Technische Universität München
Zentrum Mathematik
HVB-Stiftungsinstitut für Finanzmathematik

# On Utility-Based Investment, Pricing and Hedging in Incomplete Markets

Johannes Muhle-Karbe

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

| | | |
|---|---|---|
| Vorsitzender: | | Univ.-Prof. Dr. Rudi Zagst |
| Prüfer der Dissertation: | 1. | Univ.-Prof. Dr. Jan Kallsen<br>Christian-Albrechts-Universität zu Kiel |
| | 2. | Prof. Dr. Fred Espen Benth<br>Universitet i Oslo, Norwegen<br>(schriftliche Beurteilung) |
| | 3. | Univ.-Prof. Dr. Claudia Klüppelberg |

Die Dissertation wurde am 25.11.2008 bei der Technischen Universität eingereicht und durch die Fakultät für Mathematik am 10.04.2009 angenommen.

# Zusammenfassung

Diese Arbeit beschäftigt sich mit rationalen Investoren, die in unvollständigen Märkten ihren Erwartungsnutzen maximieren.

In Teil I betrachten wir Unvollständigkeit aufgrund von Sprüngen bzw. stochastischer Volatilität. Mithilfe von Martingalmethoden bestimmen wir zunächst in einer Vielzahl verschiedener Modelle optimale Handelsstrategien bzgl. Potenznutzenfunktionen. Weiterhin zeigen wir wie lineare Nährungen nutzenbasierter Preise und Absicherungsstrategien als Lösung eines quadratischen Hedge-Problems unter einem geeigneten Maß bestimmt werden können. Angewandt auf affine Volatilitätsmodelle ergeben sich daraufhin semi-explizite Formeln.

In Teil II behandeln wir Unvollständigkeit aufgrund von proportionalen Transaktionskosten. Wir zeigen, dass in diskreten Modellen immer ein Schattenpreis existiert, welcher innerhalb der Bid-Ask Preise des ursprünglichen Modells liegt und zum gleichen maximalen Erwartungsnutzen führt. Anschließend erläutern wir wie das klassische Merton Problem mit Transaktionskosten durch die simultane Berechnung von Schattenpreis und optimaler Strategie gelöst werden kann.

# Abstract

This thesis deals with rational investors who maximize their expected utility in incomplete markets.

In Part I, we consider models where incompleteness is induced by jumps and stochastic volatility. Using martingale methods we determine optimal investment strategies for power utility in a wide class of different models. Moreover, we show how first-order approximations of utility-based prices and hedging strategies can be computed by solving a quadratic hedging problem under a suitable measure. This representation result is then applied to affine stochastic volatility models leading to semi-explicit solutions.

In Part II, we deal with incompleteness due to proportional transaction costs. In finite discrete time we establish that there always exists a shadow price process, which lies within the bid-ask bounds of the original market with transaction costs and leads to the same maximal expected utility. We then show that this idea can also be used in actual computations. This is done by reconsidering the classical Merton problem with transaction costs and solving it by computing the shadow price and the optimal strategy simultaneously.

# Acknowledgements

Writing this thesis proved to be an amazing experience. It is therefore a great pleasure for me to thank the people who contributed to it.

First and foremost, I sincerely thank my advisor Prof. Dr. Jan Kallsen for all his encouragement and support over the last years. I benefited immensely from his candid advice as well as from numerous discussions with him. I very much appreciate what he has taught me about research in general and Mathematical Finance in particular.

I am also very grateful to Prof. Dr. Claudia Klüppelberg and Prof. Fred Espen Benth for accepting to act as referees for this thesis.

Moreover, I thank Prof. Dr. Rudi Zagst for the excellent working conditions at the HVB Stiftungsinstitut für Finanzmathematik. I am especially grateful for the opportunity to stay after Jan Kallsen moved to Kiel. It has also been a privilege to be able to attend conferences all over Europe to present my work.

I benefited a lot from brainstorming ideas with different people. In no particular order, I thank Jan Mai, Dr. Matthias Scherer, Dr. Robert Stelzer and Bernhard Vesenmayer, to whom I am also grateful for proofreading the manuscript. I also thank Dr. Arnd Pauwels for his outstanding exercise classes, which played a big part in introducing me to stochastic calculus. Special thanks are due to Richard Vierthauer for innumerable discussions as well as the very enjoyable and fruitful joint work that lead to Chapter 6 of this thesis.

The last two years would not have been as much fun as they have been, if it wasn't for the cast of people assembled at the HVB-Stiftungsinstitut. It has been a pleasure to work in this great team and for that I warmly thank all of them.

I would also like to express my gratitude to Karin Höhl and Prof. Dr. Josef Steinebach for enkindling my interest in probability theory and Mathematical Finance almost ten years ago.

Very special thanks go to my mother Dorothea for instilling so much curiosity in me and always believing I would find my own way.

Last but by no means least, I thank my girlfriend Sarah for her loving support and the great times we have shared all over the world.

# Contents

# Chapter 1

# Introduction

Economic agents trading in a securities market are faced with the following three classical problems of financial theory:

1. Suppose the investor disposes of some initial endowment. Then what is an *optimal investment strategy*, i.e. what kind of dynamic trading strategy should the investor employ so as to make the most of her money?

2. Assume the investor is approached by some other economic agent who offers her a certain premium in exchange for some specific nontraded contingent claim. Should the investor accept the deal? More generally, is there some threshold *price* such that the investor accepts the deal if she is offered more and declines otherwise?

3. If the investor accepts the deal, she receives a premium today, but is obliged to pay out the random value of the contingent claim at maturity. How should the premium be invested in order to reduce this risk, i.e. how should the contingent claim be *hedged*?

Answers to all three of these fundamentals questions can of course only be given subject to some probabilistic model for the securities market. Here, one has to consider two fundamentally different situations.

In *complete markets* every contingent claim admits a replicating portfolio composed of the traded assets which duplicates the corresponding payoff. The prime example is of course the classical model of Black & Scholes (1973), where the (logarithmized) asset returns are assumed to be stationary over time, independent of the past and distributed according to a Normal distribution. Subject to the reasonable assumption that the market does not admit arbitrage, i.e. risk-free profits do not exist, the answers to Questions 2 and 3 above are very simple. There exists a unique price for the contingent claim that does not allow for risk-free profits, namely the initial value of the replicating strategy. Moreover, the replicating strategy completely removes the risk incurred by selling the contingent claim.

Question 1 posed above turns out to be more difficult from a conceptual point of view even in complete markets. The reason is that in addition to the general No Arbitrage assumption, one also needs to specify the investor's individual preferences in order to decide what constitutes an optimal investment strategy. A classical approach in Mathematical Economics and Financial Mathematics is to consider a *rational* investor who strives to maximize the *expected utility* derived from her portfolio at some future date. Here utility is measured by a so-called *utility function*, which assigns a numerical value to the degree of satisfaction induced by a certain terminal value of the portfolio. This approach is founded on the work of von Neumann & Morgenstern (1947), who show that such a utility function exists for any economic agent who has preferences over lotteries, i.e. who can rank all random payoffs in terms of her individual preferences.

This profound conceptual distinction between Question 1 and Questions 2, 3 above is rather surprising at first glance. When examined more closely, it indeed appears quite artificial and unrealistic, since in reality risks cannot be completely removed. This is reflected in *incomplete markets*, where not every random payoff admits a replicating strategy. It turns out that most probabilistic models of securities markets are incomplete: For example, if one replaces the Normal distribution in the Black-Scholes model by any other distribution, this leads to an incomplete model due to jumps in the asset price. Moreover, completeness of a given model depends delicately on a number of other simplifying modelling assumptions. For example, the Black-Scholes model is no longer complete if trading is only allowed at a finite number of dates or if transaction costs make the implementation of hedging strategies prohibitively expensive.

Consequently, incomplete markets are not a peculiar exception but comprise most realistic models that have been proposed in the empirical literature. Let us now reconsider Questions 1-3 above in the context of incomplete markets. Since perfect hedging strategies no longer exist and many different prices are typically consistent with the absence of arbitrage now, the situation does not just become mathematically more involved, but the whole line of reasoning breaks down. In order to come up with prices and hedging strategies in incomplete markets, one therefore has to make some additional assumptions.

An economically appealing approach is to base pricing and hedging on utility maximization as well. The approach to the optimal investment problem posed in Question 1 above did not make use of completeness. Therefore incompleteness of the given model only leads to additional mathematical but not conceptual difficulties. Assuming the investor strives to maximize her expected utility, Questions 1-3 can now be answered in a consistent way as follows:

1. Invest the initial endowment so as to maximize expected utility.

2. Agree to sell the contingent claim, if the premium raises the maximal expected utility that can be obtained by dynamic trading in the market, despite having to pay out the contingent claim.

3. The difference between the optimal investment strategies in 1 and 2 represents the optimal utility-based hedging strategy.

While this approach is appealing from an economical viewpoint, it is of course only useful in applications, if the objects of interest, i.e. optimal investment strategies as well as utility-based prices and hedging strategies, can actually be computed in concrete models. This is precisely what the present thesis is concerned with.

## 1.1 Outline of the thesis

In the following we present a brief outline of the contents of this thesis. This outline only serves to present a rough overview, for a more thorough discussion of the respective contents as well as detailed references to related literature the reader is referred to the separate introductions at the beginning of each chapter.

In *Part I*, we deal with incompleteness induced by *jumps* and/or *stochastic volatility* in the asset prices. These features arise if one replaces the Black-Scholes model by more sophisticated models in order to capture some of the empirical facts observed in reality.

*Chapter 2* is based on Kallsen & Muhle-Karbe (2008a). It is concerned with (time-inhomogeneous) affine semimartingales. This class of stochastic processes encompasses many specific models put forward in the empirical literature and is appealing mathematically due to its analytical tractability. Building on Duffie et al. (2003), we characterize these processes from the point of view of semimartingale characteristics and provide easy-to-check criteria for the exponential of an affine process to be a martingale. Using these results we establish conditions for the absolute continuity of the laws of two given affine processes. Moreover, we study whether exponential moments of affine processes can be computed by solving some integro-differential equations. These results are of independent theoretical interest, but are also used repeatedly in the remainder of the thesis.

*Chapter 3* deals with the statistical estimation of asset price models allowing both for stochastic volatility and jumps of the asset price. Making use of results from Barndorff-Nielsen & Shephard (2006), we propose a moment-based estimation approach for which we establish strong consistency and asymptotic normality. This estimation algorithms is then applied to real data and tested by performing a simulation study. Moreover, we also show how to estimate the current level of volatility by using the Kalman filter. As in Chapter 2 these results are of independent interest, but are also needed in the remainder of the thesis to provide realistic parameter values for the models under consideration.

*Chapter 4* stems from Kallsen & Muhle-Karbe (2008c) and considers utility maximization in affine stochastic volatility models, i.e. Question 1 above. Inspired by the general results of Kramkov & Schachermayer (1999), we use ideas from quadratic hedging put forward in Černý & Kallsen (2007) to construct a martingale criterion that allows both for the computation of a candidate strategy and for the verification that this candidate is indeed optimal. With the help of this criterion we characterize optimal investment strategies for power utility in a wide class of affine stochastic volatility models. Using a conditioning argument, we then go on to show that this approach can be applied for models with conditionally independent increments as well.

*Chapter 5* is concerned with Questions 2 and 3 above, i.e. with the computation of utility-based prices and hedging strategies. Since these computations are typically impossible to deal with even in simple concrete models, we consider a first-order approximation for a small number of contingent claims. Drawing on results of Kramkov & Sîrbu (2006, 2007) we show that for power utility functions, these first order approximations can be represented as the solution of a mean-variance hedging problem under a suitable equivalent probability measure subject to the original numeraire.

*Chapter 6* is based on joint work with Richard Vierthauer. It is concerned with the application of the results which have been obtained in Chapter 5 for a general semimartingale framework. More specifically, we again consider affine stochastic volatility models. By piecing together results from Chapters 4 and 5 as well as results on mean-variance hedging in affine models from the forthcoming Ph.D. thesis Vierthauer (2009), we obtain semi-explicit formulas for the first-order approximations of power utility-based prices and hedging strategies. In addition, we provide some numerical examples for the model of Barndorff-Nielsen & Shephard (2001).

In *Part II* of the thesis we turn our attention to incompleteness caused by proportional transaction costs. Here we only consider Question 1, i.e. the pure investment problem without any contingent claims. In the spirit Jouini & Kallal (1995), we employ the concept of a shadow price process. This is a fictitious price process lying within the bid-ask-bounds of the original market with transaction costs, such that the solution to the utility-maximization problem for the frictionless market with the shadow price and for the original market with transaction costs coincide.

*Chapter 7* provides an elementary proof that such a shadow price always exists in finite discrete time.

*Chapter 8*, which is based on Kallsen & Muhle-Karbe (2008b), then deals with using the concept of shadow prices for the computation of optimal portfolios in the presence of transaction costs. More specifically, we reconsider the setup of Magill & Constantinides (1976), Davis & Norman (1990) and Shreve & Soner (1994), i.e. an investor trying to maximize expected utility from consumption over an infinite horizon in a Black-Scholes model with proportional transaction costs. We show that this Merton problem with transaction costs can be solved for logarithmic utility by computing the optimal strategy and the shadow price simultaneously.

This thesis relies heavily on the general theory of stochastic processes and in particular on the calculus of semimartingale characteristics. For the convenience of the reader, the main notions and results used here are summarized in *Appendix A*.

Finally, *Appendix B* contains some technical results on Moore-Penrose pseudoinverses which are needed in Chapter 5.

# Part I

# Models with jumps and stochastic volatility

# Chapter 2

# Time-inhomogeneous affine semimartingales

## 2.1 Introduction

Affine processes play an important role in stochastic calculus and its applications e.g. in Mathematical Finance (cf. Duffie et al. (2000, 2003), Chen & Filipović (2005), Kallsen (2006), Cheridito et al. (2007) and the references therein). Their popularity for modelling purposes is probably due to their combination of flexibility and mathematical tractability. More specifically, affine processes can capture some of the stylized facts observed in the data (cf. Chapter 3 below). At the same time they possess enough mathematical structure to allow for explicit solutions to diverse problems in Mathematical Finance (cf. e.g. Chapters 4, 5, 6 and the references therein).

Affine processes have been studied in great depth in the very impressive works of Duffie et al. (2003) and Filipović (2005) using the theory of Markov processes. Here, following the approach of Kallsen (2006), we characterize the important subclass of nonexplosive affine processes in terms of their *differential semimartingale characteristics* (cf. Appendix A for a brief summary and Jacod & Shiryaev (2003) (henceforth JS) for more details). This is done in Section 2.2 below. In Section 2.3, we introduce affine stochastic volatility models, which represent the main application of affine processes in this thesis. Afterwards, we proceed to study several important properties related to affine semimartingales.

1. Suppose that the exponential of an affine process is a local martingale. Under what conditions is it a true martingale?

2. Suppose that two parameter sets of affine processes are given. Do they correspond to the same process under equivalent probability measures?

3. Under what condition is the $p$-th exponential moment of an affine process given as the solution to a generalized Riccati equation?

The first question is of interest in statistics and Mathematical Finance, where such exponentials denote density and price processes. In particular, establishing that a certain local martingale is actually a true martingale plays a key role in verifying that a given candidate strategy maximizes expected utility from terminal wealth (see Chapter 4 below). General criteria as the Novikov condition or its generalization to processes with jumps in Lépingle & Mémin (1978) are generally far from necessary. Less restrictive criteria have been obtained by making subtle use of e.g. the Markovian structure of the process. In Hobson (2004) and Cheridito et al. (2007) it is shown that in the context of bivariate affine diffusions, any exponential local martingale is a true martingale. Similarly, Cheridito et al. (2005) and Wong & Heyde (2004) contain conditions for the exponential of a diffusion with and without jumps to be a martingale. Below in Section 2.4 we present weak sufficient conditions which are taylor-made for affine processes and easy to verify in concrete models.

The second question is motivated from statistics and finance as well. Applied to finance, one law plays the role of the physical probability measure whereas the other is used as a risk-neutral measure for derivative pricing. In order to be consistent with arbitrage theory, these laws must be equivalent. In Section 2.5 we derive sufficient conditions which are based on the results of Section 2.4. On the one hand, these extend the results of JS on Lévy processes. On the other hand, they resemble results of Cheridito et al. (2005) applied to the affine case, however with sometimes less restrictive moment conditions.

As a function of $t$, the characteristic function $E(\exp(iu^\top X_t))$, $u \in \mathbb{R}^d$, of an $\mathbb{R}^d$-valued affine process $X$ solves a generalized Riccati equation as it is shown in great generality in Duffie et al. (2003) and Filipović (2005). Morally speaking, the same should hold for real exponential moments $E(\exp(p^\top X_t))$, $p \in \mathbb{R}^d$. Statements in Duffie et al. (2003) suggest that this may hold for arbitrary affine processes but the paper does not seem to provide an applicable condition. We study this question in Section 2.6. Using the results from Section 2.4, we derive criteria which are again easy to verify given a specific model. Theses results turn out to be very useful in applications, since one frequently has to calculate exponential moments, respectively verify whether they exist in the first place (cf. e.g. Chapter 6).

## 2.2 Definition and existence

Often affine processes are introduced as Markov processes whose characteristic function is of exponentially affine form. We study them from the point of view of semimartingale theory. In this context they correspond to processes with affine characteristics.

Differential characteristics of Markov processes are deterministic functions of the current state of the process. This leads to the notion of a martingale problem in the following sense.

**Definition 2.1** Suppose that $P_0$ is a distribution on $\mathbb{R}^d$ and mappings $\beta : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\gamma : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$, $\kappa : \mathbb{R}^d \times \mathbb{R}_+ \times \mathscr{B}^d \to \mathbb{R}_+$ are given. We call $(\Omega, \mathscr{F}, \mathbf{F}, P, X)$ *solution* to the *martingale problem* related to $P_0$ and $(\beta, \gamma, \kappa)$ if $X$ is a semimartingale on

$(\Omega, \mathscr{F}, \mathbf{F}, P)$ such that $P^{X_0} = P_0$ and $\partial X = (b, c, K)$ with

$$b_t(\omega) = \beta(X_{t-}(\omega), t), \tag{2.1}$$
$$c_t(\omega) = \gamma(X_{t-}(\omega), t), \tag{2.2}$$
$$K_t(\omega, G) = \kappa(X_{t-}(\omega), t, G). \tag{2.3}$$

One may also call the distribution $P^X$ of $X$ *solution* to the martingale problem. Since we consider only càdlàg solutions, $P^X$ is a probability measure on the Skorohod or canonical path space $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ of $\mathbb{R}^d$-valued càdlàg functions on $\mathbb{R}_+$ endowed with its natural filtration (cf. (JS, Chapter VI)). When dealing with this space, we denote by $X$ the canonical process, i.e. $X_t(\alpha) = \alpha(t)$ for $\alpha \in \mathbb{D}^d$. In any case, *uniqueness* of the solution refers only to the law $P^X$ because processes on different probability spaces cannot reasonably be compared otherwise.

From now on, we only consider *affine martingale problems* , where the differential characteristics are affine functions of $X_{t-}$ in the following sense:

$$\beta((x_1, ..., x_d), t) = \beta_0(t) + \sum_{j=1}^{d} x_j \beta(t), \tag{2.4}$$
$$\gamma((x_1, ..., x_d), t) = \gamma_0(t) + \sum_{j=1}^{d} x_j \gamma_j(t), \tag{2.5}$$
$$\kappa((x_1, ..., x_d), t, G) = \kappa_0(t, G) + \sum_{j=1}^{d} x_j \kappa_j(t, G), \tag{2.6}$$

where $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$ are given Lévy-Khintchine triplets on $\mathbb{R}^d$. If the triplets do not depend on $t$, we are in the setting of Duffie et al. (2003), where results on affine Markov processes yield conditions for the existence of a unique solution to this problem (cf. Kallsen (2006)). In the time-inhomogeneous case we turn to the corresponding results of Filipović (2005), namely Theorems 2.13 and 2.14.

However, we require the solution process to be a semimartingale in the usual sense, i.e. with finite values for all $t \in \mathbb{R}_+$. Filipović (2005) establishes that this is the case if the Markov process in question is *conservative*, but it does not contain analogues to the criteria for the homogeneous case in Duffie et al. (2003). Therefore we extend (Duffie et al., 2003, Lemma 9.2) to the time-inhomogeneous case, which is done in Appendix A.

Unlike most results in semimartingale theory, the conditions in Filipović (2005) depend on the choice of the truncation function on $\mathbb{R}^d$. From now on, we assume it to be of the form $h = (h_1, ..., h_d)$ with

$$h_k(x) := \chi(x_k) := \begin{cases} 0 & \text{if } x_k = 0, \\ (1 \wedge |x_k|)\frac{x_k}{|x_k|} & \text{otherwise.} \end{cases}$$

**Definition 2.2** Let $m, d \in \mathbb{N}$ with $m \leq d$. Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$, are called *strongly admissible* if, for $t \in \mathbb{R}_+$,

$$\begin{aligned}
\beta_j^k(t) - \int h_k(x)\kappa_j(t, dx) &\geq 0 \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j; \\
\kappa_j\big(t, (\mathbb{R}_+^m \times \mathbb{R}^{d-m})^C\big) &= 0 \quad \text{if} \quad 0 \leq j \leq m; \\
\int h_k(x)\kappa_j(t, dx) &< \infty \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j; \\
\gamma_j^{kl}(t) &= 0 \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k, l \leq m \quad \text{unless } k = l = j; \\
\beta_j^k(t) &= 0 \quad \text{if} \quad j \geq m+1, \quad 1 \leq k \leq m; \\
\gamma_j(t) &= 0 \quad \text{if} \quad j \geq m+1; \\
\kappa_j(t, \cdot) &= 0 \quad \text{if} \quad j \geq m+1
\end{aligned}$$

and if the following continuity conditions are satisfied:

- $\beta_j(t), \gamma_j(t)$ are continuous in $t \in \mathbb{R}_+$ for $0 \leq j \leq d$,
- $h_k(x)\kappa_j(t, dx)$ is weakly continuous on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ for $0 \leq j \leq d$, $1 \leq k \leq m$ with $k \neq j$,
- $h_k(x)^2\kappa_j(t, dx)$ is weakly continuous on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ for $0 \leq j \leq d$ and $k \geq m+1$ or $k = j$,

i.e. for $s \to t \in \mathbb{R}_+$ and any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\begin{aligned}
\int f(x)h_k(x)\kappa_j(s, dx) &\to \int f(x)h_k(x)\kappa_j(t, dx) \ \text{ if } 0 \leq j \leq d, 1 \leq k \leq m, k \neq j, \\
\int f(x)h_k(x)^2\kappa_j(s, dx) &\to \int f(x)h_k(x)^2\kappa_j(t, dx) \text{ if } 0 \leq j \leq d, k \geq m+1 \text{ or } k = j.
\end{aligned}$$

**Remark 2.3** If the Lévy-Khintchine triplets do not depend on $t$, this definition is consistent with Kallsen (2006). In this case, the attribute *strongly* can and will be dropped because it refers to continuity in $t$. In particular, the choice of the truncation function does not matter. In the time-inhomogeneous case however, the continuity conditions depend on the choice of the truncation function. Nevertheless, the function $h$ defined explicitly above can be replaced by any continuous truncation function $\widetilde{h}$ satisfying $|\widetilde{h}| \geq \varepsilon > 0$ outside of some neighbourhood of 0. In particular, $h(x) = x$ can be used if $X$ is a special semimartingale.

In view of Lemma A.10, (Filipović, 2005, Theorems 2.13 and 2.14) can immediately be rephrased as an existence and uniqueness result for affine martingale problems, which extends (Kallsen, 2006, Theorem 3.1) to the time-inhomogeneous case.

**Theorem 2.4 (Affine semimartingales)** *Let $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$ be strongly admissible Lévy-Khintchine triplets and denote by $\psi_j$ the corresponding* Lévy exponents

$$\psi_j(t, u) = u^\top\beta_j(t) + \frac{1}{2}u^\top\gamma_j(t)u + \int (e^{u^\top x} - 1 - u^\top h(x))\kappa_j(t, dx).$$

*Suppose in addition that*

$$\sup_{t\in[0,T]}\int_{\{x_k>1\}} x_k\kappa_j(t,dx)<\infty \quad \textit{for } j,k=1,\dots,m, \quad \forall T\in\mathbb{R}_+. \tag{2.7}$$

*Then the affine martingale problem related to* $(\beta,\gamma,\kappa)$ *and some initial distribution* $P_0$ *on* $\mathbb{R}_+^m\times\mathbb{R}^{d-m}$ *has a solution* $P$ *on* $(\mathbb{D}^d,\mathscr{D}^d,\mathbf{D}^d)$ *such that* $X$ *is* $\mathbb{R}_+^m\times\mathbb{R}^{d-m}$*-valued. For* $0\le t\le T$ *the corresponding* conditional characteristic function *is given by*

$$E\left(e^{i\lambda^\top X_T}\middle|\mathscr{D}_t\right)=\exp\left(\Psi^0(t,T,i\lambda)+\Psi^{(1,\dots,d)}(t,T,i\lambda)^\top X_t\right), \quad \forall\lambda\in\mathbb{R}^d, \tag{2.8}$$

*where*

$$\Psi^0(t,T,u)=\int_t^T\psi_0(s,\Psi^{(1,\dots,d)}(s,T,u))ds \tag{2.9}$$

*and* $\Psi^{(1,\dots,d)}$ *solves the following* generalized Riccati equations*:*

$$\Psi^{(1,\dots,d)}(T,T,u)=u, \quad \frac{d}{dt}\Psi^j(t,T,u)=-\psi_j(t,\Psi^{(1,\dots,d)}(t,T,u)), \quad j=1,...,d. \tag{2.10}$$

*Moreover, if* $(\Omega',\mathscr{F}',\mathbf{F}',P',X')$ *is another solution to the affine martingale problem, the distributions of* $X$ *and* $X'$ *coincide, i.e.* $P^{X'}=P$.

PROOF. This follows from (Filipović, 2005, Theorems 2.13, 2.14) and Lemma A.10 below along the lines of the proof of (Kallsen, 2006, Theorem 3.1). □

**Notation 2.5** For a semimartingale $X$, affine w.r.t. strongly admissible Lévy-Khintchine triplets $(\beta_i,\gamma_i,\kappa_i)$, $i=0,\dots,m$, we write $\psi_i^X$ for the Lévy exponent corresponding to $(\beta_i,\gamma_i,\kappa_i)$.

As is well known, the stochastic exponential of a real-valued Lévy process $X$ with $\Delta X>-1$ is the exponential of another Lévy process and vice versa. A similar statement holds for components of affine processes:

**Lemma 2.6** *Let* $X$ *be an* $\mathbb{R}^d$*-valued semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets* $(\beta_j(t),\gamma_j(t),\kappa_j(t))$*,* $0\le j\le d$*,* $t\in\mathbb{R}_+$*. Let* $i\in\{1,\dots,d\}$*. Then the differential characteristics of*

$$(X,\widetilde{X}^i):=(X,\mathscr{L}(\exp(X^i)))$$

*are affine with* $\widetilde{m}=m$*,* $\widetilde{d}=d+1$*, relative to strongly admissible Lévy-Khintchine triplets* $(\widetilde{\beta}_j(t),\widetilde{\gamma}_j(t),\widetilde{\kappa}_j(t))$*,* $0\le j\le d+1$*,* $t\in\mathbb{R}_+$*, where* $(\widetilde{\beta}_{d+1}(t),\widetilde{\gamma}_{d+1}(t),\widetilde{\kappa}_{d+1}(t))=(0,0,0)$ *and*

$$\begin{aligned}
\widetilde{\beta}_j(t) &= \begin{pmatrix}\beta(t)\\ \beta_j^i(t)+\frac{1}{2}\gamma_j^{ii}(t)+\int(\chi(e^{x_i}-1)-\chi(x_i))\kappa_j(t,dx)\end{pmatrix}\\
\widetilde{\gamma}_j^{kl}(t) &= \begin{cases}\gamma_j^{kl}(t) & \textit{for } k,l=1,\dots,d,\\ \gamma_j^{il}(t) & \textit{for } k=d+1, \quad l=1,\dots,d,\\ \gamma_j^{ki}(t) & \textit{for } k=1,\dots,d, \quad l=d+1,\\ \gamma_j^{ii}(t) & \textit{for } k,l=d+1,\end{cases}\\
\widetilde{\kappa}_j(t,G) &= \int 1_G(x,e^{x_i}-1)\kappa_j(t,dx), \quad \forall G\in\mathscr{B}^{d+1}
\end{aligned}$$

*for* $0 \leq j \leq d$. *Furthermore,* $\exp(X^i) = \exp(X^i_0)\mathscr{E}(\widetilde{X}^i)$. *Conversely, if* $\Delta X^i > -1$, *the differential characteristics of*

$$(X, \widehat{X}^i) := (X, \log(\mathscr{E}(X^i))$$

*are affine with* $\widetilde{m} = m$, $\widetilde{d} = d+1$, *relative to strongly admissible Lévy-Khintchine triplets* $(\widehat{\beta}_j(t), \widehat{\gamma}_j(t), \widehat{\kappa}_j(t))$, $0 \leq j \leq d+1$, $t \in \mathbb{R}_+$, *where* $(\widehat{\beta}_{d+1}(t), \widehat{\gamma}_{d+1}(t), \widehat{\kappa}_{d+1}(t)) = (0,0,0)$ *and*

$$\begin{aligned}
\widehat{\beta}_j(t) &= \begin{pmatrix} \beta(t) \\ \beta^i_j(t) - \frac{1}{2}\gamma^{ii}_j(t) + \int (\chi(\log(1+x_i)) - \chi(x_i))\kappa_j(t,dx) \end{pmatrix} \\
\widehat{\gamma}^{kl}_j(t) &= \begin{cases} \gamma^{kl}_j(t) & \text{for } k,l = 1,\dots,d, \\ \gamma^{il}_j(t) & \text{for } k = d+1, \quad l = 1,\dots,d, \\ \gamma^{ki}_j(t) & \text{for } k = 1,\dots,d, \quad l = d+1, \\ \gamma^{ii}_j(t) & \text{for } k,l = d+1, \end{cases} \\
\widehat{\kappa}_j(t,G) &= \int 1_G(x, \log(1+x_i))\kappa_j(t,dx), \quad \forall G \in \mathscr{B}^{d+1}
\end{aligned}$$

*for* $0 \leq j \leq d$. *Moreover, we have* $\mathscr{E}(X^i) = \exp(\widehat{X}^i)$.

