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k-disjunctive cuts and cutting plane algorithms for general mixed integer linear programs

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Abstract

In this thesis we analyze cutting planes for general mixed integer linear programs from a geometric point of view and discuss some related algorithms. It is the main goal to find answers to the following two fundamental questions: First, how can the mixed integer hull of an arbitrary polyhedron be generated by cutting planes? Secondly, how can a classical cutting plane algorithm be designed which solves an arbitrary mixed integer linear program exactly in finite time.

The crucial result for dealing with these two problems is a natural generalization of the well known split cuts of Cook, Kannan, and Schrijver to cuts which are based on multi-term disjunctions. We call them k-disjunctive cuts and analyze their properties in detail. These cuts allow us to answer the first question. We also provide a way for constructing k-disjunctive cuts and show how they can be combined with classical cutting planes to obtain a finite cutting plane algorithm for rational mixed integer linear programs.

We complete our explanations with a geometric comparison of some well known cutting planes, an analysis of their properties in generating the mixed integer hull of a polyhedron, as well as a consideration of their algorithmic performance. Moreover, we give some examples and applications to illustrate our theoretical results.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Schnittebenen für allgemeine gemischt-ganzzahlige lineare Programme und zugehörigen Algorithmen. Dabei werden vor allem die beiden folgenden Fragestellungen untersucht: Wie kann die gemischt-ganzzahlige Hülle eines beliebigen Polyeders mit Hilfe von Schnittebenen bestimmt werden? Wie kann ein klassischer Schnittebenenalgorithmus konstruiert werden, der ein beliebiges gemischt-ganzzahliges lineares Programm stets in endlicher Zeit exakt löst? Grundlegend für die Untersuchung der beiden Probleme ist dabei eine geometrische Vorgehensweise.

Der zentrale Ansatz zur Lösung der beiden Problemstellungen besteht in einer Verallgemeinerung der bekannten Split Cuts von Cook, Kannan und Schrijver auf Schnittebenen, die auf Multiterm Disjunktionen beruhen. Diese Klasse von Schnittebenen wird dabei als k-disjunctive cuts bezeichnet und im Detail untersucht. Mit Hilfe dieser Schnittebenen ist es nun auf natürliche Weise möglich, eine Antwort auf die erste Frage zu finden. Ebenso eröffnet ein konstruktives Verfahren zur Berechnung von k-disjunctive cuts die Möglichkeit, einen endlichen Schnittebenenalgorithmus für rationale gemischt-ganzzahlige Programme anzugeben.

Die Ausführungen in dieser Arbeit werden dabei ergänzt durch einen geometrischen Vergleich einiger bekannter Schnittebenen sowie eine Untersuchung ihrer Eigenschaften beim Bestimmen der gemischt-ganzzahligen Hülle eines Polyeders bzw. bei Verwendung in einem Schnittebenenalgorithmus. Außerdem werden die theoretischen Ergebnisse anhand einiger Beispiele und Anwendungen illustriert.

Contents

List of Symbols

Special sets

$\emptyset$	the empty set
$\mathbb{N}$, $\mathbb{N}_0$	the set of natural numbers excluding and including 0, respectively.
$\mathbb{Z}$	the set of the integers
$\mathbb{Q}$	the set of the rational numbers
$\mathbb{R}$	the set of the real numbers
$\mathcal{B}$	the set of all bases B

Operations on sets

$A \subseteq B$	A is subset of B or $A = B$
$\mathrm{bd}\,(S)$	the boundary of a set S
$\mathrm{int}\,(S)$	the interior of a set S
$\mathrm{ext}\,(S)$	the set of all extreme points of a set S
$\mathrm{relbd}\,(S)$	the relative boundary of a set S
$\mathrm{relint}\,(S)$	the relative interior of a set S
$\mathrm{conv}\,(S)$	the convex hull of a set S
$\lvert S \rvert$	the cardinality of a set S
$\mathrm{proj}_X\,(S)$	the projection of the set $S \subseteq \mathbb{R}^{p+q}$ on the space $\mathbb{R}^p$ of x-variables
$\dim(S)$	the dimension of the smallest affine subspace that contains S

Polyhedra, closures, and disjunctions

P	polyhedron of the feasible domain of the LP relaxation of a (mixed) integer program
P_I	the (mixed) integer hull of a polyhedron P
P^i	the set of all feasible points of the LP relaxation in step i of a cutting plane algorithm

$P^{(i)}$	the i-th split closure of a (mixed integer) polyhedron P or the i-th Chvátal-Gomory closure of a (integer) polyhedron P, respectively
$P_k^{(i)}$	the i-th k-disjunctive closure of a polyhedron P
$P_{irr,k}^{(i)}$	the i-th irrational k-disjunctive closure of a polyhedron P
$D(d,\delta)$	a split disjunction $dx \leq \delta \vee dx \geq \delta + 1$
$D(k,d,\delta)$	a k-disjunction $dx \leq \delta$
$D_{irr}(k,d,\delta)$	an irrational k-disjunction $dx \leq \delta$

Miscellaneous

$\lvert x \rvert$	the absolute value of x
$\lvert\lvert x \rvert\rvert$	the norm of a vector x
$\lfloor x \rfloor$	the largest integer less or equal to x
$\lceil x \rceil$	the smallest integer greater or equal to x
$(x)^+$	the maximum of 0 and x
$(x)^-$	the maximum of 0 and $-x$
$\mathrm{sign}(x)$	the sign of x
xy	the scalar product of two vectors of equal size
$\mathcal{O}$	Landau symbol
$\wedge, \vee$	logical 'and', 'or

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1 Introduction

In this thesis we deal with cutting planes and related algorithms for general mixed integer linear programs (MILP). MILP are linear optimization problems in which some of the variables are restricted to be integral. They can be expressed in the form

$$\begin{array}{ll} \max & cx + hy \\ & Ax + Gy \leq b \\ & x \in \mathbb{Z}^p, y \in \mathbb{R}^q, \end{array}$$

which will also be the standard representation for our considerations. MILP form an interesting and important class of problems, as for example many optimization problems which arise in economic and industrial applications can be modeled as MILP.

Cutting planes are an important tool for solving MILP. Before we consider this technique in more detail, we introduce the basic approaches for solving MILP. General MILP contain several special classes of problems. In the case of $c = 0$ and $A = 0$ we obtain an usual linear program (LP)

$$\max\{hy : Gy \leq b, y \in \mathbb{R}^q\}.$$

LP are the foundation of linear optimization and can be solved efficiently. For this purpose, the simplex algorithm or interior point methods are used in practice. In the case of $h = 0$ and $G = 0$ the MILP becomes the integer linear program (ILP)

$$\max\{cx : Ax \leq b\, , x \in \mathbb{Z}^p\}.$$

Moreover, we obtain two more important special cases by restricting the integral vector x to be binary. They are mixed binary linear programs and binary linear programs, respectively.

The approaches for solving general MILP and ILP are quite similar, where solving a MILP is typically more involved than solving an ILP. This is due to the simpler structure of an ILP, although integrality constraints are imposed on all variables. We can distinguish between three basic solution strategies: An enumerative approach, a primal approach which is based on finding augmentation vectors, and thirdly a dual cutting plane approach. Besides, there are also hybrid approaches as for example branch-and-cut algorithms which combine enumerative and dual techniques. We briefly explain the ideas of the three approaches.

The solution set - or at least the subset of integral restricted components of the solution set in the case of a MILP - is discrete. If the solution set is moreover bounded, there are only finitely many possible solutions which can be evaluated. Thus selecting the solution with the best objective function value solves the given problem. This is the underlying idea of an enumerative approach. Since the number of possible solutions usually becomes extremely large, one tries to evaluate most of the solutions implicitly. A possibility is computing bounds of the objective function value for subsets of the solution set which guarantee that the optimal solution cannot be contained in these subsets. Representatives of an enumerative approach are the well known branch-and-bound algorithm and dynamic programming. An overview of these two fundamental methods can be found for example in the books of Schrijver [Sch86], Nemhauser and Wolsey [NW88], or of Bertsimas and Weismantel [BR05].

In a primal approach, it is the basic idea to start with an arbitrary feasible solution and to check whether this solution can be improved. So the following *augmentation problem* has to be solved: Find a solution with a better objective function value or decide that none such exists. An algorithm for solving an arbitrary (M)ILP is now based on solving the augmentation problem repeatedly until a optimal solution has been found. Therefore, it is a crucial point to solve the augmentation problem efficiently. In this context, an important tool is given by the so called test sets. We refer to the survey of Aardal, Weismantel, and Wolsey [AWW02] and to the thesis of Köppe [Köp02] for a detailed introduction to primal approaches for ILP and MILP.

At last we discuss the dual cutting plane approach for solving MILP which is the foundation for this thesis. It is again based on a simple idea. We start with solving the LP relaxation of the MILP - which means that we ignore the integrality constraints on the variables - using the simplex algorithm. If the related solution is also feasible for the MILP and satisfies the integrality constraints, then an optimal solution of the MILP has been found. Otherwise, we add a cutting plane to the LP relaxation which cuts off the current optimal vertex of the relaxation but no feasible point of the MILP. Here a cutting plane is an affine half-space which contains all feasible points of the MILP and separates the infeasible vertex from the feasible points of the MILP. We repeat this approach until either an optimal solution of the MILP has been found or infeasibility of the problem is detected. The basic idea of this algorithm is also described in Figure 1.1 and an easy one step example for a two dimensional problem is given in Figure 1.2.

In solving a MILP by a cutting plane algorithm there are two important issues which have to be considered. First, it is the question how to find a feasible cutting plane which cuts off the current (mixed) integer infeasible LP solution. Secondly, as it is the aim of an algorithm to solve a MILP in finite time, we have to consider if and how cutting planes can be combined to find an optimal solution as efficiently and fast as possible. Many approaches have been developed to deal with the first issue in general and for several special cases. We give a short overview of the development of cutting planes at the end of

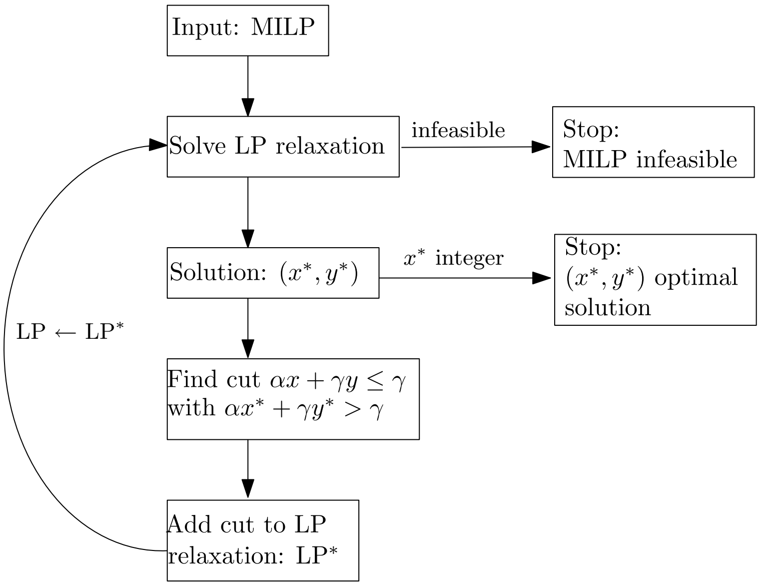
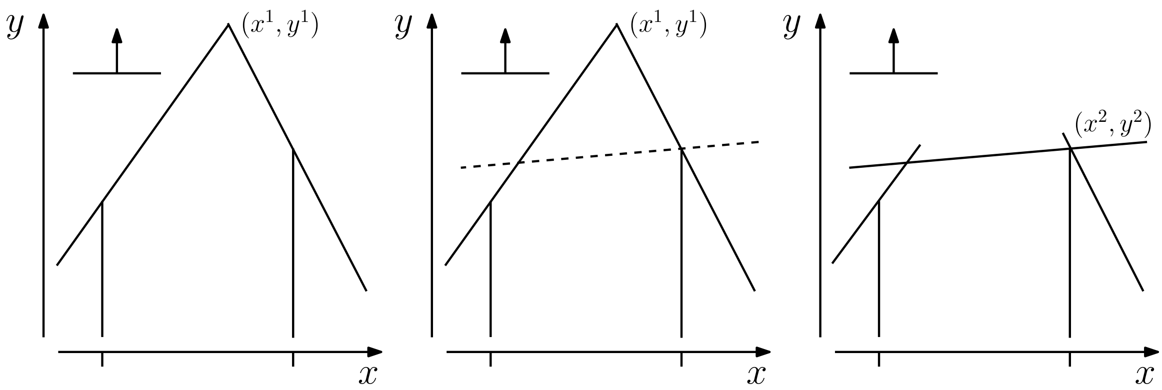


Figure 1.1: The basic principle of a cutting plane algorithm according to [Pad05]



(a) Optimal LP solution (x^1, y^1) with $x^1 \notin \mathbb{Z}$

(b) Cutting plane which cuts off (x^1, y^1) but no feasible point

(c) Optimal LP solution (x^2, y^2) with $x^2 \in \mathbb{Z}$ which is also optimal for the MILP

Figure 1.2: Example of one step of a cutting plane algorithm

this section and analyze some important representatives in detail in Chapter 2. However, unlike the computation of valid cutting planes, the second issue is more involved. In fact no finite cutting plane algorithm is known for solving a general MILP. Finite algorithms are only known for the special cases of pure integer and mixed binary programs; see for example [Cor08, MMWW02, Pad05]. This fact is supposed to be the starting point for this thesis and is discussed extensively. It is the aim both to characterize cutting planes which are required for a finite algorithm and to state an explicit finite cutting plane algorithm for solving general MILP. Here we also deal with the equivalent question how the mixed integer hull of an arbitrary polytope can be generated by cutting planes. An overview of the content and the results of this thesis is given next in Section 1.1.

We conclude this introduction with a short overview of the development of cutting planes for MILP, where we focus on more theoretical results and make no claim to be complete. In this context we refer to [Grö04, Pad05] for some very detailed summaries. We also want to refer again to the the papers [Cor08, MMWW02] and to the books [BR05, NW88, Sch86] for extensive information about various issues concerning cutting planes and algorithms.

Cutting planes for ILP and MILP have a 50-year history since the first approaches of Dantzig, Fulkerson, and Johnson [DFJ54], Beale [Bea58] and the introduction of the well known integer and mixed integer Gomory cuts [Gom58, Gom60, Gom63]. In the 1970ies, amongst others, some contributions were made by Young [You71], Garfinkel and Nemhauser [GN72], Salkin [Sal75], Jeroslow [Jer79], and Balas concerning intersection cuts and disjunctive programming [Bal71, Bal79, Bal07]. At the end of the 1980ies and in the 1990ies, the well known split cuts and mixed integer rounding inequalities were introduced by Cook, Kannan, and Schrijver [CKS90] and Nemhauser and Wolsey [NW90], respectively. Besides, lift-and-project cuts for mixed binary linear programs were introduced by Balas, Ceria, and Cornuéjols [BCC93] and Gomory cuts were revisited again [BCCN96, Cor05]. From the 1990ies to the present, a strengthening of known mixed integer cuts has been investigated by Andersen, Cornuéjols, and Li [ACL05a], Köppe and Weismantel [KW04], and Günlük and Pochet [GP01]. New properties of split and intersection cuts were found by Andersen, Cornuéjols, and Li [ACL05b]. Recently, a new type of cutting planes was introduced by Andersen, Louveaux, Weismantel, and Wolsey [ALWW07].

1.1 Overview and results

According to the motivation in the last section, the main purpose of this thesis is to investigate the following two questions:

- *What type of cutting planes is necessary to generate the mixed integer hull of an arbitrary polyhedron or how can those cutting planes be characterized?*
- *How can a classical cutting plane algorithm be designed which solves a general MILP in finite time?*

As it is the aim to find answers to these two more theoretical questions, practical considerations and applications are only mentioned marginally. Moreover, we choose a geometric approach for dealing with both problems.

We start with some definitions, the introduction of necessary notation, and some useful results in the next section. We finish this first chapter with some acknowledgments.

In Chapter 2 we discuss some known cutting planes for integer and mixed integer linear programs. We start with Chvátal-Gomory cuts for ILP in Section 2.1 and turn to cuts for MILP in Section 2.2. We explain the basic principle for deriving cuts for MILP and define split cuts as a very important class of cuts in Section 2.2.1. Afterwards, we introduce intersection cuts in Section 2.2.2 as the main tool for computing cuts explicitly. Next, we define the well known mixed integer Gomory cuts and mixed integer rounding inequalities in Section 2.2.3 and Section 2.2.4 and show that they can be seen as a special case of intersection cuts. We also state a variation of the basic mixed integer Gomory algorithm for MILP. In Section 2.2.5 we explain two concepts for finding stronger cuts than in the basic cutting plane algorithm, namely reduce-and-split cuts and cuts from a mixed integer Farkas Lemma. Finally, we present in Section 2.2.6 the idea of the recently developed cuts from two rows of the simplex tableau which do not belong to the class of split cuts.

In Chapter 3 we come to one of the main contributions of this thesis and introduce the so called k-disjunctive cuts as a natural generalization of split cuts. These cuts allow us to solve the first question stated at the beginning of this section. As we will see, the answer to this question depends on the geometric structure of the facets of the mixed integer hull. We start in Section 3.1 with the definition of k-disjunctive cuts and irrational k-disjunctive for rational and real polyhedra, respectively, and discuss some basic properties. In Section 3.2 we repeat an approximation property of split cuts and discuss some implications which is the basis for the later results. Next, we analyze in Section 3.3 which k-disjunctive cuts are necessary to generate the mixed integer hull of an arbitrary polyhedron. We distinguish between rational and real polytopes as the situation can be more involved in the later case. In Section 3.4 we deal with the issue of how deep feasible k-disjunctive cuts can be computed for a given rational polyhedron and finally, in Section 3.5, we briefly refer to some recent work of Andersen, Louveaux, and Weismantel which follows a similar approach to characterize the mixed integer hull of a general polyhedron.

In Chapter 4 we consider explicit algorithms for solving MILP and deal with the second main question of this thesis. We start in Section 4.1 with a short overview of some problems which arise in solving MILP with a classical cutting plane algorithm. Next, we discuss in Section 4.2 two approximation algorithms which find a nearly optimal feasible solution of a real MILP. We repeat an algorithm of Owen and Mehrotra in Section 4.2.1 and introduce a simple cutting plane based algorithm in Section 4.2.2. Finally, we turn to exact algorithms for MILP in Section 4.3 and introduce in Section 4.3.1 the basic form of a new cutting plane algorithm which solves a rational MILP in finite time and is based on k-disjunctive cuts. In Section 4.3.2 and Section 4.3.3 we discuss two important issues of the exact cutting plane algorithm, namely the computation of extreme rays of cones and the generation of irredundant representations of projections. We finish this chapter with an improvement of the basic exact algorithm by a sequential alternative in Section 4.3.5 and give some examples and applications in Section 4.3.5.

To conclude this thesis, we briefly summarize our results and give an outlook for possible further research in Chapter 5.

We note that some of the results of Chapter 3 and Chapter 4 have appeared previously in the preprint [Jör07].

1.2 General foundations

We introduce some notation and give some definitions and basic results which are required for this thesis. We presume that the reader is familiar with the foundations of linear and combinatorial optimization as they can be found for example in [BT97], [BR05], [NW88], or [Sch86].

For vectors $x \in \mathbb{R}^p, y \in \mathbb{R}^q$, we usually not distinguish between row and column vectors. It will always become clear in the context which form is meant. We note that for $p = q$ the product xy of two vectors of equal size denotes the usual Euclidean scalar product. Moreover, we denote by (x, y) the $p + q$ dimensional vector $(x_1, \ldots, x_p, y_1, \ldots, y_q)$.

As most of the results in this thesis are derived by a geometric reasoning we focus on *mixed integer linear programs (MILP)* in *natural form* which are defined by inequality constraints. So we consider MILP

$$\begin{array}{ll} \max & cx + hy \\ & Ax + Gy \leq b \\ & x \in \mathbb{Z}^p, \end{array} \tag{1.1}$$

with real input data given by the matrices $A \in \mathbb{R}^{m \times p}$ and $G \in \mathbb{R}^{m \times q}$, and the vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^p, h \in \mathbb{R}^q$. However, it is necessary to restrict to MILP with rational input data for the derivation of some results and algorithms. We observe that by suitable scaling all input data can also be assumed to be integral in this case. Moreover, if $q = 0$

or $p = 0$ in (1.1) we obtain an *integer linear program (ILP)* or a *linear program (LP)*, respectively.

If we omit the integrality constraint on x in (1.1) we obtain the *LP relaxation* of the (M)ILP. Its feasible domain is given by

$$P := \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}. \tag{1.2}$$

We define the *(mixed) integer hull* P_I of P as the convex hull of the feasible points of the (M)ILP, so

$$P_I := \text{conv}\,\{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q : (x, y) \in P\}. \tag{1.3}$$

For a polyhedron P which is either rational or bounded, its (mixed) integer hull P_I is again a polyhedron. This was first proven by Meyer [Mey74].

Theorem 1.1 ([Mey74])**.** *Let $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ be either rational or bounded. Then P_I is a polyhedron.* □

Throughout this thesis, the sets P, P_I always refer to the feasible domain of the LP relaxation of a (M)ILP and its (mixed) integer hull.

The pure integer hull P_I of an arbitrary polytope P is always rational. Moreover, there exists a 'rational approximation' of P.

Lemma 1.2. *Let $P \subseteq \mathbb{R}^p$ be a polytope and P_I its (pure) integer hull. Then there exists a rational polytope Q such that $P \subseteq Q$ and $P_I = Q_I$.*

Proof. As P is bounded, there exists an integer M such that

$$P \subseteq \{x \in \mathbb{R}^p : ||x||_\infty \leq M\} =: S.$$

Now for each integral vector $z \in S \setminus P$ there exists a rational half-space $\alpha_z x \leq \gamma_z$ containing P but not containing z. So we can take for Q the (finite) intersection of S with all these half-spaces. □

At some points it is more convenient to consider *MILP* which are given in *standard form* by equations and non negativity constraints on x and y. So we have

$$\begin{array}{ll} \max & cx + hy \\ & Ax + Gy = b \\ & x \geq 0, y \geq 0 \\ & x \in \mathbb{Z}^p, \end{array} \tag{1.4}$$

where $A \in \mathbb{R}^{m \times p}, G \in \mathbb{R}^{m \times q}$ and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^p, h \in \mathbb{R}^q$. Both representations of a MILP in (1.1) and (1.4) are equivalent and can be transformed into each other.

A vertex (x^*, y^*) of the LP relaxation P of a MILP (1.1) and (1.4) can be assigned to at least one *basis* B. Here, in the first case, a basis B is a subset of $p+q$ rows of the system $Ax+Gy \leq b$ such that $A_B x^* + G_B y^* = b_B$ and the submatrix $(A\ G)_B$ is regular. In the second case, a basis B is a subset of m columns such that $(x^*, y^*)_i = 0$ for $i \notin B$. We denote by $\mathcal{B}$ the set of all bases B of a vertex (x^*, y^*).

To be able to sort several solutions of a (MI)LP, we define the *lexicographic maximal solution*

$$(x^*, y^*) = \text{arglexmax}\,\{cx + hy : Ax + Gy \leq b\} \tag{1.5}$$

of a (mixed integer) linear program. Here we say that a solution (x^1, y^1) is lexicographic greater than a solution (x^2, y^2) if the first non zero entry of the difference vector

$$(cx^1 + hy^1, x^1, y^1) - (cx^2 + hy^2, x^2, y^2)$$

is greater than zero. To obtain a lexicographic maximal solution of a LP one can use for example a lexicographic version of the simplex algorithm.

A *cutting plane* for a (M)ILP is an inequality $\alpha x + \beta y \leq \gamma$ with $\alpha \in \mathbb{R}^p, \beta \in \mathbb{R}^q$ and $\gamma \in \mathbb{R}$ which is valid for P_I but not for P. Moreover, we define more general a cutting plane $\alpha x + \beta y \leq \gamma$ for a closed convex set $S \subseteq \mathbb{R}^{p+q}$ as an inequality which is not valid for S but which is valid for all $(x, y) \in S$ with $x \in \mathbb{Z}^p$.

The *projection* of polyhedra $P \subseteq \mathbb{R}^{p+q}$ on the p-dimensional subspace $\mathbb{R}^p$ of integral variables will be an important tool later in this thesis. Here the projection $\text{proj}_X(P)$ of P on the x-space $\mathbb{R}^p$ is defined by

$$\text{proj}_X(P) := \{x \in \mathbb{R}^p : \ \exists y \in \mathbb{R}^q : (x, y) \in P\}. \tag{1.6}$$

The projection $\text{proj}_X(P)$ can be described by the following lemma. Here we say that a set of vectors $\{v^1, \ldots, v^r\}$ is the set of *extreme rays* of a cone Q if Q is generated by $\{v^1, \ldots, v^r\}$.

Lemma 1.3. *Let $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ be a polyhedron. Then*

$$\text{proj}_X(P) = \{x \in \mathbb{R}^p : \ v^r Ax \leq v^r b \ \forall v^r \in R\},$$

where R is the set of extreme rays of the cone $Q := \{v \in \mathbb{R}^m : \ G^T v = 0, v \geq 0\}$.

Proof. The statement follows by applying the Farkas Lemma; see for example [NW88]. □

We add the following variants of Lemma 1.3. If $y \geq 0$ in the definition of P, the projection cone Q is given by $Q = \{v \in \mathbb{R}^m : \ G^T v \geq 0, v \geq 0\}$. If P is additionally given by equality constraints $Ax + Gy = b$, we have $Q = \{v \in \mathbb{R}^m : \ G^T v \geq 0\}$. It is crucial for the computation of the projection $\text{proj}_X(P)$ to enumerate the extreme rays of the cone Q. We discusse this issue in Section 4.3.2. We note that another way for computing the projection of a polyhedron is given by the Fourier-Motzkin elimination procedure; see [Sch86].

We also consider sequences of convex compact nested sets. Therefore, we define more general the convergence of a sequence of compact nested sets.

Definition 1.4. *Let $(S^i)_{i\in\mathbb{N}}$ be a sequence of compact nested sets. We say that the sequence S^i converges to a set S if for any $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that*

$$\min_{x\in S} ||x^k - x|| < \epsilon \quad \text{for all } k \geq k_0 \text{ and any } x^k \in S^k.$$

In this case we also write $\lim_{i\to\infty} S^i = S$.

We note that the definition of convergence in Definition 1.4 is in line with the known concept of the *Hausdorff-distance* $\delta(K, L)$ of two compact sets $K, L \subseteq \mathbb{R}^n$ which is given by

$$\delta(K, L) = \min\{\lambda \geq 0 : K \subseteq L + \lambda B^n,\ L \subseteq K + \lambda B^n\},$$

where B^n denotes the n-dimensional unit ball. In detail, as $S \subseteq S^i\ \forall i \in \mathbb{N}$ by assumption of Definition 1.4 it is

$$\delta(S, S^i) = \min\{\lambda \geq 0 : S^i \subseteq S + \lambda B^n\} = \max_{x^i\in S^i} \min_{x\in S} ||x^i - x||,$$

see [Sch93] for more details.

We state two useful criteria which deal with the convergence of a sequence S^i to a polytope P and with the convergence of a sequence of projections $\text{proj}_X(S^i)$.

Lemma 1.5. *Let $P = \{x \in \mathbb{R}^p : a_k x \leq b_k, k \in \{1, \ldots, m\}\}$ be a polytope and let $(S^i)_{i\in\mathbb{N}}$ be a sequence of compact nested sets. Then*

$$\lim_{i\to\infty} S^i = P \iff \lim_{i\to\infty} \max\{a_k x : x \in S^i\} = b_k \quad \text{for all } k \in \{1, \ldots, m\}.$$

Proof. The statement follows directly by comparing the definitions of convergence for the sequences S^i and $\max\{a_k x : x \in S^i\}$. □

Lemma 1.6. *Let $(S^i)_{i\in\mathbb{N}} \subseteq \mathbb{R}^{p+q}$ be a sequence of compact nested sets converging to a set $S \subseteq \mathbb{R}^{p+q}$. Then*

$$\lim_{i\to\infty} \text{proj}_X(S^i) = \text{proj}_X(S).$$

Proof. The statement follows directly by Definition 1.4 as for $(x^i, y^i) \in S^i, (x, y) \in S$ it is $||(x^i, y^i) - (x, y)|| \geq ||x^i - x||$. □

According to (1.1), we allow MILP with real input data. Here we note that we have to use the real *Random Access Model* to be able to deal formally with computations with real input data in algorithms. We refer to [PS85] or [Pap94] for a detailed introduction into this concept.

Finally, we discuss two assumptions which we make during this thesis. The first one deals with the objective function vector (c, h) of a MILP. We assume in the remainder of this thesis that the objective function vector (c, h) of a MILP according to (1.1) and (1.4) satisfies the property $h \neq 0$. Otherwise, the related MILP can be treated like the pure integer program

$$\max\{cx : x \in \mathbb{Z}^p, x \in \text{proj}_X(P)\}. \tag{1.7}$$

In detail, the following lemma holds.

Lemma 1.7. *Let a MILP according to* (1.1) *or (1.4) with* $h = 0$ *be given and let* x^* *be an optimal solution of (1.7). Then there exists a* $y \in \mathbb{R}^q$ *such that* $(x^*, y) \in P$*, and every* $(x^*, y) \in P$ *is an optimal solution of the MILP.*

Proof. The statement follows directly by the definition of the projection (1.6). □

Moreover, we often restrict ourselves to (M)ILP in which the feasible domain $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ of the LP relaxation is bounded. This assumption is required for some results and algorithms to achieve finiteness and to guarantee that the mixed integer hull of a real polyhedron is again a polyhedron. However, it does not form a strong restriction in theory. If an optimal solution of a (M)ILP exists, than it is always contained in a sufficiently large bounded subset of the LP relaxation P. We also avoid the case that the objective function is unbounded over the LP relaxation P. In this situation, the (M)ILP is either infeasible or the objective function is also unbounded over the (mixed) integer hull P_I; see [Mey74].

1.3 Acknowledgments

First of all, I would like to thank my thesis advisor Peter Gritzmann for inviting me to his research group, for drawing my attention to the field of mixed integer cutting planes, and for his support during my work on this thesis.

Moreover, I want to thank all current and former members of our working group "Angewandte Geometrie & Diskrete Mathematik" at Technische Universität München for their support and for the pleasant working atmosphere. Special thanks go to Oliver Bastert and Michael Ritter who always helped me with all minor and major problems during the development of this thesis in the last three years.

I also would like to thank Matthias Köppe and Quentin Louveaux for some helpful comments on the preprint [Jör07].

2 Cutting planes for MILP

In this chapter we discuss the foundations of cutting planes for MILP and introduce some representatives and algorithms. According to the motivation of this thesis, we focus on properties of cuts from a theoretical point of view and follow a geometric approach. However, remarks on their practical application are given at some points. Moreover, we only consider cuts which apply to general MILP. This means that cutting planes for special problem classes such as for example lift-and-project cuts for mixed binary programs or cover inequalities are not respected. As a supplement to the successive sections, we refer again to [Cor08] and [MMWW02] for extensive review papers on several aspects of cutting planes for MILP and to [CL01] for an overview of the relation of various cutting planes.

We start with the special case of Chvátal-Gomory cuts for ILP in Section 2.1 and turn to cuts for general MILP in Section 2.2. We explain the basic idea of deriving cuts for MILP and discuss split cuts and different cuts which arise in computing split cuts such as for example intersection cuts, mixed integer Gomory cuts, mixed integer rounding inequalities, and some alternatives. We finish this section with the recently developed so called cutting planes from two rows of the simplex tableau as an example which is no split cut.

2.1 Chvátal-Gomory cuts

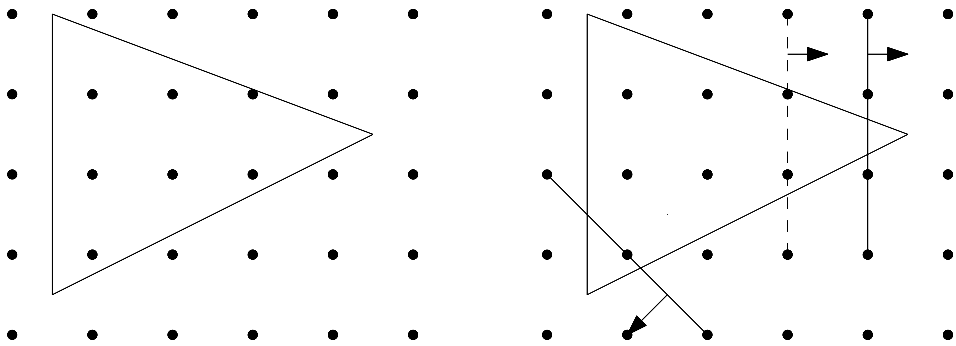
In this section we consider ILP which are given by

$$\begin{aligned} \max \quad & cx \\ & Ax \leq b \\ & x \in \mathbb{Z}^p \end{aligned} \tag{2.1}$$

as a special case of MILP according to (1.1). Cutting planes for ILP can be easily generated by the following rounding property.

