

Structured Least Squares to Improve the Performance of ESPRIT-Type High-Resolution Techniques

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Abstract — ESPRIT-type high-resolution (spatial) frequency estimation techniques, like standard ESPRIT, state space methods, matrix pencil methods, or Unitary ESPRIT, obtain their (spatial) frequency estimates from the solution of a highly-structured, overdetermined system of equations. Here, the structure is defined in terms of two selection matrices applied to a matrix spanning the estimated signal subspace. Structured least squares (SLS) is a new algorithm to solve this overdetermined system, the so called invariance equation, by preserving its structure. Simulations confirm that SLS outperforms the least squares (LS) and total least squares (TLS) solutions of this invariance equation, since the accuracy of the resulting (spatial) frequency estimates and the accuracy of the underlying signal subspace are improved significantly. Furthermore, SLS can be used to improve the accuracy of adaptive frequency estimating schemes that are based on fast adaptive subspace tracking techniques. Moreover, SLS has been extended to the two-dimensional (2-D) case to be used in conjunction with 2-D Unitary ESPRIT, an efficient ESPRIT-type algorithm that provides automatically paired 2-D (spatial) frequency estimates.

1. Introduction

Modern subspace-based high-resolution frequency or direction of arrival (DOA) estimation schemes can be classified according to the numerical procedure they exploit into [4]:

- *extrema-searching techniques*, e.g., spectral MUSIC,
- *polynomial-rooting techniques*, e.g., Pisarenko, Min-Norm, or Root-MUSIC, and
- *matrix-shifting techniques*, e.g., standard ESPRIT [8], state space methods (direct data approach or Toeplitz approximation method), matrix pencil methods, or Unitary ESPRIT [2].

In this paper, we consider the third category, i.e., matrix-shifting or ESPRIT-type techniques, based on a shift-invariant structure of the signal subspace. After calculating a basis of the estimated signal subspace, an overdetermined set of equations, the so called invariance equation, can be formed by applying two selection matrices to the basis matrix of the estimated signal subspace, cf. (2). The resulting highly-structured system of equations is usually solved via least squares (LS) or total least squares (TLS). The LS and TLS solutions, however, are not optimal, since they do not take the relationship between the entries on the left- and right-hand-side of the invariance equation into account.

In [9], a “new state space approach” has been presented that first solves a least squares problem, then constructs an error covariance matrix for the structured problem by using a first-order perturbation expansion, and finally solves for the underlying subspace in

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a weighted least squares sense. Notice that this approach is based on a first-order perturbation expansion of the SVD. In contrast to this “new state space approach”, structured least squares (SLS) is also applicable if the subspace estimate has been obtained via some fast adaptive subspace tracking scheme, e.g., [5] or [10]. Furthermore, SLS is computationally more efficient and is not restricted to state space methods, but can incorporate more complicated selection matrices, e.g., the selection matrices used in the element space [2] or DFT beamspace versions [11] of Unitary ESPRIT. Unitary ESPRIT is a recently developed ESPRIT-type algorithm for centro-symmetric array configurations, that is formulated in terms of real-valued computations throughout. It constrains the estimated phase factors to the unit circle and provides reliability information without the need for additional computations. The ability to formulate an ESPRIT-type algorithm for 1-D array structures that only requires real-valued computations from start to finish, was critically important in developing a two-dimensional (2-D) extension of Unitary ESPRIT [3]. 2-D Unitary ESPRIT is a closed-form high-resolution algorithm to provide automatically paired source azimuth and elevation angle estimates. It is also possible to extend SLS to the 2-D case such that it is applicable in conjunction with 2-D Unitary ESPRIT.

Notice that SLS uses the same approximation as the recently developed *structured total least norm* (STLN) algorithm [6, 7], namely a second order term in the expansion of the residual matrix at iteration $k + 1$ is neglected, cf. (7). STLN has been developed to compute the solution of an overdetermined linear system $Ax \approx b$ with possible errors in the matrix A and the vector b . Thereby, STLN preserves the affine structure of A , such as Toeplitz or Hankel. Although STLN has been used for linear prediction, it is not applicable to ESPRIT-type techniques.

