computational burden of MBS valuation drastically. Our approach may thus be particularly useful for risk and portfolio management purposes where portfolios of MBS have to be revaluated frequently (see also the application in Chapter 8). Our approach offers a couple of advantages compared to previous models concerned with closed-form valuation of MBS. Compared to the closed-form formula for MBS by Collin-Dufresne and Harding (1999), our model offers two major extensions. First, we are not restricted to a single stochastic factor. In addition to our 1-factor CIR interest-rate model we introduce two additional stochastic factors to model the baseline prepayment. Moreover, we do not approximate the usual S-like relation between coupon spread and refinancing-related prepayment by a purely linear functional form but introduce a piecewise linear approximation. Our results indicate that this contributes to a significant improvement of the model performance. While these two issues are also explicitly addressed in a recent extension of the Collin-Dufresne/Harding model by Rom-Poulsen (2007), we do not encounter the (theoretical) weaknesses of the quadratic interest-rate/refinancing relation in the Rom-Poulsen model for certain scenarios. Moreover, both the Collin-Dufresne/Harding and the Rom-Poulsen model require numerical techniques to solve systems of partial differential equations. These models can thus only be considered as semi-analytic. Numerical complexity is also a problem in the recently developed model by Gorovoy and Linetsky (2007), as previously mentioned. While their approach offers an 'exact' solution within a quite similar model set-up compared to our approximation, the computation of this formula involves highly complex numerical evaluations related to eigenfunction expansion techniques.

Applied to historical price data of 30yr GNMA pass-throughs traded on a TBA basis, our closed-form approximation formula proved to be able to track market price movements for a wide range of coupons with an overall average absolute pricing error of 159 basis points (with a yearly recalibration of prepayment-risk adjustment parameters as described in the previous chapter). We consider this a highly satisfactory accuracy, in particular for risk and portfolio management purposes.

A direct quantitative comparison of the performance of different MBS pricing models across the different model classes is a hardly feasible task. Such a comparison would require a well-defined criterion how model performance can be measured. A natural choice for such a criterion would of course be the models' ability to explain market prices in the spirit of our empirical evaluation in Chapter 6.2.2. Yet, the traditional econometric models do not target market prices directly, but calculate the OAS as a model output. As previously mentioned, OAS levels derived from different prepayment and interest-rate model assumptions may differ substantially and it is impossible to say what the 'correct' OAS is. Econometric models, however, aim to predict real-life prepayment behaviour. Different traditional econometric models, such as the models developed by Schwartz and Torous (1989) or Richard and Roll (1989) could thus be quantitatively compared by some purely statistical goodness-of-fit criterion applied to historically observed and fitted prepayment rates. However, prepayment-risk-neutral models like the model developed by Levin and Davidson (2005) or like our modelling approach, are not primarily meant to predict prepayment in real-life. They are in line with common derivative pricing principles, where implied parameter patterns (and thus implied prepayment patterns in our case) are obtained after calibrating the model to market data. This fact is also discussed in Levin and Davidson (2005). A pure look at the ability of a model to explain market prices (possibly in-sample and out-of-sample) may equally be too single-minded, since it would not take into account another issue which is of critical importance in practice: computational burden. The computational burden is, however, in itself not an easily comparable criterion. As previously discussed, MBS valuation models, except the ones concerned with closed-form approximations, usually rely on either Monte-Carlo simulation or on backward induction techniques on multidimensional grids. The CPU time required for a Monte-Carlo simulation depends heavily on the concrete implementation, e.g. on the variance reduction technique(s) used in the Monte-Carlo algorithm. And vice versa, the success of one particular variance reduction method will heavily depend on the model specification. While, generally speaking, backward induction methods are usually faster than Monte-Carlo simulations, a well-implemented Monte-Carlo simulation can of course be faster than a poorly implemented backward induction scheme. This also depends substantially on the number of stochastic factors in the model.

Due to the lack of comparison criteria and consistent comparison standards, a quantitative model comparison is hardly feasible. Yet, in the remainder of this section, we want to assess our modelling approach qualitatively with respect to certain criteria compared to some selected models from each of the three model classes. Of course, this assessment can only be general in nature and has to be based on subjective perceptions to a certain extent. Within the econometric model class we choose the Schwartz and Torous (1989) model. We furthermore include the Stanton (1995) model as a representative example of the option-theoretic model class and the recently developed Gorovoy and Linetsky (2007) model from the intensity-based model class into this qualitative assessment. We finally include the

Levin and Davidson (2005) and Collin-Dufresne and Harding (1999) models since we have frequently cited them in the previous chapters. The results of our qualitative assessment, based on the author's best and honest judgement according to the discussions in the Chapters 4-7, are presented in Table 7.1. As comparison criteria we consider the mathematical rigour of the pricing routine, the numerical complexity, the computational burden for the pricing of MBS pass-throughs, the flexibility of the models to include further explanatory factors or variables into the prepayment specification and finally, the flexibility with respect to extensions of the model to non-standard structures such as adjustable-rate securities and CMOs. The mathematical rigour of the Schwartz and Torous (1989) model is arguable since this model implicitly needs the assumption that a prepayment function estimated under the real-world measure can be used for pricing purposes. This is a highly problematic assumption as our empirical results in Chapter 5 indicate. The Collin-Dufresne and Harding (1999) model has the undesirable feature that refinancing prepayment can become negative in a high interest-rate environment. While we consider our closed-form approximation mathematically sound, it is still an approximation and can thus not be preferred to 'exact' approaches if mathematical rigour is the only selection criterion. We have already discussed the issue of numerical complexity extensively. Apart from the closed-form pricing approaches, all other models are rather straightforward to implement so that numerical complexity should not be an issue. The remaining three criteria are particularly important for potential uses in practice. While the closed-form approaches are of course particularly appealing with respect to computational burden of pass-through pricing, they can not easily be extended to include new factors or explanatory variables which a particular user of the model may want to include. Nor can they be used straightforwardly for the pricing of non-standard MBS structures. The same holds in general for option-theoretic models which work with pricing routines

on grids. The only group of models where these extensions are rather straightforward is the one based on Monte-Carlo simulation routines, such as our full Monte-Carlo valuation model presented in Chapter 5.

	Other models in the literature					Our models	
	Schwartz/Torous (1989)	Stanton (1995)	Gorovoy/Linetsky (2007)	Levin/Davidson (2005)	Collin-Dufresne/Harding (1999)	Full MC-simulation	Closed-form approximation
Mathematical rigour of pricing logic	−−	++	++	0	+	++	+
Numerical complexity	++	++	−−	++	−	++	++
Comput. burden of pass-through pricing	−−	−	++	−	++	−−	++
Flexibility w.r.t. new factors/variables in prepayment model	++	−	0	++	−	++	0
Flexibility w.r.t. pricing of adjustable-rate MBS & pay-through structures (CMOs)	++	−	−−	++	−−	++	−−

Table 7.1: Qualitative assessment of different agency MBS pricing models with respect to some desirable model features. Grades range from ++ (very good), 0 (some good, some problematic aspects) to −− (highly problematic).

7.2 Implications for the use in practice

After the analysis in the previous chapters, the question of which model is the 'best', in particular with a view towards the use in practice, still remains open. In the previous section we have already highlighted that the question is, in its general sense, highly problematic. This is due to the lack of a consistent optimality criterion which is needed to be able to select the 'best' among all MBS pricing models.

From the previous discussion we can conclude that the choice of a pricing model in practice should be based on the purpose, i.e. the application the MBS model is used for. With our closed-form approximation formula as presented in Chapter 6 we have developed a computationally very efficient and thus very useful tool for applications in risk and portfolio management of fixed-rate agency MBS in practice. An example of how our model can be applied to portfolio management with MBS will be given in Chapter 8. However, the closed-form formula can not be extended straightforwardly to the pricing of complex pay-through structures, i.e. of CMOs. For these securities, a computational more expensive valuation method, such as Monte-Carlo simulation based on, e.g., our model presented in Chapter 5, can not be avoided. Also, if time is not a critical factor and the valuation should be as accurate as possible, we recommend to use our full Monte-Carlo valuation model, which may be enhanced and statistically fine-tuned with additional explanatory variables, if desired.

Both, our prepayment-risk neutral valuation model as presented in Chapter 5 and the closed-form approximation in Chapter 6 offer the possibility to target market prices directly. This is a desirable feature in risk management since it eliminates the discussion of whether the OAS must be treated as a risk factor. Many risk measures commonly used in the MBS market are usually calculated with a constant OAS assumption (e.g., the calculation of the effective duration and effective convexity of a MBS with a traditional econometric valuation model). Yet, the question whether the OAS should be considered as a constant or not is of course equivalent to the question of parameter stability (in particular, w.r.t. the prepayment-risk adjustment parameters) in our modelling approach.

Finally, it should be pointed out that MBS where the mortgagors can be assumed to prepay optimally should definitely be valued with an option-theoretic model based on pricing routines for callable bonds. Optimal prepayment can, however, only be expected in the institutional market, e.g., when the mortgages are not residential but commercial. The treatment of commercial MBS is beyond the scope of this thesis.

Chapter 8

Optimal portfolios with MBS

As already mentioned in the first chapters of this thesis, mortgage-backed securities constitute a large and important segment of the US fixed-income market. The total outstanding volume of MBS in the US has surpassed by far the outstanding volume of US treasuries. MBS are thus an important asset class and are in fact very popular among institutional investors. While agency MBS feature nearly the same credit quality as US treasuries - in particular, GNMA securities have the full faith and credit of the US government- they often offer higher yields. Hence their popularity with investors.

Going back to the seminal work of Markowitz (1952), the question of how to build an optimal portfolio from a given set of different assets, considering different constraints and optimality criteria, is a well-studied problem (see, e.g., Meucci (2005) for a textbook covering a wide range of aspects of modern portfolio optimisation). In the majority of studies, however, the asset classes considered were stocks and regular bonds. More recent publications have also included alternative assets such as hedge funds or real estate investment trusts (REITs) (see, e.g., Krokhmal et al. (2002) or Höcht et al. (2007)). Yet, an inclusion of MBS into portfolio optimisation problems remains rare. One exception is the work based on Zenios (1993), Zenios and Kang (1993) and McKendall et al. (1993) who consider portfolio optimisation with MBS, mainly in an asset-liability management context. The major reason why MBS are usually not considered in portfolio optimisation problems is the computational burden associated with MBS valuation. Since most modern portfolio optimisation problems are also computationally expensive and require frequent evaluation of the assets under different scenarios, the inclusion of MBS into portfolio optimisation problems has often been infeasible. McKendall et al. (1993) use a recombining tree to create interest-rate scenarios, so that for a buy-and-hold static optimisation problem, the number of required MBS valuations is limited due to the limited number of nodes in the tree. However, when considering other correlated asset classes, such as equities, the nodes in the tree grow exponentially and the optimisation problem may again become computationally infeasible. This is particularly true if one wants to consider the possibility of asset reallocation during the investment horizon.

Our closed-form approximation approach as presented in Chapter 6 now offers a possibility to overcome the computational burden associated with portfolio optimisation with MBS. It is the main objective in this chapter to consider some selected portfolio optimisation approaches based on simulated scenarios and to extend their usual application by including fixed-rate agency-MBS with different coupons. We thereby hope to close a gap in the applied portfolio optimisation literature where MBS have long been neglected. Within a consistent simulation framework, we show how the inclusion of MBS into a classical stock/bond portfolio can enhance total returns. For this purpose we also apply our approach to real historical market data in an empirical case study. We show how optimal portfolios according to an expected utility criterion and to a conditional value-at-risk (CVaR) criterion develop and perform over time, respectively, based on a static multi-period asset allocation strategy with a rolling one year investment horizon.

In the following section we present the available assets for our study and give details of the scenario simulation procedure used for portfolio optimisation. We then consider the expected utility optimisation approach and the portfolio optimisation with CVaR constraints and apply these concepts to historical data in an empirical study.

8.1 The set-up: assets and scenarios

Of course, there are countless ways of combining possible assets and portfolio optimisation settings. Concerning the assets, we therefore concentrate on a specific set of assets which we believe to be quite representative. We assume that the investor has the choice between a total of six assets. The S&P 500 index represents the equity class, non-defaultable zero-coupon bonds with 3 and 10 years to maturity constitute the available bonds. In addition to these instruments we allow the investor to invest in three different 30yr fixed-rate GNMA pass-through MBS with coupons of 6%, 7% and 8%, respectively. While we will consider different optimisation problems in the following section, we always restrict the investment horizon to one year. We do not consider transaction costs, which is not a too unrealistic assumption since we only deal with very liquid instruments for which transaction costs for institutional investors may indeed be negligible.

Since we work with optimisation approaches based on simulated scenarios, we have to specify the distributional assumptions for the processes underlying the assets. We model the evolution of the equity index $S(t)$ with a geometric Brownian motion in the usual way, correlated to interest rates. For the interest-rate dynamics we use a one-factor CIR model as in Chapter 6 where the short-rate $r(t)$ has the dynamics (under the real-world measure Q)

$$dr(t) = (\theta_r - a_r r(t))dt + \sigma_r \sqrt{r(t)} dW(t) \tag{8.1}$$

for some positive constants θ, a_r, σ_r with $2\theta > \sigma_r^2$. The (real-world) equity index dynamics are given by

$$dS(t) = \mu S(t)dt + \sigma_S S(t)dZ(t) \tag{8.2}$$

for some positive constants μ, σ_S and we assume that the Wiener processes $W(t)$ and $Z(t)$ are correlated such that

$$Cov[dW(t), dZ(t)] = \rho dt.$$

By Cholesky decomposition (see, e.g., Brigo and Mercurio (2006), p. 886), we can rewrite (8.1) and (8.2) in the following way:

$$\begin{aligned} dr(t) &= (\theta_r - a_r r(t))dt + \sigma_r \sqrt{r(t)} d\hat{W}(t) \\ dS(t) &= \mu S(t)dt + \sigma_S \rho S(t) d\hat{W}(t) + \sigma_S S(t) \sqrt{1-\rho^2} d\hat{Z}(t) \end{aligned} \tag{8.3}$$

for two uncorrelated Wiener processes $\hat{W}(t)$, $\hat{Z}(t)$. For the estimation of the parameters in (8.3) we use historical weekly return data of the S&P 500 index from 1996 to 2005 and weekly US treasury strip rates, obtained from par rates by standard bootstrapping, of the same time period. We use a two-stage procedure for parameter estimation, similar in spirit to the approach taken in Zagst (2002b). In the first stage we determine the parameters of the interest-rate process and of the stock index process. For the parameters of the CIR interest-rate model we use the values as already reported in Table 6.1. For the estimation of the parameters of the stock index process, note that it follows from the Ito-formula that a discretised version of the dynamics of the log-returns of the stock index is given by:

$$\ln \frac{S(t+\Delta t)}{S(t)} = (\mu - \frac{1}{2}\sigma_S^2) \cdot \Delta t + \sigma_S \rho \sqrt{\Delta t} \cdot N_1 + \sigma_S \sqrt{1-\rho^2} \sqrt{\Delta t} \cdot N_2,$$

where N_1, N_2 are two independent standard-normal random variables. Thus, the log-returns of the stock index are normally distributed with

$$E\left[\ln \frac{S(t+\Delta t)}{S(t)}\right] = (\mu - \frac{1}{2}\sigma_S^2) \cdot \Delta t \tag{8.4}$$

$$Var\left[\ln \frac{S(t+\Delta t)}{S(t)}\right] = \sigma_S^2 \rho^2 \cdot \Delta t + \sigma_S^2 (\sqrt{1-\rho^2})^2 \cdot \Delta t = \sigma_S^2 \cdot \Delta t. \tag{8.5}$$

Using our historical return data sample, we can estimate the parameters μ and σ_S by simple moment estimators, i.e. equating empirical moments with the theoretical moments as given in (8.4), (8.5).

In the second stage we estimate the correlation parameter ρ. For this purpose we consider the shortest observable treasury strip rate (the 3 month rate in our sample) as a proxy for the unobservable short rate. The correlation parameter can then be estimated from the sample residuals. If we observe the 3 month rate and the log-returns of the stock index at discrete points of time t_k, $k = 0, ..., K$ and define $\Delta t_k = t_k - t_{k-1}$ we get the sample residuals

$$\begin{aligned} \varepsilon_r(k) &:= r(t_k) - r(t_{k-1}) - (\theta_r - a_r \cdot r(t_{k-1})) \cdot \Delta t_k \\ \varepsilon_S(k) &:= \ln \frac{S(t_k)}{S(t_{k-1})} - (\mu - \frac{1}{2}\sigma_S^2) \cdot \Delta t_k \end{aligned}$$

for $k = 1, ..., K$. Since (see Zagst (2002b) for a detailed discussion in the multi-asset case)

$$Cor(\varepsilon_r, \varepsilon_S) = \rho,$$

we can estimate the correlation parameter by the empirical correlation of $\varepsilon_r(k)$ and $\varepsilon_S(k)$, $k = 1, ..., K$. All parameter values are summarised in Table 8.1. With the given parameters it is now easy to generate scenarios with a

	Parameter	Estimate
Short-rate process	θ_r	0.014
	a_r	0.41
	σ_r	0.059
	$\hat{a}_r$	0.20
Stock index process	μ_S	0.089
	σ_S	0.179
Correlation parameter	ρ	0.122

Table 8.1: Parameter estimates for scenario generation. The parameter $\hat{a}_r$ is the mean-reversion speed parameter in the CIR model (8.1) if the dynamics of the short-rate is considered under the risk-neutral measure for pricing purposes.

straightforward Monte-Carlo simulation. In this study we consider a monthly discretisation and an investment horizon of one year. I.e., given a current state of the economy $S(t_0)$, $r(t_0)$, we generate Monte-Carlo paths with 12 (monthly) grid points using the discretised version of (8.3)

$$\begin{aligned} r(t_k) &= r(t_{k-1}) + (\theta - a_r r(t_{k-1}))\Delta t_k + \sigma_r \sqrt{r(t_{k-1})} \cdot \sqrt{\Delta t_k} \cdot N_1 \\ S(t_k) &= S(t_{k-1}) \cdot e^{(\mu_S - \frac{1}{2}\sigma_S^2)\cdot \Delta t_k + \sigma_S \cdot \rho \cdot \sqrt{\Delta t_k} \cdot N_1 + \sigma_S \cdot \sqrt{1-\rho^2} \cdot \sqrt{\Delta t_k} \cdot N_2}, \end{aligned}$$

for $k = 1, ..., 12$, where N_1, N_2 are again two independently drawn standard-normal random variables. Note that the discretisation method for $r(t)$ used here is the simple Euler scheme as also used in, e.g., Zagst (2002b).