Lemma 2.1. *Let $P \subseteq \mathbb{R}^p$ be a polyhedron and P_I its integer hull. If for $\alpha \in \mathbb{Z}^p$ and $\gamma \in \mathbb{R}$ the inequality $\alpha x \leq \gamma$ is valid for P then $\alpha x \leq \lfloor \gamma \rfloor$ is valid for P_I.*

Proof. Since for every integral point $x \in P_I$ the left hand side of $\alpha x \leq \gamma$ is integral, the right hand side can be rounded down to the next integer. □



(a) A polyhedron P in $\mathbb{R}^2$

(b) Two Chvátal-Gomory cuts (solid) for P. The dashed line is valid for P_I but no Chvátal-Gomory cut for P.

Figure 2.1: Examples of Chvátal-Gomory cuts

Cutting planes which are derived by this rounding property are known as *Chvátal-Gomory cuts*. We note that their derivation does not depend on the structure of P or P_I. In detail, $\alpha x \leq \lfloor \gamma \rfloor$ is a valid inequality for the integer hull P_I of every polyhedron P for which $\alpha x \leq \gamma$ is valid. We will see in the next section that the derivation of valid cuts for MILP also requires information of the structure of its feasible domain.

Chvátal-Gomory cuts turn out to be sufficient for generating the integer hull P_I of a rational or bounded polyhedron. In the following we say that $\alpha x \leq \lfloor \gamma \rfloor$ is a Chvátal-Gomory cut for a polyhedron P if it is a cutting plane and $\alpha x \leq \gamma$ is valid for P. We start with the case of rational polyhedra.

Definition 2.2. *Let $P \subseteq \mathbb{R}^p$ be a rational polyhedron. Then the intersection of P and all Chvátal-Gomory cuts for P is called the Chvátal-Gomory closure of P and denoted by $P^{(1)}$. Accordingly, for $i \in \mathbb{N}$ the i-th Chvátal-Gomory closure $P^{(i)}$ of P is defined as the Chvátal-Gomory closure of $P^{(i-1)}$.*

The second part of the last definition is particularly well defined as the Chvátal-Gomory closure of a rational polyhedron is again a polyhedron.

Theorem 2.3. *Let $P \subseteq \mathbb{R}^p$ be a rational polyhedron. Then the Chvátal-Gomory closure $P^{(1)}$ is again a polyhedron.*

Proof. A proof of this theorem can be found in [Sch86]. □

The last theorem only refers to rational polyhedra. It is open if the Chvátal-Gomory closure of a general polytope with real input data is again a polyhedron. It is only known that Theorem 2.3 remains true if P is bounded and $P^{(1)}$ has an empty intersection with the boundary of P; see [Sch80]. Therefore, we expand the definition of a Chvátal-Gomory cut and the related closure to closed convex sets.

Definition 2.4. *Let $S \subseteq \mathbb{R}^p$ be a closed convex set. Then an inequality $\alpha x \leq \lfloor \gamma \rfloor$ is a Chvátal-Gomory cut for S if it is a cutting plane and $\alpha x \leq \gamma$ is valid for S. The intersection of S and all Chvátal-Gomory cuts for S is called the Chvátal-Gomory closure of S and denoted by $S^{(1)}$. Accordingly, for $i \in \mathbb{N}$ the i-th Chvátal-Gomory closure $S^{(i)}$ of S is defined as the Chvátal-Gomory closure of $S^{(i-1)}$.*

Now we can deal with the generation of the integer hull.

Theorem 2.5. *Let $P \subseteq \mathbb{R}^p$ be a rational or bounded polyhedron. Then there exists a $k \in \mathbb{N}$ with $P^{(k)} = P_I$. k is also called the Chvátal-Gomory rank of P.*

Proof. A proof can again be found in [Sch86]. First, the statement is proven for rational polyhedra. In a second step, the result can be transferred to general polytopes by applying the result to a rational polytope Q with $P \subseteq Q$ and $P_I = Q_I$ according to Lemma 1.2. The result was shown first by Chvátal [Chv73] for rational polytopes and by Schrijver [Sch80] for the general case. □

Theorem 2.5 shows that Chvátal-Gomory cuts are sufficient to generate the integer hull of an arbitrary rational or bounded polyhedron. Next, it is the question how to find suitable cuts which cut off infeasible vertices of the LP relaxation P within a cutting plane algorithm for ILP and guarantee finiteness of the procedure. An approach to deal with these issues is given by the (pure) integer Gomory cut which only applies to rational ILP. So we assume for the remainder of this section that all input data of the ILP (2.1) is rational. This means that we can even assume by suitable scaling that the input data is integral. We discuss a cutting plane algorithm for ILP with general input data at the end of Section 2.2.3.

The set of all Chvátal-Gomory cuts for a polyhedron $P = \{x \in \mathbb{R}^p : Ax \leq b\}$ can be obtained by taking all supporting hyperplanes $\alpha x \leq \gamma$ of P with $\alpha \in \mathbb{Z}$, $\gamma \notin \mathbb{Z}$ and rounding down the right hand side. This set can be described by all inequalities

$$vAx \leq \lfloor vb \rfloor, \text{ with } v \geq 0,\ vA \in \mathbb{Z}^p,\ vb \notin \mathbb{Z}. \tag{2.2}$$

So to cut off an infeasible vertex x^* of P by a cutting plane, we need a hyperplane that supports the polyhedron P in x^* and satisfies the above conditions for α and γ. For this purpose, we have to determine a suitable multiplier v in (2.2). This can be done by the following definition of the Gomory cut for a vertex x^* with related basis B and an integral vector α with $\alpha x^* \notin \mathbb{Z}$. We note that all input data is integral by assumption.

Definition 2.6. *Let $P = \{x \in \mathbb{R}^p : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times p}, b \in \mathbb{Z}^m$ be a polyhedron and $x^* \notin \mathbb{Z}^p$ a vertex of P with related basis matrix A_B and right hand side b_B. Then the Gomory cut to the vector $\alpha \in \mathbb{Z}^p$ with $\alpha x^* \notin \mathbb{Z}$ is defined by*

$$vA_Bx \leq \lfloor vb_B \rfloor, \text{ with } v = \alpha A_B^{-1} - \lfloor \alpha A_B^{-1} \rfloor. \tag{2.3}$$

By definition, (2.3) is a valid inequality for P_I as $v \geq 0$ and $vA_B \in \mathbb{Z}^p$. Moreover, it is $vb \notin \mathbb{Z}$ if and only if $\alpha x^* \notin \mathbb{Z}$, as

$$vb = \alpha A_B^{-1} b - \lfloor \alpha A_B^{-1} \rfloor b = \alpha x^* - \lfloor \alpha A_B^{-1} \rfloor b,$$

and the second term is integral.

By Definition 2.6, we can determine a cutting plane that cuts off an infeasible vertex x^* of P if we find an arbitrary vector $\alpha \in \mathbb{Z}^p$ with $\alpha x^* \notin \mathbb{Z}$. The easiest way to satisfy this condition is to choose an unit vector u_i for which the entry x_i^* of the vertex x^* is not integral. Moreover, this choice for α is almost sufficient for a finite cutting plane algorithm. We additionally have to respect cuts with $\alpha = c$ to the objective function vector c of the ILP, only. Altogether, we obtain a finite algorithm using cuts according to (2.3) with $\alpha \in \{c, u_1, \ldots, u_p\}$. We do not give the details of the algorithm here, but refer to Algorithm 1 and Theorem 2.20 as the algorithm and the proof of convergence are identically to a special mixed integer case. Both the Gomory cut (2.3) and the related algorithm were originally developed by Gomory for ILP given in standard form (1.4); see [Gom58, Gom60, Gom63].

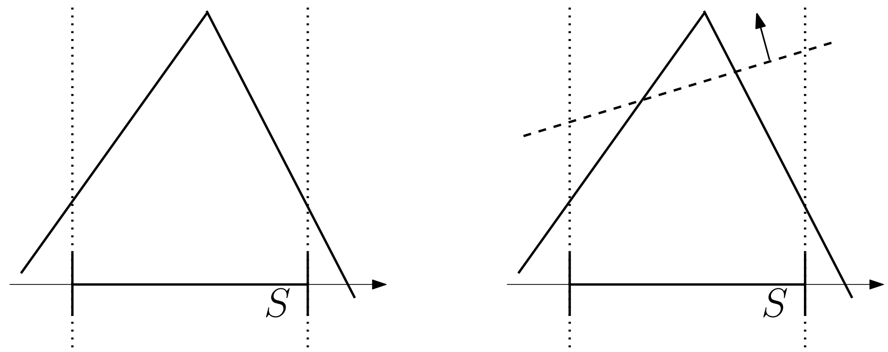
2.2 Cuts for general MILP

We now turn to cutting planes for MILP. As one can easily see, the rounding procedure of Lemma 2.1 fails in generating valid inequalities for MILP. If $\alpha x + \beta y \leq \gamma$ with $\alpha \in \mathbb{Z}^p, \beta \in \mathbb{Z}^q$ and $\gamma \in \mathbb{R}$ is valid for a polyhedron P the inequality $\alpha x + \beta y \leq \lfloor \gamma \rfloor$ is not valid for P_I in general as βy has not to be integral. Therefore, an other approach for deriving valid cuts is required. This one is based on the local properties of P at a vertex (x^*, y^*) and is not constructive at first. We explain the basic idea.

Let $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ be a polyhedron and (x^*, y^*) a vertex of P with $x^* \notin \mathbb{Z}^p$. We are looking for an inequality $\alpha x + \beta y \leq \gamma$ that is valid for P_I and cuts off (x^*, y^*). We take a closed convex set $S \subseteq \mathbb{R}^p$ with

$$x^* \in \operatorname{int}(S) \text{ and } x \notin \operatorname{int}(S) \; \forall x \in \mathbb{Z}^p, \tag{2.4}$$

that means S contains x^* in its interior but no integral vector $x \in \mathbb{Z}^p$. In this situation, every inequality $\alpha x + \beta y \leq \gamma$ which only cuts off points $(x, y) \in P$ with $x \in \operatorname{int}(S)$ is valid for P_I by construction as no vectors $(x, y) \in P$ with $x \in \mathbb{Z}^p$ are cut off.



(a) A polyhedron $P \subseteq \mathbb{R}^{1+1}$ and a set $S \subseteq \mathbb{R}^1$

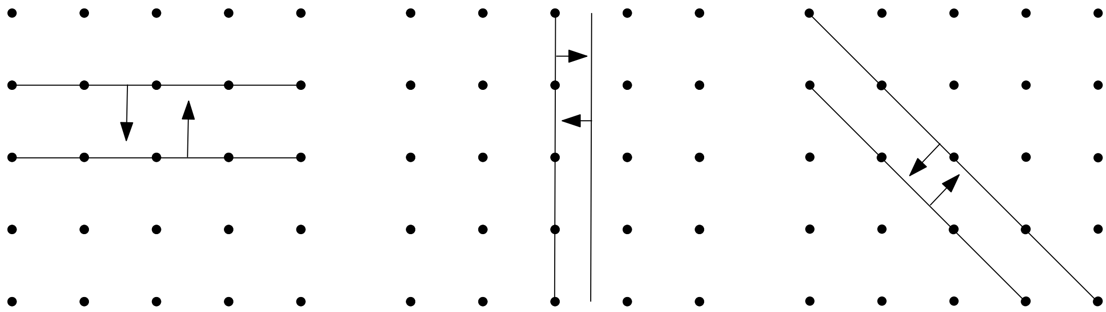
(b) A valid cut for P to S

Figure 2.2: Derivation of cutting planes for MILP

In this way the derivation of valid mixed integer cutting planes is based on two steps. First, we choose a suitable set S according to (2.4) and then compute a cut in relation to the set S in the second step. A two dimensional example of this approach is also given in Figure 2.2.

We note that all valid inequalities for P_I can be described as a cutting plane in relation to a set S according to (2.4). We prove this statement later in Lemma 3.5 in a more specific form. In the next subsections we discuss in detail how valid cuts can be computed explicitly. We deal with the issues of how a suitable set S can be found and how a cut can be computed given a fixed set S. Here it is especially the question how to determine deep cutting planes. We note that the properties of the set S to which a cut is computed even provide a way to characterize different cutting planes, for example S being polyhedral or not.

In the following we only deal with the case that S is a polyhedron and use a slightly different notation which is more convenient. Instead of taking a polyhedron S which contains no integer point in its interior, we consider disjunctions D which are defined as the closed complement of S. So a disjunction D contains every integral point $x \in \mathbb{Z}^p$ and does not contain points that are supposed to be cut off. We formally introduce disjunctions in Section 2.2.1 in a special case and in Section 3.1 in general. The most used type of a disjunction D for deriving valid cutting planes is a split disjunction which is defined by two hyperplanes. After we have defined it in the next section, the remainder of this chapter mainly deals with various approaches to compute split cuts. We introduce intersection cuts in Section 2.2.2, mixed integer Gomory cuts in Section 2.2.3, mixed integer rounding cuts in Section 2.2.4, and methods for strengthening mixed integer Gomory cuts in Section 2.2.5. In Section 2.2.6 we briefly introduce the recent approach of cutting planes from two rows of a simplex tableau as an example of a more general disjunctive cut.



(a) Split disjunction $D(u_2, 2)$

(b) Split disjunction $D(2u_1, 4)$

(c) Split disjunction $D(u_1 + u_2, 3)$

Figure 2.3: Examples of split disjunctions in $\mathbb{R}^2$

2.2.1 Split cuts

Split cuts were introduced by Cook, Kannan, and Schrijver in 1990 [CKS90]. They are disjunctive cuts to special two term disjunctions and are not defined constructively. They are easy to describe and include many special types of cutting planes. So results based on the properties of split cuts have a wide importance. We start with the definition of a split disjunction and the related split cut.

Definition 2.7. *Let $d \in \mathbb{Z}^p$ be an integral vector and $\delta \in \mathbb{Z}$ an integer. Then the inequalities*

$$dx \leq \delta \ \vee \ dx \geq \delta + 1$$

are called a split disjunction. We write $D(d, \delta)$ for the split disjunction defined by d, δ.

Definition 2.8. *Let $P \subseteq \mathbb{R}^{p+q}$ be a polyhedron and $\alpha x + \beta y \leq \gamma$ be a cutting plane. Then $\alpha x + \beta y \leq \gamma$ is called a split cut for P if there exists a split disjunction $D(d, \delta)$ with*

$$(x, y) \in P \wedge \ \alpha x + \beta y > \gamma \Longrightarrow \delta < dx < \delta + 1.$$

Every split cut for P is a valid cutting plane for P_I. This is easy to see as for every feasible solution $(x^*, y^*) \in P$ with $x^* \in \mathbb{Z}^p$ it is $dx^* \leq \delta \ \vee \ dx^* \geq \delta + 1$ and so (x^*, y^*) is contained in every possible split disjunction.

In line with the Chvátal-Gomory closure in Definition 2.2 and Definition 2.4 we define the split closure of a polyhedron P and analyze some of its properties.

Definition 2.9. *Let $P \subseteq \mathbb{R}^{p+q}$ be a rational polyhedron. Then the intersection of P and all split cuts for P is called the split closure of P and denoted by $P^{(1)}$. Accordingly, for $i \in \mathbb{N}$ the i-th split closure $P^{(i)}$ of P is defined as the split closure of $P^{(i-1)}$.*

The definition of the i-th split closure $P^{(i)}$ is again well defined as finite many cutting planes are already sufficient to generate the split closure of a rational polyhedron. This result is given next and was first proved by Cook, Kannan, and Schrijver.

Theorem 2.10 ([CKS90]). *Let $P \subseteq \mathbb{R}^{p+q}$ be a rational polyhedron. Then the split closure $P^{(1)}$ of P is a polyhedron.* □

Other proofs of this property were given by Andersen, Cornuéjols, and Li [ACL05b], Dash, Günlük, and Lodi [DGL07], and Vielma [Vie07]. We get back to this statement and its proofs in Section 3.1. As in the case of the Chvátal-Gomory closure, it is not known if the split closure of a real polytope is again a polytope. So we generalize the definition of a split cut to closed convex sets.

Definition 2.11. *Let $S \subseteq \mathbb{R}^{p+q}$ be a closed convex set and $\alpha x + \beta y \leq \gamma$ be a cutting plane. Then $\alpha x + \beta y \leq \gamma$ is called a split cut for S if there exists a split disjunction $D(d, \delta)$ with*

$$(x, y) \in S \wedge \alpha x + \beta y > \gamma \Longrightarrow \delta < dx < \delta + 1.$$

The intersection of S and all split cuts for S is called the split closure of S and denoted by $S^{(1)}$. Accordingly, for $i \in \mathbb{N}$ the i-th split closure $S^{(i)}$ of S is defined as the split closure of $S^{(i-1)}$.

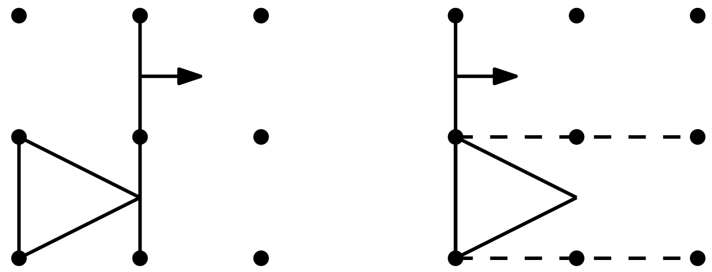
We have used the same notation for the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^p$ in the integral case and for the split closure of a polyhedron $P \subseteq \mathbb{R}^{p+q}$ in the mixed integer case. In this context we note that the Chvátal-Gomory closure of a polyhedron is contained in its split closure. This is easy to see as every Chvátal-Gomory cut $\alpha x \leq \gamma$ for a polyhedron $P \subseteq \mathbb{R}^p$ according to Lemma 2.1 is a split cut to the disjunction $D(\alpha, \gamma)$ by definition. Conversely, not every split cut for P has to be a Chvátal-Gomory cut. This can be seen in the following

Example 2.12. *Let $P \subseteq \mathbb{R}^2$ the polyhedron defined as the convex hull of the vertices*

$$(0, 0), (0, 1), \left(1, \frac{1}{2}\right).$$

The (pure) integer hull is given by $P_I = \operatorname{conv}\{(0, 0), (0, 1)\}$. The cut $x_1 \leq 0$ is no Chvátal-Gomory cut to P, as the inequality $x_1 \leq 1$ supports P. However, $x_1 \leq 0$ is a split cut for P to the disjunction $D(u_2, 0)$, see Figure 2.4;

Due to the last considerations we could interpret split cuts as a straightforward generalization of the Chvátal-Gomory cuts for MILP. However, unlike the pure integer case split cuts are not sufficient for generating the mixed integer hull of an arbitrary polyhedron in general. This can be seen in the following 'classical' example of Cook, Kannan, and Schrijver [CKS90] which has also been discussed before in similar form in [Whi61], [Pad], and [Sal75].



(a) $x_1 \leq 0$ is no Chvátal-Gomory cut

(b) $x_1 \leq 0$ is a split cut to the split disjunction $D(u_2, 0)$

Figure 2.4: To Example 2.12

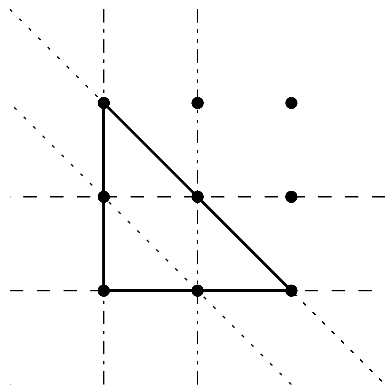


Figure 2.5: The integer hull P_I of Example 2.13 projected on the x_1x_2-space. It is not possible to cover all vertices of P_I with the boundary of an arbitrary split disjunction.

Example 2.13 ([CKS90])**.** *Let $P \subseteq \mathbb{R}^{2+1}$ the polyhedron defined as the convex hull of the four vectors*

$$(0,0,0), (2,0,0), (0,2,0), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Then the integer hull P_I is the convex hull of the vectors

$$(0,0,0), (2,0,0), (0,2,0).$$

On the other hand, the valid inequality $y \leq 0$ is no split cut for all polyhedra $P^{(i)}, i \in \mathbb{N}_0$. A proof of this statement in a more general setting is given in Lemma 3.20.

The last example has also direct consequences for cutting plane algorithms. It is obvious that an arbitrary algorithm which is based on split cuts does not solve a MILP in finite time in general. We will get back later to this basic result.

At last we want to note two positive results. Split cuts can be used to approximate the mixed integer hull of an arbitrary polytope arbitrarily exact. This enables us to obtain the mixed integer hull of a rational polytope by combining split cuts with certain rounding cuts based on the numerical properties of the input data. We deal with this property more detailed in Section 3.2. Moreover, in the special case of mixed binary programs, split cuts are sufficient for generating the mixed interger hull P_I of a polyhedron P; see for example [Cor08].

2.2.2 Intersection cuts

Intersection cuts were introduced by Balas in 1971 [Bal71]. They provide a possibility to compute a valid cutting plane to a given disjunction D and are derived from a basis solution (x^*, y^*) of the LP relaxation P. Moreover, they have an easy geometric interpretation. Although intersection cuts can be applied to compute cuts to general disjunctions, we only consider the case of split disjunctions in the following as this is the most important case in applications. However, the principle can also be transferred to more general disjunctions. We presume again a MILP (1.1) and describe the idea of the derivation of the cut.

Let again $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ be a polyhedron and (x^*, y^*) a vertex of P with $x^* \notin \mathbb{Z}^p$ and let $D(d, \delta)$ be a split disjunction which does not contain x^*, so $\delta < dx^* < \delta + 1$. Given a basis submatrix $(A\ G)_B$ of $(A\ G)$ to the vertex (x^*, y^*), the extreme rays $(r_j, s_j) \in \mathbb{R}^{p+q}$ of the related basis cone are given by the columns of the matrix $-(A\ G)_B^{-1}$. We take for all $j \in \{1, \dots, p+q\}$ the intersection points $z_j \in \mathbb{R}^{p+q}$ of the rays

$$(x^*, y^*) + \lambda_j (r_j, s_j),\ \lambda_j \geq 0$$

with the hyperplanes $dx = \delta$ and $dx = \delta + 1$ of the split disjunction $D(d, \delta)$. If one of the rays (r_j, s_j) intersects neither of the two hyperplanes, it is parallel to them and we can choose an arbitrary point z_j on this ray.

Now we consider the inequality $\alpha x + \beta y \leq \gamma$ which is defined by the $p + q$ points z_j and satisfies $\alpha x^* + \beta y^* > \gamma$. This inequality is valid for P_I as it is a split cut by construction. Moreover, this cut can also be computed by a closed formula.

Definition 2.14. *Let (x^*, y^*) be a vertex of a polyhedron $P \subseteq \mathbb{R}^{p+q}$, $(A\ G)_B$ a related basis submatrix, (r_j, s_j) the columns of $-(A\ G)_B^{-1}$, and $D(d, \delta)$ a split disjunction that does not contain x^*. Let for $j = 1, \dots, p + q$*

$$\lambda_j := \begin{cases} \frac{\delta - dx^*}{dr_j}, & \text{if } dr_j < 0 \\ \frac{\delta - dx^* + 1}{dr_j}, & \text{if } dr_j > 0 \\ \infty, & \text{if } dr = 0. \end{cases} \tag{2.5}$$

and define the vector $l \in \mathbb{R}^{p+q}$ by $l_j := \frac{1}{\lambda_j}$ (with $0 =: \frac{1}{\infty}$). Then the intersection cut to the basis B and the disjunction D is given by $\alpha x + \beta y \leq \gamma$ with

$$(\alpha, \beta) := l(A\ G)_B \quad \text{and} \quad \gamma := lb_B - 1. \tag{2.6}$$

Lemma 2.15. *$\alpha x + \beta y \leq \gamma$ with α, β, γ according to (2.6) is a valid cutting plane for P_I.*

Proof. We show that the intersection cut is a split cut to the split disjunction $D(d, \delta)$. We suppose that there exists a vector (v, w) with $A_B v + G_B w \leq b$ and $\alpha v + \beta w > \gamma$ which is contained in the disjunction $D(d, \delta)$. (v, w) has an unique representation as a positive linear combination of the rays (r_j, s_j) of the basis cone of B, so

$$(v, w) = (x^*, y^*) + \sum_{j=1}^{p+q} \mu_j (r_j, s_j), \text{ with } \mu_j \geq 0,\ j \in \{1, \ldots, p+q\}.$$

If $dr_j = 0$ for a $j \in \{1, \ldots, p+q\}$ we have

$$\alpha r_j + \beta s_j = l(A_B r_j + G_B s_j) = -\frac{1}{\lambda_j} u_j = 0,$$

so we can assume without loss of generality that $\mu_j = 0$ in the representation of (v, w).

Define for $j \in \{1, \ldots, p+q\}$ with $dr_j \neq 0$ the points

$$z_j := (x^*, y^*) + \lambda_j (r_j, s_j)$$

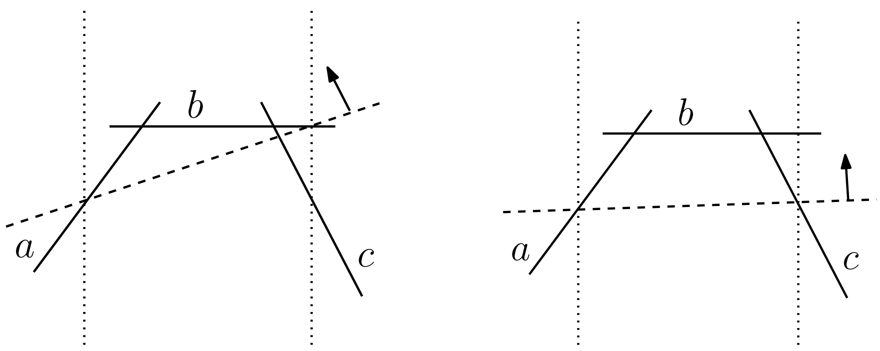
with λ_j according to (2.5). It is easy to see that the points z_j define the intersection of the rays (r_j, s_j) with the disjunctive hyperplanes $dx = \delta$ and $dx = \delta + 1$. Moreover, it is

$$(\alpha\ \beta) z_j = l(A_B x^* + G_B y^*) + l\lambda_j (A_B r_j + G_B s_j) = lb - 1 = \gamma.$$

Now we consider the set

$$S := \{(x, y) \in \mathbb{R}^{p+q} : (x, y) = (x^*, y^*) + \sum_{j=1, dr_j \neq 0}^{p+q} \nu_j (r_j, s_j), \nu_j \geq 0 \wedge \alpha x + \beta y > \gamma\}$$

of vectors that is cut off by the intersection cut and can be represented by rays with $dr_j \neq 0$. It follows that each vector $(x, y) \in S$ is a convex combination of (x^*, y^*) and the intersection points z_j and satisfies $\delta < dx < \delta + 1$. So $(v, w) \notin S$ in contradiction to the assumption. $\square$



(a) Intersection cut to the basis a, b

(b) Intersection cut to the (infeasible) basis a, c

Figure 2.6: Examples of intersection cuts

A graphical example of an intersection cut is given in Figure 2.6. The derivation of the intersection cut is based on the geometry of the basis cone, so it is independent of the representation of the system $A_B x + G_B y \leq b_B$. However, the intersection cut depends of course on the selection of the basis B and of the split disjunction $D(d, \delta)$ to which the cut is computed. We will see in the remainder of this chapter that intersection cuts play a vital role in computing valid cutting planes. Many cuts are actually equal to or dominated by an intersection cut to a certain split disjunction.

By our construction, every intersection cut is a split cut. But also the converse direction is true in this way that a finite set of intersection cuts is sufficient to generate the split closure $P^{(1)}$ of a rational polyhedron $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$. Here also intersection cuts to infeasible bases of the polyhedron P have to be respected. This means that we also derive intersection cuts according to Definition 2.14 for vertices (x^*, y^*) with basis B which are only valid for a subsystem of $Ax + Gy \leq b$. The connection between split cuts and intersection cuts was investigated in detail by Andersen, Cornuéjols, and Li [ACL05b]. We state one result explicitly which deals with the computation of cuts to a single split disjunction. We note that adding all cuts to a split disjunction $D(d, \delta)$ to a polyhedron P leads to the new polyhedron

$$P_D := \operatorname{conv}\{(x, y) \in P : dx \leq \delta \vee dx \geq \delta + 1\}. \tag{2.7}$$

Theorem 2.16 ([ACL05b]). *Let $D(d, \delta)$ be a split disjunction, $P \subseteq \mathbb{R}^{p+q}$ be a polyhedron and P_D according to (2.7). Moreover, let $\mathcal{B}$ the set of all (feasible and infeasible) bases of P and let $\alpha_B x + \beta_B y \leq \gamma_B$ be the intersection cut to the disjunction D and basis B. Then*

$$P_D = \bigcap_{B \in \mathcal{B}} (P \cap \{(x, y) : \alpha_B x + \beta_B \leq \gamma_B\}).$$

□

The last theorem becomes very simple when P is a basis cone. In this case, every split cut to the disjunction D is equal to or dominated by the only possible intersection cut. Andersen, Cornuéjols, and Li also use the last result to give a further proof of the polyhedrality of the split closure of a rational polyhedron (Theorem 2.10).

Finally, we get back to intersection cuts to general disjunctions. It is obvious that Definition 2.14 can be simply modified and that Lemma 2.15 remains true accordingly in the more general case. However, similar results for Theorem 2.16 cannot be stated in general. This is considered in Example 3.4.

2.2.3 Mixed integer Gomory cuts

The mixed integer Gomory cut was one of the first cutting planes for MILP. It was introduced by Gomory at the end of the 1950ies [Gom58, Gom60, Gom63]. The cut applies to a rational mixed integer set that is defined by a single equation with non negative variables. So let the set

$$S := \{(x, y) \in \mathbb{Z}^p_+ \times \mathbb{R}^q_+ : ax + gy = b\}. \tag{2.8}$$

with rational vectors $a \in \mathbb{Q}^p, g \in \mathbb{Q}^q$ and a non integral right hand side $b \in \mathbb{Q} \setminus \mathbb{Z}$ be given.

Definition 2.17. *Let S be defined according to* (2.8) *and let $b = \lfloor b \rfloor + f_0,\ a_j = \lfloor a_j \rfloor + f_j$, for $j = 1, \dots, p$. Then the mixed integer Gomory cut for S is defined by*

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \ge 1. \tag{2.9}$$

Theorem 2.18. *Inequality* (2.9) *is a valid cut for S.*

Proof. We show that (2.9) is a split cut to the disjunction $D(d, \delta)$ with $\delta = \lfloor b \rfloor$ and

$$d_i = \begin{cases} \lfloor a_i \rfloor, & \text{if } f_j \le f_0 \\ \lceil a_i \rceil, & \text{else.} \end{cases} \tag{2.10}$$

First, let (x, y) be in the LP relaxation of S with $dx \le \delta$. This inequality is by definition equivalent to

$$\sum_{f_j \le f_0} (a_j x_j - f_j x_j) + \sum_{f_j > f_0} (a_j x_j + (1 - f_j) x_j) \le \lfloor b \rfloor \iff$$
$$ax - \sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} (1 - f_j) x_j \le \lfloor b \rfloor.$$

Multiplying the last inequality by -1 and inserting in $gy = b - ax$ gives

$$gy \geq b - \lfloor b \rfloor - \sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} (1 - f_j) x_j \Longleftrightarrow$$
$$\sum_{f_j \leq f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + gy \geq f_0 \Longleftrightarrow$$
$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j + \sum_{g_j < 0} \frac{g_j}{f_0} y_j \geq 1.$$

Since the second and the last term of the left hand side of the last inequality are negative, we can see that all $(x, y) \in S$ with $dx \leq \delta$ also satisfy (2.9) as $x, y \geq 0$.