2. Review of Standard ESPRIT

Consider d narrowband, planar wavefronts with common wavelength λ and distinct directions of arrival (DOAs) θ_i , $1 \leq i \leq d$, impinging on a sensor array of M elements. For simplicity, we will assume that all sensors have identical characteristics. The array consists of two identical, possibly overlapping, subarrays. Let Δ denote the distance between the two subarrays. The signals arriving at the M sensors at time t are denoted as $x(t) = As(t) + n(t) \in \mathbb{C}^M$, where A is the array steering matrix and $s(t)$ the d -dimensional vector of impinging wavefronts. The additive noise vector $n(t)$ is taken from a zero-mean, spatially uncorrelated random process, which is also uncorrelated with the signals. Since every row of A corresponds to an element of the sensor array, a particular subarray configuration can be described by selection matrices, that choose m elements of $x(t) \in \mathbb{C}^M$, where $m, d \leq m < M$, is the number of

elements in each subarray. Let J_{μ_1} and J_{μ_2} be $m \times M$ selection matrices that assign elements of $x(t)$ to the subarrays 1 and 2, respectively. In case of a maximum overlap situation, J_{μ_1} picks the first $m = M - 1$ rows of A , while J_{μ_2} selects the last $m = M - 1$ rows of the array steering matrix.

ESPRIT-type algorithms are based on the following invariance property of the array steering matrix A ,

$$J_{\mu_1} A \Phi = J_{\mu_2} A, \quad \text{where } \Phi = \text{diag} \{ e^{j\mu_i} \}_{i=1}^d \quad (1)$$

is a unitary diagonal $d \times d$ matrix with spatial frequencies given by $\mu_i = \frac{2\pi}{\lambda} \Delta \sin \theta_i$. Let X denote the $M \times N$ complex data matrix composed of N snapshots $x(t_n)$, $1 \leq n \leq N$. Then, a signal subspace estimate can be obtained via an eigenvalue decomposition (EVD) of the (scaled) sample covariance matrix XX^H , a singular value decomposition (SVD) of the noise-corrupted data matrix X , or some fast subspace estimation technique that approximates either the EVD or the SVD. Assume that the columns of $U_s \in \mathbb{C}^{M \times d}$ span the estimated signal subspace. By applying the two selection matrices to the signal subspace matrix, the following overdetermined set of equations is formed:

$$J_{\mu_1} U_s \Psi \approx J_{\mu_2} U_s \in \mathbb{C}^{m \times d} \quad (2)$$

Notice that the invariance equation (2) is highly-structured if overlapping subarray configurations are used. Usually (2) is solved by using least squares (LS) or total least squares (TLS) [8]. Then, the eigenvalues of the resulting solution $\Psi \in \mathbb{C}^{d \times d}$ are estimates of the phase factors $e^{j\mu_i}$. Thus, estimates of the spatial frequencies μ_i and the corresponding DOAs θ_i are easily obtained.

3. Structured Least Squares

Let us take a closer look at equation (2). Its LS solution Ψ_{LS} satisfies

$$J_{\mu_1} U_s \Psi_{\text{LS}} = J_{\mu_2} U_s + \Delta U_{s2} \quad (3)$$

such that the Frobenius-norm of ΔU_{s2} is minimized. Thus, it is assumed that $J_{\mu_1} U_s$ is known without error and that only the right-hand-side of (2), i.e., $J_{\mu_2} U_s$, is subject to error, clearly an assumption that is not satisfied in our case.

The TLS solution Ψ_{TLS} , however, satisfies

$$(J_{\mu_1} U_s + \Delta U_{s1}) \Psi_{\text{TLS}} = J_{\mu_2} U_s + \Delta U_{s2} \quad (4)$$

such that the Frobenius-norm of $[\Delta U_{s1} \quad \Delta U_{s2}]$ is minimized. Notice that the TLS solution is appropriate if the two subarrays do not share any elements, i.e., the entries of $J_{\mu_1} U_s$ and $J_{\mu_2} U_s$ are independent. For overlapping subarray configurations, however, *structured least squares* (SLS) should be preferred, since SLS accounts for the specific relationship between the entries of $J_{\mu_1} U_s$ and $J_{\mu_2} U_s$.

