Subspace Estimation Using Unitary Schur-Type Methods

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Abstract — **This paper presents efficient Schur-type algorithms for estimating the column space (signal subspace) of a low rank data matrix corrupted by additive noise. Its computational structure and complexity are similar to that of an LQ-decomposition, except for the fact that plane *and* hyperbolic rotations are used. Therefore, they are well suited for a parallel (systolic) implementation. The required rank decision, i.e., an estimate of the number of signals, is automatic, and updating as well as downdating are straightforward. The new scheme computes a matrix of minimal rank which is γ-close to the data matrix in the matrix 2-norm, where γ is a threshold that can be determined from the noise level. Since the resulting approximation error is not minimized, critical scenarios lead to a certain loss of accuracy compared to SVD-based methods. This loss of accuracy is compensated by using *Unitary ESPRIT* in conjunction with the Schur-type subspace estimation scheme. *Unitary ESPRIT* represents a simple way to constrain the estimated phase factors to the unit circle and provides a new reliability test. Due to the special algebraic structure of the problem, all required factorizations can be transformed into decompositions of real-valued matrices of the same size. The advantages of *Unitary ESPRIT* dramatically improve the resulting subspace estimates, such that the performance of *Unitary Schur ESPRIT* is comparable to that of SVD-based methods, at a fraction of the computational cost. Compared to the original Schur method, *Unitary Schur ESPRIT* yields improved subspace estimates with a reduced computational load, since it is formulated in terms of real-valued computations throughout.**

1. Introduction

The estimation of unknown parameters from a given data matrix $\boldsymbol{X} \in \mathbb{C}^{M \times n}$ is encountered in many signal processing applications. Modern high resolution parameter estimation techniques usually consist of two major steps. The first step is the computation of the signal and the noise subspace from a given data matrix. Then, the unknown parameters are extracted from one of the subspaces using algorithms like ESPRIT or MUSIC. The subspaces can be determined via a singular value decomposition (SVD) of the data matrix, since the SVD is known to be the most reliable tool for the subspace separation task. It is, however, computationally expensive and yields more information than necessary to separate the signal from the noise subspace. Therefore, different computationally more efficient algorithms have been presented, which can be interpreted as "approximate SVDs", like the rank revealing QR-decomposition [1], the URV-decomposition [9], SVD-updating algorithms [8], and a recently proposed Schur-type method [4, 11]. The Schur-type method is the cheapest approach in terms of its computational expense and can easily be implemented in a parallel fashion. It is similar to an LQ-decomposition, except for the fact that plane *and* hyperbolic rotations are used. Moreover, updating and downdating are straightforward. It requires, however, the knowledge of a threshold γ. This threshold γ can be estimated from the noise level. A simple formula for γ is given in this paper.

Recently, there have been efforts to improve the subspace based ESPRIT method using the SVD as the basic tool for the subspace separation task (e.g., total least squares ESPRIT or *Unitary ESPRIT* [5]). *Unitary ESPRIT* yields improved parameter estimates at a lower computational cost by taking the unitarity of the phase factors into account. Constraining the phase factors to the unit circle improves the performance significantly, especially if the sources are correlated, cf. section 4. *Unitary ESPRIT* provides a very simple solution to this task and retains an ESPRIT-like structure, except for the fact that it is formulated in terms of real-valued computations throughout. Since the 1D version of *Unitary ESPRIT* is a completely real-valued algorithm, *Unitary ESPRIT* can easily be extended to the two-dimensional (2D) case, yielding efficient closed-form algorithms in element space [6] and reduced dimension DFT beamspace [12] to provide automatically paired source azimuth and elevation angle estimates.

In this paper, *Unitary ESPRIT* is combined with the Schur-type subspace estimation method to compensate a certain loss of accuracy inherent in the computationally efficient Schur-type method. Compared to the Schur-type methods presented in [10, 4, 11], *Unitary Schur ESPRIT* yields improved results at a lower computational cost. Its performance even attains the performance of SVD-based methods. Moreover, *Unitary Schur ESPRIT* outperforms the SVD-based standard ESPRIT algorithm in the case of correlated sources, cf. section 4. Furthermore, *Unitary Schur ESPRIT* requires only real arithmetic and can easily be implemented in a parallel fashion.

