MULTIWAVELET TRANSFORMS BASED ON SEVERAL SCALING FUNCTIONS

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ABSTRACT

An algebraic method for the design of discrete wavelet transforms based on several scaling functions is presented. Solving systems of partly nonlinear equations is necessary to compute the discrete coefficients. Wavelet transforms based on several scaling functions enable properties that are impossible in the single-wavelet case. Wavelet transforms based on several scaling functions can also be designed wavelet-like, what leads to a better approximation of the continuous bases. In this paper we show how to construct the discrete wavelet transforms based on several scaling functions with the algebraic design method and discuss the properties of the resulting wavelet bases.

1. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in many application fields, e.g. signal processing [4], solving differential and integral equations [3]. Different variations of wavelet bases (orthogonal, biorthogonal, multiwavelets) have been presented and the design of the corresponding wavelet and scaling functions has been adressed [5, 6, 8, 10]. In [1] a purely algebraic approach to the design of these discrete wavelet transforms was taken, while the different properties of the wavelet bases were discussed with respect to orthogonality, approximation properties and symmetry. In this paper we extend the algebraic design method to multiwavelet transforms based on several scaling functions. Multiwavelets using several scaling functions enable to fullfil properties, that are impossible in the single-wavelet case. Using for example 2 scaling functions and 2 wavelets enables orthogonality and symmetry at the same time [11]. Furthermore it is possible to construct nonoverlapping bases with arbitrary approximation order (not possible with one scaling function [1, 7]).
An alternative to the standard wavelet transform is the use of wavelet-like transforms. Those multiwavelet-like transforms are mostly based on as many scaling functions as wavelets [2, 3]. They are called multiwavelet-like, since the wavelet coefficients used in the different stages of the wavelet transform vary from stage to stage.

All these wavelet transforms can be computed with an algebraic design method by solving systems of partly nonlinear equations. How to gain the discrete equations for the algebraic design of the multiwavelet transforms based on several scaling functions from the properties of the continuous basis functions is shown in section 2. Designing multiwavelet-like transforms requires a different basis matrix for each stage of transform. How to compute these different stages, and the differences to standard wavelet transforms (these use the same coefficients for all stages) is discussed in section 3.

2. MULTIWAVELET TRANSFORMS BASED ON SEVERAL SCALING FUNCTIONS

The transform matrix $\mathbf{U}$ of a discrete wavelet transform is the product of the different stages $\mathbf{U}_j$, as it is shown in [1]: $\mathbf{U} = \mathbf{U}_l \dots \mathbf{U}_2\mathbf{U}_1$. Each stage of transform is composed out of the basis matrix $\mathbf{W}$, which can be divided in an upper part $\mathbf{W}^U$ representing the scaling functions (the number of rows of $\mathbf{W}^U$ is equal to the amount of scaling functions that are used) and a lower part $\mathbf{W}^L$ representing the wavelet functions (the number of rows of $\mathbf{W}^L$ is equal to the amount of wavelets that are used). For convenience the algebraic design method is presented for an example with 2 scaling functions and 2 wavelets, i.e. $\mathbf{W}$ is of size 4xn and $\mathbf{W}^L$ and $\mathbf{W}^U$ are of size 2xn. The matrix $\mathbf{W}$ can be divided into h matrices $\mathbf{A}_\nu$ of size 4x4, with n=4h. The coefficients of the rows of $\mathbf{W}$ are named a_i, b_i, c_i, d_i as it is shown in the following equation:

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}^U \\ \mathbf{W}^L \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \dots & a_{4h-1} \\ b_0 & b_1 & \dots & b_{4h-1} \\ c_0 & c_1 & \dots & c_{4h-1} \\ d_0 & d_1 & \dots & d_{4h-1} \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_h \end{pmatrix};$$

Φ_1 and Φ_2 as well as the multiwavelets Ψ_1 and Ψ_2 can be represented by a linear combination of dilated and translated versions of the scaling functions Φ_1 and Φ_2. The coefficients with even index, namely $a_{2k}, b_{2k}, c_{2k}, d_{2k}$ belong to Φ_1, the coefficients with odd index, namely a_{2k+1}, b_{2k+1}, c_{2k+1}, d_{2k+1} belong to the scaling function Φ_2.