Now let (x, y) be in the LP relaxation of S with $dx \geq \delta + 1$. Repeating the above approach we obtain finally the valid inequality

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{g_j > 0} \frac{g_j}{1 - f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \geq 1.$$

As now the first and the third term of the left hand side of the last inequality are negative, we can see that all $(x, y) \in S$ with $dx \geq \delta + 1$ again satisfy (2.9). □

Mixed integer Gomory cuts can be used easily in a cutting plane algorithm for solving MILP. We assume for the moment that a MILP

$$\max\{cx + hy : (x, y) \in \mathbb{Z}^p \times \mathbb{R}^q : Ax + Gy = b, \ x, y \geq 0\}$$

in standard form is given. We solve the LP relaxation of the MILP with the simplex algorithm. If the optimal solution (x^*, y^*) of the relaxation does not satisfy the integrality constraint $x^* \in \mathbb{Z}^p$, we can apply the cut (2.9) to a row of the optimal simplex tableau which corresponds to an integral restricted basis variable x_i with $x_i^* \notin \mathbb{Z}$. Therewith, we cut off the LP solution (x^*, y^*) and can iterate this approach with the new LP relaxation. In this way we obtain an algorithm using Gomory cuts to integral rows of the simplex tableau. We give more details later.

At first, we get back to MILP (1.1) in natural form. Let a basis solution $(x^*, y^*), x^* \in \mathbb{Z}^p$ of the polyhedron $P = \{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q : Ax + Gy \leq b\}$ with basis B be given. To apply the mixed integer Gomory cut to the polyhedron P, it has to be transformed to standard form using slack variables and respecting the non negativity constraints. Here it is enough to restrict to the system $A_B x + G_B y \leq b_B$ of constraints which are part of the basis B. We obtain the system

$$A_B x^+ + G_B y^+ - A_B x^- - G_B y^- + t = b_B, \ x^+, x^-, y^+, y^-, t \geq 0.$$

with basis solution $((x^*)^+, (y^*)^+, (x^*)^-, (x^*)^-, 0)$. Next, we can apply the mixed integer Gomory cut to a row of the simplex tableau of the basis system. In the proof of Theorem 2.18 we have shown that the Gomory cut is a split cut. In fact, we will see now that the mixed integer Gomory cut to the ith-row of the simplex tableau is equivalent to the intersection cut to the split disjunction $D(u_i, \lfloor x_i^* \rfloor)$ and the basis B.

Theorem 2.19. *Let (x^*, y^*) be a basis solution of the polyhedron P, $(A\ G)_B$ a related basis submatrix, and $x_i^* \notin \mathbb{Z}$. Then the intersection cut to the split disjunction $D(u_i, \lfloor x_i^* \rfloor)$ and the basis B is equal to the mixed integer Gomory cut to the i-th component.*

Proof. Transforming the basis subsystem $A_B x + G_B y \leq b_B$ of P to standard form yields the system

$$A_B x^+ + G_B y^+ - A_B x^- - G_B y^- + t = b_B,\ x^+, x^-, y^+, y^-, t \geq 0 \tag{2.11}$$

We assume at first that the original basis solution satisfies $(x^*, y^*) \geq 0$. In this case, we obtain as basis solution of the transformed system $(x^*, y^*, 0, 0, 0)$. The matrix of the related simplex tableau is given by $(I\ \ -I\ \ (A\ G)_B^{-1})$. We now apply the mixed integer Gomory cut (2.9) to the i-th row of this tableau and obtain

$$\sum_{z_j > 0} \frac{z_j}{f_0} t_j - \sum_{z_j < 0} \frac{z_j}{1 - f_0} t_j \geq 1, \tag{2.12}$$

where

$$z_j = (A\ G)_{B,ij}^{-1} \text{ and } f_0 = ((A\ G)_B^{-1} b)_i - \lfloor ((A\ G)_B^{-1} b)_i \rfloor = x_i^* - \lfloor x_i^* \rfloor.$$

Inserting $t = b_B - A_B x - G_B y$ in (2.12) gives

$$\left(\sum_{z_j > 0} \frac{z_j}{f_0} (A\ G)_{B,j} - \sum_{z_j < 0} \frac{z_j}{1 - f_0} (A\ G)_{B,j} \right) (x, y) \leq \sum_{z_j > 0} \frac{z_j}{f_0} b_j - \sum_{z_j < 0} \frac{z_j}{1 - f_0} b_j - 1; \tag{2.13}$$

Now comparing (2.13) with the definition of the intersection cut to the disjunction $D(u_i, \lfloor x_i^* \rfloor)$ in Definition 2.14 proves the statement as $z_j = -u_i(r_j, s_j) = -dr_j$.

For $(x^*, y^*) \not\geq 0$, the statement follows in the same way by a suitable permutation of the columns of A^+ and of A^-, and of G^+ and of G^- in (2.11). Then the basis solution of the transformed system has still the form $(x^*, y^*, 0, 0, 0)$ and the related cut is equal to (2.12). □

The last theorem gives an easy geometric interpretation of the mixed integer Gomory cut if we apply it to MILP in natural form. So it is appropriate to state a cutting plane algorithm for MILP in natural form using intersection cuts directly. This approach also provides the opportunity to deal with MILP with real input data. According to the basic algorithmic approach that we have briefly discussed after Theorem 2.18, we use intersection cuts to split disjunctions $D(d, \delta)$, where d is defined by a unit vector $u_1, \ldots, u_p$. To obtain uniqueness in the algorithm we use the following two rules. First, we compute in every step of the algorithm the lexicographic maximal solution (x^*, y^*) of the LP relaxation. Secondly, we add cuts by a least index rule. So an intersection cut to the disjunction $D(u_i, \lfloor x_i^* \rfloor)$ is added, where $i = \text{argmin}\{j \in \{1, \ldots, p\} : x_j^* \notin \mathbb{Z}\}$. Moreover, we do not add only one intersection cut but intersection cuts to all bases B of (x^*, y^*) to the disjunction $D(u_i, \lfloor x_i^* \rfloor)$. The formal algorithm is stated in Algorithm 1.

Algorithm 1 Cutting plane algorithm - basic form

```
procedure BasicAlgorithm
    Input: MILP (1.1);
    Output: "optimal solution (x*, y*)" or "problem infeasible" if no solution exists;

    P ← {(x, y) : Ax + Gy ≤ b};
    (x*, y*) ← arglexmax {cx + hy : (x, y) ∈ P};
    𝓑 ← set of all bases B of (x*, y*);

    if P = ∅ then
        "problem infeasible"; break
    end if
    if x* ∈ ℤ^p then
        "optimal solution (x*, y*)"; break
    end if

    while x* ∉ ℤ^p do
        i ← argmin {j ∈ {1, ..., p} : x*_j ∉ ℤ};
        Compute intersection cuts α_B x + β_B y ≤ γ_B to P, D(u_i, ⌊x*_i⌋) and all B ∈ 𝓑.
        P ← P ∩ {(x, y) : α_B x + β_B y ≤ γ_B, B ∈ 𝓑};
        (x*, y*) ← arglexmax {cx + hy : (x, y) ∈ P};
        𝓑 ← set of all bases B of (x*, y*);

        if P = ∅ then
            "problem infeasible"; break
        end if
        if x* ∈ ℤ^p then
            "optimal solution (x*, y*)"; break
        end if
    end while
end procedure
```

Reviewing the results of Section 2.2.1 and of Example 2.13 especially, it is evident that Algorithm 1 does not find an optimal solution of a MILP within a finite number of steps in general. However, under certain assumptions a finite convergence of the algorithm can be guaranteed.

Theorem 2.20. *Let a MILP* (1.1) *be given, where the polyhedron* $P = \{(x, y) : Ax + Gy \leq b\}$ *is bounded. Then Algorithm 1 finds an optimal solution of the MILP or detects infeasibility after a finite number of steps under the following assumption:*

It can be assumed that the optimal objective function value is integral. Whenever possible, the inequality $cx + hy \leq \lfloor cx^* + hy^* \rfloor$ *to the objective function is added additionally in Algorithm 1.*

Proof. We first assume that the MILP has an optimal solution. By definition of Algorithm 1, a sequence $(x^k, y^k)_{k \in \mathbb{N}}$ of lexicographic decreasing vectors is computed. Since P is bounded, the sequence $(\gamma^k)_{k \in \mathbb{N}} := (cx^k + hy^k)_{k \in \mathbb{N}}$ of objective function values is bounded and therewith convergent. As all possible inequalities $cx + hy \leq \lfloor cx^k + hy^k \rfloor$ are added additionally in Algorithm 1, it follows that there exists a $k_0 \in \mathbb{N}$ such that γ^k remains constant and integral for all $k \geq k_0$.

Next, we consider the sequence $(x_1^k)_{k \geq k_0}$ of the first entry of (x^k, y^k). As P is bounded, this sequence is again convergent with limit $\widetilde{x}_1$. So there is a $k_1 \geq k_0$ such that

$$\lfloor \widetilde{x}_1 \rfloor \leq x_1^{k_1} < \lfloor \widetilde{x}_1 \rfloor + 1.$$

If $\lfloor \widetilde{x}_1 \rfloor = x_1^{k_1}$, there is nothing to show. Otherwise, we add intersection cuts to all bases $B \in \mathcal{B}$ of (x^{k_1}, y^{k_1}) to the disjunction $D(u_1, \lfloor x_1^{k_1} \rfloor)$. By the first part of the proof, the sequence γ^k of objective function values remains constant and integral for $k \geq k_1 > k_0$ and the inequality $cx + hy \leq \gamma^{k_0}$ has been added to the polyhedron P within the algorithm. This means especially that there exists a basis $B \in \mathcal{B}$ of (x^{k_1}, y^{k_1}) which contains the inequality $cx + hy \leq \gamma^{k_0}$. So it follows by Theorem 2.16 and the fact that (x^{k_1}, y^{k_1}) is lexicographic maximal that $x_1^k = \lfloor x_1^{k_1} \rfloor$ for all $k \geq k_1$. This shows that x_1^k remains constant and integral for $k > k_1$. This argument can now be repeated for the remainder of the integral restricted variables $x_2, \ldots, x_p$. It follows inductively that an optimal solution of the MILP is found after a finite number of steps.

Suppose now that no optimal solution of the MILP exists, so $P_I = \emptyset$. As P is bounded, there exists

$$\gamma := \min\{cx + hy : \ Ax + Gy \leq b\}.$$

It follows by the above proof and the fact that all possible inequalities $cx + hy \leq \lfloor cx^k + hy^k \rfloor$ are added within the algorithm that the sequence γ^k of objective function values decreases to the value $\lfloor \gamma \rfloor - 1$ in finite time. As $P \cap \{(x, y) : cx + hy \leq \lfloor \gamma \rfloor - 1\}$ is empty, the algorithm terminates finitely.

Moreover, by construction no feasible point is cut off during the algorithm and the proof is complete. □

We note that the rule for adding cuts in line 18 of Algorithm 1 can be weakened. Actually, we do not have to add intersection cuts to all bases $B \in \mathcal{B}$ of the current LP solution (x^k, y^k) at once. Even a sequential addition of intersection cuts to different bases $B \in \mathcal{B}$ until the next solution (x^{k+1}, y^{k+1}) is integral in the related component is sufficient for convergence.

Moreover, Theorem 2.20 shows that we can check with Algorithm 1 in finite time if there is a mixed integer feasible point in a polytope with a given objective function value. We can consider the modified MILP

$$\max\{x_1 : (x, y) \in \mathbb{Z}^p \times \mathbb{R}^q : Ax + Gy \leq b,\, cx + hy = \gamma\}$$

for this purpose. Here we can assume that the optimal objective function value of the modified MILP is integral. Additional rounding cuts to the objective function according to the assumption of Theorem 2.20 are not necessary in this case, as cuts to disjunctions $D(u_1, \lfloor x_1^* \rfloor)$ are part of Algorithm 1.

Corollary 2.21. *Let a MILP* (1.1) *with the additional constraint* $cx + hy = \gamma$ *be given, where the polyhedron* $P = \{(x, y) : Ax + Gy \leq b\}$ *is bounded. Then Algorithm 1 terminates in finite time with a feasible solution of the MILP or detects infeasibility.*

□

Finally, we add some remarks. The classic form of Algorithm 1 and Theorem 2.20 based on MILP in standard form (1.4) and on mixed integer Gomory cuts (2.9) was stated by Gomory in [Gom58]. So Algorithm 1 is a variation of the Gomory algorithm for MILP.

Mixed integer Gomory cuts or intersection cuts to split disjunctions, respectively, can of course also be applied to ILP. As already discussed in Example 2.12, intersection cuts to split disjunctions are in general stronger than pure integer Gomory cuts (2.2). However, applying intersection cuts to ILP can have the disadvantage that the structure of the ILP changes. Applying a pure integer Gomory cut according to Definition 2.6 to an ILP with integral data gives a cutting plane with integral coefficients. So the related modified ILP is again given by integral data. This is not true in general if we apply an intersection cut to the ILP. Moreover, this means that adding an intersection cut $\alpha x \leq \gamma$ with $\alpha \in \mathbb{R}^p, \gamma \in \mathbb{R}$ to an ILP

$$\max\{cx : Ax = b, x \geq 0\, x \in \mathbb{Z}^p\}$$

in standard form can have the consequence that the modified program has to be treated as MILP. This follows as the transformation of the cut $\alpha x \leq \gamma$ to standard form gives

$\alpha x + s = \gamma, s \geq 0$, where the slack variable s does not have to be integral if α and γ are not integral.

On the other hand, intersection cuts to split disjunctions can be used to deal with ILP with irrational input data. In this case, the derivation of pure integer Gomory cuts according to Definition 2.6 is not possible as there exist no integral representation of the polyhedron $P = \{x \in \mathbb{R}^p : Ax \leq b\}$ in general. If the objective function vector c of the ILP is rational, we can also assume by suitable scaling that c is integral. So the optimal objective function value of the ILP $\max\{cx : Ax \leq b, x \in \mathbb{Z}^p\}$ is integral. This means that Algorithm 1 with the modification of Theorem 2.20 can be used to solve the ILP.

If the objective function vector c is irrational, the ILP cannot be solved directly by Algorithm 1. We can compute a feasible approximative solution with Algorithm 1 by considering a suitable rational approximation $\widetilde{c}$ of the objective function vector c in this case. Moreover, it is also possible to compute an exact optimal solution of the original ILP by taking a sufficiently close rational approximation of the objective function vector c. This follows as for an arbitrary bounded polyhedron $P = \{x \in \mathbb{R}^p : Ax \leq b\}$, its (pure) integer hull P_I is a rational polyhedron; see Lemma 1.2.

2.2.4 Mixed integer rounding inequalities

Mixed integer rounding (MIR) inequalities were introduced by Nemhauser and Wolsey in 1990 [NW90]. The cut is derived from a rational mixed integer set that is defined by a single inequality with non negative variables. So let the set

$$T := \{(x, y) \in \mathbb{Z}_+^p \times \mathbb{R}_+^q : ax + gy \leq b\} \tag{2.14}$$

with $a \in \mathbb{Q}^p, g \in \mathbb{Q}^q$, and $b \in \mathbb{Q}$ be given.

Definition 2.22. *Let T be defined according to* (2.14), $b = \lfloor b \rfloor + f_0,\ a_j = \lfloor a_j \rfloor + f_j$, *and* $(x)^+ := \max\{0, x\}$ *for* $x \in \mathbb{R}$. *Then the mixed integer rounding cut for T is defined by*

$$\sum_{j=1}^{n} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{g_j < 0} g_j y_j \leq \lfloor b \rfloor. \tag{2.15}$$

Like the mixed integer Gomory cut, the MIR inequality is again a split cut to the split disjunction $D(d, \lfloor b \rfloor)$, where $d_i = \lfloor a_i \rfloor$ if $f_j \leq f_0$ and $d_i = \lceil a_i \rceil$ if $f_j > 0$ according to (2.10). Moreover, even the following statement is true.

Theorem 2.23. *The MIR inequality* (2.15) *for T is identical to the related mixed integer Gomory cut* (2.9).

Proof. To be able to apply the mixed integer Gomory cut to the set T according to (2.14) we transform the representation of T to standard form by introducing a non negative slack variable s. This gives the equation $ax + gy + s = b$. We obtain the mixed integer Gomory cut

$$\begin{aligned}
&\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j + \frac{1}{f_0} s \ge 1 \iff \\
- &\sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} f_0 \frac{1 - f_j}{1 - f_0} x_j - \sum_{g_j > 0} g_j y_j + \sum_{g_j < 0} f_0 \frac{g_j}{1 - f_0} y_j - s \le -f_0
\end{aligned}$$

We substitute $s = b - ax - gy$ to return to a representation in the original space. This gives

$$\begin{aligned}
&\sum_{f_j \le f_0} (a_j x_j - f_j x_j) + \sum_{f_j > f_0} \left(a_j x_j - f_0 \frac{1 - f_j}{1 - f_0} x_j \right) + \sum_{g_j < 0} \left(f_0 \frac{g_j}{1 - f_0} y_j + g_j y_j \right) \le \lfloor b \rfloor \iff \\
&\sum_{f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{f_j > f_0} \frac{a_j - a_j f_0 - f_0 + f_0 f_j}{1 - f_0} x_j + \frac{1}{1 - f_0} \sum_{g_j < 0} g_j y_j \le \lfloor b \rfloor \iff \\
&\sum_{f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{f_j > f_0} \left(\frac{\lfloor a_j \rfloor - f_0 (a_j - f_j)}{1 - f_0} + \frac{f_j - f_0}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{g_j < 0} g_j y_j \le \lfloor b \rfloor \iff \\
&\sum_{j=1}^{n} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{g_j < 0} g_j y_j \le \lfloor b \rfloor .
\end{aligned}$$

□

As every MIR inequality is a mixed integer Gomory cut, we do not discuss more details or algorithms concerning MIR cuts but add some remarks, only.

The definition of the MIR inequality in Definition 2.22 is not unique. There are some slightly different cutting planes which are all known under the name of mixed integer rounding inequalities. All of these modified cutting planes are also mixed integer Gomory cuts. We note in this context that the converse direction of Theorem 2.23 is only true using a stronger version of (2.15). So not every mixed integer Gomory cut can be expressed as MIR cut given in (2.15). For details we refer to [NW88], [Wol98], and [DGL07].

We have mentioned at the end of Section 2.2.1 that split cuts are sufficient for generating the mixed integer hull of a polyhedron in the case of a mixed binary linear program. Nemhauser and Wolsey have shown in [NW90] that MIR inequalities are already sufficient for generating the mixed binary hull of a polyhedron. Moreover, many strong practical cutting planes such as flow cover and integer cover inequalities are MIR inequalities; see for example [MW01]. A further approach to obtain valid inequalities consists in mixing MIR cuts. Günlük and Pochet [GP01] introduce a mixing procedure and discuss how strong inequalities for several practical problems can be generated.

2.2.5 Strengthening of mixed integer cuts

In this section we introduce two approaches for obtaining deep split cuts for MILP with rational input data. By solving MILP with Algorithm 1, we add in every step intersection cuts to split disjunctions of the form $D(u_i, \lfloor x_i^* \rfloor)$. This is the easiest way to find a disjunction $D(d, \delta)$ with $dx^* > \delta$ for $x^* \notin \mathbb{Z}^p$. Cuts to these disjunctions, however, do not have to be deep in general. Therefore, it is an interesting question how to find disjunctions to which deep cuts can be derived.

The first idea to answer this question is known under the name of reduce-and-split cuts and was introduced by Andersen, Cornuéjols, and Li [ACL05a]. It is based on the representation of the mixed integer Gomory cut. So we consider again the set $S := \{(x, y) \in \mathbb{Z}_+^p \times \mathbb{R}_+^q : ax + gy = b\}$ with rational input data and the mixed integer Gomory cut

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \ge 1$$

according to (2.9). By the proof of Theorem 2.18, the cut is a split cut to the disjunction $D(d, \delta)$ with $\delta = \lfloor b \rfloor$ and

$$d_i = \begin{cases} \lfloor a_i \rfloor, & \text{if } f_j \le f_0 \\ \lceil a_i \rceil, & \text{else.} \end{cases}$$

This means that we can change the split disjunction to which we cut if we multiply the equation defining S by an arbitrary factor k. Now to obtain a stronger cut, we observe that the coefficients f_0, f_j in the formula of the mixed integer Gomory cut always remain between 0 and 1, while the coefficients g_j can take arbitrary values. Moreover, the cut gets weaker as the norm of the coefficients g_j increases. So it is the idea to multiply the equation defining S by a suitable multiplier k such that the norm of g decreases. The Gomory cut is then derived from the modified set in a second step.

This strategy can now also be applied within a cutting plane algorithm for MILP in standard form (1.4). We take more generally integral combinations of non integral basis rows of the simplex tableau to generate an equation with a reduced norm of the vector g. This equation is then used for deriving the mixed integer Gomory cut. Here we do not go into further details how this approach can be implemented. However, the procedure is similar to Lovász's basis reduction algorithm; see [LLL82]. Andersen, Cornuéjols, and Li also tested their approach on the problems of the MIPLIB 3.0 [BCMS98] and showed that the performance of cutting plane algorithms can often be improved by reduce-and-split cuts; see [ACL05a].

The second idea for generating stronger cuts was developed independently by Köppe and Weismantel [KW04]. The starting point is given by a mixed integer Farkas Lemma.

Theorem 2.24 ([KW04]). *Let $A \in \mathbb{Z}^{m\times p}, G \in \mathbb{Z}^{m\times q}, b \in \mathbb{Z}^m$ be given such that the rank of $(A\ G)$ is equal to m. Then either the system*

$$Ax + Gy = b,\ x \in \mathbb{Z}^p,\ y \in \mathbb{R}^q$$

has a solution, or the system

$$z(A\ G) \in \mathbb{Z}^p \times \{0\}^q,\ zb \notin \mathbb{Z},\ z \in \mathbb{R}^m$$

has a solution. □

Theorem 2.24 can now be used to derive valid cutting planes for MILP (1.1) in natural form. We define for a vertex (x^*, y^*) of the rational polyhedron $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ with basis B the lattice

$$\mathcal{L}^* := \{z \in \mathbb{Q}^m : z(A\ G) \in \mathbb{Z}^p \times \{0\}^q, z_i = 0\ \forall i \notin B\}. \tag{2.16}$$

Moreover, let $x^- := \max\{0, -x\}$ for $x \in \mathbb{R}$ and denote for $z \in \mathbb{R}^p$ by z^- the vector with components z_i^-.

Theorem 2.25 ([KW04]). *Let (x^*, y^*) be a vertex of the polyhedron $P = \{(x, y) : Ax + Gy \leq b\}$ and B a related basis. Moreover, let $z \in \mathcal{L}^*$ with $zb \notin \mathbb{Z}$ and $\mathcal{L}^*$ according to* (2.16). *Define $z(A\ G) =: (\alpha, 0) \in \mathbb{Z}^p \times \{0\}^q$, $\gamma := zb$, and $f_0 := \gamma - \lfloor\gamma\rfloor$. Then*

$$\alpha x + \frac{1}{1-f_0}\lceil z^-\rceil(Ax + Gy) \leq \lfloor\gamma\rfloor + \frac{1}{1-f_0}\lceil z^-\rceil b \tag{2.17}$$

is a valid inequality for P_I that cuts off (x^*, y^*). □

We note that the assumptions in Theorem 2.25 are well defined. Köppe and Weismantel show that $\mathcal{L}^*$ is indeed a lattice and that for any lattice base $\{z^1, \dots, z^k\}$ of $\mathcal{L}^*$ it is

$$z^l b \in \mathbb{Z} \text{ for all } l \in \{1, \dots, k\} \text{ if and only if } x^* \in \mathbb{Z}^p.$$

So there always exists a $z \in \mathcal{L}^*$ according to the assumptions in Theorem 2.25.

The derivation of the cutting plane (2.17) to a vector $z \in \mathcal{L}^*$ is based on the MIR procedure discussed in Section 2.2.4. Therefore, it is proximate that (2.17) is again a split cut. We show this property in the next lemma. We also refer to [Vie07] for some more details concerning results about cutting planes from lattices.

Lemma 2.26. *Inequality* (2.17) *is a split cut for P to the disjunction $D(\alpha, \lfloor\gamma\rfloor)$.*

Proof. First let $(\widetilde{x}, \widetilde{y}) \in P$ with $\alpha\widetilde{x} \leq \lfloor\gamma\rfloor$. By definition, it is $1 - f_0 > 0$, $\lceil z^- \rceil \geq 0$, and $A\widetilde{x} + G\widetilde{y} \leq b$. So it is

$$\frac{1}{1 - f_0}\lceil z^- \rceil(A\widetilde{x} + G\widetilde{y}) \leq \frac{1}{1 - f_0}\lceil z^- \rceil b,$$

and it follows that (2.17) is valid for $(\widetilde{x}, \widetilde{y})$.

Now let $(\widetilde{x}, \widetilde{y}) \in P$ with $\alpha\widetilde{x} \geq \lfloor\gamma\rfloor + 1$. (2.17) is equivalent to

$$\alpha x - \lfloor\gamma\rfloor - f_0\alpha x + f_0\lfloor\gamma\rfloor \leq \lceil z^- \rceil(b - Ax - Gy); \tag{2.18}$$

We define $\widetilde{z} = z + \lceil z^- \rceil$, so $\widetilde{z} \geq 0$ and $\lceil z^- \rceil = \widetilde{z} - z$. By inserting the last equation in (2.18), we obtain

$$\begin{aligned}
\alpha x - \lfloor\gamma\rfloor - f_0\alpha x + f_0\lfloor\gamma\rfloor &\leq \widetilde{z}(b - Ax - Gy) - z(b - Ax - Gy) \Longleftrightarrow \\
\alpha x - \lfloor\gamma\rfloor - f_0\alpha x + f_0\lfloor\gamma\rfloor &\leq \widetilde{z}(b - Ax - Gy) - \gamma + \alpha x \Longleftrightarrow \\
f_0(\lfloor\gamma\rfloor + 1 - \alpha x) &\leq \widetilde{z}(b - Ax - Gy)
\end{aligned}$$

By case assumption, the left hand side of the last inequality is non positive, so it follows that (2.17) is valid for $(\widetilde{x}, \widetilde{y})$ as $A\widetilde{x} + G\widetilde{y} \leq b$. □

Inequality (2.17) is a split cut to $D(\alpha, \lfloor\gamma\rfloor)$, where $(\alpha, 0) = z(A\ G)$ for $z \in \mathcal{L}^*$. Since a split disjunction becomes larger if the norm of α becomes smaller, the selection of a suitable lattice basis is crucial for deriving deep cutting planes. Köppe and Weismantel suggest to compute a Lovász-reduced basis of $\mathcal{L}^*$. It is the idea that such a basis may also lead to short vectors α. At this point we can draw a parallel to the reduce-and-split cuts which we have discussed before, as both approaches use the idea of basis reduction to compute strong cuts.

2.2.6 Cuts from two rows of the simplex tableau

We conclude this chapter with a short reference to a recent approach of Andersen, Louveaux, Weismantel, and Wolsey [ALWW07] for deriving valid cutting planes. In contrast to the results of the last sections, these cutting planes are in general no split cuts but derived from more general disjunctions. This has the advantage that deeper cuts can be found. For example, the approach is sufficient for generating the mixed integer hull of the polyhedron in Example 2.13.

Both the derivation of the mixed integer Gomory cut (2.9) and of the mixed integer rounding inequality (2.15) was based on finding valid inequalities for mixed integer sets which are defined by a single constraint. These sets were $S := \{(x, y) \in \mathbb{Z}^p_+ \times \mathbb{R}^q_+ : ax + gy = b\}$ in the first case and $T := \{(x, y) \in \mathbb{Z}^p_+ \times \mathbb{R}^q_+ : ax + gy \leq b\}$ in the second case. For the derivation of valid cuts for a MILP these basic cuts were now applied to a suitable row of the simplex tableau. It can be seen as a proximate generalization of the prior work to consider sets which are defined by two equations and to derive valid inequalities for them. This is exactly the point of departure of the work of Andersen, Louveaux, Weismantel, and Wolsey. The main contribution in [ALWW07] consists in characterizing geometrically all facets of the mixed integer set

$$U := \operatorname{conv}\left(\{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j=1}^{n} s_j r^j\}\right),$$

where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $r^j \in \mathbb{Q}^2$.

They show that every facet of the polyhedron U can be derived as intersection cut, see Section 2.2.2, to certain two dimensional disjunctions. Moreover, they investigate the various possibilities which can arise for the geometric structure of the facets of U and distinguish three different cases. They prove that facets derived in two of the three cases are never split cuts. Cutting planes which are derived from the set U can now also be used in a cutting plane algorithm by applying them to sets defined by two rows of the simplex tableau.

We will get back to these cuts in Section 3.5, where we relate them to k-disjunctive cuts which we introduce in the next chapter.

3 k-disjunctive cuts

We have seen in the last chapter that split cuts are in general not sufficient for generating the mixed integer hull of a polyhedron. On the other hand, the cuts from two rows of a simplex tableau in Section 2.2.6 show that more general disjunctions can be used to compute deeper cuts. Therefore, we now consider cutting planes which are derived from arbitrary polyhedral disjunctions and which are characterized by the number of disjunctive inequalities.

In detail, we introduce the so called k-disjunctive cuts in Section 3.1 and discuss some basic properties. In Section 3.2 we repeat the approximation property of split cuts. Next, we analyze in Section 3.3 which type of cuts is required to generate the mixed integer hull of a polyhedron. In Section 3.4 we turn to the problem of how deep cuts for a rational polyhedron to a general disjunction can be computed. At last we give a short overview of some related work in Section 3.5.

3.1 Basic definitions and properties

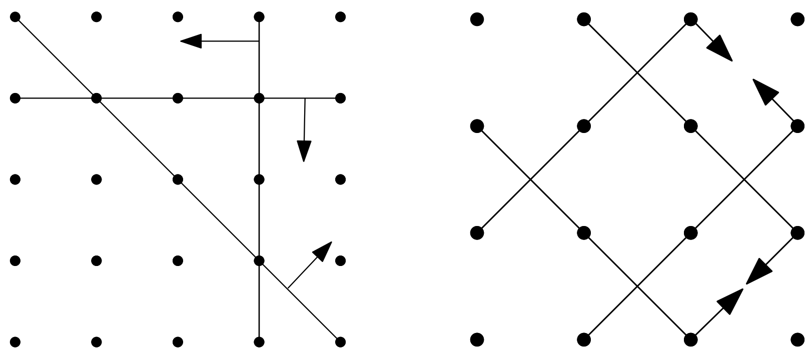
According to the definition of a split cut based on a split disjunction we now define a k-disjunctive cut based on a k-disjunction which contains every integral vector.

Definition 3.1. *Let $k \geq 2$ be a natural number, $d^1, \ldots, d^k \in \mathbb{Z}^p$ integral vectors and $\delta^1, \ldots, \delta^k \in \mathbb{Z}$. Then we call the system of inequalities $d^1x \leq \delta^1, \ldots, d^kx \leq \delta^k$ a k-disjunction if for all $x \in \mathbb{Z}^p$ there is an $i \in \{1, \ldots, k\}$ with $d^ix \leq \delta^i$. In this case we write $D(k, d, \delta)$ with $d = (d^1, \ldots, d^k), \delta = (\delta^1, \ldots, \delta^k)$ for the k-disjunction.*

We note that we do not require the vectors d^i, δ^i to be different. So every l-disjunction is also a k-disjunction for $l < k$. Especially, every split disjunction is also a k-disjunction. Moreover, every k-disjunction is a cover of $\mathbb{Z}^p$ by definition.

Definition 3.2. *Let $S \subseteq \mathbb{R}^{p+q}$ be a closed convex set and $\alpha x + \beta y \leq \gamma$ be a cutting plane. Then $\alpha x + \beta y \leq \gamma$ is called a k-disjunctive cut for S if there exists a k-disjunction $D(k, d, \delta)$ with*

$$(x, y) \in S: \; \alpha x + \beta y > \gamma \Longrightarrow d^ix > \delta^i \; \forall i \in \{1, \ldots, k\}.$$



(a) a 3-disjunction (b) a 4-disjunction

Figure 3.1: Examples of k-disjunctions in $\mathbb{R}^2$

Every k-disjunctive cut for a polyhedron P is valid for P_I by definition. We go on with the definition of the k-disjunctive closure of a closed convex set S as a generalization of the split closure introduced in Definition 2.8 and Definition 2.11.