2. Schur-Type Methods for Subspace Estimation

If the SVD of the data matrix $\boldsymbol{X} \in \mathbb{C}^{M \times n}$ is given by $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$, an eigenvalue decomposition (EVD) of its Gramian $\boldsymbol{G} = \boldsymbol{X}\boldsymbol{X}^H$ equals $\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^H$, where the eigenvalues of $\boldsymbol{G}$ are the square of the singular values of $\boldsymbol{X}$, i.e., $\lambda_i(\boldsymbol{G}) = \sigma_i^2(\boldsymbol{X}) \geq 0$. Now, suppose the noise level and, therefore, the threshold γ are known. A method for finding a good estimate for γ is proposed in section 3.3. It is easy to show that a spectral shift of the Gramian $\boldsymbol{G}$ by $\gamma^2\boldsymbol{I}_M$ yields

$$\gamma^2\boldsymbol{I}_M - \boldsymbol{G} = \boldsymbol{U}\left(\gamma^2\boldsymbol{I}_M - \boldsymbol{\Sigma}^2\right)\boldsymbol{U}^H \tag{1}$$

and, therefore, its matrix sign function [3] equals

$$\operatorname{sign}\left(\gamma^2\boldsymbol{I}_M - \boldsymbol{G}\right) = \boldsymbol{U}\begin{bmatrix} \boldsymbol{I}_{M-d} & \\ & -\boldsymbol{I}_d \end{bmatrix}\boldsymbol{U}^H. \tag{2}$$

Computing the matrix sign function of $\gamma^2\boldsymbol{I}_M - \boldsymbol{G}$ is sufficient to separate the subspaces, since the column space corresponding to the negative eigenvalues of $\gamma^2\boldsymbol{I}_M - \boldsymbol{G}$ represents an estimate of

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the signal subspace. Using the signature matrix

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_M & \\ & -\boldsymbol{I}_n \end{bmatrix},$$

the "shifted Gramian" can be expressed as

$$\gamma^2 \boldsymbol{I}_M - \boldsymbol{G} = \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix} \boldsymbol{J} \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix}^H.$$

In the sequel, assume that $\boldsymbol{\Theta}$ is $\boldsymbol{J}$-unitary, i.e., $\boldsymbol{J} = \boldsymbol{\Theta}\boldsymbol{J}\boldsymbol{\Theta}^H$. Therefore,

$$\begin{aligned} &\begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix} \boldsymbol{J} \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix}^H \\ &= \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix} \boldsymbol{\Theta} \boldsymbol{J} \boldsymbol{\Theta}^H \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix}^H \\ &= \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \boldsymbol{J} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^H \\ &= \boldsymbol{C}\boldsymbol{C}^H - \boldsymbol{D}\boldsymbol{D}^H, \end{aligned}$$

where $\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix} \boldsymbol{\Theta}$.

A well known method for obtaining efficient algorithms from covariance based algorithms is the use of square root techniques, i.e., working with the Cholesky factor (square root) of the Gramian instead of the Gramian itself. A square root version of (3) can be derived by taking the signature of the columns into account,

$$\begin{matrix} + & - \\ \begin{bmatrix} \gamma \boldsymbol{I}_M & \boldsymbol{X} \end{bmatrix} \end{matrix} \widetilde{\boldsymbol{\Theta}} = \begin{matrix} +/- & +/- \\ \begin{bmatrix} \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \end{matrix} = \begin{matrix} + & - \\ \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \end{matrix} \boldsymbol{P}, \quad (3)$$

$$\text{where} \quad \begin{matrix} & M-d & d \\ \boldsymbol{C} = M & \begin{bmatrix} \boldsymbol{C}' & \boldsymbol{0} \end{bmatrix} \end{matrix}, \quad \begin{matrix} & d & n-d \\ \boldsymbol{D} = M & \begin{bmatrix} \boldsymbol{D}' & \boldsymbol{0} \end{bmatrix} \end{matrix},$$