$$\Phi_1(t) = \sum_{k=0}^{2h-1} a_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} a_{2k+1}\Phi_2(2t-k);$$

$$\Phi_2(t) = \sum_{k=0}^{2h-1} b_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} b_{2k+1}\Phi_2(2t-k);$$

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$$\Psi_1(t) = \sum_{k=0}^{2h-1} c_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} c_{2k+1}\Phi_2(2t-k);$$

$$\Psi_2(t) = \sum_{k=0}^{2h-1} d_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} d_{2k+1}\Phi_2(2t-k);$$

Orthogonality

A sufficient condition for orthogonality of the transform is, that the matrix $\mathbf{W}$ fullfils the orthogonality and the shifted orthogonality conditions:

$$\mathbf{W}\mathbf{W}^T = \mathbf{I}; \quad \sum_{i=1}^{j} \mathbf{A}_i \mathbf{A}_{h+i-j} = \mathbf{0}, \quad j = 1, 2 \ldots l-1;$$

Approximation

The continuous wavelet functions have to fullfil the equations of vanishing moments for a maximum approximation order p.

$$0 = \int_{-\infty}^{+\infty} t^j \Psi_1(t)\,dt; \quad 0 = \int_{-\infty}^{+\infty} t^j \Psi_2(t)\,dt; \quad 0 \le j \le p.$$

In order to formulate these equations quite easily, the moments of the scaling functions Φ_1 and Φ_2 are computed.

$$I_j = \int_{-\infty}^{+\infty} t^j \Phi_1(t)\,dt; \quad J_j = \int_{-\infty}^{+\infty} t^j \Phi_2(t)\,dt; \quad 0 \le j \le p.$$

The additional parameters I_j and J_j appear in the system of equations and have influence on the smoothness of the generated bases.

The most important step of the algebraic design method is the conversion of the conditions for the continuous functions to equations for the discrete coefficients of the matrix $\mathbf{W}$. We proceed from the equation of vanishing moments and insert the dilation equations. The result is a set of equations for the coefficients c_i of the matrix $\mathbf{W}$ and the parameters I_j and J_j.

$$0 = \int_{-\infty}^{+\infty} t^j \Psi_1(t)\,dt, \qquad j = 0, \ldots p-1$$

$$0 = \int_{-\infty}^{+\infty} t^j \left(\sum_{k=0}^{2h-1} c_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} c_{2k+1}\Phi_2(2t-k) \right) dt$$

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k c_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k c_{2k+1} k^{j-r}$$

The algebraic design method yields similar equations for Ψ_2 (d_i), Φ_1 (a_i) and Φ_2 (b_i). The moments of Φ_1 and Φ_2 do not vanish, what leads to the parameters I_j and J_j.

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k d_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k d_{2k+1} k^{j-r}$$

$$2^{j+1} I_j = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k a_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k a_{2k+1} k^{j-r}$$

$$2^{j+1} J_j = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k b_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k b_{2k+1} k^{j-r}$$

Those approximation equations together with the orthogonality conditions have to be solved to design the multiwavelets and the corresponding multiscaling functions.

An advantage of wavelet transforms based on several scaling functions is, that properties can be fullfilled, which are absolutely impossible with single-wavelet transforms. Strang has shown [11], that it is possible to compute an orthogonal multiwavelet transform with 2 wavelets and 2 scaling functions fullfilling the property of symmetry. For single-wavelet bases it is only possible to compute symmetric wavelets by releasing the orthogonality [4, 6]. The example of Strang, that offers symmetry and orthogonality, is a special case for an approximation order p=2, that has not yet been generalized. The algebraic design method enables the computation of Strang's multiwavelets by solving a system of partly nonlinear equations after formulation of the marginal conditions of symmetry. Figure 1 shows the 2 scaling functions (upper row) and the 2 multiwavelets (lower row) to the computed orthogonal matrix $\mathbf{W}_S$, that includes the symmetric rows. Plotted are the corresponding rows of the matrix $\mathbf{U}$. As these plots only approximate the continuous functions, they show some discontinuities. A possibility to avoid these small saw teeth is discussed in the next section.