PROOF. The characteristics can be computed with Proposition A.3 and A.4. Strong admissibility of the triplets $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\kappa}_j)$ and $(\widehat{\beta}_j, \widehat{\gamma}_j, \widehat{\kappa}_j)$ follows immediately from strong admissibility of $(\beta_j, \gamma_j, \kappa_j)$ because the mappings $x \mapsto \frac{\chi(e^{x_i}-1)-\chi(x_i)}{\chi(x_i)^2}$ and $x \mapsto \frac{\chi(\log(1+x_i))-\chi(x_i)}{\chi(x_i)^2}$ are bounded and continuous on $(\mathbb{R}^m_+ \times \mathbb{R}^{d-m})\backslash\{0\}$. $\square$

## 2.3 Affine stochastic volatility models

One application of affine semimartingales is given by affine stochastic volatility models (cf. Carr et al. (2003), Carr & Wu (2003), Kallsen (2006) and the references therein for an overview), which allow to capture many empirically observed phenomena by modelling the logarithmized stock price and the market volatility as a bivariate affine process (cf. Chapter 3 below for more details).

**Definition 2.7 (Affine stochastic volatility models)** Let $S$ be some positive asset price process and denote by $X = \log(S/S_0)$ the corresponding logarithmized asset price. A bivariate stochastic process $(y, X)$ is called *affine stochastic volatility model*, if it is an affine semimartingale with $m = 1$ relative to admissible triplets $(\beta_i, \gamma_i, \kappa_i)$, $i = 0,1,2$ on $\mathbb{R}^2$. More specifically, this means that the differential characteristics $(b^{(y,X)}, c^{(y,X)}, K^{(y,X)}, I)$ of $(y, X)$ are of the form

$$b^{(y,X)} = \begin{pmatrix} \beta^1_0 + \beta^1_1 y_- \\ \beta^2_0 + \beta^2_1 y_- + \beta^2_2 X_- \end{pmatrix}, \quad c^{(y,X)} = \begin{pmatrix} \gamma^{11}_1 y_- & \gamma^{12}_1 y_- \\ \gamma^{12}_1 y_- & \gamma^{22}_0 + \gamma^{22}_1 y_- \end{pmatrix}, \tag{2.1}$$

$$K^{(y,X)}(G) = \kappa_0(G) + \kappa_1(G) y_-, \quad \forall G \in \mathscr{B}^2, \tag{2.2}$$

for Lévy-Khintchine triplets $(\beta_i, \gamma_i, \kappa_i)$, $i = 0, 1, 2$ on $\mathbb{R}^2$. Since $y$ governs the magnitude of the dynamics of the asset price, it can be interpreted as the *stochastic volatility* of $X$.

**Remark 2.8** It is of course possible to construct affine models for several assets driven by multiple factors as well. In order to simplify the exposition in the following sections, we do not follow this path here and instead refer the reader to Vierthauer (2009).

We now consider some examples that have been proposed in the literature.

### 2.3.1 Heston (1993)

The most prominent example of a *continuous* affine stochastic volatility model is given by the *Heston model* introduced by Heston (1993) as the solution to the following pair of stochastic differential equations (SDEs):

$$\begin{aligned} dy_t &= (\vartheta - \lambda y_t)dt + \sigma\sqrt{y_t}dZ_t, \quad & y_0 > 0, \\ dX_t &= (\mu + \delta y_t)dt + \sqrt{y_t}dB_t, \quad & X_0 = 0. \end{aligned} \tag{2.3}$$

Here, $\vartheta \geq 0$, $\mu$, $\delta$, $\lambda$, $\sigma$ denote constants and $Z$, $B$ Wiener processes with constant correlation $\varrho \in [-1, 1]$. An application of Propositions A.2 and A.3 shows that $(y, X)$ is an affine stochastic volatility model relative to the triplets $(\beta_i, \gamma_i, \kappa_i)$, $i = 0, 1, 2$ given by

$$\begin{aligned} (\beta_0, \gamma_0, \kappa_0) &= \left( \begin{pmatrix} \vartheta \\ \mu \end{pmatrix}, 0, 0 \right), \\ (\beta_1, \gamma_1, \kappa_1) &= \left( \begin{pmatrix} -\lambda \\ \delta \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma\varrho \\ \sigma\varrho & 1 \end{pmatrix}, 0 \right), \\ (\beta_2, \gamma_2, \kappa_2) &= (0, 0, 0). \end{aligned}$$

### 2.3.2 Barndorff-Nielsen and Shephard (2001)

If the square-root process (2.3) is replaced with a *Lévy-driven Ornstein-Uhlenbeck* (OU) process, one obtains the model proposed by Barndorff-Nielsen & Shephard (2001) (henceforth BNS). More specifically, it is given as the solution to

$$\begin{aligned} dy_t &= -\lambda y_{t-}dt + dZ_{\lambda t}, \quad & y_0 > 0, \\ dX_t &= (\mu + \delta y_{t-})dt + \sqrt{y_{t-}}dB_t, \quad & X_0 = 0, \end{aligned}$$

where $\mu$, $\delta$, $\lambda > 0$ denote constants, $B$ a Wiener process and $Z$ a *subordinator* (i.e. an increasing Lévy process) with Lévy-Khintchine triplet $(b^Z, 0, K^Z)$. From Propositions A.2 and A.3 it follows that $(y, X)$ is an affine stochastic volatility model relative to

$$\beta_0 = \begin{pmatrix} \lambda b^Z \\ \mu \end{pmatrix}, \quad \gamma_0 = 0, \quad \kappa_0(G) = \int 1_G(z, 0)\lambda K^Z(dz) \quad \forall G \in \mathscr{B}^2,$$

$$\begin{aligned} (\beta_1, \gamma_1, \kappa_1) &= \left( \begin{pmatrix} -\lambda \\ \delta \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \\ (\beta_2, \gamma_2, \kappa_2) &= (0, 0, 0). \end{aligned}$$

### 2.3.3 Carr et al. (2003)

Carr et al. (2003) generalize both the Heston model and the BNS model by introducing jumps in the asset price. As discussed in Kallsen (2006), one has to consider time-changed Lévy models instead of SDEs in this case in order to preserve the affine structure. Here, we restrict our attention to the following generalization of the BNS model:

$$\begin{aligned} dy_t &= -\lambda y_{t-}dt + dZ_{\lambda t}, \quad && y_0 > 0, \\ dY_t &= y_{t-}dt, && Y_0 = 0, \\ X_t &= \mu t + B_{Y_t}. \end{aligned}$$

Here, $\mu$ and $\lambda > 0$ are constants, whereas $B$ and $Z$ denote a Lévy process with triplet $(b^B, c^B, K^B)$ and an independent subordinator with triplet $(b^Z, 0, K^Z)$, respectively. In view of (Kallsen, 2006, Section 4.4), $(y, X)$ is affine stochastic volatility model relative to the triplets

$$\beta_0 = \begin{pmatrix} \lambda b^Z \\ \mu \end{pmatrix}, \quad \gamma_0 = 0, \quad \kappa_0(G) = \int 1_G(z,0)\lambda K^Z(dz) \quad \forall G \in \mathscr{B}^2,$$

$$\beta_1 = \begin{pmatrix} -\lambda \\ b^B \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & c^B \end{pmatrix}, \quad \kappa_1(G) = \int 1_G(0,x)K^B(dx) \quad \forall G \in \mathscr{B}^2,$$

$$(\beta_2, \gamma_2, \kappa_2) = (0,0,0).$$

Note that we recover the BNS model, if $B$ is chosen to be a Brownian motion with drift, i.e. with Lévy-Khintchine triplet $(b^B, c^B, K^B) = (\delta, 1, 0)$.

## 2.4 Exponentially affine martingales

In this section we provide criteria for the exponential of a component of an affine process to be a martingale. We start with a general sufficient condition which is proved in Section 2.4.2. In Sections 2.4.3 and 2.4.4 we apply this general result to the time-homogeneous case and to processes with independent increments, respectively.

### 2.4.1 Time-inhomogeneous exponentially affine martingales

Let $X$ be an $\mathbb{R}^d$-valued semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $0 \le j \le d$, $t \in \mathbb{R}_+$. The following result is proved in Section 2.4.2.

**Theorem 2.9** *Suppose that for some* $1 \le i \le d$ *and* $T \in \mathbb{R}_+$ *the following holds:*

1. $\kappa_j(t, \{x \in \mathbb{R}^d : x_i < -1\}) = 0$ *for* $j = 0, \dots, m,$ $\forall t \in [0,T],$
2. $\int_{\{x_i > 1\}} x_i \kappa_j(t, dx) < \infty$ *for* $j = 0, \dots, m,$ $\forall t \in [0,T],$

3. $\beta_j^i(t) + \int (x_i - h_i(x))\kappa_j(t, dx) = 0$ *for* $j = 0, \dots, d,$ $\forall t \in [0, T],$

4. *the measure* $h_k(x)x_i\kappa_j(t, dx)$ *on* $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ *is weakly continuous in* $t \in [0, T]$ *for* $j = 1, \dots, m$ *and* $k = 1, \dots, d$ *.*

5. $\sup_{t\in[0,T]} \int_{\{x_k>1\}} x_k(1 + x_i)\kappa_j(t, dx) < \infty$ *for* $j, k = 1, \dots, m.$

*Then the stopped process* $\mathscr{E}(X^i)^T$ *is a martingale.*

Condition 1 ensures that $\mathscr{E}(X^i)$ does not jump to negative values. Condition 2 is needed for the integral in Condition 3 to be finite. Condition 3 in turn means that $\mathscr{E}(X^i)^T$ has zero drift, i.e. that it is a local martingale. The continuity condition 4 is needed to apply the results of Filipović (2005). It holds automatically in the time-homogeneous case (cf. Corollary 2.17). The crucial nontrivial assumption is the last one. The origin of this moment condition is discussed in Section 2.4.2.

From Theorem 2.9 we can obtain a similar result on the entire real line:

**Corollary 2.10** *Suppose that for some* $1 \le i \le d$ *and all* $t \in \mathbb{R}_+$ *the following holds:*

1. $\kappa_j(t, \{x \in \mathbb{R}^d : x_i < -1\}) = 0$ *for* $j = 0, \dots, m,$ $\forall t \in \mathbb{R}_+,$

2. $\int_{\{x_i>1\}} x_i\kappa_j(t, dx) < \infty$ *for* $j = 0, \dots, m,$ $\forall t \in \mathbb{R}_+,$

3. $\beta_j^i(t) + \int (x_i - h_i(x))\kappa_j(t, dx) = 0$ *for* $j = 0, \dots, d,$ $\forall t \in \mathbb{R}_+,$

4. *the measure* $h_k(x)x_i\kappa_j(t, dx)$ *on* $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ *is weakly continuous in* $t$ *for* $j = 1, \dots, m$ *and* $k = 1, \dots, d.$

5. $\sup_{t\in[0,T]} \int_{\{x_k>1\}} x_k(1 + x_i)\kappa_j(t, dx) < \infty$ *for* $j, k = 1, \dots, m,$ $\forall T \in \mathbb{R}_+.$

*Then* $\mathscr{E}(X^i)$ *is a martingale.*

PROOF. By Theorem 2.9, $\mathscr{E}(X^i)^T$ is a martingale for all $T \in \mathbb{R}_+$, which implies that $\mathscr{E}(X^i)$ is a martingale as well. □

**Example 2.11** If $X$ is continuous, Conditions 1–5 above reduce to $\beta_j^i = 0$, $j = 0, \dots, d$, i.e. essentially to assuming that $\mathscr{E}(X^i)$ is a local martingale. This applies e.g. to Heston's stochastic volatility model from Section 2.3.1 above.

We also obtain an analogue of Theorem 2.9 for ordinary exponentials:

**Corollary 2.12** *Suppose that for some* $1 \le i \le d$ *and* $T \in \mathbb{R}_+$ *the following holds:*

1. $E(e^{X_0^i}) < \infty,$

2. $\int_{\{x_i>1\}} e^{x_i}\kappa_j(t, dx) < \infty,$ $j = 0, \dots, m,$ $\forall t \in [0, T],$

3. $\beta_j^i(t) + \frac{1}{2}\gamma_j^{ii}(t) + \int (e^{x_i} - 1 - h_i(x))\kappa_j(t, dx) = 0,$ $j = 0, \dots, d,$ $\forall t \in [0, T],$

4. *the measure* $h_k(x)(e^{x_i}-1)\kappa_j(t,dx)$ *on* $(\mathbb{R}^m_+\times\mathbb{R}^{d-m})\backslash\{0\}$ *is weakly continuous in* $t\in[0,T]$ *for* $j=1,\dots,m$ *and* $k=1,\dots,d$,

5. $\sup_{t\in[0,T]}\int_{\{x_k>1\}} x_k e^{x_i}\kappa_j(t,dx)<\infty \quad$ *for* $j,k=1,\dots,m$ .

*Then the stopped process* $(e^{X^i})^T$ *is a martingale.*

PROOF. By Proposition A.4 and Lemma A.8 the process $\exp(X^i)^T$ is a $\sigma$-martingale. From Proposition A.9 it follows that is a supermartingale, in particular it is integrable. We have $\exp(X^i)=e^{X^i_0}\mathscr{E}(\widetilde{X}^i)$ for $\widetilde{X}^i$ as in Lemma 2.6. $\mathscr{E}(\widetilde{X}^i)^T$ is a martingale by Theorem 2.9. Since $e^{X^i_0}$ is integrable, we have

$$E\big(e^{X^i_t}\big)=E\big(e^{X^i_0}E\big(\mathscr{E}(\widetilde{X}^i)_t|\mathscr{F}_0\big)\big)=E\big(e^{X^i_0}\big)<\infty.$$

This yields that $e^{X_i}$ is a martingale as well. □

Of course an analogue of Corollary 2.10 holds for ordinary exponentials as well.

### 2.4.2 Proof of Theorem 2.9

Set $M:=\mathscr{E}(X^i)^T$. Condition 1 implies $\Delta X^i\geq -1$ on $[0,T]$, which in turn yields $M\geq 0$. Since any nonnegative $\sigma$-martingale is a supermartingale by Proposition A.9, it remains to show that $E(M_T)=1$. Since this property only depends on the law of $X$, we can assume w.l.o.g. that $X$ is the canonical process on the canonical path space.

If $M$ is a martingale, we can use it as the density process of a locally absolutely continuous measure change and employ Girsanov's theorem to calculate the characteristics of the canonical process under this new measure. In this proof the fundamental idea is to work in the opposite direction: we *define* the triplets as motivated by Girsanov and prove that there is a probability measure $Q$ that endows the canonical process with these characteristics. There we need the crucial moment condition 5. Next, we establish that this new measure is locally absolutely continuous with respect to the original probability measure, by using a certain uniqueness property of the martingale problems in question. Hence a density process exists. The final step of the proof is to show that this density process coincides with $M$. Related approaches are taken e.g. in Cheridito et al. (2005), Cheridito et al. (2007), Hobson (2004) and Wong & Heyde (2004).

**Lemma 2.13** *For* $j=0,\dots,d$ *and* $t\in\mathbb{R}_+$ *set*

$$\beta^*_j(t) = \beta_j(t\wedge T)+\gamma^{\cdot i}_j(t\wedge T)+\int x_i h(x)\kappa_j(t\wedge T,dx), \tag{2.1}$$

$$\gamma^*_j(t) = \gamma_j(t\wedge T), \tag{2.2}$$

$$\kappa^*_j(t,G) = \int 1_G(x)(1+x_i)\kappa_j(t\wedge T,dx), \quad \forall G\in\mathscr{B}^d. \tag{2.3}$$

*Under Conditions 1–4 of Theorem 2.9 this defines strongly admissible Lévy-Khintchine triplets. If Condition 5 holds as well, then there is a unique solution* $Q$ *to the corresponding affine martingale problem on* $(\mathbb{D}^d,\mathscr{D}^d,\mathbf{D}^d)$ *with any fixed initial distribution* $Q_0$ *on* $\mathbb{R}^m_+\times\mathbb{R}^{d-m}$.

PROOF. In view of Condition 5 and Theorem 2.4, it suffices to show that $(\beta_j^*(t), \gamma_j^*(t), \kappa_j^*(t))$ are strongly admissible Lévy-Khintchine triplets. Let $0 \le t \le T$. By Condition 2 the integral in (2.1) exists. The equivalence of $\kappa_j^*(t,dx)$ and $\kappa_j(t,dx)$ implies $\kappa_j^*(\{0\}) = 0$ and we have

$$\int (1 \wedge |x|^2)\kappa_j^*(t,dx) = \int (1 \wedge |x|^2)\kappa_j(t \wedge T, dx) + \int (1 \wedge |x|^2) x_i \kappa_j(t \wedge T, dx) < \infty$$

because $\kappa_j(t)$ is a Lévy measure and by Condition 2. Therefore $(\beta_j^*(t), \gamma_j^*(t), \kappa_j^*(t))$ are Lévy-Khintchine triplets. Now let $0 \le j \le m$, $1 \le k \le m$, $k \ne j$. Then

$$\beta_j^{*k}(t) - \int h_k(x)\kappa_j^*(t,dx) = \beta_j^k(t \wedge T) - \int h_k(x)\kappa_j(t,dx) \ge 0$$

because of the first and fourth admissibility condition for the original triplets $(\beta_j, \gamma_j, \kappa_j)$. From the second admissibility condition and by equivalence of $\kappa_j(t,dx)$ and $\kappa_j^*(t,dx)$ we obtain $\kappa_j^*(t, (\mathbb{R}_+^m \times \mathbb{R}^{d-m})^C) = 0$. Moreover, Condition 2 and the third condition on the original triplets yield

$$\int h_k(x)\kappa_j^*(t,dx) = \int h_k(x)(1+x_i)\kappa_j(t \wedge T, dx) < \infty.$$

We have thus established the first three admissibility conditions, the remaining four being obvious. Since the mapping $t \mapsto t \wedge T$ is continuous, $\gamma^*$ and, due to Condition 4, also $\beta^*$ are continuous in $t$. Finally, Condition 4 and the continuity conditions for the original triplets imply weak continuity of

$$h_k(x)\kappa_j^*(t,dx) = h_k(x)\kappa_j(t \wedge T, dx) + h_k(x) x_i \kappa_j(t \wedge T, dx)$$

for $1 \le k \le m$, $k \ne j$, and of

$$h_k(x)^2\kappa_j^*(t,dx) = h_k(x)^2\kappa_j(t \wedge T, dx) + h_k(x)^2 x_i \kappa_j(t \wedge T, dx)$$

for $k \ge m+1$ or $k = j$. Therefore $(\beta_j^*, \gamma_j^*, \kappa_j^*)$ are strongly admissible. □

The next step is to work towards local absolute continuity of $Q$ with respect to $P$. In view of (JS, Lemma III.3.3), we do this by constructing a localizing sequence $(T_n)_{n\in\mathbb{N}}$ for $M$ under $P$ such that $T_n \uparrow \infty$ holds under $Q$ as well. In the continuous case this can always be achieved by considering the hitting times $T_n = \inf\{t \in \mathbb{R}_+ : |M_t| \ge n\}$. This approach does not work in the presence of jumps, yet here a similar explicit construction is possible.

**Lemma 2.14** *Let $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $j = 0, \ldots, d$, $t \in \mathbb{R}_+$ be strongly admissible Lévy-Khintchine triplets. Assume that a solution $P$ to the corresponding affine martingale problem on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ exists. Then the stopping times $(T_n)_{n\in\mathbb{N}}$ given by*

$$T_n = \inf\{t > 0 : |X_{t-}| \ge n \text{ or } |X_t| \ge n\}$$

*satisfy $T_n \uparrow \infty$ $P$-almost surely. If Condition 4 in Theorem 2.9 holds and $M = \mathscr{E}(X^i)^T$ is a local martingale for some $1 \le i \le d$ and $T \in \mathbb{R}_+$, then $(T_n)_{n\in\mathbb{N}}$ is a localizing sequence for $M$.*

PROOF. $T_n \uparrow \infty$ follows immediately from the càdlàg property of $X$. Since $\mathscr{M}_{\mathrm{loc}}$ is stable under stopping, we know that $M^{T_n} \in \mathscr{M}_{\mathrm{loc}}$. By (JS, I.1.47c) it remains to show that $M^{T_n}$ is of class (D), i.e. $\{M_S^{T_n} : S \text{ finite stopping time}\}$ is uniformly integrable. It suffices to show

$$E\left(\sup_{t\in[0,T]} |M_{T_n\wedge t}|\right) < \infty \tag{2.4}$$

because $M_t^{T_n}$ is constant for $t \geq T$. Let $(B, C, \nu)$ be the characteristics of $M$. By Lemma A.7 the stopped process $M^{T_n}$ admits the stopped characteristics $(B^{T_n}, C^{T_n}, \nu^{T_n})$. Since it is a local martingale, (JS, II.2.38) yields its canonical decomposition

$$\begin{aligned} M^{T_n} &= M_0^{T_n} + (M^{T_n})^c + x * (\mu^{T_n} - \nu^{T_n}) \\ &= M_0^{T_n} + (M^{T_n})^c + (x1_{\{|x|\leq 1\}}) * (\mu^{T_n} - \nu^{T_n}) + (x1_{\{|x|>1\}}) * (\mu^{T_n} - \nu^{T_n}). \end{aligned}$$

The definition of $T_n$ and (JS, I.4.61) yield

$$\sup_{t\in[0,T]} M_{t-}^{T_n} \leq \sup_{t\in[0,T]} \exp\big((X^i)_{t-}^{T_n}\big) \leq e^n. \tag{2.5}$$

For the jump at $t$ we obtain

$$\Delta M_t^{T_n} = \Delta\left(x1_{\{|x|\leq 1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t + \Delta\left(x1_{\{|x|>1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t \tag{2.6}$$

because $(M^{T_n})^c$ is continuous and $M_0^{T_n}$ is constant. By (JS, II.1.27) we have

$$\sup_{t\in[0,T]} \Delta\left(x1_{\{|x|\leq 1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t = \sup_{t\in[0,T]} \Delta M_t^{T_n} 1_{\{|\Delta M_t^{T_n}|\leq 1\}} \leq 1. \tag{2.7}$$

Furthermore, we obtain

$$\sup_{t\in[0,T]} \Delta\left(x1_{\{|x|>1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t \leq \sum_{t\leq T} |\Delta M_t^{T_n}| 1_{\{|\Delta M_t^{T_n}|>1\}} = |x| 1_{\{|x|>1\}} * \mu_T^{T_n}.$$

By (JS, II.1.8) we have

$$E\left(|x|1_{\{|x|>1\}} * \mu_T^{T_n}\right) = \int_0^{T_n\wedge T} \int_{\{|x|>1\}} |x| K_t^M(dx)dt,$$

where $K_t^M$ denotes the local Lévy measure of $M$. We can compute the differential characteristics of $M$ through Proposition A.3. With $G_t = \{x \in \mathbb{R}^d : M_{t-}|x_i| > 1\}$ and the definition of $T_n$ this yields

$$\begin{aligned} \int_0^{T_n\wedge T} \int_{\{|x|>1\}} |x| K_t^M(dx)dt = &\int_0^{T_n\wedge T} \int_{G_t} M_{t-}|x_i|\kappa_0(t,dx)dt \\ &+ \sum_{j=1}^m \int_0^{T_n\wedge T} \int_{G_t} M_{t-}|x_i|\kappa_j(t,dx) X_{t-}^j dt \\ \leq &ne^n \sum_{j=0}^m \int_0^{T_n\wedge T} \int_{\{|x_i|>\frac{1}{n}\}} |x_i|\kappa_j(t,dx)dt. \end{aligned}$$

Since $|1/h_i|$ is bounded on $\{|x_i| > \frac{1}{n}\}$ and since it has a positive, bounded and continuous extension $\widetilde{h}$ to $\mathbb{R}^d$, it follows from Condition 4 in Theorem 2.9 that

$$\sup_{t\in[0,T]} \int_{\{|x_i|>\frac{1}{n}\}} |x_i|\kappa_j(t,dx) \leq \sup_{t\in[0,T]} \int_{\mathbb{R}^d} \widetilde{h}(x)|h_i(x)||x_i|\kappa_j(t,dx) < \infty$$

for $j = 0, \ldots, m$. Combining the above results yields

$$E\left(\sup_{t\in[0,T]} \Delta\left(x1_{\{|x|>1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t\right) < \infty. \tag{2.8}$$

In view of $M_t^{T_n} = M_{t-}^{T_n} + \Delta M_t^{T_n}$ and (2.5–2.8) we have that (2.4) holds as well. This proves the assertion. □

If we apply the previous result to $P$ and $Q$ we get the following

**Corollary 2.15** *Under the assumptions of Theorem 2.9, $(T_n)_{n\in\mathbb{N}}$ defined as in Lemma 2.14 is a localizing sequence for $M$ under $P$ and we have $T_n \uparrow \infty$ $Q$-a.s.*

PROOF. $M$ is a $\sigma$-martingale by Conditions 2 and 3 in Theorem 2.9. Since it is nonnegative by Condition 2, it is a supermartingale and in particular a special semimartingale. Hence it is a local martingale by (Kallsen, 2004, Corollary 3.1). The claim then follows immediately from Condition 4 in Theorem 2.9 and from Lemmas 2.13 and 2.14. □

Now we can prove that $Q|_{\mathscr{D}_T^0}$ is locally absolutely continuous with respect to $P|_{\mathscr{D}_T^0}$. Here, $\mathscr{D}_t^0$ denotes the $\sigma$-field generated by all maps $\alpha \mapsto \alpha(s)$, $s \leq t$ on $\mathbb{D}^d$. The filtration $(\mathscr{D}_t^0)_{t\in\mathbb{R}_+}$ is needed to apply (JS, III.2.40).

**Lemma 2.16** *Under the assumptions of Theorem 2.9 we have $Q|_{\mathscr{D}_T^0} \ll P|_{\mathscr{D}_T^0}$.*

PROOF. Since $M_0 = 1$, $M \geq 0$ and $(T_n)_{n\in\mathbb{N}}$ is a localizing sequence for $M \in \mathscr{M}_{\text{loc}}$ under $P$, we can define probability measures $Q^n \ll P$, $n \in \mathbb{N}$ with density processes $M^{T_n}$. We now show that the stopped canonical process $X^{T_n\wedge T}$ has differential characteristics $(b^*1_{[\![0,T_n\wedge T]\!]}, c^*1_{[\![0,T_n\wedge T]\!]}, K^*1_{[\![0,T_n\wedge T]\!]})$ under both $Q$ and $Q^n$, where $(b^*, c^*, F^*)$ are defined in (2.1–2.3), (2.4–2.6) but relative to $(\beta_j^*, \gamma_j^*, \kappa_j^*)$ instead of $(\beta_j, \gamma_j, \kappa_j)$.

By construction and Lemma A.7, $X^{T_n\wedge T}$ has the required characteristics under $Q$. Since $Q^n \ll P$, we can use (Kallsen, 2006, Proposition 4) to calculate the characteristics of $X^{T_n\wedge T}$ under $Q^n$. By $X^i \in \mathscr{M}_{\text{loc}}$ and (JS, II.2.38) we have

$$X^i = X_0^i + e_i \bullet X^c + x_i * (\mu^X - \nu^X)$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the $i$-th unit vector. (Kallsen, 2006, Proposition 4) yields that $X^{T_n\wedge T}$ has the desired characteristics under $Q^n$ as well.

The martingale problem corresponding to $(b^*, c^*, K^*)$ and arbitrary initial law on $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ has a unique solution by Lemma 2.13. Since the solution process is Markovian and by (JS, III.2.40), local uniqueness in the sense of (JS, III.2.37) is implied by uniqueness of the martingale problem. (JS, VI.2.10) yields that the stopping times $T_n \wedge T$, $n \in \mathbb{N}$ are strict in the sense of (JS, III.2.35). Hence $Q^n|_{\mathscr{D}^0_{T_n\wedge T}} = Q|_{\mathscr{D}^0_{T_n\wedge T}}$. By construction we have $Q^n|_{\mathscr{D}^0_{T_n\wedge T}} \ll P|_{\mathscr{D}^0_{T_n\wedge T}}$, which implies $Q|_{\mathscr{D}^0_{T_n\wedge T}} \ll P|_{\mathscr{D}^0_{T_n\wedge T}}$. Let $A \in \mathscr{D}^0_T$ with $P(A) = 0$. From

$$A \cap \{T_n > T\} \in \mathscr{D}^0_{T_n} \cap \mathscr{D}^0_T = \mathscr{D}^0_{T_n\wedge T}$$

it follows that $Q(A \cap \{T_n > T\}) = 0$ for all $n \in \mathbb{N}$ and hence $Q(A) = 0$ by Corollary 2.15. This proves the claim. □

If $Q^n$ denotes the probability measure with density process $M^{T_n}$ as in the proof of Lemma 2.16, we have $M_{T_n} = \frac{dQ^n}{dP}$. Since $M_{T_n} = M_{T_n\wedge T}$ is $\mathscr{D}^0_{T_n\wedge T}$-measurable, it is also the density on the smaller $\sigma$-field $\mathscr{D}^0_{T_n\wedge T}$, i.e. we have

$$M_{T_n} = \frac{dQ^n|_{\mathscr{D}^0_{T_n\wedge T}}}{dP|_{\mathscr{D}^0_{T_n\wedge T}}} = \frac{dQ|_{\mathscr{D}^0_{T_n\wedge T}}}{dP|_{\mathscr{D}^0_{T_n\wedge T}}} =: Z_n,$$

where the second equality is shown in the previous proof. Now notice that $(Z_n)_{n\in\mathbb{N}}$ is the martingale generated by $Z_\infty := dQ|_{\mathscr{D}^0_T}/dP|_{\mathscr{D}^0_T}$ on the discrete-time probability space $(\mathbb{D}^d, \mathscr{D}^0_T, (\mathscr{D}^0_{T_n\wedge T})_{n\in\mathbb{N}}, P)$. The martingale convergence theorem yields $M_{T_n} = Z_n \to Z_\infty$ a.s. for $n \to \infty$. Since we have $M_{T_n} = M_{T_n\wedge T} \to M_T$ a.s. for $n \to \infty$, it follows that $E(M_T) = E(Z_\infty) = 1$, which proves Theorem 2.9.

### 2.4.3 Time-homogeneous exponentially affine martingales

We now apply the results of Section 2.4.1 to the homogeneous case. Throughout, let $X^i$ with $1 \le i \le d$ be a component of an $\mathbb{R}^d$-valued semimartingale $X$ admitting affine differential characteristics relative to admissible Lévy-Khintchine triplets $(\beta_j, \gamma_j, \kappa_j)$, $j = 0, ..., d$ which do not depend on $t$. Corollary 2.10 now reads as follows:

**Corollary 2.17** *$\mathscr{E}(X^i)$ is a martingale if the following conditions hold:*

1. $\kappa_j(\{x \in \mathbb{R}^d : x_i < -1\}) = 0, \quad j = 0, \dots, m,$
2. $\int_{\{x_i>1\}} x_i \kappa_j(dx) < \infty, \quad j = 0, \dots, m,$
3. $\beta^i_j + \int (x_i - h_i(x))\kappa_j(dx) = 0, \quad j = 0, \dots, d,$
4. $\int_{\{x_k>1\}} x_k(1 + x_i)\kappa_j(dx) < \infty, \quad j, k = 1, \dots, m.$

Of course a counterpart to Corollary 2.12 can be derived similarly.