Definition 3.3. *Let $S \subseteq \mathbb{R}^{p+q}$ be a closed convex set. Then the intersection of S and all k-disjunctive cuts for S is called the k-disjunctive closure of S and denoted by $S_k^{(1)}$. Accordingly, for $i \in \mathbb{N}$ the i-th k-disjunctive closure $S_k^{(i)}$ of S is defined as the k-disjunctive closure of $S_k^{(i-1)}$. In the special case of $k = 2$ we also write $S^{(i)}$ instead of $S_2^{(i)}$ as denoted in Definition 2.8 and Definition 2.11.*

It is open if the k-disjunctive closure $P_k^{(1)}$ of a polyhedron P is again a polyhedron for $k \geq 3$, even if we restrict ourselves to rational polyhedra. The special case of this statement for $k = 2$ and a rational polyhedron was stated in Theorem 2.10. Unfortunately, the various proofs of the polyhedrality of the split closure $P^{(1)}$ of a rational polyhedron by Andersen,Cornuéjols, and Li [ACL05b], Cook, Kannan, and Schrijver [CKS90], Dash, Günlük, and Lodi [DGL07], and Vielma [Vie07] cannot be applied to the more general case. Andersen, Cornuéjols, and Li use for their proof that each split cut can be generated by intersection cuts to suitable basis solutions of the polyhedron. We will see in Example 3.4 that this property is not satisfied by disjunctive cuts with $k \geq 3$. The proof of Cook, Kannan, and Schrijver is based on a geometric property of split disjunctions that is not valid for general k-disjunctions. Dash, Günlük, and Lodi show the property for the MIR closure and Vielma uses an algebraic characterization of split cuts for his constructive approach. However, our results in the remainder of this chapter are independent of the polyhedrality of $P_k^{(1)}$. We note that we show in Theorem 3.8 that $P_{2^p}^{(1)} = P_I$ for a rational polyhedron P and so it follows that $P_k^{(1)}$ is again a polyhedron for $k \geq 2^p$ and a rational polyhedron.

A valid cut to a given k-disjunction $D(k, d, \delta)$ can be computed as intersection cut to any basis solution (x, y) of the polyhedron P that is not contained in the disjunction;

see Section 2.2.2. We have stated in Theorem 2.16 a result of Andersen, Cornuéjols, and Li that intersection cuts are sufficient to describe all cuts to a given split disjunction. However, the generalization of this result to general k-disjunctions with $k > 2$ is not true. Not every valid k-disjunctive cut to a given disjunction is equal to or dominated by a set of intersection cuts. This can be seen in the following

Example 3.4. *We consider the polyhedral cone* $C \subseteq \mathbb{R}^{2+1}$ *with apex* $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ *that is defined by the system*

$$\begin{array}{rcl} -x_1 + y & \leq & 0 \\ -x_2 + y & \leq & 0 \\ x_1 + y & \leq & 1 \\ x_2 + y & \leq & 1. \end{array} \tag{3.1}$$

The inequality $y \leq 0$ *is a* 4*-disjunctive cut for* C *to the* 4*-disjunction*

$$D := \{x_1 + x_2 \geq 2, x_1 - x_2 \geq 1, -x_1 + x_2 \geq 1, -x_1 - x_2 \geq 0\};$$

see Figure 3.2. The set $\mathcal{B}$ *of all bases of* C *is given by any three inequalities of* (3.1).

We consider the intersection cut to D *and the basis* B *which consists of the first three inequalities of* (3.1)*. The basis submatrix* $(A\ G)_B$ *is given by*

$$(A\ G)_B = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \textit{with} \quad (A\ G)_B^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

So the extreme rays of the basis cone are generated by the three vectors

$$v^1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), v^2 = (0, 1, 0), v^3 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right).$$

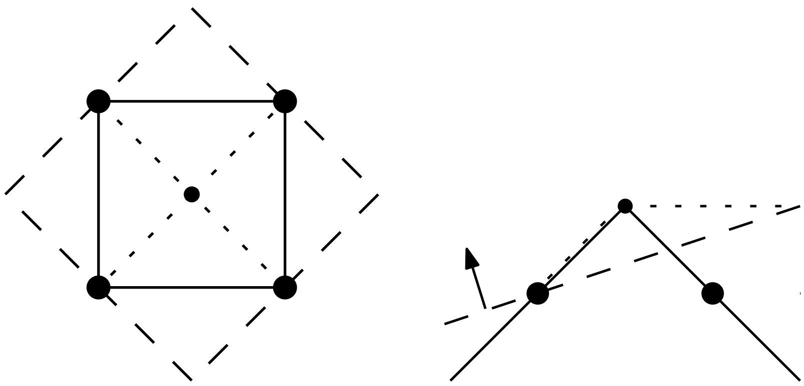
The intersection of the half rays

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \lambda_i v^i,\ \lambda_i \geq 0 \ \textit{for}\ i = 1, 2, 3$$

with the disjunction D *is given by the three points*

$$(1, 0, 0), \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right), (0, 0, 0).$$

This yields the intersection cut $-x_2 + 3y \leq 0$*. It follows by symmetry that the point* $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\right)$ *is valid for all four intersection cuts to all bases* $B \in \mathcal{B}$ *of* C *and so the inequality* $y \leq 0$ *dominates the intersection of all four intersection cuts.*



(a) Projection of $C \cap \{y \geq 0\}$ on the x_1x_2-space (solid) and the disjunction D (dashed).

(b) Projection of C on the x_2y-space (solid) and one intersection cut (dashed) to a basis B (dotted).

Figure 3.2: To Example 3.4

Although the properties of general k-disjunctive cuts are more involved than the properties of split cuts, we will see in the next sections that analyzing these cuts is useful. At first we show that every cutting plane for a rational polyhedron P which is valid for P_I is a k-disjunctive cut for some k.

Lemma 3.5. *Let $P \subseteq \mathbb{R}^{p+q}$ be a rational polyhedron and $\alpha x + \beta y \leq \gamma$ be a valid rational cutting plane for P_I. Then $\alpha x + \beta y \leq \gamma$ is a k-disjunctive cut for some $k \in \mathbb{N}$.*

Proof. Let P be a rational polyhedron and $\alpha x + \beta y \leq \gamma$ be a valid rational cutting plane for P_I. The set of points that is cut off by the cutting plane is given by

$$M := \{(x, y) \in P : \alpha x + \beta y > \gamma\}.$$

By Lemma 1.3, the projection of M on the x-space can be written in the form

$$\mathrm{proj}_X(M) = \{x \in \mathbb{R}^p : A^e x \leq b^e,\ A^l x < b^l\}. \tag{3.2}$$

As M contains no point of P_I, it is $x \notin \mathbb{Z}^p$ for all $x \in \mathrm{proj}_X(M)$.

Since P is rational, also the projection $\mathrm{proj}_X(M)$ is rational. So we can choose without loss of generality integral matrices A^e, A^l in (3.2) for the representation of $\mathrm{proj}_X(M)$. We enlarge the projection $\mathrm{proj}_X(M)$ by setting

$$\begin{aligned} \widetilde{b}_i^e &:= \lfloor b_i^e \rfloor + 1, \\ \widetilde{b}_i^l &:= \lceil b_i^l \rceil. \end{aligned}$$

It follows that $\alpha x + \beta y \leq \gamma$ is a k-disjunctive cut to the disjunction $D(k, -(A^e, A^l), -(\widetilde{b}^e, \widetilde{b}^l))$, where k is equal to the sum of rows of the matrices A^e and A^l. □

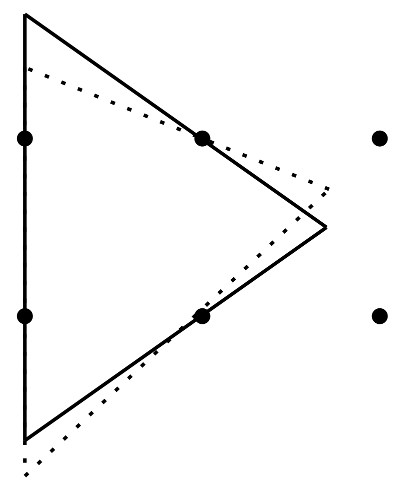


Figure 3.3: The polyhedron P of Example 3.6 projected on the x_1x_2-space and a rational 3-disjunction (dotted).

Lemma 3.5 is in general not true for polyhedra with irrational input data. Not every valid cutting plane for polyhedra with irrational input data is a k-disjunctive cut. This can be seen in the following

Example 3.6. *Let $P \subseteq \mathbb{R}^{2+1}$ be the irrational polyhedron defined as the convex hull of the vertices*

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \left(0,-\frac{\sqrt{2}}{2},0\right), \left(0,1+\frac{\sqrt{2}}{2},0\right), \left(1+\frac{\sqrt{2}}{2},\frac{1}{2},0\right).$$

It follows that $P_I = \operatorname{conv}\{0,(1,0,0),(1,1,0),(0,1,0)\}$ and so $y \leq 0$ is a valid cutting plane for P_I. However, as two facets of P are irrational and contain a relative interior integral point, $y \leq 0$ is no k-disjunctive cut for any $k \in \mathbb{N}$ as there does not exists a k-disjunction $D(k,d,\delta)$ such that the interior $\operatorname{int}(\operatorname{proj}_X(P))$ of the projection $\operatorname{proj}_X(P)$ of P on the x_1x_2-space is disjoint to $D(k,d,\delta)$. This can be seen in Figure 3.3.

To be able to deal with polytopes with real input data we generalize the definition of a k-disjunctive cut. For the derivation of cuts we now also allow k-disjunctions $D(k,d,\delta)$ with irrational inequalities $d^ix \leq \delta^i$. However, as we will see later in Section 3.3, it is enough for the characterization of the mixed integer hull of an arbitrary bounded polyhedron to take k-disjunctions $D(k,d,\delta)$ in which only one of the inequalities $d^ix \leq \delta^i$ is irrational.

Definition 3.7. *Let $k \geq 2$ be a natural number, $d^1,\dots,d^{k-1} \in \mathbb{Z}^p$, $d^k \in \mathbb{R}^p$ and $\delta^1,\dots,\delta^{k-1} \in \mathbb{Z}$, $\delta^k \in \mathbb{R}$. Then we call the system of inequalities $d^1x \leq \delta^1,\dots,d^kx \leq \delta^k$ an irrational k-disjunction if for all $x \in \mathbb{Z}^p$ there is an $i \in \{1,\dots,k\}$ with $d^ix \leq \delta^i$. In* *this case we write $D_{irr}(k,d,\delta)$ with $d=(d^1,\dots,d^k),\delta=(\delta^1,\dots,\delta^k)$ for the irrational k-disjunction.*

We define an irrational k-disjunctive cut for a closed convex set S according to Definition 3.2 and define for $i\in\mathbb{N}$ the i-th irrational k-disjunctive closure $S^{(i)}_{irr,k}$ of a closed convex set S according to Definition 3.3.

We note that by Definition 3.7 every k-disjunctive cut is also an irrational k-disjunctive cut.

It is our goal to compute the mixed integer hull of a given polyhedron with k-disjunctive cuts. This should be done as efficient as possible. That means that both the maximal number k of disjunctive hyperplanes defining an (irrational) k-disjunctive cut and the number of iterations in a cutting plane procedure should be small. At least the latter property can be easily realized for a rational polyhedron.

Theorem 3.8. *Let $P\subseteq\mathbb{R}^{p+q}$ be a rational polyhedron. Then $P_I=P^{(1)}_{2^p}$.*

Proof. We show that every valid (rational) inequality $\alpha x+\beta y\le\gamma$ for P_I is a 2^p-disjunctive cut for P. This is sufficient for the theorem. By Lemma 3.5, $\alpha x+\beta y\le\gamma$ is a k-disjunctive cut to a disjunction $D(k,d,\delta)$. So the statement follows if $k\le 2^p$. Otherwise, the number k of inequalities of the disjunction $D(k,d,\delta)$ can be reduced to 2^p. Since $D(k,d,\delta)$ is a k-disjunction we have

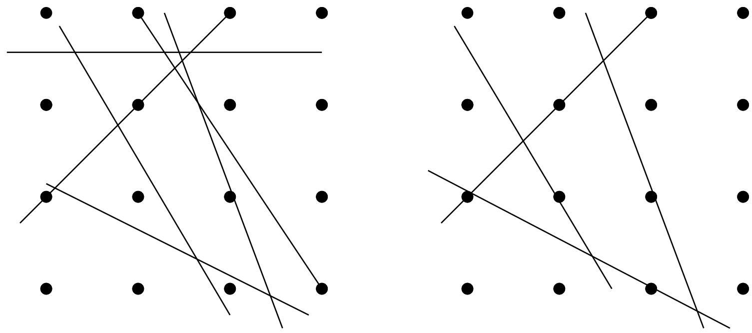
$$\forall x\in\mathbb{Z}^p\ \exists i\in\{1,\dots,k\}:\ d^ix\le\delta^i.$$

We consider successively the sets S^i of all integral vectors $x\in\mathbb{Z}^p$ with the property $d^ix=\delta^i$ for a fixed $i\in\{1,\dots,k\}$. Now it exists either a vector $\bar{x}\in S^i$ with $d^i\bar{x}=\delta^i$ and $d^j\bar{x}>\delta^j\ \forall j\in\{1,\dots,k\}\setminus\{i\}$, or we can expand the disjunction by decreasing the right hand side of the inequality to δ^i-1 and repeating this consideration. This may also lead to the case that the inequality $d^ix\le\delta^i$ can be left out, see Figure 3.4. Therewith we can restrict ourselves without loss of generality to disjunctions $D(k,d,\delta)$ with the additional condition

$$\forall i\in\{1,\dots,k\}\ \exists x^i\in\mathbb{Z}^p:\ d^ix^i=\delta^i\wedge d^jx^i>\delta^j\ \forall j\in\{1,\dots,k\}\setminus\{i\};$$

The set $\operatorname{conv}(\{x^1,\dots,x^k\})$ contains except for its k vertices $\{x^1,\dots,x^k\}$ no more integral vector $y\in\mathbb{Z}^p$ by construction. Otherwise, y would be a convex combination of some vertices $x^i, i\in I\subseteq\{1,\dots,k\}$ and it would follow that there exists a disjunctive inequality $d^jx\le\delta^j$ with $d^jy=d^jx^i=\delta^j\ \forall i\in I$ in contradiction to the properties of the set $\{x^1,\dots,x^k\}$.

If now $k>2^p$ then the set $\operatorname{conv}(\{x^1,\dots,x^k\})$ contains two vertices v,w with the additional property that each component $v_i,w_i,\ i\in\{1,\dots,p\}$ of both vectors is either even or odd. So the sum v_i+w_i is even for all $i\in\{1,\dots,p\}$. This means that $\frac{1}{2}(v+w)$ is an integral vector which is contained in $\operatorname{conv}(\{x^1,\dots,x^k\})$. This is a contradiction. □



(a) A k-disjunction in $\mathbb{R}^2$ given by the projection.

(b) Reduced k-disjunction after expanding and dropping

Figure 3.4: To the proof of Theorem 3.8

We consider an easy example to see that in general 2^p-disjunctive cuts are required to compute the mixed integer hull of a polyhedron in one step.

Example 3.9. *Let $C = [0;1]^p$ be the p-dimensional unit cube and define the polyhedron Q by*

$$Q = \{x \in \mathbb{R}^p : ax \leq \max_{x \in C} ax,\ a \in \{-1,1\}^p\};$$

see Figure 3.5. Next we embed Q in the $\mathbb{R}^{p+1}$ and define the polyhedron

$$P = \left\{(x,y) \in \mathbb{R}^{p+1} : (x,y) \in \operatorname{conv}\left\{(x,0), \frac{1}{2}\mathbf{1},\ x \in Q\right\}\right\}.$$

It is $P_I = C$ and the only k-disjunction to derive the valid cut $y \leq 0$ is defined by the 2^p facets of Q itself. This can be seen as each facet of Q contains a relative interior integral point, namely a vertex of the unit cube.

As Theorem 3.8 shows, the mixed integer hull of a rational polyhedron can in theory be generated in one step with 2^p-disjunctive cuts. However, using cuts to k-disjunctions at which k is exponential in the dimension p of the integral variables becomes extremely costly in practical applications for increasing p. Moreover, we have not yet found a way to generate the mixed integer hull of a general bounded polyhedron with real input data. Therefore, we deal in the following with the problem which type of k-disjunctive cuts is at least required in computing the mixed integer hull of a bounded polyhedron if cuts can also be added successively.

Concluding, we transfer the results of the proof of Theorem 3.8 to irrational k-disjunctive cuts.

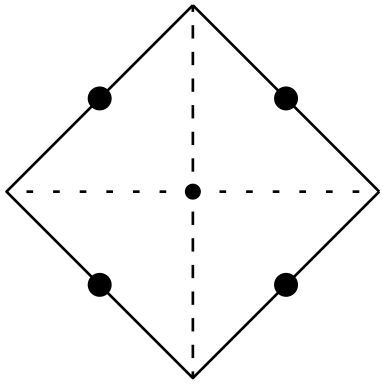


Figure 3.5: To Example 3.9. The polytope Q for $p = 2$ with the projection of the apex and the edges of P.

Corollary 3.10. *Let $P \subseteq \mathbb{R}^{p+q}$ be a bounded polyhedron and $\alpha x + \beta y \leq \gamma$ be an irrational k-disjunctive cut. Then $\alpha x + \beta y \leq \gamma$ is an irrational k-disjunctive cut with $k \leq 2^p$.*

Proof. Let $\alpha x + \beta y \leq \gamma$ be an irrational k-disjunctive cut to the irrational k-disjunction $D_{irr}(k, d, \delta)$ with irrational inequality $d^k x \leq \delta^k$. We differ two cases and suppose at first that there exists a $x \in \mathbb{Z}^p$ with $d^k x = \delta^k$ and $d^j x > \delta^j$ for all $j \in \{1, \dots, k-1\}$. In this case we can directly apply the proof of Theorem 3.8.

Now we suppose on the contrary that there does not exist a $x \in \mathbb{Z}^p$ with $d^k x = \delta^k$ and $d^j x > \delta^j$ for all $j \in \{1, \dots, k-1\}$. It follows in this case by Lemma 1.2 that we can replace the irrational inequality $d^k x \leq \delta^k$ by some rational inequalities to obtain a rational l-disjunction $D(l, \widetilde{d}, \widetilde{\delta})$ to which $\alpha x + \beta y \leq \gamma$ is a l-disjunctive cut. Now we can apply Theorem 3.8 to the rational disjunction $D(l, \widetilde{d}, \widetilde{\delta})$. □

3.2 Approximation property of split cuts

Before we further analyze which cuts we need to generate the mixed integer hull of an arbitrary polyhedron, we deal with the approximation properties of k-disjunctive cuts. Here we repeat that split cuts are already sufficient to approximate the mixed integer hull P_I of any polytope P arbitrarily exact as stated in Theorem 3.11. This was shown by Owen and Mehrotra [OM01].

Theorem 3.11 ([OM01])**.** *Let $P \subseteq \mathbb{R}^{p+q}$ be a bounded polyhedron, then*

$$\lim_{i \to \infty} P^{(i)} = P_I.$$

□

The proof of this statement by Owen and Mehrotra is based on a repeated variable disjunction, so only cuts to split disjunctions $D(d, \delta)$ with $d = u_i, i \in \{1, \dots, p\}$ are used. In [OM01] they also present an algorithm for computing approximative solutions of a general MILP. We take a closer look at this algorithm in Section 4.2.1. It also provides a constructive proof of Theorem 3.11.

According to Lemma 1.5, we can state Theorem 3.11 also in a different form by considering the sequence $(\gamma^{(i)})_{i\in\mathbb{N}}$ of objective function values that is given by

$$\gamma^{(i)} := \max\{cx + hy : (x, y) \in P^{(i)}\} \tag{3.3}$$

for an arbitrary vector $(c, h) \in \mathbb{R}^{p+q}$.

Corollary 3.12. *Let $P \subseteq \mathbb{R}^{p+q}$ be a bounded polyhedron, $\gamma^* = \max\{cx + hy : (x, y) \in P_I\} > -\infty$, and let $\gamma^{(i)}$ be according to (3.3). Then for all $\epsilon > 0$ there is an $i_0 \in \mathbb{N}$ with*

$$|\gamma^{(i)} - \gamma^*| < \epsilon \text{ for all } i \geq i_0.$$

If $P_I = \emptyset$ then there is an $i_0 \in \mathbb{N}$ with $P^{(i_0)} = \emptyset$.

□

Since split cuts are sufficient for approximating the optimal objective function value of any MILP arbitrarily exact, k-disjunctive cuts only become necessary for computing exact solutions of a MILP or determining the exact mixed integer hull of a polytope. Here we note that for rational polyhedra P the mixed integer hull P_I can be generated by combining split cuts with certain rounding cuts based on a discretization of the continuous variable y; see [CKS90]. In detail, let the MILP

$$\max\{cx + hy : Ax + Gy \leq b\}$$

be given, where A, G, b and c, h are integral and let (x^*, y^*) be an optimal solution of the MILP. We denote by $M \in \mathbb{Z}$ the absolute value of the product of all regular subdeterminants of G. As (x^*, y^*) is an optimal solution of the MILP, y^* is an optimal solution of the LP

$$\max\{hy : Gy \leq b - Ax^*\}.$$

It follows by Cramer's rule that My^* is integral, so the optimal objective function value of $\max\{Mcx + Mhy : Ax + Gy \leq b\}$ is integral. This means that for each inequality $cx + hy \leq \gamma$ that is valid for the MILP also the inequality

$$cx + hy \leq \frac{\lfloor M\gamma \rfloor}{M}$$

is valid for the MILP.

However, although the above rounding cut approach provides an opportunity to obtain the mixed integer hull of a rational polyhedron, we want to characterize the mixed integer hull by 'classical' cutting planes and find a related finite cutting plane algorithm. This means that cuts are supposed to be derived directly by the representation of P and the objective function without any prior numerical transformation of the problem. Moreover, rounding cuts cannot be applied to MILP with irrational input data. Therefore, a further consideration of k-disjunctive cuts is vital.

At last we want to note that in practical applications already optimizing over the first split closure often gives a good approximation of the optimal objective function value. This was investigated in detail by Balas and Saxena [BS08] for instances from the MIPLIB 3.0 and several other classes of structured MILP.

3.3 Characterization of the mixed integer hull

We get back to the question which type of cuts is required to obtain the mixed integer hull of an arbitrary polyhedron exactly. For this purpose, we analyze how the optimal objective function value γ^* of a general MILP with objective function $\max cx + hy$ can be computed. According to the formulation of Corollary 3.12, we consider again the sequence

$$\gamma^{(i)} := \max\{cx + hy : \ (x, y) \in P_k^{(i)}\} \tag{3.4}$$

of objective function values. An important issue for solving MILP exactly is given by the structure of the projection $\text{proj}_X\left(\{(x, y) \in P_I : cx + hy = \gamma^*\}\right)$ of the set of optimal solutions on the x-space of integral variables. For example, if the solution set of a rational MILP consists of an unique vertex (x^*, y^*), the MILP can be solved by split cuts; see Theorem 3.13. In general, however, the number k of disjunctive hyperplanes defining a k-disjunctive cut has to be chosen exponential in the dimension p of the integral space to determine exact optimal solutions of a MILP; see Theorem 3.21. Our approach also provides a natural way to characterize the faces of the mixed integer hull P_I of a polytope P.

We start with the special case that the projection $\text{proj}_X\left(\{(x, y) \in P_I : cx + hy = \gamma^*\}\right)$ of the solution set contains a relative interior integral point. Moreover, we restrict ourselves in the remainder of this section to bounded polyhedra P with $P_I \neq \emptyset$.

Theorem 3.13. *Let $P \subseteq \mathbb{R}^{p+q}$ be a rational polytope, $(c, h) \in \mathbb{Q}^{p+q}$, and $\gamma^* = \max\{cx + hy : \ (x, y) \in P_I\}$. If*

$$\text{relint}\left(\text{proj}_X\left(\{(x, y) \in P_I : cx + hy = \gamma^*\}\right)\right) \cap \mathbb{Z}^p \neq \emptyset,$$

then there is an $i \in \mathbb{N}$ with $\max\{cx + hy : \ (x, y) \in P^{(i)}\} = \gamma^*$.

Proof. If $\max\{cx + hy : (x, y) \in P\} = \gamma^*$ there is nothing to show, so let $\max\{cx + hy : (x, y) \in P\} > \gamma^*$ and let $M := \{(x, y) \in P_I : cx + hy = \gamma^*\}$ denote the solution set. Moreover, let $x^* \in \operatorname{relint}(\operatorname{proj}_X(M)) \cap \mathbb{Z}^p$ according to the assumption. By

$$M_>^{(i)} := \{(x, y) \in P^{(i)} : cx + hy > \gamma^*\} \text{ and}$$
$$M_\geq^{(i)} := \{(x, y) \in P^{(i)} : cx + hy \geq \gamma^*\}$$

we denote for $i \in \mathbb{N}_0$ and $P^{(0)} := P$ the set of all $(x, y) \in P^{(i)}$ which have to cut off and its closure, respectively. To prove the claim, we have to show that $cx + hy \leq \gamma^*$ is a split cut for one of the sets $P^{(i)}$, $i \in \mathbb{N}$. We divide the proof in three parts.

1) We show that $\dim(\operatorname{proj}_X(M)) \leq p - 1$. By Lemma 1.3, we can write

$$\operatorname{proj}_X(M_\geq^{(i)}) = \left\{x \in \mathbb{R}^p : v^r \begin{pmatrix} A \\ -c \end{pmatrix} x \leq v^r \begin{pmatrix} b \\ -\gamma^* \end{pmatrix} \ \forall v^r \in R\right\}, \tag{3.5}$$

where R is the set of extreme rays of the cone

$$Q = \left\{v \in \mathbb{R}^{m+1} : v \begin{pmatrix} G \\ -h \end{pmatrix} = 0, \ v \geq 0\right\}.$$

The right hand side in the representation of $\operatorname{proj}_X(M_\geq^{(i)})$ in (3.5) depends on the value of γ^*, while the left hand side does not. Suppose now that $\dim(\operatorname{proj}_X(M)) = p$. In this case it would be $x^* \in \operatorname{int}(\operatorname{proj}_X(M))$ and so $x^* \in \operatorname{int}(\operatorname{proj}_X(M_\geq^{(i)}))$. Now by (3.5) it would exist a $\epsilon > 0$ such that $x^* \in \operatorname{proj}_X(\{(x, y) \in P : cx + hy \geq \gamma^* + \epsilon\})$ and the inequality $cx + hy \leq \gamma^*$ would be no valid cut for P_I in contradiction to the assumption.

2) We construct a suitable split disjunction to derive the cut $cx + hy \leq \gamma^*$ for a split closure $P^{(i)}, i \in \mathbb{N}$. It follows by definition and as $x^* \in \operatorname{relint}(\operatorname{proj}_X(M))$ that

$$\operatorname{proj}_X(M) \cap \operatorname{proj}_X(M_>^{(i)}) = \emptyset \text{ for all } i \in \mathbb{N}_0.$$

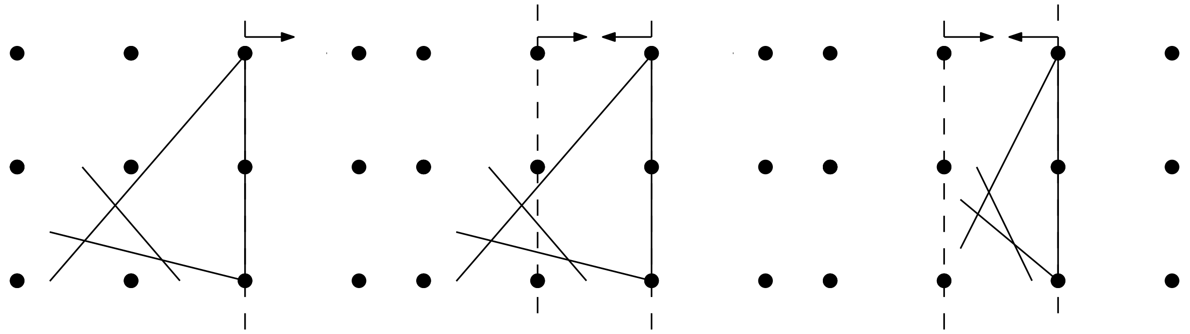
Moreover, it is $M_\geq^{(i+1)} \subseteq M_\geq^{(i)}$ for all $i \in \mathbb{N}_0$, so it follows further that

$$\operatorname{proj}_X(M) \subseteq \operatorname{relbd}(\operatorname{proj}_X(M_\geq^{(i)})) \text{ for all } i \in \mathbb{N}_0. \tag{3.6}$$

As P is rational, also $M, M_>^{(i)}$, $M_\geq^{(i)}$ and their projections on the x-space are rational. Therewith, it follows altogether by part 1) and by (3.6) that there exists an inequality $dx \leq \delta$ with $d \in \mathbb{Z}^p, \delta \in \mathbb{Z}$ which supports $\operatorname{proj}_X(M_\geq^{(0)})$ in x^* and satisfies

$$dx = \delta \ \forall x \in \operatorname{proj}_X(M). \tag{3.7}$$

According to (3.7) we define the split disjunction $D(d, \delta - 1)$.



(a) The set $\text{proj}_X\,(M_{\geq}^{(0)})$ and the inequality $dx \leq \delta$

(b) The set $\text{proj}_X\,(M_{\geq}^{(0)})$ and the split disjunction $D(d, \delta - 1)$

(c) The set $\text{proj}_X\,(M_{\geq}^{(i)})$ for sufficiently large i. $\text{proj}_X\,(M_{>}^{(i)})$ is now disjoint to the disjunction.

Figure 3.6: To the proof of Theorem 3.13

3) We show that $cx + hy \leq \gamma^*$ is a split cut to the disjunction $D(d, \delta - 1)$ for a split closure $P^{(i)}$, $i \in \mathbb{N}$. It follows by Theorem 3.11 that $\lim_{i\to\infty} M_{\geq}^{(i)} = M$, so it is $\lim_{i\to\infty} \text{proj}_X\,(M_{\geq}^{(i)}) = \text{proj}_X\,(M)$ by Lemma 1.6. This means by Corollary 3.12 that

$$\lim_{i\to\infty} \max\{-dx : x \in \text{proj}_X\,(M_{\geq}^{(i)})\} = -\delta. \tag{3.8}$$

(3.8) shows that there exists a $i_0 \in \mathbb{N}$ such that

$$\delta - 1 < dx < \delta \quad \text{for all } x \in M_{>}^{(i_0)},$$

so $M_{>}^{(i_0)}$ is disjoint to the disjunction $D(d, \delta - 1)$. This proves the theorem. □

Before we discuss some implications of Theorem 3.13, we turn to real polytopes. In this case, the theorem is not to true in general. Here the crucial point in the proof of Theorem 3.13 is in part 2). If P is not rational, also $\text{proj}_X\,(M_{\geq}^{(i)})$ might not be rational anymore. So it can be impossible to find a rational supporting hyperplane of $\text{proj}_X\,(M_{\geq}^{(i)})$ which satisfies condition (3.7) in the case that $\dim(\text{proj}_X\,(M)) \leq p - 2$. To solve this problem we consider a suitable irrational $(p+2)$-disjunction instead of a split disjunction for the derivation of the cut $cx + hy \leq \gamma^*$.

Theorem 3.14. *Let* $P \subseteq \mathbb{R}^{p+q}$ *be a polytope,* $(c, h) \in \mathbb{R}^{p+q}$, *and* $\gamma^* = \max\{cx + hy :\ (x, y) \in P_I\}$. *If*

$$\text{relint}\,(\text{proj}_X\,(\{(x, y) \in P_I : cx + hy = \gamma^*\})) \cap \mathbb{Z}^p \neq \emptyset,$$

then there is an $i \in \mathbb{N}$ *with* $\max\{cx + hy :\ (x, y) \in P_{irr,p+2}^{(i)}\} = \gamma^*$.

Proof. The proof is quite analog to the proof of Theorem 3.13, so we assume again that $\max\{cx + hy : (x, y) \in P\} > \gamma^*$ and let $M := \{(x, y) \in P_I : cx + hy = \gamma^*\}$ denote the solution set. Moreover, let $x^* \in \text{relint}\,(\text{proj}_X(M)) \cap \mathbb{Z}^p$ according to the assumption. To prove the statement for real polytopes, we have to show that $cx + hy \leq \gamma^*$ is an irrational $(p + 2)$-disjunctive cut for one of the sets $P^{(i)}$, $i \in \mathbb{N}$.