Given a state of the short rate $r(t)$, the price

$$P(t,T) := E_{\widetilde{Q}}[e^{-\int_t^T r(s)ds} | \mathcal{F}_t]$$

of a zero-coupon bond with maturity T at time t and notional 1 is given in the CIR interest-rate model by (3.23). For the pricing of the mortgage-backed securities and for the generation of prepayment scenarios we use the closed-form approximation approach developed in Chapter 6. Due to its computational efficiency, the closed-form pricing approach makes an integration of MBS into a scenario-based portfolio optimisation feasible within a reasonable time frame. For the prepayment scenarios, we first have to generate scenarios for the (annualised, continuous) baseline prepayment process $\pi_0(t)$. We obtain from (6.17) after Euler discretisation:

$$\begin{aligned} \pi_0(t_k) &= \pi_0(t_{k-1}) + (\theta_p + b_{pw} w(t_k) - a_p \pi_0(t_{k-1}))\Delta t_k + \sigma_p \cdot \sqrt{\Delta t_k} \cdot N_3, \\ w(t_k) &= w(t_{k-1}) + (\theta_w - a_w w(t_{k-1}))\Delta t_k + \sigma_w \cdot N_4, \end{aligned}$$

where N_3, N_4 are two independently drawn standard-normal random variables. In addition to the baseline prepayment, we obtain the (annualised, continuous) refinancing prepayment speed

$$\pi_{refi}(t) = 5.6 \cdot \max(\min(WAC - R_{10}(t), 0.04), 0),$$

where WAC is the security's weighted average coupon and $R_{10}(t)$ is the 10-year treasury zero rate calculated according to the CIR interest-rate model. The overall prepayment speed

$$\pi(t) = \pi_{refi}(t) + \pi_0(t)$$

has to be converted into the usual constant prepayment rate (CPR) by

$$CPR(t) = e^{\pi(t)} - 1$$

and we finally obtain the overall single monthly mortality by

$$SMM(t) = 1 - (1 - CPR(t))^{\frac{1}{12}}.$$

With the parameters as given in Table 6.1 the single monthly mortalities vary in the range between 1% and 6% for the three different MBS in 1000 simulated interest-rate and prepayment scenarios (if we take the short-rate level as of March 2001 and the mean-reversion level of the baseline prepayment as starting values for the scenario generation). Recall that since the pricing approach targets market prices directly, no OAS input is required for the MBS valuation. We calibrate the prepayment-risk-adjustment parameters of our closed-form approximation model to market prices of GNMA securities with different coupons, traded on a to-be-announced (TBA) basis, on the start date of the scenario generation.

8.2 Scenario-based portfolio optimisation with MBS

For the empirical part of this study we focus on the time horizon from spring 2001 to spring 2003, since this was an interesting time period in the US fixed-income market in general and in the MBS market in particular. Interest rates decreased sharply during this time period, the 1yr treasury zero-rate dropped from levels around 4.2% in March 2001 to about 1.1% in March 2003. The 10yr rate slid from 5.3% to 4.1% during the same time period. This decrease in interest rates triggered a significant increase in prepayments (see also Figure 2.1). MBS which were traded in the discount area in spring 2001 became premiums in the subsequent months. E.g., a 6% coupon GNMA security was traded at 98.14% in March 2001 and at 103.94% in March 2003 (see Figure 8.1). The development of the S&P 500 index and of the 3 month treasury strip rate during the same time period is shown in Figure 8.2. By leaving the universe of available instruments for our portfolio optimisation procedure unchanged, we can see the effects of these changes in the economic environment on the optimal portfolios. We select an optimal portfolio each month, between March 2001 and February 2003 with an investment horizon of one year in any case. Our empirical case study is thus equivalent to an optimal portfolio allocation strategy with monthly rebalancing, based on a rolling one year investment horizon from March 2001 to March 2003.

The returns of the assets along each scenario path are calculated by taking into account the prices, amortisation factors and cash flows of the instruments. Similar to McKendall et al. (1993) we calculate the (scenario-dependent) total return of an investment of 1 in the j-th instrument over the

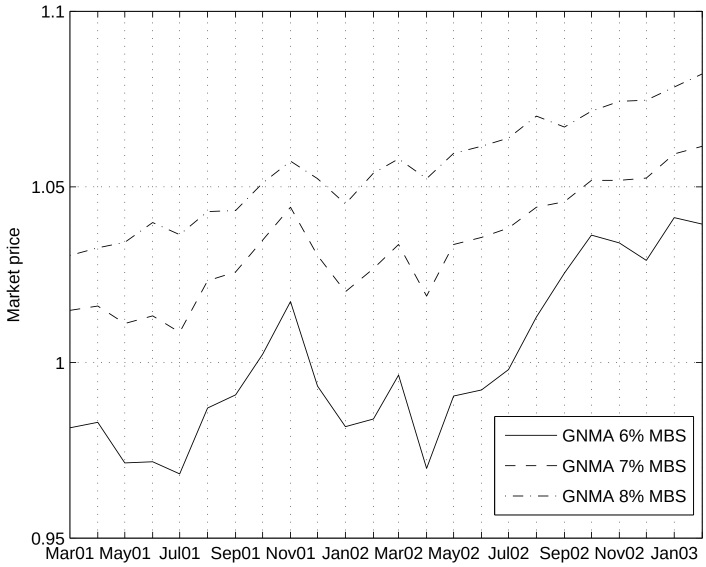

Figure 8.1: Market prices of GNMA pass-through MBS with different coupons, traded on a TBA basis, from 2001 to 2003.

1-year horizon by

$$TR_s^j := \frac{\alpha^j(s) \cdot V_{final}^j(s) + \kappa^j(s)}{V_{initial}^j}, \tag{8.6}$$

where $V_{initial}^j$ denotes the initial value of the j-th instrument , $V_{final}^j(s)$ is the value of the instrument at the end of the 1-year horizon, $\alpha^j(s)$ is the amortisation factor and $\kappa^j(s)$ is the cash flow factor in the scenario s, $s = 1, ..., S$. The amortisation factor is the fraction of the notional which remains outstanding at the end of the investment horizon. Of course, $\alpha^j(s) = 1$ for the stock index investment and for the zero-coupon bonds, for all scenarios s. Moreover, for these instruments $\kappa^j(s) = 0$ for all s since neither the stock index nor the zero-coupon bonds generate any cash flows during the investment horizon. For the MBS, however, the amortisation factor has to be calculated according to the prepayment model, taking into account prepayments and scheduled principal repayments along each scenario path. The cash flow factor is calculated accordingly, taking into account principal repayments and prepayments and all interest received by the investor. All cash flows occurring prior to the end of the 1-year investment period are assumed to earn the

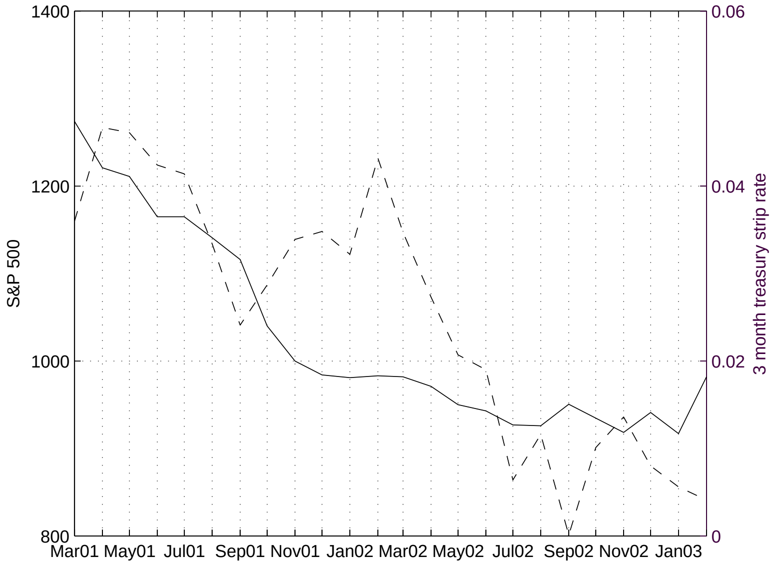

Figure 8.2: Development of the S&P 500 index (dotted line, left-hand scale) and of the 3 month treasury strip rate (solid line, right-hand scale) from 2001 to 2003.

risk-free rate from a money-market account until the end of the investment horizon. Figure 8.3 shows the histograms of the total returns of our portfolio instruments based on 1000 simulated scenarios with starting values as of March 2001. Consistent with basic asset pricing theory, those instruments with higher expected returns also feature a higher return variance. This holds particularly for the stock index, but also for the lowest coupon MBS, which offers a slightly higher expected total return than the MBS with higher coupons. Furthermore, the return distributions of the MBS instruments are asymmetric, as expected. In particular, the 7% and 8% GNMA securities feature more downside risk (with respect to deviations from the expected return) than upside potential. This is a typical feature of MBS caused by the prepayment option inherent in the underlying mortgages and also explains the negative convexities of MBS, which is a commonly observed and well-studied characteristic of MBS as already discussed in Chapters 5 and 6.

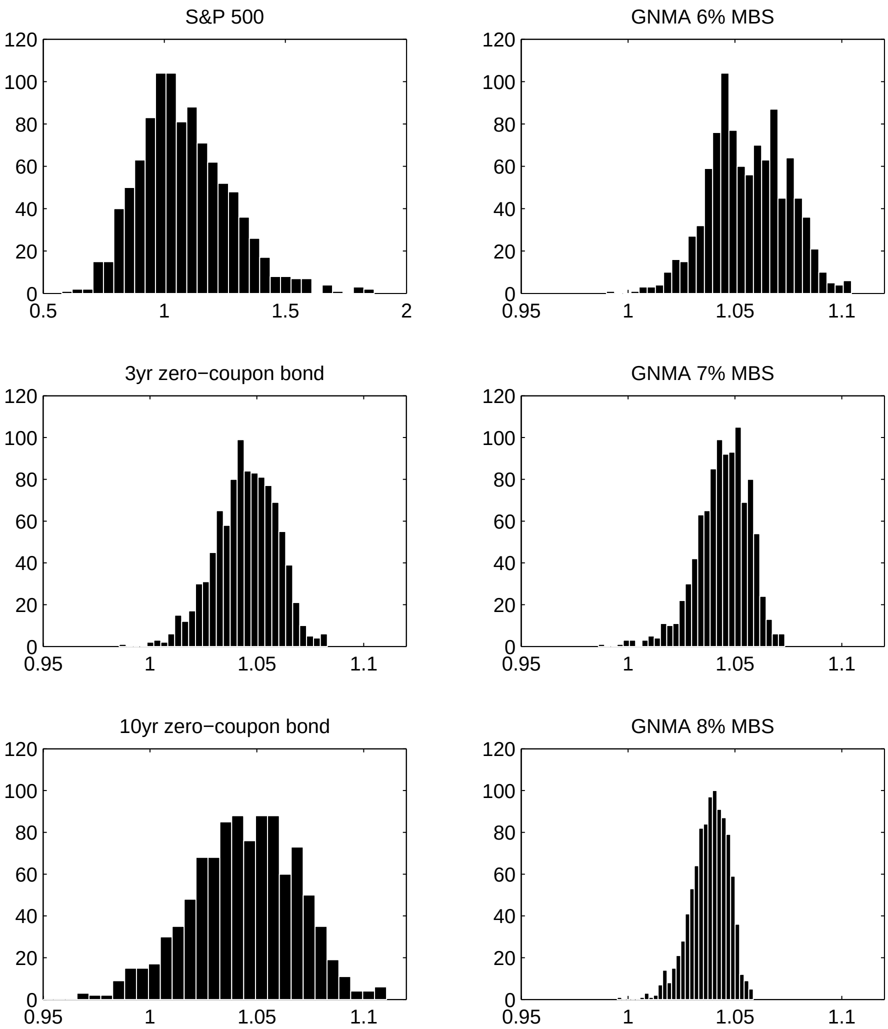

Figure 8.3: Histograms of the total returns of our portfolio instruments based on 1000 scenarios.

8.2.1 Expected utility approach

The first optimality criterion for our portfolio selection problem which we want to consider in this study is the expected utility criterion. This concept is well-known and is discussed in detail in, e.g., Meucci (2005). Expected utility was also used by McKendall et al. (1993). In this study we use the exponential utility function

$$U(v) := -e^{-\gamma \cdot v},$$

for some constant γ, which belongs to the class of CARA-utility functions (Constant Absolute Risk Aversion). If we denote by J the total number of available assets, by $x = (x_1, ..., x_J)$ the portfolio weight vector and by $TR_s(x)$ the total return of the portfolio x in the scenario s, the expected utility $\xi^U(x)$ of the portfolio x is given by

$$\xi^U(x) := \sum_{s=1}^{S} q(s) U(TR_s(x)), \tag{8.7}$$

where $s = 1, ..., S$ is the set of scenarios and $q(s)$ is the probability of scenario s. Since in this study we work with a Monte-Carlo simulation for scenario generation we assume that all scenarios have the same probability, i.e. (8.7) reduces to

$$\xi^U(x) = \frac{1}{S} \sum_{s=1}^{S} U(TR_s(x)). \tag{8.8}$$

In this study we do not want to allow short-selling and we limit the maximum investment in any particular asset to 80% of the portfolio value. I.e., we require that $0.8 \geq x_j \geq 0$ for all j. The portfolio optimisation problem within the expected utility framework is thus given by

$$(P1) \quad : \quad \begin{cases} -\xi^U(x) \longrightarrow \min\limits_{x} \\ \\ \sum_{j=1}^{J} x_j = 1 \\ \\ 0.8 \geq x_j \geq 0 \end{cases}$$

(P1) can be solved with standard optimisation software, e.g. with the *fmincon* function in Matlab.

Figure 8.4 shows the expected utilities ξ^U and the variances σ^2 of the optimal stock/bond/MBS portfolios for different values of the risk aversion parameter γ. For lower values of γ, the investor is less risk averse and the variance of the total return of the optimal portfolio increases. The ξ^U-σ^2 line can be compared to the efficient frontier in the classical mean-variance concept as introduced by Markowitz (1952). Figure 8.5 shows this efficient frontier for the portfolio optimisation with the stock index and bonds only and the efficient frontier for the portfolio optimisation problem with the stock index, bonds and MBS. The picture confirms that the inclusion of MBS into the portfolio optimisation problem can substantially enhance the expected utility of the optimal portfolio for the same degree of risk aversion. Figures 8.6 and 8.7 show how the composition of the optimal portfolio changes over time for γ set equal to 2. Note that it is not surprising that the optimal portfolio weight of the stock index increases from March 2001 to October 2001 and remains at its maximum level afterwards, since interest rates slid to historical lows during this time period. When the short rate is very low the CIR interest-rate model implies that the short rate has a strong upward drift back to the long-term mean-reversion level. Thus, most of the interest-rate paths generated with low starting values of the short rate end with substantially higher rates, resulting in losses for bond and MBS investors. Both bonds and MBS therefore become less attractive compared to stocks. Finally, Figure 8.8 shows how the wealth of an investor develops over time, if the initial wealth is 1 and the investor follows our optimal asset allocation strategy from March 2001 to March 2003. Since the amount of prepayment for the MBS depends on the individual pools delivered to the investor in the TBA trades it is unclear which historical prepayment rates to use for this historical study (average realised GNMA prepayment rates would be an option here, but unfortunately aggregated data were not available for this study). We thus assume for simplicity that the prepayment rates are equal to the expected values according to the prepayment model. Consistent with our previous results, the optimal asset allocation strategy with MBS outperforms the strategy with the stock index and bonds only. Of course, the overall performance of both strategies is negative due to the unfavourable market conditions during the time horizon considered in this study, in particular with respect to the stock index.

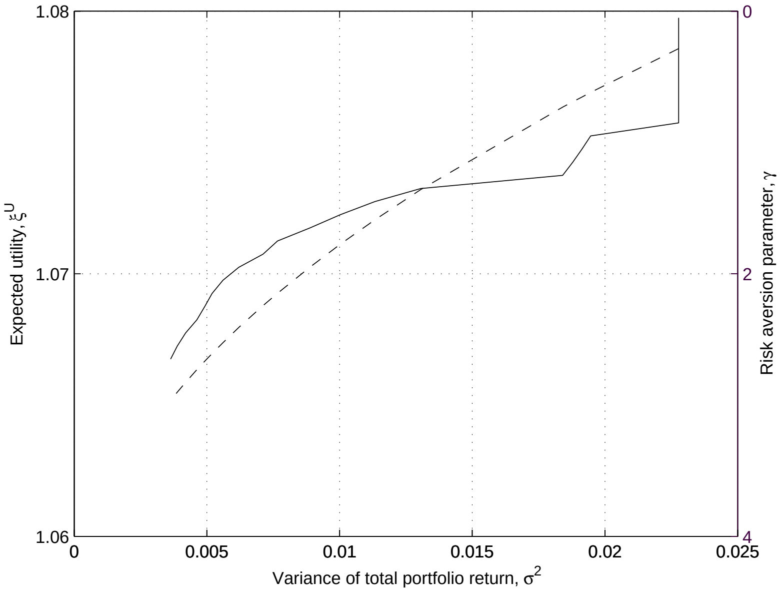

Figure 8.4: Expected utility-variance efficient frontier of a stock/bond/MBS portfolio (dotted line, left-hand scale) and relation to the risk-aversion parameter γ (solid line, right-hand scale).