$$\mathbf{W}_S = \begin{pmatrix} 6\sqrt{2} & 16 & 6\sqrt{2} & 0 & 0 & 0 & 0 \\ -1 & -3\sqrt{2} & 9 & 10\sqrt{2} & 9 & -3\sqrt{2} & -1 \\ -1 & -3\sqrt{2} & 9 & -10\sqrt{2} & 9 & -3\sqrt{2} & -1 \\ \sqrt{2} & 6 & -9\sqrt{2} & 0 & 9\sqrt{2} & -6 & -\sqrt{2} \end{pmatrix} /20$$

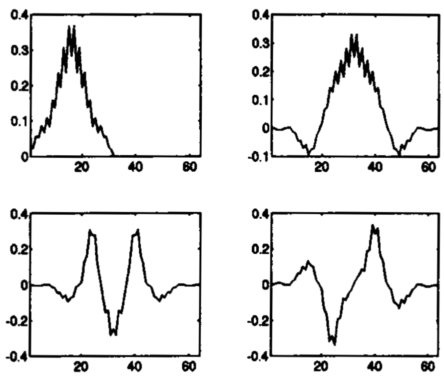

Figure 1. Plots of the transform matrix based on Strang's bases (p=2)

Another restriction of single-wavelets is, that it is impossible to compute nonoverlapping bases of approximation order p>1 (Haar basis) [1]. Uncoupled wavelet bases find application in solving differential and integral equations [3]. Using more than only one scaling functions offers the possibility of generating uncoupled wavelets of any approximation order. For an approximation order p, p scaling functions and p wavelets are needed. The algebraic design yields a basis matrix $\mathbf{W}$ of size $2p$x$2p$. Remarkable is the fact, that the multiwavelets can be chosen symmetric, while the multiscaling functions cannot. This is shown for p=3, i.e. 3 scaling functions (lower row) and 3 multiwavelets (upper row) in Figure 2.

3. WAVELET-LIKE TRANSFORMS BASED ON SEVERAL SCALING FUNCTIONS

The only nonoverlapping wavelets known up to now have been designed by Alpert et al. [2, 3]. Having the same

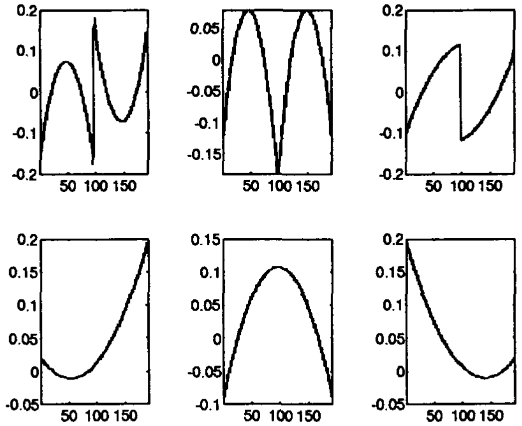

Figure 2. Uncoupled Multiwavelet transform of p=3

areas of application and the same possibilities of implementation, as the uncoupled wavelets of the previous section, there is the fact, that these bases are only wavelet-like. The difference between a wavelet transform and a wavelet-like transform is, that wavelet-like transforms need a new basis matrix $\mathbf{W}_i$ for each stage i of the transform.