1) The first part of the proof of Theorem 3.13 is independent of rational input data, so we have again $\dim(\text{proj}_X(M)) < p$.

2) We construct a suitable irrational $p+2$-disjunction $D_{irr}(p+2, d, \delta)$ to derive the cut $cx + hy \leq \gamma^*$ for a split closure $P^{(i)}, i \in \mathbb{N}$. The sets

$$M_>^{(i)} := \{(x, y) \in P^{(i)} : cx + hy > \gamma^*\} \text{ and}$$
$$M_\geq^{(i)} := \{(x, y) \in P^{(i)} : cx + hy \geq \gamma^*\}$$

and $\text{proj}_X(M_>^{(i)})$, $\text{proj}_X(M_\geq^{(i)})$ are in general not rational anymore, whereas $\text{proj}_X(M)$ is still rational. Except for these modifications, the consequences of the second part in the proof of Theorem 3.13 remain correct, so there exists a possibly irrational inequality $zx \leq \zeta$ with $z \in \mathbb{R}^p, \zeta \in \mathbb{R}$ which supports $\text{proj}_X(M_\geq^{(0)})$ in x^* and satisfies

$$zx = \zeta \;\forall x \in \text{proj}_X(M). \tag{3.9}$$

We take a rational p-dimensional simplex S with vertices $s_1, \ldots, s_{p+1}$ which satisfies the properties

- $M \subseteq \text{int}\,(S)$ and
- $zs_1 < \zeta$ and $zs_i > \zeta_i$ for $i \in \{2, \ldots, p + 1\}$.

As S is bounded, there are only finite many integral vectors $x_1, \ldots, x_r \in \mathbb{Z}^p$ such that $x_i \in \text{int}\,(S)$ and $zx_i < \zeta$.

Next, we define the set $A := \text{conv}\,(s_1, x_1, \ldots, x_r)$. A is rational by construction and disjoint to $\text{proj}_X(M)$. As $\text{proj}_X(M)$ is also rational, there exists a rational affine half-space $wx \leq \omega$ such that $wx > \omega$ for all $x \in A$ and $wx < \omega$ for all $x \in \text{proj}_X(M)$. Therewith we define the polyhedron

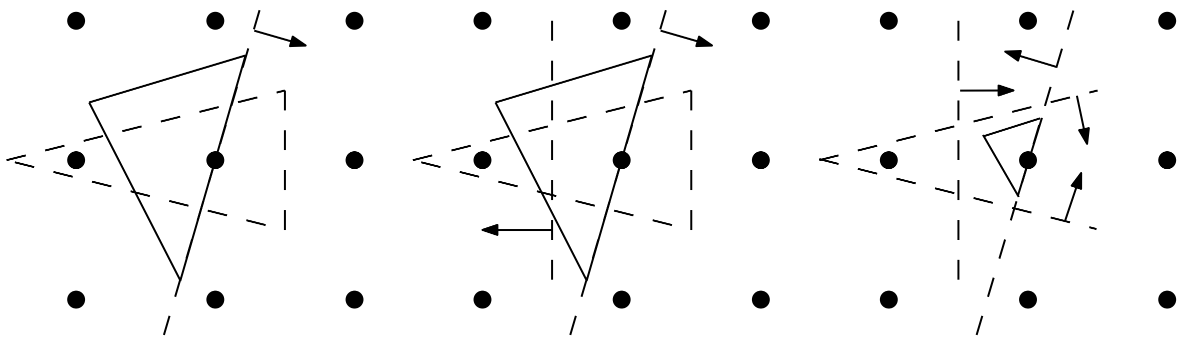
$$T := S \cap \{x \in \mathbb{R}^p : zx \leq \zeta, wx \leq \omega\}.$$

By construction, T is a polyhedron defined by $p + 1$ rational inequalities and the inequality $zx \leq \zeta$. So we can write

$$T = \{x \in \mathbb{R}^p : tx \leq \tau, zx \leq \zeta\}$$

with $t \in \mathbb{Z}^{(p+1)\times p}, \tau \in \mathbb{Z}^{p+1}$. Now we define the irrational $(p + 2)$-disjunction $D_{irr}(p + 2, d, \delta)$ by

$$d = (-t, -z) \text{ and } \delta = (-\tau, -\zeta).$$



(a) The set $\text{proj}_X\,(M_{\geq}^{(0)})$, the inequality $zx \leq \zeta$, and the simplex S.

(b) The set $\text{proj}_X\,(M_{\geq}^{(0)})$ and the polyhedron T

(c) The set $\text{proj}_X\,(M_{\geq}^{(i)})$ for sufficiently large i. $\text{proj}_X\,(M_{\geq}^{(i)})$ is now disjoint to the irrational $(p+2)$-disjunction D.

Figure 3.7: To the proof of Theorem 3.14

3) According to the third part of the proof of Theorem 3.13 it follows that

$$\lim_{i\to\infty} \max\{t_j x : x \in \text{proj}_X\,(M_{\geq}^{(i)})\} \leq \tau - 1.$$

So $cx + hy \leq \gamma^*$ is an irrational $(p+2)$-disjunctive cut for a split closure $P^{(i)}$. □

An important special case of Theorem 3.13 and Theorem 3.14 is certainly given if the projection $\text{proj}_X\,(M)$ of the solution set M is 0-dimensional, so especially if M is a vertex of P_I. In this case, the optimal objective function value γ^* and therewith by Corollary 2.21 also an optimal solution (x^*, y^*) of a MILP can always be computed by split cuts or by irrational $(p+2)$-disjunctive cuts, respectively. Moreover, as the set of optimal solutions always contains a vertex if the feasible domain $P_I \neq \emptyset$ is given by a polytope, we can in theory find an optimal solution of every MILP by split cuts or by irrational $(p+2)$-disjunctive cuts if we consider if necessary a MILP with a suitable, sufficiently small perturbation of the objective function.

Corollary 3.15. *Let a MILP* (1.1) *be given, where the feasible domain P of the LP relaxation is bounded, and let $\gamma^* := \max\{cx + hy : (x,y) \in P_I\}$. Then there exists an objective function $\max \widetilde{c}x + \widetilde{h}y$ and a $(x^*, y^*) \in P_I$ with the following properties:*

1. *(x^*, y^*) is the unique optimal solution of $\max\{\widetilde{c}x + \widetilde{h}y : (x,y) \in P_I\}$ with $\widetilde{\gamma} := \widetilde{c}x^* + \widetilde{h}y^*$.*

2. *It is $cx^* + hy^* = \gamma^*$.*

3. *If P and (c, h) are rational there is a $n \in \mathbb{N}$ with* $\max\{\widetilde{c}x + \widetilde{h}y : (x, y) \in P^{(n)}\} = \widetilde{\gamma}$. *Otherwise, there is a $n \in \mathbb{N}$ with* $\max\{\widetilde{c}x + \widetilde{h}y : (x, y) \in P^{(n)}_{irr,p+2}\} = \widetilde{\gamma}$. □

As an easy application of Corollary 3.15, we consider a slight variation of Example 2.13.

Example 3.16. *Let the polyhedron $P \subseteq \mathbb{R}^{2+1}$ be defined by the system*

$$\begin{aligned} -x_1 + y &\leq 0 \\ -x_2 + y &\leq 0 \\ x_1 + x_2 + y &\leq 2, \end{aligned}$$

and consider the MILP $\max\{y : (x, y) \in P, x \in \mathbb{Z}^2\}$.

The optimal objective function value is given by $y = 0$ and the set of optimal solutions of the MILP is the facet of P_I which is given as the convex hull of the vectors $(0, 0, 0), (2, 0, 0), (0, 2, 0)$. *Moreover, it is*

$$\max\{y : (x, y) \in P^{(k)}\} > 0 \ \forall k \in \mathbb{N};$$

see Example 2.13. The vector $(0, 0, 0)$ is an optimal solution of the MILP and the perturbated objective function vector $(-\epsilon, -\epsilon, 1-\epsilon)$ with $0 < \epsilon < \frac{1}{3}$ satisfies the conditions of Corollary 3.15 with $(x^*, y^*) = (0, 0, 0)$. So we can obtain an optimal solution of the MILP within a finite number of split closures by considering the MILP

$$\max\{-\epsilon x_1 - \epsilon x_2 + (1 - \epsilon)y : (x, y) \in P, x \in \mathbb{Z}^2\}.$$

We note that applying Corollary 3.15 practically yields some problems. First, it is not obvious how a suitable perturbated objective function which satisfies the conditions of the corollary can be found in general. Secondly, Corollary 3.15 is based on the split closure of a polytope P. So it is open which split cuts or which irrational $(p + 2)$-disjunctive cuts have to be added to obtain an optimal solution (x^*, y^*) of a MILP. For the latter issue we especially remind that it is open if the split closure or the irrational k-disjunctive closure is again a polytope for a given real polytope.

Theorem 3.13 shows that we can obtain all vertices of the mixed integer hull P_I of a polytope P by an repeated application of split cuts or irrational $(p+2)$-disjunctive cuts. So we can see that these two types of cuts are sufficient to find a $\mathcal{V}$-representation of P_I. We compare this property later with the characterization of a $\mathcal{H}$-representation of P_I.

After we have dealt with the characterization of special faces of P_I, we turn to the general situation. Here we have already seen in Theorem 3.8 that it is $P_I = P^{(1)}_{2^p}$ for rational polyhedra. The idea for finite convergence using k-disjunctive cuts is based on the principle that there has to exist a k-disjunction $D(k, d, \delta)$ such that the relative interior $\operatorname{relint}(\operatorname{proj}_X(M))$ of the projection of the solution set M is disjoint to the set of all points satisfying the disjunction D. We will see that if no such k-disjunction exists for all closures $P_k^{(i)}$, then we cannot achieve a finite algorithm by k-disjunctive cuts with $k < 2^p$ in general.

On the other hand, if this condition is satisfied for each face of the solution set, finite convergence can be obtained.

Theorem 3.17. *Let $P \subseteq \mathbb{R}^{p+q}$ be a rational polyhedron, $(c, h) \in \mathbb{Q}^{p+q}$, $\gamma^* = \max\{cx + hy : (x, y) \in P_I\}$ and $M_0 := \{(x, y) \in P_I : cx + hy = \gamma^*\}$. Moreover, denote by $M_1, \ldots, M_F$ the faces of M_0. If there exists for all sets $M_i, i \in \{0, \ldots, F\}$ with* $\operatorname{relint}(\operatorname{proj}_X(M_i)) \cap \mathbb{Z}^p = \emptyset$ *a k-disjunction $D_i(k, d_0, \delta_0)$ with the property*

$$x \in \operatorname{relint}(\operatorname{proj}_X(M_i)) \implies d_0 x > \delta_0,$$

then there exists a $n \in \mathbb{N}$ with $\max\{cx + hy : (x, y) \in P_k^{(n)}\} = \gamma^*$.

Proof. We prove the theorem by induction on the dimension $l := \dim(M_0)$ of the solution set $M_0 = \{(x, y) \in P_I : cx + hy = \gamma^*\}$. For $l = 0$, the result is a special case of Theorem 3.13 and $k = 2$, that means split cuts, can be chosen. The same is true if $\operatorname{relint}(\operatorname{proj}_X(M_0)) \cap \mathbb{Z}^p \neq \emptyset$. So let the statement be true for $l - 1, l \in \mathbb{N}$ and let $\operatorname{relint}(\operatorname{proj}_X(M_0)) \cap \mathbb{Z}^p = \emptyset$.

Now let $\dim(M_0) = l$ and let for $k \in \mathbb{N}$ exist k-disjunctions $D_i(k, d_i, \delta_i)$ for M_0 and all of its faces $M_1, \ldots, M_F$ according to the assumption. We prove that $cx + hy \leq \gamma^*$ is a k-disjunctive cut to the disjunction D_0 for a set $P_k^{(n)}$, so we have to show that there exists a $n \in \mathbb{N}$ such that $d_0 x > \delta_0$ for all x with $\{(x, y) \in P_k^{(n)} : cx + hy > \gamma^*\}$.

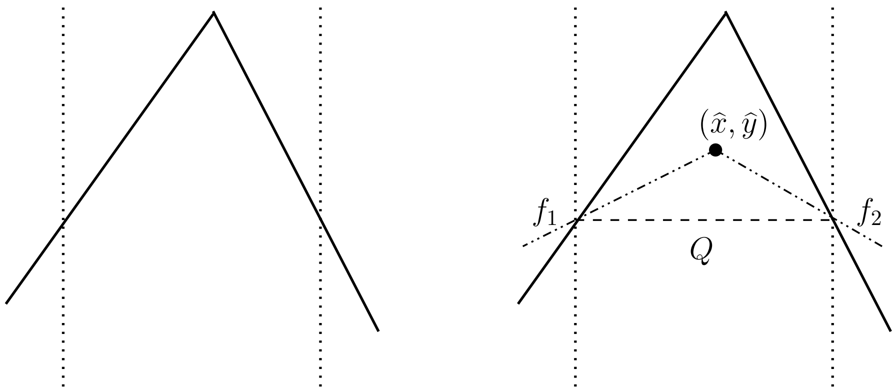
Given the disjunction D_0, we define the polyhedron

$$Q := \{(x, y) \in \mathbb{R}^{p+q} : d_M x \geq \delta_M \wedge cx + hy = \gamma^*\}$$

as the closure of all points $(x, y) \in \mathbb{R}^{p+q}$ which are not contained in the disjunction D_0 and satisfy $cx + hy = \gamma^*$. It follows by assumption for the disjunction D_0 that $\operatorname{relint}(M_0) \subseteq \operatorname{relint}(Q)$.

Now let $(\widehat{x}, \widehat{y}) \in \mathbb{R}^{p+q}$ be a point with $c\widehat{x} + h\widehat{y} > \gamma^*$ and $d\widehat{x} > \delta$. We define for each facet f of Q the inequality $c_f x + h_f y \leq \gamma_f$ such that $c_f x + h_f y = \gamma_f$ for all $x \in f$, $c_f \widehat{x} + h_f \widehat{y} = \gamma_f$, and $c_f x + h_f y \leq \gamma_f$ for all $x \in Q$; see Figure 3.8.

As $cx + hy = \gamma^*$ is a supporting hyperplane of P_I, the point $(\widehat{x}, \widehat{y})$ can be chosen such that each of the inequalities $c_f x + h_f y \leq \gamma_f$ is valid for P_I and supports M_0 at most in one of its faces $M_1, \ldots, M_F$. For the construction of $(\widehat{x}, \widehat{y})$ we can take for example a relative interior point $(\tilde{x}, \tilde{y})$ of Q and set $(\widehat{x}, \widehat{y}) = (\tilde{x}, \tilde{y} + \epsilon h)$ for a sufficiently small ϵ. We note that $h \neq 0$ by assumption made at the end of Section 1.2.



(a) A polyhedron $P \subseteq \mathbb{R}^{1+1}$ and the disjunction D_M (dotted)

(b) The polyhedron Q, the point $(\hat{x}, \hat{y})$, and two inequalities $c_f x + h_j y \leq \gamma_f$

Figure 3.8: To the proof of Theorem 3.17.

It follows either by induction hypothesis or by Theorem 3.11 that the inequalities $c_f x + h_f y \leq \gamma_f$ are valid for a closure $P_k^{(n)}$. Therewith, the condition $d_0 x > \delta_0$ for all x with $\{(x, y) \in P_k^{(n)} : cx + hy > \gamma^*\}$ is satisfied. □

For real polytopes we obtain the following analog statement.

Theorem 3.18. *Let $P \subseteq \mathbb{R}^{p+q}$ be a polytope, $(c, h) \in \mathbb{R}^{p+q}$, $\gamma^* = \max\{cx + hy : (x, y) \in P_I\}$ and $M_0 := \{(x, y) \in P_I : cx + hy = \gamma^*\}$. Moreover, denote by $M_1, \ldots, M_F$ the faces of M_0. If there exists for all sets M_i, $i \in \{0, \ldots, F\}$ with $\operatorname{relint}(\operatorname{proj}_X(M_i)) \cap \mathbb{Z}^p = \emptyset$ a k-disjunction $D_i(k, d_i, \delta_i)$ with the property*

$$x \in \operatorname{relint}(\operatorname{proj}_X(M_i)) \implies d_i x > \delta_i,$$

then there exists a $n \in \mathbb{N}$ with $\max\{cx + hy : (x, y) \in P_{irr,j}^{(n)}\} = \gamma^*$, where $j = \max\{k, p + 2\}$.

Proof. The proof is quite analog to the proof of Theorem 3.17. For the base case of the induction we now need irrational $(p+2)$-disjunctive cuts according to Theorem 3.14. For the remainder of the proof rational k-disjunctions are actually sufficient as all projections $\operatorname{proj}_X(M_i)$, $i \in \{0, \ldots, F\}$ are rational. □

We note that it is actually necessary to involve not only the solution set M_0 but also all of its faces M_i, $i \in \{0, \ldots, F\}$ in the assumptions of Theorem 3.17 and Theorem 3.18. This can be seen in the following example. We 'duplicate' the mixed integer hull of Example 2.13 and embed it in a higher dimensional space.

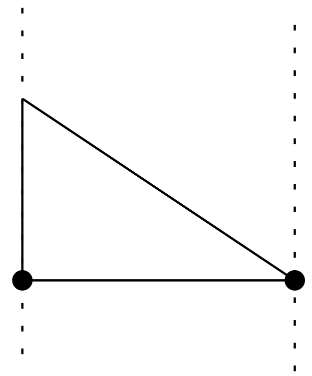


Figure 3.9: To Example 3.19. Projection of P on the x_3y-space and the disjunction $D(u_3, 0)$

Example 3.19. *We define the polyhedron $P \subseteq \mathbb{R}^{3+1}$ as the convex hull of the vertices*

$$\begin{aligned}&(0,0,0,0),(2,0,0,0),(0,2,0,0),\\&(0,0,1,0),(2,0,1,0),(0,2,1,0),\\&\left(\frac{2}{3},\frac{2}{3},0,\frac{2}{3}\right).\end{aligned}$$

and consider the MILP $\max\{y : (x, y) \in P, x \in \mathbb{Z}^3\}$. *It is*

$$P_I = \operatorname{conv}\{(0,0,0,0),(2,0,0,0),(0,2,0,0),(0,0,1,0),(2,0,1,0),(0,2,1,0)\},$$

so $\max\{y : (x, y) \in P_I\} = 0$. *The relative interior* $\operatorname{relint}(P_I)$ *is not contained in the split disjunction* $D(u_3, 0) = \{x \in \mathbb{Z}^3 : x_3 \leq 0 \vee x_3 \geq 1\}$. *However, the inequality* $y \leq 0$ *is no split cut to the disjunction* $D(u_3, 0)$ *as the vertex* $\left(\frac{2}{3}, \frac{2}{3}, 0, \frac{2}{3}\right)$ *satisfies* $x_3 = 0$*; see Figure 3.9.*

We go on and deal now with the question which cuts we need to solve a MILP in general. We define special polytopes which can arise as solution sets of MILP. The idea is based on a generalization of Example 2.13.

Lemma 3.20. *Let $P \subseteq \mathbb{R}^{p+q}$ be a polytope,*

$$\gamma^* = \max\{cx + hy : (x, y) \in P_I\} < \max\{cx + hy : (x, y) \in P\}$$

and let $M := \{(x, y) \in P_I : cx + hy = \gamma^*\}$. *If* $\operatorname{proj}_X(M)$ *is full dimensional and has* $k \leq 2^p$ *facets with each containing a relative interior integral point, then*

$$\max\{cx + hy : (x, y) \in P_{k-1}^{(i)}\} > \gamma^* \quad \textit{for all } i \in \mathbb{N}.$$

Proof. Let $\text{proj}_X(M)$ with at least k facets be given and let $x^1, \dots, x^k \in \mathbb{Z}^p$, where x^i is a relative interior point of facet i. Then it is

$$x^{ij} := \frac{1}{2}(x^i + x^j) \in \text{int}\,(\text{proj}_X(M)) \text{ for all } i, j,\ i \neq j$$

as $\dim(\text{proj}_X(M)) = p$ by assumption. Since $\max\{cx + hy : (x, y) \in P\} > \gamma^*$ for all x^{ij} there exists a $y^{ij} \in \mathbb{R}^q$ such that $(x^{ij}, y^{ij}) \in P$ and $cx^{ij} + hy^{ij} > \gamma^*$.

For each $(k-1)$-disjunction $D(k-1, d, \delta)$ there exists an inequality $d^l x \leq \delta^l$ of the disjunction D with $d^l x^i \leq \delta^l$ and $d^l x^j \leq \delta^l$ for two of the k relative interior integral points $x^1, \dots, x^k$, as k integral points have to be covered by $k-1$ inequalities. This means that also the point x^{ij} is contained in the disjunction D, so $d^l x^{ij} \leq \delta^l$. Therewith each cut to the disjunction D is valid for the set

$$Q^{ij} := \text{conv}\,\{(x^{ij}, y^{ij}), P_I\}.$$

As every $(k-1)$-disjunctive cut is valid for one of the sets Q^{ij}, we obtain that

$$Q := \bigcap_{i \neq j} Q^{ij} \subseteq P^{(1)}_{k-1}.$$

Moreover, we can see that Q also contains a point $(x, y) \in P$ with $cx + hy > \gamma^*$. As $Q \subseteq P^{(1)}_{k-1}$ satisfies again all presumptions of the lemma and the solution set M does not change, the proof follows by induction. □

We can now show that the generation of the mixed integer hull of a polytope in general requires k-disjunctive cuts, where k is exponential in the dimension p. For this purpose, we construct a suitable polytope that satisfies the assumption of Lemma 3.20.

Theorem 3.21. *Let $P \subseteq \mathbb{R}^{p+q}$ be a polyhedron and $\gamma^* = \max\{cx + hy : (x, y) \in P_I\}$. Then in general*

$$\max\{cx + hy : (x, y) \in P^{(i)}_{2^{p-1}+1}\} > \gamma^* \quad \textit{for all } i \in \mathbb{N}.$$

Proof. By Lemma 3.20, it is sufficient to give a full dimensional integral polytope $Q \subseteq \mathbb{R}^p$ with at least $2^{p-1} + 2$ facets and which contains no interior integral point but a relative interior integral point in each facet. Then the polytope Q can be embedded in the $\mathbb{R}^{p+1}$ and easily be completed to a polyhedron P which satisfies the assumption of Lemma 3.20 with $M = Q$.

For $x \in \mathbb{R}^{p-1}, x_p \in \mathbb{R}$ we define Q as the set of all $(x, x_p) \in \mathbb{R}^p$ which satisfy the inequalities

$$\begin{aligned} &ax - \pi(a)x_p \leq 1,\ a \in \{\pm 1\}^{p-1} \text{ and} \\ &0 \leq x_p \leq 2, \end{aligned} \tag{3.10}$$

where $\pi(a) := |\{i \in \{1,\dots,p-1\} : a_i = 1\}| - 1$. We prove that Q satisfies the properties of Lemma 3.20. First, we show that Q can be described as the convex hull of two $(p-1)$-dimensional cross polytopes.

For $x_p = 0$, the related polytope given by (3.10) is the $(p-1)$-dimensional cross polytope. For $x_p = 2$, the related polytope is the convex hull of the vertices $\mathbf{1} \pm (p-2)u_i$. This follows from the fact that an inequality $ax - \pi(a) \cdot 0 \leq 1$ in (3.10) is satisfied with equality for an unit vector $\pm u_i \subseteq \mathbb{R}^{p-1}$ if and only if $ax \leq 1 + 2\pi(a)$ is satisfied with equality for the vector $\mathbf{1} \pm (p-2)u_i$. In detail, it is

$$\begin{aligned}
&a(\mathbf{1} \pm (p-2)u_i) = \\
&= |\{i \in \{1,\dots,p-1\} : a_i = 1\}| - |\{i \in \{1,\dots,p-1\} : a_i = -1\}| \pm (p-2)au_i = \\
&= \pi(a) + 1 - (p - 1 - (\pi(a)+1)) \pm (p-2)au_i = \\
&= 2\pi(a) + 3 - p \pm (p-2)au_i = \\
&= 2\pi(a) + 1, \quad \text{for } \pm au_i = 1.
\end{aligned}$$

We just have shown by this calculation that an inequality $ax \leq 1 + x_p\pi(a)$ supports a vertex $(\pm u_i, 0)$ if and only if it supports the vertex $(\mathbf{1} \pm (p-2)u_i, 2)$. So it follows that Q has no more vertices and we have

$$Q = \operatorname{conv}\{(\pm u_i, 0), (\mathbf{1} \pm (p-2)u_i, 2),\ i \in \{1,\dots,p-1\}\},$$

and Q is integral.

Now let $(z,1) \in \mathbb{Z}^p$ be given. We take the inequality

$$ax \leq 1 + 1 \cdot \pi(a) \;\text{ with } a_i = 1 \Longleftrightarrow z_i > 0,\; i \in \{1,\dots,p-1\} \tag{3.11}$$

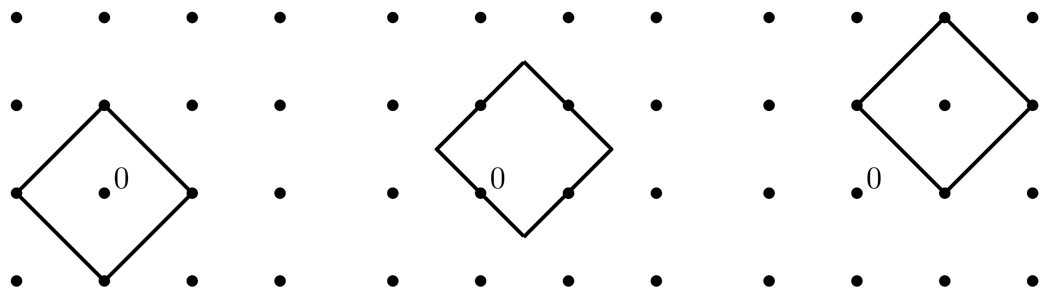
and obtain

$$az = \sum_{1 \leq i \leq p-1} |z_i| \geq \sum_{z_i > 0} z_i \geq |\{i \in \{1,\dots,p-1\} : a_i = 1\}| = 1 + \pi(a). \tag{3.12}$$

This shows that $(z,1)$ is no interior point of Q and so Q contains no integral interior points.

Finally, we show that each facet of Q contains a relative interior integral point. First, we can see that $(\mathbf{0},0)$ and $(\mathbf{1},2)$ are relative interior points of the facets $x_p = 0$ and $x_p = 2$. Next, the point $(z,1)$ with $z \in \{0,1\}^{p-1}$ is a relative interior point of the facet $ax \leq 1 + x_p\pi(a)$ according to (3.11). Here it follows by (3.12) that $az = 1 + \pi(a)$ and it is $az < 1 + \pi(a)$ for the remaining inequalities by definition of z.

So Q satisfies all properties of Lemma 3.20 and Q has $2^{p-1} + 2$ facets. □



(a) $x_3 = 0$ (b) $x_3 = 1$ (c) $x_3 = 2$

Figure 3.10: The set Q according to Theorem 3.21 for p = 3, projected on the x_3-space

We have proven that in general at least $2^{p-1}+2$-disjunctive cuts are required to generate the mixed integer hull of a polyhedron or to solve a MILP exactly. By Theorem 3.8, we have an upper bound of $k = 2^p$. Here it is open if the upper bound can be improved to $k = 2^{p-1} + 2$. Theorem 3.21 moreover shows that not only general $2^{p-1} + 2$-disjunctive cuts but one special $2^{p-1}+2$-disjunctive cut can be necessary to obtain the mixed integer hull P_I of a given polyhedron.

Altogether, we can see that an exact cutting plane algorithm becomes very costly in general as a large number of disjunctive hyperplanes has to be determined for cut generation. We also have not yet discussed how to determine a deep cut to a given k-disjunction. As Example 3.4 has shown, intersection cuts to basic solutions of the polyhedron fail here in general. Therefore, we deal with this issue in the next section, where we have to restrict ourselves to rational polytopes.

On the other hand, we have seen that - depending on the geometric structure of the projection of the solution set on the integral space - a wide class of problems can even be solved by split cuts. In this context it is interesting to see that a repeated application of cuts to small disjunctions cannot replace a single cut to a large disjunction in general. This means that also facets of the mixed integer hull of a polyhedron can be only obtained by k-disjunctive cuts with an exponential k in general. So determining a $\mathcal{H}$-representation of P_I requires much more effort than a $\mathcal{V}$-representation.

At last we note that due to our observations it also seems to be reasonable to use information of projections for cut generation and related algorithms.

3.4 Computing k-disjunctive cuts

In this section we introduce an alternative for computing deep valid k-disjunctive cuts for rational polyhedra. Therefore, we assume for the remainder of this section that all input data is rational.

Unlike the 'usual way' of computing valid cuts for MILP, we need a vector (c, h) to which we want to cut as additional input in our approach. We restrict ourselves at the moment to polyhedral cones $P = \{(x, y) : Ax + Gy \leq b\}$ with apex (x^*, y^*), $x^* \notin \mathbb{Z}^p$ and assume that $cx + hy$ attains its unique maximum at (x^*, y^*) with function value γ^*. Therewith, we can describe (x^*, y^*) as polyhedron given by

$$P_{\gamma^*} := \left\{ (x, y) \in \mathbb{R}^{p+q} : \begin{pmatrix} A \\ -c \end{pmatrix} x + \begin{pmatrix} G \\ -h \end{pmatrix} y \leq \begin{pmatrix} b \\ -\gamma^* \end{pmatrix} \right\}. \tag{3.13}$$

By Lemma 1.3, the projection of this polytope on the x-space - which is equal to x^* - is given by

$$\text{proj}_X(P_{\gamma^*}) = \left\{ x \in \mathbb{R}^p : v^r \begin{pmatrix} A \\ -c \end{pmatrix} x \leq v^r \begin{pmatrix} b \\ -\gamma^* \end{pmatrix} \ \forall v^r \in R \right\}, \tag{3.14}$$

where R is the set of extreme rays of the cone

$$Q = \left\{ v \in \mathbb{R}^{m+1} : v \begin{pmatrix} G \\ -h \end{pmatrix} = 0, \ v \geq 0 \right\}. \tag{3.15}$$

As the cone Q is rational we can assume that the extreme rays v^r are elements of the additive group

$$\mathcal{G}_{P_{\gamma^*}} := \left\{ w \in \mathbb{Q}^{m+1} : w \begin{pmatrix} A \\ -c \end{pmatrix} \in \mathbb{Z}^{m+1} \right\}. \tag{3.16}$$

We can now use the polyhedral description of $\text{proj}_X(P_{\gamma^*})$ to define a valid k-disjunction for P which does not contain the apex (x^*, y^*). We do this by rounding up the right hand sides of the inequalities which define $\text{proj}_X(P_{\gamma^*})$ in (3.14).

Lemma 3.22. *Let* $P, (x^*, y^*), (c, h), \gamma^*$, *and* $\text{proj}_X(P_{\gamma^*})$ *be according to* (3.13) - (3.16). *Moreover, define for* $v^r \in R$

$$\begin{aligned} d^r &:= v^r_{1,\dots,m} A - v^r_{m+1} c \\ \delta^r &:= \lfloor v^r_{1,\dots,m} b - v^r_{m+1} \gamma^* \rfloor + 1 \end{aligned}$$

with $v^r = (v^r_{1,\dots,m}, v^r_{m+1})$. *Then* $D(|R|, -d, -\delta)$ *is a valid* $|R|$*-disjunction that does not contain* (x^*, y^*).

Proof. By definition of P, it is $\max\{cx + hy : (x, y) \in P_I\} < \gamma^*$. So there is an $\epsilon > 0$ such that $cx + hy \leq \gamma^* - \epsilon$ is valid but not optimal for P_I. Moreover, $\text{proj}_X(P_{\gamma^* - \epsilon})$ is defined by the system

$$\begin{aligned} v^r_{1,\dots,m} Ax - v^r_{m+1} cx &\leq v^r_{1,\dots,m} b - v^r_{m+1} \gamma^* + v^r_{m+1} \epsilon \ \forall v^r \in R \iff \\ d^r x &\leq v^r_{1,\dots,m} b - v^r_{m+1} \gamma^* + v^r_{m+1} \epsilon \ \forall v^r \in R \end{aligned}$$

The set $\operatorname{proj}_X(P_{\gamma^*-\epsilon})$ contains no integral points, so for all $x \in \mathbb{Z}^p$ there is a $v^r \in R$ with

$$d^r x > v^r_{1,\dots,m} b - v^r_{m+1}\gamma^*.$$

It follows that the polyhedron $\{x \in \mathbb{R}^p : dx \leq \delta\}$ contains no integral point in its interior. This is equivalent to the fact that $D(|R|, -d, -\delta)$ is a valid $|R|$-disjunction. As the right hand side of each inequality which defines the projection $\operatorname{proj}_X(P_{\gamma^*})$ has been enlarged by the definition of the $|R|$-disjunction D, the vertex (x^*, y^*) is not contained in the disjunction.