**Example 2.18** Consider the affine stochastic volatility model of Carr et al. (2003) from Section 2.3.3. The corresponding triplets are admissible with $m = 1$. If the moment condition

$$\int_{\{y>1\}} e^y F^B(dy) < \infty$$

and the drift conditions

$$0 = \mu, \quad 0 = b^B + \frac{1}{2}c^B + \int (e^y - 1 - h(y))F^B(dy)$$

are satisfied, Corollary 2.12 yields that $e^X$ is a martingale. These conditions are equivalent to $e^B$ and $e^{\mu I}$ being martingales, where $I$ denotes the identity process $I_t = t$.

The following example shows that even in the homogeneous case with $\Delta X^i > -1$, Corollary 2.17 does not generally hold without the crucial moment condition 4.

**Example 2.19** Let

$$(\beta_0, \gamma_0, \kappa_0) := (0,0,0),$$
$$\beta_1 := \begin{pmatrix} \frac{1}{2\sqrt{\pi}} \int_0^\infty h(y) y^{-\frac{3}{2}}(1+y)^{-1}dy \\ \frac{1}{2\sqrt{\pi}} \int_0^\infty (h(y) - y) y^{-\frac{3}{2}}(1+y)^{-1}dy \end{pmatrix}, \quad \gamma_1 := 0,$$
$$\kappa_1(G) := \frac{1}{2\sqrt{\pi}} \int_0^\infty 1_G(y,y) y^{-\frac{3}{2}}(1+y)^{-1}dy, \quad \forall G \in \mathscr{B}^2,$$
$$(\beta_2, \gamma_2, \kappa_2) := (0,0,0).$$

This defines admissible Lévy-Khintchine triplets on $\mathbb{R}^2$ satisfying (2.7), but violating Condition 4 in Corollary 2.17 for $i = 2$. By Theorem 2.4 there exists a probability measure $P$ on $(\mathbb{D}^2, \mathscr{D}^2, \mathbf{D}^2)$ such that $X$ is a semimartingale with affine differential characteristics relative to these triplets and $X_0 = (1,1)$ $P$-almost surely. Computing the differential characteristics $(b^M, c^M, K^M)$ of $M = \mathscr{E}(X^2)$ with Proposition A.3 yields

$$b^M = \int (h(x) - x)K^M(dx) \quad \text{and} \quad \int_{\{|x|>1\}} |x| K^M(dx) < \infty.$$

By Lemma A.8 it follows that $M$ is a positive local martingale. Now suppose $M$ was a true martingale. In view of Lemma A.11 we could then define a probability measure $Q \overset{\text{loc}}{\ll} P$ with density process $M$. Since $M = \mathscr{E}(x_2 * (\mu^X - \nu^X))$, an application of (Kallsen, 2006, Proposition 4) yields the differential characteristics $\partial X^1 = (b, c, K)$ of $X^1$ under $Q$, namely

$$b_t = \int h(x) F_t(dx), \quad c_t = 0, \quad K_t(G) = \frac{X^1_{t-}}{2\sqrt{\pi}} \int_{G \cap (0,\infty)} x^{-\frac{3}{2}} dx \quad \forall G \in \mathscr{B}.$$

Hence $X^1$ coincides in law under $Q$ with the process in (Duffie et al., 2003, Example 9.3), which explodes in $[0,1]$ with strictly positive probability. Since this contradicts $Q|_{\mathscr{D}^2_1} \ll P|_{\mathscr{D}^2_1}$, we conclude that $M = \mathscr{E}(X^2)$ is not a martingale.

Recall that Conditions 1–3 in Corollary 2.17 essentially mean that $\mathscr{E}(X^i)$ is a non-negative local martingale. Condition 4, on the other hand, is not needed for strong admissibility of $(\beta_j^*, \gamma_j^*, \kappa_j^*)$ in (2.1–2.3). Hence we know from (Duffie et al., 2003, Theorem 2.7) that there exists a unique Markov process whose conditional characteristic function satisfies

(2.8) with respect to $(\beta_j^*, \gamma_j^*, \kappa_j^*)$. But in order to ensure that it does not explode in finite time and hence is a semimartingale in the usual sense, we must also require this process to be conservative (cf. (Duffie et al., 2003, Theorem 2.12)). To establish conservativeness, one generally has to resort to the sufficient but not necessary criteria in (Duffie et al., 2003, Proposition 9.1 and Lemma 9.2), which is precisely what is done in the proof of Theorem 2.9.

### 2.4.4 Processes with independent increments

Instead of time-homogeneity we consider now deterministic characteristics. The following result slightly generalizes a parallel statement in the proof of (Eberlein et al., 2005, Proposition 4.4) by dropping the assumption of absolutely continuous characteristics. Hence we also incorporate processes with fixed times of discontinuity.

**Proposition 2.20** *Let $X$ be a semimartingale with independent increments (a PII in the sense of JS) satisfying $\Delta X > -1$. Then $\mathscr{E}(X)$ is a martingale if and only if it is a local martingale.*

PROOF. $\Rightarrow$: This is obvious.

$\Leftarrow$: W.l.o.g. $X_0 = 0$. Denote the characteristics of $X$ by $(B, C, \nu)$. From $X \in \mathscr{M}_{\text{loc}}$, (Kallsen, 2004, Lemma 3.1) and (JS, II.5.2) it follows that there exists a PII $Y$ with triplet $(B^*, C^*, \nu^*)$ given by

$$B_t^* = B_t + C_t + xh(x) * \nu_t, \quad C_t^* = C_t, \quad \nu^*(dt, dx) = (1 + x)\nu(dt, dx).$$

Its law is uniquely determined. We now choose $Q$ equal to the law of $Y$ and proceed almost literally as in the proof of Thereom 2.9: Lemma 2.16 is derived as above by using (JS, III.3.24) or Proposition A.5 rather than (Kallsen, 2006, Proposition 4). Moreover, the proof of Lemma 2.14 must be slightly modified. □

## 2.5 Locally absolutely continuous change of measure

In the context of measure changes, Theorem 2.9 can be used to derive a sufficient condition for local absolute continuity of the law of one affine process relative to another, similar to (JS, IV.4.32) for processes with independent increments.

**Theorem 2.21** *Let $Y$ and $Z$ be $\mathbb{R}^d$-valued semimartingales admitting affine differential characteristics relative to triplets $(\beta_j(t), \gamma_j(t), \kappa_j(t))$ and $(\widetilde{\beta}_j(t), \widetilde{\gamma}_j(t), \widetilde{\kappa}_j(t))$, $j = 0, \dots, d$, $t \in \mathbb{R}_+$, which satisfy the conditions in Theorem 2.4. We have $P^Z \overset{\text{loc}}{\ll} P^Y$ if there exist continuous functions $H : \mathbb{R}_+ \to \mathbb{R}^d$ and $W : \mathbb{R}_+ \times \mathbb{R}^d \to [0, \infty)$ such that, for $j = 0, \dots, d$ and all $t \in \mathbb{R}_+$,*

1. $\int_0^t \int (1 - \sqrt{W(s, x)})^2 \kappa_j(s, dx) ds < \infty$,

2. $\widetilde{\kappa}_j(t,G) = \int 1_G(x)W(t,x)\kappa_j(t,dx), \quad \forall G \in \mathscr{B}^d$,
3. $\int |h(x)(W(t,x)-1)|\kappa_j(t,dx) < \infty$,
4. $\widetilde{\beta}_j(t) = \beta_j(t) + H_t^\top \gamma_j(t) + \int h(x)(W(t,x)-1)\kappa_j(t,dx)$,
5. $\widetilde{\gamma}_j(t) = \gamma_j(t)$,
6. *the measure* $\chi(W(t,x)-1)(W(t,x)-1)\kappa_j(t,dx)$ *is weakly continuous in* $t$.

PROOF. As before, we denote the canonical process by $X$. Condition 1 implies that the measure in Condition 6 is finite. Condition 1 and (JS, II.1.33) with the stopping times from Lemma 2.14 yield $W-1 \in G_{\mathrm{loc}}(\mu^X)$ under $P^Y$. Since $H$ is continuous, it follows that

$$N = H \bullet X^c + (W-1) * (\mu^X - \nu^X)$$

is a well-defined local martingale. The differential characteristics of $(X,N)$ under $P^Y$ are affine relative to

$$\widehat{\beta}_j(t) = \begin{pmatrix} \beta_j(t) \\ \int(\chi(W(t,x)-1) - W(t,x)+1)\kappa_j(t,dx) \end{pmatrix},$$

$$\widehat{\gamma}_j(t) = \begin{pmatrix} \gamma_j(t) & \gamma_j(t)H_t \\ H_t^\top \gamma_j(t) & H_t^\top \gamma_j(t) H_t \end{pmatrix},$$

$$\widehat{\kappa}_j(t,G) = \int 1_G(x, W(t,x)-1)\kappa_j(t,dx), \quad \forall G \in \mathscr{B}^{d+1} \setminus \{0\}, \quad 0 \le j \le d,$$

$$(\widehat{\beta}_{d+1}, \widehat{\gamma}_{d+1}, \widehat{\kappa}_{d+1}) = 0.$$

These triplets are strongly admissible: the first seven admissibility conditions are obviously satisfied, the eighth follows from Condition 6, the weak continuity conditions for $\kappa_j$ and the continuity of $H$. The ninth condition is clear and the last is again a consequence of Condition 6. Moreover, Conditions 1–5 in Theorem 2.9 hold for $i = d+1$: Condition 4 in Theorem 2.9 is a consequence of the strong admissibility of $(\beta_j, \gamma_j, \kappa_j)$, $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\kappa}_j)$ and the continuity of $H$. Condition 1 above implies Condition 2 in Theorem 2.9 and Condition 3 is obviously satisfied. Condition 5 in Theorem 2.9 holds by

$$\int_{\{x_k>1\}} x_k(1+x_{d+1})\widehat{\kappa}_j(t,dx) = \int_{\{x_k>1\}} x_k W(t,x)\kappa_j(t,dx) = \int_{\{x_k>1\}} x_k \widetilde{\kappa}_j(t,dx),$$

which is uniformly bounded on $[0,T]$ by Condition (2.7) in Theorem 2.4.

By Theorem 2.9 we have that $\mathscr{E}(N)$ is a martingale. Since it is positive, we can use it as a density process to define a probability measure $Q \overset{\mathrm{loc}}{\ll} P^Y$ on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ (cf. Lemma A.11). By (Kallsen, 2006, Proposition 4) the differential characteristics of the canonical process under $Q$ and $P^Z$ coincide. Therefore Theorem 2.4 yields $Q = P^Z$, which proves the claim. □

Conditions 1–5 also appear in the necessary and sufficient theorem for PII in (JS, IV.4.32). We base our proof on the results of Filipović (2005). Since the latter are only formulated for continuous triplets, we require the additional continuity condition 6. This property holds in the time-homogeneous case. Except for Assumption (2.7) in Theorem 2.4 the remaining conditions for each triplet coincide with those for Lévy processes in (JS, IV.4.39).

**Corollary 2.22** *Let $Y$ and $Z$ be $\mathbb{R}^d$-valued semimartingales with affine differential characteristics relative to triplets $(\beta_j, \gamma_j, \kappa_j)$ and $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\kappa}_j)$, $j = 0, \ldots, d$, respectively, which satisfy the conditions of Theorem 2.4. Then $P^Z \overset{\text{loc}}{\ll} P^Y$ if there exist $H \in \mathbb{R}^d$ and a Borel function $W : \mathbb{R}^d \to [0, \infty)$ such that, for $0 \leq j \leq d$, we have*

1. $\int (1 - \sqrt{W(x)})^2 \kappa_j(dx) < \infty$ ,
2. $\widetilde{\kappa}_j(G) = \int 1_G(x) W(x) \kappa_j(dx), \quad \forall G \in \mathscr{B}^d$,
3. $\int |h(x)(W(x) - 1)| \kappa_j(dx) < \infty$,
4. $\widetilde{\beta}_j = \beta_j + H^\top \gamma_j + \int h(x)(W(x) - 1)\kappa_j(dx)$,
5. $\widetilde{\gamma}_j = \gamma_j$.

Similar results could be derived from (Cheridito et al., 2005, Theorem 2.4) applied to the affine case. Due to our heavy use of Filipović (2005), we end up with continuity conditions in the time-inhomogeneous case, whereas Cheridito et al. (2005) only require measurability and a certain uniform boundedness for $H$ and $W$. However, our moment conditions are sometimes less restrictive than the criterion in (Cheridito et al., 2005, Remark 2.5).

**Example 2.23** As in Example 2.18, we consider the stochastic volatility model of Carr et al. (2003). From Corollary 2.22 with $H \in \mathbb{R}^2$, $W(x) = e^{H^\top x}$ we obtain that the distribution corresponding to the transformed triplets is locally equivalent to the original one if we have

$$\int_{\{|x|>1\}} e^{H^\top x} \kappa_j(dx) < \infty, \quad j = 0, 1.$$

For the application of Cheridito et al. (2005), one needs the slightly stronger moment condition

$$\int_{\{|x|>1\}} (H^\top x) e^{H^\top x} \kappa_j(dx) < \infty, \quad j = 0, 1.$$

## 2.6 Exponential Moments

Let $X$ be a semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \kappa_j(t))$, $j = 0, \ldots, d$, $t \in \mathbb{R}_+$.

In (Duffie et al., 2003, Propositions 6.1 and 6.4) (respectively (Filipović, 2005, Propositions 4.1 and 4.3) for the time-inhomogeneous case), it is shown that a solution to the generalized Riccati equations from Theorem 2.4 always exists for initial values $u \in \mathbb{C}^m_- \times i\mathbb{R}^{d-m}$. (Duffie et al., 2003, Theorem 2.16) then asserts that if there exists an analytic extension of this solution to an open convex set containing $p \in \mathbb{R}^d$, the exponential moment

$E(\exp(p^\top X_T))$ can be obtained by inserting the value $p$ into the formula for the characteristic function.

The existence of this extension, however, may be difficult to verify, even for models without jumps. Using the results from Section 2.4, we show that $E(\exp(p^\top X_T))$ or, more generally, $E(\exp(p^\top X_T)|\mathscr{F}_t)$ can typically be obtained by solving the generalized Riccati equations (2.9, 2.10) with initial value $p$.

**Theorem 2.24** *Let $p \in \mathbb{R}^d$ and $T \in \mathbb{R}_+$. Suppose that $\Psi^0 \in C^1([0,T],\mathbb{R})$ and $\Psi^{(1,\dots,d)} = (\Psi^1,\dots,\Psi^d) \in C^1([0,T],\mathbb{R}^d)$ satisfy*

1. $\int_{\{|x|>1\}} e^{\Psi^{(1,\dots,d)}(t)^\top x}\kappa_j(t,dx) < \infty, \quad j = 0,\dots,d, \quad \forall t \in [0,T],$
2. $\Psi^{(1,\dots,d)}(T) = p, \quad \frac{d}{dt}\Psi^j(t) = -\psi_j(t,\Psi^{(1,\dots,d)}(t)), \quad j = 1,\dots,d,$
3. $\Psi^0(t) = \int_t^T \psi_0(s,\Psi^{(1,\dots,d)}(s))ds, \quad \forall t \in [0,T],$
4. $E(\exp(\Psi^{(1,\dots,d)}(0)^\top X_0)) < \infty,$
5. $\sup_{t\in[0,T]} \int_{\{x_k>1\}} x_k e^{\Psi^{(1,\dots,d)}(t)^\top x}\kappa_j(t,dx) < \infty, \quad 1 \le j,k \le m.$

*Then we have*

$$E\big(e^{p^\top X_T}\big|\mathscr{F}_t\big) = \exp\big(\Psi^0(t) + \Psi^{(1,\dots,d)}(t)^\top X_t\big), \quad \forall t \le T. \tag{2.1}$$

PROOF. By Condition 1 we have $\psi_j(t,\Psi^{(1,\dots,d)}(t)) < \infty$ for all $t \in [0,T]$. Define

$$N_t := \Psi^0(t) + \Psi^{(1,\dots,d)}(t)^\top X_t.$$

Since $\Psi^{(1,\dots,d)}$ is continuously differentiable, all $\Psi^j$ are of finite variation. Hence $[\Psi^j, X^j] = 0$ and

$$\begin{pmatrix} X - X_0 \\ N - N_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Psi^{(1,\dots,d)}(I) & \frac{d}{dt}\Psi^0(I) + X^\top \frac{d}{dt}\Psi^{(1,\dots,d)}(I) \end{pmatrix} \bullet \begin{pmatrix} X \\ I \end{pmatrix}.$$

by the fundamental theorem of calculus and partial integration in the sense of (JS, I.4.45). From this representation we obtain the differential characteristics $\partial(X,N)$ by using Proposition A.3. They are affine relative to time-inhomogeneous triplets $(\hat\beta_j,\hat\gamma_j,\hat\kappa)_j$ given by

$$\hat\beta_j(t) = \begin{pmatrix} \beta_j(t) \\ \frac{d}{dt}\Psi^j(t) + \Psi^{(1,\dots,d)}(t)^\top\beta_j(t) + \\ +\int(h(\Psi^{(1,\dots,d)}(t)^\top x) - \Psi^{(1,\dots,d)}(t)^\top h(x))\kappa_j(t,dx) \end{pmatrix},$$

$$\hat\gamma_j(t) = \begin{pmatrix} \gamma_j(t) & \gamma_j(t)\Psi^{(1,\dots,d)}(t) \\ \Psi^{(1,\dots,d)}(t)^\top\gamma_j(t) & \Psi^{(1,\dots,d)}(t)^\top\gamma_j(t)\Psi^{(1,\dots,d)}(t) \end{pmatrix},$$

$$\hat\kappa_j(t,G) = \int 1_G(x,\Psi^{(1,\dots,d)}(t)^\top x)\kappa_j(t,dx), \quad \forall G \in \mathscr{B}^{d+1}$$

for $j = 0,\dots,d$ and

$$(\hat\beta_{d+1},\hat\gamma_{d+1},\hat\kappa_{d+1}) = (0,0,0).$$

From admissibility of the original triplets $(\beta_j, \gamma_j, \kappa_j)$ and continuity of $\Psi^j$, $j = 0, \ldots, d$, we infer that $(\hat{\beta}_j, \hat{\gamma}_j, \hat{\kappa}_j)$ are strongly admissible. The prerequisites of Corollary 2.12 are satisfied for $i = d + 1$: the first follows immediately from Condition 4. The second is a consequence of Condition 1 and the fact that all $\kappa_j$ are Lévy measures, while the third follows from the definition of $\Psi^0$, $\Psi^{(1,\ldots,d)}$. The fourth prerequisite of Corollary 2.12 follows again from the continuity of $\Psi^{(1,\ldots,d)}$ while the fifth is just Condition 5. Therefore $\exp(N^T)$ is a martingale. For $t \leq T$ the martingale property yields

$$E(e^{p^\top X_T} | \mathscr{F}_t) = E(\exp(N_T) | \mathscr{F}_t) = \exp(N_t) = \exp(\Psi^0(t) + \Psi^{(1,\ldots,d)}(t)^\top X_t),$$

which proves the claim. □

Condition 1 is only needed for the ordinary differential equation in Condition 2 to be defined. It is automatically satisfied if the Lévy measures $\kappa_j$ have compact support, i.e. if $X$ has bounded jumps. Condition 2 and 3 mean that $\Psi^0$ and $\Psi^{(1,\ldots,d)}$ solve equations (2.9, 2.10) with initial value $p$. In the common situation that $X_0$ is deterministic, Condition 4 obviously holds. The moment condition 5 is crucial. It holds e.g. if the Lévy measures $\kappa_j$ have compact support or if $\kappa_1, \ldots, \kappa_m$ are concentrated on the set $\{x \in \mathbb{R}^d : x_1 = \ldots = x_m = 0\}$. This is the case for many affine stochastic volatility models as e.g. the time-changed Lévy models proposed by Carr et al. (2003).

As a side remark, the proof of Theorem 2.24 shows that the theory of time-inhomogeneous affine processes can become useful even in the study of time-homogeneous processes.

# Chapter 3

# Statistical estimation

## 3.1 Introduction

It is well known that most financial time series exhibit certain distinct features, usually called *stylized facts*. In particular, one usually encounters the following phenomena (cf. e.g. (Cont & Tankov, 2004, Chapter 7) and the references therein):

1. *Gain/Loss asymmetry*, i.e. returns are negatively skewed.
2. *Heavy tails* of the returns compared to the Normal distribution.
3. *Conditionally heavy tails*, i.e. heavy tails even after correcting for volatility clustering.
4. *Absence of autocorrelation* of asset returns, but *volatility clustering*, i.e. significant autocorrelation of the squared returns.
5. *Leverage effect*, i.e. a negative crosscorrelation between returns and squared returns.

Consequently, there exists a growing literature on different models trying to recapture these empirical observations. In continuous time the first three characteristica are typically tackled by allowing for jumps in the asset price process (cf. e.g. Madan & Seneta (1990), Eberlein & Keller (1995), Barndorff-Nielsen (1998)), whereas the last two are usually accounted for by introducing some kind of stochastic volatility (cf. e.g. Heston (1993), Barndorff-Nielsen & Shephard (2001), Carr et al. (2003) for models where the volatility is driven by an additional stochastic process and Klüppelberg et al. (2004), Klüppelberg et al. (2006) for continuous time GARCH models). Overviews on the subject can also be found in Schoutens (2003) and Cont & Tankov (2004).

For applications in Mathematical Finance finding a suitable statistical model for the data under consideration is of course only one part of the story. Indeed, one prefers models that are able to explain at least some of the stylized facts, but at the same time one needs enough mathematical structure to allow for the solution of financial problems. One class of models that fits these requirements surprisingly well is given by *affine stochastic volatility models* (cf. Section 2.3 above and Schoutens (2003), Kallsen (2006) for an overview). Since the stochastic volatility $y$ and the logarithmized asset price $X$ are modelled as a bivariate affine process in these models, the joint conditional characteristic function can be computed with Theorem 2.4 by solving some generalized Riccati equations. This opens the door to explicit solutions of many classical problems, some of which we will consider in Chapters 4 and 6. In the present chapter, we introduce an estimation algorithm for a particular subclass of time-changed Lévy models introduced by Carr et al. (2003). In these the asset price is modelled as $S_t = S_0 \exp(X_t)$ with $S_0 \in \mathbb{R}$ and

$$\begin{aligned} X_t &= \mu t + B_{Y_t}, \\ Y_t &= \int_0^t y_s ds, \end{aligned} \tag{3.1}$$

where $\mu \in \mathbb{R}$ and $B$ denotes a Lévy process, whereas $y$ is assumed to be positive, stationary and independent of $X$. These models can capture several stylized facts observed in the data, nevertheless they are quite tractable from an analytical point of view.

When performing statistical estimation, it is typically assumed that the time series under consideration is mean adjusted, i.e. $\mu$ is set equal to 0 and $B$ is assumed to be a martingale in Equation 3.1. By Barndorff-Nielsen & Shephard (2006), it is straightforward to estimate $\mu$ from the mean adjustment if $B$ is a martingale, since different values for $\mu$ do not change any of the higher centered moments or the second order dependence structure. If on the other hand, we do not require $B$ to be a martingale, the situation becomes much more involved (cf. Barndorff-Nielsen & Shephard (2006)).

In applications in Mathematical Finance, the situation is completely different though. Here, many problems can only be solved if the parameter $\mu$ is set equal to zero, thus requiring a non-martingale $B$ to model the drift of the asset under consideration (cf. e.g. Kraft (2005), Vierthauer (2009) and Chapter 4 for examples when this condition is necessary). Some problems can also be solved for arbitrary values of $\mu$ (cf. e.g. Benth et al. (2003) and Chapter 4 for portfolio optimization), but in general it is an important problem in Financial Mathematics to deal with the non-martingale case for $B$ as well.

Statistical estimation of stochastic volatility models typically falls into one of the following two broad categories:

1. *Simulation based techniques*: See e.g. Andersen et al. (2002), Chernov et al. (2003), Eraker et al. (2003) and the references therein for applications to affine jump-diffusion models, which correspond to choosing $B$ in Equation (3.1) to be the sum of a standard Brownian motion and a compound Poisson process. These approaches could also be used in the more general setup considered here. However, they require lengthy computations and are tedious to implement for the non-specialist. Furthermore, consistency and asymptotic normality are typically only assured under regularity conditions that are not easily checked in concrete models (cf. e.g. Hansen (1982), Duffie & Singleton (1993), Gallant & Tauchen (1996) for more details).

2. *Approaches using exact formulas for moments of the model*: Barndorff-Nielsen & Shephard (2006) calculate the moments and second order dependence structure of

model (3.1), exactly in the case where $B$ is a martingale and approximately for frequent observations in the general case. They proceed to construct a quasi-maximum likelihood (QML) estimator in the case where $B$ is a martingale with symmetric marginal distributions and note that it would also possible to argue approximately otherwise. Again easy-to-check regularity conditions ensuring good asymptotic properties are missing. Furthermore, QML estimation involves nonlinear minimization and is also not robust with respect to model misspecification.

This last drawback is avoided by performing a direct (generalized-) method of moment estimation, matching theoretical moments of the model to the corresponding empirical moments of the data. For affine jump-diffusions this approach has been considered by Jiang & Knight (2002) in the case where $B$ is a martingale. They use the first four moments of the returns as well as some autocorrelations of the squared returns to exemplarily estimate the Heston model. However, asymptotic results are once more only obtained subject to regularity conditions (cf. Hansen (1982)) that may be difficult to check in concrete models. On the contrary, Haug et al. (2007), who use a similar moment based approach for the COGARCH model, only impose conditions on the parameters of the model that are easily verified for a concrete specification.

The aim of this Chapter is fourfold. First, we extend the method of moments algorithms used by Jiang & Knight (2002), Haug et al. (2007) to the setup considered here (which encompasses pure jump driving processes of infinite activity like the popular Normal Inverse Gaussian process, for example), drawing on the results of Barndorff-Nielsen & Shephard (2006). In particular, we consider the case where $B$ is possibly skewed and not necessarily a martingale. No simulation is required and all estimators are given explicitly, which makes straightforward implementation for diverse models possible. Inspired by Haug et al. (2007), we then present exact asymptotic results if $B$ is assumed to be a martingale and approximate asymptotic results if this assumption is dropped, only imposing conditions that are easily verified in concrete models. Thirdly, we analyze the small sample behavior of our estimation algorithms by fitting parametrized versions of the models to real data and performing simulation studies with the parameters obtained in this way. Finally, we also show how to estimate the current level of volatility by using a Kalman filter (if $B$ is a martingale) respectively an extended Kalman filter (for general $B$).

The remainder of this chapter is organized as follows. In Section 3.2, we introduce the model and supply the formulas for its moments obtained by Barndorff-Nielsen & Shephard (2006). In Section 3.3, we deal with estimation in the case where $B$ is a martingale. We provide an estimation algorithm before proving that the sequence of returns is strongly mixing with exponentially decreasing rate, which then implies strong consistency and asymptotic normality of our estimators as the number of observations tends to infinity. Furthermore, we fit the model to real data and test the small sample behavior of our estimators via a simulation study. Finally, we show how to estimate the current level of volatility by using a Kalman filter. In Section 3.4, we deal with estimating the model if the martingale assumption on $B$ is dropped. Using approximate moments obtained by Barndorff-Nielsen &

Shephard (2006), we construct estimators and prove that they are strongly consistent and asymptotically normal as the number of observations goes to infinity *and* the space between subsequent observations tends to zero. Finally, we present another simulation study using parameters obtained by fitting the model to real data and show how the current level of volatility can be estimated by using an approximate extended Kalman filter.

## 3.2 Moments and dependence structure of time-changed Lévy models

We consider time changed Lévy-models of the following form as proposed by Carr et al. (2003):

$$\begin{aligned} X_t &= \mu t + B_{Y_t}, \\ Y_t &= \int_0^t y_s ds. \end{aligned} \tag{3.2}$$

Here, $\mu \in \mathbb{R}$ and $B$ denotes a real valued Lévy process with Lévy-Khintchine triplet $(b^B, c^B, K^B)$ and Lévy exponent

$$\psi^B(u) = ub^B + \frac{1}{2}uc^B u + \int (e^{ux} - 1 - uh(x))K^B(dx),$$

whereas $y$ is assumed to be positive, stationary and independent of $B$.

**Example 3.1** If $y$ is chosen to be a Lévy-driven OU process, i.e.

$$dy_t = -\lambda y_{t-}dt + dZ_{\lambda t},$$

for $\lambda > 0$ and an increasing Lévy process $Z$ independent of $B$, (3.2) leads to the generalized BNS model of Carr et al. (2003) from Section 2.3.3. In particular, $(y, X)$ is an affine stochastic volatility model in this case. However, notice that the Heston model with correlation (i.e. $\varrho \neq 0$ in Section 2.3.1) is not a special case of (3.2).

To use the generalized method of moments for parameter estimation, one needs to calculate sufficiently many moments of the model under consideration. For time-changed Lévy models this has been done by Barndorff-Nielsen & Shephard (2006) by conditioning on the time-change $Y$. More specifically, let $\Delta > 0$ be some grid size and define the discrete increments $X_{(n)}$ of the log-price $X$ as

$$X_{(n)} := X_{n\Delta} - X_{(n-1)\Delta}, \quad n \in \mathbb{N}^*. \tag{3.3}$$

Barndorff-Nielsen & Shephard (2006) relate the moments and dependence structure of $(X_{(n)})_{n\in\mathbb{N}^*}$ to the moments and dependence structure of $y$ as well as the *cumulants* of $B$, given by

$$c_n := \frac{\partial^n}{\partial u^n}\psi^B(u)\big|_{u=0}, \quad n \in \mathbb{N}^*.$$

Summing up results from Barndorff-Nielsen & Shephard (2006), the following holds.