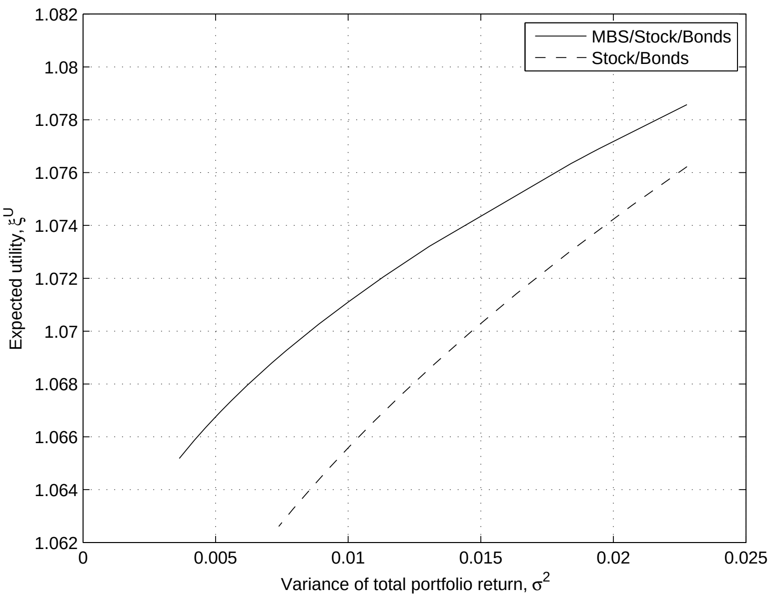

Figure 8.5: Expected utility-variance efficient frontier for a stock/bond portfolio (dotted line) and for a stock/bond/MBS portfolio (solid line).

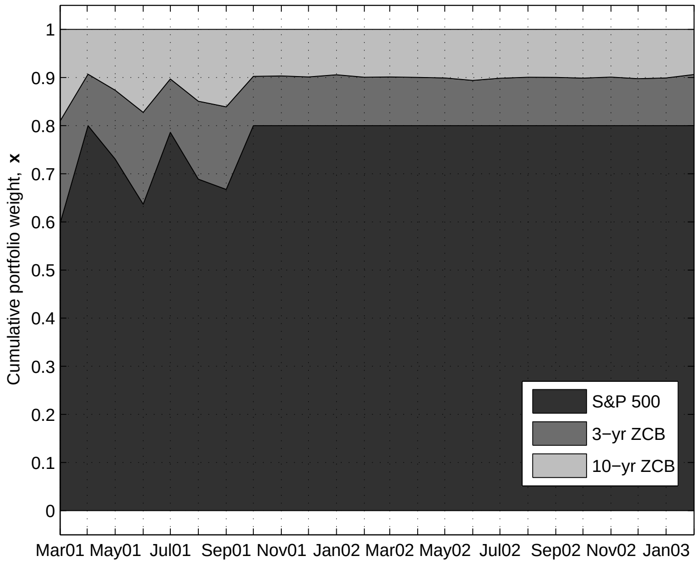

Figure 8.6: Optimal portfolio weights from 2001-2003 in a stock/bond portfolio.

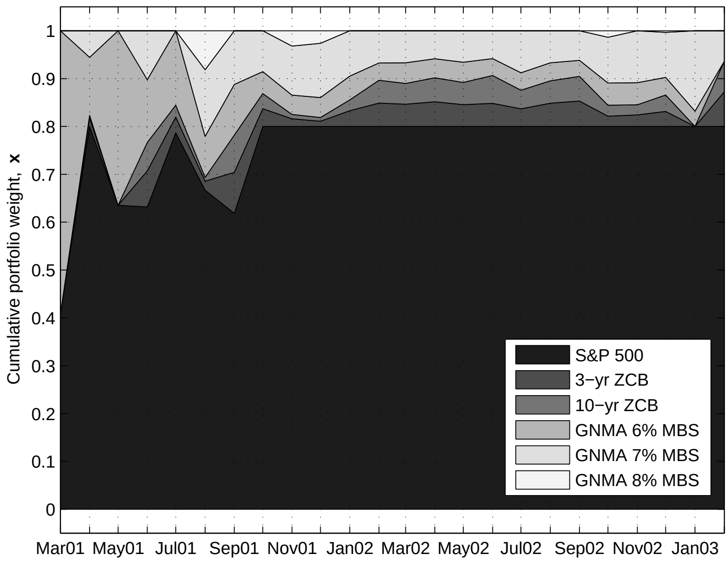

Figure 8.7: Optimal portfolio weights from 2001-2003 in a stock/bond/MBS portfolio.

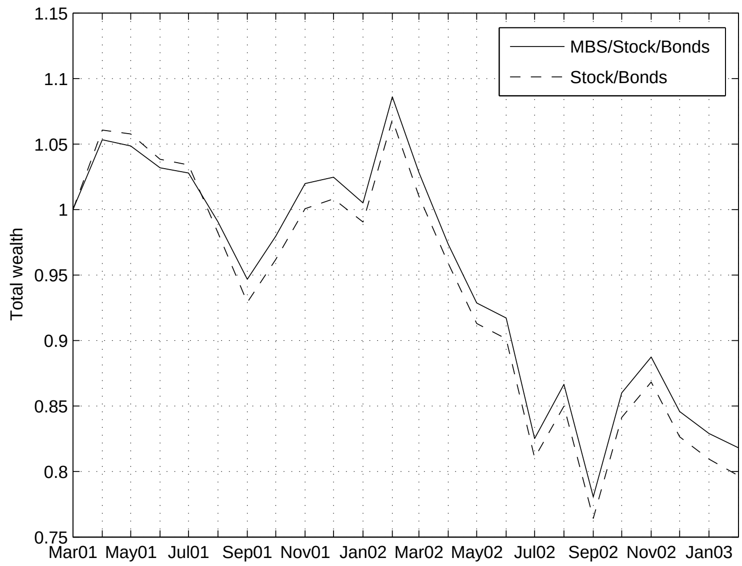

Figure 8.8: Development of the total wealth according to our multi-period optimal asset allocation strategy from 2001-2003 for a stock/bond portfolio (dotted line) and for a stock/bond/MBS portfolio (solid line).

8.2.2 Portfolio optimisation with CVaR constraints

The concept of portfolio optimisation with Conditional Value-at-Risk (CVaR, sometimes also called Tail Conditional Expectation or Expected Shortfall), was originally introduced by Krokhmal, Palmquist, and Uryasev (2002) and is summarised in, e.g., Krokhmal, Uryasev, and Zrazhevsky (2002). In our set-up, let $f(x, y)$ be the loss function defined by

$$f(x, y_s) := 1 - TR_s(x)$$

where the stochastic vector y with probability density function $q(y)$ contains all uncertainty factors regarding the portfolio's total return, and let $\Psi(x, \zeta)$ be the cumulative distribution function defined by

$$\Psi(x, \zeta) := Q(f(x, y) \leq \zeta) = \int_{f(x,y) \leq \zeta} q(y) \quad dy.$$

Similar to Krokhmal, Uryasev, and Zrazhevsky (2002), we define the value-at-risk $VaR(\alpha, x)$ with respect to the confidence level α by

$$VaR(\alpha, x) := \min\{\zeta \in \mathbb{R} : \Psi(x, \zeta) \geq \alpha\}$$

and the CVaR as the α-tail expectation of the loss function, i.e.

$$CVaR(\alpha, x) := \frac{1}{1-\alpha} \cdot \int_{f(x,y) \geq VaR(\alpha,x)} f(x, y) \cdot q(y) \quad dy.$$

Denoting by

$$\xi(x) := \frac{1}{S} \sum_{s=1}^{S} TR_s(x) \tag{8.9}$$

the expected total return of the portfolio x and by ω some pre-specified CVaR limit (e.g. a certain fraction of the initial portfolio value), our general portfolio optimisation problem can be written as

$$(P2) \quad : \quad \begin{cases} -\xi(x) \longrightarrow \min\limits_{x} \\ CVaR(\alpha, x) \leq \omega \\ \sum_{j=1}^{J} x_j = 1 \\ 0.8 \geq x_j \geq 0 \end{cases} \tag{8.10}$$

It is a major result of Krokhmal, Palmquist, and Uryasev (2002) that (8.10) is equivalent to a much easier-to-handle optimisation problem in which the function $F(\alpha, x, \zeta)$ defined by

$$F(\alpha, x, \zeta) := \zeta + \frac{1}{1-\alpha} \cdot \int \max[f(x,y) - \zeta, 0] \cdot q(y) \quad dy$$

on $(0,1) \times \mathbf{X} \times \mathbb{R}$ plays a crucial role. Its most important properties are collected in the following theorem.

Theorem 8.1. *As a function of ζ, $F(\alpha, x, \zeta)$ is convex and continuously differentiable. The Conditional Value-at-Risk with respect to the level α of the loss associated with any $x \in \mathbf{X}$ can be determined from the formula*

$$CVaR(\alpha, x) = \min_{(x,\zeta)\in\mathbf{X}\times\mathbb{R}} F(\alpha, x, \zeta).$$

In this formula, the set consisting of the value of ζ for which the minimum is attained, namely

$$A(\alpha, x) := \operatorname*{argmin}_{\zeta\in\mathbb{R}} F(\alpha, x, \zeta)$$

is a non-empty, closed, bounded interval (perhaps reducing to a single point), and the $VaR(\alpha, x)$ is given by

$$VaR(\alpha, x) = \textit{left endpoint of } A(\alpha, x).$$

In particular, one always has

$$VaR(\alpha, x) \in \operatorname*{argmin}_{\zeta\in\mathbb{R}} F(\alpha, x, \zeta)$$

and

$$CVaR(\alpha, x) = F(\alpha, x, VaR(\alpha, x)).$$

Proof. The theorem is taken from Krokhmal, Palmquist, and Uryasev (2002). A proof can be found in Rockafellar and Uryasev (2000). □

The following theorem gives the theoretical justification why one can use the function $F(\alpha, x, \zeta)$ instead of $CVaR(\alpha, x)$ for the purpose of determining a vector x that yields the minimum $CVaR(\alpha, x)$.

Theorem 8.2. *Minimising the Conditional Value-at-Risk with respect to the level* α *of the loss associated with* x *over all* $x \in \mathbf{X}$ *is equivalent to minimising* $F(\alpha, x, \zeta)$ *over all* $(x, \zeta) \in \mathbf{X} \times \mathbb{R}$*, in the sense that*

$$\min_{x \in \mathbf{X}} CVaR(\alpha, x) = \min_{(x,\zeta) \in \mathbf{X} \times \mathbb{R}} F(\alpha, x, \zeta),$$

where moreover a pair (x^*, ζ^*) *achieves the right hand side minimum if and only if* x^* *achieves the left hand side minimum and* $\zeta^* \in A(\alpha, x^*)$*. In particular, therefore, in circumstances where the interval* $A(\alpha, x^*)$ *reduces to a single point (as is typical), the minimisation of* $F(\alpha, x, \zeta)$ *over* $(x, \zeta) \in \mathbf{X} \times \mathbb{R}$ *produces a pair* (x^*, ζ^*)*, not necessarily unique, such that* x^* *minimises the Conditional Value-at-Risk and* ζ^* *gives the corresponding* α*-Value-at-Risk. Furthermore,* $F(\alpha, x, \zeta)$ *is convex with respect to* (x, ζ) *and* $CVaR(\alpha, x)$ *is convex with respect to* x *when* $f(x, y)$ *is convex with respect to* x*, in which case, if the constraints are such that* $\mathbf{X}$ *is a convex set, the joint minimisation is an instance of convex programming.*

Proof. The theorem is also taken from Krokhmal, Palmquist, and Uryasev (2002) and proven in Rockafellar and Uryasev (2000). □

We finally obtain the useful result:

Theorem 8.3. *The minimisation problem* $(P2)$ *as given in (8.10) is equivalent to the minimisation problem*

$$(P3) \quad : \quad \begin{cases} -\xi(x) \longrightarrow \min\limits_{x,\zeta} \\ F(\alpha, x, \zeta) \leq \omega \\ \sum_{j=1}^{J} x_j = 1 \\ 0.8 \geq x_j \geq 0 \end{cases} \tag{8.11}$$

in the sense that their objectives achieve the same minimum values. Moreover, a pair (x^*, ζ^*) *achieves the minimum of* $(P3)$ *if and only if* x^* *achieves the minimum of* $(P2)$ *and* $\zeta^* \in A(\alpha, x^*)$*. In particular, when the interval* $A(\alpha, x^*)$ *reduces to a single point, the minimisation of* $-\xi(x)$ *produces a pair* (x^*, ζ^*) *such that* x^* *maximises the expected return and* ζ^* *gives the corresponding Value-at-Risk.*

Proof. See Krokhmal, Palmquist, and Uryasev (2002). □

If we use the Monte-Carlo approximation of the function $F(\alpha, x, \zeta)$, defined by

$$\widetilde{F}(\alpha, x, \zeta) := \zeta + \frac{1}{1-\alpha} \cdot \frac{1}{S} \sum_{s=1}^{S} \max[f(x, y_s) - \zeta, 0],$$

where y_s denotes the realisation of the uncertainty vector y in scenario s we can rewrite $(P3)$ as

$$(P3') \quad : \quad \begin{cases} -\xi(x) \longrightarrow \min\limits_{x,\zeta} \\ \widetilde{F}(\alpha, x, \zeta) \leq \omega \\ \sum_{j=1}^{J} x_j = 1 \\ 0.8 \geq x_j \geq 0 \end{cases} \tag{8.12}$$

If we finally introduce the dummy-variables ϕ_s, $s = 1, ..., S$ and replace $\widetilde{F}(\alpha, x, \zeta)$ by the linear function

$$\bar{F}(\alpha, \phi, \zeta) := \zeta + \frac{1}{1-\alpha} \cdot \frac{1}{S} \sum_{s=1}^{S} \phi_s,$$

and the constraints

$$\phi_s \geq f(x, y_s) - \zeta, \quad \phi_s \geq 0$$

for all $s = 1, ..., S$, we obtain the equivalent linear optimisation problem

$$(P4) \quad : \quad \begin{cases} -\xi(x) \longrightarrow \min\limits_{x,\zeta} \\ \bar{F}(\alpha, \phi, \zeta) \leq \omega \\ \phi_s \geq f(x, y_s) - \zeta, \quad \phi_s \geq 0, \quad s = 1, ..., S \\ \sum_{j=1}^{J} x_j = 1 \\ 0.8 \geq x_j \geq 0, \quad \zeta \in \mathbb{R} \end{cases} \tag{8.13}$$

which can be solved with standard software. E.g., with the *linprog* function in Matlab.

In the empirical examples of our portfolio optimisation approach with

CVaR constraints we set $\alpha = 0.95$. Figure 8.11 shows the expected returns and the variances of the optimal stock/bond portfolios and of the optimal stock/bond/MBS portfolios for different values of ω. Note that the higher ω, the higher the willingness to accept risk for higher expected returns. The ξ-σ^2 line can again be compared to the efficient frontier in the classical mean-variance concept of Markowitz (1952). Similar to the expected utility maximisation approach of the previous subsection, the expected return of the optimal portfolio can be substantially enhanced by including MBS with the same CVaR constraints. Figures 8.9 and 8.10 show the optimal portfolio composition over time when $\omega = 0.1$. The portfolio weight of the stock index is about 50% in March 2001, at the beginning of the time period considered. This is roughly the same as in the expected utility maximisation approach (see Figures 8.6 and 8.7). Yet, in contrast to the maximum utility approach, the stock index portfolio weight remains almost constant over time. This can be explained by the nature of the CVaR as a portfolio risk measure. Even if interest rates decline and the risk/return profile of fixed-income securities becomes less attractive compared to an investment in equities, the stock index is still the most risky asset and very large portfolio losses can almost exclusively be caused by losses from the equity investment. Thus, the CVaR constraint implies a limit for the portfolio weight of the stock index which does not change substantially over time. It is also interesting to note that, in contrast to the maximum utility approach, for almost all points of time the optimal portfolio does not contain any zero-coupon bonds at all if MBS are allowed.

Finally, Figure 8.8 shows how the wealth of an investor develops over time, if the investor follows our optimal asset allocation strategy from March 2001 to March 2003 according to the portfolio optimisation approach with CVaR constraints. The optimal asset allocation strategy with MBS again outperforms the optimal stock/bond strategy clearly. The overall performance looks of course better than the maximum utility equivalent in Figure 8.8 due to the lower equity exposure.

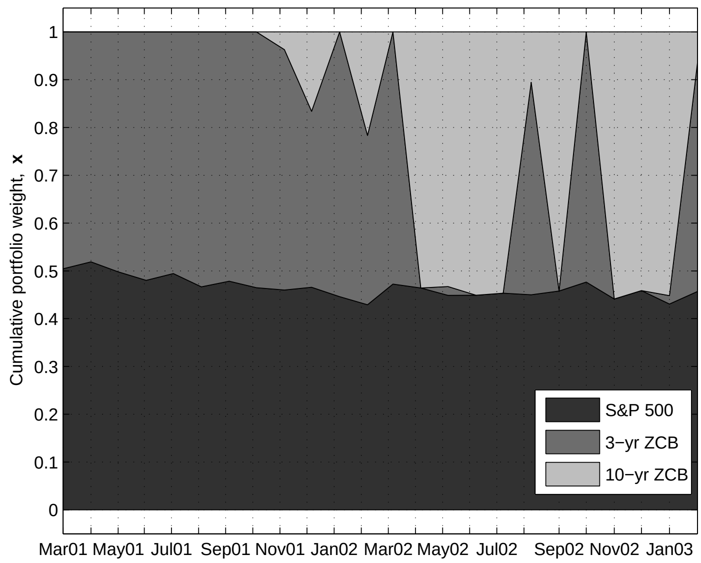

Figure 8.9: Optimal portfolio weights from 2001-2003 in a stock/bond portfolio.