We note that the above approach is similar to the proof of Lemma 3.5. □

We have constructed a k-disjunction which can be used to cut off the infeasible LP solution (x^*, y^*). As we have mentioned at the beginning of this section, we want to cut to the vector (c, h). We can do this now by using the values of the right hand sides δ^r of the disjunction $D(|R|, -d, -\delta)$ which we have introduced in Lemma 3.22.

As for $v^r_{m+1} > 0$ the value of δ^r depends on the objective function value, we can compute the objective function value γ^r that corresponds to the value of δ^r. Here the inequalities $-d^r x \leq -\delta^r$ of the disjunction D at which the right hand side δ^r is independent of the value of γ can be omitted. These are exactly those inequalities $-d^r x \leq -\delta^r$ which are related to an extreme ray $v^r \in R$ with $v^r_{m+1} = 0$. In detail, we obtain for an extreme ray $v^r \in R$ with $v^r_{m+1} > 0$ the value γ^r from the equation

$$v^r_{1,\dots,m} b - v^r_{m+1}\gamma^r = \lfloor v^r_{1,\dots,m} b - v^r_{m+1}\gamma^* \rfloor + 1 = \delta^r, \tag{3.17}$$

this yields

$$\gamma^r = \frac{\delta^r - v^r_{1,\dots,m} b}{-v^r_{m+1}}.$$

Taking the maximum $\widehat{\gamma}$ of the objective function values γ^r for all constraints $-d^r x \leq -\delta^r$ with $v^r_{m+1} > 0$ gives us the inequality $cx + hy \leq \widehat{\gamma}$ as valid cut to the vector (c, h). Since the disjunction $D(|R|, -d, -\delta)$ does not contain x^*, we can ensure that the infeasible vertex (x^*, y^*) is cut off. We summarize the result in the next

Theorem 3.23. *Let* $P, (x^*, y^*), (c, h), \gamma^*, \operatorname{proj}_X(P_{\gamma^*})$*, and* $D(|R|, -d, -\delta)$ *be according to* (3.13) - (3.16) *and to Lemma 3.22. Define for* $v^r \in R$ *with* $v^r_{m+1} > 0$

$$\begin{aligned} \gamma^r &:= \frac{\delta^r - v^r_{1,\dots,m} b}{-v^r_{m+1}} \quad \text{and} \\ \widehat{\gamma} &:= \max\{\gamma^r : r \in R,\ v^r_{m+1} > 0\}. \end{aligned}$$

Then $cx + hy \leq \widehat{\gamma}$ *is a valid inequality for* P_I *and* $cx^* + hy^* > \widehat{\gamma}$*.*

Proof. The validity of the inequality $cx + hy \leq \widehat{\gamma}$ follows directly by Lemma 3.22 and (3.17), as it is a $|R|$-disjunctive cut to the disjunction $D(|R|, -d, -\delta)$ by definition. Moreover, it is $cx^* + hy^* > \widehat{\gamma}$ as (x^*, y^*) is not contained in the disjunction D and so $\widehat{\gamma} < \gamma^*$. □

We consider again Example 3.4 as an example for our approach.

Example 3.24. *Let again the polyhedral cone $C \in \mathbb{R}^{2+1}$ defined by the system*

$$\begin{aligned} -x_1 + y &\leq 0 \\ -x_2 + y &\leq 0 \\ x_1 + y &\leq 1 \\ x_2 + y &\leq 1. \end{aligned}$$

with apex $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be given. Additionally, let $(0, 0, 1)$ be the vector to which we cut. The LP relaxation $\max\{y : (x, y) \in C\}$ attains its unique maximum at the apex of C with value $\gamma^* = \frac{1}{2}$. So the presumptions of Lemma 3.22 and Theorem 3.23 for applying the $|R|$-disjunctive cut are satisfied. Next, we compute the extreme rays of the projection cone

$$Q = \{(1\ 1\ 1\ 1\ \ -1)v = 0, v \geq 0\}$$

according to (3.15). As result, which also satisfies condition (3.16), we obtain the four vectors

$$(1, 0, 0, 0, 1), (0, 1, 0, 0, 1), (0, 0, 1, 0, 1), (0, 0, 0, 1, 1).$$

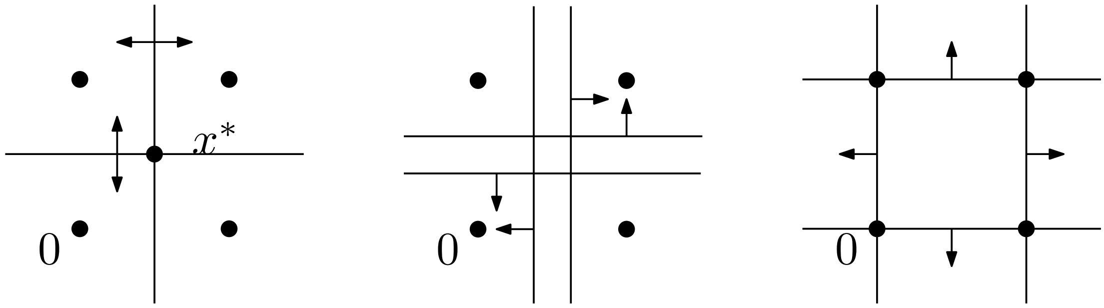
We can now apply Lemma 3.22 to determine the valid 4-disjunction $D(4, -d, -\delta)$ and obtain for the inequalities $d^r x \leq \delta^r$

$$\begin{aligned} -x_1 &\leq \lfloor 0 - \frac{1}{2} \rfloor + 1 = 0 \\ -x_2 &\leq \lfloor 0 - \frac{1}{2} \rfloor + 1 = 0 \\ x_1 &\leq \lfloor 1 - \frac{1}{2} \rfloor + 1 = 1 \\ x_2 &\leq \lfloor 1 - \frac{1}{2} \rfloor + 1 = 1. \end{aligned}$$

Now we can compute a valid cut according to Theorem 3.23. It is

$$\gamma^1 = \gamma^2 = \frac{0 - 0}{-1} = 0 \text{ and } \gamma^3 = \gamma^4 = \frac{1 - 1}{-1} = 0,$$

and so $\widehat{\gamma} = 0$. Therefore, the inequality $y \leq 0$ is valid for C; see also Figure 3.11.



(a) For $\gamma = \frac{1}{2}$ (b) For $\gamma = \frac{1}{2} - \epsilon$ (c) For $\gamma = 0$

Figure 3.11: To Example 3.24. Projection of $C \cap \{(x, y) : y \geq \gamma\}$ on the x_1x_2-space depending on γ

We discuss some further properties of the cutting plane approach. The computation of a cut according to Lemma 3.22 and Theorem 3.23 is not independent of the representation of the polyhedron P and the extreme rays v^r of the cone Q. For example, if we scale in Example 3.24 each of the extreme rays of the cone Q by 4, we obtain the vectors

$$(4, 0, 0, 0, 4), (0, 4, 0, 0, 4), (0, 0, 4, 0, 4), (0, 0, 0, 4, 4).$$

This leads to the smaller 4-disjunction

$$\begin{aligned}
-4x_1 &\leq \lfloor 0 - 4\frac{1}{2} \rfloor + 1 = -1 \iff x_1 \geq \frac{1}{4} \\
-4x_2 &\leq \lfloor 0 - 4\frac{1}{2} \rfloor + 1 = -1 \iff x_2 \geq \frac{1}{4} \\
4x_1 &\leq \lfloor 4 - 4\frac{1}{2} \rfloor + 1 = 3 \iff x_1 \leq \frac{3}{4} \\
4x_2 &\leq \lfloor 4 - 4\frac{1}{2} \rfloor + 1 = 3 \iff x_2 \leq \frac{3}{4}
\end{aligned}$$

and the weaker cut $y \leq \frac{1}{4}$. The same effect can be observed if the extreme rays remain unchanged but the inequalities defining the polyhedron C and the objective function vector are scaled by the factor 4 in Example 3.24.

By using the projection $\text{proj}_X (P_{\gamma^*})$ according to (3.14) as k-disjunction, we solve the problem of how to find a suitable k-disjunction $D(k, d, \delta)$ for deriving cuts. This relates to the selection of both the number k and the defining inequalities $dx \leq \delta$ of the disjunction. Moreover, we have seen in the last section that information on the projection of the polyhedron P can be useful for deriving strong cuts. On the other hand, there are two issues which have to be considered. First, the projection which defines the k-disjunction depends on the predisposed cut vector (c, h). So the selection of a suitable k-disjunction has been shifted to the selection of the cut vector. Here it is particularly not obvious how to choose the cut vector (c, h) to obtain deep cuts in general. However, we will see in Section 4.3 that this approach leads to a finite cutting plane algorithm for rational MILP if we use the objective function vector of the MILP for cutting.

Secondly, the number k of disjunction hyperplanes is equal to the number of extreme rays of the cone $Q = \left\{ v \in \mathbb{R}^{m+1} : v \begin{pmatrix} G \\ -h \end{pmatrix} = 0,\ v \geq 0 \right\}$ according to (3.15). This means that k is only determined by the properties of the $(m+1) \times q$ matrix G and is therewith independent of the number p of integral variables. So there is no direct connection between the results concerning the properties of k-disjunctive cuts which are required to solve a MILP in the last section and the way we choose the k-disjunction $D(k, d, \delta)$ here. Moreover, it also follows that in general the number k of disjunctive inequalities $d^i x \leq \delta^i$, $i \in \{1, \ldots, k\}$ is larger than necessary for the derivation of a deep cut.

At last we note that we have not yet discussed the complexity of our approach nor any further details which are vital for the approach as for example a procedure for computing extreme rays of cones. We also deal with these issues in Section 4.3 in the context of the finite cutting plane algorithm.

3.5 Related work

To conclude this chapter, we give a brief overview of a similar approach for general disjunctive cuts. It was developed independently from our work by Andersen, Louveaux, and Weismantel [ALW07a], [ALW07b] and has an other starting point, namely a generalization of the classical integer Farkas Lemma - Theorem 2.24 in the case of $G = 0$ - of Kronecker [Kro84] to systems of equations and inequalities. In detail, it is their aim to give certificates for rational systems of the form

$$\begin{aligned} Ax &= b \\ Cx &\leq d \\ x &\in \mathbb{Z}^p. \end{aligned}$$

They start with a geometric interpretation of the integer Farkas Lemma and show that there exists a connection to split disjunctions. To deal with equations and inequalities in the Farkas Lemma, they generalize split disjunctions to certain polyhedra which are lattice point free in their interior and call these polyhedra split bodies. Actually, split bodies can be seen as an alternative to our way of denoting general disjunctions in Definition 3.1. They are defined as follows.

Definition 3.25 ([ALW07b]). *A set $L \subseteq \mathbb{R}^p$ is a split body if*

- $\dim(L) = p$;
- $\text{int}\,(L) \cap \mathbb{Z}^p = \emptyset$;
- *each facet F of L contains an integral point and can be represented by an integral vector $(\pi, \pi_0) \in \mathbb{Z}^{p+1}$ as $F = \{x \in L : \pi x = \pi_0\}$.*
- *L can be represented as the orthogonal Minkowski-sum of a polytope plus a linear space, in especially, there exist affinely independent vectors $v^1, \ldots, v^s$ and linear independent vectors $w^1, \ldots, w^d \in \mathbb{Z}^p$ such that $w^i v^j = 0$ for all $i = 1, \ldots, d$ and $j = 1, \ldots, s$ such that*

$$L = \{x \in \mathbb{R}^p : x = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{d} \mu_j w^j,\ \sum_{i=1}^{s} \lambda_i = 1, \lambda_i \geq 0\}.$$

The split dimension of L is defined to be $p - d$.

According to the definition, a split body L satisfies at first more conditions than a k-disjunction according to Definition 3.1. It is required that L is full dimensional and that each of its facets contains an integral point. Moreover, a split body is not characterized by the number of facets but by its dimension given by $p - d$.

However, every split body can be seen as the closed complement of a k-disjunction as it is a polyhedron which contains no integral point in its interior. Besides, each k-disjunction which is used for the derivation of a cutting plane always satisfies the first condition of Definition 3.25 due to Definition 3.2. Likewise, each k-disjunctive cut can be derived from a disjunction which satisfies the third condition of the above definition. This can be seen by expanding a given disjunction $D(k, d, \delta)$ by the rules of the proof of Theorem 3.8. It follows that each k-disjunction can be linked to a related split body. Andersen, Louveaux, and Weismantel also show that every lattice point free polyhedron is contained in a split body of appropriate dimension.

Now by the concept of split bodies, they can state the following certificate for systems of equations and inequalities. The underlying idea is that a system $\{Ax = b, Cx \leq d, x \in \mathbb{Z}^p\}$ is infeasible if it is contained in a certain split body of appropriate dimension.

Theorem 3.26 ([ALW07b]). *Let $A \in \mathbb{Z}^{m \times p}, C \in \mathbb{Z}^{n \times p}$ and let $l = rank(C)$. For integral vectors b and d, the primal system*

$$\begin{aligned} Ax &= b \\ Cx &\leq d \\ x &\in \mathbb{Z}^p \end{aligned}$$

is empty if and only if there exist rational vectors $y^1, \dots, y^t \in \mathbb{Q}^m \times \mathbb{Q}^n_+$*, and at most* l *linearly independent integral vectors* $v^1, \dots, v^l \in \mathbb{Z}^p$ *such that*

$$y^k \begin{pmatrix} A \\ C \end{pmatrix} = \sum_{i=1}^{l} \lambda_i^k v^i \in \mathbb{Z}^p \text{ with } \lambda_i^k \in \mathbb{Z} \text{ for all } i = 1, \dots, l, k = 1, \dots, t.$$

Introducing variables z_i*,* $i \in \{1, \dots, l\}$ *(representing* $v^i x$*), the following system of* t *inequalities in* l *variables has no integral solution.*

$$\sum_{j=1}^{l} \lambda_j^k z_j \le y_k \begin{pmatrix} b \\ d \end{pmatrix} \text{ for all } k = 1, \dots, t. \tag{3.18}$$

□

The application of split bodies can now be generalized to consider mixed integer systems; see [ALW07a]. This can be done as a polyhedron P contains mixed integer feasible points if and only if its projection $\text{proj}_X(P)$ on the integral space contains integral points. This also leads to a natural generalization of the mixed integer Farkas Lemma (Theorem 2.24) to systems with equations and inequalities in line with the pure integer case given in Theorem 3.26. Moreover, split bodies can now also be used to derive cutting planes for MILP. Here, very recently Andersen, Louveaux, and Weismantel have also given a characterization of the faces of the mixed integer hull of a polyhedron based on cuts to split bodies; see [ALW07a].

Finally, we briefly get back to the cuts from two rows of the simplex tableau of Andersen, Louveaux, Weismantel, and Wolsey which we have introduced in Section 2.2.6. The cuts characterize the facets of the set

$$U := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j=1}^{n} s_j r^j\}$$

and are derived as intersection cuts from two dimensional split bodies. This means that they can also be seen as 3- and 4-disjunctive cuts according to our definition. These cuts can also be linked to Theorem 3.26 to obtain a full characterization of the certificate (3.18) for $l = rank(C) = 2$. In this case, (3.18) always consists of 3 or 4 inequalities; see again [ALW07a].

4 Algorithms for MILP

In Chapter 3 we have analyzed in detail which cuts are required for a finite cutting plane algorithm for solving a MILP. In this chapter we now want to apply these theoretical results in the design of algorithms. We distinguish exact and approximation algorithms, at which we understand by an approximation algorithm a procedure that computes a feasible solution of a MILP with a ϵ-optimal objective function value.

First, we point out some problems which arise by solving a MILP with cutting planes in Section 4.1. Next, we discuss in Section 4.2 some finite approximation algorithms which are based on cuts to simple split disjunctions and give some extensions. Beside these algorithms we present at last in Section 4.3 a new cutting plane algorithm which also uses more general k-disjunctive cuts with $k > 2$ and always finds an exact optimal solution of a rational MILP in finite time. We also discuss some theoretical and practical issues which are linked with this algorithm and its application.

4.1 Basic algorithm and problems

To get a first picture of the difficulties which can arise by solving a MILP with a cutting plane algorithm we start with a further analysis of Algorithm 1. We have already discussed in Section 2.2.3 that this algorithm does not solve a MILP in finite time in general. However, as Theorem 3.11 shows, split cuts are sufficient for an arbitrary approximation of the optimal objective function value. So we can ask if Algorithm 1 provides an implementation of this result. This is the first requirement for an approximation algorithm, as otherwise it is not possible to check the quality of the objective function value of any feasible solution. Unfortunately, Algorithm 1 fails in this point. This was shown by Owen and Mehrotra [OM01] by an example which we discuss now.

We consider a slight modification of Algorithm 1 which effects that more cuts are added in a step of the algorithm than in the original case. In detail, we add for an optimal solution (x^l, y^l), $x_i^l \notin \mathbb{Z}$ of the LP relaxation P^l in step l not only intersection cuts to all bases B of (x^l, y^l), but intersection cuts to all (feasible and infeasible) bases B of the polyhedron P^l to the disjunction $D(u_i, \lfloor x_i^l \rfloor)$. It follows by Theorem 2.16 that the feasible domain P^l of the LP relaxation in step l of the algorithm is given by

$$P^l = \text{conv}\left(\{(x, y) \in P^{l-1} : \ x_i \leq \lfloor x_i^{l-1} \rfloor \vee x_i \geq \lfloor x_i^{l-1} \rfloor + 1\}\right).$$

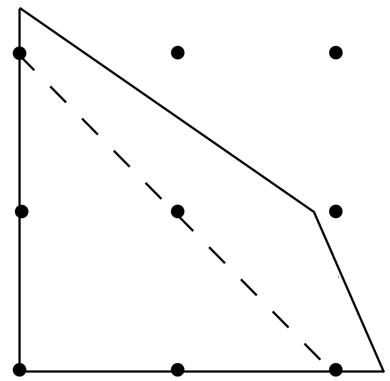


Figure 4.1: The polyhedron P of Example 4.1. The dashed line gives the missing facet of the integer hull P_I.

Now we look at the following

Example 4.1 ([OM01]). *Let the ILP*

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 8x_1 + 12x_2 \leq 27 \\ & 8x_1 + 3x_2 \leq 18 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

be given. The set of all feasible points of the LP relaxation is given by the polyhedron

$$P = \{x \in \mathbb{R}^2 : \, 8x_1 + 12x_2 \leq 27, \, 8x_1 + 3x_2 \leq 18, \, x_1, x_2 \geq 0\},$$

and $\left(\frac{15}{8}, 1\right)$ *is the unique optimal solution of* $\max\{x_1 + x_2 : \, x \in P\}$*. The set of optimal solutions of the ILP is given by* $(2, 0), (1, 1)$*, and* $(0, 2)$ *which satisfy the equation* $x_1 + x_2 = 2$*; see Figure 4.1.*

Now Owen and Mehrotra show that neither of the vertices $\left(\frac{9}{4}, 0\right) \in P$ *and* $\left(0, \frac{9}{4}\right) \in P$ *is cut off by Algorithm 1 or its above modification. This is due to the fact that the optimal LP solution* $\left(x_1^l, x_2^l\right)$ *in step* l *of the algorithm always satisfies one of the conditions*

$$x_1 \in \{1, 2\} \wedge x_2 \notin (2, 3) \text{ or } x_1 \notin (2, 3) \wedge x_2 \in \{1, 2\}.$$

As the points $\left(\frac{9}{4}, 0\right)$ *and* $\left(0, \frac{9}{4}\right)$ *are not cut off, all LP solutions* $\left(x_1^l, x_2^l\right)$ *satisfy the inequality* $x_1 + x_2 \geq \frac{9}{4}$ *and the approximation of the optimal objective function value* $\gamma^* = 2$ *fails. We note that the sequence* $\gamma^l := x_1^l + x_2^l$ *of objective function values tends to the limit* $\frac{9}{4}$*.*

We analyze the reasons why Algorithm 1 and its modification do not approximate the optimal objective function value even in the case of an ILP as in Example 4.1. We note that it is $P^{(1)} = P_I$ for the split closure $P^{(1)}$ of P and $P^{(\infty)} = P_I$ applying only simple split cuts to disjunctions $D(d, \delta)$ with $d = u_i$ to the polytope P. As we have already mentioned in Example 4.1, a crucial point for generating the integer hull P_I is to cut off the points $\left(\frac{9}{4}, 0\right)$ and $\left(0, \frac{9}{4}\right)$. For this purpose, split cuts to the disjunctions $D(u_1, 2)$ and $D(u_2, 2)$ are required but not added in the algorithm due to the disadvantageous properties of the sequence $\left(x_1^l, x_2^l\right)$ of optimal LP solutions. This means that a selection of split cuts according to the local information of the optimum of the current LP relaxation fails. For these reasons we need a criterion for the selection of cuts which is based on more global information. We discuss two possibilities for this issue in the next section.

Moreover, it remains open if we can design an approximation algorithm that is still based on the local information of the current LP optimum, but adds cuts to more general split or k-disjunctions. We can see that we obtain an optimal solution of the ILP in Example 4.1 if we compute a cut to the split disjunction $D((1, 1), 2)$ given by the objective function. Of course, for pure ILP as in this example, rounding cuts to the objective function can always be used for cutting to obtain even an exact solution in finite time in combination with cuts to unit vectors $u_1, \ldots, u_p$ as we have already discussed in Section 2.1. Additionally, it is open how an exact algorithm for solving MILP based on k-disjunctive cuts can be designed.

4.2 Approximation algorithms

We discuss two algorithms which overcome the problems of Algorithm 1 and always find an ϵ-optimal solution of a MILP. Both algorithms are again based on simple split cuts. The first one is an algorithm of Owen and Mehrotra [OM01] and was - to the best of our knowledge - the first cutting plane approximation algorithm for MILP. This algorithm also provides a constructive proof of Theorem 3.11. As an alternative we introduce in Section 4.2.2 a second algorithm which combines the results of Theorem 2.20 with cuts that might not be valid for all feasible points of the MILP but for some ϵ-optimal solutions, only. This approach will especially be the basis for an exact algorithm in Section 4.3.

4.2.1 Algorithm of Owen and Mehrotra

As we have discussed in the last section, Algorithm 1 fails in approximating the optimal objective function value in general as cuts are only added according to local information. Therefore, Owen and Mehrotra suggest to add cuts not only to the optimal vertex of the LP relaxation but also to all 'nearly optimal' vertices. Here a vertex is nearly optimal if its objective function value does not differ more than a predefined amount from the optimal objective function value of the LP relaxation. This strategy leads to a finite approximation algorithm. We discuss the algorithm in more detail.

The input is given by a MILP (1.1), where the feasible domain P of the LP relaxation is bounded, the optimality tolerance $\epsilon > 0$, a parameter $\nu > 0$ which defines nearly optimal solutions, and a parameter $\bar{\epsilon} \in (0, 5]$ which is used for integer rounding within the procedure. The algorithm starts as usual with solving the LP relaxation $\max\{cx + hy : (x, y) \in P\}$ of the MILP and computing the related objective function value γ^*.

We determine the set Ω of all nearly optimal vertices of P. These are all vertices with an objective function value not less than $\gamma^* - \nu$. It is now searched for ϵ-optimal solutions in the following way. The integral restricted components $\hat{x}_i$, $i \in \{1, \ldots, p\}$ of all vertices $(\hat{x}, \hat{y}) \in \Omega$ are first rounded to the next integer $d_i = \lfloor \hat{x}_i \rfloor$ or $d_i = \lceil \hat{x}_i \rceil$. This is only done if the distance between each component $\hat{x}_i$ of the vertex and the nearest integer d_i is not too large. Here the maximum feasible distance is defined by the parameter $\bar{\epsilon}$.

Next, we check for each vertex $(\hat{x}, \hat{y}) \in \Omega$ if there exists a feasible point $(d, y) \in P$ with integral components d_i as computed in the rounding operation and an objective function value greater or equal to $\gamma^* - \epsilon$. If there exists such a point (d, y) satisfying both conditions, a ϵ-optimal solution of the MILP has been found. Otherwise, the set of nearly optimal vertices is used to check if the MILP is infeasible. We test for each vertex $(\hat{x}, \hat{y}) \in \Omega$ and each component $\hat{x}_i \notin \mathbb{Z}$ if the set

$$P \cap \{(x, y) : x_i \leq \lfloor \hat{x}_i \rfloor \vee x_i \geq \lfloor \hat{x}_i \rfloor + 1\}$$

is empty. If this should be the case for a vertex $(\hat{x}, \hat{y})$ and a disjunction $D(u_i, \lfloor \hat{x}_i \rfloor)$ the MILP is infeasible. Otherwise, we go on and add a cut to each possible disjunction $D(u_i, \lfloor \hat{x}_i \rfloor)$ at each vertex $(\hat{x}, \hat{y}) \in \Omega$. After all cuts have been added, one step of the algorithm is complete and we start over.

We note that Owen and Mehrotra do not use intersection cuts in their algorithm, but cuts to split disjunctions $D(u_i, \lfloor \hat{x}_i \rfloor)$ which are given by a certain subgradient. In detail, let $(\hat{x}, \hat{y})$ be a vertex of P with $\hat{x}_i \notin \mathbb{Z}$ and let (x^*, y^*) be an optimal solution of the separation problem

$$\min \{||(x, y) - (\hat{x}, \hat{y})|| : \ (x, y) \in P \cap \{(x, y) : x_i \leq \lfloor \hat{x}_i \rfloor \vee x_i \geq \lfloor \hat{x}_i \rfloor + 1\}\}.$$

Then the inequality

$$\alpha x + \beta y \geq \alpha x^* + \beta y^*, \tag{4.1}$$

where (α, β) is a subgradient of $||(x, y) - (\hat{x}, \hat{y})||$ at (x^*, y^*), is a split cut for P to the disjunction $D(u_i, \lfloor \hat{x}_i \rfloor)$. We refer to [OM01] for more details.

The formal algorithm is stated in Algorithm 2. It is slightly modified to our representation as we consider maximization problems. Owen and Mehrotra prove that the algorithm always finds an ϵ-optimal solution.

Algorithm 2 Algorithm of Owen and Mehrotra

procedure AOM
    Input: MILP (1.1), where P is bounded, ϵ, $\bar{\epsilon}$, ν;
    Output: ϵ-optimal solution (x^*, y^*) or "problem infeasible" if no solution exists;

    $P \leftarrow \{(x, y) : Ax + Gy \leq b\};\ i \leftarrow 0;\ S^0 \leftarrow P;$

    while $S^i \neq \emptyset$ **do**
        $\gamma^i \leftarrow \max\{cx + hy : (x, y) \in S^i\};$
        $\Omega^i \leftarrow \{(x, y) \in \text{ext}(S^i) : \gamma^i - cx - hy \leq \nu\};$

        $\Omega_{\bar{\epsilon}} \leftarrow \{(x, y) \in \Omega^i : \min\{(x_j - \lfloor x_j \rfloor), (\lceil x_j \rceil - x_j) < \bar{\epsilon} \forall j \in \{1, \ldots, p\}\};$
        if $\Omega_{\bar{\epsilon}} \neq \emptyset$ **then**
            for each $(x, y) \in \Omega_{\bar{\epsilon}}$ **do**
                for each $j \in \{1, \ldots, p\}$ **do**
                    if $(x_j - \lfloor x_j \rfloor < \epsilon)$ **then** $d_j \leftarrow \lfloor x_j \rfloor;$
                    else $d_j \leftarrow \lceil x_j \rceil;$
                    end if
                end for

                $\bar{S} \leftarrow \{x \in S^i : x_j = d_j \forall j = 1, 2, \ldots, p\};$
                if $\bar{S} \neq \emptyset$ **then**
                    $(\bar{x}, \bar{y}) \leftarrow \text{argmax}\,\{cx + hy : (x, y) \in \bar{S}\};$
                    if $\gamma^i - cx - hy < \epsilon$ **then**
                        "ϵ-optimal solution $(\bar{x}, \bar{y})$"; **break**
                    end if
                end if
            end for
        end if

        $S^{i+1} \leftarrow S^i;$
        for each $(\bar{x}, \bar{y}) \in \Omega^i$ and $j \in \{1, \ldots, p\}$ such that $\bar{x}_j \notin \mathbb{Z}$ **do**
            if $\{(x, y) \in S^i : x_j \leq \lfloor \bar{x}_j \rfloor\} = \{(x, y) \in S^i : x_j \geq \lceil \bar{x}_j \rceil\}$ **then**
                "problem infeasible"; **break**
            end if

            Compute cutting plane $\alpha x + \beta y \leq \gamma$ to $S^i, D(u_j, \lfloor \bar{x}_j \rfloor)$ according to (4.1);
            $S^{i+1} \leftarrow \{(x, y) \in S^{i+1} : \alpha x + \beta y \leq \gamma)\};$
        end for
        $i \leftarrow i + 1;$
    end while
    "problem infeasible"; **break**
end procedure

Theorem 4.2 ([OM01]). *Let a MILP (1.1) be given, where the feasible domain P of the LP relaxation is bounded. Then Algorithm 2 either finds a ϵ-optimal solution of the MILP or detects infeasibility in a finite number of steps.* □

We refer to [OM01] for more details regarding Algorithm 2 such as the choice of the parameter ν. Finally, we illustrate the procedure by considering again Example 4.1 which we have used for the motivation at the beginning of this chapter.

Example 4.3. *Let again the ILP*

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 8x_1 + 12x_2 \leq 27 \\ & 8x_1 + 3x_2 \leq 18 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

be given. We have seen in Example 4.1 that the sequence (x_1^l, x_2^l) of LP optima satisfies the condition $x_1 + x_2 \geq \frac{9}{4}$, the sequences γ^l of objective function value tends to $\frac{9}{4}$, and that it was crucial for the failure of Algorithm 1 that the vertices $\left(\frac{9}{4}, 0\right)$ and $\left(0, \frac{9}{4}\right)$ of the LP relaxation P are not cut off.

Now as the the sequence γ^l of objective function values tends to $\frac{9}{4}$, each of the crucial vertices of P becomes nearly optimal for an arbitrary choice of the parameter ν and for $\epsilon < \frac{1}{4}$ in Algorithm 2. So both vertices are cut off in a step of Algorithm 2 by cuts to the disjunctions $D(u_1, 2)$ and $D(u_2, 2)$, respectively. Moreover, one can see that Algorithm 2 even finds one of the exact optimal solutions which are given by $(2, 0), (1, 1)$, and $(0, 2)$, as for $\epsilon < 1$ each feasible solution is also optimal in this example.

4.2.2 Searching for feasible points

We present another approximation algorithm which is motivated by Theorem 2.20. By this result, we can especially check if an arbitrary polytope contains a mixed integer feasible point by solving a MILP with the integral objective function $\max x_1$; see also Corollary 2.21 for a similar result. As we have shown, Algorithm 1 can get stuck before converging to the optimal objective function value. To avoid this problem we now additionally cut with the objective function in such a way that at most ϵ-optimal solutions become infeasible. Afterwards, we check if a feasible point has been cut off. If this should be the case, a ϵ-optimal solution has been found. Otherwise, we have added a valid cutting plane and go on with the algorithm. The finiteness of this procedure follows as we can guarantee that the sequence of objective function values always decreases by a fixed ϵ within a finite number of steps.

We describe the approach in more detail and start with the procedure FEASIBLE POINT, see Algorithm 3, which checks if feasible points have been cut off. The input is given by a polytope P and an objective function $cx + hy$ with value γ, where all points of P with a value greater than γ are cut off. The output is given either by a feasible point $(x, y) \in P$ with $x \in \mathbb{Z}^p$ and $cx + hy \geq \gamma$ or the information that none such point exists. The procedure FEASIBLE POINT uses as subroutine the procedure BASISALGORIHTM stated in Algorithm 1.