**Theorem 3.2** *Let $B$ be a Lévy process with $c_4 < \infty$ and suppose $y$ is stationary with $E(y_t^4) < \infty$, $E(y_t) =: \xi$ and $\mathbb{V}\mathrm{ar}(y_t) =: \omega^2$ for all $t \in \mathbb{R}_+$. Let $r_y$ be the autocorrelation function of $y$ and define*

$$r_y^{**}(t) := \int_0^t \int_0^v r_y(u) du\, dv.$$

*Then, if $\mu = 0$, the following holds:*

$$\begin{aligned}
&E(X_{(n)}) = c_1 \Delta \xi, \\
&E(X_{(n)}^2) = c_2 \Delta \xi + c_1^2 \left(2\omega^2 r_y^{**}(\Delta) + (\Delta\xi)^2\right), \\
&E(X_{(n)}^3) = c_3 \Delta \xi + 3c_1 c_2 \left(2\omega^2 r_y^{**}(\Delta) + (\Delta\xi)^2\right) + c_1^3 E(Y_\Delta^3), \\
&E(X_{(n)}^4) = c_4 \Delta \xi + (4c_1c_3 + 3c_2^2) \left(2\omega^2 r_y^{**}(\Delta) + (\Delta\xi)^2\right) + 6c_1^2 c_2 E(Y_\Delta^3) + c_1^4 E(Y_\Delta^4),
\end{aligned}$$

*as well as, for $s \in \mathbb{N}^*$,*

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(X_{(n)}, X_{(n+s)}) =& c_1^2 \mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}), \\
\mathbb{C}\mathrm{ov}(X_{(n)}, X_{(n+s)}^2) =& c_1 c_2 \mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}) + c_1^3 \mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}^2), \\
\mathbb{C}\mathrm{ov}(X_{(n)}^2, X_{(n+s)}^2) =& c_2^2 \mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}) + c_1^2 c_2 \mathbb{C}\mathrm{ov}(y_{n\Delta}^2, y_{(n+s)\Delta}) \\
&+ c_2 c_1^2 \mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}^2) + c_1^4 \mathbb{C}\mathrm{ov}(y_{n\Delta}^2, y_{(n+s)\Delta}^2).
\end{aligned}$$

*Moreover,*

$$\mathbb{C}\mathrm{ov}(y_{n\Delta}, y_{(n+s)\Delta}) = \omega^2 (r_y^{**}((s+1)\Delta) - 2r_y^{**}(s\Delta) + r_y^{**}((s-1)\Delta)).$$

PROOF. (Barndorff-Nielsen & Shephard, 2006, Propositions 2, 5). □

**Example 3.3** If $y$ is either a stationary OU process or a stationary square-root process, the autocorrelation function $r_y$ of $y$ is given by

$$r_y(u) = e^{-\lambda u}, \quad u \in \mathbb{R}_+.$$

A proof of this result can be found e.g. in (Cont & Tankov, 2004, Chapter 15). Consequently, by (Barndorff-Nielsen & Shephard, 2001, Example 4) we have

$$r_y^{**}(u) = \frac{1}{\lambda^2} \left(e^{-\lambda u} - 1 + \lambda u\right), \quad u \in \mathbb{R}_+.$$

**Remark 3.4** To make the conditioning argument of Barndorff-Nielsen & Shephard (2006) work, it is crucial that the driver $B$ of the asset price is independent of the volatility process $y$. This explains why this approach does not work for e.g. the Heston model with correlation. For affine models one can in principle instead differentiate the characteristic function, which is often known in closed form. However, this typically leads to extremely complicated expressions that only yield estimators via the solution of a system of nonlinear equations. Nevertheless, this approach has been applied by Jiang & Knight (2002) for the Heston model in the case where $B$ is a martingale. However, desirable asymptotic properties such as strong consistency and asymptotic normality can only be established subject to additional assumptions in this case (cf. e.g. Hansen (1982) for more details). Moreover, the much more involved case when $B$ is not a martingale is not treated in Jiang & Knight (2002).

## 3.3 Moment estimation if *B* is a martingale

By Theorem 3.2 above, we know the moments and second-order dependence structure of the time-changed Lévy model. We now use these to set up a generalized method of moments estimator, extending similar approaches used by Jiang & Knight (2002) and Haug et al. (2007) to estimate affine jump diffusion models and the COGARCH(1,1) model, respectively. Estimation is done subject to the following assumptions on the model:

(A1) For time horizon $T > 0$ and grid size $\Delta > 0$ we have equally spaced observations $X_{j\Delta}, j = 0, ..., \lfloor T/\Delta \rfloor$ leading to returns $X_{(j)} = X_{j\Delta} - X_{(j-1)\Delta}, j = 1, ..., \lfloor T/\Delta \rfloor$.

(A2) The cumulants $c_j$ of $B$ satisfy $c_1 = 0$, $c_2 = 1$ and $c_4 < \infty$.

(A3) $y$ is a stationary OU or square-root process with mean reversion $\lambda > 0$, mean $\xi > 0$ and variance $\infty > \omega^2 > 0$.

**Remark 3.5** The condition $c_4 < \infty$ holds for most Lévy processes typically used in the literature, e.g. Variance Gamma (VG) and Normal Inverse Gaussian (NIG) processes (cf. e.g. Schoutens (2003)). The normalization $c_2 = 1$ just leads to a rescaling of the time change and therefore can be assumed without leading to a loss of generality in the model (cf. e.g. Pauwels (2007)). The final parameter restriction $c_1 = 0$ is equivalent to $B$ being a martingale. It is commonly made in the literature (see e.g. Barndorff-Nielsen & Shephard (2001), Haug et al. (2007), Pigorsch & Stelzer (2008)), because it drastically simplifies the moment and dependence structure of the model (cf. Theorem 3.2 above). We will discuss the case $c_1 \neq 0$ in Section 3.4 below.

For given $\Delta > 0$, denote by $m_{i,\Delta}$ and $\mu_{i,\Delta}$, $i \in \mathbb{N}$, the $i$-th uncentered and centered moments of $X_{(n)}$, respectively. Furthermore, let $\gamma_\Delta(s) := \mathbb{C}\mathrm{ov}(X^2_{(n)}, X^2_{(n+s)})$ for $n, s \in \mathbb{N}^*$ and define $\gamma_{\Delta,d} := (\gamma_\Delta(1), \ldots, \gamma_\Delta(d))$ for $d \in \mathbb{N}^*$. Assuming (A1)-(A3), Theorem 3.2 then reads as follows.

**Corollary 3.6** *Assume (A1)-(A3) hold. Then for any $\mu \in \mathbb{R}$, we have*

$$m_{1,\Delta} = \mu\Delta, \quad \mu_{2,\Delta} = \Delta\xi, \quad \mu_{3,\Delta} = c_3\Delta\xi,$$
$$\mu_{4,\Delta} = c_4\Delta\xi + 6\frac{\omega^2}{\lambda^2}(e^{-\lambda\Delta} - 1 + \lambda\Delta) + 3(\Delta\xi)^2,$$
$$\gamma_\Delta(s) = \omega^2\frac{\left(1 - e^{-\lambda\Delta}\right)^2}{\lambda^2}e^{-\lambda\Delta(s-1)}, \quad s \in \mathbb{N}^*.$$

### 3.3.1 The estimation procedure

We begin by showing that the unknown model parameters $\mu, c_3, c_4, \lambda, \xi, \omega^2$ are uniquely determined as a continuously differentiable function of the first four moments of the returns as well as the autocovariance function of the squared returns.

**Proposition 3.7** *Let (A1)-(A3) be satisfied and $k_\Delta, p > 0$ such that, for $s \in \mathbb{N}^*$,*

$$\gamma_\Delta(s) = k_\Delta e^{-p\Delta(s-1)}.$$

*Then $\mu, c_3, c_4, \lambda, \xi, \omega^2$ are uniquely determined by $m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, k_\Delta, p$ as*

$$(\mu, c_3, c_4, \lambda, \xi, \omega^2) = H_\Delta(m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, k_\Delta, p)$$

*with $H_\Delta : \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}^3 \times \mathbb{R}_{++} \to \mathbb{R}^6$ defined as*

$$H_\Delta(m_1, \mu_2, \mu_3, \mu_4, k, p)) := \left( \frac{m_1}{\Delta}, \frac{\mu_3}{\mu_2}, \frac{\mu_4}{\mu_2} - 3\mu_2 - \frac{6k_\Delta(e^{-p\Delta} - 1 + p\Delta)}{\mu_2(1-e^{-p\Delta})^2}, p, \frac{\mu_2}{\Delta}, \frac{p^2 k_\Delta}{(1-e^{-p\Delta})^2} \right).$$

*Furthermore, $H_\Delta$ is continuously differentiable in $(m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, k_\Delta, p)$.*

PROOF. Follows immediately from Theorem 3.2 and Corollary 3.6 above. □

Proposition 3.7 motivates the following estimation algorithm, which estimates $\mu$, $c_3$, $c_4$, $\lambda$, $\xi$, $\omega^2$ by matching the first four moments of the model to the corresponding empirical moments of the data and fitting the logarithmized autocovariance function of the model to its empirical counterpart via linear regression.

**Algorithm 3.8** *1. Calculate the moment estimators*

$$\widehat{m}_{1,\Delta,T} := \frac{1}{\lfloor T/\Delta \rfloor} \sum_{j=1}^{\lfloor T/\Delta \rfloor} X_{(j)}, \quad \widehat{\mu}_{i,\Delta,T} := \frac{1}{\lfloor T/\Delta \rfloor} \sum_{j=1}^{\lfloor T/\Delta \rfloor} (X_{(j)} - \widehat{m}_{1,\Delta,T})^i,\ i = 2,3,4,$$

*and for $d \geq 2$ the empirical autocovariances $\widehat{\gamma}_{\Delta,T,d} := (\widehat{\gamma}_{\Delta,T}(1), \ldots, \widehat{\gamma}_{\Delta,T}(d))$ as*

$$\widehat{\gamma}_{\Delta,T}(s) := \frac{1}{\lfloor T/\Delta \rfloor} \sum_{j=1}^{\lfloor T/\Delta \rfloor - s} \left(X_{(j)}^2 - \widehat{\mu}_{2,\Delta,T}\right)\left(X_{(j+s)}^2 - \widehat{\mu}_{2,\Delta,T}\right), \quad h = 1, \ldots, d.$$

2. *For fixed $d \geq 2$ define the mapping $K_\Delta : \mathbb{R}^d_{++} \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$K_\Delta(\gamma, k, p) := \sum_{s=1}^{d} \left(\log(\gamma(h)) - \log(k) + p\Delta s\right)^2,$$

*and compute the least square estimator*

$$(\widehat{k}_\Delta(\widehat{\gamma}_{\Delta,T,d}), \widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d})) := \operatorname*{arg\,min}_{(k,p)\in\mathbb{R}^2} K_\Delta(\widehat{\gamma}_{\Delta,T,d}, k, p),$$

*which is given by*

$$\widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d}) = -\frac{\sum_{s=1}^d \left(\log(\widehat{\gamma}_{\Delta,T,d}(s)) - \overline{\log(\widehat{\gamma}_{\Delta,T,d})}\right)(s - \frac{d+1}{2})}{\Delta \sum_{s=1}^d (s - \frac{d+1}{2})^2},$$

$$\widehat{k}_\Delta(\widehat{\gamma}_{\Delta,T,d}) = \exp\left(\overline{\log(\widehat{\gamma}_{\Delta,T,d})} + \frac{d+1}{2}\widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d})\right),$$

*with $\overline{\log(\widehat{\gamma}_{\Delta,T,d})} := \frac{1}{d}\sum_{s=1}^d \log(\widehat{\gamma}_{\Delta,T,d}(s))$.*

3. *Define the mapping* $J_\Delta : \mathbb{R}^{d+4} \to \mathbb{R}^6$ *by*

$$J_\Delta(m_1,\mu_2,\mu_3,\mu_4,\gamma) := \begin{cases} H_\Delta(m_1,\mu_2,\mu_3,\mu_4,\widehat{k}_\Delta(\gamma),\widehat{p}_\Delta(\gamma)) & \text{if } \mu_2,\gamma,\widehat{p}_\Delta(\gamma) > 0,\\ (0,0,0,0,0,0) & \text{otherwise}, \end{cases}$$

*and compute the estimator*

$$(\widehat{\mu}_{\Delta,T},\widehat{c}_{3,\Delta,T},\widehat{c}_{4,\Delta,T},\widehat{\lambda}_{\Delta,T},\widehat{\xi}_{\Delta,T},\widehat{\omega^2}_{\Delta,T}) := J_\Delta(\widehat{m}_{1,\Delta,T},\widehat{\mu}_{2,\Delta,T},\widehat{\mu}_{3,\Delta,T},\widehat{\mu}_{4,\Delta,T},\widehat{\gamma}_{\Delta,T,d}).$$

**Remark 3.9** In view of Corollary 3.6 and Assumption (A3), we have $\mu_{2,\Delta} > 0$ as well as $p > 0$ and $\gamma_\Delta(s) > 0$ for all $s \in \mathbb{N}^*$. However, this does not necessarily mean that the corresponding estimators are strictly positive. Nevertheless, we will show in Corollary 3.14 below that all estimators are strongly consistent, which implies that all estimators will be almost surely well defined for sufficiently large samples.

Similarly, $\widehat{c}_{4,\Delta,T} < 0$ is possible depending on the data. On the other hand, we have $c_4 = 0$ if $B$ is chosen as a Brownian motion as well as $c_4 > 0$ for all other Lévy process $B$ with jumps. Hence we take $\widehat{c}_{4,\Delta,T} < 0$ as a strong indication that the data is too light tailed to be suitably modeled by the class of (semi-) heavy tailed models considered here.

**Remark 3.10** If one considers the special case where $B$ is chosen to be a Brownian motion, i.e. the BNS model, we have $c_3 = c_4 = 0$. Hence one can still use Algorithm 3.8 above by simply neglecting the moments of order 3 and 4 and setting $\widehat{c}_{3,\Delta,T} = \widehat{c}_{4,\Delta,T} = 0$. All asymptotic considerations in Section 3.3.2 below remain true.

**Remark 3.11** As in Haug et al. (2007), we fit the model to the logarithms of the empirical autocovariances rather than the covariances themselves, because this leads to a linear regression and allows to compute the least squares estimator explicitly. Using the empirical covariances as proposed by Barndorff-Nielsen & Shephard (2001), one is lead to a nonlinear least squares problem. Consequently, the existence of a unique solution, which depends on the model parameters in a continuously differentiable way, is no longer obvious and can only be guaranteed under additional assumptions (c.f. e.g. Hansen (1982)). Nevertheless this approach seems to work fine in practice and is the natural choice when considering superpositions of OU processes (cf. Barndorff-Nielsen & Shephard (2001)) of the form $y = \sum_{j=1}^m y^{(j)}$, where $y^{(j)}$, $j = 1,\dots,m$ denote independent stationary OU processes. If each $y^{(j)}$ has mean reversion $\lambda_j$ and IG$(w_j a, b)$ or $\Gamma(w_j a, b)$ marginals with $\sum_{j=1}^m w_j = 1$, Barndorff-Nielsen & Shephard (2001) show that

$$\gamma_\Delta(s) = \omega^2 \sum_{j=1}^m \frac{w_j}{\lambda_j^2}\left(1 - e^{-\lambda_j\Delta}\right)^2 e^{-\lambda_j\Delta(s-1)}, \quad s \in \mathbb{N}^*,$$

which can be used to fit the parameters $\lambda_j, w_j, j = 1,\dots,m$ to the empirical autocovariances via a nonlinear least squares regression.

### 3.3.2 Asymptotic properties of the estimator

Since all estimators in Algorithm 3.8 are continuously differentiable functions of empirical moments, strong consistency and asymptotic normality will follow from ergodicity of the process $(X_{(n)})_{n\in\mathbb{N}^*}$. For stochastic volatility models with stock prices driven by Brownian motion, it has been shown independently by Genon-Catalot et al. (2000) and Sørensen (2000) that the return sequence $(X_{(n)})_{n\in\mathbb{N}^*}$ is $\alpha$-mixing (and hence ergodic), if $y$ is $\alpha$-mixing and further that the mixing coefficents for returns are smaller than or equal to the mixing coefficients of $y$. An inspection of the arguments in Genon-Catalot et al. (2000) shows that this remains true for time-changed Lévy models.

**Theorem 3.12** *Suppose the process $y$ is strictly stationary and $\alpha$-mixing with mixing coefficients $(\alpha_y(k))_{k\in\mathbb{R}_+}$. Then $(X_{(n)})_{n\in\mathbb{N}^*}$ is also strictly stationary and $\alpha$-mixing with mixing coefficients $(\alpha_X(n))_{n\in\mathbb{N}^*}$ satisfying*

$$\alpha_X(n) \le \alpha_y(n), \quad \forall n \in \mathbb{N}^*.$$

*In particular, $(X_{(n)})_{n\in\mathbb{N}^*}$ is ergodic and if $y$ is $\alpha$-mixing with exponentially decreasing rate, then $(X_{(n)})_{n\in\mathbb{N}^*}$ is $\alpha$-mixing with exponentially decreasing rate, too.*

PROOF. We generalize the arguments of (Genon-Catalot et al., 2000, Sections 3.1, 3.2) to time-changed Lévy models. In view of (Genon-Catalot et al., 2000, Proposition 3.1) it is enough to check the prerequisites of (Genon-Catalot et al., 2000, Definition 3.1). The first property of (Genon-Catalot et al., 2000, Definition 3.1) follows as in (Genon-Catalot et al., 2000, Theorem 3.1) if the space of continuous functions and its Borel $\sigma$-algebra associated with the uniform topology are replaced with the Skorokhod space $\mathbb{D}$ and its Borel $\sigma$-algebra $\mathscr{D}$ associated with the Skorokhod topology (cf. (JS, Chapter VI) and in particular Theorem VI.1.14 for more details), because the mapping

$$T : \mathbb{D} \to \mathbb{R}^2; \quad (f(t))_{t\in\mathbb{R}_+} \mapsto \left( \int_0^\Delta f(s)ds, f(\Delta) \right)$$

is $\mathscr{D}$-$\mathscr{B}(\mathbb{R})$ measurable. The other two properties of (Genon-Catalot et al., 2000, Definition 3.1) follow literally as in (Genon-Catalot et al., 2000, Theorem 3.1) by applying (Jacod & Shiryaev, 2003, II.4.15), because $X$ has independent increments on $[\![0, n\Delta]\!]$ conditional on $\sigma(y_s, s \le n\Delta)$. □

Theorem 3.12 is applicable in our setup because of the following well known fact.

**Lemma 3.13** *Let $y$ be a strictly stationary OU process such that $E(|y_t|^p) < \infty$ for some $p > 0$. Then $y$ is $\alpha$-mixing with exponentially decreasing rate.*

PROOF. By (Masuda, 2004, Theorem 4.3) the process $y$ is $\beta$-mixing with exponentially decreasing rate, hence also $\alpha$-mixing with exponentially decreasing rate (cf. e.g. (Genon-Catalot et al., 2000, Section 2.3)). □

By Birkhoff's ergodic theorem (cf. (Shiryaev, 1995, Theorem V.3.1)) all moments estimators in Algorithm 3.8 are strongly consistent.

**Corollary 3.14** *Assuming that (A1)-(A3) hold, we have, for $T \to \infty$,*

$$\widehat{m}_{1,\Delta,T} \xrightarrow{\text{a.s.}} m_{1,\Delta}, \quad \widehat{\mu}_{i,\Delta,T} \xrightarrow{\text{a.s.}} \mu_{i,\Delta}, \quad i = 2,3,4, \quad \widehat{\gamma}_{\Delta,T}(s) \xrightarrow{\text{a.s.}} \gamma_\Delta(s), \quad s = 1,\dots,d.$$

Next we turn to asymptotic normality, which can be obtained by applying a central limit theorem for strongly mixing processes under the following additional assumption.

(A4) $E(X_{(n)}^{8+\varepsilon}) < \infty$ for some $\varepsilon > 0$.

**Remark 3.15** Since $E(B_1) = 0$, condition (A4) holds e.g. if $E(B_1^{10}) < \infty$ and $E(|y_1|^5) < \infty$, since this implies $E(B_{Y_t}^{10}) < \infty$ and hence $E(X_{(n)}^{10}) < \infty$. This can be seen by conditioning on the time-change $Y$ and differentiating the characteristic function of $B$.

**Lemma 3.16** *Let (A1)-(A4) be satisfied. Then, for $T \to \infty$,*

$$\sqrt{\left\lfloor \frac{T}{\Delta} \right\rfloor} \Big( (\widehat{m}_{1,\Delta,T}, \widehat{\mu}_{2,\Delta,T}, \widehat{\mu}_{3,\Delta,T}, \widehat{\mu}_{4,\Delta,T}, \widehat{\gamma}_{\Delta,T,d}) - (m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, \gamma_{\Delta,d}) \Big) \xrightarrow{d} N_{d+4}(\mathbf{0}, \Sigma),$$

*where the covariance matrix $\Sigma$ has components*

$$\Sigma_{k,l} = \mathbb{C}\text{ov}(G_{1,k}, G_{1,l}) + 2 \sum_{j=1}^{\infty} \mathbb{C}\text{ov}(G_{1,k} G_{1+j,l}),$$

*with*

$$G_n := \big( X_{(n)}, (X_{(n)} - m_{1,\Delta})^2, (X_{(n)} - m_{1,\Delta})^3, (X_{(n)} - m_{1,\Delta})^4, \\ (X_{(n)}^2 - \mu_{2,\Delta})(X_{(n+1)}^2 - \mu_{2,\Delta}), \dots, (X_{(n)}^2 - \mu_{2,\Delta})(X_{(n+d)}^2 - \mu_{2,\Delta}) \big).$$

PROOF. Since $(X_{(n)})_{n \in \mathbb{N}^*}$ is strongly mixing with exponentially decreasing rate, the claim follows from the Ibragimov central limit theorem for strongly mixing processes (cf. (Ibragimov & Linnik, 1971, Theorem 18.5.3)) along the lines of the proof of (Haug et al., 2007, Proposition 3.7). □

Summing up, we have the following result.

**Theorem 3.17** *Assume (A1)-(A3) hold. Then, for $T \to \infty$,*

$$(\widehat{\mu}_{\Delta,T}, \widehat{c}_{3,\Delta,T}, \widehat{c}_{4,\Delta,T}, \widehat{\lambda}_{\Delta,T}, \widehat{\xi}_{\Delta,T}, \widehat{\omega^2}_{\Delta,T}) \xrightarrow{\text{a.s.}} (\mu, c_3, c_4, \lambda, \xi, \omega^2)$$

*If additionally (A4) holds, then, for $T \to \infty$,*

$$\sqrt{\left\lfloor \frac{T}{\Delta} \right\rfloor} \Big( (\widehat{\mu}_{\Delta,T}, \widehat{c}_{3,\Delta,T}, \widehat{c}_{4,\Delta,T}, \widehat{\lambda}_{\Delta,T}, \widehat{\xi}_{\Delta,T}, \widehat{\omega}^2_{\Delta,T}) - (\mu, c_3, c_4, \lambda, \xi, \omega^2) \Big) \xrightarrow{\text{d}} \nabla J_\Delta(m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, \gamma_{\Delta,d}) N_{d+4}(0, \Sigma),$$

*where $\Sigma$ is defined as in Lemma 3.16.*

PROOF. The strong consistency follows from Corollary 3.14 by the continuous mapping theorem (cf. (van der Vaart, 1998, Theorem 2.3)) and the asymptotic normality is a consequence of Lemma 3.16 and the delta method (cf. (van der Vaart, 1998, Theorem 3.1)), because $J_\Delta$ is continuously differentiable in $(m_{1,\Delta}, \mu_{2,\Delta}, \mu_{3,\Delta}, \mu_{4,\Delta}, \gamma_{\Delta,d})$. □

### 3.3.3 Estimation results for real data

Using Algorithm 3.8 proposed above, we now fit the time-changed Lévy-model to real data. As in e.g. Andersen et al. (2002), Chernov et al. (2003), Eraker et al. (2003) we consider a long time series of daily returns, since this provides rich information about the conditional and unconditional distribution of the returns while allowing us to sidestep the seasonality issues inherent in high frequency data, which are beyond our scope here.

Consequently, we use a daily time series of the German industrial index *Dax* $S$ spanning from the 14th of June in 1988 to the 10th of April in 2008 (i.e. $T = 20$, $\Delta = 1/250$ and $T/\Delta = 5000$ returns). The paths of the returns $(X_{(n)})_{n\in\mathbb{N}^*}$ and the logarithmized price $(X_t)_{t\in\mathbb{R}_+}$ with $X_t = \log(S_t/S_0)$ are depicted in Figure 3.1 below.

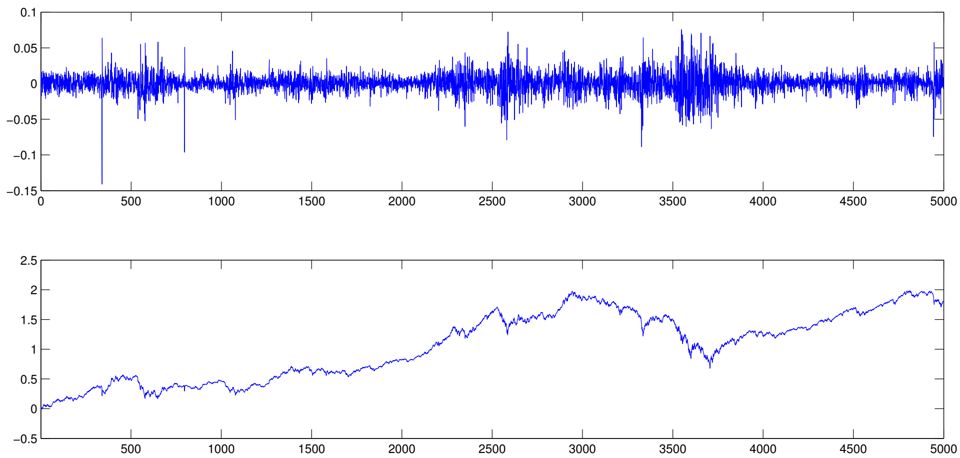


Figure 3.1: log-returns $(X_{(n)})_{n\in\mathbb{N}^*}$ (first) and log-price $X$ (second) of the Dax

Following Haug et al. (2007), we use $d \approx \sqrt{\lfloor T/\Delta \rfloor}$, i.e. $d = 70$ for $T = 20$ and $\Delta = 1/250$. The results are shown in Table 3.1 below.

| $\widehat{\mu}_{1/250,20}$ | $\widehat{c}_{1,1/250,20}$ | $\widehat{c}_{3,1/250,20}$ | $\widehat{c}_{4,1/250,20}$ | $\widehat{\lambda}_{1/250,20}$ | $\widehat{\xi}_{1/250,20}$ | $\widehat{\omega}^2_{1/250,20}$ |
|---|---|---|---|---|---|---|
| 0.0894 | 0 | -0.00549 | 0.000445 | 2.54 | 0.0485 | 0.00277 |

Table 3.1: Estimation results based on Algorithm 3.8 with $d = 70$.

**Remark 3.18** Many applications in Mathematical Finance require a model for the stock price discounted by a bond $S_t^0 = e^{rt}$ with constant interest rate $r$. If we use the average $0.0456$ of the 6-month EURIBOR from its inception as a proxy for $r$ and estimate the parameters of the discounted model using Algorithm 3.8, we obtain the results shown in Table 3.2. Only the estimate of $\mu$ changes, since all other estimators use centered moments.

| $\widehat{\mu}_{1/250,20}$ | $\widehat{c}_{1,1/250,20}$ | $\widehat{c}_{3,1/250,20}$ | $\widehat{c}_{4,1/250,20}$ | $\widehat{\lambda}_{1/250,20}$ | $\widehat{\xi}_{1/250,20}$ | $\widehat{\omega}^2_{1/250,20}$ |
|---|---|---|---|---|---|---|
| 0.0438 | 0 | -0.00549 | 0.000445 | 2.54 | 0.0485 | 0.00277 |

Table 3.2: Estimation results for the discounted stock price with Algorithm 3.8.

The fitted model accounts for the skewness of $-0.3943$ and the kurtosis of $8.8210$ exhibited by our data set, i.e. both for asymmetry and heavy tails. For the returns and squared returns, the empirical autocorrelation functions and their theoretical counterparts are shown in Figure 3.2 below, indicating that the dependency structure is fit quite well, too.

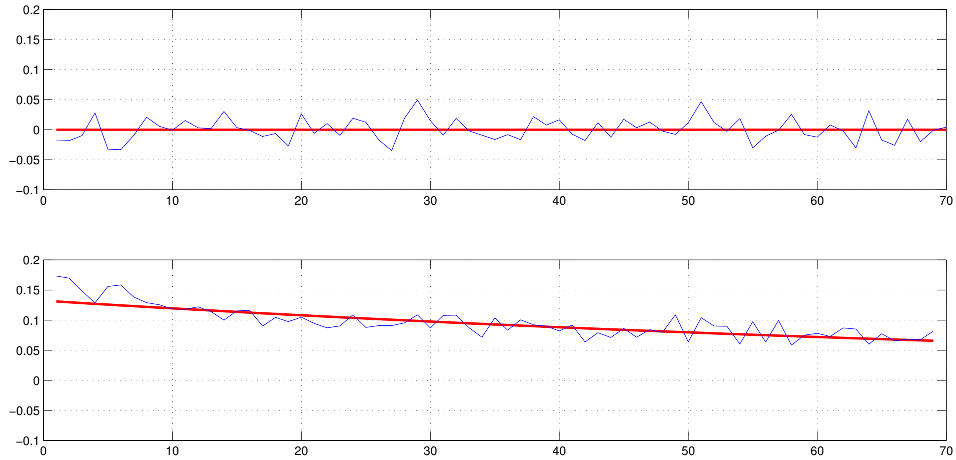


Figure 3.2: Empirical (blue) and fitted (red) autocorrelation functions of the log returns (first) and the squared log returns (second)

**Remark 3.19** An inspection of the crosscorrelation between the returns and the squared returns reveals that the leverage effect is present in our data set as well.

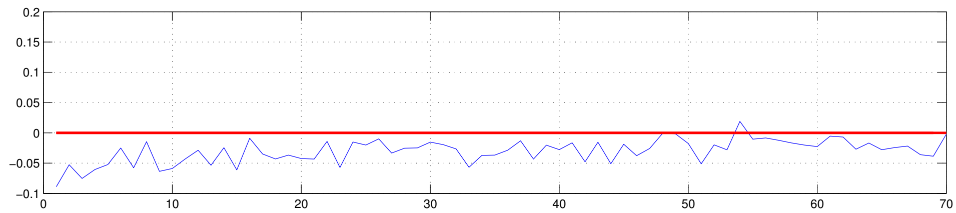


Figure 3.3: Empirical (blue) and fitted (red) crosscorrelation function of the log returns and the squared log returns

Assuming $y$ is an OU process driven by a subordinator $Z$, this effect can be accounted for by introducing a leverage term and generalizing the model to

$$X_t = \mu t + B_{Y_t} + \varrho Z_{\lambda t}, \quad \varrho \in [-1, 1].$$

For the BNS model, this is discussed in detail by Barndorff-Nielsen & Shephard (2001), who also calculate the resulting second order dependence structure. These results can be extended to cover the present setup, however this class of models is not very tractable from the point of view of Mathematical Finance. Hence we do not go into more details here.