$\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta}\boldsymbol{P}$, and $\boldsymbol{L}$ is a lower triangular $M \times M$ matrix. $\boldsymbol{P}$ is a permutation matrix sorting the columns according to their signature, such that $\boldsymbol{C}$ and $\boldsymbol{D}$ contain the columns of $\begin{bmatrix} \boldsymbol{L} & \boldsymbol{0} \end{bmatrix}$ with positive and negative signature, respectively. The $\boldsymbol{J}$-unitary matrix $\widetilde{\boldsymbol{\Theta}}$ is composed of circular and hyperbolic plane rotations, where the type of rotation as well as the signature of the resulting columns depends on the signature of the *rotated* columns [4, 11]. Equation (3) represents a hyperbolic $\boldsymbol{L}\boldsymbol{\Theta}$-decomposition [11]. It can also be interpreted as a generalized Schur method, i.e., the extension of the algebraic version of Schur's algorithm for positive definite matrices to the indefinite case [2].

Thus, a very simple estimate of the signal subspace is given by the range of $\boldsymbol{U}_{S1} = \boldsymbol{D}' \in \mathbb{C}^{M \times d}$. For non-critical scenarios (high SNR, well separated signals) the range of $\boldsymbol{U}_{S1}$ is a reliable estimate of the signal subspace. It has been shown [10, 11], that

$$\boldsymbol{U}_{S2} = \boldsymbol{D}' - \boldsymbol{C}' \left(\boldsymbol{\Theta}_{11}^{-1} \boldsymbol{\Theta}_{12} \right)_{11} \quad (4)$$

represents an improved estimate, which should be used in more critical situations and can be computed via a Schur complement formula [4]. Here, $\left(\boldsymbol{\Theta}_{11}^{-1} \boldsymbol{\Theta}_{12} \right)_{11}$ denotes the leading $(M-d) \times d$ block of $\boldsymbol{\Theta}_{11}^{-1} \boldsymbol{\Theta}_{12}$, where $\boldsymbol{\Theta}_{11}$ and $\boldsymbol{\Theta}_{12}$ are the respective blocks of $\boldsymbol{\Theta}$, having dimension $M \times M$ and $M \times n$, respectively. The 2-norm approximants corresponding to $\boldsymbol{U}_{S1}$ and $\boldsymbol{U}_{S2}$ do not necessarily minimize the norm difference [11]. This leads to some performance degradation compared to the "optimal" SVD-based scheme. *Unitary ESPRIT* provides a way to compensate for this loss of accuracy by exploiting additional information inherent in the rotational invariance structure of the signal subspace.

3. Unitary Schur ESPRIT

Since *Unitary ESPRIT* [5] includes forward-backward averaging, all required decomposition, i.e., the hyperbolic $\boldsymbol{L}\boldsymbol{\Theta}$-decomposition as well as the subsequent least squares problem and the final eigendecomposition, can be transformed into real-valued decompositions of the same size. This is achieved by constructing invertible transformations that map centro-Hermitian matrices to real matrices [7]. *Unitary ESPRIT* also provides additional reliability information and yields improved subspace estimates (especially for correlated sources).

3.1. Standard ESPRIT Scenario

Consider the standard *ESPRIT* scenario, i.e., an M-element sensor array composed of m pairs of pairwise identical, but displaced sensors (doublets). Let Δ denote the distance between the two subarrays. Incident on both subarrays are d narrow-band noncoherent planar wavefronts $s_k(t_n)$, $1 \le k \le d \le m$, with common wavelength λ. The d impinging signals are combined to a signal vector $\boldsymbol{s}(t_n)$. In the noiseless case, the array measurements are given by $\boldsymbol{x}(t_n) = \boldsymbol{A}\,\boldsymbol{s}(t_n) \in \mathbb{C}^M$, where every row of the array steering matrix $\boldsymbol{A}$ corresponds to an element of the sensor array. With additive noise, we get $\widetilde{\boldsymbol{x}}(t_n) = \boldsymbol{x}(t_n) + \boldsymbol{n}(t_n)$, where the noise vector $\boldsymbol{n}(t_n)$ is assumed to be spatially white and uncorrelated with the signals. If, however, the spatial covariance matrix of the additive noise $\boldsymbol{R}_{nn}$ is known up to a scalar factor, the identity matrix $\boldsymbol{I}_M$ in (3) should be replaced by a lower triangular matrix $\boldsymbol{L}_n$, such that $\boldsymbol{R}_{nn} = \boldsymbol{L}_n \boldsymbol{L}_n^H$. Notice that this "generalized" Schur method corresponds the Generalized or Quotient SVD (QSVD), which is often used in the case of correlated noise.