Algorithm 3 Search for feasible points

procedure FEASIBLE POINT
    Input: Polytope P, $cx + hy$, γ;
    Output: feasible point $(x^*, y^*) \in P$ with $cx^* + hy^* \geq \gamma$ or
    "problem infeasible" if no solution exists;

    $P \leftarrow P \cap \{(x, y) : cx + hy \geq \gamma\}$;
    Set MILP: $\max\{x_1 : (x, y) \in P, x \in \mathbb{Z}^p\}$;
    BASICALGORITHM(MILP);
end procedure

We can now state the whole algorithm in the procedure ϵ-APPROXIMATION; see Algorithm 4. The input is given by a MILP (1.1), where the feasible domain P of the LP relaxation is bounded, and the parameter ϵ which defines the quality of the approximation. We obtain as output of Algorithm 4 either a ϵ-optimal solution (x^*, y^*) of the MILP or the information that no feasible solution exists.

Theorem 4.4. *Let a MILP* (1.1) *be given, where the feasible domain P of the LP relaxation is bounded. Then Algorithm 4 either generates a ϵ-optimal solution of the MILP* (1.1) *or detects infeasibility in a finite number of steps.*

Proof. The procedure FEASIBLE POINT (Algorithm 3) always gives the correct output within a finite number of steps by Theorem 2.20, as a MILP with an integral objective function is solved over a polytope. Likewise, either a optimal solution (x^*, y^*) or a new objective function value $\gamma < \gamma^*$ is computed in finite time in the inner while-loop (l. 18-31) of Algorithm 4 by Corollary 2.21.

If we do not obtain an improvement of the objective function value γ of at least ϵ, we ensure this progress by the procedure FEASIBLE POINT (Algorithm 3) in the if-loop (l. 33-41) of Algorithm 4. So we can always guarantee an ϵ-improvement of the objective function value γ^* in finite time and the algorithm terminates in a finite number of steps. Moreover, it follows by construction that Algorithm 4 always finds a ϵ-optimal solution if one exists. □

Algorithm 4 Cutting plane algorithm - ϵ-approximation

procedure ϵ-APPROXIMATION
    Input: MILP (1.1), where P is bounded, ϵ;
    Output: ϵ-optimal solution (x^*, y^*) or "problem infeasible" if no solution exists;

    $P \leftarrow \{(x, y) : Ax + Gy \leq b\}$; $(x^*, y^*) \leftarrow \text{arglexmax}\,\{cx + hy : (x, y) \in P\}$;
    $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*);

    if $P = \emptyset$ **then**
        "problem infeasible"; **break**
    end if
    if $x^* \in \mathbb{Z}^p$ **then**
        "optimal solution (x^*, y^*)"; **break**
    end if

    while $x^* \notin \mathbb{Z}^p$ **do**
        $\gamma^* \leftarrow cx^* + hy^*$; $\gamma \leftarrow \gamma^*$;

        while $\gamma = \gamma^*$ **do**
            $i \leftarrow \text{argmin}\,\{j \in \{1, \ldots, p\} : x_j^* \notin \mathbb{Z}\}$;
            Compute intersection cuts $\alpha_B x + \beta_B y \leq \eta_B$ to $P, D(u_i, \lfloor x_i^* \rfloor)$, all $B \in \mathcal{B}$;
            $P \leftarrow P \cap \{(x, y) : \alpha_B x + \beta_B y \leq \eta_B,\, B \in \mathcal{B}\}$;
            $(x^*, y^*) \leftarrow \text{arglexmax}\,\{cx + hy : (x, y) \in P\}$;
            $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*);
            if $P = \emptyset$ **then**
                "problem infeasible"; **break**
            end if
            if $x^* \in \mathbb{Z}^p$ **then**
                "optimal solution (x^*, y^*)"; **break**
            end if
            $\gamma \leftarrow cx^* + hy^*$
        end while

        if $\gamma^* - \gamma < \epsilon$ **then**
            FEASIBLE POINTS$(P, cx + hy, \gamma - \epsilon)$;
            if Output: (x^*, y^*) **then**
                "optimal solution (x^*, y^*)"; **break**
            end if
            $P \leftarrow P \cap \{(x, y) : cx + hy \leq \gamma - \epsilon\}$;
            $(x^*, y^*) \leftarrow \text{arglexmax}\,\{cx + hy : (x, y) \in P\}$;
            $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*);
        end if
    end while
end procedure

We note that Algorithm 4 is no 'classical' cutting plane algorithm, as we also add cuts that might not be valid for the mixed integer hull P_I. However, by checking if we have cut off any feasible points in a second step, we implicitly connect the requirement of a global approach for adding cuts with the search for feasible points.

According to the basic idea, Algorithm 4 can also be diversified. We apply the procedure FEASIBLE POINT (l. 33-41) in Algorithm 4 if adding cuts to the whole polytope at the optimal vertex (x^*, y^*) of the current LP relaxation does not provide an ϵ-improvement of the objective function value γ^*. We could also change this approach and use the procedure FEASIBLE POINT in Algorithm 4 exclusively. This would mean to discard the inner while loop (l. 18-31) of Algorithm 4. However, the original way that we have chosen has some advantages. The cuts which are computed in the inner while loop (l. 18-31) can be applied to the whole polytope and reduce the feasible domain of the MILP in the remainder of the computation. By contrast, the cuts which are computed in the procedure FEASIBLE POINT are only valid for the related subproblem and cannot be used later. Moreover, the improvement of the objective function value is typically large - relatively to ϵ - at the beginning of Algorithm 1. So it might save some effort in general to start at first with the basic cutting plane approach.

We state two examples to illustrate the basic principle of Algorithm 4. We start again with a short consideration of the ILP of Example 4.1 and Example 4.3.

Example 4.5. *Let again the ILP of Example 4.1 be given. We show that Algorithm 4 cuts off again the crucial vertices* $\left(0, \frac{9}{4}\right), \left(\frac{9}{4}, 0\right)$ *of* P.

Suppose that the sequence $\gamma^i, i \in \mathbb{N}$ *of objective function values that is computed in Algorithm 4 converges to* $\frac{9}{4}$. *Then the objective function value* γ^i *would satisfy* $\gamma^i < \frac{9}{4} + \epsilon$ *and its progress would be less than* ϵ *in an iteration step of Algorithm 4. In this case, the polytope*

$$P^i \cap \{x \in \mathbb{R}^2 : x_1 + x_2 \geq \gamma^i - \epsilon\}$$

is searched for feasible points. Depending on the choice of ϵ, *either one of the feasible solutions* $(2,0), (1,1), (0,2)$ *is found, or the cut*

$$x_1 + x_2 \leq \gamma^i - \epsilon$$

is added. So either the ILP is solved or the crucial vertices $\left(0, \frac{9}{4}\right), \left(\frac{9}{4}, 0\right)$ *are cut off. One can see that Algorithm 4 also finds an exact optimal solution in the latter case as for* $\epsilon < 1$ *each approximation solution is also an exact optimal solution.*

Secondly, we take up again Example 2.13 and Example 3.16.

Example 4.6. *So let the MILP*

$$\begin{aligned} \max \quad & y \\ & -x_1 + y \le 0 \\ & -x_2 + y \le 0 \\ & x_1 + x_2 + y \le 2 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

with the optimal LP solution $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ *be given. Solving the MILP with Algorithm 1 yields the optimal LP solution* (x^l, y^l) *after* l *iterations*

$$x_1^l = x_2^l = \frac{2l+2}{2l+3}, \; y^l = \frac{2}{2l+3}$$

and the related polyhedron

$$P^l := \{(x,y) : -x_1 + (l+1)y \le 0, \; -x_2 + (l+1)y \le 0, \; x_1 + x_2 + y \le 2\};$$

see [Pad05]. The improvement of the objective function in step l *is given by*

$$y^{l-1} - y^l = \frac{4}{(2l+1)(2l+3)},$$

which decreases in each step and tends to 0*. Applying Algorithm 4 to this example, we search with Algorithm 3 for* ϵ*-optimal solutions in the set* $P^l \cap \{(x,y) \in \mathbb{R}^2 : y \ge y^l - \epsilon\}$ *as soon as* $y^{l-1} - y^l < \epsilon$*. In this case it is*

$$\frac{y^l}{\epsilon} < \frac{y^l}{y^{l-1} - y^l} = l + \frac{1}{2}.$$

This means that we need at most l *calls of Algorithm 3 to solve the MILP in this example, where* l *behaves like* $\mathcal{O}\left(\sqrt{\left(\frac{1}{\epsilon}\right)}\right)$*.*

Finally, we comment on solving MILP exactly. Both Algorithm 2 and Algorithm 4 can also be used to solve a MILP with rational input data exactly. This follows from the fact that for a sufficiently small ϵ, each ϵ-optimal solution (x^*, y^*) of a MILP is already an exact solution. The reasoning is similar to the rounding cut approach for MILP at the end of Section 3.2. We assume that the input data of the MILP is integral and let $M \in \mathbb{Z}$ be an upper bound on the product of determinants of all regular submatrices of G. Moreover, we only consider those solutions (x^*, y^*) of the MILP which cannot be improved for fixed x^*, that means y^* is an optimal solution of the LP

$$\max\{hy : Gy \le b - Ax^*\}. \tag{4.2}$$

In this case, we can choose $\epsilon < \frac{1}{M}$ to find an exact optimal solution of the MILP with both approximation algorithms. This follows by comparing all feasible solutions (x^*, y^*) of the MILP which satisfy (4.2) and by Cramer's rule.

We note that we can also obtain an exact algorithm for solving a rational MILP by combining Algorithm 2 or Algorithm 4 with the well known method of binary search; see for example [BR05] or [Sch86].

4.3 Exact algorithm

After we have discussed two approximation algorithms in the last section, we introduce an exact cutting plane algorithm which generalizes Algorithm 4. This approach was based on a combination of simple split cuts and cuts to the objective function which did not have to be valid for the mixed integer hull of a polytope. By the results of Section 3.4, we now design an algorithm which adds valid cuts to the objective function. In detail, we state a basic form of the exact algorithm in Section 4.3.1. Afterwards, we deal with the problem of computing extreme rays of cones in Section 4.3.2 and with irredundant representations of projections in Section 4.3.3. In Section 4.3.4 we describe a sequential form of the exact algorithm and consider some examples in Section 4.3.5.

4.3.1 Basic form

We present an algorithmic application of the results on general k-disjunctive cuts in Section 3.3 and Section 3.4 and give a basic form of an exact cutting plane algorithm which solves a rational MILP in finite time. It is based on a sequence of simple split cuts which are supplemented with certain k-disjunctive cuts to the objective function. As the assumptions which we have made in Section 3.4 for the derivation of k-disjunctive cuts are not satisfied in general, we have to do some slight modifications. So we define k-disjunctive cuts for general polytopes P and for an arbitrary cut vector (c, h).

The description of the projection $\text{proj}_X(P_{\gamma^*})$ in equations (3.13), (3.14), (3.15), (3.16) in Section 3.4 also holds for a polytope P and an arbitrary value γ^* of the objective function. Even the derivation of a valid $|R|$-disjunction and a valid $|R|$-disjunctive cut in Lemma 3.22 and Theorem 3.23, respectively, remains correct if the value γ^* of the objective function $\max cx + hy$ is not optimal for P_I. However, for the application in the algorithm we define a slightly weaker version of the $|R|$-disjunctive cut which does not always cut off the current LP solution, but can be used more general.

Theorem 4.7. *Let P be a rational polytope and γ^* such that $cx + hy \leq \gamma^*$ is valid for P_I but not for P. Define the $|R|$-disjunction $D(|R|, -d, -\delta)$ by equations* (3.13), (3.14), (3.15), (3.16) *with*

$$\begin{aligned} d^r &:= v^r_{1,\ldots,m} A - v^r_{m+1} c \\ \delta^r &:= \lceil v^r_{1,\ldots,m} b - v^r_{m+1} \gamma^* \rceil \end{aligned}$$

and let $\widehat{\gamma} = \max\{\gamma^r : v^r \in R,\, v^r_{m+1} > 0\}$ with

$$\gamma^r := \frac{\delta^r - v^r_{1,\ldots,m} b}{-v^r_{m+1}}, \text{ for } v^r_{m+1} > 0$$

according to Theorem 3.23. Then $cx + hy \leq \widehat{\gamma}$ is a valid cutting plane for P_I.

Proof. By assumption, $\text{proj}_X(P_{\gamma^*})$ does not contain an integral point $x \in \mathbb{Z}^p$ in its interior. Otherwise, the inequality $cx + hy \leq \gamma^*$ was not valid for P_I. As $d^r \in \mathbb{Z}^p$ by (3.16), rounding up the right hand sides of the inequalities of the projection (3.14) to δ^r gives the valid $|R|$-disjunction $D(|R|, -d, -\delta)$. Therefore, $cx + hy \leq \widehat{\gamma}$ is a valid cutting plane in line with the proof of Theorem 3.23. □

We explain the single steps of the algorithm. We start with Algorithm 1 as long as we obtain either an optimal solution of the MILP or the objective function value of the LP relaxation decreases. This happens in finite time by Corollary 2.21 as the feasible domain P of the LP relaxation is a polytope. Now we can compute a valid $|R|$-disjunctive cut to the objective function according to Theorem 4.7. Next, we repeat the approach and apply again Algorithm 1 to the modified program. In this way we obtain an algorithm which terminates in finite time with an optimal solution of the MILP or detects infeasibility. The formal algorithm is stated in Algorithm 5.

Theorem 4.8. *Let a rational MILP* (1.1) *be given, where the feasible domain P of the LP relaxation is bounded. Then Algorithm 5 either finds an optimal solution of the MILP or detects infeasibility in a finite number of steps.*

Proof. The proof follows immediately by the following two facts: The inner while loop (l. 18 - 34) of Algorithm 5 always breaks after finite many iterations by Corollary 2.21 as P is a polytope by assumption. Similarly, the outer while loop (l. 15 - 40) breaks after finite many iterations as the number of different values $\widehat{\gamma}$ is finite. This follows as the set $\{\gamma^r : v^r \in R, v_{m+1} > 0\}$ of all possible objective function values is discrete and P is bounded. Moreover, no feasible points of P_I are cut off by construction. □

We give two examples of Algorithm 5 and consider again the MILP of Example 2.13, Example 3.16, and Example 4.6 and the ILP of Example 4.1, Example 4.3, and Example 4.5.

Algorithm 5 Exact cutting plane algorithm

procedure EXACTCUTTING
    Input: rational MILP (1.1), where P is bounded;
    Output: optimal solution (x^*, y^*) or "problem infeasible" if no solution exists;

    $P \leftarrow \{(x, y) : Ax + Gy \leq b\}$, $(x^*, y^*) \leftarrow \text{argmax}\,\{cx + hy : (x, y) \in P\}$;
    $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*), $P^0 \leftarrow P$;

    if $P = \emptyset$ **then**
        "problem infeasible"; **break**
    end if
    if $x^* \in \mathbb{Z}^p$ **then**
        "optimal solution (x^*, y^*)"; **break**
    end if

    while $x^* \notin \mathbb{Z}^p$ **do**
        $\gamma^* \leftarrow cx^* + hy^*$, $\gamma \leftarrow \gamma^*$;

        while $\gamma = \gamma^*$ **do**
            $i \leftarrow \text{argmin}\,\{j \in \{1, \ldots, p\} : x_j^* \notin \mathbb{Z}\}$;
            Compute intersection cuts $\alpha_B x + \beta_B y \leq \eta_B$ to $P, D(u_i, \lfloor x_i^* \rfloor)$, all $B \in \mathcal{B}$;
            $P \leftarrow P \cap \{(x, y) : \alpha_B x + \beta_B y \leq \eta_B, B \in \mathcal{B}\}$;
            $(x^*, y^*) \leftarrow \text{arglexmax}\,\{cx + hy : (x, y) \in P\}$;
            $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*);

            if $P = \emptyset$ **then**
                "problem infeasible"; **break**
            end if

            if $x^* \in \mathbb{Z}^p$ **then**
                "optimal solution (x^*, y^*)"; **break**
            end if

            $\gamma \leftarrow cx^* + hy^*$;
        end while

        Compute $\widehat{\gamma} = \max\{\gamma^r : r \in R, v_{m+1}^r > 0\}$ according to Theorem 4.7
        for $P^0, (c, h), \gamma^*$;
        $\gamma^* \leftarrow \widehat{\gamma}$;

    end while
end procedure

Example 4.9. *Let again the MILP*

$$\begin{aligned} \max \quad & y \\ & -x_1 + y \leq 0 \\ & -x_2 + y \leq 0 \\ & x_1 + x_2 + y \leq 2 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

with optimal LP solution $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ *be given. We add the intersection cut to the vertex* $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ *and the disjunction* $D(u_1, 0)$*. As the optimal objective function value* γ *of the LP relaxation decreases, we can compute* $\widehat{\gamma}$ *according to Theorem 4.7. The extreme rays of the projection cone*

$$Q = \{\begin{pmatrix} 1 & 1 & 1 & -1 \end{pmatrix} y = 0, \ y \geq 0\}$$

are the three vectors

$$(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1),$$

so the projection $\operatorname{proj}_X(P_\gamma)$ *is defined by the system*

$$\begin{aligned} -x_1 &\leq 0 - \gamma \\ -x_2 &\leq 0 - \gamma \\ x_1 + x_2 &\leq 2 - \gamma. \end{aligned}$$

We insert the current objective function value γ *with* $0 < \gamma < \frac{2}{3}$*. Rounding and computing the new objective function value* $\widehat{\gamma}$ *according to Theorem 4.7 gives*

$$\widehat{\gamma} = \max_{r=1,2,3} \gamma^r = \max\{0, 0, 0\} = 0.$$

Adding the related cut $y \leq 0$ *and solving the modified LP relaxation gives the point* $(2, 0, 0)$ *and so the algorithm breaks with an optimal solution of the MILP after one step.*

Example 4.10. *Let the ILP*

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 8x_1 + 12x_2 \leq 27 \\ & 8x_1 + 3x_2 \leq 18 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

with optimal LP solution $(\frac{15}{8}, 1)$ *be given. After adding the first cut to the disjunction* $D(u_1, 1)$*, the value* γ *of the objective function decreases and we can go to the second* *step of Algorithm 5. As we have an ILP it is* $\text{proj}_X(P_\gamma) = P_\gamma$*, where* P_γ *is given by the system*

$$\begin{aligned} -x_1 - x_2 &\leq -\gamma \\ 8x_1 + 12x_2 &\leq 27 \\ 8x_1 + 3x_2 &\leq 18 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Inserting γ *with* $2 < \gamma < 3$ *and rounding yields*

$$-x_1 - x_2 \leq \lceil -\gamma \rceil = -2 = -\gamma^1 = -\widehat{\gamma}.$$

This gives the valid inequality $x_1 + x_2 \leq 2$ *which relates to the optimal objective function value. So we obtain the optimal solution* $(2, 0)$ *of the ILP in the next step of Algorithm 5.*

The result of this example is typical for applying Algorithm 5 to ILP. In this case, we have to presume that all input data is integral and the k*-disjunctive cut to the objective function reduces to the Chvátal-Gomory cut to the objective function vector according to Lemma 2.1.*

We have seen in the last example that for an ILP the k-disjunctive cut to the objective function reduces to a Chvátal-Gomory cut to the objective function. So Algorithm 5 can be seen as a variant of the pure integer Gomory algorithm in this case. The crucial fact for finite convergence of the integral algorithm is the possibility to add cuts both to the objective function and to each integral variable. Using k-disjunctive cuts to the objective function, we can now add cuts to the objective function to MILP as well. In this way we obtain a convergent algorithm in analogy to the integral case.

The complex part of Algorithm 5 consists in computing the $|R|$-disjunctive cut as the number $|R|$ of extreme rays v^r of the cone Q grows exponentially in the number q of continuous variables and the number m of hyperplanes defining the polyhedron P. So an efficient algorithm for computing the extreme rays of the related cone is required. Moreover, we have to ensure that the computed rays satisfy the integrality constraints, what means that they have to be contained in the group $\mathcal{G}_{P_{\gamma^*}}$ according to (3.16). We should assume in applications that the coefficients of the matrix A and the vector c are integral. Then the integrality constraints are satisfied if all components of the extreme rays v^r are integral. We deal with the computation of extreme rays in the next subsection. Furthermore, we also discuss two approaches to reduce the number of extreme rays of the cone Q in Section 4.3.3 and Section 4.3.4. The first one deals with an irredundant representation of the projection $\text{proj}_X(P_{\gamma^*})$, the second one is a sequential cutting plane algorithm which derives k-disjunctive cuts based on basis relaxations of the polyhedron P.

At last we present another interpretation of Algorithm 5. We assume that the feasible domain P of the relaxation is a full dimensional polytope. One can see that we can always choose an optimal solution (x^*, y^*) of the MILP such that at least q inequalities in the representation of P are active in this case. This follows by considering the LP

$$\max\{hy : Gy \leq b - Ax^*\}$$

for a given $x^* \in \mathbb{Z}^p$. So the solution (x^*, y^*) is contained in a at most $(p+q) - q = p$-dimensional face of the polytope P. This means that we can solve the MILP by solving the set of all MILP which are given by the original objective function and a p-dimensional face of P as feasible domain, and comparing the related optimal objective function values.

Moreover, the set of feasible solutions in each p-dimensional face is discrete in general. So solving a MILP over a p-dimensional face of P can be interpreted as solving an ILP as we could apply a suitable affine transformation. In this way we can consider solving a MILP as parallel solving of several ILP. Especially, every valid cutting plane for P_I is even valid for each of the discrete subproblems. Therefore, we need information of the p-dimensional discrete subproblems if we want to generate strong valid cuts. As the number of p-dimensional faces of P grows exponentially in m and q, this interpretation also gives another reasoning for the need of k-disjunctive cuts with a large k to solve a general MILP. Within Algorithm 5 we can assign the p-dimensional subproblems to the facets of the polyhedron $\text{proj}_X(P_{\gamma^*})$, where the values of the right hand sides δ^i of the disjunctive inequalities in the $|R|$-disjunction $D(|R|, d, \delta)$ can be related to the objective function values of the subproblems.

4.3.2 Computing extreme rays of cones

Computing k-disjunctive cuts based on the projection $\text{proj}_X(P_{\gamma^*})$ according to (3.14) as described in Section 3.4 and in Section 4.3.1 requires the computation of extreme rays of the projection cone

$$Q = \{v \in \mathbb{R}^m : vG = 0, v \geq 0\}$$

according to (3.15). As this is a crucial point of our cutting plane approach, we deal with this issue now. We start with a short overview of some algorithms and the complexity of the problem and introduce secondly a well known algorithm for computing extreme rays, namely the so called double description method of Motzkin, Raiffa, Thompson, and Thrall [MRTT53].

Overview

The extreme ray enumeration problem for an arbitrary cone Q given in $\mathcal{H}$-representation has been studied by a multiplicity of people, see for example [Sch86] and [DBS97]. We also refer to the latter paper for a detailed overview and a comparison of different algorithms, complexity results, and related issues. We note that the extreme ray enumeration problem is closely related to the vertex enumeration problem for general polyhedra. In detail, the second one is the non homogeneous version of the first problem and both problems can be transformed into each other.

We can in principle distinguish two types of algorithms for computing vertices or extreme rays, namely graph traversal or pivoting algorithms and incremental algorithms; see [DBS97]. The first class of algorithms starts with an arbitrary vertex or extreme ray and then attempts to identify other vertices or extreme rays by traversing their basis representation. This idea corresponds to a pivot step in the simplex algorithm. Representatives of this class are for example the gift wrapping algorithm of Chand and Kapur [CK70], Seidel's algorithm [Sei86], or the reverse search algorithm of Avis and Fukuda [AF92]. The second class of algorithms computes the vertices or extreme rays, respectively, by adding inequalities of the $\mathcal{H}$-representation of P sequentially and updating the vertices or extreme rays which have been obtained so far. An example of this class is the double description method of Motzkin, Raiffa, Thompson, and Thrall [MRTT53] which is also the basis of the beneath and beyond method of Seidel [Sei81], the randomized algorithm of Clarkson and Shore [CS88], and the derandomized algorithm of Chazelle [Cha93].

Given a polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}$, an upper bound for the number of vertices is given by

$$\binom{m - \lfloor \frac{(n+1)}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} + \binom{m - \lfloor \frac{(n+2)}{2} \rfloor}{\lfloor \frac{(n-1)}{2} \rfloor}, \tag{4.3}$$

see McMullen [McM70], where this bound is tight. It follows that also the number of extreme rays of an arbitrary cone Q can be exponential in m and n. This fact has to be considered by analyzing the performance of vertex or extreme ray enumeration algorithms. In this context it is an outstanding problem if there exists an algorithm that is polynomial in m and in the number of vertices or extreme rays. Moreover, the exponential growth of the number of extreme rays may cause the problem that it is not possible to compute all extreme rays of a given cone in applications.

Double description method

We introduce a basic form of the double description method which was introduced by Motzkin, Raiffa, Thompson, and Thrall in 1953 [MRTT53]. We are mostly geared to the representation of Fukuda and Prodon [FP95]. The algorithm applies to a cone C given by inequality constraints, so let

$$C = \{x \in \mathbb{R}^p : Ax \leq 0\}, \tag{4.4}$$

where $A \in \mathbb{Q}^{m \times p}$. We assume for simplicity that the cone C is pointed and that its $\mathcal{H}$-representation is irredundant. We are looking for the extreme rays $\{r_1, \dots, r_l\}$ which generate the cone C. Let now R denote the matrix with columns $(r_1, \dots, r_l)$. So we have to determine the matrix R for a matrix A giving the $\mathcal{H}$-representation of the cone C. The pair (A, R) is called a double description pair as it gives both a $\mathcal{H}$- and a $\mathcal{V}$-representation of C.

We describe the details of the procedure. Let S be an arbitrary subset of $\{1, \dots, m\}$ and let A_S denote the submatrix of A consisting of these rows a_i with $i \in S$. The algorithm is incremental, so let for a set $S \subseteq \{1, \dots, m\}$ a double description pair (A_S, R_S) be given. We add one row a_i with $i \notin S$ to A_S and construct a double description pair (A_{S+i}, R_{S+i}). Adding the row a_i leads to a three partition of the space $\mathbb{R}^p$ and a related partition of the set J_S of column indices of the matrix R_S. In detail, we obtain the sets

$$\begin{aligned} H^- &:= \{x \in \mathbb{R}^p : a_i x < 0\}, & J^- &:= \{j \in J_S : r_j \in H^-\} \\ H^0 &:= \{x \in \mathbb{R}^p : a_i x = 0\}, & J^0 &:= \{j \in J_S : r_j \in H^0\} \\ H^+ &:= \{x \in \mathbb{R}^p : a_i x > 0\}, & J^+ &:= \{j \in J_S : r_j \in H^+\}. \end{aligned}$$

We have to replace the columns r_k of the matrix R_S with indices $k \in J^+$ and define $|J^-| \times |J^+|$ new rays $r_{jj'}$ which are generated by appropriate positive combinations of a 'negative' ray $r_j, j \in J^-$ and a 'positive' ray $r_{j'}, j' \in J^+$. The rays $r_{jj'}$ have to be contained in the 'cut' hyperplane $a_i x = 0$ and generate the cone
$\{x \in \mathbb{R}^p : \ A_S x \leq 0\,, a_i x = 0\}$. We define

$$r_{jj'} = (a_i r_{j'}) r_j - (a_i r_j) r_{j'} \ \forall (j, j') \in J^- \times J^+. \tag{4.5}$$

The new rays $r_{jj'}$ satisfy the condition $A_{S+i} r_{jj'} \leq 0$, as $r_{jj'}$ is a positive linear combination of the rays $r_j, r_{j'}$ and

$$a_i r_{jj'} = (a_i r_{j'}) a_i r_j - (a_i r_j) a_i r_{j'} = 0;$$

Moreover, we directly obtain a new double description pair (A_{S+i}, R_{S+i}) by this construction.

Lemma 4.11. *Let (A_S, R_S) be a double description pair, $i \notin S$, and $J_{S+i} := J^+ \cup J^0 \cup (J^- \times J^+)$. Then (A_{S+i}, R_{S+i}) is a double description pair, where R_{S+i} is the $p \times |J_{S+i}|$ matrix with column vectors r_j, $j \in J^+ \cup J^0$ and column vectors $r_{jj'}$ according to* (4.5).

Proof. A proof of this statement can be found in [FP95]. □

A repeated application of (4.5) and Lemma 4.11 yields an easy algorithm for computing all extreme rays of the cone C. We start with an initial double description pair (A_S, R_S),

Algorithm 6 Double description method - basic form

procedure DOUBLEDESCRIPTION
  Input: $A \in \mathbb{Z}^{m \times p}$ with $C = \{x \in \mathbb{R}^p : Ax \leq 0\}$;
  Output: Set R of extreme rays r_j of C;

  Let (A_S, R_S) be an initial double description pair;
  while $S \neq \{1, \ldots, m\}$ **do**
    Select an $i \in \{1, \ldots, m\} \setminus S$;
    Construct (A_{S+i}, R_{S+i}) by Lemma 4.11;
    $S \leftarrow S \cup \{i\}$;
  end while
end procedure

for example with a basis cone of C with $|S| = p$, and apply Lemma 4.11 until $S = \{1, \ldots, m\}$. The complete procedure is stated in Algorithm 6.

We add that Algorithm 6 is only a very basic form of the double description method. It is not suitable for a practical implementation as for example many redundant rays are computed by applying Lemma 4.11. One can show that this problem can be overcome by computing rays $r_{jj'}$ only for those indices $(j, j') \in J^- \times J^+$ for which the related rays r_j and $r_{j'}$ are adjacent in the cone C. Here we say that two rays are adjacent in a cone C if the minimal face of C containing both rays contains no other extreme rays. Therewith we can replace (4.5) by

$$r_{jj'} = (a_i r_j) r_{j'} - (a_i r_{j'}) r_j \ \forall (j, j') \in J^- \times J^+ \text{ and } r_j, r_{j'} \text{ are adjacent in } C. \tag{4.6}$$

Moreover, there are some more issues that have to be considered for a practical application of the procedure. For example, the double description method is very sensitive to the order in which the rows of the input matrix A are added. On the other hand, this approach also provides the possibility to decompose problems into smaller subproblems. This feature rises the chance to treat even large problem instances. We refer again to [FP95] for a further discussion of these and other issues. An implementation of the double description method is given by the cdd and cdd+ package of Fukuda [Fuk93]. We will also use this code for some preliminary tests on computing extreme rays of projection cones related to k-disjunctive cuts in Section 4.3.5.

At last, we comment on applying the double description method on computing k-disjunctive cuts. We have to compute the extreme rays of the cone Q according to (3.15) and satisfy the integrality constraint (3.16). The latter requirement does not need to be satisfied by the output of the double description method, so one eventually has to scale the rays of the matrix R additionally. Moreover, the representation of the projection cone Q in (3.15) differs from the one of the cone C according to (4.4) which is used for the double description method. Therefore, a suitable transformation either of the projection cone Q or of the underlying MILP has to be done first.

4.3.3 Irredundant representation of projections

We consider again the basic derivation of a k-disjunctive cut according to Section 3.4 and Section 4.3.1. We compute the projection $\text{proj}_X(P_\gamma)$ by the extreme rays of the related projection cone Q according to (3.15). Then each of the extreme rays of Q corresponds to one inequality in the representation of the projection $\text{proj}_X(P_\gamma)$. However, these inequalities do not correspond to the facets of $\text{proj}_X(P_\gamma)$ in general. It is in fact often the case that a large number of redundant inequalities is generated, even if the representation of the original polyhedron P is irredundant. We have seen that enumerating all extreme rays of a cone is often difficult due to the exponential growth of the number of rays. So avoiding the computation of redundant extreme rays is an interesting task for computing k-disjunctive cuts in applications. Moreover, even if all rays can be enumerated, redundant inequalities in the representation of the projection $\text{proj}_X(P_\gamma)$ can cause a weaker k-disjunctive cut as more function values for the cut vector have to be considered in Lemma 3.22, Theorem 3.23, and Theorem 4.7 respectively. We start with an easy example for an illustration.

Example 4.12. *Let the polyhedron $P \subseteq \mathbb{R}^{1+1}$ defined by the system*

$$\begin{aligned} x + y &\leq 1, & x - y &\leq 1, \\ -x + y &\leq 1, & -x - y &\leq 1, \\ 2x &\leq 1, & -2x &\leq 1. \end{aligned}$$

The projection $\text{proj}_X(P)$ of P on the x-space can be described by the two inequalities

$$2x \leq 1, \; -2x \leq 1;$$

see Figure 4.2. By computing the projection according to Lemma 1.3, we first have to determine the extreme rays of the cone

$$Q = \{v \in \mathbb{R}^6 : (1 \; -1 \; 1 \; -1 \; 0 \; 0)v = 0, \; v \geq 0\}.$$

Since Q has six extreme rays, we obtain the following representation of the projection $\text{proj}_X(P)$:

$$\begin{aligned} 2x &\leq 1, & -2x &\leq 1, \\ 2x &\leq 2, & -2x &\leq 2, \\ 0x &\leq 2, & 0x &\leq 2. \end{aligned}$$

We see that the first two inequalities define $\text{proj}_X(P)$, while the second two inequalities are redundant as well as the the last two inequalities which are always satisfied.