The algebraic design of these wavelet-like basis matrices differs a little bit from the computation of the previous wavelet bases. As Alpert's bases are nonoverlapping, orthogonality of $\mathbf{W}_i$ is ensured without fullfilling any shifted orthogonality conditions, i.e.

$$\mathbf{W}_i\mathbf{W}_i^T = \mathbf{I}$$

suffice for orthogonality. The ultimate transform matrix $\mathbf{U}$ of Alpert [2, 3] has only symmetric rows, so we have to choose $\mathbf{W}_1$ symmetric. The odd rows are symmetic and the even rows are antisymmetric. For convenience we restrict our presentation to p=2, i.e. $\mathbf{W}_1$ is

$$\mathbf{W}_1 = \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_1 & a_0 \\ b_0 & b_1 & -b_1 & -b_0 \\ c_0 & c_1 & c_1 & c_0 \\ d_0 & d_1 & -d_1 & -d_0 \end{pmatrix}.$$

The multiwavelets have to fullfil approximation criteria.

$$0 = \sum_i i^j c_i \qquad 0 = \sum_i i^j d_i \qquad j = 0, 1, \ldots, p-1;$$

The moments of the scaling functions lead to the parameters I_j and J_j, while in order to get a definite solution the ith row of Alpert's start matrix $\mathbf{W}_1$ fullfills $i-1$ approximation equations, what leads to $J_0 = 0$ in our example.

$$I_j = \sum_i i^j a_i \qquad J_j = \sum_i i^j b_i \qquad j = 0, 1, \ldots, p;$$

The matrices $\mathbf{W}_i$ $(2 \le i \le l)$ show no symmetry, they only guarantee symmetry of the rows of the ultimate $\mathbf{U}$, and therefore have a definite form. We restrict our considerations to $\mathbf{W}_2$:

$$\mathbf{W}_2 = \begin{pmatrix} e_i \\ f_i \\ g_i \\ h_i \end{pmatrix} = \begin{pmatrix} e_0 & e_1 & e_0 & -e_1 \\ f_0 & f_1 & -f_0 & f_1 \\ g_0 & g_1 & g_0 & -g_1 \\ h_0 & h_1 & -h_0 & h_1 \end{pmatrix}$$

Also, the matrices $\mathbf{W}_i$ fullfil no approximation criteria, they only guarantee, that the rows of $\mathbf{U}$ fullfil the approximation equations. The matrix $\mathbf{W}_2$ is computed with the algebraic design method in the same way as in section 2. The only difference is, that I_j and J_j (j=0,1;) are no more free parameters, but predetermined by the startmatrix $\mathbf{W}_1$.

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k g_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k g_{2k+1} k^{j-r}$$

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k h_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k h_{2k+1} k^{j-r}$$

The matrices $\mathbf{W}_{i\geq 3}$ are computed in the same manner as $\mathbf{W}_2$. Of course, since we have to compute all the matrices $\mathbf{W}_i$, many more equations have to be solved for these wavelet-like transforms than for standard transforms. The matrices $\mathbf{W}_i$, however, converge rapidly with increasing i, and the complexity of unknowns is reduced to one matrix $\mathbf{W}_1$ and 1 or 2 further matrices $\mathbf{W}_i$ [1]. Figure 3 shows the wavelet-like bases for p=3. Plotted are the last rows of the matrix $\mathbf{U}$ that correspond to the 3 wavelets (upper row) and to the 3 scaling functions (lower row).

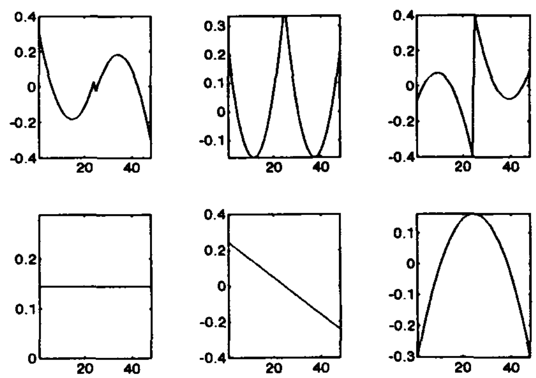

Figure 3. Alpert's scaling functions and wavelets for p

Remark: Alpert's scaling functions are well known as Legendre polynomials in literature. He computes his nonoverlapping bases not by the algebraic design method. The matrix $\mathbf{W}_1$ is computed by a QR decomposition of the moment matrix [2, 3, 9]. Clearly, the QR decomposition with its linear operations can be computed with less efforts than solving a system of partly nonlinear equations. The algebraic design, however, is a general tool for computing all locally supported bases (including Alpert's). □