A particular subarray configuration can be described by selection matrices, that choose m elements of $\boldsymbol{x}(t_n) \in \mathbb{C}^M$, where $m < M$ is the number of elements in each subarray. Let $\boldsymbol{J}_{\mu 1}$ and $\boldsymbol{J}_{\mu 2}$ be $m \times M$ selection matrices that assign elements of $\boldsymbol{x}(t_n)$ to the subarrays 1 and 2, respectively. The two selection matrices are chosen to be centro-symmetric with respect to one another, i.e.,

$$\boldsymbol{J}_{\mu 2} = \boldsymbol{\Pi}_m \boldsymbol{J}_{\mu 1} \boldsymbol{\Pi}_M, \quad (5)$$

where $\boldsymbol{\Pi}_p$ is the $p \times p$ exchange matrix with ones on its antidiagonal and zeros elsewhere. By collecting $N \ge d$ snapshots from each sensor, $1 \le n \le N$, a measurement matrix $\boldsymbol{X}$ is formed, obeying

$$\begin{bmatrix} \boldsymbol{J}_{\mu 1} \\ \boldsymbol{J}_{\mu 2} \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} \boldsymbol{J}_{\mu 1} \\ \boldsymbol{J}_{\mu 2} \end{bmatrix} \boldsymbol{A}\,\boldsymbol{S} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_1 \boldsymbol{\Phi} \end{bmatrix} \boldsymbol{S}, \quad (6)$$

where $\boldsymbol{S} \in \mathbb{C}^{d \times N}$ is the signal matrix. $\boldsymbol{A}_1 \in \mathbb{C}^{m \times d}$ denotes the steering matrix of the first subarray, while $\boldsymbol{\Phi} = \operatorname{diag}\{\phi_k\}_{k=1}^d$ is a diagonal matrix of the phase delays between the sensor doublets for the d wavefronts. Its diagonal elements, the phase factors ϕ_k, are given by $\phi_k = e^{\mathrm{j}\frac{2\pi}{\lambda}\Delta \sin\theta_k} = e^{\mathrm{j}\mu_k}$, $1 \le k \le d$, where $\theta_k = 0$ corresponds to the direction perpendicular to the direction of the displacement Δ. Equation (6) implies that the measurement matrix $\boldsymbol{X}$ is rank-deficient, namely rank $\boldsymbol{X} = d$.

Thus, if $\widehat{\boldsymbol{U}} \in \mathbb{C}^{M \times d}$ spans the column space of a rank d approximant of the noise-corrupted measurement matrix $\widetilde{\boldsymbol{X}}$, $\boldsymbol{\Phi}$ can be estimated by solving the overdetermined set of equations

$$\boldsymbol{J}_{\mu 1} \widehat{\boldsymbol{U}}\,\boldsymbol{\Psi} \approx \boldsymbol{J}_{\mu 2} \widehat{\boldsymbol{U}} \quad (7)$$

and subsequently computing an eigendecomposition of the (total) least squares solution $\boldsymbol{\Psi} = \boldsymbol{T}\,\boldsymbol{\Phi}\,\boldsymbol{T}^{-1}$.