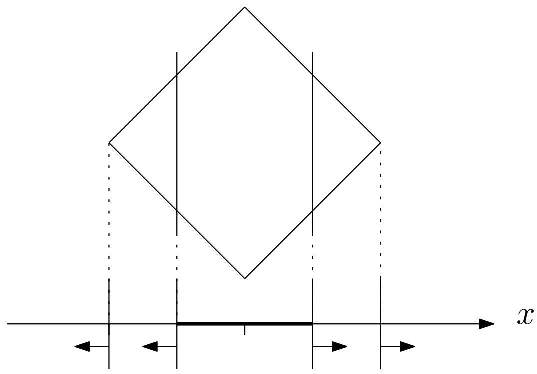


Figure 4.2: The polyhedron P of Example 4.12 and its projection $\operatorname{proj}_X(P)$

In the remainder of this section we show that it is possible to transform the representation of an arbitrary polyhedron P such that each extreme ray of the projection cone Q corresponds to a facet of the projection $\operatorname{proj}_X(P)$. This approach was introduced by Balas; see [Bal97]. Its key feature is determining a new representation of P in which the coefficient matrix A of x is the identity matrix plus possibly some zero rows, while the right hand side b is the unit vector with entry one in the last position.

Now let a polyhedron $P := \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ be given. For a more compact representation of the procedure, we restrict ourselves to the case that $A \in \mathbb{R}^{m\times p}$ has full column rank, so $\operatorname{rank}(A) = p$ and $m \geq p$, and refer to [Bal97] for the general case. We set $A = \binom{A_p}{A_{m-p}}$ and assume without loss of generality that A_p is nonsingular. The procedure requires as input a quadratic matrix A with full rank. Therefore, we complement the matrix A with columns and define the matrix

$$\widetilde{A} := \begin{pmatrix} A_p & 0 \\ A_{m-p} & I_{m-p} \end{pmatrix} \in \mathbb{R}^{m\times m},$$

where I_{m-p} denotes the $(m-p)$-dimensional unit matrix. $\widetilde{A}$ is quadratic and invertible. We consider the polyhedron $\widetilde{P}$ defined by the system

$$\begin{aligned} \widetilde{A}^{-1}s + \widetilde{A}^{-1}Gy - \widetilde{A}^{-1}by_o + \begin{pmatrix} I_p \\ 0_{m-p} \end{pmatrix} x &= 0 \\ y_0 &= 1 \\ s &\geq 0, \end{aligned} \tag{4.7}$$

where $(s, y, y_o, x) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^p$ and 0_{m-p} denotes the $(m-p)$-dimensional zero matrix. Multiplying the first equation of (4.7) by $\widetilde{A}$ yields

$$I_m s + I_m Gy - I_m by_0 + Ax = 0,$$

so the polyhedra P and $\widetilde{P}$ are equal as $s \geq 0$ and $y_0 = 1$.

We project $\widetilde{P}$ on the x-space. According to the remark after Lemma 1.3, the projection cone $\widetilde{Q}$ is given by the system

$$\begin{aligned} (v,w)\widetilde{A}^{-1} &\geq 0 \\ (v,w)\widetilde{A}^{-1}G &= 0 \\ (v,w)\widetilde{A}^{-1}b + v_0 &= 0, \end{aligned} \tag{4.8}$$

where $(v, w, v_0) \in \mathbb{R}^p \times \mathbb{R}^{m-p} \times \mathbb{R}$. So the projection $\text{proj}_X(\widetilde{P})$ is given by

$$\text{proj}_X(\widetilde{P}) = \text{proj}_X(P) = \{x \in \mathbb{R}^p : v^r x \leq v_0^r \text{ for all } (v^r, w^r, v_0^r) \in \widetilde{R}\}, \tag{4.9}$$

where $\widetilde{R}$ is the set of all extreme rays of the projection cone $\widetilde{Q}$ according to (4.8).

Additionally, we consider the projection $\text{proj}_{(v,v_0)}(\widetilde{Q})$ of the cone $\widetilde{Q}$ on the (v, v_0)-space and obtain

Theorem 4.13 ([Bal97]). *Let* $\text{proj}_X(P)$ *be full dimensional. Then an inequality*

$$v^r x \leq v_0^r \quad \textit{in the system } (4.9)$$

defines a facet of $\text{proj}_X(P)$ *if and only if* (v^r, v_0^r) *is an extreme ray of the cone* $\text{proj}_{(v,v_0)}(\widetilde{Q})$.

□

Therewith we have found a way to generate an irredundant representation of $\text{proj}_X(P)$. However, there are some issues that have to be respected before applying this approach practically.

The first one concerns the computation of the transformation. While the generation of the sets $\widetilde{P}$ and $\widetilde{Q}$ is easy, the computation of $\text{proj}_{(v,v_0)}(\widetilde{Q})$ is not. There is in general no bijective correspondence between the rays of the cones $\widetilde{Q}$ and the rays of $\text{proj}_{(v,v_0)}(\widetilde{Q})$, and it is non trivial to determine whether (v^r, v_0^r) is an extreme ray of $\text{proj}_{(v,v_0)}(\widetilde{Q})$ given a ray $(v^r, w^r, v_0^r) \in \widetilde{R}$ of $\widetilde{Q}$. This can be seen in Example 4.14. The problem can be avoided by generating $\text{proj}_{(v,v_0)}(\widetilde{Q})$ explicitly; see [Bal97]. However, this approach also requires some effort.

Secondly, we note that the transformation can destroy special structures of the input matrices that make computation of extreme rays of the cone Q easy. So even if a large number of irredundant rays is generated by the original representation, this might be faster than applying the transformation. Nevertheless, the transformation might be useful in applications if a direct computation of the projection fails.

At last we show how the approach applies to Example 4.12 from the beginning of this section.

Example 4.14. *Let again the polyhedron P of Example 4.12 be given. Applying the above transformation on P, we obtain at first the system defining $\widetilde{P}$ according to* (4.7) *by*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} s + \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \\ -2 \\ 2 \end{pmatrix} y + \begin{pmatrix} -1 \\ 0 \\ -2 \\ -2 \\ 1 \\ -3 \end{pmatrix} y_0 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} x = 0$$

and $y_o = 1,\ s \geq 0.$

This yields the cone $\widetilde{Q}$ according to (4.8) *given by*

$$(v,w)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \geq 0,\ (v,w)\begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \\ -2 \\ 2 \end{pmatrix} = 0,\ (v,w)\begin{pmatrix} -1 \\ 0 \\ -2 \\ -2 \\ 1 \\ -3 \end{pmatrix} + v_0 = 0;$$

The projection cone $\widetilde{Q}$ has six extreme rays $(v^r, w^r, v_0^r) \in \mathbb{R} \times \mathbb{R}^5 \times \mathbb{R}$, namely

$$(2,1,0,0,0,0,2), (0,0,0,1,0,0,2), (-2,0,1,1,0,0,2),$$
$$(0,1,1,0,0,0,2), (2,0,0,0,1,0,1), (-2,0,0,0,0,1,1).$$

By projecting these rays on the (v, v_0)-space, we obtain the generators of the cone $\text{proj}_{(v,v_0)}(\widetilde{Q})$ *and see that* $\text{proj}_{(v,v_0)}(\widetilde{Q})$ *is already generated by the two rays*

$$(2,1) \text{ and } (-2,1).$$

So $\text{proj}_X(P)$ *is defined by the two inequalities*

$$2x \leq 1 \wedge -2x \leq 1.$$

Moreover, we can see in this example how the extreme rays defining redundant inequalities of the projection are dropped out in the last step of the procedure.

4.3.4 Sequential algorithm

We discuss another type of redundancy which can appear by applying Algorithm 5 and can cause an arbitrarily large additional computational effort. It is based on the fact that in solving a MILP many facets which describe the polyhedron P of the LP relaxation do not have any influence on the optimal solution of the MILP and its objective function value. This effect does not have to be linked to a redundancy of the representation of P itself and can be seen in the next

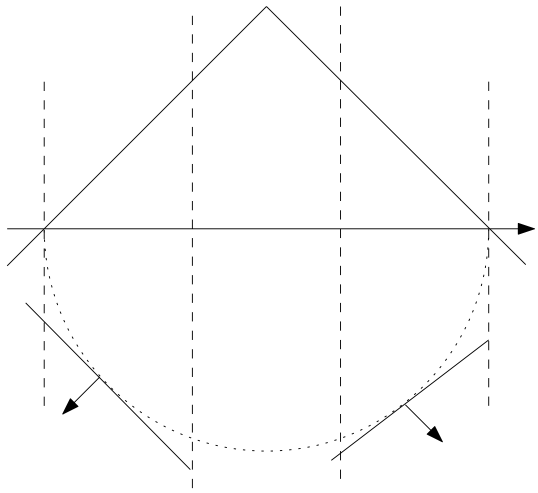


Figure 4.3: The polyhedron of Example 4.15 with two additional 'redundant' hyperplanes.

Example 4.15. *Let the two dimensional MILP*

$$\begin{aligned} \max \quad & y \\ & x + y \leq 4 \\ & -x + y \leq -1 \\ & x \in \mathbb{Z} \end{aligned}$$

with optimal LP solution $(2.5, 1.5)$ *be given. The MILP has the two optimal solutions* $(2, 1)$ *and* $(3, 1)$. *We consider the half circle with center* $(2.5, 0)$ *and radius* 1.5 *with negative y values. Now changing the feasible domain of the MILP by adding any supporting hyperplane of the half circle to* $P = \{(x, y) \in \mathbb{R}^{1+1} : x + y \leq 4, -x + y \leq -1\}$ *does not change the set of optimal solutions of the MILP and the representation of* P *remains irredundant; see Figure 4.3. However, adding new inequalities to* P *leads of course to a higher dimensional projection cone* Q *with new extreme rays associated with irredundant inequalities of the projection* $\text{proj}_X (P_{\gamma^*})$.

As the example shows, it is in general not advantageously to respect all facets of $P = \{(x, y) \in \mathbb{R}^{p+q} : Ax + Gy \leq b\}$ in computing a k-disjunctive cut according to Theorem 4.7. Therefore, we now suggest a sequential algorithm that modifies the basic procedure of Algorithm 5. It is based on the idea to respect only those inequalities in the representation of the polytope P for the derivation of the k-disjunctive cut which have already been active. So the k-disjunctive cuts are derived from a relaxation $\widetilde{P}$ of P which is updated during the algorithm.

In detail, we start with all inequalities defining P which support the optimal solution

(x^*, y^*) of the initial LP relaxation. So we have

$$\widetilde{P} := \{(x,y) : a_j x + g_j y \le b_j, j \in J\},$$

where $J := \{j \in \{1,\dots,m\} : a_j x^* + g_j y^* = b_j\}$. Every time a new inequality $a_i x + g_i y \le b_i$ gets active in the algorithm, this means it satisfies $a_i x^* + g_i y^* = b_i$ for the current LP optimum (x^*, y^*), we add it to $\widetilde{P}$ and compute a new k-disjunctive cut according to Theorem 4.7 based on the updated relaxation. This means that the present extreme rays of the projection cone $\widetilde{Q}$ have to be complemented with rays belonging to the new columns of the updated projection cone.

Except for this modification, the algorithm is identical to Algorithm 5. We state the procedure in Algorithm 7.

Theorem 4.16. *Let a rational MILP* (1.1) *be given, where the feasible domain P of the LP relaxation is bounded. Then Algorithm 7 either finds an optimal solution of the MILP or detects infeasibility in a finite number of steps.*

Proof. The proof is quite analog to the proof of Theorem 4.8 for Algorithm 5. The modifications of Algorithm 5 neither change its properties nor are feasible points now cut off. We note that every k-disjunctive cut for $\widetilde{P}$ is also valid for P as $P \subseteq \widetilde{P}$. □

Due to the updates of $\widetilde{Q}$, extreme rays of the projection cone $\widetilde{Q}$ have to be computed several times in Algorithm 7. It is not obvious how this can be done efficiently given the present rays. This issue can become important if many updates of $\widetilde{P}$ and $\widetilde{Q}$ are necessary in the algorithm. In this case, the effort for a repeated computation of the extreme rays should be compared with the singular effort which is necessary for determining the rays of the whole projection cone Q in Algorithm 5. However, Algorithm 7 avoids the redundancy which we have discussed in Example 4.15 and requires less information at the beginning of the algorithm. Therefore, one could choose Algorithm 7 as the standard alternative of both exact algorithms.

4.3.5 Examples and applications

At the end of this chapter we present some more examples and applications of the exact cutting plane procedure. We consider two classes of problems for which k-disjunctive cuts to the objective function can be easily computed. In detail, we investigate rational MILP which either have only one continuous variable or which are defined by a single knapsack constraint. Furthermore, we deal with the computation of k-disjunctive cuts to the objective function for some small examples from the MIPLIB.

Algorithm 7 Exact cutting plane algorithm - sequential form

procedure SequentialCutting
  Input: rational MILP (1.1), where P is bounded;
  Output: optimal solution (x^*, y^*) or "problem infeasible" if no solution exists;
  $P \leftarrow \{(x,y) : Ax + Gy \leq b\},\ (x^*, y^*) \leftarrow \text{argmax}\,\{cx + hy : (x,y) \in P\}$;
  $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*), $J := \{j \in \{1,\ldots,m\} : a_j x^* + g_j y^* = b_j\}$;
  $P^0 := \{(x,y) : a_j x + g_j y \leq b_j, j \in J\}$;
  if $P = \emptyset$ **then**
    "problem infeasible"; **break**
  end if
  if $x^* \in \mathbb{Z}^p$ **then**
    "optimal solution (x^*, y^*)"; **break**
  end if
  while $x^* \notin \mathbb{Z}^p$ **do**
    Set $\gamma^* \leftarrow cx^* + hy^*$, $\gamma \leftarrow \gamma^*$;
    while $\gamma = \gamma^*$ **do**
      $i \leftarrow \text{argmin}\,\{k \in \{1,\ldots,p\} : x_k^* \notin \mathbb{Z}\}$;
      Compute intersection cuts $\alpha_B x + \beta_B y \leq \eta_B$ to $P, D(u_i, \lfloor x_i^* \rfloor)$, all $B \in \mathcal{B}$;
      $P \leftarrow P \cap \{(x,y) : \alpha_B x + \beta_B y \leq \eta_B, B \in \mathcal{B}\}$;
      $(x^*, y^*) \leftarrow \text{arglexmax}\,\{cx + hy : (x,y) \in P\}$;
      $\mathcal{B} \leftarrow$ set of all bases B of (x^*, y^*);
      $J \leftarrow J \cup \{j \in \{1,\ldots,m\} : a_j x^* + g_j y^* = b_j\}$;
      $P^0 \leftarrow \{(x,y) : a_j x + g_j y \leq b_j, j \in J\}$;
      if $P = \emptyset$ **then**
        "problem infeasible"; **break**
      end if
      if $x^* \in \mathbb{Z}^p$ **then**
        "optimal solution (x^*, y^*)"; **break**
      end if
      $\gamma \leftarrow cx^* + hy^*$
    end while
    Compute $\widehat{\gamma} = \max\{\gamma^r : r \in R, v_{m+1}^r > 0\}$ according to Theorem 4.7
    for $P^0, (c,h), \gamma^*$;
    $\gamma^* \leftarrow \widetilde{\gamma}$;
  end while
end procedure

MILP with one continuous variable

We consider rational MILP with one continuous variable and show how Algorithm 5 works for this class of problems. As the projection cone Q has only one row, the computation of extreme rays becomes easy in this case. So let a MILP

$$\begin{aligned} \max \quad & cx + hy \\ & Ax + gy \leq b \\ & x \in \mathbb{Z}^p \end{aligned}$$

with $h \in \mathbb{Z} \setminus \{0\}$, $g \in \mathbb{Z}^m$ be given. We assume for simplicity that also the remaining input data is integral. For the computation of a k-disjunctive cut to the objective function vector (c, h) we have to determine the extreme rays of the projection cone Q according to (3.15). Here we have

$$Q = \{v \in \mathbb{R}^{m+1} : (g \;\; -h)v = 0,\ v \geq 0\}. \tag{4.10}$$

Without loss of generality, we can presume that each component of the vector g is unequal to 0. Otherwise, the related row of the system $Ax + gy \leq b$ does not have to be considered in the derivation of the k-disjunctive cut. Moreover, we only have to respect extreme rays v^r of Q with the property $v^r_{m+1} > 0$.

Now one can see that each relevant extreme ray of Q given in (4.10) has only two non-zero components which correspond to two coefficients of the vector $(g \; -h)$ with different sign. Therefore, there are at most m extreme rays v^r of Q which are relevant for the derivation of the k-disjunctive cut and each extreme ray corresponds to an inequality of the system $Ax + gy \leq b$. In detail, the inequality $a_i x + g_i y \leq b_i$ relates to an extreme ray of Q if and only if $\mathrm{sign}(g_i) = \mathrm{sign}(h)$. In this case, we obtain the extreme ray $v^{r,i}$ by

$$v^{r,i}_i = |h|,\ v^{r,i}_{m+1} = |g_i| \text{ and } v^{r,i}_j = 0 \text{ for } j \neq i, m+1. \tag{4.11}$$

By (4.11), the relevant inequalities of the projection $\mathrm{proj}_X\,(P_{\gamma^*})$ according to (3.14) are given by

$$|h|a_i x - |g_i|cx \leq |h|b_i - |g_i|\gamma^*, \text{ for } v^{r,i} \neq 0.$$

We apply Theorem 4.7 and obtain the valid objective function value $\widehat{\gamma}$ by

$$\widehat{\gamma} = \max\{\gamma^{r,i} : v^{r,i} \neq 0\} \quad \text{with } \gamma^{r,i} = \frac{\lceil |h|b_i - |g_i|\gamma^* \rceil - |h|b_i}{-|g_i|}. \tag{4.12}$$

We illustrate the above approach and consider a generalization of Example 2.13, which we have also considered several other times in this thesis, to arbitrary dimension. We have seen in Example 4.9 that Algorithm 5 solves the 3-dimensional problem instance in one step. This also remains true for higher dimensions.

Example 4.17. *Let the MILP*

$$\begin{aligned} \max \quad & y \\ & -x_i + y \leq 0, \ i = 1, \ldots, p \\ & \sum_{i=1}^{p} x_i + y \leq p \\ & x \in \mathbb{Z}^p \end{aligned}$$

for $p \in \mathbb{N}$ be given. The optimal solution of the LP relaxation is given by $(\frac{p}{p+1}, \ldots, \frac{p}{p+1})$. The projection cone Q for the derivation of the k-disjunctive cut is given by

$$Q = \{v \in \mathbb{R}^{p+2} : (1 \ 1 \ \cdots \ 1 \ -1)v = 0, v \geq 0\}$$

with extreme rays

$$(1, 0, \ldots, 0, 1), \ldots, (0, \ldots, 0, 1, 1).$$

So the projection $\operatorname{proj}_X(P_\gamma)$ of the polyhedron P_γ on the x-space is given by the system

$$\begin{aligned} -x_i &\leq 0 - \gamma, \ i = 1, \ldots, p \\ \sum_{i=1}^{p} x_i &\leq p - \gamma. \end{aligned}$$

As the optimal objective function value of the LP relaxation is given by $\gamma^* = \frac{p}{p+1} < 1$, it is $\lceil 0 - \gamma^* \rceil - 0 = 0$ and $\lceil p - \gamma^* \rceil - p = 0$. This yields the optimal objective function value $\gamma = 0$ after one step of the algorithm.

MILP with a single knapsack constraint

As a further special case for the application of Algorithm 5 we consider the basic mixed integer knapsack problem

$$\begin{aligned} \max \quad & cx + hy \\ & ax + gy \leq b \\ & x, y \geq 0 \\ & x \in \mathbb{Z}^p, \end{aligned}$$

with $a \in \mathbb{Z}^p, g \in \mathbb{Z}^q$, where we assume again that all input data is integral. For the computation of the k-disjunctive cut to the objective function vector we have to determine again the extreme rays of the cone Q according to (3.15). We transform the MILP to the form (1.1) and write the non-negativity constraints in the form $-x \leq 0$, $-y \leq 0$. Therewith we obtain the cone

$$Q = \{v \in \mathbb{R}^{q+2} : (g \ -I_q \ -h)v = 0, v \geq 0\}. \tag{4.13}$$

We start with the classical knapsack problem, where all components of the vectors c, h, a, g are positive. We can see that we have only one relevant extreme ray v^r of Q with $v^r_{q+2} > 0$ in this case. We set

$$\lambda := \max_{i=1,\ldots,q} \frac{h_i}{g_i} \tag{4.14}$$

and define the ray v^r by

$$v^r_1 = \lambda,\ v^r_{q+2} = 1,\ \text{and } v^r_{1+i} = \lambda g_i - h_i,\ \text{for } i = 1,\ldots,q. \tag{4.15}$$

By (4.15), we obtain as only constraint for computing a valid objective function value according to Theorem 4.7 the inequality

$$\lambda ax - v^r_{2,\ldots,q+1} I_p x - cx \leq \lambda b - \gamma^*. \tag{4.16}$$

If we assume that all vectors in (4.16) are integral, we finally obtain the valid cutting plane $cx + hy \leq \widehat{\gamma}$ with

$$\widehat{\gamma} = \frac{\lceil \lambda b - \gamma^* \rceil - \lambda b}{-1} = \lambda b - \lceil \lambda b - \gamma^* \rceil; \tag{4.17}$$

However, to satisfy the integrality constraint (3.16) in (4.16), we eventually have to scale the inequality $\lambda ax - v^r_{2,\ldots,q+1} I_p x - cx \leq \lambda b - \gamma^*$ with the coefficient g_{i_0}, where $i_0 = \operatorname{argmax}\{\frac{h_i}{g_i} : i = 1,\ldots,q\}$. This yields the valid inequality $cx + hy \leq \widehat{\gamma}$ with

$$\widehat{\gamma} = \frac{\lceil g_{i_0} \lambda b - g_{i_0} \gamma^* \rceil - g_{i_0} \lambda b}{-g_{i_0} 1} = \lambda b - \frac{\lceil g_{i_0}(\lambda b - \gamma^*) \rceil}{g_{i_0}}; \tag{4.18}$$

As for $x \in \mathbb{R}$ and $a \in \mathbb{N}$ it is $\lceil x \rceil \geq \frac{\lceil ax \rceil}{a}$, a necessary scaling of (4.16) gives a weaker cut in general.

We shortly remark about the general case which arises by removing the positivity constraints from the vectors c, h, a, g and consider again the cone (4.13). In this case, each relevant extreme ray v^r of Q has to satisfy the constraint $v^r_1 g - v^r_{q+2} h \geq 0$ with $v^r_1, v^r_{q+2} > 0$. We can set $v^r_{q+2} = 1$ without loss of generality and obtain the constraints

$$v^r_1 g_i \geq h_i,\ i = 1,\ldots,q. \tag{4.19}$$

Depending on the sign of the coefficient g_i, we consider

$$v^r_1 \geq \frac{h_i}{g_i},\quad \text{for } g_i > 0 \quad \text{and} \quad v^r_1 \leq \frac{h_i}{g_i},\quad \text{for } g_i < 0$$

and set

$$\lambda_{max} := \max_{i=1,\ldots,q: g_i > 0} \frac{h_i}{g_i} \text{ and } \lambda_{min} := \min_{i=1,\ldots,q: g_i < 0} \frac{h_i}{g_i}. \tag{4.20}$$

If $0 \leq \lambda_{max} \leq \lambda_{min}$ the system (4.19) can be satisfied and relevant extreme rays exist. Moreover, if $\lambda_{max} < \lambda_{min}$ there exists in general more than one relevant ray. After the value of the first coefficient v^r_1 has been determined, the remaining coefficients can be computed analog to (4.15). We omit the details here.

Small examples from MIPLIB

Finally, we want to give a short overview of the derivation of k-disjunctive cuts to the objective function for some small instances from the MIPLIB 3.0 [BCMS98] and the MIPLIB 2003 [AKM06]. In this way, we can get a first impression of a practical application of the results which we have introduced from a theoretical point of view. As this is the crucial point, we only consider the computation of the set of extreme rays of the projection cone Q according to (3.15). After the extreme rays of Q have been determined, the computation of a valid cut is easily done by comparing the related objective function values according to Theorem 4.7. Moreover, we follow the most general way of computing k-disjunctive cuts as described at the beginning of Section 4.3.1 and do not respect any possible variations as for example discussed in Section 4.3.4.

We compute the extreme rays with the `cdd`-package, version 061, a C-implementation of Fukuda [Fuk93] of the double description method which we have introduced in Section 4.3.2 in a basic form. We refer to the `cdd` documentation for details of the implementation. As we have introduced the procedure in Section 4.3.2, the double description method applies to cones $Q = \{x \in \mathbb{R}^p : Ax \leq b\}$. This form differs from the one which we have considered in (3.15) for the derivation of a k-disjunctive cut. Therefore, we start with MILP given in standard form (1.4) instead of MILP in natural form. According to the remark after Lemma 1.3, the projection cone Q is then given in the form $Q = \{x \in \mathbb{R}^p : Ax \leq b\}$. Of course, the remainder of our cutting plane approach remains unaffected by this change.

We consider eleven instances from the MIPLIB; see Figure 4.4. The table shows the number of rows and columns of the projection cone Q, the number of extreme rays v^r, and the number of relevant extreme rays v^r with $v^r_{m+1} > 0$ which have to be respected for the derivation of the cut. We note that we have made some modifications of the input data, such as deleting pure integer equations and fixed variables to avoid some redundant computation.

The instances `egout,qnet1,qnet1_o,flugpl,blend2` are only part of the MIPLIB 3.0, whereas all other instances can be found in both databases. `blend2, flugpl, qnet1, qnet1_o` are real mixed integer problems while the remaining problems are mixed binary, and except for `markshare1` and `markshare2` all problems are easy to solve in relation to the time needed to solve the problem with a commercial solver. The computation of the rays was made within seconds for all instances.

We look at the results of the computation. We can see in Figure 4.4 that the number of rays v^r with $v^r_{m+1} > 0$ which are relevant for the computation of the k-disjunctive cut is small in general, especially compared to the total number of extreme rays. An exception is given by the problem `pk1`, where all extreme rays are relevant for the derivation of the cut. The instances `mas74` and `mas76` have only one continuous variable, and are

Name	Rows	Columns	Rays	Relevant
markshare1	7	7	7	1
markshare2	8	8	8	1
mas74	15	14	26	13
mas76	14	13	24	12
pk1	62	46	61	61
egout	111	99	69	4
qnet1	125	126	127	1
fixnet6	1001	601	479	1
qnet1_o	125	126	127	1
flugpl	20	15	10	1
blend2	189	189	189	1

Figure 4.4: Computation of extreme rays of the projection cone for some examples from the MIPLIB

defined by a system of 13 and 12 inequalities, respectively. So the number of extreme rays corresponds with the theoretical results at the beginning of this section.

It is interesting to see that already in these small examples a large number of redundant extreme rays is computed. We note that this redundancy does not necessarily have to be related to one of the problems which we have discussed in Section 4.3.3 and Section 4.3.4, as we do not consider the redundancy of the projection $\text{proj}_X(P_\gamma)$ itself, but redundancy in relation to extreme rays with v^r_{m+1} which are required for the derivation of the k-disjunctive cut. However, as the simple computations of Figure 4.4 show, it can also become an important issue in applications to avoid this type of redundancy. We also note that the computation of extreme rays of the projection cone for most instances of the MIPLIB is not possible - at least not using the basic approach which we have used here - due to the exponential growth of the number of extreme rays. Thus it could be interesting to design an algorithm which computes only these extreme rays of a projection cone which are relevant for the derivation of a k-disjunctive cut. A first approach for this issue could be the use of a vertex enumeration algorithm which is applied to the polyhedron

$$S := Q \cap \{x \in \mathbb{R}^{m+1} : x_{m+1} = 1\}.$$

Each vertex of S except for the origin then corresponds in a natural way to a relevant extreme ray of Q by taking the extreme rays defined by the origin and a vertex of S.

5 Discussion

We have dealt with the problems of describing the mixed integer hull of an arbitrary polytope and of designing a related finite cutting plane algorithm for rational MILP. First, we have analyzed and compared the properties of several known cutting planes in Chapter 2 and have shown why these approaches are not sufficient for solving both issues. In Chapter 3 we have introduced general k-disjunctive cuts and have discussed which type of these cuts is required for obtaining the mixed integer hull of a rational and an arbitrary polytope. Moreover, we have also provided a method for constructing k-disjunctive cuts for rational polyhedra and have used this approach to design a finite cutting plane algorithm for rational MILP in Chapter 4. Finally, we now choose three issues and discuss some ideas for possible further research. We consider our results concerning split cuts, the computation of k-disjunctive cuts, and their practical application.

Split cuts and rational MILP

We have seen in Theorem 3.13 that split cuts are sufficient to obtain all vertices of the mixed integer hull of a rational polytope. As mentioned in Corollary 3.15, this property can also be used to solve a rational MILP by considering a suitable perturbated objective function. It would be interesting to see if this theoretical statement can be applied practically and if an exact cutting plane algorithm for rational MILP based on split cuts can be designed.

For this purpose, we have to find a way to give a suitable perturbated objective function according to Corollary 3.15 based on the original objective function of the MILP and the facets of the polytope P of the LP relaxation. It could be a first approach to determine a bundle of perturbated objective function vectors which includes a suitable vector in any case. Moreover, it is open which split cuts would have to be added in a related algorithm. To answer this question one could start with analyzing a cutting plane algorithm for a MILP with a suitable objective function in which cutting planes are added according to the approach of the approximation algorithm of Owen and Mehrotra; see Section 4.2.1.

Computation of k-disjunctive cuts

In Section 3.4 we have introduced a method for computing k-disjunctive cuts for rational polyhedra based on projections and a given cut vector. This approach was also influenced by the results of Section 3.3. However, as already mentioned, it is a fundamental problem of this approach that the computation of the related projection is very complex in general. This effect can make it impossible to compute a certain cut in practical applications. Therefore, it would be interesting to investigate some easier alternatives for deriving k-disjunctive cuts, even if finiteness of a related algorithm cannot be guaranteed anymore. An example could be the introduction of some type of standard k-disjunctive cut which is derived as intersection cut to a given standard k-disjunction such as a p-dimensional simplex. The last approach would also provide an opportunity for deriving k-disjunctive cuts for non rational polyhedra.

Beside a different approach for deriving k-disjunctive cuts, one should also deal with the issue of how projections for deriving cuts according to our approach in Section 3.4 can be computed more efficiently. This concerns avoiding redundancies both in the computation of the underlying k-disjunction, see Section 4.3.2 and Section 4.3.3, and in the computation of inequalities of the projection which are required for the derivation of the cut according to Theorem 3.23, see Section 4.3.4 and Section 4.3.5. Therewith even larger problem instances could be handled.

Furthermore, it would also be interesting to analyze the depth of a k-disjunctive cut depending on the given cut vector and the underlying polytope. So it could be more efficient to choose a vector different from the objective function for cutting in applications.

Practical application and implementation

An implementation of the basic exact cutting plane algorithm as introduced in Section 4.3.1 and its sequential alternative given in Section 4.3.4, respectively, and tests on some elementary examples are the foundation for a further analysis on how useful k-disjunctive cuts can be in practical applications. This could also include an implementation of some alternatives of k-disjunctive cuts which can be computed more easily as discussed above.

We note that we have always presumed an exact arithmetic for the derivation of rational k-disjunctive cuts and the exact cutting plane algorithm. This fact has also to be respected in a practical implementation. Otherwise, one has to analyze which problems could arise in each step of the computation of a k-disjunctive cut to ensure that only valid cuts are computed.

The combination of cutting planes and an enumerative approach is in general more powerful in solving a MILP than a pure cutting plane approach. Therefore, it would also be interesting to analyze the gain of k-disjunctive cuts within a branch-and-cut framework. Here k-disjunctive cuts based on the projection approach and derived from standard disjunctions can be investigated again. We add that in theory the derivation of deep cuts becomes the easier the more integral variables have been fixed in the branch-and-bound tree. This follows from our results in Section 3.3.

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