SLS assumes that the entries of the matrix U_s in (2) are subject to error. Recall that the columns of U_s only span a noise-corrupted estimate of the unknown signal subspace. Therefore, we can allow for a small change ΔU_s of the basis of the estimated signal subspace. Let $U = U_s + \Delta U_s$ denote a basis matrix of an improved signal subspace estimate. This improved signal subspace should be determined such that the Frobenius-norm of the resulting residual matrix

$$R(U, \Psi) = J_{\mu_1} U \Psi - J_{\mu_2} U \quad (5)$$

is minimized. At the same time, the Frobenius-norm of matrix representing the subspace change ΔU_s should be kept as small as possible. Given an initial basis for the signal subspace U_s , SLS determines the matrices ΔU_s and Ψ such that they minimize the following expression:

$$\min_{\Delta U_s, \Psi} \left\| \begin{bmatrix} R(U, \Psi) \\ \kappa \cdot \Delta U_s \end{bmatrix} \right\|_F \quad (6)$$

Here, $\kappa = \sqrt{m/(\alpha M)}$ is a weighting factor that provides a normalization such that the minimization is independent of the two block matrix sizes in (6). Furthermore, $\alpha > 1$ accounts for the fact that the entries of the residual matrix $R(U, \Psi)$ should be smaller than the entries of ΔU_s .

Let us derive an algorithm that solves (6) in an iterative fashion by linearizing $R(U, \Psi)$. To this end, assume that the residual matrix at the k th iteration step is given by $R(U_k, \Psi_k) = J_{\mu_1} U_k \Psi_k - J_{\mu_2} U_k$. Therefore, the residual matrix at iteration $k + 1$ can be written as

$$\begin{aligned} R(U_{k+1}, \Psi_{k+1}) &= R(U_k + \Delta U_k, \Psi_k + \Delta \Psi_k) \\ &= J_{\mu_1} [U_k + \Delta U_k] [\Psi_k + \Delta \Psi_k] - J_{\mu_2} [U_k + \Delta U_k] \\ &\cong R(U_k, \Psi_k) + J_{\mu_1} [U_k \Delta \Psi_k + \Delta U_k \Psi_k] - J_{\mu_2} \Delta U_k. \end{aligned} \quad (7)$$

Here, the second order term in ΔU_k and $\Delta \Psi_k$, i.e., $J_{\mu_1} \Delta U_k \Delta \Psi_k$, has been neglected.

Let $\text{vec}\{R\}$ denote a vector-valued function that maps an $m \times d$ matrix R into an md -dimensional column vector by stacking the columns of the matrix. In the sequel, we will use the following important property of the vec -operator. Given the matrices $Y_1 \in \mathbb{C}^{y_1 \times y_2}$, $Y_2 \in \mathbb{C}^{y_2 \times y_3}$, and $Y_3 \in \mathbb{C}^{y_3 \times y_4}$,

$$\text{vec}\{Y_1 Y_2 Y_3\} = (Y_3^T \otimes Y_1) \text{vec}\{Y_2\}. \quad (8)$$

As usual, the symbol \otimes denotes the Kronecker matrix product. Applying the vec -operator to equation (7) and taking property (8) into account yields

$$\begin{aligned} \text{vec}\{R(U_{k+1}, \Psi_{k+1})\} &= \text{vec}\{R(U_k, \Psi_k)\} + [I_d \otimes (J_{\mu_1} U_k)] \text{vec}\{\Delta \Psi_k\} \\ &\quad + [\Psi_k^T \otimes J_{\mu_1}] \text{vec}\{\Delta U_k\} - [I_d \otimes J_{\mu_2}] \text{vec}\{\Delta U_k\}. \end{aligned} \quad (9)$$