3.2. Unitary ESPRIT

To derive the completely real-valued implementation of *Unitary ESPRIT*, let us define left Π-real matrices [7, 5], i.e., matrices $\boldsymbol{Q} \in \mathbb{C}^{p \times q}$ satisfying $\boldsymbol{\Pi}_p \overline{\boldsymbol{Q}} = \boldsymbol{Q}$. Throughout this paper, an overbar denotes complex conjugation without transposition. The unitary matrix

$$\boldsymbol{Q}_{2n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \mathrm{j}\boldsymbol{I}_n \\ \boldsymbol{0}^T & \sqrt{2} & \boldsymbol{0}^T \\ \boldsymbol{\Pi}_n & \boldsymbol{0} & -\mathrm{j}\boldsymbol{\Pi}_n \end{bmatrix}, \tag{8}$$

for example, is left Π-real of odd order. A unitary left Π-real matrix of size $2n \times 2n$ is obtained from (8) by dropping its center row and center column. *Unitary ESPRIT* starts by forming the real matrix

$$\mathcal{T}(\widetilde{\boldsymbol{X}}) \triangleq \boldsymbol{Q}_M^H \begin{bmatrix} \widetilde{\boldsymbol{X}} & \boldsymbol{\Pi}_M \overline{\widetilde{\boldsymbol{X}}} \boldsymbol{\Pi}_N \end{bmatrix} \boldsymbol{Q}_{2N}. \tag{9}$$

If $\boldsymbol{Q}$ denotes the left Π-real matrix defined in (8), an efficient computation of $\mathcal{T}(\widetilde{\boldsymbol{X}}) \in \mathbb{R}^{M \times 2N}$ from the data matrix $\widetilde{\boldsymbol{X}}$ only requires $M \times 2N$ real additions [5]. Let $\boldsymbol{U}_S \in \mathbb{R}^{M \times d}$ denote the resulting signal subspace estimate obtained from a rank-revealing factorization of $\mathcal{T}(\widetilde{\boldsymbol{X}})$, cf. section 2.

It was shown in [5] that the overdetermined complex-valued set of equations (7) can be replaced by the following real-valued set of equations

$$\boldsymbol{K}_{\mu 1} \boldsymbol{U}_S \boldsymbol{\Upsilon} \approx \boldsymbol{K}_{\mu 2} \boldsymbol{U}_S, \tag{10}$$

where the real-valued selection matrices

$$\begin{aligned} \boldsymbol{K}_{\mu 1} &= \boldsymbol{Q}_m^H (\boldsymbol{J}_{\mu 1} + \boldsymbol{J}_{\mu 2}) \boldsymbol{Q}_M \\ \boldsymbol{K}_{\mu 2} &= \boldsymbol{Q}_m^H \mathrm{j} (\boldsymbol{J}_{\mu 1} - \boldsymbol{J}_{\mu 2}) \boldsymbol{Q}_M \end{aligned}$$

are sparse. Furthermore, the total least squares (TLS) solution of the complex-valued system (7) $\boldsymbol{\Psi}^{(\mathrm{TLS})}$ and the TLS solution of the real-valued system (10) $\boldsymbol{\Upsilon}^{(\mathrm{TLS})}$ are related via the linear fractional transformation

$$f(x) = -\frac{x - \mathrm{j}}{x + \mathrm{j}}, \tag{11}$$

which is analytic for $x \neq -\mathrm{j}$, namely $\boldsymbol{\Psi}^{(\mathrm{TLS})} = f\left(\boldsymbol{\Upsilon}^{(\mathrm{TLS})}\right)$. To achieve additional computational savings, the TLS solution of (10) $\boldsymbol{\Upsilon}^{(\mathrm{TLS})}$ might be replaced by its least squares (LS) solution, which is a simplification of the algorithm that does not affect the accuracy of the resulting estimates [5].

Let $\boldsymbol{\Upsilon} = \boldsymbol{T} \boldsymbol{\Omega}_\mu \boldsymbol{T}^{-1}$ be an eigendecomposition of the real matrix $\boldsymbol{\Upsilon}$, i.e., the LS or TLS solution of (10). Then, the eigenvalues of $\boldsymbol{\Psi}$ can be obtained through the same linear fractional transformation, that is

$$\boldsymbol{\Phi} = f(\boldsymbol{\Omega}_\mu) \quad \text{with} \quad \boldsymbol{\Omega}_\mu = \operatorname{diag}\{\omega_{\mu_i}\}_{i=1}^d \tag{12}$$

and $\omega_{\mu_i} \neq -\mathrm{j}$. The associated eigenvectors of $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Psi}$ are the same. Notice also that solving (12) for ω_{μ_i} yields

$$\omega_{\mu_i} = \frac{1}{\mathrm{j}} \frac{e^{\mathrm{j}\mu_i} - 1}{e^{\mathrm{j}\mu_i} + 1} = \tan\left(\frac{\mu_i}{2}\right), \tag{13}$$

which provides an easy way to determine the DOAs from the eigenvalues of the real matrix $\boldsymbol{\Upsilon}$.