**Remark 3.20** As discussed in Remark 3.11 above, it is also possible to consider superpositions of OU processes and fit them to the empirical autocovariances. Using the MATLAB nonlinear least squares routine `lsqnonlin`, this approach yields the following set of parameter estimates for the superposition of two independent OU-processes with mean reversion $\lambda_j$, mean $w_j\xi$ and variance $w_j\omega^2$, $j = 1, 2$:

$$\widehat{\xi} = 0.0485, \quad \widehat{\omega}^2 = 0.00402, \quad \widehat{w}_1 = 0.446, \quad \widehat{\lambda}_1 = 32.5, \quad \widehat{w}_2 = 0.554, \quad \widehat{\lambda}_2 = 1.38.$$

The corresponding fitted autocorrelation function for the superposition of two OU processes is shown alongside its counterpart for one OU process in Figure 3.4 below.

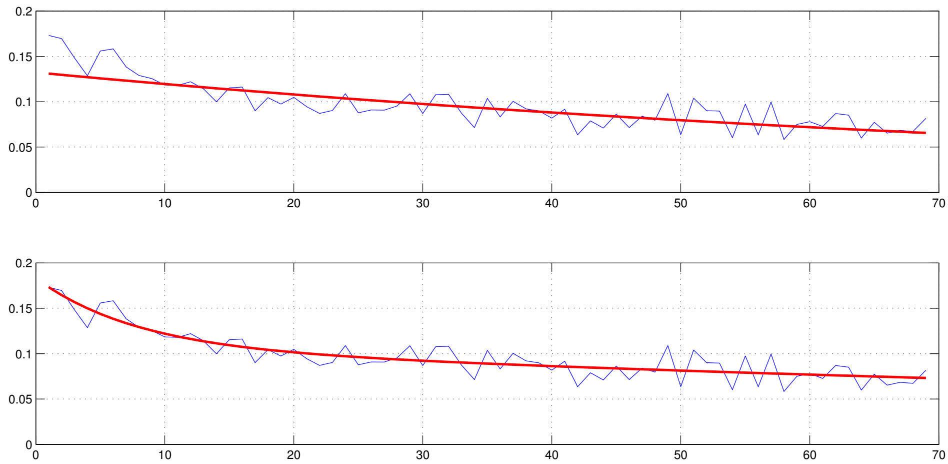


Figure 3.4: Empirical (blue) and fitted (red) autocorrelation functions of the squared log returns for a superposition of one (first) and two (second) OU processes.

Clearly, the fit is improved considerably for short lags, although the overall effect is not too big for our daily data. If one moves to more highly frequent data, however, several OU processes become indispensable to model dependencies on different time scales.

So far these results are really of semiparametric nature, since we have not completely specified the processes $B$ and $y$ yet. We now present some examples of parametric models commonly used in the literature.

**Example 3.21 (IG-OU process, Gamma-OU process)** Suppose $y$ follows a stationary IG-OU process (cf. e.g. Schoutens (2003)) with $\text{IG}(a, b)$ marginals. Then $a = \sqrt{\xi^3/\omega^2}$ and

$b = \sqrt{\xi/\omega^2}$, hence strongly consistent and asymptotically normal estimators are given by

$$\widehat{a}_{1/250,20} = \sqrt{\frac{\widehat{\xi}^3_{1/250,20}}{\widehat{\omega}^2_{1/250,20}}} = 0.203, \quad \widehat{b}_{1/250,20} = \sqrt{\frac{\widehat{\xi}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}}} = 4.1835.$$

If $y$ follows a stationary Gamma-OU process with $\Gamma(a, b)$ marginals, the corresponding estimators are

$$\widehat{a}_{1/250,20} = \frac{\widehat{\xi}^2_{1/250,20}}{\widehat{\omega}^2_{1/250,20}} = 0.8483, \quad \widehat{b}_{1/250,20} = \frac{\widehat{\xi}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}} = 17.5013.$$

**Example 3.22 (BNS model)** In the BNS model, $B$ is chosen to be a Brownian motion with drift. In this case, $c_1 = 0$ and $c_2 = 1$ imply that $B$ is a Standard Brownian Motion.

If the BNS model is estimated using Algorithm 3.8 (which closely resembles the approach of Barndorff-Nielsen & Shephard (2001) in this case), the third and fourth moments of the model are not fitted to the data. More specifically, Theorem 3.2 yields that the fitted BNS model has skewness $0$ and kurtosis $6.52$ compared with the values $-0.39$ and $8.82$ observed in our data set. This shows that stochastic volatility without jumps in the asset price cannot explain the skewness in the data and can only account for a part of the heavy tails. To show the full flexibility of the class of models considered here, we now consider a Lévy process $B$ with jumps. More specifically, we assume that $B$ is modelled as an NIG process, which is a popular model for stock prices itself (cf. e.g. Barndorff-Nielsen (1997, 1998), Rydberg (1997)).

**Example 3.23 (NIG process)** Let $B$ be a NIG process with characteristic function

$$E\left(e^{iuB_t}\right) = \exp\left(t\left(iu\delta + \vartheta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right)\right)\right),$$

where $\delta \in \mathbb{R}$, $\alpha, \vartheta > 0$ and $\beta \in (-\alpha, \alpha)$. Then by e.g. (Schoutens, 2003, Section 5.3.8),

$$c_1 = \delta + \frac{\vartheta\beta}{\sqrt{\alpha^2-\beta^2}}, \quad c_2 = \frac{\alpha^2\vartheta}{(\alpha^2-\beta^2)^{3/2}}, \quad c_3 = \frac{3\beta\alpha^2\vartheta}{(\alpha^2-\beta^2)^{5/2}}, \quad c_4 = \frac{3\alpha^2\vartheta(\alpha^2+4\beta^2)}{(\alpha^2-\beta^2)^{7/2}}.$$

Hence Conditions (A1)-(A4) are satisfied for $\vartheta = (\alpha^2-\beta^2)^{3/2}\alpha^{-2}$ and $\delta = -\beta(\alpha^2-\beta^2)\alpha^{-2}$. Solving for $\alpha, \beta, \delta, \vartheta$, this leads to the following estimators, which are strongly consistent and asymptotically normal by Theorem 3.17 above:

$$\widehat{\beta}_{\Delta,T} := \frac{\widehat{c}_{3,\Delta,T}}{\widehat{c}_{4,\Delta,T} - 5\widehat{c}^2_{3,\Delta,T}/3}, \quad \widehat{\alpha}_{\Delta,T} := \sqrt{\widehat{\beta}^2_{\Delta,T} + 3\widehat{\beta}_{\Delta,T}/\widehat{c}_{3,\Delta,T}},$$

$$\widehat{\vartheta}_{\Delta,T} := \frac{(\widehat{\alpha}^2_{\Delta,T} - \widehat{\beta}^2_{\Delta,T})^{3/2}}{\widehat{\alpha}^2_{\Delta,,T}}, \quad \widehat{\delta}_{\Delta,T} := \frac{-\widehat{\vartheta}_{\Delta,T}\widehat{\beta}_{\Delta,T}}{\sqrt{\widehat{\alpha}^2_{\Delta,T} - \widehat{\beta}^2_{\Delta,T}}},$$

For our data set, this yields

$$\widehat{\beta}_{1/250,20} = -13.9,\ \widehat{\alpha}_{1/250,20} = 88.3,\ \widehat{\vartheta}_{1/250,20} = 85.0,\ \widehat{\delta}_{1/250,20} = 13.6.$$

### 3.3.4 Simulation study

To investigate the small sample behavior of our estimation algorithm, we now assume that $X$ is given by a NIG-IG-OU process, i.e. $y$ is chosen to be a stationary IG-OU process with mean reversion $\lambda$ and marginal IG$(\sqrt{\xi^3/\omega^2}, \sqrt{\xi/\omega^2})$ distributions, whereas the Lévy process $B$ is assumed to be a NIG process.

As for parameters, we use the estimates obtained from our daily Dax time series in Examples 3.21, 3.23 above. Sample paths of an NIG-IG-OU process can easily be simulated using algorithms found in (Schoutens, 2003, Sections 8.4.5, 8.4.7), examples being shown in Figure 3.5 below.

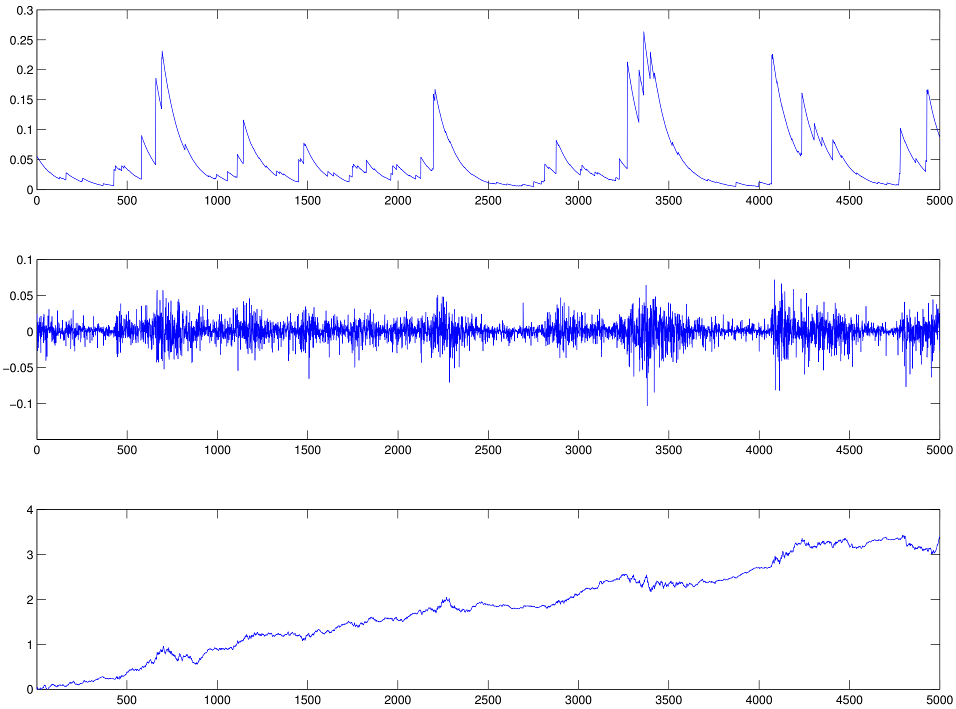


Figure 3.5: Sample paths of the volatility $y$ (first), the returns $X_{(n)}$ (second) and the $\log$-asset price $X$ (third) for a NIG-IG-OU process with parameters as in Examples 3.21, 3.23.

We simulate $1000$ samples of equidistant observations of returns $X_{(n)}$ for $\Delta = 1/250$ and $T = 20$ and $T = 40$, where we first work on a finer grid with 80 steps per day and then only use the returns on the original grid to minimize discretization errors.

The results are shown in Table 3.3 below. As above, we have chosen $d \approx \sqrt{\lfloor T/\Delta \rfloor}$, i.e. $d = 70$ for $T = 20$ and $d = 100$ for $T = 40$ here. Note that we measure the estimation error relative to the true values of the parameters in order to account for the different sizes of the parameters under consideration.

| | $\mu$ | $c_3$ | $c_4$ | $\lambda$ | $\xi$ | $\omega^2$ |
|---|---|---|---|---|---|---|
| True Value | 0.0894 | -0.00549 | 0.000445 | 2.54 | 0.0485 | 0.00277 |
| $T = 20$ | $\widehat{\mu}_{1/250,T}$ | $\widehat{c}_{3,1/250,T}$ | $\widehat{c}_{4,1/250,T}$ | $\widehat{\lambda}_{1/250,T}$ | $\widehat{\xi}_{1/250,T}$ | $\widehat{\omega}^2_{1/250,T}$ |
| Mean | 0.0856 | -0.0543 | 0.000454 | 3.01 | 0.0478 | 0.00250 |
| AAPE | 0.427 | 0.354 | 0.307 | 0.339 | 0.174 | 0.454 |
| $T = 40$ | $\widehat{\mu}_{1/250,T}$ | $\widehat{c}_{3,1/250,T}$ | $\widehat{c}_{4,1/250,T}$ | $\widehat{\lambda}_{1/250,T}$ | $\widehat{\xi}_{1/250,T}$ | $\widehat{\omega}^2_{1/250,T}$ |
| Mean | 0.0910 | -0.00547 | 0.000450 | 2.82 | 0.0484 | 0.00272 |
| AAPE | 0.311 | 0.271 | 0.231 | 0.242 | 0.125 | 0.347 |

Table 3.3: Estimated mean and average absolute percentage error for the parameters $\widehat{\mu}_{\Delta,T}$, $\widehat{c}_{3,\Delta,T}$, $\widehat{c}_{4,\Delta,T}$, $\widehat{\lambda}_{\Delta,T}$, $\widehat{\xi}_{\Delta,T}$ and $\widehat{\omega}^2_{\Delta,T}$ estimated with Algorithm 3.8.

The estimators seem to be fairly consistent for the sample size under consideration, the only notable exception being the mean reversion parameter $\lambda$ which is markedly biased to the right. We also find that the average error for the drift rate $\mu$ is substantially larger than for the mean volatility $\xi$, an effect which is well known from estimation of the classical Black-Scholes model.

Moving from $T = 20$ to $T = 40$ we observe that the mean absolute errors decrease by factors of roughly $\sqrt{2}$ as would be expected from the Ibragimov central limit theorem.

### 3.3.5 Estimation of the current level of volatility

The current value of the volatility process $v$ is needed in many applications in Mathematical Finance, e.g. portfolio optimization (c.f. Benth et al. (2003) and Section 4.5) or variance-optimal hedging (cf. Pauwels (2007)). Since it cannot be observed directly, it has to be filtered from the given returns. Assuming $y$ follows an OU process and $c_1 = 0$, we can proceed along the lines of (Barndorff-Nielsen & Shephard, 2001, Section 5.4.3), and obtain a linear state space representation which allows to use the Kalman filter (cf. Harvey (1989) for more details), to provide a best linear (based on $X_{(n)}$ and $X^2_{(n)}$) predictor of $y$. More specifically, it follows from Corollary 3.6 and (Barndorff-Nielsen & Shephard, 2001, Section 5.4.3) that a linear state space representation of $(X_{(n)}, X^2_{(n)})$ is given by

$$\begin{pmatrix} X_{(n)} \\ X^2_{(n)} \end{pmatrix} = \begin{pmatrix} \mu\Delta \\ \mu^2\Delta^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} \lambda(Y_{n\Delta} - Y_{(n-1)\Delta}) \\ y_{n\Delta} \end{pmatrix} + u_n,$$

where the vector martingale difference sequence $u_n$ satisfies

$$\mathbb{V}\mathrm{ar}(u_{1n}) = \Delta\xi, \quad \mathbb{C}\mathrm{ov}(u_{1n}, u_{2n}) = 2\mu\Delta^2\xi + c_3\Delta\xi,$$
$$\mathbb{V}\mathrm{ar}(u_{2n}) = 4\mu^2\Delta^3\xi + 4\frac{\omega^2}{\lambda^2}\left(e^{-\lambda\Delta} - 1 + \lambda\Delta\right) + 2\xi^2\Delta^2 + c_4\Delta\xi + 4\mu\Delta^2 c_3\xi,$$

and

$$\begin{pmatrix} \lambda(Y_{(n+1)\Delta} - Y_{n\Delta}) \\ y_{(n+1)\Delta} \end{pmatrix} = \begin{pmatrix} 0 & 1 - e^{-\lambda\Delta} \\ 0 & e^{-\lambda\Delta} \end{pmatrix} \begin{pmatrix} \lambda(Y_{n\Delta} - Y_{(n-1)\Delta}) \\ y_{n\Delta} \end{pmatrix} + w_n,$$

with IID noise $w_n$ (uncorrelated with $u_n$) satisfying

$$E(w_n) = \xi \begin{pmatrix} e^{-\lambda\Delta} - 1 + \lambda\Delta \\ 1 - e^{-\lambda\Delta} \end{pmatrix},$$

$$\mathbb{V}\mathrm{ar}(w_n) = 2\omega^2 \begin{pmatrix} \lambda\Delta - 2(1-e^{-\lambda\Delta}) + \frac{1}{2}(1-e^{-2\lambda\Delta}) & \frac{1}{2}(1-e^{-\lambda\Delta})^2 \\ \frac{1}{2}(1-e^{-\lambda\Delta})^2 & \frac{1}{2}(1-e^{-\lambda\Delta}) \end{pmatrix}.$$

In Figure 3.6 below we show the results of applying the Kalman filter to the simulated returns, suggesting it is possible to obtain decent estimates of the volatility in this way.

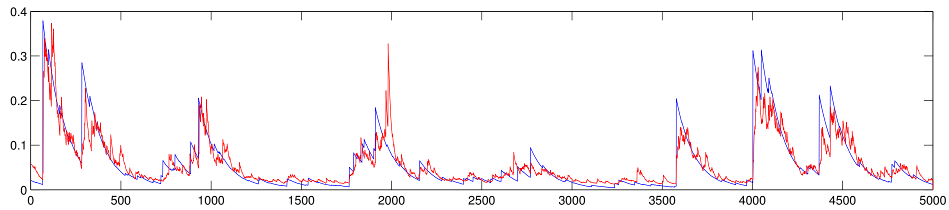


Figure 3.6: Sample paths of an IG-OU process (blue) with parameters as in Example 3.21 and the Kalman filter estimate (red) obtained from the corresponding NIG-IG-OU process with parameters as in Examples 3.23, 3.21.

Notice that if the marginal distribution of $B$ is known (as e.g. for VG or NIG processes), it is also possible to use a particle filter (cf. e.g. Pitt & Shephard (1999) for details). Since Barndorff-Nielsen & Shephard (2001) have noted that estimates obtained from the Kalman filter and the particle filter are close together in the BNS model, we restrict ourselves to the simpler Kalman filter approach here and leave an application of particle filters for future research.

## 3.4 Moment estimation for arbitrary *B*

We now consider the case where $\mu = 0$ and the Lévy process $B$ is not necessarily assumed to be a martingale, i.e. $c_1 \neq 0$. Estimation is done subject to the following assumptions:

(B1) For time horizon $T > 0$ and grid size $\Delta > 0$ we have equally spaced observations $X_{j\Delta}$, $j = 0, ..., \lfloor T/\Delta \rfloor$ leading to returns $X_{(j)} = X_{j\Delta} - X_{(j-1)\Delta}$, $j = 1, ..., \lfloor T/\Delta \rfloor$.

(B2) $\mu = 0$ and the cumulants of $B$ satisfy $c_2 = 1$ and $c_4 < \infty$.

(B3) $y$ is a stationary OU or CIR process with mean reversion $\lambda > 0$, mean $\xi > 0$, variance $\omega^2 > 0$ and existing fourth moments.

(B4) $E(X_{(n)}^{8+\varepsilon}) < \infty$ for some $\varepsilon > 0$.

As above, for given grid size $\Delta > 0$ we write $\mu_{i,\Delta}$ and $m_{i,\Delta}$ for the $i$-th centered and uncentered moment of $X_{(n)}$, set $\gamma_\Delta(s) := \mathbb{C}\mathrm{ov}(X_{(n)}, X_{(n+s)})$ for $s \in \mathbb{N}^*$ and define $\gamma_{\Delta,d_\Delta} := (\gamma_\Delta(1), \ldots, \gamma_\Delta(d_\Delta))$ for $d_\Delta \in \mathbb{N}^*$.

### 3.4.1 Approximate moments

The key to the estimation algorithms proposed below is the following observation made by Barndorff-Nielsen & Shephard (2006).

**Lemma 3.24** *Assume that (B1)-(B3) hold. Then for $\Delta \downarrow 0$,*

$$m_{3,\Delta} = c_3 \Delta \xi + 3c_1 \left( \frac{2\omega^2}{\lambda^2}(e^{-\lambda\Delta} - 1 + \lambda\Delta) + \Delta^2\xi^2 \right) + O(\Delta^3),$$
$$m_{4,\Delta} = c_4 \Delta \xi + (3 + 2c_1c_3) \left( \frac{2\omega^2}{\lambda^2}(e^{-\lambda\Delta} - 1 + \lambda\Delta) + \Delta^2\xi^2 \right) + O(\Delta^3),$$

*as well as , for $s \in \mathbb{N}^*$ and $\Delta \downarrow 0$,*

$$\gamma_\Delta(h) = \omega^2 \frac{(1 - e^{-\lambda\Delta})^2}{\lambda^2} e^{-\lambda\Delta(s-1)} + O(\Delta^3) = \omega^2\Delta^2 e^{-\lambda\Delta(s-1)} + O(\Delta^3).$$

PROOF. (Barndorff-Nielsen & Shephard, 2006, Propositions 4, 5). □

### 3.4.2 The estimation procedure

By neglecting all terms of order $\Delta^3$ or higher in Lemma 3.24, we obtain the following approximations of the model parameters by moments of the returns and the autocovariance function of the squared returns.

**Lemma 3.25** *Assume (B1)-(B3) and let $k, p \in \mathbb{R}_{++}$ be constants such that, for fixed $D \in \mathbb{N}^*$ and $\Delta \downarrow 0$,*

$$\gamma_\Delta(s) = k\Delta^2 e^{-p\Delta(s-1)} + O(\Delta^3), \quad s \in \left\{ 1, \ldots, \left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor + 1 \right\}. \tag{3.4}$$

*Then, for sufficiently small $\Delta$, there exists a largest solution $x_\Delta > 0$ to*

$$0 = \mu_{2,\Delta} x^2 - \Delta x^3 - m_{1,\Delta}^2 k,$$

*and we have, for $\Delta \downarrow 0$,*

$$\lambda = p + O(\sqrt{\Delta}/D), \quad \omega^2 = k + O(\Delta), \quad \xi = x_\Delta + O(\Delta^2), \quad c_1 = \frac{m_{1,\Delta}}{\Delta x_\Delta} + O(\Delta^2),$$
$$c_3 = \frac{m_{3,\Delta}}{\Delta x_\Delta} - 3m_{1,\Delta}\left(1 + \frac{k}{x_\Delta^2}\right) + O(\Delta^2),$$
$$c_4 = \frac{m_{4,\Delta}}{\Delta x_\Delta} - \left\{ \frac{3\Delta}{x_\Delta} + \frac{2m_{1,\Delta}}{x_\Delta^2}\left( \frac{m_{3,\Delta}}{\Delta x_\Delta} - 3m_{1,\Delta}\left(1 + \frac{k}{x_\Delta^2}\right)\right)\right\} (x_\Delta^2 + k) + O(\Delta^2).$$

PROOF. Inserting $s = 1$ into Equation (3.4) and applying Lemma 3.24 yields

$$k\Delta^2 = \omega^2 \frac{(1 - e^{-\lambda\Delta})^2}{\lambda^2} + O(\Delta^3) = \omega^2\Delta^2 + O(\Delta^3),$$

and hence $\omega^2 = k + O(\Delta)$ for $\Delta \downarrow 0$. Using this formula, Lemma 3.24 and Equation (3.4) with $s - 1 = \lfloor D/\sqrt{\Delta} \rfloor$ we obtain

$$\left| e^{-p\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor} - e^{-\lambda\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor} \right| \leq C_1 \Delta$$

for some constant $C_1 > 0$ and sufficiently small $\Delta$. Since $\Delta\lfloor D/\sqrt{\Delta}\rfloor$ is bounded from above for $\Delta \downarrow 0$ this implies that there exists a constant $C_2 > 0$ such that

$$\begin{aligned} |\lambda - p| &= \frac{|\log(e^{-p\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor}) - \log(e^{-\lambda\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor})|}{\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor} \\ &\leq \frac{C_2}{\lfloor D/\sqrt{\Delta} \rfloor} \frac{|e^{-p\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor} - e^{-\lambda\Delta\left\lfloor \frac{D}{\sqrt{\Delta}} \right\rfloor}|}{\Delta} \leq \frac{C_2 C_1}{\lfloor D/\sqrt{\Delta} \rfloor}, \end{aligned}$$

and hence $\lambda = p + O(\sqrt{\Delta}/D)$ for $\Delta \downarrow 0$. By Theorem 3.2 we have $m_{2,\Delta} = \Delta\xi + c_1^2(2\omega^2 r_y^{**}(\Delta) + \Delta^2\xi^2)$. Inserting $c_1 = m_{1,\Delta}/\Delta\xi$ as well as the expressions for $\omega^2$ and $r_y^{**}$ and rearranging terms shows that

$$0 = \mu_{2,\Delta}\xi^2 - \Delta\xi^3 - m_{1,\Delta}^2 k + O(\Delta^3). \tag{3.5}$$

Differentiation shows that the mapping

$$f_\Delta : x \mapsto \mu_{2,\Delta} x^2 - \Delta x^3$$

attains a unique maximum at $x_{0,\Delta} = 2\mu_{2,\Delta}/3\Delta$ with $f_\Delta(x_{0,\Delta}) = 4\mu_{2,\Delta}^3/27\Delta^2$, and is strictly decreasing on $(x_{0,\Delta}, \infty)$. Note that $m_{1,\Delta}^2 k = O(\Delta^2)$ whereas $f_\Delta(x_{0,\Delta}) = O(\Delta)$, hence $f_\Delta(x_{0,\Delta}) > m_{1,\Delta}^2 k$ for sufficiently small $\Delta$. Together with $\lim_{x\to\infty} f_\Delta(x) = -\infty$ and the continuity of $f_\Delta$ this implies that for $\Delta \downarrow 0$ there exists a unique solution $x_\Delta$ to $f_\Delta(x) = m_{1,\Delta}^2 k$ on $(x_{0,\Delta}, \infty)$.
Since we have $\mu_{2,\Delta} = \Delta\xi + O(\Delta^2)$ for $\Delta \downarrow 0$ by Theorem 3.2, $f_\Delta(x_\Delta) = m_{1,\Delta}^2 k$ and Equation (3.5) yield $\Delta x_\Delta^2(x_\Delta - \xi) = O(\Delta^2)$, and hence $\xi = x_\Delta + O(\Delta)$, because $x_\Delta > x_{0,\Delta} = 2\xi/3 + O(\Delta) > \xi/3 > 0$ for $\Delta \downarrow 0$ by Theorem 3.2.

As $f_\Delta$ is strictly decreasing on $(x_{0,\Delta}, \infty)$ and hence on $\mathscr{U}_\Delta := [x_\Delta \wedge \xi, x_\Delta \vee \xi]$ for sufficiently small $\Delta$, the inverse mapping $f_\Delta^{-1}$ is well-defined on $\mathscr{U}_\Delta$ and continuously differentiable in the interior of $\mathscr{U}_\Delta$. Since $f'_\Delta$ is decreasing on $\mathscr{U}_\Delta$ for sufficiently small $\Delta$, the mean value theorem implies

$$|x_\Delta - \xi| = |f_\Delta^{-1}(f_\Delta(x_\Delta)) - f_\Delta^{-1}(f_\Delta(\xi))| \leq \max\left\{ \left| \frac{1}{f'(x_\Delta)} \right|, \left| \frac{1}{f'(\xi)} \right| \right\} |f_\Delta(x_\Delta) - f_\Delta(\xi)|, \tag{3.6}$$

where we have used $(f_\Delta^{-1})'(f_\Delta(x)) = 1/f'_\Delta(x)$. Now notice that $\mu_{2,\Delta} = \Delta\xi + O(\Delta^2)$ and $\xi = x_\Delta + O(\Delta)$ yield $f'_\Delta(x_\Delta) = -\xi^2\Delta + O(\Delta^2)$ and $f'_\Delta(\xi) = -\xi^2\Delta + O(\Delta^2)$. Combining this with $f_\Delta(x_\Delta) - f_\Delta(\xi) = O(\Delta^3)$ and (3.6) shows $\xi = x_\Delta + O(\Delta^2)$ for $\Delta \downarrow 0$ as claimed.

The remaining assertions now follow from Theorem 3.2 and Lemma 3.24. □

**Remark 3.26** In view of Lemma 3.25 the model parameters can be identified by the first four moments and the autocovariance function up to an error term vanishing as the grid size $\Delta$ approaches zero and the number of autocovariance lags taken into account tends to infinity.

Lemma 3.25 motivates the following estimation Algorithm. In view of Theorem 3.28 below, all estimators will again be almost surely well-defined for sufficiently small $\Delta$ and sufficiently large samples.

**Algorithm 3.27** *1. Calculate the moment estimators*

$$\widehat{m}_{i,\Delta,T} := \frac{1}{\lfloor T/\Delta \rfloor} \sum_{j=1}^{\lfloor T/\Delta \rfloor} X_{(n)}^i, \quad i = 1,2,3,4,$$

*as well as for fixed* $D \in \mathbb{N}^*$ *and* $d_\Delta := \lfloor D/\sqrt{\Delta} \rfloor + 1$ *the empirical autocovariances* $\widehat{\gamma}_{\Delta,T,d_\Delta} := (\widehat{\gamma}_{\Delta,T}(1), \ldots, \widehat{\gamma}_{\Delta,T}(d_\Delta))$*, as*

$$\widehat{\gamma}_{\Delta,T}(s) := \frac{1}{\lfloor T/\Delta \rfloor} \sum_{j=1}^{\lfloor T/\Delta \rfloor - s} \left( X_{(j)}^2 - \widehat{m}_{2,\Delta,T} \right) \left( X_{(j+s)}^2 - \widehat{m}_{2,\Delta,T} \right), \quad s = 1, \ldots, d_\Delta.$$

2. *Define the mapping* $K_\Delta : \mathbb{R}_{++}^d \times \mathbb{R}^2 \to \mathbb{R}$ *by*

$$K_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta}, k, p) := \sum_{s=1}^{d_\Delta} \left( \log(\widehat{\gamma}_{\Delta,T,d_\Delta}(s)) - \log(\Delta^2 k) + p\Delta s \right)^2,$$

*and compute the least square estimator*

$$(\widehat{k}_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta}), \widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta})) := \underset{(k,p)\in\mathbb{R}^2}{\arg\min}\, K_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta}, k, p),$$

*which is given by*

$$\widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta}) = -\frac{\sum_{s=1}^{d_\Delta} \left( \log(\widehat{\gamma}_{\Delta,T,d_\Delta}(s)) - \overline{\log(\widehat{\gamma}_{\Delta,T,d_\Delta})} \right) (s - \frac{d_\Delta+1}{2})}{\Delta \sum_{s=1}^{d_\Delta} (s - \frac{d_\Delta+1}{2})^2)},$$
$$\widehat{k}_{\Delta,T}(\widehat{\gamma}_{\Delta,T,d_\Delta}) = \Delta^{-2} \exp\left( \overline{\log(\widehat{\gamma}_{\Delta,T,d_\Delta})} + \frac{d_\Delta + 1}{2} \widehat{p}_\Delta(\widehat{\gamma}_{\Delta,T,d_\Delta}) \right),$$

*with* $\overline{\log(\widehat{\gamma}_{\Delta,T,d_\Delta})} := \frac{1}{d_\Delta} \sum_{s=1}^{d_\Delta} \log(\widehat{\gamma}_{\Delta,T,d_\Delta}(s))$.