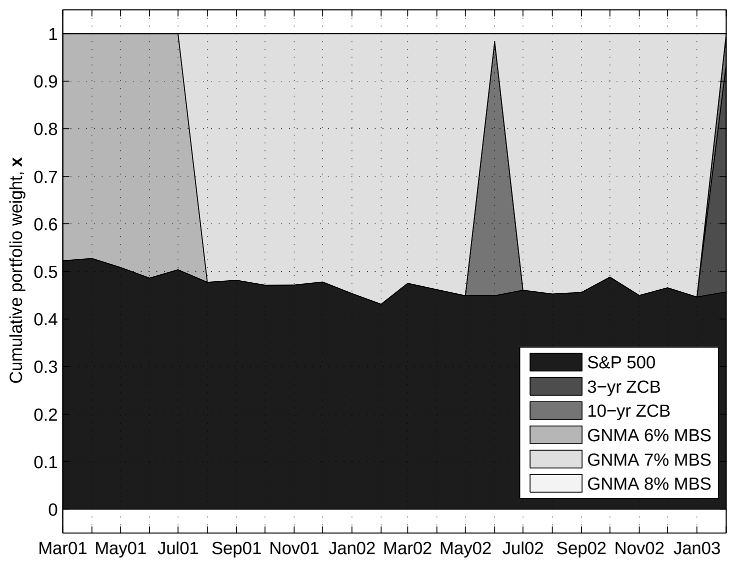

Figure 8.10: Optimal portfolio weights from 2001-2003 in a stock/bond/MBS portfolio.

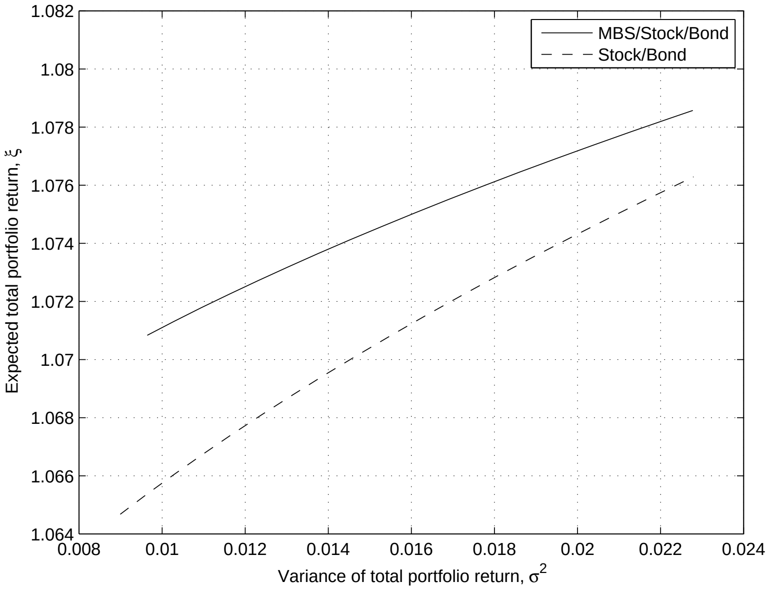

Figure 8.11: Expected return-variance efficient frontier for a stock/bond portfolio (dotted line) and for a stock/bond/MBS portfolio (solid line).

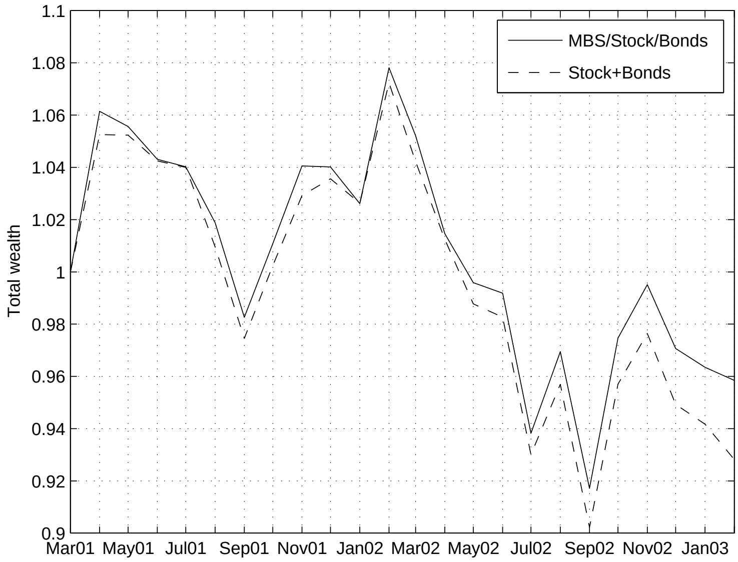

Figure 8.12: Development of the total wealth according to our multi-period optimal asset allocation strategy from 2001-2003 for a stock/bond portfolio (dotted line) and for a stock/bond/MBS portfolio (solid line).

Chapter 9

Valuation and Pricing of Reverse Mortgages

We have already introduced reverse mortgages in Chapter 2 and we have already pointed out that the risk that the total amount of the loan may exceed the house value at termination of the contract is crucial for the valuation and pricing of reverse mortgages. Yet, the question of how to take into account this risk for a consistent valuation approach is not trivial. In a recent paper, Wang et al. (2007) address this issue and consider survivor bonds and survivor swaps for reverse mortgages within an actuarial approach. It is the main contribution of this chapter to develop a consistent framework for the pricing of reverse mortgages with different characteristics under (limited) default risk. Limited default risk means that, making reasonable model assumptions, we calculate the maximum payment(s) to the mortgagor such that the mortgage lender does not experience any losses with a given probability or that the losses of the mortgage lender do not exceed a given amount with a certain probability. These considerations may justify the pricing of a reverse mortgage contract with a particular interest-rate curve (e.g. the EURIBOR/Swap curve, which we use in the empirical examples).

Our mathematical model is again based on a stochastic intensity framework, which we have also used for the valuation of mortgage-backed securities in the previous chapters. In the following we adapt this modelling framework to reverse mortgages and develop, in a first step, formulas for the valuation of fixed-rate and adjustable-rate reverse mortgage contracts in a default-free setting. By default-free valuation setting we mean that, for pricing purposes, we do not take into account the possibility that the total loan amount outstanding at termination of the contract may exceed the house value. Subsequently, we extend these concepts for certain contract specifications and consider the general case where the losses which a mortgage lender may suffer are explicitly taken into account. We determine the maximum payment(s) to the mortgagor such that the total loan amount does not exceed the house value at termination of the reverse mortgage contract with a prespecified probability. Alternatively, we propose a Credit-Value-at-Risk criterion in order to determine the maximum loan amounts. We finally illustrate our model with some examples directed towards the German market and investigate the sensitivity of the results with respect to some model parameters, e.g., with respect to the drift parameter of the house price appreciation process.

9.1 The default-free modelling framework

For the sake of simplicity of the presentation, we will assume in the following that the reverse mortgage is associated with a single person and that there is a maximum age this person can attain. We explicitly consider two basic types of reverse mortgages. The payment for the first type of reverse mortgage, denoted by upfront-payment reverse mortgage in the following, is made as a lump sum at origination of the mortgage contract. The second type of reverse mortgage we consider is a lifetime annuity, where the mortgagor receives regular periodic payments until the contract is terminated. Of course, our concept can be readily applied to combinations of these two basic types and can be easily extended to other forms of reverse mortgages, e.g. to lines of credit, which have become popular in the US. We do not consider any upfront fees in the following. However, it is straightforward to incorporate upfront fees into our modelling framework by simply considering them as an upfront payment which the mortgagor never receives. In any case, we assume that the reverse mortgage contract is only terminated when the mortgagor dies or sells the house and that all accrued interest is added to the outstanding loan amount. For both basic types of reverse mortgages we consider fixed interest-rate agreements and adjustable-rate agreements, denoted by FRRM and ARRM respectively.

We assume that we have a probability space $(\Omega, \mathcal{G}, Q)$ with the three filtrations $\{\mathcal{G}_t\}_{t\geq 0}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathcal{F}^{\mathcal{N}}{}_t\}_{t\geq 0}$ as already introduced in Chapter 3.3. In this chapter we use again a Hull-White interest-rate model. I.e., the short-rate dynamics under the risk-neutral measure $\widetilde{Q}$ are given by

$$dr(t) = (\theta_r(t) - \hat{a}_r r(t))dt + \sigma_r d\widetilde{W}_r \tag{9.1}$$

where $\widetilde{W}_r$ is a $\widetilde{Q}$-Wiener process, $\hat{a}_r := a_r + \lambda_r \sigma_r^2$ with the market price of risk parameter λ_r and a_r, σ_r are some positive constants. The short-rate dynamics under the real-world measure Q are given by

$$dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r. \tag{9.2}$$

Recall from Chapter 3.2.2 that the zero-coupon bond prices

$$P(t,T) := E_{\widetilde{Q}}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t]$$

in the Hull-White model are given by:

$$P(t,T) = e^{\hat{A}(t,T)-\hat{B}(t,T)\cdot r(t)} \tag{9.3}$$

where

$$\begin{aligned}\hat{A}(t,T) &= \int_t^T (\frac{1}{2}\sigma_r^2\hat{B}(l,T) - \theta_r(l)\hat{B}(l,T))dl \\ \hat{B}(t,T) &= \frac{1}{\hat{a}_r}(1-e^{-\hat{a}_r(T-t)}).\end{aligned} \tag{9.4}$$

Now let us assume that until the maximum age which the mortgagor can attain (in the calculations later we assume a maximum age of 125 years), there are K remaining mortgage payment dates. For simplicity let the payment dates be equidistant (e.g., there are regular monthly payments) and denote them by $t_1, ..., t_K$ with $\Delta t := t_{k+1} - t_k$ for all $k \geq 1$. We also assume that the reverse mortgage contract is written in such a way that when the mortgagor dies or sells the house, the contract is only terminated on the next payment date. At the time of termination of the reverse mortgage contract τ, i.e. when the mortgagor dies or sells the house, the outstanding notional plus all accrued interest has to be paid back in a lump sum. We denote this amount by $A(\tau, i)$, where i is the contract rate, determining the accrued interest at the time of termination of the contract. The total amount which has to be paid back is, however, capped at the house sale proceeds (i.e. the house value) which we denote by $H(\tau)$. Thus, all cash flows depend on the time of termination τ. If we denote by $p(t)$ the risk-neutral (annualised, continuous) termination intensity process, we can recall from (3.29) that

$$\widetilde{Q}(\tau \in (t, t+dt]|\mathcal{G}_t) = p(t)dt \tag{9.5}$$

for an arbitrarily small interval dt, if the contract has not been terminated prior to time t. Furthermore, it follows from (3.30) that the probability that the contract is still alive at time t_k (given that the contract has not been terminated prior to time t) can be calculated by

$$\widetilde{Q}(\tau > t_k|\mathcal{F}_t) = E_{\widetilde{Q}}[e^{-\int_t^{t_k} p(s)ds}|\mathcal{F}_t] \cdot 1_{\{\tau>t\}}$$

Applying this result, we can conclude that

$$1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\left[\int_{t_{k-1}}^{t_k} p(u)\cdot e^{-\int_t^u p(s)ds}du\middle|\mathcal{F}_t\right] = \widetilde{Q}(\tau \in (t_{k-1}, t_k]|\mathcal{G}_t) \tag{9.6}$$

is the probability of termination between t_{k-1} and t_k, resulting in the 'recovery payment' at the payment date t_k. Thus, the value $V(t)$ of the mortgage contract at time $t_0 \leq t \leq t_1$ admits the representation

$$\begin{aligned} V(t) &= 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Big[\sum_{k=1}^{K}\Big(- M(t_k)\cdot \Delta t \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds} \\ &\quad + e^{-\int_t^{t_k} r(s)ds} \cdot min(H(t_k), A(t_k,i)) \cdot \int_{t_{k-1}}^{t_k} p(u)\cdot e^{-\int_t^u p(s))ds} du\Big)\Big|\mathcal{F}_t\Big], \end{aligned} \tag{9.7}$$

where $M(t_k)\cdot\Delta t$ is the payment to the mortgage borrower on the payment date t_k. Note that Formula (9.7) is an application of (3.35) to reverse mortgage contracts.
Approximating the integral

$$\int_{t_{k-1}}^{t_k} p(u)e^{-\int_t^u p(s)ds}du \approx p(t_k)\cdot e^{-\int_t^{t_k} p(s)ds}\cdot \Delta t \tag{9.8}$$

and noting that

$$min(H(t), A(t,i)) = A(t,i) - max(A(t,i) - H(t), 0),$$

(9.7) can be written in the form

$$\begin{aligned} V(t) &\approx 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Big[\sum_{k=1}^{K}\Big(- M(t_k)\cdot \Delta t \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds} \\ &\quad + \Delta t \cdot min(H(t_k), A(t_k,i)) \cdot p(t_k) \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Big)\Big|\mathcal{F}_t\Big] \end{aligned} \tag{9.9}$$

$$\begin{aligned} &= 1_{\{\tau>t\}} \cdot \Big\{ E_{\widetilde{Q}}\Big[\sum_{k=1}^{K} -M(t_k)\cdot\Delta t\cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Big|\mathcal{F}_t\Big] \\ &\quad + E_{\widetilde{Q}}\Big[\sum_{k=1}^{K}\Delta t\cdot A(t_k,i)\cdot p(t_k)\cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Big|\mathcal{F}_t\Big] \\ &\quad - E_{\widetilde{Q}}\Big[\sum_{k=1}^{K}\Delta t\cdot max(A(t_k,i) - H(t_k),0)\cdot p(t_k) \\ &\qquad \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Big|\mathcal{F}_t\Big]\Big\} \end{aligned} \tag{9.10}$$

$$\begin{aligned} &=: 1_{\{\tau>t\}} \cdot (V_{app}^{ND}(t) - V_{app}^{D}(t)) \\ &=: V_{app}(t), \end{aligned} \tag{9.11}$$

where $V_{app}^{ND}(t)$ is the value of a reverse mortgage contract without the possibility of loss for the mortgage lender and $V_{app}^{D}(t)$ is the term capturing the 'value of the default risk'. The approximation error in (9.9) is again of the kind R_{t_k} as already introduced in (3.36) and discussed in detail in the appendix. We will concentrate on the reverse mortgage value $V_{app}(t)$ in the following since the difference between (9.7) and (9.11) is negligible for reasonably small Δt (e.g., for $\Delta t = 1/12$, indicating monthly payments). In the remainder of this section we will assume that the last term in (9.10), i.e. the 'value of the default risk' $V_{app}^{D}(t)$, is small enough so that

$$\begin{aligned} V_{app}^{ND}(t) &= 1_{\{\tau>t\}} \cdot E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K}\Bigg(- M(t_k)\cdot \Delta t \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds} \\ &\quad + A(t_k,i)\cdot p(t_k)\cdot \Delta t \cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Bigg)\Bigg|\mathcal{F}_t\Bigg] \qquad (9.12) \end{aligned}$$

is a good approximation of $V(t)$. Evidently, this is only justified if the probability that the house value is smaller than the total loan amount outstanding at termination of the contract is negligibly low. We will extensively discuss this topic and, in particular, the implications for the maximum payments which can be made to the homeowner later in the subsequent section. The more general case, where we have to evaluate (9.11) will also be discussed in the subsequent section for the analytically tractable contract specifications.

We decompose the overall termination intensity $p(t)$ into the two independent parts, $p_d(t)$ and $p_s(t)$ with

$$p(t) = p_d(t) + p_s(t),$$

where $p_d(t)$ denotes the "death intensity process" and $p_s(t)$ denotes the home sale intensity. The separation of the overall contract termination intensity into two independent parts was already used in Chapters 5 and 6 in the context of prepayment modelling for MBS. While p_d can be extracted from life expectancy tables (and is thus assumed to be deterministic in the following), we assume that the home sale intensity process follows a Vasicek process, with dynamics

$$dp_s(t) = (\theta_p - \hat{a}_p p_s(t))dt + \sigma_p d\widetilde{W}_p(t) \qquad (9.13)$$

under the risk-neutral pricing measure $\widetilde{Q}$ and by

$$dp_s(t) = (\theta_p - a_p p_s(t))dt + \sigma_p dW_p(t) \qquad (9.14)$$

under the real-world measure Q, where $\widetilde{W}_p$ (W_p) is a $\widetilde{Q}$ (Q)- Wiener process (independent of the previously defined $\widetilde{W}_r$ (W_r)). With these model specifications we can explicitly calculate the expectations in (9.11).

Lemma 9.1. *In the model set-up as previously introduced it holds that*

$$\begin{aligned} P^d(t,T) &:= E_{\widetilde{Q}}[e^{-\int_t^T (r(s)+p(s))ds}|\mathcal{F}_t] \\ &= P(t,T)\cdot e^{-\int_t^T p_d(s)ds}\cdot e^{\hat{A}^d(t,T)-\hat{C}^d(t,T)\cdot p_s(t)} \end{aligned}$$

with

$$\begin{aligned} \hat{A}^d(t,T) &= \int_t^T \frac{1}{2}\sigma_p^2 C^d(l,T)^2 - \theta_p \hat{C}^d(l,T)dl, \\ &= \left(\frac{\theta_p}{\hat{a}_p} - \frac{\sigma^2}{2\hat{a}_p^2}\right)[\hat{C}^d(t,T) - T + t] - \frac{\sigma^2}{4\hat{a}_p}\cdot \hat{C}^d(t,T)^2 \\ \hat{C}^d(t,T) &= \frac{1}{\hat{a}_p}\left(1 - e^{-\hat{a}_p(T-t)}\right), \end{aligned}$$

Proof. Since the short-rate process $r(t)$ is independent of the termination intensity $p(t)$ and since $p_d(t)$ is deterministic, the lemma follows from the bond pricing formulas in the Vasicek model (see (3.12) and (3.13)). □

Lemma 9.2. *In the model set-up as previously introduced it holds that*

$$E_{\widetilde{Q}}[p(T)\cdot e^{-\int_t^T (r(s)+p(s))ds}|_{\mathcal{F}t}] = P^d(t,T)\cdot f^d(t,T)$$

where

$$f^d(t,T) := -\frac{\partial}{\partial T}\ln\frac{P^d(t,T)}{P(t,T)}$$

is the "termination spread forward rate".