Furthermore, let us define $\Delta U_{s_k} = \sum_{i=1}^{k-1} \Delta U_i$ as the subspace change at the k th iteration step, such that

$$U_k = U_s + \Delta U_{s_k}. \quad (10)$$

With (9) the linearized minimization problem (6) becomes:

$$\min_{\Delta U_k, \Delta \Psi_k} \left\| Z \cdot \begin{bmatrix} \text{vec}\{\Delta \Psi_k\} \\ \text{vec}\{\Delta U_k\} \end{bmatrix} + \begin{bmatrix} \text{vec}\{R(U_k, \Psi_k)\} \\ \kappa \cdot \text{vec}\{\Delta U_{s_k}\} \end{bmatrix} \right\|_2$$

The resulting overdetermined least squares problem can efficiently be solved by a QR decomposition of the $(M + m)d \times (M + d)d$ matrix

$$Z = \begin{bmatrix} I_d \otimes (J_{\mu_1} U_k) & [\Psi_k^T \otimes J_{\mu_1}] - [I_d \otimes J_{\mu_2}] \\ 0 & \kappa \cdot I_{Md} \end{bmatrix}.$$

Notice that Z is block-upper triangular, which can, for instance, be exploited if Givens rotations are used to compute the QR decomposition of Z . Furthermore, one could exploit the fact that $I_d \otimes (J_{\mu_1} U_k)$

is a block-diagonal matrix with identical diagonal blocks. Finally, the vector $[\text{vec}\{\Delta\Psi_k\}^T \text{vec}\{\Delta U_k\}^T]^T$ can be obtained via back substitution [1].

SLS needs an initial estimate of the matrix Ψ in addition to the initial basis of the estimated signal subspace $U_1 = U_s$ for $k = 1$. One simple choice would be the LS solution of (2), i.e., $\Psi_1 = \Psi_{LS}$. In fast adaptive frequency tracking applications, like FSD-ESPRIT, based on the fast subspace decomposition (FSD) [10], or URV ESPRIT, based on the URV decomposition [5], the LS solution of the *current* system of equations should simply be replaced by the solution of (2) computed at the *previous* time step. Thereby, one iteration of SLS improves the “updated” signal subspace estimate significantly, cf. Fig. 6. Thus, the performance of the whole adaptive frequency tracking scheme will be improved.

4. Simulations

Simulations were conducted employing a uniform linear array of $M = 10$ sensors with maximum overlap ($m = 9$) and $\Delta = \lambda/2$. A given trial run involved $N = 20$ snapshots. For SLS the weighting factor α was set to $\alpha = 10$.

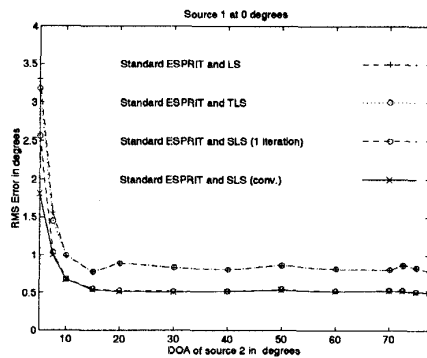


Fig. 1: RMSE (in degrees) of the estimated DOA for source 1 at $\theta_1 = 0^\circ$ as a function of θ_2 ($M = 10$ sensors, $N = 20$, 1000 trial runs, SNR = 0 dB).

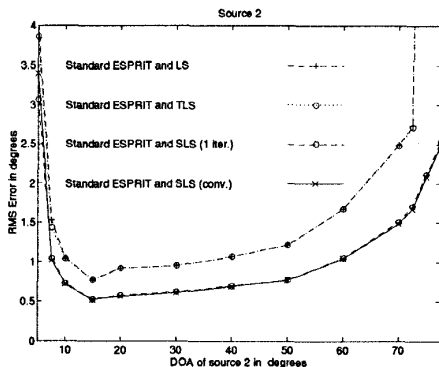


Fig. 2: RMSE (in degrees) of the estimated DOA for source 2 as a function of its DOA ($M = 10$ sensors, $N = 20$, 1000 trial runs, SNR = 0 dB).