3. *Compute*

$$\widehat{x}_\Delta(\widehat{m}_{1,\Delta,T}, \widehat{m}_{2,\Delta,T}, \widehat{\gamma}_{\Delta,T}) := \max\left\{ x \in \mathbb{R} : \widehat{\mu}_{2,\Delta,T} x^2 - \Delta x^3 - \widehat{m}_{1,\Delta,T}^2 \widehat{k}_n(\widehat{\gamma}_{\Delta,T,d_\Delta}) = 0 \right\}.$$

4. *Define the mapping* $H_\Delta : \mathbb{R}_{++} \times \mathbb{R}^4 \times \mathbb{R}_{++} \to \mathbb{R}^6$ *by*

$$H_\Delta(x, m_1, m_3, m_4, k, p) := \Bigg( \frac{m_1}{\Delta x}, \frac{m_3}{\Delta x} - 3m_1\left(1 - \frac{k}{x^2}\right),$$
$$\frac{m_4}{\Delta x} - \left\{ \frac{3\Delta}{x} + \frac{2m_1}{x^2}\left(\frac{m_3}{\Delta x} - 3m_1\left(1 - \frac{k}{x^2}\right)\right)\right\}\left(x^2 + k\right), p, x, k \Bigg).$$

5. *Define the mapping* $J_\Delta : \mathbb{R}^4 \times \mathbb{R}_+^{d_\Delta} \to \mathbb{R}^6$ *by*

$$J_\Delta(m_1, m_2, m_3, m_4, \gamma)$$
$$:= \begin{cases} H(\widehat{x}_\Delta(m_1,\gamma), m_1, m_2, m_3, m_4, \widehat{k}_\Delta(\gamma), \widehat{p}_\Delta(\gamma) & \text{if } \gamma, \widehat{x}_\Delta(m_1, m_2, \gamma), \widehat{p}_\Delta(\gamma) > 0, \\ (0,0,0,0,0,0) & \text{otherwise}, \end{cases}$$

*and compute the estimator*

$$(\widehat{c}_{1,\Delta,T}, \widehat{c}_{3,\Delta,T}, \widehat{c}_{4,\Delta,T}, \widehat{\lambda}_{\Delta,T}, \widehat{\xi}_{\Delta,T}, \widehat{\omega}^2_{\Delta,T}) = J_\Delta(\widehat{m}_{1,\Delta,T}, \widehat{m}_{2,\Delta,T}, \widehat{m}_{3,\Delta,T}, \widehat{m}_{4,\Delta,T}, \widehat{\gamma}_{\Delta,T,d_\Delta}).$$

**Remarks.**

1. Note that the mapping $J_\Delta$ is continuously differentiable in the true parameter values $(m_{1,\Delta}, m_{2,\Delta}, m_{3,\Delta}, m_{4,\Delta}, \gamma_{\Delta,d_\Delta})$, because the implicit function theorem shows that $\widehat{x}_\Delta$ is continuously differentiable in $(m_{1,\Delta}, m_{2,\Delta}, \gamma_{\Delta,d_\Delta})$.

2. As above, $\widehat{c}_{4,\Delta,T} < 0$ is possible depending on the data, which we once again take as a strong indication that the data is too light tailed to be suitably modeled by the class of (semi-) heavy tailed models considered here.

3. Notice that the fitted model only recaptures the first four moments of the data up to an error of order $\Delta^3$. Likewise, the true logarithmized autocovariance function of the model differs from the results of the linear regression in Algortihm 3.27 by an error term of order $\Delta^3$. We will comment on the size of these errors in Section 3.4.6 below.

### 3.4.3 Asymptotic properties of the estimator

In the construction of the estimation algorithms in Section 3.4.2 we had to resort to approximate moments with an error term vanishing only as $\Delta \downarrow 0$. Consequently, strong consistency and asymptotic normality of these algorithms only hold up to this error term as well.

**Theorem 3.28** *Define* $\widehat{c}_{1,\Delta,T}$, $\widehat{c}_{3,\Delta,T}$, $\widehat{c}_{4,\Delta,T}$, $\widehat{\lambda}_{\Delta,T}$, $\widehat{\xi}_{\Delta,T}$, $\widehat{\omega^2}_{\Delta,T}$ *as in Algorithm 3.27 and assume (B1)-(B3) hold. Then for* $\Delta \downarrow 0$, *we have*

$$\lim_{T\to\infty} \left( (\widehat{c}_{1,\Delta,T}, \widehat{c}_{3,\Delta,T}, \widehat{c}_{4,\Delta,T}, \widehat{\lambda}_{\Delta,T}, \widehat{\xi}_{\Delta,T}, \widehat{\omega^2}_{\Delta,T}) - \left((c_1, c_3, c_4, \lambda, \xi, \omega^2) + \varepsilon_\Delta\right) \right) \overset{\text{a.s.}}{=} 0,$$

*and if additionally (B4) holds, then as $T \to \infty$,*

$$\sqrt{\lfloor T/\Delta \rfloor}\Big( (\widehat{c}_{1,\Delta,T}, \widehat{c}_{3,\Delta,T}, \widehat{c}_{4,\Delta,T}, \widehat{\lambda}_{\Delta,T}, \widehat{\xi}_{\Delta,T}, \widehat{\omega^2}_{\Delta,T}) - \big((c_1, c_3, c_4, \lambda, \xi, \omega^2) + \varepsilon_\Delta\big)\Big) \\ \stackrel{\mathrm{d}}{\to} \nabla J_\Delta(m_1, m_2, m_3, m_4, \gamma_{\Delta,d_\Delta}) N_{d+4}(0, \Sigma),$$

*where*

$$\varepsilon_\Delta = (O(\Delta^2), O(\Delta^2), O(\Delta^2), O(\sqrt{\Delta}), O(\Delta^2), O(\Delta)) \quad \textit{for } \Delta \downarrow 0,$$

*and the covariance matrix $\Sigma$ has components*

$$\Sigma_{k,l} = \mathbb{C}\mathrm{ov}(G_{1,k}, G_{1,l}) + 2\sum_{j=1}^{\infty} \mathbb{C}\mathrm{ov}(G_{1,k} G_{1+j,l}),$$

*for*

$$G_n := \Big( X_{(n)}, X_{(n)}^2, X_{(n)}^3, X_{(n)}^4, \\ (X_{(n)}^2 - m_{2,\Delta})(X_{(n+1)}^2 - m_{2,\Delta}), \ldots, (X_{(n)}^2 - m_{2,\Delta})(X_{(n+d)}^2 - m_{2,\Delta}) \Big).$$

PROOF. Set

$$\varepsilon_\Delta := (c_1, c_3, c_4, \lambda, \xi, \omega^2) - J_\Delta(m_{1,\Delta}, m_{2,\Delta}, m_{3,\Delta}, m_{4,\Delta}, \gamma_{\Delta,d_\Delta}),$$

where $d_\Delta = \lfloor \sqrt{D/\Delta} \rfloor$ for some $D \in \mathbb{R}_+$. By Lemma 3.25 and the definition of $J_\Delta$ in Algorithm 3.27, we have

$$\varepsilon_\Delta = (O(\Delta^2), O(\Delta^2), O(\Delta^2), O(\sqrt{\Delta}), O(\Delta^2), O(\Delta)) \quad \text{for } \Delta \downarrow 0.$$

Notice that the proof of Theorem 3.12 also holds in the present setup. Hence, for fixed $\Delta > 0$, the series $(X_{(n)})_{n\in\mathbb{N}^*}$ is ergodic and Birkoff's ergodic theorem yields that for $T \to \infty$, we have

$$\widehat{m}_{i,\Delta,T} \stackrel{\text{a.s.}}{\to} m_{i,\Delta}, \quad i = 1, 2, 3, 4, \qquad \widehat{\gamma}_{\Delta,T,d_\Delta} \stackrel{\text{a.s.}}{\to} \gamma_{\Delta,d_\Delta}.$$

By the continuous mapping theorem (cf. van der Vaart (1998), Theorem 2.3), this implies

$$J_\Delta(\widehat{m}_{1,\Delta,T}, \widehat{m}_{2,\Delta,T}, \widehat{m}_{3,\Delta,T}, \widehat{m}_{4,\Delta,T}, \widehat{\gamma}_{\Delta,T,d_\Delta}) \stackrel{\text{a.s.}}{\to} J_\Delta(\widehat{m}_{1,\Delta}, \widehat{m}_{2,\Delta}, \widehat{m}_{3,\Delta}, \widehat{m}_{4,\Delta}, \widehat{\gamma}_{\Delta,d_\Delta}),$$

as $T \to \infty$, because $J_\Delta$ is continuous in $(m_{1,\Delta}, m_{2,\Delta}, m_{3,\Delta}, m_{4,\Delta}, \gamma_{\Delta,d_\Delta})$. This shows the first statement. The second now follows analogously from the Ibragimov central limit theorem by an application of the delta method (cf. the proofs of Lemma 3.16 and Theorem 3.17 for more details). □

### 3.4.4 Estimation results for real data

We now apply Algorithm 3.27 to the same set of daily DAX data used in Section 3.3 above. The results are shown in Table 3.4 below.

| $\widehat{\mu}$ | $\widehat{c}_{1,1/250,20}$ | $\widehat{c}_{3,1/250,20}$ | $\widehat{c}_{4,1/250,20}$ | $\widehat{\lambda}_{1/250,20}$ | $\widehat{\xi}_{1/250,20}$ | $\widehat{\omega}^2_{1/250,20}$ |
|---|---|---|---|---|---|---|
| 0 | 1.85 | -0.00675 | 0.000448 | 2.54 | 0.0485 | 0.00277 |

Table 3.4: Estimation results based on Algorithm 3.27.

**Remark 3.29** As in Remark 3.18, one can again discount by a constant deterministic interest rate $r = 0.0456$ first and then apply the estimation Algorithm 3.27. Since uncentered moments are used for the estimation of all parameters in Algorithm 3.27, all parameters are potentially affected by this. However, the results shown in Table 3.5 suggest that the effect is quite small for all parameters except for the drift $c_1$.

| $\widehat{\mu}$ | $\widehat{c}_{1,1/250,20}$ | $\widehat{c}_{3,1/250,20}$ | $\widehat{c}_{4,1/250,20}$ | $\widehat{\lambda}_{1/250,20}$ | $\widehat{\xi}_{1/250,20}$ | $\widehat{\omega}^2_{1/250,20}$ |
|---|---|---|---|---|---|---|
| 0 | 0.904 | -0.00610 | 0.000444 | 2.54 | 0.0485 | 0.00278 |

Table 3.5: Estimation results for the discounted stock price based on Algorithm 3.27.

**Example 3.30 (IG-OU process, Gamma-OU process)** Suppose $y$ follows a stationary IG-OU process with IG$(a,b)$ marginals. Plugging in $\widehat{\xi}$ and $\widehat{\omega}^2$ obtained from our data set with Algorithm 3.27, we obtain the following estimators for $a, b$, which are approximately consistent and asymptotically normal for small $\Delta$:

$$\widehat{a}_{1/250,20} = \sqrt{\frac{\widehat{\xi}^{\,3}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}}} = 0.203, \quad \widehat{b}_{1/250,20} = \sqrt{\frac{\widehat{\xi}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}}} = 4.182.$$

For a stationary Gamma-OU process $y$ with $\Gamma(a,b)$ marginals we obtain

$$\widehat{a}_{1/250,20} = \frac{\widehat{\xi}^{\,2}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}} = 0.847, \quad \widehat{b}_{1/250,20} = \frac{\widehat{\xi}_{1/250,20}}{\widehat{\omega}^2_{1/250,20}} = 17.5.$$

Note that these are practically the same parameters as in Section 3.3.

**Example 3.31 (BNS model)** If $B$ is given by a Brownian motion with drift $\delta \in \mathbb{R}$ and volatility $\sigma \in \mathbb{R}_+$, we have $\delta = c_1$ and $\sigma^2 = c_2$. Consequently, $\sigma = 1$ and the estimator $\widehat{\delta}_{1/250,20} = \widehat{c}_{1,1/250,20} = 1.85$ is approximately consistent and asymptotically normal for small $\Delta$. If one considers data discounted with the constant deterministic interest rate $r = 0.0456$, the corresponding estimator is given by $\widehat{\delta} = 0.904$.

**Example 3.32 (NIG process)** Suppose that $B$ is given by an NIG process. Plugging $c_2 = 1$ and the estimates for $c_1, c_3, c_4$ given in Table 3.4 above into

$$\beta = \frac{c_3}{c_4 - 5c_3^2/3}, \quad \alpha = \sqrt{\beta^2 + 3\beta/c_3}, \quad \vartheta = \frac{(\alpha^2-\beta^2)^{3/2}}{\alpha^2}, \quad \delta = c_1 - \frac{\vartheta\beta}{\sqrt{\alpha^2-\beta^2}},$$

yields estimators $\widehat{\beta}_{1/250,20}, \widehat{\alpha}_{1/250,20}, \widehat{\vartheta}_{1/250,20}, \widehat{\delta}_{1/250,20}$ for the parameters $\beta, \alpha, \vartheta, \delta$ of the NIG process, which are approximately consistent and asymptotically normal for small $\Delta$:

$$\widehat{\beta}_{1/250,20} = -18.2, \quad \widehat{\alpha}_{1/250,20} = 91.7, \quad \widehat{\vartheta}_{1/250,20} = 86.3, \quad \widehat{\delta}_{1/250,20} = 17.4.$$

For discounted data, we obtain

$$\widehat{\beta}_{1/250,20} = -16.0, \quad \widehat{\alpha}_{1/250,20} = 90.1, \quad \widehat{\vartheta}_{1/250,20} = 85.9, \quad \widehat{\delta}_{1/250,20} = 15.5.$$

### 3.4.5 Simulation study

We now investigate the performance of Algorithm 3.27 by performing the same simulation study as for Algorithm 3.8 in Section 3.3.4 above.

Consequently, we assume once more that $X$ is given by an NIG-IG-OU process. We again simulate 1000 samples of equidistant observations of returns $X_{(n)}$, $n = 1, \dots, \frac{T}{\Delta}$ for $\Delta = 1/250$ as well as $T = 20$ and $T = 40$, first working on a finer grid with 80 steps per day to minimize discretization errors. As for parameters we use the values given in Examples 3.30 and 3.32, respectively. Simulated sample paths are shown in Figure 3.7 below.

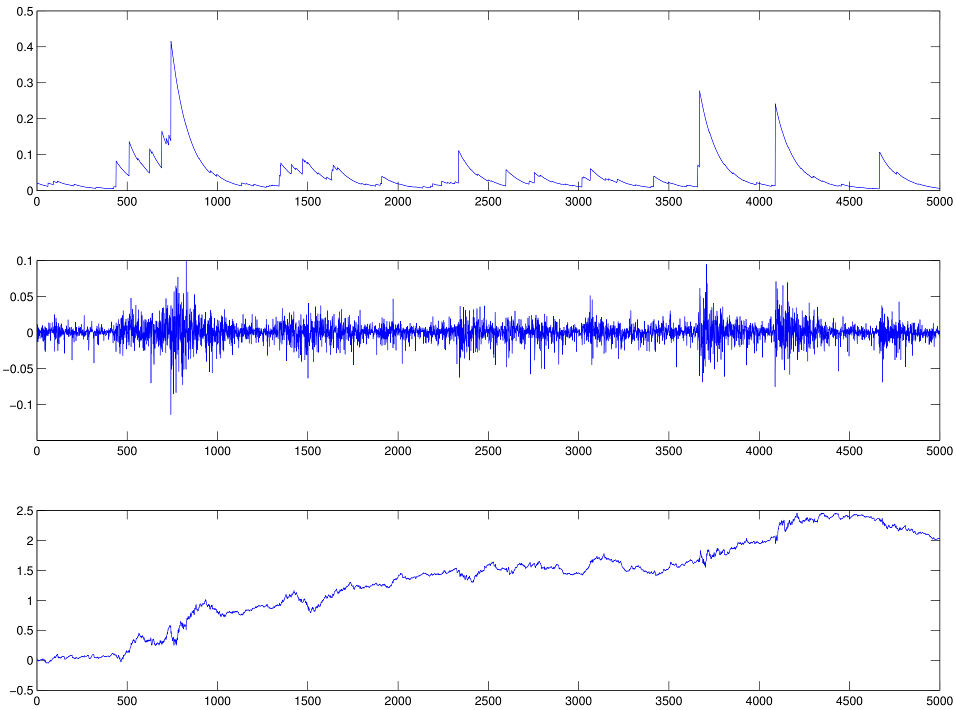


Figure 3.7: Sample paths of the volatility $v$ (first), the returns $y$ (second) and the logarithmized asset price $X$ (third) for an NIG-IG-OU process with parameters as in Examples 3.30, 3.32.

The results of our simulation study are shown in Table 3.6 below. As above we have chosen $d \approx \sqrt{\lfloor T/\Delta \rfloor}$, i.e. $d = 70$ for $T = 20$ and $d = 100$ for $T = 40$.

| | $c_1$ | $c_3$ | $c_4$ | $\lambda$ | $\xi$ | $\omega^2$ |
|---|---|---|---|---|---|---|
| True Value | 1.84 | -0.00675 | 0.000448 | 2.54 | 0.0485 | 0.00278 |
| $T = 20$ | $\widehat{c}_{1,1/250,T}$ | $\widehat{c}_{3,1/250,T}$ | $\widehat{c}_{4,1/250,T}$ | $\widehat{\lambda}_{1/250,T}$ | $\widehat{\xi}_{1/250,T}$ | $\widehat{\omega}^2_{1/250,T}$ |
| Mean | 1.90 | -0.00667 | 0.000452 | 3.00 | 0.0482 | 0.00253 |
| AAPE | 0.463 | 0.278 | 0.335 | 0.347 | 0.176 | 0.467 |
| $T = 40$ | $\widehat{c}_{1,1/250,T}$ | $\widehat{c}_{3,1/250,T}$ | $\widehat{c}_{4,1/250,T}$ | $\widehat{\lambda}_{1/250,T}$ | $\widehat{\xi}_{1/250,T}$ | $\widehat{\omega}^2_{1/250,T}$ |
| Mean | 1.84 | -0.00673 | 0.000445 | 2.83 | 0.0486 | 0.00264 |
| AAPE | 0.318 | 0.189 | 0.270 | 0.243 | 0.123 | 0.357 |

Table 3.6: Estimated mean and average absolute percentage error for the parameters $\widehat{c}_{1,\Delta,T}$, $\widehat{c}_{3,\Delta,T}$, $\widehat{c}_{4,\Delta,T}$, $\widehat{\lambda}_{\Delta,T}$, $\widehat{\xi}_{\Delta,T}$ and $\widehat{\omega}^2_{\Delta,T}$.

Comparing these results with Table 3.3, we find that the use of the approximate moments entails virtually no loss in the quality of the estimators for our daily data. This suggests that the approximation errors resulting from the use of the approximate moment are rather small compared to the variance of our estimators.

### 3.4.6 Computation of the approximation error

The results of our simulation studies suggest that the errors resulting from the use of approximate moments are quite small. However, it is generally difficult to quantify them without resorting to large scale Monte-Carlo simulations. For affine models however, it is sometimes possible to explicitly calculate the joint characteristic function of the returns $X_{(n)}$ and $X_{(n+s)}$ for $n, s \in \mathbb{N}^*$. Differentiation and evaluation at zero via MATLAB's symbolic toolbox then lead to exact equations for moments and autocovariances. These equations do not yield any favorable estimation algorithms, because they are extremely complicated and hideously nonlinear. However, they can comfortably be used for an a posteriori error estimation. We have the following general result from Kallsen (2006):

**Lemma 3.33** *Suppose $B$ has characteristic exponent $\psi^B$ and $y$ is an OU-process driven by a subordinator with characteristic exponent $\psi^Z$. Then for $n, s \in \mathbb{N}^*$, the joint characteristic function of the returns $X_{(n)}$ and $X_{(n+s)}$ is given by*

$$E\left(e^{iu_1 X_{(n)} + iu_2 X_{(n+s)}}\right) = e^{\Psi^0(\Delta,0,iu_2)+\Psi^0((s-1)\Delta,\Psi^1(\Delta,0,iu_2),0)+\Psi^0(\Delta,\Psi^1((s-1)\Delta,\Psi^1(\Delta,0,iu_2),0),iu_1)} \times E\left(e^{\Psi^1(\Delta,\Psi^1((s-1)\Delta,\Psi^1(\Delta,0,iu_2),0),iu_1) y_{(n-1)\Delta}}\right),$$

*where*

$$\Psi^1(t,u_1,u_2) := u_1 e^{-\lambda t} + \frac{1-e^{-\lambda t}}{\lambda}\psi^B(u_2), \quad \Psi^0(t,u_1,u_2) := \int_0^t \psi^Z(\Psi^1(t,u_1,u_2))ds.$$

PROOF. Follows from (Kallsen, 2006, Corollaries 3.2, 3.1). □

If $y$ is chosen to be a stationary Gamma-OU process, all terms can be determined explicitly (cf. e.g. Nicolato & Venardos (2003) for similar formulas in the case where $B$ is a Brownian motion).

**Corollary 3.34** *Suppose $B$ has characteristic exponent $\psi_B$ and $y$ is a $\Gamma(\frac{\xi^2}{\omega^2}, \frac{\xi}{\omega^2})$-OU-process. Then for $s \in \mathbb{N}^*$, the joint conditional characteristic function of the returns $X_{(n)}$ and $X_{(n+s)}$ is given by*

$$E\left(e^{iu_1 X_{(n)} + iu_2 X_{(n+s)}}\right) = e^{\Psi^0(\Delta,0,iu_2)+\Psi^0((s-1)\Delta,\Psi^1(\Delta,0,iu_2),0)+\Psi^0(\Delta,\Psi^1((s-1)\Delta,\Psi^1(\Delta,0,iu_2),0),iu_1)}$$
$$\times\left(1 - \frac{\omega^2}{\xi}\Psi^1(\Delta, \Psi^1((s-1)\Delta, \Psi^1(\Delta, 0, iu_2), 0), iu_1)\right)^{-\xi^2/\omega^2},$$

*where*

$$\Psi^1(t, u_1, u_2) := u_1 e^{-\lambda t} + \frac{1 - e^{-\lambda t}}{\lambda}\psi_B(u_2),$$
$$\Psi^0(t, u_1, u_2) := \frac{\frac{\xi^2}{\omega^2}\left(\frac{\xi}{\omega^2}\log\left(\frac{\xi/\omega^2 - \Psi^1(t,u_1,u_2)}{\xi/\omega^2 - u_1}\right) + \psi_B(u_2)t\right)}{\xi/\omega^2 - \psi_B(u_2)/\lambda}.$$

*Here* log *denotes the* distinguished logarithm *in the sense of (Sato, 1999, Lemma 7.6).*

PROOF. Since $\Psi^1$ is $\mathbb{C}_-$-valued by (Duffie et al., 2003, Propositions 6.1, 6.4), the first formula follows from Lemma 3.33 by inserting the analytic continuation of the characteristic function of the $\Gamma(\frac{\xi^2}{\omega^2}, \frac{\xi}{\omega^2})$-distribution to $\mathbb{C}_-$. By e.g. (Schoutens, 2003, Section 7.1.1) we have $\psi^Z(u) = \frac{\lambda(\xi^2/\omega^2)u}{\xi/\omega^2 - u}$ for the given stationary Gamma-OU process. Substitution into Lemma 3.33 and integration using partial fractions yield the assertion. □

The results of using MATLAB's symbolic toolbox to differentiate and evaluate the characteristic function given in Corollary 3.34 are given in Table 3.7 below.

| | $m_{1,1/250}$ | $m_{2,1/250}$ | $m_{3,1/250}$ | $m_{4,1/250}$ |
|---|---|---|---|---|
| Data | $3.5777 \times 10^{-4}$, | $1.9400 \times 10^{-4}$ | $-8.5630 \times 10^{-7}$, | $3.3018 \times 10^{-7}$ |
| Fitted Model | $3.5777 \times 10^{-4}$ | $1.9404 \times 10^{-4}$ | $-8.5680 \times 10^{-7}$ | $3.2683 \times 10^{-7}$ |
| RAE | $< 10^{-11}\%$ | $< 0.02\%$ | $< 0.06\%$ | $< 1.02\%$ |

Table 3.7: Empirical moments of data, exact theoretical moments of the model fitted with Algorithm 3.27 and the corresponding relative absolute errors.

Clearly, the first four moments are still fit very well despite the approximation errors involved. We can also compute the exact autocorrelation and crosscorrelation functions of the returns and squared returns . They are plotted together with the corresponding approximations and their empirical counterparts in Figure 3.8 below.

Again, the approximation errors involved turn out to be negligible compared to the variance of the corresponding estimators. Furthermore, it is clearly visible that while the positive autocorrelation of the returns and the positive crosscorrelation between the returns and the squared returns are of course negative features of the model from a theoretical point of view, the size of these effects is very small. Hence we can conclude that the second-order structure of the data is still fit satisfactorily for practical purposes.

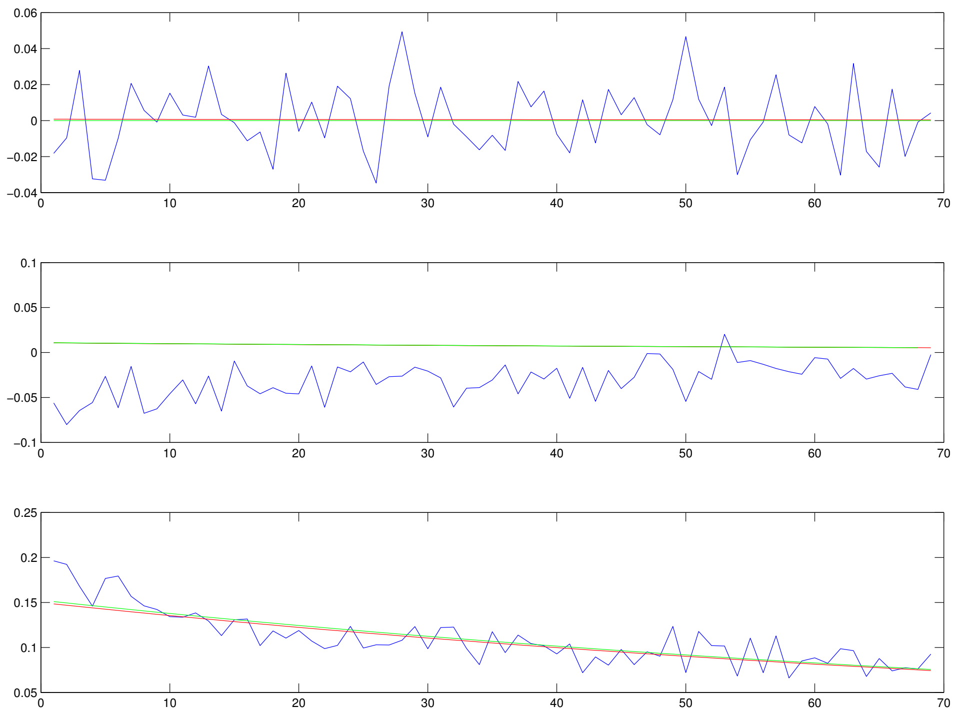

Figure 3.8: Empirical (blue), approximate (red) and exact (green) autocorrelation functions of the log returns (first), crosscorrelation function of the returns and the squared returns (second) autocorrelation function of the squared log returns (third).

Similar formulas for the joint characteristic function of the returns can also be obtained if $y$ is chosen to be an IG-OU process (cf. Nicolato & Venardos (2003) for similar formulas). In this case however, one encounters numerical problems when evaluating the derivatives of the characteristic function near zero.

### 3.4.7 Estimation of the current level of volatility

We now propose an approach to estimate the current level of volatility in the case $c_1 \neq 0$. Assuming $\mu = 0$ and $y$ follows an OU process, (Barndorff-Nielsen & Shephard, 2001, Section 5.4.3) and Theorem 3.2 yield the following state-space representation of $(X_{(n)}, X_{(n)}^2)$:

$$\begin{pmatrix} X_{(n)} \\ X_{(n)}^2 \end{pmatrix} = \begin{pmatrix} c_1(Y_{n\Delta} - Y_{(n-1)\Delta}) \\ (Y_{n\Delta} - Y_{(n-1)\Delta}) + c_1^2(Y_{n\Delta} - Y_{(n-1)\Delta})^2 \end{pmatrix} + u_n,$$

where the vector martingale difference sequence $u_n$ satisfies, for $\Delta \downarrow 0$,

$$\mathbb{V}\mathrm{ar}(u_{1n}) = \Delta\xi, \quad \mathbb{C}\mathrm{ov}(u_{1n}, u_{2n}) = c_3\Delta\xi + 2c_1\Delta^2\left(\omega^2 + \xi^2\right) + O(\Delta^3),$$
$$\mathbb{V}\mathrm{ar}(u_{2n}) = c_4\Delta\xi + (4c_1c_3 + 2)\Delta^2\left(\omega^2 + \xi^2\right) + O(\Delta^3),$$

and

$$\begin{pmatrix} \lambda(Y_{(n+1)\Delta} - Y_{n\Delta}) \\ y_{(n+1)\Delta} \end{pmatrix} = \begin{pmatrix} 0 & 1 - e^{-\lambda\Delta} \\ 0 & e^{-\lambda\Delta} \end{pmatrix} \begin{pmatrix} \lambda(Y_{n\Delta} - Y_{(n-1)\Delta}) \\ y_{n\Delta} \end{pmatrix} + w_n,$$

with IID noise $w_n$ (uncorrelated with $u_n$) satisfying

$$E(w_n) = \xi \begin{pmatrix} e^{-\lambda\Delta} - 1 + \lambda\Delta \\ 1 - e^{-\lambda\Delta} \end{pmatrix},$$
$$\mathbb{V}\mathrm{ar}(w_n) = 2\omega^2 \begin{pmatrix} \lambda\Delta - 2(1 - e^{-\lambda\Delta}) + \frac{1}{2}(1 - e^{-2\lambda\Delta}) & \frac{1}{2}(1 - e^{-\lambda\Delta})^2 \\ \frac{1}{2}(1 - e^{-\lambda\Delta})^2 & \frac{1}{2}(1 - e^{-\lambda\Delta}) \end{pmatrix}.$$

While the nonlinearity of this representation prohibits the use of the Kalman filter, it is still possible to use the extended Kalman filter by neglecting terms of order $O(\Delta^3)$ or higher once again. Despite the approximations involved, the results shown in Figure 3.9 below suggest that it is still possible to obtain decent estimates of the volatility in this way.