The fact that the eigenvalues of a real matrix can either be real or occur in complex conjugate pairs gives rise to a new *reliability test*, provided by *Unitary ESPRIT* without the need for additional computations [5]. If the eigenvalues of $\boldsymbol{\Upsilon}$ are real, all the $f(\omega_{\mu_i})$ have unit magnitude. Therefore, they provide a reliable estimate of the phase factors $e^{\mathrm{j}\mu_i}$, cf. (12). Otherwise, i.e., if the ω_{μ_i} occur in complex conjugate pairs, the *Unitary ESPRIT reliability test* has "failed", and the algorithm has to be restarted with more or more reliable measurements.

3.3. Choosing the Threshold γ

To illustrate the proper selection of the parameter γ, consider the *Unitary ESPRIT* scenario and assume ergodicity. Let, furthermore, the SVD of the extended data matrix $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{\Pi}_M \overline{\boldsymbol{X}} \end{bmatrix}$ be equal to $\boldsymbol{Z} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^H$, where $\boldsymbol{\Sigma} = \operatorname{diag}\{\sigma_i\}_{i=1}^M$ and $\sigma_i \equiv 0$, $\forall\, d+1 \leq i \leq M$, since $\boldsymbol{Z}$ represents the noiseless data matrix. Without loss of generality, assume also that the additive noise is uncorrelated with variance σ^2. Then, the covariance matrix of the noise-corrupted measurements $\widetilde{\boldsymbol{Z}}$ takes the form

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{2N} \widetilde{\boldsymbol{Z}} \widetilde{\boldsymbol{Z}}^H &= \lim_{N \to \infty} \frac{1}{2N} \boldsymbol{Z} \boldsymbol{Z}^H + \sigma^2 \boldsymbol{I}_M \\ &= \lim_{N \to \infty} \frac{1}{2N} \boldsymbol{U} \left(\boldsymbol{\Sigma}^2 + 2N\sigma^2 \boldsymbol{I}_M \right) \boldsymbol{U}^H. \end{aligned}$$

Therefore, for large values of N, the SVD of $\widetilde{\boldsymbol{Z}}$ approximately equals $\widetilde{\boldsymbol{Z}} \approx \boldsymbol{U} \left(\boldsymbol{\Sigma}^2 + 2N\sigma^2 \boldsymbol{I}_M \right)^{\frac{1}{2}} \widetilde{\boldsymbol{V}}^H$ for some unitary matrix $\widetilde{\boldsymbol{V}}$. Thus, the $M - d$ smallest singular values of $\widetilde{\boldsymbol{Z}}$ are approximately given by $\sigma\sqrt{2N}$. Since the left singular vectors of $\boldsymbol{Z}$ remain unchanged, the corresponding signal subspace can be determined from the first d columns of $\boldsymbol{U}$. In this case, an appropriate value for the threshold γ of the Schur-type subspace estimation method is chosen according to $\gamma > \sigma\sqrt{2N}$.

4. Simulations

Assume that three equi-powered signals are impinging on a uniform linear array (ULA). Their DOAs are $\theta_1 = 10°$, $\theta_2 = 20°$, and $\theta_3 = 30°$. The presented results are averaged over 500 trials, while a given trial run involves $N = 20$ snapshots.