4.1. Standard ESPRIT

In the first experiment, $d = 2$ equi-powered, uncorrelated sources were used to evaluate the performance of algorithms based on the *standard ESPRIT* scheme. More specifically, the accuracy of the

DOA estimates resulting from different solutions of equation (2), i.e., LS, TLS, and SLS, was compared. The results were averaged over 1000 trials. While the DOA of the first source was fixed at $\theta_1 = 0^\circ$, the DOA of the second source θ_2 was varied from 5° to 77.5° for an SNR of 0 dB. Figs. 1 and 2 depict the resulting RMS error (in degrees) of the estimated DOA as a function of θ_2 for the first and second source, respectively. Notice first that there is hardly any difference between the LS and TLS solutions, whereas SLS improves the performance of *standard ESPRIT* considerably. Furthermore, SLS has already converged after its first iteration. SLS achieves quite a substantial improvement of the estimation accuracy of the first source, which is almost independent of the value of θ_2 . As expected, the RMS error of the estimated value of θ_2 gets larger for increasing values of θ_2 . Notice, however, that the improvement obtained by SLS with respect to LS and TLS also increases significantly, cf. Fig. 2.

4.2. 1-D Unitary ESPRIT

In the second experiment, LS, TLS, and SLS were used in conjunction with 1-D Unitary ESPRIT [2]. Here, three equi-powered, uncorrelated sources were impinging from $\theta_1 = 0^\circ$, $\theta_2 = 10^\circ$, and $\theta_3 = 20^\circ$. The SNR was varied from -6 dB to 10 dB.

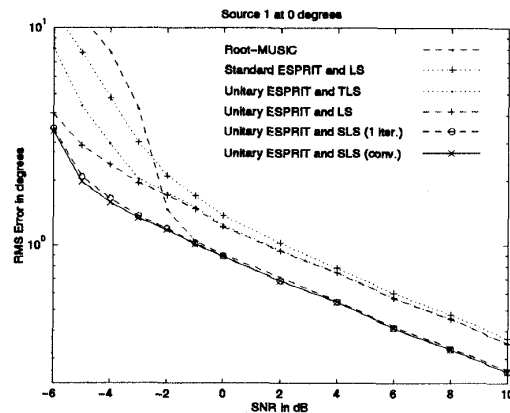


Fig. 3: RMSE (in degrees) of the estimated DOA for source 1 at $\theta_1 = 0^\circ$ as a function of the SNR ($M = 10$ sensors, $N = 20$, 4000 trial runs).

Figs. 3, 4, and 5 show the RMS error (in degrees) of the estimated DOAs for sources 1, 2, and 3, respectively. The results were averaged over 4000 trial runs. For comparative purposes, we have also plotted the performance curves for standard ESPRIT using the LS solution of (2) and Root-MUSIC. Bear in mind that Root-MUSIC is computationally more demanding than ESPRIT-type techniques and that all algorithms based on 1-D Unitary ESPRIT only involve *real-valued* computations. Therefore, algorithms based on Unitary ESPRIT require even less computations than similar algorithms based on standard ESPRIT.

Figs. 3, 4, and 5 show that *Unitary ESPRIT and LS* always outperforms *standard ESPRIT and LS*, especially for low SNRs. For low SNRs, *Unitary ESPRIT and LS* also performs better than *Unitary ESPRIT and TLS*, a somewhat surprising result, since TLS is computationally more expensive than LS. Except for very low SNRs (≤ -5 dB), the SLS solution achieves a constant improvement (on a semi-logarithmic scale) over the LS solution. This improvement is larger for the outer sources at $\theta_1 = 0^\circ$ and $\theta_3 = 20^\circ$ than for the

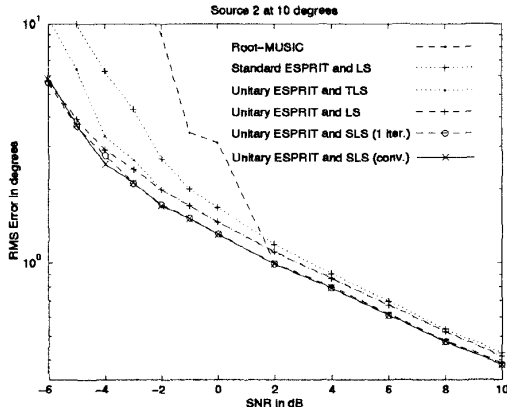


Fig. 4: RMSE (in degrees) of the estimated DOA for source 2 at $\theta_2 = 10^\circ$ as a function of the SNR ($M = 10$ sensors, $N = 20$, 4000 trial runs).