Finally we want to adress why it makes sense to use wavelet-like transforms, when standard wavelet transforms can be done with less effort, i.e. only one basis matrix $\mathbf{W}$ for all stages. In the case of a multiwavelet transform based on several scaling functions, the discrete coefficients of the dilation equations do not always approximate the continuous bases. Though the moments of the continuous wavelet functions vanish, the discrete wavelet coefficients do not fullfil these approximation equations (see Strang's bases, where adding the coefficients of the third row of $\mathbf{W}_S$, the sum differs from zero). In the case of a multiwavelet-like transform, the matrix $\mathbf{W}_1$, used as a start matrix (1st stage of transform), approximates the continuous bases. Therefore, the discrete wavelet coefficients are chosen in order to fullfill approximation equations. For the higher stages again the matrices $\mathbf{W}_i$ are needed to transform the bases from the coarse to the next finer resolution (dilation equation). The matrices $\mathbf{W}_i$ do not have to fullfil approximation criteria, they only have to conserve them. As the coefficients of $\mathbf{W}_i$ appear in the dilation equations, this would cause only one matrix $W_{i=2}$, but as the coefficients of $\mathbf{W}_1$ only approximate the continuous functions, and as this approximation is getting better with each stage, a matrix $\mathbf{W}_i$ is needed for each stage. Therefore, it is also clear, that the matrices $\mathbf{W}_i$ converge quite fast.

The wavelet transform based on Strang can be modified, in order to design a 'Strang-like' wavelet transform. The structure of the nonzero elements of the matrices $\mathbf{W}_1$ and $\mathbf{W}_i$ stays the same. The third and forth row of $\mathbf{W}_1$ representing the multiwavelets have to fullfil approximation equations.

$$\sum i^j c_i = 0 \qquad \sum i^j d_i = 0 \qquad j = 0, 1;$$

Then the matrices $\mathbf{W}_{i \geq 2}$ preserve the symmetry and the approximation properties. Comparing the matrix $\mathbf{W}_2$ with Strang's basis matrix $\mathbf{W}_S$ shows only small differences, what is coherent. The matrices $\mathbf{W}_i$ converge to W_S with increasing i. The Strang-like discrete wavelet transform is based on the same continuous basis functions as the standard transform computed with $\mathbf{W}_S$. In the Strang-like case not only the continuous wavelet functions fullfil approximation criteria, but also each row of the discrete transform matrix. Therefore, the discontinuities (Figure 3) can be avoided. Figure 4 shows the plots of the the rows of the Strang-like transform matrix $\mathbf{U}$. The Strang-like transform converges to the exact continuous functions for ∞ stages.

4. CONCLUSION

In this paper we have extended the algebraic design method to multiwavelet transforms based on several scaling functions. At first the properties of the continuous functions Φ_i and Ψ_i are formulated. Inserting the dilation equations enables to convert these equations to equations for discrete coefficients. Solving the resulting system of partly nonlinear equations yields these discrete coefficients and the discrete transform matrix can be composed. Using several scaling functions enables to compute orthogonal, overlapping and nonoverlapping bases, while also in the nonoverlapping case symmetric bases can be generated, what is impossible in the single-wavelet case. Using several scaling functions the discrete transform can be designed wavelet-like, what leads to a better approximation of the continuous basis functions.

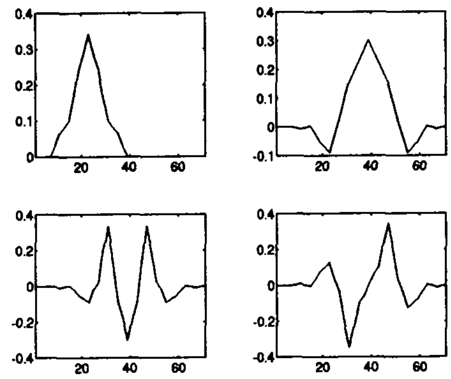

Figure 4. Plots based on a Strang-like transform

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