In the first experiment, a ULA with $M = 9$ element is used, the three sources are uncorrelated, and the SNR is varied from 0 dB to 30 dB in steps of 5 dB. Fig. 1 shows the largest principle angle between the estimated and the "true" signal subspace as a function of the SNR. All results are plotted for the two signal subspace estimates that are based on the Schur method, i.e., $\boldsymbol{U}_{S1}$ (dotted line $\cdots$) as well as $\boldsymbol{U}_{S2}$ (dashed line - - -). They are compared to the more expensive SVD–based signal subspace estimate (solid line —). Notice that the both Schur methods estimate the number of signals d automatically, while d is assumed to be known for the SVD implementation. For each of the three methods, *Unitary ESPRIT* ("U") is compared to the standard ESPRIT algorithm ("S"). Recall that *Unitary ESPRIT* reduces the computational load of the standard ESPRIT algorithm, since it is formulated in terms of real computations throughout. Furthermore, *Unitary ESPRIT* provides additional reliability information. Its superior performance is demonstrated by the fact that, throughout all experiments, the top curves correspond to the standard ESPRIT algorithm while the bottom curves correspond to *Unitary ESPRIT*. The RMS error of the estimated angles, which corresponds to the largest principle angles in Fig. 1, is depicted in Fig. 2. Notice that *Unitary Schur ESPRIT* based on the "improved" estimate $\boldsymbol{U}_{S2}$ performs as well as *Unitary SVD ESPRIT* for moderate and high signal to noise ratios.

Then, sources 1 and 2 are correlated and the correlation coefficient ρ_{12} is varied from 0.0 to 1.0. Fig. 3 shows the largest

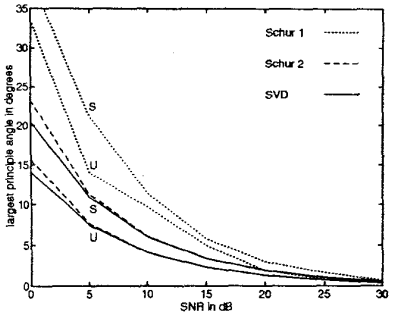

Fig. 1: Largest principle angle θ_p between the estimated and the "true" signal subspace ($0^\circ \leq \theta_p \leq 90^\circ$) as a function of the SNR for $\theta_1 = 10^\circ$, $\theta_2 = 20^\circ$, and $\theta_3 = 30^\circ$ ($M = 9$ sensors, $N = 20$, 500 trial runs). S: Standard version, U: Unitary version.

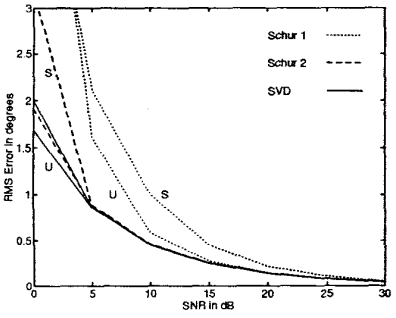

Fig. 2: RMS error of the estimated angles as a function of the SNR for $\theta_1 = 10^\circ$, $\theta_2 = 20^\circ$, and $\theta_3 = 30^\circ$ ($M = 9$ sensors, $N = 20$, 500 trial runs), S: Standard version, U: Unitary version.

principle angle between the estimated and the "true" signal subspace as a function of ρ_{12}, while Fig. 4 depicts the RMS error of the estimated angles as a function of ρ_{12}. It can be seen that the performance of *Unitary ESPRIT* is independent of the correlation between signals 1 and 2, whereas the performance of the standard ESPRIT methods deteriorates dramatically with increasing ρ_{12}. Notice also that the improved (second) version of *Unitary Schur ESPRIT* yields better results than the much more expensive SVD-based standard ESPRIT algorithm. It has the same performance as the SVD-based *Unitary ESPRIT* scheme.

5. Concluding Remarks

A new ESPRIT-like parameter estimation scheme, *Unitary Schur ESPRIT*, was presented in the paper. It uses an efficient Schur-type algorithm, namely a hyperbolic $L\Theta$-decomposition, for the computation of a signal subspace estimate. Moreover, the number of signals d is estimated automatically. The performance of the original Schur-type method can be improved significantly by combining it with *Unitary ESPRIT*, yielding not only improved estimation accuracy, but also a completely real-valued algorithm. Furthermore, *Unitary Schur ESPRIT* can easily be implemented in a sequential or a parallel fashion, since only real-valued plane or hyperbolic rotations are required. Simulations have shown that *Unitary Schur ESPRIT* achieves a performance that is comparable to SVD-based methods with a tremendous reduction of the computational complexity.