Proof. The lemma follows directly from the well-known result (see, e.g., Schmid (2004), p. 243) saying that

$$E_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl} r(T)|\mathcal{F}_t\right] = -E_{\widetilde{Q}}\left[e^{-\int_t^T r(l)dl}|\mathcal{F}_t\right]\cdot\frac{\partial}{\partial T}\ln P(t,T),$$

which we have already used earlier in this thesis. □

As a further ingredient of our pricing model we need to model the collateral, i.e. the house value, in order to be able to determine the maximum monthly payment to the mortgagor so that the mortgage is adequately collateralised and the pricing with the default-free valuation formula (9.12) is justified. Let us assume that the house price $H(t)$ has the dynamics

$$dH(t) = \mu_H H(t)dt + \sigma_H H(t)dW_H(t) \tag{9.15}$$

where μ_H and σ_H are some positive constants, $W_H(t)$ is a Wiener process (w.r.t. the real-world measure Q), independent of the previously defined W_r, W_p, and $H(0) = H_0$ is the value of the house at origination of the reverse mortgage contract. Thus, the house price $H(t)$ follows a geometric Brownian motion and it follows that the distribution of $H(t)/H_0$ is lognormal with parameters $(\mu_H - \frac{1}{2}\sigma_H^2) \cdot t$ and $\sigma_H^2 \cdot t$, i.e.

$$H(t) \sim H_0 \cdot LN((\mu_H - \frac{1}{2}\sigma_H^2) \cdot t, \sigma_H^2 \cdot t),$$

where $LN(\cdot, a, b)$ denotes the cdf of the lognormal distribution with parameters a and b.

9.1.1 Fixed-rate reverse mortgages

If the interest-rate agreement of the reverse mortgage contract is a fixed-rate agreement with interest rate i (expressed as annual rate with discrete compounding), we obtain:

Theorem 9.3. *The value at time $t_0 \leq t \leq t_1$ of a "default-free" upfront-payment FRRM with initial payment M_0 and fixed interest rate i is given by:*

$$\begin{aligned} V_{FRRM}^{UP}(t,i) &= 1_{\{\tau>t\}} \cdot \Bigg(- M_0 \cdot P^d(t,t_1) \\ &\quad + M_0 \cdot \sum_{k=1}^{K} (1 + i \cdot \Delta t)^{k-1} \cdot \Delta t \cdot P^d(t,t_k) \cdot f^d(t,t_k) \Bigg). \end{aligned}$$

Proof. Noting that

$$A(t_k, i) = M_0 \cdot (1 + i \cdot \Delta t)^{k-1}$$

for $k = 1, ..., K$, the theorem follows directly from (9.12). □

Theorem 9.4. *The value at time $t_0 \leq t \leq t_1$ of a "default-free" lifetime annuity FRRM with a periodic payment of $M \cdot \Delta t$ and fixed interest rate i is given by:*

$$\begin{aligned} V_{FRRM}^{LA}(t;i) &= 1_{\{\tau>t\}} \cdot M \cdot \Delta t \cdot \sum_{k=1}^{K} \Bigg(- P^d(t,t_k) \\ &\quad + \frac{(1 + i \cdot \Delta t)^k - 1}{i} \cdot P^d(t,t_k) \cdot f^d(t,t_k) \Bigg). \end{aligned}$$

Proof. Noting that

$$\begin{aligned} A(t_k, i) &= M \cdot \Delta t \cdot \sum_{j=0}^{k-1} (1 + i \cdot \Delta t)^j \\ &= M \cdot \Delta t \cdot \frac{(1 + i \cdot \Delta t)^k - 1}{i \cdot \Delta t}, \end{aligned} \tag{9.16}$$

the theorem follows again directly from (9.12). □

9.1.2 Adjustable-rate reverse mortgages

In an adjustable-rate contract, the interest rate is not constant, but is adjusted at given fixing dates, so that it reflects prevailing market conditions during the whole lifetime of the product. For simplicity, we assume in the following that the fixing dates coincide with the payment dates and that on the payment date t_k the interest rate is fixed for the subsequent interest-rate period from t_k to t_{k+1}. Usually, for adjustable-rate contracts the reference index rate is a LIBOR (or EURIBOR) rate. LIBOR rates are simple money-market interest rates and can be considered as short-term riskless rates. We denote by $L(t_k) := L(t_k, t_{k+1})$ the LIBOR rate for the time period from t_k to t_{k+1}, by s^L some spread on the LIBOR rate. In the following, we use the approximation

$$(1 + (L(t_k) + s^L) \cdot \Delta t) \approx e^{(r(t_k)+s^L)\cdot \Delta t}. \tag{9.17}$$

Theorem 9.5. *The value at time $t_0 \leq t \leq t_1$ of a 'default-free' adjustable-rate reverse mortgage with initial payment M_0, i.e. $A(t_1) = M_0$, and fixed spread s^L on the (simple) reference index rate is given by*

$$\begin{aligned} V^{UP}_{ARRM}(t, \Delta t, s^L) &\approx 1_{\{\tau > t\}} \cdot \Bigg(- M_0 \cdot P^d(t, t_1) + M_0 \cdot \Delta t \cdot P(t, t_1) \\ &\quad \cdot \sum_{k=1}^{K} f^d(t, t_k) \cdot \frac{P^d(t, t_k)}{P(t, t_k)} \cdot e^{s^L \cdot (t_k - t_1)} \Bigg). \end{aligned}$$

Proof. Note that using (9.17) we get

$$\begin{aligned} A(t_k, s^L) &= M_0 \cdot \prod_{j=1}^{k-1} (1 + (L(t_j) + s^L) \cdot \Delta t) \\ &\approx M_0 \cdot e^{\sum_{j=1}^{k-1} (r(t_j)+s^L)\cdot \Delta t)} \\ &\approx M_0 \cdot e^{\int_{t_1}^{t_k} r(s)ds} \cdot e^{s^L \cdot (t_k - t_1)}. \end{aligned}$$

Then, it follows from (9.12) that

$$
\begin{aligned}
V_{ARRM}^{UP}(t,\Delta t,s^L) &= 1_{\{\tau>t\}}\cdot\Bigg(-M_0\cdot P^d(t,t_1)+E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K}M_0 \\
&\cdot\Bigg(\prod_{j=1}^{k-1}(1+(L(t_j)+s^L)\cdot\Delta t)\Bigg)\cdot p(t_k)\cdot\Delta t\cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Bigg|\mathcal{F}_t\Bigg]\Bigg) \\
&\approx 1_{\{\tau>t\}}\cdot\Bigg(-M_0\cdot P^d(t,t_1)+E_{\widetilde{Q}}\Bigg[\sum_{k=1}^{K}M_0\cdot\Delta t \\
&\cdot e^{\int_{t_1}^{t_k}r(s)ds}\cdot e^{s^L\cdot(t_k-t_1)}\cdot p(t_k)\cdot e^{-\int_t^{t_k}(r(s)+p(s))ds}\Bigg|\mathcal{F}_t\Bigg]\Bigg),
\end{aligned}
$$

which further simplifies to

$$
\begin{aligned}
V_{ARRM}^{UP}(t,\Delta t,s^L) &\approx 1_{\{\tau>t\}}\cdot\Bigg(-M_0\cdot P^d(t,t_1) \\
&+M_0\cdot\Delta t\cdot\sum_{k=1}^{K}\Bigg\{e^{s^L\cdot(t_k-t_1)}\cdot E_{\widetilde{Q}}\left[e^{\int_t^{t_1}r(s)ds}|\mathcal{F}_t\right] \\
&\cdot E_{\widetilde{Q}}\left[p(t_k)\cdot e^{-\int_t^{t_k}p(s)ds}|\mathcal{F}_t\right]\Bigg\}\Bigg) \\
&= 1_{\{\tau>t\}}\cdot\Bigg(-M_0\cdot P^d(t,t_1)+M_0\cdot\Delta t\cdot P(t,t_1) \\
&\cdot\sum_{k=1}^{K}f^d(t,t_k)\cdot\frac{P^d(t,t_k)}{P(t,t_k)}\cdot e^{s^L\cdot(t_k-t_1)}\Bigg).
\end{aligned}
$$

□

Theorem 9.6. *The value at time $t_0\le t\le t_1$ of a 'default-free' adjustable-rate reverse mortgage with a periodic payment of $M\cdot\Delta t$ and a fixed spread s^L on the (simple) reference index rate is given by*

$$
\begin{aligned}
V_{ARRM}^{LA}(t,\Delta t,s^L) &= 1_{\{\tau>t\}}\cdot\sum_{k=1}^{K}\Bigg(-M\cdot\Delta t\cdot P^d(t,t_k)+P(t,t_k)\cdot M\cdot\Delta t \\
&\cdot\sum_{m=k}^{K}\Delta t\cdot e^{s^L\cdot(t_m-t_k)}\cdot f^d(t,t_m)\cdot\frac{P^d(t,t_m)}{P(t,t_m)}\Bigg).
\end{aligned}
$$

Proof. For notational convenience, we define

$$V(t) := V^{LA}_{ARRM}(t, \Delta t, s^L).$$

Then, according to (9.12)

$$\begin{aligned}
V(t) &= 1_{\{\tau>t\}} \cdot \sum_{k=1}^{K} \Bigg\{ -M \cdot \Delta t \cdot P^d(t, t_k) \\
&\quad + E_{\widetilde{Q}} \Bigg[\Bigg(\sum_{l=0}^{k-1} M \cdot \Delta t \prod_{j=k-l}^{k-1} (1 + (L(t_j) + s^L) \cdot \Delta t) \Bigg) \\
&\quad \cdot \Delta t \cdot p(t_k) \cdot e^{-\int_t^{t_k} (r(s)+p(s))ds} \Bigg| \mathcal{F}_t \Bigg] \Bigg\} \\
&\approx 1_{\{\tau>t\}} \cdot \sum_{k=1}^{K} \Bigg\{ -M \cdot \Delta t \cdot P^d(t, t_k) + M \cdot \Delta t \\
&\quad \cdot \sum_{l=0}^{k-1} P(t, t_{k-l}) \cdot e^{s^L \cdot (t_k - t_{k-l})} \cdot E_{\widetilde{Q}} \left[\Delta t \cdot p(t_k) \cdot e^{-\int_t^{t_k} p(s)ds} | \mathcal{F}_t \right] \Bigg\}.
\end{aligned}$$

By applying Lemma 9.2 we get

$$\begin{aligned}
V(t) &= 1_{\{\tau>t\}} \cdot \sum_{k=1}^{K} \Bigg(-M \cdot \Delta t \cdot P^d(t, t_k) + M \cdot \Delta t \\
&\quad \cdot \sum_{l=0}^{k-1} P(t, t_{k-l}) \cdot e^{s^L \cdot (t_k - t_{k-l})} \cdot f^d(t, t_k) \cdot \frac{P^d(t, t_k)}{P(t, t_k)} \cdot \Delta t \Bigg).
\end{aligned}$$

Noting that

$$\sum_{k=1}^{K} \sum_{l=0}^{k-1} a_{k-l,k} = \sum_{k=1}^{K} \sum_{m=k}^{K} a_{k,m}$$

finally yields

$$\begin{aligned}
V(t) &= 1_{\{\tau>t\}} \cdot \sum_{k=1}^{K} \Bigg(-M \cdot \Delta t \cdot P^d(t, t_k) + M \cdot \Delta t \\
&\quad \sum_{m=k}^{K} P(t, t_k) \cdot e^{s^L \cdot (t_m - t_k)} \cdot \Delta t \cdot f^d(t, t_m) \cdot \frac{P^d(t, t_m)}{P(t, t_m)} \Bigg).
\end{aligned}$$

□

Note that it follows directly from Theorem 9.5 and Theorem 9.6 that if $s^L = 0$, i.e. if there is no spread on the index rate, the value of the reverse mortgage contract is equal to the outstanding loan amount on any (next) payment and fixing date t_1 (up to a very small discretisation error resulting from the discretisation in (9.11)). In order to see this, note that with $t = t_1$ the term

$$\sum_{k=1}^{K} P(t, t_1) \cdot f^d(t, t_k) \cdot \frac{P^d(t, t_k)}{P(t, t_k)} \cdot \Delta t \cdot e^{s^L \cdot (t_k - t_1)}$$

reduces to

$$\sum_{k=1}^{K} f^d(t_1, t_k) \cdot \frac{P^d(t_1, t_k)}{P(t_1, t_k)} \cdot \Delta t \approx \sum_{k=1}^{K} \widetilde{Q}(\tau = t_k | \mathcal{F}_{t_1}) = 1,$$

given that $\tau > t_0$. Furthermore, note that with $s^L = 0$ and $t = t_1$ for the lifetime annuity ARRM, the term

$$\sum_{k=1}^{K} P(t, t_k) \cdot M \cdot \Delta t \cdot \sum_{m=k}^{K} \Delta t \cdot e^{s^L \cdot (t_m - t_k)} \cdot f^d(t, t_m) \cdot \frac{P^d(t, t_m)}{P(t, t_m)}$$

reduces to

$$\begin{aligned} &\sum_{k=1}^{K} P(t_1, t_k) \cdot M \cdot \Delta t \cdot \sum_{m=k}^{K} \widetilde{Q}(\tau = t_m | \mathcal{F}_{t_1}) \\ &= \sum_{k=1}^{K} P(t_1, t_k) \cdot M \cdot \Delta t \cdot \widetilde{Q}(\tau \geq t_k | \mathcal{F}_{t_1}) \\ &= \sum_{k=1}^{K} P^d(t_1, t_k) \cdot M \cdot \Delta t, \end{aligned}$$

given that $\tau > t_1$. I.e., the net present value of the sum of all future payments is equal to 0 and it follows that

$$V_{ARRM}^{LA}(t_j, \Delta t, 0) = 0$$

The fact that the present value of an ARRM with $s^L = 0$ is equal to the outstanding loan amount on any fixing date is perfectly consistent with the valuation principles of ordinary floating rate notes. The value of any floating-rate note (independent of the maturity) must always be equal to the outstanding notional on any fixing date, as long as the total notional is fully recovered at termination, i.e. as long as the floating-rate note is default-free.

9.2 Introducing default risk

In the first part of this section we address the question of how much a mortgagor can borrow, so that the "default-free" considerations are justified. Recall that this is only the case if the probability that the total outstanding loan amount exceeds the house value at termination of the contract is negligibly low. Let us assume that the house value exceeds the total loan amount outstanding at termination of the contract with a probability of at least $q \in (0,1)$. Given such a probability q, at the time of origination of the contract t_0, the mortgage lender has to limit the initial payment M_0 (for an upfront-payment FRRM) or the periodic payment $M \cdot \Delta t$ (for a lifetime annuity FRRM) in such a way that

$$Q(H(\tau) < A(\tau,i)) = \sum_{k=1}^{K} Q(H(\tau) < A(\tau,i)|\tau = t_k) \cdot Q(\tau = t_k) \quad \leq \quad 1-q \tag{9.18}$$

holds (we will call this criterion 'q-criterion' in the following). Since

$$Q(H(\tau) < A(\tau,i)|\tau = t_k) = LN(\frac{A(t_k,i)}{H_0}; (\mu_H - \frac{1}{2}\sigma_H^2)\cdot t_k, \sigma_H^2 \cdot t_k),$$

it follows that (9.18) is equivalent to

$$\sum_{k=1}^{K} LN(\frac{A(t_k,i)}{H_0}; (\mu_H - \frac{1}{2}\sigma_H^2)\cdot t_k, \sigma_H^2 \cdot t_k) \cdot Q(\tau = t_k) \quad \leq \quad 1-q. \tag{9.19}$$

We can calculate $Q(\tau = t_k)$ using the following theorem.

Theorem 9.7. *In the model set-up as previously introduced it holds that*

$$\begin{aligned} Q(\tau = t_k) \quad \approx \quad & \Big(\Delta t \cdot p_d(t_k) \cdot e^{-\int_t^{t_k} p_d(s)ds} \cdot P^s(t,t_k) \\ & -\Delta t \cdot e^{-\int_t^{t_k} p_d(s)ds} \cdot P^s(t,t_k) \cdot \frac{\partial}{\partial t_k} \ln P^s(t,t_k)\Big) \cdot 1_{\{\tau > t\}}, \end{aligned}$$

where

$$P^s(t,t_k) := E_Q[e^{-\int_t^{t_k} p_s(u)du}] = e^{A^d(t,T) - C^d(t,T)\cdot p_s(t)}$$

with

$$\begin{aligned} A^d(t,T) &= \left(\frac{\theta_p}{a_p} - \frac{\sigma^2}{2a_p^2}\right)[C^d(t,T) - T + t] - \frac{\sigma^2}{4a_p} \cdot C^d(t,T)^2 \\ C^d(t,T) &= \frac{1}{a_p}\left(1 - e^{-a_p(T-t)}\right) \end{aligned}$$

Proof. First, recall (9.6) and the approximation (9.8). Noting that $p_d(t)$ is deterministic we have to calculate $E_Q[e^{-\int_t^{t_k} p_s(u)du}]$, where the expectation is taken under the real-world measure Q. This expectation can directly be calculated using solely the distribution of the home sale intensity implied by its real-world dynamics as given in (9.14). Since the distribution of $p_s(t)$ is normal, it can be shown that the distribution of $\int_t^{t_k} p_s(u)du$ is also normal. Thus, the distribution of $e^{-\int_t^{t_k} p_s(u)du}$ is lognormal and its expectation follows from the expectation and variance of $\int_t^{t_k} p_s(u)du$, which yields the formulas as stated above (for a detailed derivation see, e.g., Mamon (2004)). Furthermore, if we assume as usual that $E_Q[e^{-\int_t^{t_k} p_s(u)du}] < \infty$ and note that $e^{-\int_t^{t_k} p_s(u)}$ is continuously differentiable w.r.t. t_k (for almost all $\omega \in \Omega$, it holds that

$$\begin{aligned}\frac{\partial}{\partial t_k} \ln P^s(t,t_k) &= \frac{1}{P^s(t,t_k)} \cdot E_Q\left[\frac{\partial}{\partial t_k} e^{-\int_t^{t_k} p_s(u)du}\right] \\ &= -\frac{1}{P^s(t,t_k)} \cdot E_Q[p_s(t_k) \cdot e^{-\int_t^{t_k} p_s(u)du}].\end{aligned}$$

I.e., we obtain

$$E_Q[p_s(t_k) \cdot e^{-\int_t^{t_k} p_s(u)du}] = -P^s(t,t_k) \cdot \frac{\partial}{\partial t_k} \ln P^s(t,t_k).$$

Note that we have already used a similar result earlier for the short-rate process under the risk-neutral martingale measure. This completes the proof. □

Since the left-hand side of (9.19) is a strictly increasing function in M_0 and M for an upfront-payment FRRM and a lifetime annuity FRRM respectively, we get the maximum initial payment M_0^* and the maximum periodic payment M^* if we replace "$\leq$" with "$=$" in (9.19) and solve the equation for M_0 and M respectively. This can be done by standard methods, e.g. with the *fzero* function in Matlab.