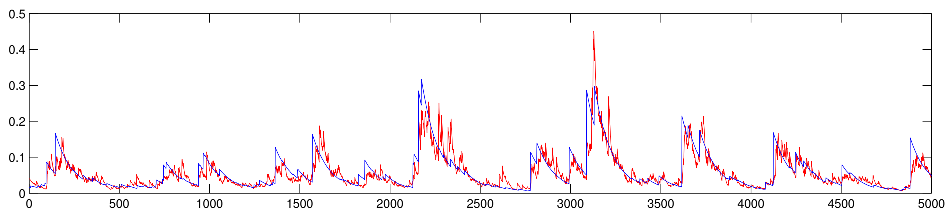


Figure 3.9: Sample paths of IG-OU process (blue) with parameters as in Examples 3.30 and the approximate extended Kalman filter estimate obtained from the corresponding NIG-IG-OU process with parameters as in Examples 3.30, 3.32.

As above it is of course also possible to use a particle filter if the marginal distribution of $B$ is known, but this is beyond our scope here.

# Chapter 4

# Power utility maximization in incomplete markets

## 4.1 Introduction

A classical problem in Mathematical Finance is to consider the investment decisions of an economic agent who tries to maximize her *expected utility from terminal wealth* in a securities market (cf. Karatzas & Shreve (1998), Korn (1997) for an overview). This is often called the *Merton problem*, since it was first solved in a continuous-time setting in the seminal papers of Merton (1969, 1971, 1973) (also cf. Mossin (1968), Samuelson (1969), Hakansson (1970) for related pioneering work in discrete time). Using methods from the theory of optimal stochastic control, Merton derived a nonlinear partial differential equation, the so-called *Hamilton-Jacobi-Bellman equation*, for the value function of the utility maximization problem in a Markovian Itô process setting. Moreover, he also solved this equation in closed form for logarithmic, power and exponential utility. This control theoretic approach has since been studied and applied extensively, as it is flexible enough to accommodate diverse problems. In particular, a candidate solution can often be constructed by heuristic means, even though the ensuing verification procedure is generally rather tedious.

*Martingale methods* represent a rather different approach to utility maximization, based on duality relationships between optimal strategies and equivalent martingale measures. For *complete* markets, where the set of equivalent martingale measures is a singleton, this approach was put forward by Pliska (1986), Karatzas et al. (1987) as well as Cox & Huang (1989, 1991). In varying degree of generality it was shown that the marginal utility of the optimal portfolio is — up to a constant — equal to the density of the equivalent martingale measure. This leads to the optimal terminal payoff which in turn allows to compute the corresponding optimal strategy.

The case of *incomplete markets* is considerably more involved, since there no longer exists a unique martingale measure in this case. Utility maximization in incomplete markets using martingale methods has been studied by He & Pearson (1991a) in finite discrete time, by He & Pearson (1991b), Karatzas et al. (1991), Cvitanić & Karatzas (1992) in diffusion-type settings and by Foldes (1990, 1992), Kramkov & Schachermayer (1999), Schachermayer (2001), Kramkov & Schachermayer (2003) in the general semimartingale case. The key idea is to relate the optimal portfolio to the solution of a suitable *dual minimization problem*. In finite discrete time, the solution to this problem is always given by an equivalent martingale measure, whereas one has to pass to equivalent local martingale measures or even supermartingale densities in more general settings.

The martingale duality approach is very general, e.g. in contrast to the stochastic control approach no Markovian structure is required. Moreover, since it allows to employ the powerful *théorie générale de processus stochastiques*, this method often allows for short proofs once the underlying structure of the problem at hand is understood. On the other hand, it is usually quite difficult to come up with candidate solutions in the first place, unless the market is time-homogeneous (see Kallsen (2000)) or the logarithm is used as the utility function (cf. e.g. Goll & Kallsen (2000) and the references therein).

In this chapter, we show that similar explicit results can be obtained for power utility in quite complex models featuring jumps as well as stochastic volatility. The key idea is to represent the optimal strategy in terms of an *opportunity process* as it is used in Černý & Kallsen (2007) (henceforth ČK) for quadratic hedging problems. After presenting a brief account of the general duality theory in Section 4.2, we introduce the notion of an opportunity process for power utility maximization in Section 4.3. We then use this concept to characterize optimal strategies in a fairly general class of affine stochastic volatility models in Section 4.4, using the results on affine semimartingales developed in Chapter 2. This extends earlier results for Lévy processes (cf. Framstad et al. (1999), Kallsen (2000), Benth et al. (2001b)), the Heston model (cf. Kraft (2005)) and the Barndorff-Nielsen-Shephard model (cf. Benth et al. (2003)). In Section 4.5 we then go on to show that by a conditioning argument a similar approach can also be used in quite general models whose increments are independent conditional on some stochastic factor process. This generalizes previous results obtained by Benth et al. (2003) and Delong & Klüppelberg (2008).

Summing up, the goal of this chapter is threefold. Firstly, we solve the power utility maximization problem in a rather complex setup allowing for some of the stylized facts observed in real data. Secondly, we indicate that the combination of a martingale approach, the notion of an opportunity process, and the calculus of semimartingale characteristics turns out to be very useful both for deriving candidate solutions and for verification. Thirdly, we lay the foundation for the computation of utility-based prices and hedging strategies in Chapter 6.

## 4.2 Existence, uniqueness and duality

Here and in the remainder of Part I of this thesis, our mathematical framework for a frictionless market model is as follows. Fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ in the sense of (JS, I.1.2). For ease of exposition, we also assume that $\mathscr{F}_T = \mathscr{F}$ and $\mathscr{F}_0 = \{\varnothing, \Omega\}$, i.e. all $\mathscr{F}_0$-measurable random variables are a.s. constant.

We consider a securities market which consists of $d+1$ assets, one bond and $d$ stocks. As is common in Mathematical Finance, we work in *discounted* terms. That means we suppose that the bond has constant value 1 and denote by $S = (S^1, \ldots, S^d)$ the *discounted price process* of the $d$ stocks in terms of multiples of the bond. The process $S$ is assumed to be an $\mathbb{R}^d$-valued semimartingale. In this financial model, we consider an investor who disposes of an *initial endowment* $v \in (0, \infty)$ and tries to maximize utility from terminal wealth.

**Definition 4.1** A (self-financing) *trading strategy* is an $\mathbb{R}^d$-valued predictable stochastic processes $\phi = (\phi^1, \ldots, \phi^d) \in L(S)$, where $\phi^i_t$ denotes the number of shares of stock $i$ in the investor's portfolio at time $t$. A trading strategy is called *admissible* for initial endowment $v \in (0, \infty)$, if the corresponding (discounted) value process $V(\phi) := v + \phi \bullet S$ is nonnegative. The set of admissible strategies is denoted by

$$\Theta(v) := \{\phi \in L(S) : v + \phi \bullet S \geq 0\}.$$

**Remark 4.2** The investors initial endowment $v$ admits two possible interpretations. On the one hand, one can consider it to be the initial cash position disposed of by the economic agent. Alternatively, it can be interpreted as the initial cash position augmented by the discounted future earnings of the investor. In view of the definition of admissibility used here, no debts are allowed at all for the first interpretation, whereas the investor can borrow up to the value of his future income for the second one.

We suppose that the investor's preferences are modelled by a utility function on $\mathbb{R}_+$ in the following sense.

**Definition 4.3** A mapping $u : (0, \infty) \to \mathbb{R}$ is called *utility function* if it is strictly increasing, strictly concave, differentiable, satisfies the *Inada conditions* $\lim_{x \to 0} u'(x) = \infty$, $\lim_{x \to \infty} u'(x) = 0$ and is of *reasonable asymptotic elasticity* in the sense of (Kramkov & Schachermayer, 1999, Definition 2.2), i.e. $\limsup_{x \to \infty} \frac{xu'(x)}{u(x)} < 1$.

The investors goal is to make the most of her initial endowment in the following sense.

**Definition 4.4** An admissible trading strategy $\varphi$ is called *optimal* for $u$ given initial endowment $v$, if it maximizes

$$\phi \mapsto E\big(u(V_T(\phi))\big)$$

over all $\phi \in \Theta(v)$, with the convention that $E(u(V_T(\phi))) = -\infty$ if $E(u(V_T(\phi))^-) = \infty$.

**Remark 4.5** Let us briefly discuss what happens if one does not assume that the value of the bond is normalized to 1. In this case, the *undiscounted price process* $\widehat{S} = (\widehat{S}^0, \widehat{S}^1, \ldots, \widehat{S}^d)$ of the bond and the $d$ stocks is modelled as an $\mathbb{R}^{d+1}$-valued semimartingale. A trading strategy is then defined as an $\mathbb{R}^{d+1}$-valued predictable process $\widehat{\phi} \in L(\widehat{S})$ and is called self-financing for (undiscounted) initial endowment $\widehat{v} \in (0, \infty)$, if its (undiscounted) value process is given by $\widehat{V}(\widehat{\phi}) := \widehat{\phi}^\top \widehat{S} = \widehat{v} + \widehat{\phi} \bullet \widehat{S}$. Let $\phi = (\phi^1, \ldots, \phi^d)$ be a self-financing trading strategy in the sense of Definition 4.1 above and suppose that $\widehat{S}^0, \widehat{S}^0_- > 0$. It then follows along the lines of (Pauwels, 2007, Lemma 1.4) that there exists a unique $\mathbb{R}$-valued predictable process $\phi^0$ given by

$$\phi^0 := v + \phi \bullet S_- - \phi^\top S_-,$$

such that $\widehat{\phi} := (\phi^0, \phi^1, \ldots, \phi^d)$ is self-financing in the market with price process $\widehat{S}$. If one identifies $\phi$ with $\widehat{\phi}$, this shows that the set of self-financing strategies remains invariant under this change of numeraire. Since $\widehat{S}^0 > 0$, the same holds true for the set of admissible strategies. Note that utility maximization in the sense of Definition 4.4 refers to *discounted utility*, i.e. utility in terms of units of the reference asset 0. If the underlying undiscounted price process $\widehat{S}^0$ is deterministic, this is equivalent to maximizing expected *undiscounted utility* for the utility function $\widehat{u} : x \mapsto u(x/\widehat{S}^0_T)$. For random $\widehat{S}^0$ though, the two notions typically differ.

In the following, $u$ denotes a general utility function. Later we will only consider *power utility functions* of the form $u(x) = x^{1-p}/(1-p)$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$ or alternatively *logarithmic utility* $u(x) = \log(x)$.

**Remark 4.6** For power or logarithmic utility, discounted utility $u(V(\phi))$ and undiscounted utility $u(\widehat{V}(\phi))$ only differ by a constant factor respectively by a constant, if $\widehat{S}^0$ is deterministic. Hence working in discounted terms entails no loss of generality in this case.

Throughout, we make the following weak assumption. By the *Fundamental Theorem of Asset Pricing* (cf. (Delbaen & Schachermayer, 1998, Theorem 1.1)) and (Becherer, 2001, Proposition 2.3), it is equivalent to *No Free Lunch with Vanishing Risk* (NFLVR) for the given financial market (cf. Delbaen & Schachermayer (1994, 1998) for more details).

**Assumption 4.7** *There exists an* equivalent weak local martingale measure*, i.e. a probability measure $Q \sim P$ such that $V(\phi)$ is a local $Q$-martingale for any admissible $\phi$.*

Subject to Assumption 4.7, admissible strategies can be alternatively represented by the *numbers $-\widetilde{u}$ of shares per unit of wealth* invested into each of the stocks. If the discounted stock price $S$ is strictly positive, one can alternatively use the *fractions $\theta$ of wealth* invested into the stocks.

**Lemma 4.8** *Suppose Assumption 4.7 holds and let $\phi \in \Theta(v)$. Then there exists $-\widetilde{u} \in L(S)$ such that*

$$V(\phi) = v\mathscr{E}(-\widetilde{u} \bullet S).$$

*If additionally $S > 0$, there exists $\theta \in L((\mathscr{L}(S^1), \ldots, \mathscr{L}(S^d)))$ such that*

$$V(\phi) = v\mathscr{E}(\theta \bullet (\mathscr{L}(S^1), \ldots, \mathscr{L}(S^d))).$$

*Consequently, we may assume w.l.o.g. that $\phi = -\widetilde{u}v\mathscr{E}(-\widetilde{u} \bullet S)_-$ respectively $\phi^i = \theta^i v\mathscr{E}(\theta \bullet (\mathscr{L}(S^1), \ldots, \mathscr{L}(S^d)))_-/S^i_-$ for $i = 1, \ldots, d$ if $S > 0$.*

PROOF. Since $V(\phi)$ is a local martingale under the equivalent weak local martingale measure $Q$ from Assumption 4.7, (JS, I.2.27) and $V(\phi) \geq 0$ imply $V(\phi) = 0$ up to indistinguishability on the predictable set $\{V_- = 0\}$. Hence we can assume w.l.o.g. that $\phi = 0$ on $\{V_-(\phi) = 0\}$ as well, because we can otherwise consider $\widetilde{\phi} := 1_{\{V_-(\phi)>0\}}\phi$ instead without changing the value process. Consequently, we can write $\phi^i = -\widetilde{u}^i V_-(\phi)$ for some predictable process $-\widetilde{u}$, which belongs to $L(S)$ by (Goll & Kallsen, 2000, Proposition A.1). This yields

$$V(\phi) = v + \phi \bullet S = v + V_-(\phi) \bullet (-\widetilde{u} \bullet S)$$

and hence $V(\phi) = v\mathscr{E}(-\widetilde{u} \bullet S)$ by the definition of the stochastic exponential. For $S > 0$, Assumption 4.7 implies that $S$ is a local $Q$-martingale. Hence it follows from (JS, I.2.27) and $S > 0$, that $S_- > 0$ as well. Therefore we can write $\phi^i = -\theta^i V_-(\phi)/S^i_-$ for predictable processes $\theta^i \in L(\mathscr{L}(S^i))$, where the stochastic logarithms $\mathscr{L}(S^i)$ are well-defined for $i = 1, \dots, d$, since $S, S_- > 0$. Hence

$$V(\phi) = v + V_-(\phi) \bullet \left(\theta \bullet \left(\left(\frac{1}{S^1_-} \bullet S^1, \dots, \frac{1}{S^d_-} \bullet S^d\right)\right)\right)$$

for $\theta = (\theta^1, \dots, \theta^d)$ and it follows that $V(\phi) = v\mathscr{E}(\theta \bullet (\mathscr{L}(S^1), \dots, \mathscr{L}(S^d)))$ by the definition of the stochastic exponential and the stochastic logarithm. □

In Section 4.1 we referred to the *general principle* that a self-financing trading strategy $\varphi$ is optimal for terminal wealth if and only if $u'(V_T(\varphi))$ is — up to a constant — the density of an equivalent martingale measure. This *Fundamental Theorem of Utility Maximization* only holds true literally in finite discrete time, i.e. if both $\Omega$ and the time set $\{0, 1, \dots, T\}$ are finite (cf. (Kallsen, 2002, Corollary 2.7)). For arbitrary $\Omega$ and in continuous time the situation becomes more involved. The general semimartingale case has been thoroughly investigated by Kramkov & Schachermayer (1999). Apart from NFLVR, they require that the maximal expected utility in the given financial market is finite. This property is often difficult to check even in concrete models, see Sections 4.4 and 4.5. However, it is satisfied automatically if $u$ is bounded from above as e.g. for $u(x) = x^{1-p}/(1-p)$, $p \in (1, \infty)$.

**Assumption 4.9**

$$U(v) := \sup_{\phi \in \Theta(v)} E(u(V_T(\phi))) < \infty.$$

**Remark 4.10** Even in concrete models, Assumption 4.9 is generally not a consequence of Assumption 4.7. For example, the minimal entropy martingale measure exists for some parametrizations of the Heston model, that nevertheless allow for infinite expected power utility (cf. Vierthauer (2009) and Section 4.4.2 below). Conversely, there are models with finite maximal expected utility, that nevertheless do not satisfy NFLVR (cf. e.g. (Goll & Kallsen, 2003, Example 5.1)). The issue of whether or not the maximal expected utility is finite is of profound importance in utility maximization. If it is finite, the optimal value process is guaranteed to be unique due to the strict concavity of the utility function (cf. e.g. (Kallsen, 2000, Lemma 2.5)). For infinite expected utility this uniqueness ceases to hold. E.g. for power or logarithmic utility, shifting half of the stock investments to the bank account will also lead to infinite utility in this case. Consequently, two different strategies can both be optimal, even though one of the respective value processes dominates the other in an almost-sure sense.

Subject to Assumptions 4.7 and 4.9 we now have the following duality result due to Kramkov & Schachermayer (1999), which is a precise statement of the Fundamental Theorem of Utility Maximization in the general semimartingale case.

**Theorem 4.11** *Let $u$ be a utility function, $v \in (0, \infty)$ and suppose Assumptions 4.7 and 4.9 are satisfied. Then an optimal strategy exists and the corresponding value process is unique. Moreover, for any admissible strategy $\varphi$ the following are equivalent:*

1. *There exists a positive supermartingale $Z$ such that $ZV(\phi)$ is a supermartingale for any admissible $\phi$, $ZV(\varphi)$ is a martingale and $Z_T = u'(V_T(\varphi))$.*
2. *$\varphi$ is optimal for $u$ and initial endowment $v \in (0, \infty)$.*

PROOF. Existence and uniqueness as well as the implication $2. \Rightarrow 1.$ are established in (Kramkov & Schachermayer, 1999, Theorem 2.2).

$1. \Rightarrow 2.$ Let $\phi$ be any competing admissible strategy. Since $u$ is concave, we have

$$u(V_T(\phi)) \leq u(V_T(\varphi)) + u'(V_T(\varphi))(V_T(\phi) - V_T(\varphi)).$$

As $u'(V_T(\varphi))V_T(\phi)$ and $u'(V_T(\varphi))V_T(\varphi)$ coincide with the terminal values of the supermartingale $ZV(\phi)$ and the martingale $ZV(\varphi)$, respectively, this proves the assertion. □

**Remark 4.12** By (Kramkov & Schachermayer, 1999, Theorem 2.2) the supermartingale $Z$ solves a dual minimization problem and is therefore referred to as the *dual minimizer*. Subject to the assumptions of Theorem 4.11 the optimal value process $V(\varphi)$ and the dual minimizer $Z$ are both strictly positive (cf. (Kramkov & Schachermayer, 1999, Theorem 2.2).

## 4.3 The opportunity process in power utility maximization

For power utility the dependence between the initial endowment $v$ and the corresponding optimal strategy and dual minimizer can easily be quantified. Moreover, the maximal expected utility can be determined explicitly as well.

**Corollary 4.13** *Let $u = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0, 1\}$ be a power utility function and assume Assumption 4.7 is satisfied. Then for any strategy $\varphi \in \Theta(1)$, i.e. $1 + \varphi \bullet S \geq 0$, the following are equivalent:*

1. *There exists a positive supermartingale $Z$ such that $Z(1+\phi \bullet S)$ is a supermartingale for any $\phi \in \Theta(1)$, $Z(1 + \varphi \bullet S)$ is a martingale and $Z_T = (1 + \varphi \bullet S_T)^{-p}$.*

2. *For $u$ and initial endowment $v \in (0,\infty)$, the strategy $v\varphi$ is optimal, the corresponding maximal expected utility is finite and the dual minimizer is given by $v^{-p}Z$.*

*Moreover, the maximal expected utility is given by $U(v) = \frac{v^{1-p}}{1-p} Z_0$ in this case.*

PROOF. 1. $\Rightarrow$ 2. As in the proof of Theorem 4.11 it follows that $\varphi$ is optimal for $u$ and initial endowment 1. Since $\phi \in \Theta(v)$ implies $\phi/v \in \Theta(1)$ and vice versa, this shows that for any $\phi \in \Theta(v)$, we have

$$E(u(V_T(\phi)) = v^{1-p} E(u(1 + \phi/v \bullet S_T)) \leq v^{1-p} E(u(1 + \varphi \bullet S_T)) = E(u(V_T(v\varphi))),$$

which shows that $v\varphi$ is optimal for $u$ and initial endowment $v$. Moreover, as $Z(1 + \varphi \bullet S)$ is a martingale with terminal value $Z_T(1 + \varphi \bullet S_T) = (1 + \varphi \bullet S_T)^{1-p}$, we obtain

$$U(v) = E(u(V_T(\varphi))) = v^{1-p} E(u(1 + \varphi \bullet S_T)) = \frac{v^{1-p}}{1-p} Z_0 < \infty.$$

Since $u'(V_T(v\varphi)) = v^{-p} u'(1 + \varphi \bullet S_T)$, the formula for the dual minimizer is obvious.

2. $\Rightarrow$ 1. Since $v\varphi$ is optimal, it follows as above that $\varphi$ is optimal for $u$ and initial endowment 1 with finite expected utility. Hence Assumption 4.9 holds and the claim follows from Theorem 4.11 for $v = 1$. $\square$

**Remark 4.14** For power utility $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0,1\}$, the process $Z/Z_0$ minimizes the $L^q$*-distance* $E(-\mathrm{sgn}(q)(Y_T)^q)$ for $q := 1 - \frac{1}{p} \in (-\infty, 1)$ over the set

$$\mathscr{Y}(1) := \{Y \geq 0 : Y_0 = 1 \text{ and } YV(\phi) \text{ is a supermartingale for all admissible } \phi\},$$

which in particular contains the densities of all equivalent martingale measures. If $Z$ is actually a martingale, then $Z/Z_0$ represents the density process of the so-called *q-optimal martingale measure* $Q_0$, whose terminal value minimizes the $L^q$-distance over the densities of equivalent martingale measures in this case.

Since $u'$ and the optimal value process $V(\varphi)$ are strictly positive (cf. Remark 4.12) and the dual minimizer $Z$ has terminal value $u'(V_T(\varphi))$, we can represent the supermartingale $Z$ as $Z = Lu'(V(\varphi))$ for some strictly positive semimartingale $L$ with $L_T = 1$. Corollary 4.13 then reads as follows.

**Proposition 4.15** *Let $u(x) = \frac{x^{1-p}}{1-p}$ for some $p \in \mathbb{R}_+ \backslash \{0,1\}$ and $v \in (0,\infty)$. Fix an admissible strategy $\varphi$ and suppose that Assumption 4.7 holds. Then the following are equivalent.*

1. *There exists a strictly positive semimartingale $L$ with $L_T = 1$ such that $LV(\varphi)^{-p}V(\phi)$ is a $\sigma$-supermartingale for any admissible $\phi$ and $LV(\varphi)^{1-p}$ is a martingale.*

2. *$\varphi$ is optimal for $u$ and initial endowment $v$ with finite expected utility.*

*The corresponding maximal expected utility is given by $U(v) = \frac{v^{1-p}}{1-p} L_0$.*

PROOF. Follows immediately from Corollary 4.13 by inserting $L := ZV(\varphi)^p$ and using that any nonnegative $\sigma$-supermartingale is a supermartingale by Proposition A.9. □

**Remark 4.16** Let $S$ be strictly positive. Then this supermartingale criterion allows to incorporate *convex constraints* into the utility maximization problem. More specifically, let $C \subset \mathbb{R}^d$ be some nonempty convex set and define

$$\Theta(v, C) := \{\phi \in \Theta(v) : V(\phi) = v\mathscr{E}(\theta \bullet \mathscr{L}(S)) \text{ with some } C\text{-valued } \theta\}.$$

The most prominent example is the set $\Theta(v, [0, 1]^d)$ of admissible strategies involving neither shortselling and nor leverage. Suppose there exists a trading strategy $\varphi \in \Theta(v, C)$ and a positive semimartingale $L$ with $L_T = 1$, such that $LV(\varphi)^{-p}V(\phi)$ is a supermartingale for all $\phi \in \Theta(v, C)$ and such that $LV(\varphi)^{1-p}$ is a martingale. Then it follows literally as in the proof of $1. \Rightarrow 2.$ in Theorem 4.11 that $\varphi$ maximizes expected utility from terminal wealth over all $\phi \in \Theta(v, C)$.

The idea to state optimality in terms of a process $L$ as in Corollary 4.15 is inspired by a similar approach of ČK in the context of quadratic hedging, where $L$ is called *opportunity process*. It makes sense to use the same terminology here. Indeed, we have

$$\begin{aligned} E(u(V_T(\varphi))|\mathscr{F}_t) &= \frac{1}{1-p} E(L_T V_T(\varphi)^{1-p}|\mathscr{F}_t) \\ &= \frac{1}{1-p} L_t V_t(\varphi)^{1-p} \end{aligned} \tag{4.1}$$

and hence

$$L_t = (1-p) E\left( u\left( \frac{V_T(\varphi)}{V_t(\varphi)} \right) \middle| \mathscr{F}_t \right). \tag{4.2}$$

The optimal strategy $\varphi$ has value $V_t(\varphi)$ at time $t$. One easily verifies that on $[\![t, T]\!]$, $\varphi$ is the $V_t(\varphi)$-fold of the investment strategy $\phi$ which starts with initial endowment $1$ at time $t$ and maximizes the expected utility at $T$. In view of (4.2) this means that $L_t$ stands — up to a factor $1-p$ — for the maximal utility from trading between $t$ and $T$ with initial endowment $1$. The parallel statement for quadratic utility inspired the term *opportunity process* in ČK. Moreover, (4.1) means that $LV(\varphi)^{1-p}/(1-p)$ corresponds to the *value function* used in stochastic control theory.

In view of Corollary 4.15 our approach for finding optimal strategies consists of three steps. The first is to make an appropriate ansatz for $L$ and $\varphi$ up to some yet unknown parameters or deterministic functions. In view of Lemma A.8 the $\sigma$-supermartingale respectively $\sigma$-martingale properties of $LV(\varphi)^{-p}V(\phi)$ and $LV(\varphi)^{1-p}$ can be viewed as drift conditions, which are used to determine the unknown parameters in a second step. Finally, one verifies that the obtained candidate processes $L$ and $\varphi$ indeed meet all conditions of Proposition 4.15, in particular that the $\sigma$-martingale $LV(\varphi)^{1-p}$ is in fact a true martingale.

**Remark 4.17** Opportunity processes can also be used for the computation and verification of optimal strategies for *exponential utility* $u(x) = 1-\exp(-px)$, $p \in (0, \infty)$ (cf. Vierthauer (2009) for more details).

## 4.4 Solution in affine stochastic volatility models

In this section we consider a single risky asset (i.e. $d = 1$), but the results extend to multiple stocks in a straightforward manner. For the application of the optimality criterion in Proposition 4.15 two problems have to be solved. First, one needs an appropriate ansatz for the optimal strategy $\varphi$ and the opportunity process $L$. Having chosen parameters such that the drift rates of $LV(\varphi)^{-p}V(\phi)$ and $LV(\varphi)^{1-p}$ are nonpositive respectively vanish, one must then establish that the $\sigma$-martingale $LV^{1-p}$ is a true martingale. Both problems can be solved in a number of affine stochastic volatility models by using the results on exponentially affine martingales established in Chapter 2.

Let $(y, X)$ be an affine stochastic volatility model such that the discounted stock price $S = S_0\mathscr{E}(X)$ is strictly positive. In the case where $X$ is a Lévy process, the optimal strategy is known to invest a constant fraction of current wealth in the risky security, i.e. $\varphi_t = \eta\frac{V_{t-}(\varphi)}{S_{t-}}$ for some constant $\eta \in \mathbb{R}$ (cf. Kallsen (2000)). We replace the constant $\eta$ by some deterministic function $\eta \in L(X)$ for the more general class of models considered here. This leads to

$$V(\varphi) = v + \left(\eta\frac{V_-(\varphi)}{S_-}\right) \bullet S = v + V_-(\varphi) \bullet (\eta \bullet X) = v\mathscr{E}(\eta \bullet X). \tag{4.1}$$

Since $\eta$ is assumed to be deterministic, the processes $(y, \mathscr{L}(V(\varphi)^{-p}))$, $(y, \mathscr{L}(V(\varphi)^{1-p}))$ turn out to be time-inhomogeneous affine semimartingales in the sense of Chapter 2. We guess that the opportunity process $L$ is of exponentially affine form as well, more specifically

$$L_t = \exp(\alpha_0(t) + \alpha_1(t)y_t)$$

with deterministic functions $\alpha_0, \alpha_1 : [0, T] \to \mathbb{R}$. In order to have $L_T = 1$ we need $\alpha_0(T) = \alpha_1(T) = 0$. Up to the concrete form of $\eta$, $\alpha_0$, $\alpha_1$, we have specified candidate processes $\varphi$, $L$. The functions are chosen such that the required $\sigma$-martingale respectively $\sigma$-supermartingale properties hold (cf. the proof of Theorem 4.20). In order to show that the $\sigma$-martingale $LV(\varphi)^{1-p}$ is a true martingale, we use the results developed in Section 2.4, which state that exponentially affine $\sigma$-martingales are martingales under weak assumptions.

**Remark 4.18** In the literature, the asset price is sometimes modelled as an ordinary exponential $S_t = S_0 \exp(X_t)$ with some bivariate affine process $(y, X)$. In this case we have $S_t = S_0\mathscr{E}(\widetilde{X}_t)$ with some bivariate affine process $(y, \widetilde{X})$ (cf. Lemma 2.6). Hence we are in the setup considered here.

The optimality criterion Proposition 4.15 is necessary and sufficient for models satisfying NFLVR and admitting only finite maximal expected utility. However, the present approach of computing the optimal strategy only works if the optimal strategy $\varphi$ and the opportunity process $L$ are of the form proposed above. It turns out that this is the case only if the dynamics of $X$ are proportional to the volatility $y$ with no additional constant part.

**Assumption 4.19** *The differential characteristics* $(b^{(y,X)}, c^{(y,X)}, K^{(y,X)}, I)$ *of* $(y,X)$ *are of the form*

$$b^{(y,X)} = \begin{pmatrix} \beta_0^1 + \beta_1^1 y_- \\ \beta_1^2 y_- \end{pmatrix}, \quad c^{(y,X)} = \begin{pmatrix} \gamma_1^{11} & \gamma_1^{12} \\ \gamma_1^{12} & \gamma_1^{22} \end{pmatrix} y_-,$$
$$K^{(y,X)}(G) = \int 1_G(z_1,0)\kappa_0(dz) + \int 1_G(0,x_2)\kappa_1(dx)y_-, \quad \forall G \in \mathscr{B}^2,$$

*for given admissible Lévy-Khintchine triplets* $(\beta_i, \gamma_i, \kappa_i)$, $i = 0,1$ *on* $\mathbb{R}^2$.

For models satisfying NFLVR, we then have the following general result.

**Theorem 4.20** *Let* $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+\backslash\{0,1\}$ *and* $v \in (0,\infty)$. *Suppose Assumptions 4.7 and 4.19 hold and there exist mappings* $\eta, \alpha_1 \in C^1([0,T],\mathbb{R})$ *such that the following conditions are satisfied up to a* $dt$*-null set on* $[0,T]$.