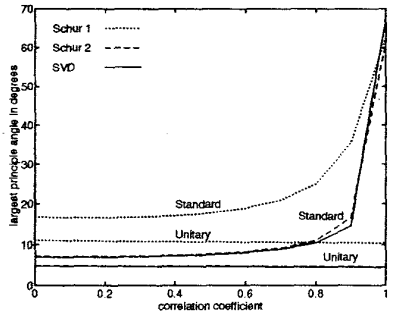

Fig. 3: Largest principle angle θ_p between the estimated and the "true" signal subspace ($0^\circ \leq \theta_p \leq 90^\circ$) as a function of the correlation coefficient ρ_{12} for $\theta_1 = 10^\circ$, $\theta_2 = 20^\circ$, and $\theta_3 = 30^\circ$ ($M = 8$ sensors, SNR = 10 dB, $N = 20$, 500 trial runs).

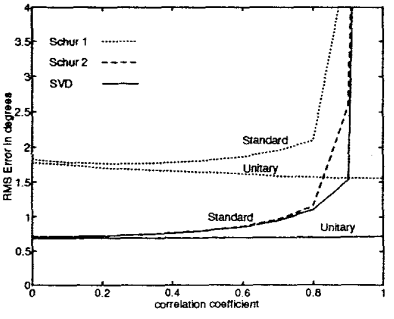

Fig. 4: RMS error of the estimated angles as a function of the correlation coefficient ρ_{12} for $\theta_1 = 10^\circ$, $\theta_2 = 20^\circ$, and $\theta_3 = 30^\circ$ ($M = 8$ sensors, SNR = 10 dB, $N = 20$, 500 trial runs).

References

[1] T. F. Chan, "Rank revealing QR factorization", *Linear Algebra and its Applications*, vol. 89, pp. 67–82, 1987.

[2] K. Diepold and R. Pauli, "A recursive algorithm for lossless embedding of nonpassive systems", in *Challenges of a Generalized System Theory*, P. Dewilde, M. Kaashoek, and M. Verhaegen, Eds., pp. 209–222, North Holland, 1993.

[3] J. D. Gardiner and A. J. Laub, "A generalization of the matrix-sign-function for algebraic Ricatti equations", *Int. J. Control*, vol. 44, no. 3, pp. 823–832, 1986.

[4] J. Götze and A.-J. van der Veen, "On-line subspace estimation using a Schur-type method", *IEEE Trans. on Signal Processing*, 1993, submitted for publication.

[5] M. Haardt and J. A. Nossek, "Unitary ESPRIT: How to obtain increased estimation accuracy with a reduced computational burden", *IEEE Trans. Signal Processing*, vol. 43, May 1995, scheduled to appear.

[6] M. Haardt, M. D. Zoltowski, C. P. Mathews, and J. A. Nossek, "2D Unitary ESPRIT for efficient 2D parameter estimation", in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Detroit, MI, May 1995.

[7] A. Lee, "Centrohermitian and skew-centrohermitian matrices", *Linear Algebra and its Applications*, vol. 29, pp. 205–210, 1980.

[8] M. Moonen, P. van Dooren, and J. Vandewalle, "A singular value decomposition updating algorithm for subspace tracking", *SIAM J. Matrix Anal. Appl.*, vol. 13, pp. 1015–1038, 1992.

[9] G. W. Stewart, "An updating algorithm for subspace tracking", *IEEE Trans. on Signal Processing*, vol. 40, pp. 1535–1541, 1992.

[10] A.-J. van der Veen, "A Schur method for low-rank matrix approximation", *SIAM J. Matrix Anal. Appl.*, 1994, accepted for publication.

[11] A.-J. van der Veen and J. Götze, "On-line subspace estimation using a generalized Schur method", in *Proc. 7th IEEE SP Workshop on Statistical Signal & Array Processing*, pp. 87–90, Quebec City, Canada, June 1994.

[12] M. D. Zoltowski, M. Haardt, and C. P. Mathews, "Closed-form 2D angle estimation with rectangular arrays via DFT beamspace ESPRIT", in *Proc. 28th Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Nov. 1994.