The question of how much a mortgagor can borrow so that (9.19) holds for a certain probability q, is much more difficult for ARRM contracts than for FRRM contracts since in a contract with a floating interest rate the quantity $A(t_k)$ is stochastic. Within our Hull-White interest-rate model framework, the quantity $A(t_k)$ in a lifetime annuity ARRM is basically a sum of correlated lognormal random variables. We would thus have to compare a sum of correlated lognormal random variables with the house value, which is also lognormally distributed. It is a well known fact, e.g. from the pricing of arithmetic average Asian options that the sum of lognormal random variables is not lognormally distributed. While some closed-form approximations have been suggested in the literature (see, e.g., Milevsky and Posner (1998)), these can not be used for our problem since we would finally have to find a distribution for the difference between the lognormally distributed house value and the approximate distribution of $A(t_k)$, which is infeasible. Thus, for a lifetime annuity ARRM, the only way to determine the maximum periodic payment M^* is by simulation. The situation looks better for an upfront-payment ARRM. For an upfront-payment ARRM we can reasonably approximate the outstanding loan amount at time t_k by

$$A(t_k) \approx M_0 \cdot e^{\int_0^{t_k} r(s)ds} \cdot e^{s^L \cdot t_k}.$$

We already know from Lemma 3.4 that the distribution of $\int_0^{t_k} r(s)ds$ is normal in the Hull-White model and we obtain:

$$\int_0^{t_k} r(s)ds \quad \sim \quad N(a_k; V(0, t_k)), \tag{9.20}$$

where

$$\begin{aligned} a_k &:= -\ln P(0, t_k) + \frac{\sigma_r^2}{2a_r^2} \cdot [t_k - 2 \cdot B(0, t_k) \\ &\quad + \frac{1}{2a_r}(1 - e^{-2a_r \cdot t_k})] \\ V(0, t_k) &:= \frac{\sigma_r^2}{a_r^2}\left(t_k + \frac{2}{a_r}e^{-a_r t_k} - \frac{1}{2a_r}e^{-2a_r t_k} - \frac{3}{2a_r}\right) \end{aligned} \tag{9.21}$$

and $B(0, t_k)$ is as given in (3.17).

Now, note that (9.18) is equivalent to

$$\sum_{k=1}^{K} Q\left(\log \frac{H(\tau)}{H_0} - \log \frac{A(\tau)}{M_0} < \log M_0 - \log H_0 \middle| \tau = t_k\right) \cdot Q(\tau = t_k) \quad \leq \quad 1 - q.$$

Thus, in order to get the maximum initial payment M_0^* we have to solve the equation

$$\sum_{k=1}^{K} \Phi\left(\frac{\log M_0 - \log H_0 - ((\mu_H - \frac{1}{2}\sigma_H^2) \cdot t_k - a_k - s^L \cdot (t_k - t_1))}{\sqrt{\sigma_H^2 \cdot t_k + V(0, t_k)}}\right) \cdot Q(\tau = t_k) = 1 - q,$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution and $Q(\tau = t_k)$ can be calculated as stated in Theorem 9.7.

All calculations so far have been based on "default-free" considerations. This, as previously explained, is only justified if the maximum payments to the homeowner are sufficiently low. This, however, may make a reverse mortgage unattractive to some homeowners. When taking into account the potential losses the mortgage lender faces if the total loan amount exceeds the house value at termination of the contract, we have to work with the exact (up to the very small discretisation error) valuation formula (9.11). The following considerations are, however, only analytically well tractable for fixed-rate contracts. We will thus focus on these contracts and point out that for adjustable-rate contracts formula (9.11) can, of course, always be evaluated by simulation[7].

If we assume that the house price dynamics under the risk-neutral pricing measure are given by

$$dH(t) = r(t)dt + \sigma_H d\widetilde{W}_H(t) \tag{9.22}$$

we obtain the following versions of Theorems 9.3 and 9.4:

Theorem 9.8. *The value at time $t_0 \leq t \leq t_1$ of an upfront-payment FRRM with initial payment M_0, fixed interest rate i and recovery capped at the house value is given by:*

$$\begin{aligned} V^{UP}_{dFRRM}(t,i) &= 1_{\{\tau>t\}} \cdot \Bigg(- M_0 \cdot P^d(t,t_1) + \sum_{k=1}^{K} \Bigg\{ M_0 \cdot (1 + i \cdot \Delta t)^{k-1} \\ &\quad \cdot \Delta t \cdot P^d(t,t_k) \cdot f^d(t,t_k) \\ &\quad - \Delta t \cdot \frac{P^d(t,t_k)}{P(t,t_k)} \cdot f^d(t,t_k) \cdot PUT(t,t_k,A(t_k,i)) \Bigg\} \Bigg), \end{aligned}$$

[7] An alternative to simulation is the use of another approximation. The term

$$E_{\widetilde{Q}}[max(A(t_k) - H(t_k), 0) \cdot p(t_k) \cdot e^{\int_t^{t_k}(r(s)+p(s))ds} | \mathcal{F}_t]$$

can be written as

$$E_{\widetilde{Q}}[max(A(t_k) - H(t_k), 0) \cdot e^{\int_t^{t_k} r(s)ds} | \mathcal{F}_t] \cdot E_{\widetilde{Q}}[p(t_k) \cdot e^{\int_t^{t_k} p(s)ds} | \mathcal{F}_t],$$

where the first expectation can be calculated using Margrabe's exchange option formula, if one assumes that the interest rates for discounting are independent of those used for determining the outstanding loan amount $A(t_k)$ (see Margrabe (1978)) and the second expectation can be calculated using previously developed results.

where

$$\begin{aligned} PUT(t,t_k,A(t_k,i)) &:= A(t_k,i)\cdot P(t,t_k) \\ &\quad \cdot\Phi\left(-\frac{\ln\frac{H(t)}{A(t_k,i)\cdot P(t,t_k)}-\frac{1}{2}v^2(t,t_k)}{v(t,t_k)}\right) \\ &\quad -H(t)\cdot\Phi\left(-\frac{\ln\frac{H(t)}{A(t_k,i)\cdot P(t,t_k)}+\frac{1}{2}v^2(t,t_k)}{v(t,t_k)}\right), \\ v(t,t_k) &:= \hat{V}(t,t_k)+\sigma_H^2(t_k-t) \end{aligned}$$

and $\hat{V}(t,t_k)$ is similar to $V(t,t_k)$ as defined in (9.21) if a_r is replaced with $\hat{a}_r$.

Proof. Recalling Lemma 9.2 and Theorem 9.3 it remains to show that

$$E_{\widetilde{Q}}[max(A(t_k,i)-H(t_k),0)\cdot e^{-\int_t^{t_k} r(s)ds}] = PUT(t,t_k,A(t_k,i))$$

with $PUT(t,t_k,A(t_k,i))$ as defined above. The expression $max(A(t_k,i)-H(t_k),0)$ is, however, simply the payoff of a European put option w.r.t. the underlying $H(t)$, strike price $A(t_k,i)$ and maturity t_k. We can thus apply the put option formula in the Black-Scholes framework with stochastic interest rates (see, e.g., Brigo and Mercurio (2006), p. 888, for the case when interest rates follow the Hull-White 1-factor model, as we assume in our model). □

Theorem 9.9. *The value at time $t_0 \le t \le t_1$ of a lifetime annuity FRRM with a periodic payment of $M\cdot\Delta t$, fixed interest-rate i and recovery capped at the house value is given by:*

$$\begin{aligned} V_{dFRRM}^{LA}(t;i) &= 1_{\{\tau>t\}}\cdot\sum_{k=1}^{K}\Bigg\{-M\cdot\Delta t\cdot P^d(t,t_k) \\ &\quad +M\cdot\Delta t\cdot\frac{(1+i\cdot\Delta t)^k-1}{i}\cdot P^d(t,t_k)\cdot f^d(t,t_k) \\ &\quad -\Delta t\cdot\frac{P^d(t,t_k)}{P(t,t_k)}\cdot f^d(t,t_k)\cdot PUT(t,t_k,A(t_k,i))\Bigg\}, \end{aligned}$$

where $PUT(t,t_k,A(t_k,i))$ is defined as in Theorem 9.8.

Proof. The Theorem follows directly from Theorems 9.4 and 9.8. □

If we explicitly take into account the losses the mortgage lender possibly suffers, it seems appropriate to limit the maximum payments to the home-owner in such a way that the lender's losses do not exceed a certain amount with a given probability. In some sense, this is equivalent to setting a Credit Value-at-Risk (CreditVaR) limit.

The CreditVaR of a reverse mortgage contract could, e.g., simply be a fraction of the initial house value. If we set this fraction to, say, 0.1 and the CreditVaR-probability level to α we have to limit the payments to the homeowner in such a way that

$$\begin{aligned} Q(A(\tau,i)-H(\tau)>0.1\cdot H_0) &= \\ \sum_{k=1}^{K} Q(A(\tau,i)-H(\tau)>0.1\cdot H_0|\tau=t_k)\cdot Q(\tau=t_k) &\leq 1-\alpha. \end{aligned}$$

I.e. we have to solve

$$\begin{aligned} \sum_{k=1}^{K} LN(\frac{max(0,A(t_k,i)-0.1\cdot H_0)}{H_0};(\mu_H-\frac{1}{2}\sigma_H^2)\cdot t_k,\sigma_H^2\cdot t_k) & \\ \cdot Q(\tau=t_k) &= 1-\alpha \end{aligned}$$

to obtain the maximum initial payment $M_{0,d}^*$ or the maximum periodic payment M_d^*.

In the last part of this section we briefly investigate the difference between the valuation formulas in the "default-free" setting of Chapter 9.1 and the formulas in this section where we explicitly take into account the possibility of losses for the mortgage lender. This difference is, of course, the quantity

$$V_{app}^D(t)=\sum_{k=1}^{K}\Delta t\cdot\frac{P^d(t,t_k)}{P(t,t_k)}\cdot f^d(t,t_k)\cdot PUT(t,t_k,A(t_k,i))$$

which we have labelled "value of the default risk". Note that, assuming a complete market where the dynamics of the house price process under the risk-neutral pricing measure are given by (9.22), the value of the PUT options does not depend on the parameter μ_H. However, it does depend on the volatility of the house price process, i.e. on σ_H. If the volatility σ_H increases, as in the current housing turmoil in the US, the value of the default risk increases. We will exemplarily illustrate this in the following section.

9.3 Results and implications

In this section we discuss the available data and shortly comment on the techniques which we need to estimate the model parameters. With the obtained parameter estimates (and assumptions where there is no reliable data in order to estimate the parameters statistically) we then illustrate the previously introduced modelling approach.

Let us start with the "default-free" model set-up as discussed in Chapter 9.1 and with the probability q. Recall that q has to be chosen close to 1 in order to justify the "default-free" considerations. As risk-free interest-rate curve we use the LIBOR/Swap curve in this paper. In practice loans are often priced with the LIBOR/Swap curve since this is the common benchmark curve for a bank's or mortgage lender's liabilities. The Global Average One-Year Rating Transition Rates for global corporates as of 2006 (with data from 1981-2005; Source: Standard & Poor's CreditPro 7.02) yield that a 25-year loan of a AAA-rated company defaults with a probability of approximately 4%. Thus, we can conclude for our purposes that by setting $q = 96.5\%$ a reverse mortgage loan can be considered as virtually default-free and that the use of the LIBOR/Swap or the EURIBOR/Swap curve as risk free curve is adequate. We have historical EURO EURIBOR/Swap rates from 1999 to 2007 for maturities from 6 months to 30 years for the estimation of the parameters in the Hull-White interest-rate model. We estimate the parameters with Kalman filter techniques as described in Chapter 5. By using the Kalman filter with historical data we can estimate the dynamics of the short-rate under the real-world measure Q and under the risk-neutral pricing measure $\widetilde{Q}$ simultaneously. The results are reported in Table 9.1.

The mortality rates which we need to determine the (deterministic) mortality intensity process can easily be extracted from life expectancy tables. In this paper we use the 2004 R mortality rates of the first order published by the *Deutsche Aktuarsvereinigung* [association for actuaries in Germany] which are the current standard for pension insurance calculations in Germany. We thus obtain a piecewise constant mortality intensity $p_d(l)$ from year l to $l+1$ given by

Parameter	Estimate
a_r	0.0518
$\hat{a}_r$	0.0177
σ_r	0.0065
h_r	0.0003

Table 9.1: Estimates of the interest-rate model parameters where h_r is the measurement std. error.

$$p_d(l) = \log(1 - Q(\tau_d \in (l, l+1] | \tau_d > l)), \qquad (9.23)$$

where τ_d is the stopping time associated with the death of the mortgagor and the probability on the right hand side in (9.23) is the value published by the *Deutsche Aktuarsvereinigung*. So far, we have only considered reverse mortgages for a single mortgagor. If the reverse mortgage is associated with two persons, all conclusions remain valid if we consider the supremum of the stopping times $\tau_d^{(1)}$ and $\tau_d^{(2)}$, associated with the death of person 1 and person 2 respectively, instead of τ_d for a single person. The survival and termination probabilities associated with $sup(\tau_d^{(1)}, \tau_d^{(2)})$ are straightforward to calculate from the same mortality table as long as $\tau_d^{(1)}$ can be assumed to be independent of $\tau_d^{(2)}$. This is certainly not a too restrictive assumption when the reverse mortgage is, e.g., related to a couple.

In order to fully specify the overall intensity process $p(t)$ we need to have values for the parameters θ_p, a_p, $\hat{a}_p$ and σ_p, which determine the dynamics of the home sale intensity process (under the real-world measure and under the risk-neutral pricing measure). Unfortunately, we do not have any reliable data available to estimate these parameters for the German market. For the US market, some rough numbers for mobility rates of the elderly are available (see Davidoff and Welke (2005) and the references therein, who report an annual mobility rate of approximately 4% among older single women). However, whether any of these data can be used to estimate the house sale intensity process in Germany is highly questionable. We therefore have to resort to reasonable assumptions at this stage. Noting that the lower the house sale intensity, the higher the probability that the mortgage lender experiences a loss, we make conservative assumptions for the parameters and set $\theta_p = 0.01$, $a_p = \hat{a}_p = 0.5$ and $\sigma_p = 0.002$. These parameter values yield a mean-reversion level of $\theta_p/a_p = 0.02$, i.e. we assume that the long-term average of the house sale intensity is just 2%. Note also that by setting $a_p = \hat{a}_p$ we assume that the house sale intensity process has the same dynamics under the real-world and under the risk-neutral pricing measure.

In general, the parameters μ_H and σ_H of the house value process can be expected to vary according to the specific characteristics and location of the house. It is thus highly recommended that these parameters incorporate expert assumptions based on the individual property. In this paper, we estimate the parameters from the quarterly house price index of newly constructed single-family homes in Germany from 2000-2006, published by the German Federal Statistical Office. The simple moment estimators yield $\mu_H = 0.0037$ and $\sigma_H = 0.0047$ (note that house price appreciation has been very low in Germany since the mid-1990s). It is possible to combine these parameter estimates obtained from general data with individual expert assumptions, e.g., via a Bayesian approach or with the well-known method proposed by Black and Litterman (1992).

With the previously given parameter values we will discuss some examples in the following. Table 9.2 shows how a reverse mortgage contract could look like for a female and for a male person at the age of 65 and 70. In this example our calculations are based on the EURIBOR/Swap curve as of 05-Feb-2007 (with money market/par rates from approximately 3.78% for the 3-month EURIBOR to 4.40% for the 30yr Swap rate). Furthermore, we assume that the house, which the reverse mortgage contract is related to, is worth EUR 500,000 at the time of origination of the mortgage contract. In Table 9.2, i_0^* and i^* denote the interest rate, which would make the reverse mortgage contract "fair" according to the "default-free" formulas as developed in Chapter 9.1, i.e. for which $V_{FRRM}^{UP}(0, i_0^*) = 0$ and $V_{FRRM}^{LA}(0, i^*) = 0$ hold respectively. Hence, i^* can be considered in some sense as the internal rate of return of the contract. M_0^* and M^* are the maximum payments as defined previously (with $q = 96.5\%$ and $H_0 = 500,000$) in the default-free setting using the internal rate of return of the corresponding contract plus a spread of 150 basis points. Note that M_0^* is rounded to the nearest multiple of EUR 5,000 below the actual value and that M^* is rounded to the nearest multiple of 10 below the actual value. Of course, the maximum payments are sensitive to the model parameters and inputs. Sensitivity tests have shown that the maximum payments depend in particular on the expected house price appreciation (i.e. on the parameter μ_H) and on the initial yield curve. This result is not surprising and perfectly in line with Eschtruth et al. (2006) who report that a 65-year old reverse mortgagor in the US could expect to receive only 5% of the house value as a lump sum in 1981, when interest rates were at historical highs, and as much as 51% in 2002 when interest rates dropped to historical lows. In order to illustrate the sensitivity of the maximum payments with respect to μ_H and with respect to the initial yield curve at origination of the contract, we consider the case of an upfront-payment FRRM for a 70-year old female homeowner more closely. We recalculate the maximum payment M_0^* in this case for different values of μ_H and for different parallel shifts of the initial yield curve. The results are shown in Figures 9.1 and 9.2. E.g., for an expected yearly house price appreciation of 3% and an initial yield curve between 2.8% and 3.4% (corresponding to a 100 basis points downward shift of the yield curve) the 70-year old homeowner could get a maximum upfront-payment of 252,500, i.e. more than 50% of the current house value.

We finally calculate the value of the contract at time 0 with the maximum payments (corresponding to the originally estimated parameters and initial yield curve) if the interest rate is chosen to be 150 basis points above the "fair" interest rate, which is a common profit margin for retail mortgage products in Germany. This quantity can, in some sense, be considered as the mortgage lender's (marked-to-market) profit for a contract with such an interest-rate agreement. In Figure 9.3 we show how the loan amount develops over time in the examples as given in Table 9.2 and compare it to one exemplary (simulated) path of the house price process.