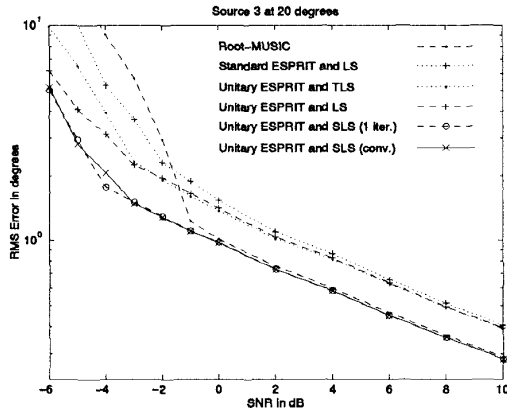


Fig. 5: RMSE (in degrees) of the estimated DOA for source 3 at $\theta_3 = 20^\circ$ as a function of the SNR ($M = 10$ sensors, $N = 20$, 4000 trial runs).

middle source at $\theta_2 = 10^\circ$. Once again, SLS has almost converged after the first iteration. Notice that Root-MUSIC is the worst algorithm for very low SNRs, while it almost attains the performance of *Unitary ESPRIT and SLS* for SNRs that are greater or equal than 2 dB. To evaluate the quality of the updated subspace estimate (10), produced by SLS, Fig. 6 depicts the largest principal angle θ_p between the estimated and the "true" signal subspace (spanned by the columns of \mathbf{A}) as a function of the SNR. Here, the *largest principal angle* θ_p ($0^\circ \leq \theta_p \leq 90^\circ$) between two subspaces spanned by the columns of \mathbf{U}_s and \mathbf{A} is defined as

$$\cos \theta_p = \sigma_{\min}(\mathbf{W}_1^H \mathbf{W}_2), \quad (11)$$

where $\mathbf{W}_1 = \text{orth}(\mathbf{U}_s)$ and $\mathbf{W}_2 = \text{orth}(\mathbf{A})$ are unitary basis matrices for the subspaces and $\sigma_{\min}(\mathbf{Y})$ denotes the smallest singular value of the matrix \mathbf{Y} [1]. Although Unitary ESPRIT already produces a better signal subspace estimate than standard ESPRIT, SLS used in conjunction with Unitary ESPRIT improves this subspace estimate considerably, cf. Fig. 6.

5. Concluding Remarks

Structured least squares (SLS) is a new structure-preserving algorithm, used instead of LS or TLS, to improve the performance of

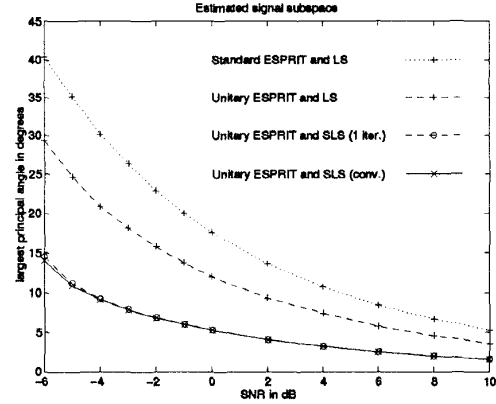


Fig. 6: Largest principal angle θ_p ($0^\circ \leq \theta_p \leq 90^\circ$) between the estimated and the "true" signal subspace (in degrees) as a function of the SNR of the three sources ($M = 10$ sensors, $N = 20$, 4000 trial runs).

ESPRIT-type high-resolution techniques, if overlapping subarray configurations are used. Although SLS was designed as an iterative scheme, simulations indicate that this new algorithm only requires one iteration to achieve convergence. This is also true for the two-dimensional extension of SLS, which can be used in conjunction with 2-D Unitary ESPRIT [3]. Moreover, SLS and 2-D SLS provide efficient ways to improve the performance of adaptive frequency estimation schemes that are based on fast adaptive subspace tracking techniques.

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