For comparison, we calculate the same quantities as in Table 9.2 if we use the valuation formulas in Chapter 9.2, explicitly taking into account the possibility of losses for the lender, with a 5%-CreditVaR limit set to EUR $0.1 \cdot 500,000 =$ EUR 50,000. The results are shown in Table 9.3.

In order to illustrate the value of the default risk, i.e. the difference between the "default-free" valuation formulas and the valuation formulas with the PUT options, we re-visit the example of the 70-year old female homeowner. Let us assume that the homeowner receives EUR 250,000.- in an upfront-payment FRRM. This huge quantity evidently implies a considerable risk of losses for the lender. The probability that the outstanding loan amount exceeds the house value at termination of the contract is approx. 65% and the critical point for the lender is reached after 144 months (with the previously simulated path of the house price process), i.e. when the mortgagor is 82 years old. If we calculate with the interest rate $i_0^* = 4.37\%$ and add a spread of 150 basis points, the value of the default risk V_{app}^D is EUR 15,016.- in this example, yielding a net present value of 63,734.- for this contract (compared to EUR 78,750.- for the same contract priced in the default-free setting). The "fair" interest rate in this example, which would make the contract value equal to 0 at origination is 4.42%, i.e. 5 basis points above the fair interest rate in the default-free setting. If the house price volatility parameter σ_H increases from 0.47% to 5% (which may be a realistic value in the current US market environment), the value of the default risk rises to EUR 19,551.- in the previous example.

Let us now consider an adjustable-rate contract in the default-free model setting. As previously discussed, the outstanding loan amount of an ARRM is stochastic. If interest rates rise, the outstanding loan amount of an ARRM will grow faster than the outstanding loan amount of a fixed-rate agreement. It is thus not surprising that the maximum payments M_0^* and M^* (according to the q-criterion) are smaller for an ARRM compared to the respective quantities for a FRRM. We will give some examples for an upfront-payment ARRM, where we can calculate all quantities analytically. If we leave all parameters as in the examples for the fixed-rate agreements above and choose the 1-month EURIBOR rate as reference index rate, we obtain the results as given in Table 9.4. The spread s^{L*}, which makes the contract "fair" at origination is of course 0, as shown in Chapter 9.1. In Figure 9.4 we illustrate the risk of rising interest rates. We show two different simulated paths of

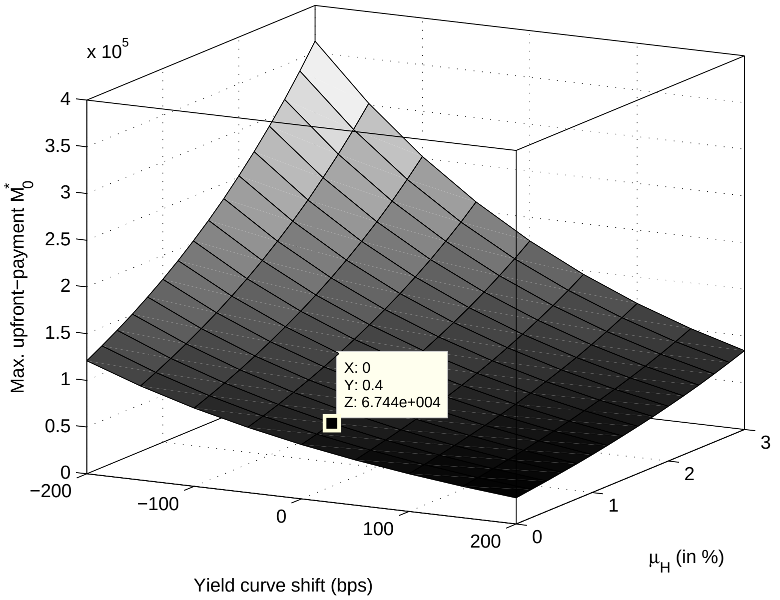

Figure 9.1: Sensitivity of the maximum upfront-payment M_0^* for a 70-year old female homeowner with respect to μ_H and with respect to parallel shifts of the initial yield curve in our example calculations. The highlighted grid point corresponds to the base situation as used in Tables 9.2 and 9.3.

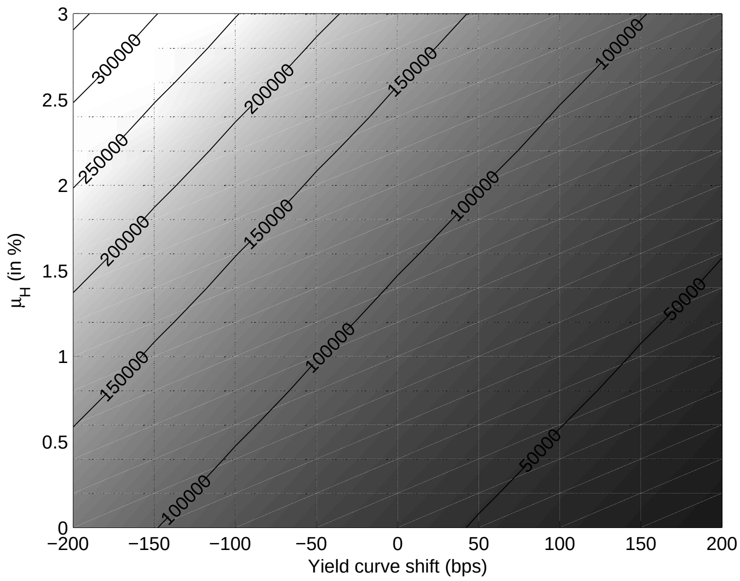

Figure 9.2: Contour plot of the maximum upfront-payment M_0^* in the example of Figure 9.1.

	Male	Female
Age 65	$i_0^* = 4.37\%$	$i_0^* = 4.39\%$
	$i^* = 4.36\%$	$i^* = 4.36\%$
	$M_0^* = 60,000$	$M_0^* = 50,000$
	$M^* = 330$	$M^* = 260$
	$V_{FRRM}^{UP}(0, i_0^* + 0.015) = 19,918$	$V_{FRRM}^{UP}(0, i_0^* + 0.015) = 19,001$
	$V_{FRRM}^{LA}(0, i^* + 0.015) = 10,973$	$V_{FRRM}^{LA}(0, i^* + 0.015) = 10,412$
Age 70	$i_0^* = 4.35\%$	$i_0^* = 4.37\%$
	$i^* = 4.35\%$	$i^* = 4.35\%$
	$M_0^* = 80,000$	$M_0^* = 65,000$
	$M^* = 450$	$M^* = 370$
	$V_{FRRM}^{UP}(0, i_0^* + 0.015) = 21,754$	$V_{FRRM}^{UP}(0, i_0^* + 0.015) = 20,475$
	$V_{FRRM}^{LA}(0, i^* + 0.015) = 10,923$	$V_{FRRM}^{LA}(0, i^* + 0.015) = 11,119$

Table 9.2: Examples of possible fixed-rate reverse mortgage contracts in the 'default-free' model setting with $q = 0.965$ and $H_0 = 500,000$.

	Male	Female
Age 65	$i_{0,d}^* = 4.37\%$	$i_{0,d}^* = 4.39\%$
	$i_d^* = 4.36\%$	$i_d^* = 4.36\%$
	$M_{0,d}^* = 70,000$	$M_{0,d}^* = 55,000$
	$M_d^* = 400$	$M_d^* = 320$
	$V_{dFRRM}^{UP}(0, i_{0,d}^* + 0.015) = 23,153$	$V_{dFRRM}^{UP}(0, i_{0,d}^* + 0.015) = 20,797$
	$V_{dFRRM}^{LA}(0, i_d^* + 0.015) = 12,487$	$V_{dFRRM}^{LA}(0, i_d^* + 0.015) = 11,923$
Age 70	$i_{0,d}^* = 4.35\%$	$i_{0,d}^* = 4.37\%$
	$i_d^* = 4.35\%$	$i_d^* = 4.35\%$
	$M_{0,d}^* = 95,000$	$M_{0,d}^* = 80,000$
	$M_d^* = 550$	$M_d^* = 440$
	$V_{dFRRM}^{UP}(0, i_{0,d}^* + 0.015) = 25,784$	$V_{dFRRM}^{UP}(0, i_{0,d}^* + 0.015) = 25,127$
	$V_{dFRRM}^{LA}(0, i_d^* + 0.015) = 12,757$	$V_{dFRRM}^{LA}(0, i_d^* + 0.015) = 12,482$

Table 9.3: Examples of possible fixed-rate reverse mortgage contracts in the model setting of Section 3 with a 5%-CreditVaR limit of 50,000 and $H_0 = 500,000$.

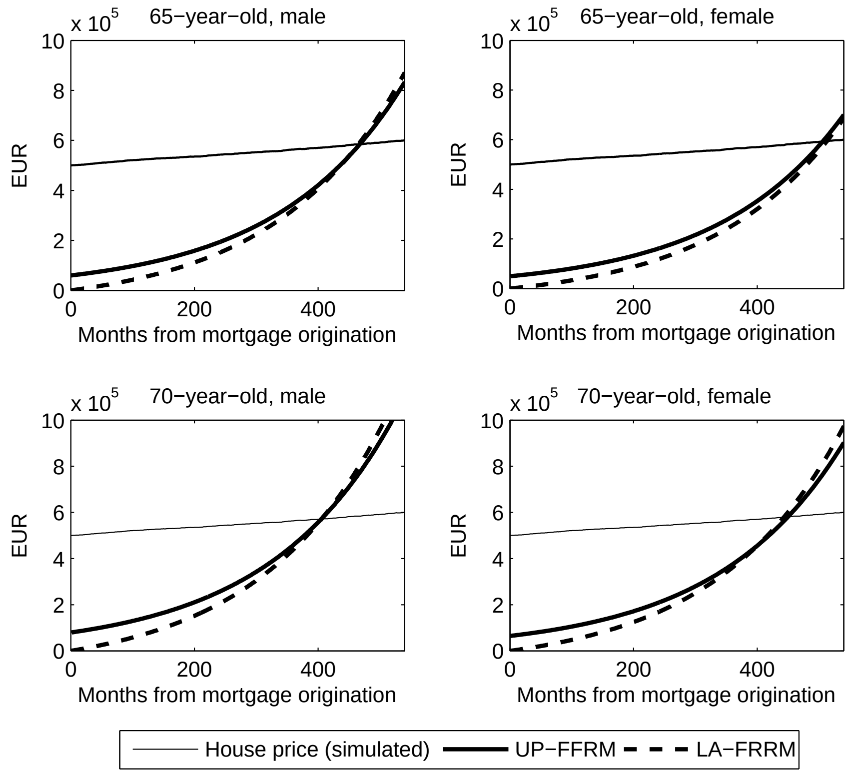

Figure 9.3: Evolution of the outstanding loan amount and of an exemplary path of the house price process in the examples of Table 9.2.

the 1-month EURIBOR rate together with the outstanding loan amount. In scenario 1, the loan amount grows rapidly and the loan amount of a contract for a 65-year-old male person exceeds EUR 500,000 (i.e. the house value at origination) after 351 months, i.e. when the mortgagor is 94 years old.

Finally we want to discuss the implications for the contract design from

	Male	Female
Age 65	$M_0^* = 35,000$	$M_0^* = 25,000$
	$V_{ARRM}^{UP}(0, 1/12, 0.015) = 11,775$	$V_{ARRM}^{UP}(0, 1/12, 0.015) = 9,601$
Age 70	$M_0^* = 55,000$	$M_0^* = 40,000$
	$V_{ARRM}^{UP}(0, 1/12, 0.015) = 15,048$	$V_{ARRM}^{UP}(0, 1/12, 0.015) = 12,668$

Table 9.4: Examples of possible upfront payment ARRM contracts.

a mortgage lender's point of view. First, a mortgage lender has to decide

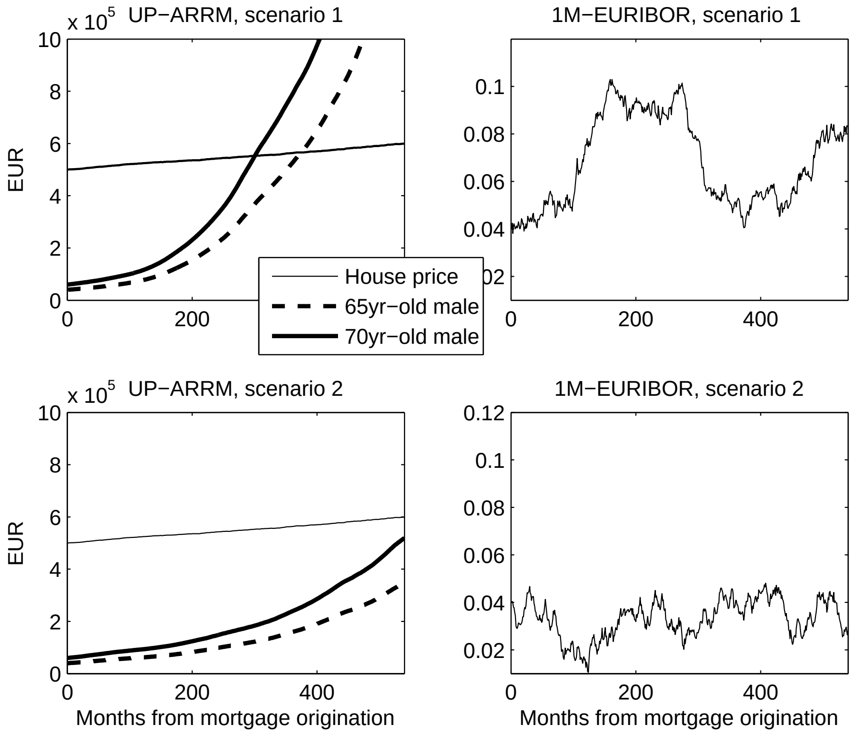

Figure 9.4: Evolution of the outstanding loan amount and of an exemplary path of the house price process for two different interest-rate scenarios and for the contract examples of the right-hand side of Table 9.4.

which criterion to choose in order to determine the maximum payment(s) to the homeowner. Of course, the CreditVaR criterion and the "q-criterion", which stems from a possible requirement that a reverse mortgage should be virtually "default-free" in the previously discussed sense, can not be compared directly. However, if one allows reasonable CreditVaR levels it can be expected that, in general, the maximum payments can be higher than those determined by the "default-free" criterion (if the contract spreads are identical). This follows directly from the fact that the amount of the mortgage lender's potential loss depends primarily on the survival probabilities which decrease as the mortgagor gets older, resulting in a low probability for large losses. Tables 9.2 and 9.3 also indicate this result.

Furthermore, we have seen that the possible maximum payments for fixed-rate agreements are higher than those for adjustable-rate agreements. Consequently, fixed-rate agreements may be more appealing to potential new mortgagors. This is particularly true when long-term interest rates are low. Moreover, due to the higher maximum payments, the mortgage lender's profit opportunities are larger (compare the examples in Tables 9.2 and 9.4). However, these advantages of fixed-rate contracts do have a price: interest-rate risk. A mortgage lender issuing potentially long-term fixed-rate loans is obviously exposed to considerable interest-rate risk when marking-to-market the mortgage contracts. The hedging of this interest-rate risk is not straightforward since the time of termination of the contracts is not known. A hedging strategy based on duration may constitute an easy first approach. Since a detailed discussion of this topic is beyond the scope of this thesis, we just add at this point that from a mortgage lender's point of view, the interest-rate risk associated with fixed-rate contracts must be monitored carefully.

So far, we have excluded the possibility of systematic prepayment. We have already explained why this is not a restriction, particularly in the German market. Yet, we would like to mention at this point that if an explicit prepayment option is a desired contract feature, the valuation of the reverse mortgage contract is still feasible, yet much more complex. The prepayment intensity would, in this case, depend on the refinancing rates and a relationship between (long-term) interest rates and prepayment would have to be established. The valuation of reverse mortgage contracts would then follow the same lines as the intensity-based valuation of regular mortgages and mortgage-backed securities (see Chapters 5 and 6). Depending on the assumed relation between interest rates and prepayment, the modelling framework would, however, lose a considerable part of its analytical tractability.

Chapter 10

Summary and conclusion

In this thesis we have first developed new valuation techniques for mortgage-backed securities. MBS constitute a tremendously important segment of the US fixed-income market and are widely accepted as an interesting asset class among institutional investors, although their valuation and risk management is in general a challenging and computationally expensive task.

In Chapter 5 we have presented a new valuation model based on a stochastic prepayment-intensity specification, which extends the more traditional econometric MBS models based on proportional hazard techniques. Our model explicitly accounts for the general economic environment by the incorporation of a factor which is fitted to the GDP growth rate. Applied to a series of GNMA MBS with different coupons we were able to successfully explain market prices across different coupons. While recognizing that a 1-factor model for the baseline prepayment process in the spirit of Kau et al. (2004) also produces good empirical results, we have found that the GDP factor adds explanatory power to our model when applied to market prices. Both the risk of refinancing understatement (for premiums) and the risk of turnover overstatement (for discounts) are accounted for in our prepayment-risk-neutral setting by three prepayment-risk adjustment parameters which can be calibrated to market data. The intensity-based modelling framework offers the necessary mathematical rigour to target MBS market prices directly, without the need of any OAS input. The OAS has become subject to criticism in the academic literature lately due to its lack of a theoretical foundation. Yet, it remains perfectly possible to conduct a classical OAS analysis within our modelling framework by simply setting the prepayment-risk adjustment parameters to those values for which the prepayment intensity dynamics are identical under the real-world and under the risk-neutral pricing measure. This may be a particularly appealing characteristic of our model with respect to its use in practice where the OAS is still a widely accepted quantity. It is also important to note that our model is of course flexible enough to allow for further statistical fine-tuning by incorporating additional exogenously given covariates in a straightforward way. This may be an interesting path to follow for further research, particularly with a view towards applications in practice. Furthermore, we have shown that our modelling approach can be easily adapted for the valuation of adjustable-rate MBS and CMOs. The valuation of more complex CMO structures, possibly incorporating both prepayment and default risk (e.g., CMOs without agency guarantees) may also be a fruitful field for further research.

We have then presented a closed-form formula which approximates the value of fixed-rate MBS in Chapter 6. While the model set-up is slightly different for our closed-form approximation model in order to achieve a better analytical tractability, the approach is based on the same stochastic prepayment intensity techniques as the model presented in Chapter 5. Tackling some of the inconveniences of previous approaches to calculate MBS prices in closed or semi-analytic form, our model proves to be able to explain major market price movements successfully for a wide range of coupons. The overall average absolute pricing error is 159 basis points in our sample (with a yearly recalibration of prepayment-risk adjustment parameters). The time periods where deviations between model and market prices were rather large were in some cases characterised by unusually flat or even inverse shapes of the yield curve. A way to further improve the empirical performance of our closed-form approximation may thus be the incorporation of a second interest-rate factor, representing long-term rates, into the CIR model. With an overall of four stochastic factors in this case, however, one would have to carefully analyse and weigh up the potential improvement in the empirical fit against the danger of model overfitting. Obviously, an overparameterised model may cause problems in the model calibration procedure and produce less reliable out-of-sample results. We leave this point for further research.

The closed-form formula is computationally highly efficient and reduces the computational burden of MBS valuation drastically. It may thus be particularly useful in risk and portfolio management. In Chapter 8 we have considered selected scenario-based portfolio optimisation problems and extended their usual application by including prepayment-sensitive fixed-rate agency MBS into the universe of available assets. In a case study with historical data from an interesting time period in the US fixed-income markets from 2001 to 2003, when rates dropped to 40-year lows, we have empirically tested our optimal asset allocation strategies. Our results indicate that a portfolio with MBS is indeed able to outperform a classical stock/bond portfolio significantly. We have therefore provided further empirical evidence for the attractiveness of MBS from a quantitatively-oriented investor's point of view.

Finally, we have presented a consistent framework for the valuation of reverse mortgages in Chapter 9. Within our modelling framework we have also calculated explicitly the probability that, at termination of the contract, the outstanding loan amount exceeds the house value. This probability can be considered as 'default probability' for the mortgage lender and by determining the maximum payment(s) to the homeowner in such a way that this default probability is very low, the pricing of the contract with a default-free model set-up and a riskless benchmark curve can be justified. We have shown that for all fixed-rate reverse mortgages and for upfront-payment adjustable-rate reverse mortgages the 'default probabilities' in a default-free setting can be calculated analytically. In the case of a fixed-rate contract this is also possible if the valuation framework is extended to account for default and loss given default. In this extended set-up the maximum payment(s) to the homeowner can be determined by a CreditVaR criterion. Applied to data from the German market, we have provided a couple of examples of how reverse mortgage contracts could look like in practice. While, due to the limited availability of data, some assumptions are necessary, we have also pointed out that it is possible to combine individual expert assumptions with available data. By comparing different types of reverse mortgages and interest-rate agreements we have also discussed implications for the design of reverse mortgage contracts from a lender's perspective.

In a nutshell, the overall contribution of this thesis and its relation to the recent academic literature may be regarded as threefold: First, we have further developed and justified the intensity-based approach in the modelling of prepayment and in the valuation of MBS (introduced recently by, e.g., Kau et al. (2006), Goncharov (2005)), taking into account the particularities of a simultaneous consideration of discount and premium securities. In the spirit of an earlier model presented by Levin and Davidson (2005), we have targeted prices directly by the introduction of prepayment-risk adjustment parameters. Second, we have presented a new concept to approximate the value of fixed-rate MBS in closed-form. This has become an active field of research recently with some notable contributions by, e.g., Rom-Poulsen (2007) (extending a concept developed earlier by Collin-Dufresne and Harding (1999)) and Gorovoy and Linetsky (2007). Our approach offers a computationally easy-to-handle alternative to these approaches which rely on numerically complex techniques to evaluate semi-analytic pricing formulas. We have applied this approach in some selected asset allocation case studies, which would not have been feasible with more traditional, computationally expensive pricing routines. Third, this thesis aims to make a contribution to a better understanding of valuation, pricing and risk issues associated with reverse mortgages. Despite some publications and discussions in the economic and practitioner-oriented literature, a mathematically rigorous treatment of reverse mortgages has so far remained scarce (with the possible exception of the recent contribution by Wang et al. (2007)). We thus hope that this thesis may help to spread the popularity of reverse mortgages. Despite the obvious economic benefits of reverse mortgages, they are not yet available in many European countries such as Germany.

Appendix A

A Monte-Carlo algorithm

In the following we present the major implementation steps needed in a Monte-Carlo simulation to evaluate (5.8) at time $t_0 = 0$ for a 30yr fixed-rate agency-MBS pass-through security with monthly payment dates. Antithetic paths are used as a method of variance reduction (see, e.g., Glasserman (2004) for a discussion of variance reduction techniques in general and antithetic paths in particular).

Algorithm 2. *Monte Carlo simulation*

(i) Determine the number K of payment dates/remaining months (excluding the settlement month) until maturity of the MBS.

(ii) Calculate:

$$burnout(0) := \ln \frac{PF(0)}{A(360-K)},$$

where PF is the current pool factor and A is the outstanding notional of the security according to the original amortisation schedule without any prepayments.

(iii) Let Z denote the (even) number of Monte-Carlo paths. Draw $K \cdot \frac{Z}{2}$ iid r.v. from a standard normal distribution and arrange them arbitrarily in a $K \times Z/2$ matrix N_{IR}.

(iv) Use antithetic interest-rate paths for variance reduction in the MC simulation and define the $K \times Z$ matrix

$$N_{AP} := [N_{IR} \ , -N_{IR}].$$

(v) Moreover, draw a further $K \cdot 2Z$ iid r.vs. from a standard normal distribution and also arrange them arbitrarily in a $K \times 2Z$ matrix N_{base}.

(vi) *Get* $r(0)$, $R_{par}(0,10)$ *from current yield curve, set* $w(0)$ *equal to quarterly GDP growth rate 6 months ago. Calculate*

$$spread(0) := WAC - R_{par}(0,10),$$

$$p_0(0) = \frac{\theta_p + b_{pw} \cdot w(0)}{\hat{a}_p}$$

and $p(0)$ *according to (5.2). Also, calculate the discount rates* $R(0,t_{settle})$ *and* $R(0,t_1)$ *from the current yield curve, where* t_{settle} *and* t_1 *correspond to the settlement date and the first payment date respectively.*

(vii) *Set the option-adjusted spread to the value* OAS *to be used in the valuation routine.*

(viii) *Calculate the discount factors*

$$DF(i) = e^{-(R(0,t_i)+OAS)\cdot t_i} \quad i = settle, 1$$

(ix) *Start MC simulation:*
for z=1:Z
for k=1:K

- *Get* $f(0,t_k)$, $f_\tau(0,\tau)|_{\tau=t_k}$ *from current yield curve, calculate* $\theta_r(t_k)$ *according to (3.11).*
- *With* $\Delta t_k := t_k - t_{k-1}$, *calculate*

$$\begin{aligned} r(t_k) &= e^{-\hat{a}_r \cdot \Delta t_k} \cdot r(t_{k-1}) + \frac{\theta_r(t_k)}{\hat{a}_r} \cdot (1 - e^{-\hat{a}_r \cdot \Delta t_k}) \\ &\quad + \sqrt{\frac{\sigma_r^2}{2\hat{a}_r} \cdot (1 - e^{-2\cdot\hat{a}_r\cdot\Delta t_k})} \cdot N_{AP}(k,z) \end{aligned}$$

- *Calculate* $R(t_k, t_k+\tau)$ *for* $\tau = 0.5, 1, 1.5, ..., 10$ *and the 10yr par-yield* $R_{par}(t_k, t_k+10)$ *from* $R(t_k, t_k+\tau)$ *as given in, e.g., Hull (2003), p. 96.*
- *From* $r(t_k)$, *calculate the discount rate* $R(t_k, t_k + \Delta t_k)$ *and the discount factor from* t_{k+1} *to* 0:

$$DF(t_{k+1}) = e^{-(R(t_k,t_k+\Delta t_k)+OAS)\cdot\Delta t_k} \cdot DF(t_k)$$

- *Calculate*

$$spread(t_k) = WAC - R_{par}(t_k, t_k+10)$$

- *Calculate $PF(t_k)$ given $PF(t_{k-1})$ and $p(t_{k-1})$ according to the formulas as stated in Chapter 2.2.3 and calculate*

$$burnout(t_k) = \ln \frac{PF(t_k)}{A(360 - K + k)}.$$

- *Calculate*

$$\begin{aligned} w(t_k) &= e^{-\hat{a}_w \cdot \Delta t_k} \cdot w(t_{k-1}) + \frac{\theta_w}{\hat{a}_w} \cdot (1 - e^{-\hat{a}_w \cdot \Delta t_k}) \\ &\quad + \sqrt{\frac{\sigma_w^2}{2\hat{a}_w} \cdot (1 - e^{-2 \cdot \hat{a}_w \cdot \Delta t_k})} \cdot N_{base}(k, z) \end{aligned}$$

- *Calculate*

$$\begin{aligned} p_0(t_k) &= e^{-\hat{a}_p \cdot \Delta t_k} \cdot p_0(t_{k-1}) + \frac{\theta_{p0} + b_{pw} \cdot w(t_k)}{\hat{a}_w} \cdot (1 - e^{-\hat{a}_p \cdot \Delta t_k}) \\ &\quad + \sqrt{\frac{\sigma_p^2}{2\hat{a}_p} \cdot (1 - e^{-2 \cdot \hat{a}_p \cdot \Delta t_k})} \cdot N_{base}(k, Z + z) \end{aligned}$$

- *Calculate $p(t_k)$ according to (5.2) and (5.9)*
  end (loop over k)
- *With prepayment vector $p = (p(1), ..., p(t_{K-1}))'$ calculate MBS cash flows (e.g., with mbscfamounts - function in Matlab), including accrued interest, for a face amount of 1.*
- *Discount cash flows occurring at dates $t_{settle}, t_1, ..., t_K$ with the respective discount factors and calculate the sum of the discounted cash flows to obtain the value $V_z(0)$ of the MBS.*
  end (loop over z)

(x) Average over MC paths to obtain the value of the MBS

$$V(0) = \frac{1}{Z} \cdot \sum_{z=1}^{Z} V_z(0)$$

Appendix B

The moving block bootstrap

The bootstrap technique is a computer-intensive resampling method, which is usually applied to statistical inference problems for level-two (or higher-level) parameters, such as, e.g., the standard error or bias of an estimator for a parameter of interest in a statistical model. In many cases, standard analytic methods, for example based on likelihood theory, can be applied to obtain estimates of the model parameters. Yet, these standard methods often become infeasible for an assessment of the accuracy or the quality of the inference based on the estimator. Bootstrap techniques were originally introduced by Efron (1979) and have since then been applied to many statistical problems, e.g. to the estimation of level-two parameters in time-series models. In this thesis we have applied a moving block bootstrap to estimate standard errors of maximum likelihood parameter estimators in state space models. In this section we will give a brief overview of the moving block bootstrap algorithm. For further details on the moving block bootstrap and on bootstrap methods in general, the reader is referred to Lahiri (2003).

Suppose we have a sequence of random variables $Y_1, Y_2, ...$ and that we observe a realisation of the first T variables $\{Y_1, ..., Y_T\}$. We assume that the process $(Y_t)_{t=1,...,T}$ is stationary and features weak dependence. Weak dependence means that the process has limited memory. At an informal level, the limited memory condition can be said to be satisfied if the dependence between Y_t and Y_{t+h} vanishes as h becomes large (for a more formal treatment of memory properties and weak dependence of time series see, e.g., Beran (1994), p.6ff.). The moving block bootstrap is a resampling technique which resamples blocks of consecutive observations $(Y_i, Y_{i+1}, ..., Y_{i+l-1})$, where l is the block length. Let

$$\mathcal{B}_i := (Y_i, ..., Y_{i+l-1}) \tag{B.1}$$

denote the block of length l starting with the i-th observation for $i = 1, ..., N$ where $N := T - l + 1$. The moving block bootstrap sample of size $m = k \cdot l$

is then obtained by concatenating the blocks $\mathcal{B}_1^*, ..., \mathcal{B}_k^*$ which are drawn with replacement from the collection $\{\mathcal{B}_1, ..., \mathcal{B}_N\}$. Usually, the sample size m is chosen to be of the same order as the original sample size T, i.e. $m \approx T$ (see Lahiri (2003), p. 26). The resampled sequence of random variables is denoted by $\{Y_1^*, ..., Y_m^*\}$. Now, let

$$\hat{\theta}_{l,m}^* := t(Y_1^*, ..., Y_m^*) \tag{B.2}$$

denote one bootstrap realisation of the estimator of the parameter of interest θ, where $t(\cdot)$ is some function/statistic of the data used as an estimator for θ. Repeating the resampling procedure B times, let

$$\hat{\theta}_{l,m}^{*,(j)} := t(Y_1^{*,(j)}, ..., Y_m^{*,(j)}), \tag{B.3}$$

$j = 1, ..., B$, denote the bootstrap replicates of $\hat{\theta}_{l,m}^*$. Applying the Monte-Carlo principle, an estimate $\hat{\theta}$ of the quantity of interest θ is finally obtained by averaging over the bootstrap replicates:

$$\hat{\theta}_T = \frac{1}{B} \sum_{j=1}^{B} \theta_{l,m}^{*,(j)}. \tag{B.4}$$

The moving block bootstrapping procedure can be summarised in the following algorithm:

Algorithm 3. *(Moving block bootstrap)*

(i) Given a data sample $\mathbf{Y}_T := (Y_1, ..., Y_T)$ of size T, choose the block length l and the number of blocks k to be concatenated in the resampling procedure such that $m = k \cdot l \approx T$.

(ii) Randomly draw k blocks from the collection $\{\mathcal{B}_1, ..., \mathcal{B}_N\}$ (with replacement) with $\mathcal{B}_i$ as defined in (B.1) to obtain a bootstrap sample $\{\mathcal{B}_1^*, ..., \mathcal{B}_k^*\}$. Calculate $\hat{\theta}_{l,m}^*$ as given in (B.2).

(iii) Repeat (ii) B times to obtain bootstrap replicates $\theta_{l,m}^{*,(j)}$, $j = 1, ..., B$, of the quantity of interest.

(iv) Average over the bootstrap replicates to obtain the estimate of the quantity of interest as given in (B.4).

For a theoretical justification of the moving block bootstrap, the reader is referred to Lahiri (2003). Yet, it has to be mentioned that the moving block bootstrap principle is partly based on heuristic arguments only. In addition to this, the theoretical foundation holds only asymptotically (i.e. as the number of observations $T \to \infty$) and it is hard to assess the quality of the approximation for finite samples. The choice of the block length, which potentially optimises the bootstrap procedure, is also a non-trivial problem. Lahiri (2003) discusses all these topics and further challenges concerning bootstrap techniques in detail. Since we have only used the bootstrap principle in order to obtain estimates for level-two parameters (which is not our main concern in this thesis), we rely on the results generated by this method while keeping in mind that they should only be considered as rough estimates.

Appendix C

Discussion of approximation errors

The error term R_{t_k}

In (3.36) we have introduced the error terms R_{t_k}, $k = 1, ..., K$ resulting from the approximation of an integral by sums. From (3.35) and (3.36) it follows that for $k = 1, ..., K$:

$$\begin{aligned}
|R_{t_k}| &= \left| \int_{t_{k-1}}^{t_k} \gamma(u) e^{-\int_t^u \gamma(s)ds} du - \gamma(t_k) \cdot e^{-\int_t^{t_k} \gamma(s)ds} \cdot \Delta t_k \right| \\
&= \left| e^{-\int_t^{t_{k-1}} \gamma(s)ds} - (1 + \gamma(t_k) \cdot \Delta t_k) \cdot e^{-\int_t^{t_k} \gamma(s)ds} \right| \\
&= e^{-\int_t^{t_k} \gamma(s)ds} \cdot \left| e^{\int_{t_{k-1}}^{t_k} \gamma(s)ds} - (1 + \gamma(t_k) \cdot \Delta t_k) \right| \\
&= e^{-\int_t^{t_k} \gamma(s)ds} \cdot \left| \int_{t_{k-1}}^{t_k} \gamma(s)ds - \gamma(t_k) \cdot \Delta t_k + O\left[\left(\int_{t_{k-1}}^{t_k} \gamma(s)ds \right)^2 \right] \right| \\
&\leq e^{-\int_t^{t_k} \gamma(s)ds} \cdot R_k^{\Delta t_k},
\end{aligned}$$

where

$$R_k^{\Delta t_k} := \max_{\xi \in [t_{k-1}, t_k]} |\gamma(\xi) - \gamma(t_k)| \cdot \Delta t_k + O\left[\left(\int_{t_{k-1}}^{t_k} \gamma(s)ds \right)^2 \right].$$

Note that $R_k^{\Delta t_k} \to 0$ and consequently $|R_{t_k}| \to 0$ as $\Delta t_k \to 0$.

The error terms $\widetilde{R}$, $\bar{R}$

In Theorem 6.2 we have first introduced the error terms

$$\widetilde{R}(\Delta t, v_k^2, w_k^2, z_k \cdot v_k, z_k \cdot w_k, v_k \cdot w_k), \quad k = 1, ..., K.$$

These terms contain residual terms of the order $O(\Delta t)$ resulting from the approximation of integrals by sums, similar to the error term R_{t_k} as discussed in the previous section, with

$$\widetilde{R}(\Delta t, v_k^2, w_k^2, z_k \cdot v_k, z_k \cdot w_k, v_k \cdot w_k) \rightarrow \bar{R}(v_k^2, w_k^2, z_k \cdot v_k, z_k \cdot w_k, v_k \cdot w_k)$$

for $\Delta t \rightarrow 0$. $\bar{R}(v_k^2, w_k^2, z_k \cdot v_k, z_k \cdot w_k, v_k \cdot w_k)$ contains the residual terms from the series expansion of the exponential function in (6.12). The size of this error term primarily depends on v_k and w_k. Following the definition of these quantities in Theorem 6.2, v_k can be expected to be small for premiums and w_k can be expected to be small for discounts. The empirical results as discussed in Chapter 6 indicate that our closed-form approximation model performs slightly better for premiums than for discounts.

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List of Figures