1. $\kappa_1(\{x \in \mathbb{R}^2 : 1 + \eta(t)x_2 \le 0\}) = 0$.
2. $\int |x_2(1+\eta(t)x_2)^{-p} - h(x_2)|\kappa_1(dx) < \infty$.
3. $$\beta_1^2 + \gamma_1^{12}\alpha_1(t) - p\gamma_1^{22}\eta(t) + \int \left(\frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2)\right)\kappa_1(dx) \ge 0$$
   *if there exists* $\theta < \eta(t)$ *such that* $\kappa_1(\{x \in \mathbb{R}^2 : 1 + \theta x_2 < 0\}) = 0$ *and*
   $$\beta_1^2 + \gamma_1^{12}\alpha_1(t) - p\gamma_1^{22}\eta(t) + \int \left(\frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2)\right)\kappa_1(dx) \le 0$$
   *if there exists* $\theta > \eta(t)$ *such that* $\kappa_1(\{x \in \mathbb{R}^2 : 1 + \theta x_2 < 0\}) = 0$.
4. $\alpha_1(T) = 0$ *and*
   $$\begin{aligned}\alpha_1'(t) =& (p-1)\eta(t)\beta_1^2 + \frac{p(1-p)\gamma_1^{22}}{2}\eta^2(t) + ((p-1)\gamma_1^{12}\eta(t) - \beta_1^1)\alpha_1(t) - \frac{\gamma_1^{11}}{2}\alpha_1^2(t) \\ &- \int \left((1+\eta(t)x_2)^{1-p} - 1 - (1-p)\eta(t)h(x_2)\right)\kappa_1(dx)\end{aligned}$$
5. $\int_0^T \int_{\{z_1>1\}} e^{\alpha_1(t)z_1}\kappa_0(dz)dt < \infty$.

*Then* $\varphi_t := \eta(t)v\mathscr{E}(\eta \bullet X)_{t-}/S_{t-}$ *is optimal for* $u$ *and initial endowment* $v$ *with value process* $V(\varphi) = v\mathscr{E}(\eta \bullet X)$. *The corresponding maximal expected utility is finite and given by* $E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p}L_0$ *with the opportunity process*

$$L_t = \exp(\alpha_0(t) + \alpha_1(t)y_t) \quad \text{where} \quad \alpha_0(t) := \int_t^T \psi_0^{(y,X)}(\alpha_1(s),0)ds.$$

PROOF. As in the proof of Lemma 4.8 it follows that $V(\varphi) = v\mathscr{E}(\eta \bullet X)$. Hence

$$E\left(\sum_{t\leq T} 1_{(-\infty,0]}(1+\eta(t)\Delta X_t)\right) = E\left(1_{(-\infty,0]}(1+\eta x) * \mu^X_T\right)$$
$$= E\left(1_{(-\infty,0]}(1+\eta x) * \nu^X_T\right)$$
$$= E\left(\int_0^T \int 1_{(-\infty,0]}(1+\eta(t)x_2)\kappa_1(dx)y_{s-}ds\right) = 0$$

by (JS, II.1.8) and Condition 1. Consequently $P(\exists t \in [0,T] : \eta(t)\Delta X_t \leq -1) = 0$. By (JS, I.4.61) this implies that $V(\varphi) = v\mathscr{E}(\eta \bullet X) > 0$. Therefore $\varphi$ is admissible.

Notice that Condition 2 and a second order Taylor expansion show that all integrals in Conditions 3 and 4 are well-defined. Moreover, $\alpha_0$ is well-defined and in $C^1([0,T])$ as well by Condition 5.

Let $\phi$ be any competing admissible strategy. In view of Lemma 4.8, the corresponding value process can be written as $V(\phi) = v\mathscr{E}(\theta \bullet X)$ for some predictable process $\theta$. The admissibility of $\theta$ implies $\theta_t\Delta X_t \geq -1$ which in turn yields

$$\kappa_1(\{x \in \mathbb{R}^2 : 1+\theta_t x_2 < 0\}) = 0 \tag{4.2}$$

outside some $dP \otimes dt$-null set. Since the identity process $I_t = t$ is continuous and of finite variation, $\partial(y,X,I)$ are given by

$$b^{(y,X,I)} = \begin{pmatrix} \beta^1_0 + \beta^1_1 y_- \\ \beta^2_1 y_- \\ 1 \end{pmatrix}, \quad c^{(y,X,I)} = \begin{pmatrix} \gamma^{11}_1 & \gamma^{12}_1 & 0 \\ \gamma^{12}_1 & \gamma^{22}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_-,$$
$$K^{(y,X,I)}(G) = \int 1_G(z_1,0,0)\kappa_0(dz) + \int 1_G(0,x_2,0)\kappa_1(dx)y_- \quad \forall G \in \mathscr{B}^3.$$

The fundamental theorem of calculus and integration by parts in the sense of (JS, I.4.45) yield

$$\alpha_0(I) + \alpha_1(I)y - \alpha_0(0) - \alpha_1(0)y_0 = (\alpha_0'(I) + \alpha_1'(I)y) \bullet I + \alpha_1(I) \bullet y,$$

therefore we can compute the differential characteristics of $(y, V(\varphi), V(\phi), L)$ in the following steps:

$$\partial \begin{pmatrix} y \\ X \\ I \end{pmatrix} \overset{\text{Prop. A.3}}{\longrightarrow} \partial \begin{pmatrix} y \\ V(\varphi) \\ V(\phi) \\ \alpha_0(I) + \alpha_1(I)y \end{pmatrix} \overset{\text{Prop. A.4}}{\longrightarrow} \partial \begin{pmatrix} y \\ V(\varphi) \\ V(\phi) \\ L \end{pmatrix}.$$

Since $V(\varphi) > 0$, $(b^{LV(\varphi)^{-p}V(\phi)}, c^{LV(\varphi)^{-p}V(\phi)}, K^{LV(\varphi)^{-p}V(\phi)})$ can now be derived by applying Proposition A.4. In particular, for $G \in \mathscr{B}$, we have

$$K^{LV(\varphi)^{-p}V(\phi)}(G) = \int 1_G\left(L_-V_-(\varphi)^{-p}V_-(\phi)\left(e^{\alpha_1 z_1}-1\right)\right)\kappa_0(dz)$$
$$+ \int 1_G\left(L_-V_-(\varphi)^{-p}V_-(\phi)\left(\frac{1+\theta x_2}{(1+\eta x_2)^p}-1\right)\right)\kappa_1(dx)y_-.$$

From Conditions 2 and 5 it follows that

$$\int_{\{|x|>1\}} |x| K^{LV(\varphi)^{-p}V(\phi)}(dx) < \infty \tag{4.3}$$

holds outside some $dP \otimes dt$-null set. Moreover, by inserting the definition of $\alpha_0$ and the formula for $\alpha_1'$ from Condition 4, we obtain

$$\begin{aligned} b_t^{LV(\varphi)^{-p}V(\phi)} &= \int (h(x) - x) K_t^{LV^{-p}V(\phi)}(dx) + L_{t-}V_{t-}(\varphi)^{-p} V_{t-}(\phi) y_{t-}(\theta_t - \eta(t)) \\ &\times \left( \beta_1^2 + \gamma^{12}\alpha_1(t) - p\gamma^{22}\eta(t) + \int \left( \frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2) \right) \kappa_1(dx) \right) \end{aligned}$$

after tedious but straightforward calculations. In view of Condition 3 and (4.2) this yields

$$b^{LV(\varphi)^{-p}V(\phi)} + \int (x - h(x)) K^{LV(\varphi)^{-p}V(\phi)}(dx) \le 0,$$

outside some $dP \otimes dt$-null set, which combined with (4.3) shows that $LV(\varphi)^{-p}V(\phi)$ is a $\sigma$-supermartingale by Lemma A.8. Moreover, by setting $\phi = \varphi$ we obtain that $LV(\varphi)^{1-p}$ is a $\sigma$-martingale. Once more applying Proposition A.3, it follows that $(y, \mathscr{L}(LV(\varphi)^{1-p}))$ is a bivariate time-inhomogeneous affine semimartingale relative to the time-dependent triplets

$$\begin{aligned} \beta_0(t) &= \begin{pmatrix} \beta_0^1 \\ \int (h(e^{\alpha_1(t)z_1} - 1) - (e^{\alpha_1(t)z} - 1))\kappa_0(dz) \end{pmatrix}, \quad \gamma_0(t) = 0, \\ \kappa_0(t, G) &= \int 1_G(z_1, e^{\alpha_1(t)z_1} - 1)\kappa_0(dz), \quad \forall G \in \mathscr{B}^2, \\ \beta_1(t) &= \begin{pmatrix} \beta_1^1 \\ \int (h((1+\eta(t)x_2)^{1-p} - 1) - ((1+\eta(t)x_2)^{1-p} - 1))\kappa_1(dx) \end{pmatrix}, \\ \gamma_1(t) &= \begin{pmatrix} \gamma_1^{11} & \gamma_1^{11}\alpha_3(t) + (1-p)\gamma_1^{12}\eta(t) \\ \gamma_1^{11}\alpha_1(t) + (1-p)\gamma_1^{12}\eta_1(t) & \gamma_1^{11}\alpha_1^2(t) + 2(1-p)\gamma_1^{12}\alpha_1(t)\eta(t) + (1-p)^2\gamma_1^{22}\eta^2(t) \end{pmatrix}, \\ \kappa_1(t, G) &= \int 1_G(0, (1+\eta(t)x_2)^{1-p} - 1)\kappa_1(dx), \quad \forall G \in \mathscr{B}^2. \end{aligned}$$

The martingale property of $LV(\varphi)^{1-p}$ can now be established by verifying the sufficient conditions of Theorem 2.9. It is easy to see that the triplets are strongly admissible in the sense of Definition 2.2. Indeed, the continuity conditions follow from the continuity of $\eta$ and $\alpha_1$ as well as dominated convergence. The remaining assumptions of Theorem 2.9 are also satisfied as can be easily checked. Hence $LV(\varphi)^{1-p}$ is a martingale and the assertion follows from Corollary 4.15. □

**Remarks.**

1. Condition 1 is needed to ensure that the value process $V(\varphi)$ is strictly positive. In models where the asset price can jump to arbitrary positive values, it rules out short-selling and leverage for the optimal strategy. Conditions 2 and 5 ensure that $\alpha_0$ and the integrals in Conditions 3, 4 are well-defined. The crucial Conditions are 3 and 4 which represent $\eta$ and $\alpha_1$ as the solution to a differential algebraic inequality.

2. Obviously, Condition 3 is satisfied in particular if

$$\beta_1^2 + \gamma^{12}\alpha_1(t) - p\gamma^{22}\eta(t) + \int \left( \frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2) \right) \kappa_1(dx) = 0$$

holds outside some $dt$-null set. In this case it follows by inserting the admissible strategies $\phi = 1$ and $\phi = 0$ that both $LV(\varphi)^{-p}$ and $LV(\varphi)^{-p}S$ are $\sigma$-martingales. As in the proof of Theorem 4.20 it follows that they are also exponentials of time-inhomogeneous affine processes and hence true martingales by Theorem 2.9. This shows that $Z := \frac{LV(\varphi)^{-p}}{L_0 v^{-p}}$ is the density process of the $q$-optimal equivalent martingale measure in this case. In particular, this implies that the model under consideration satisfies the NFLVR Assumption 4.7. However, this condition is often not general enough in concrete applications (cf. Examples 4.24 and 4.28 below).

3. In general, it is not clear whether or not the differential algebraic inequalities for $\eta$ and $\alpha_1$ admit a solution. In Sections 4.4.2, 4.4.3 below it will turn out that in concrete models, they typically admit a solution if the maximal expected utility in the model is finite, in particular when $u$ is bounded from above for $p \in (1, \infty)$. Similarly, for *exponential utility* a solution to a similar system of equations is shown to exist under weak assumptions in Vierthauer (2009). This leads us to conjecture that an analogous result holds for power utility, if the maximal utility is finite. However, a thorough investigation of this issue is beyond our scope here.

4. Let $C \subset \mathbb{R}$ be convex. If one considers the constrained problem of maximizing expected utility over the set $\Theta(v, C)$ from Remark 4.16, $\eta$ has to be $C$-valued. On the other hand, one can replace Condition 3 with the weaker requirement

   3'. Suppose that

   $$\beta_1^2 + \gamma_1^{12}\alpha_1(t) - p\gamma_1^{22}\eta(t) + \int \frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2)\kappa_1(dx) \geq 0$$

   if there exists $\theta \in (-\infty, \eta(t)) \cap C$ s.t. $\kappa_1(\{x \in \mathbb{R}^2 : 1 + \theta x_2 < 0\}) = 0$ and

   $$\beta_1^2 + \gamma_1^{12}\alpha_1(t) - p\gamma_1^{22}\eta(t) + \int \frac{x_2}{(1+\eta(t)x_2)^p} - h(x_2)\kappa_1(dx) \leq 0$$

   if there exists $\theta \in (\eta(t), \infty) \cap C$ s.t. $\kappa_1(\{x \in \mathbb{R}^2 : 1 + \theta x_2 < 0\}) = 0$.

We now consider some examples where the differential algebraic inequality in Theorem 4.20 admits a solution.

### 4.4.1 Exponential Lévy models

Suppose the asset price is modelled as a strictly positive process of the form $S = S_0\mathscr{E}(X)$ for some $\mathbb{R}$-valued Lévy process $X$ with Lévy-Khintchine triplet $(b^X, c^X, K^X)$.

Put differently, this means that the volatility process $y$ is constant and equal to one, which in turn implies that $\partial(y,X)$ can be written as

$$b^{(y,X)} = \begin{pmatrix} 0 \\ b^X \end{pmatrix} y, \quad c^{(y,X)} = \begin{pmatrix} 0 & 0 \\ 0 & c^X \end{pmatrix} y, \quad K^{(y,X)}(G) = \int 1_G(0,x) K^X(dx) y,$$

for all $G \in \mathscr{B}^2$. In this case, Conditions 1, 2 and 3 in Theorem 4.20 neither depend on $\alpha_1$ nor $t$. Moreover, if there exists $\eta \in \mathbb{R}$ satisfying these conditions, $\alpha_1$ is given as the solution to a constant ODE $\alpha_1' = a$ for suitable $a \in \mathbb{R}$ by Condition 4.

**Corollary 4.21** *Let $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0,1\}$ and $v \in (0,\infty)$. Suppose Assumption 4.7 is satisfied and there exists $\eta \in \mathbb{R}$ such that the following conditions hold.*

1. $K^X(\{x \in \mathbb{R} : 1 + \eta x \leq 0\}) = 0$.
2. $\int |(x(1+\eta x)^{-p} - h(x)| K^X(dx) < \infty$.
3. $$b^X - pc^X \eta + \int \left( \frac{x}{(1+\eta x)^p} - h(x) \right) K^X(dx) \geq 0$$
   *if there exists $\theta < \eta$ such that $K^X(\{x \in \mathbb{R} : 1 + \theta x < 0\}) = 0$ and*
   $$b^X - pc^X \eta + \int \left( \frac{x}{(1+\eta x)^p} - h(x) \right) K^X(dx) \leq 0$$
   *if there exists $\theta > \eta$ such that $K^X(\{x \in \mathbb{R} : 1 + \theta x < 0\}) = 0$.*

*Then $\varphi_t := \eta v \mathscr{E}(\eta X)_{t-}/S_{t-}$ is optimal for $u$ and initial endowment $v \in (0,\infty)$ with value process $V(\varphi) = v\mathscr{E}(\eta X)$. Moreover, the corresponding maximal expected utility is finite and given by $E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p} L_0$ for the opportunity process $L_t = \exp(\alpha_1(t))$ with*

$$\alpha_1(t) = (t-T) \times \left( (p-1)b^X \eta + \frac{p(1-p)}{2} c^X \eta^2 - \int (1+\eta x)^{1-p} - 1 - (1-p)\eta h(x) K^X(dx) \right).$$

PROOF. Follows immediately from Theorem 4.20. □

**Remarks.**

1. In view of (Cont & Tankov, 2004, Proposition 9.9) and (Kallsen, 2004, Lemma 3.3), the NFLVR Assumption 4.7 is satisfied for all Lévy processes $X$ that are neither a.s. increasing nor a.s. decreasing.

2. The optimal strategy is *myopic*, i.e. only depends on the local dynamics of $X$ respectively $S$. If the Lévy process $X$ is continuous, i.e. a Brownian motion with drift, Condition 1 and 2 are obviously always satisfied. Moreover, Condition 3 yields that the classical Merton solution
   $$\eta = \frac{b^X}{pc^X} = \frac{b^X}{p\tilde{c}^X}.$$

is optimal in this case, i.e. the optimal fraction $\eta$ of stocks is proportional to the infinitesimal drift rate $b^X$, the inverse of the infinitesimal variance $\tilde{c}^X$ and the inverse of the investor's risk aversion $p$. If $X$ has jumps, it is still optimal to invest a constant fraction $\eta$ in the stock, where the optimal value now has to be determined by finding the root of a real-valued function. If $h(x) = x$ can be used as the truncation function and Condition 3 is satisfied with equality, a second-order Taylor expansion yields that

$$\frac{x}{(1+\eta x)^p} - x = -p\eta x^2 + o(x^2)$$

for small $x$ and hence

$$\eta \approx \frac{b^X}{p(c^X + \int x^2 K^X(dx))} = \frac{b^X}{p\tilde{c}^X}$$

if most of the mass of the Lévy measure $K$ is located in the vicinity of $0$. This shows that the Black-Scholes strategy can serve as a good proxy for the true optimal strategy, if the frequency of large jumps is sufficiently small (cf. Example 4.24 below for a specific parametric example).

3. The existence of an optimal investment strategy with finite expected utility only depends on the model parameters $(b^X, c^X, K^X)$ and the investor's risk aversion $p$, but not on the time horizon $T$. This will turn out to be different for stochastic volatility models (cf. Sections 4.4.2 and 4.4.3 below).

4. As already pointed out by Samuelson (1969), the fact that the same fraction of wealth is optimal for all time horizons $T \in \mathbb{R}_+$ seems to contradict the common wisdom that it is beneficial to hold a higher percentage of stocks in the long-run than for short-term investments. However, Corollary 4.21 can be reconciled with this if the second interpretation of the initial endowment $v$ from Remark 4.2 is used: Since the investor typically receives more future earnings over a longer time-horizon, she will take this into account by using a higher $v$ to calculate her investment decision. By Corollary 4.21, the optimal fraction of stocks relative to the augmented initial endowment remains the same, but the optimal fraction of stocks relative to the investors initial cash position increases for a longer time horizon in this case.

The following Proposition shows that the Conditions of Corollary 4.21 are satisfied for many specific Lévy processes $X$ considered in the literature.

**Proposition 4.22** *Suppose that $S = S_0\mathscr{E}(X)$ for a Lévy process $X$ with Lévy-Khintchine triplet $(b^X, c^X, K^X)$ satisfying*

1. $K^X((-1,b)), K^X((a,\infty)) > 0$ *for any* $b \in (-1,0)$ *and* $a \in (0,\infty)$.

2. $\int_\varepsilon^\infty xK^X(dx) < \infty$ *and* $\int_{-1}^{-\varepsilon} \frac{-x}{(1+x)^p} K^X(dx) < \infty$ *for some* $\varepsilon \in (0,1)$.

*Then there exists a unique $\eta \in [0,1]$ such that the conditions of Corollary 4.21 are satisfied. If $S = S_0 \exp(\widetilde{X})$ for some Lévy process $\widetilde{X}$ with Lévy-Khintchine triplet $(b^{\widetilde{X}} c^{\widetilde{X}}, K^{\widetilde{X}})$ the statement remains true if Condition 1 and 2 are replaced with*

*1'. $K^{\widetilde{X}}((-\infty,-a)), K^{\widetilde{X}}((a,\infty)) > 0$ for any $a \in (0,\infty)$.*

*2'. $\int_\varepsilon^\infty e^x K^{\widetilde{X}}(dx) < \infty$ and $\int_{-\infty}^{-\varepsilon} e^{-px} K^{\widetilde{X}}(dx) < \infty$ for some $\varepsilon > 0$.*

PROOF. We begin with the first assertion. By (Cont & Tankov, 2004, Proposition 9.9) and (Sato, 1999, Theorem 21.5), Condition 1 implies that the NFLVR Assumption 4.7 holds. Let $\phi$ be any admissible strategy. Then by Lemma 4.8 we have $V(\phi) = v\mathscr{E}(\theta \bullet X)$ for some predictable process $\theta$. Admissibility of $\phi$ implies $\theta \Delta X \geq -1$ and hence $K^X(\{x \in \mathbb{R} : 1 + \theta x < 0\}) = 0$. In view of Condition 1, this yields that $\theta$ is $[0,1]$-valued. Consequently, Condition 1 of Corollary 4.21 holds for $\theta$. Moreover, since $\theta$ takes values in $[0,1]$, it follows from Condition 2 that Condition 2 of Corollary 4.21 is satisfied for $\theta$ as well. Hence

$$f : [0,1] \to \mathbb{R}; \quad \theta \mapsto b^X - pc^X\theta + \int \left( \frac{x}{(1+\theta x)^p} - h(x) \right) K^X(dx)$$

is well-defined. $\theta \mapsto x(1+\theta x)^{-p} - h(x)$ is strictly decreasing on $[0,1]$, by Condition 1 the same holds for $f$. Hence there exists a unique $\eta \in [0,1]$ such that Condition 3 of Corollary 4.21 is satisfied. The second assertion now follows from (Kallsen, 2000, Lemma 4.2). □

**Remark 4.23** Conditions 1 respectively $1'$ in Proposition 4.22 mean that the asset price $S$ can jump to arbitrary positive values. In particular, $1'$ is satisfied for most Lévy processes typically considered in the literature, as e.g. the models of Merton (1976) and Kou (2002) as well as NIG and VG processes, since all of these have unbounded positive and negative jumps. Condition $2'$ then amounts to checking whether the Lévy measure $K^{\widetilde{X}}$ has sufficient exponential integrability in the tails.

Corollary 4.21 is a generalization of (Kallsen, 2000, Theorem 3.2) to the case where the dual minimizer is not necessarily an equivalent martingale measure (cf. Hurd (2004) for similar results). This effect can arise for realistic parameter values as is exemplified by the following example, which also considers the impact of jumps on the investors portfolio choice (also cf. Benth et al. (2001a) and Øksendal & Sulem (2005) for a similar discussion).

However, it is important to note here and in the other examples for portfolio optimization below that the actual numbers should be interpreted with caution. This is because they are typically proportional to the drift rate of the asset under consideration, which can only be estimated reliably over prohibitively long time series. For this reason, one should be careful when making quantitative rather than qualitative interpretations here.

**Example 4.24** Let $S = S_0 \exp(\widetilde{X})$ for some Lévy process $\widetilde{X}$ with Lévy-Khintchine triplet $(b^{\widetilde{X}}, c^{\widetilde{X}}, K^{\widetilde{X}})$. Then $S$ is strictly positive and by (Kallsen, 2000, Lemma 4.2) we have

$S = S_0\mathscr{E}(X)$ for the Lévy process $X$ with Lévy-Khintchine triplet $(b^X, c^X, K^X)$ given by

$$b^X = b^{\widetilde{X}} + \frac{c^{\widetilde{X}}}{2} + \int (h(e^x - 1) - h(x))K^{\widetilde{X}}(dx), \quad c^X = c^{\widetilde{X}}, \tag{4.4}$$

$$K^X(G) = \int 1_G(e^x - 1)K^{\widetilde{X}}(dx), \quad \forall G \in \mathscr{B}. \tag{4.5}$$

First consider the Black-Scholes model, i.e. let $(b^{\widetilde{X}}, c^{\widetilde{X}}, K^{\widetilde{X}}) = (\mu, \sigma^2, 0)$ for $\mu \in \mathbb{R}$ and $\sigma > 0$. As for parameters, we choose $\mu = 0.0438$ and $\sigma^2 = 0.0485$ so as to fit the first two moments of the DAX time series from Chapter 3. By inserting into Corollary 4.21, we obtain that

$$\eta^{BS} = \frac{b^X}{pc^X} = \frac{\mu + \sigma^2/2}{p\sigma^2} = \frac{1.404}{p}$$

represents the optimal fraction of wealth to be invested into the stock. Therefore we have $\eta^{BS} = 2.808$ for $p = \frac{1}{2}$, $\eta^{BS} = 0.702$ for $p = 2$ and $\eta^{BS} = 0.00936$ for $p = 150$. Notice that the first two choices of $p$ most likely correspond to initial endowment without future income, whereas $p = 150$ seems more suitable if considerable future earnings have to be factored in. Now denote by $K_1$ the modified Bessel function of the third kind with index $1$ and consider an NIG process with Lévy-Khintchine triplet

$$(b^{\widetilde{X}}, c^{\widetilde{X}}, K^{\widetilde{X}}) = \left(\delta + \frac{\vartheta\beta}{\sqrt{\alpha^2 - \beta^2}}, 0, \frac{\alpha\vartheta}{\pi} e^{\beta x} \frac{K_1(\alpha|x|)}{|x|} dx\right) \tag{4.6}$$

relative to the truncation function $h(x) = x$ which can be used since $X$ is a special semimartingale by Proposition A.2 and (JS, II.2.29). The following parameters are obtained by matching the first four moments to the discounted DAX time series considered in Chapter 3:

$$\alpha = 53.0, \quad \beta = -5.09, \quad \vartheta = 2.53, \quad \delta = 0.288.$$

Since $K^{\widetilde{X}}$ is absolutely continuous w.r.t. the Lebesgue measure with strictly positive density, Condition $1'$ of Proposition 4.22 is satisfied. Moreover, $\int_{\{|x|>1\}} \exp(mx)K^{\widetilde{X}}(dx)$ is finite for $-47.8 \leq m \leq 58$, hence Condition $2'$ of Proposition 4.22 holds for $p = 2$ and $p = \frac{1}{2}$. Consequently, there exists a unique $\eta^{NIG} \in [0, 1]$ such that Condition 3 of Corollary 4.21 is satisfied. Using the MATLAB solver `fsolve`, we obtain that for $p = 2$ both inequalities in Condition 3 are satisfied with equality for $\eta^{NIG} = 0.701$, which therefore represents the optimal fraction of wealth. Note that while this investment is marginally more prudent than for the Black-Scholes model, the difference is negligible compared to the variance of the moment estimators for the model parameters. Now let $p = 1/2$. Then it turns out that

$$b^X + \int \left(\frac{x}{(1 + \theta x)^p} - x\right) K^X(dx) > 0$$

for all $\theta \in [0, 1]$. In this case, (Kallsen, 2000, Theorem 3.2) is not applicable, but Corollary 4.21 yields that $\eta^{NIG} = 1$ is optimal, i.e. the investor buys the largest admissible fraction of stocks. For $p = 150$ the conditions of Proposition 4.22 are not satisfied, but one easily verifies that the conditions of Corollary 4.21 are satisfied for $\eta^{NIG} = 0.00936 \in (0, 1)$.

**Remark 4.25** These results indicate that as long as neither leverage nor shortselling is optimal, the optimal strategy in the Black-Scholes model serves as an excellent proxy for the true optimal investment strategy in pure jump Lévy models. This resembles results of Hubalek et al. (2006) on quadratic hedging strategies, where the Black-Scholes hedging strategy turns out to be very similar to the variance-optimal hedge in Lévy models with jumps. When leverage or shortselling is optimal in the Black-Scholes model, the optimal strategy for a Lévy model with unbounded jumps seems to resemble the optimal strategy for the constrained problem without shortselling or leverage, i.e. $C = [0, 1]$ in Remark 4.16.

### 4.4.2 Heston (1993)

If both $y$ and $X$ in Theorem 4.20 are chosen to be continuous (i.e. $\kappa_0 = \kappa_1 = 0$), the differential characteristics $(b^{(y,X)}, c^{(y,X)}, K^{(y,X)}, I)$ of $(y, X)$ can be written as

$$b^{(y,X)} = \begin{pmatrix} \vartheta - \lambda y \\ \delta y \end{pmatrix}, \quad c^{(y,X)} = \begin{pmatrix} \sigma^2 & \sigma\varrho \\ \sigma\varrho & 1 \end{pmatrix} y, \quad K^{(y,X)} = 0,$$

with constants $\vartheta \geq 0$, $\lambda$, $\delta$, $\sigma$, $\varrho$. Hence the most general class of continuous affine stochastic volatility models that fit the structure condition of Theorem 4.20 is given by the Heston model from Section 2.3.1 with constant drift rate $\mu = 0$.

In this case, Conditions 1, 2 and 5 of Theorem 4.20 are satisfied, since $\kappa_0 = \kappa_1 = 0$. Moreover, Condition 3 always determines $\eta$ as a function of $\alpha_1$. Insertion into Condition 4 then leads to a Riccati ODE of the form $\alpha_1'(t) = a\alpha_1^2(t) + b\alpha_1(t) + c$ for suitable $a, b, c \in \mathbb{R}$. Hence the existence of a $C^1([0, T])$-solution $\alpha_1$ (and in turn of an optimal strategy $\varphi_t = \eta(t)V_t(\varphi)/S_t$) depends both on the model parameters and the time horizon $T$.

**Corollary 4.26** *Let* $u(x) = \frac{x^{1-p}}{1-p}$, $p \in \mathbb{R}_+\backslash\{0, 1\}$ *and set*

$$a := -\frac{\sigma^2}{2} - \frac{1-p}{2p}\sigma^2\varrho^2, \quad b := \lambda - \frac{1-p}{p}\sigma\varrho\delta, \quad c := -\frac{1-p}{2p}\delta^2,$$
$$D := b^2 - 4ac = \lambda^2 - \frac{1-p}{p}\left(2\lambda\sigma\varrho\delta + \sigma^2\delta^2\right).$$

*Case 1: If* $D > 0$*, define*

$$\alpha_1(t) := -2c\frac{e^{\sqrt{D}(T-t)} - 1}{e^{\sqrt{D}(T-t)}(b + \sqrt{D}) - b + \sqrt{D}}.$$

*Case 2: If* $D = 0$ *and either* $b > 0$ *or* $b < 0$, $T < -2/b$*, define*

$$\alpha_1(t) := \frac{1}{a(T - t + 2/b)} - \frac{b}{2a}.$$

*Case 3: If* $D < 0$ *and either* $b > 0$, $T < \frac{2}{\sqrt{-D}}(\pi - \arctan(\frac{\sqrt{-D}}{b}))$*, or* $b = 0$, $T < \frac{\pi}{\sqrt{-D}}$*, or* $b < 0$, $T < \frac{2}{\sqrt{-D}}\arctan(\frac{\sqrt{-D}}{-b})$*, define*

$$\alpha_1(t) := -2c\frac{\sin(\frac{\sqrt{-D}}{2}(T-t))}{\sqrt{-D}\cos(\frac{\sqrt{-D}}{2}(T-t)) + b\sin(\frac{\sqrt{-D}}{2}(T-t))}.$$

*Then* $\varphi_t := \eta(t)v\mathscr{E}(\eta \bullet X)_t/S_t$ *with*

$$\eta(t) := \frac{\delta + \sigma\varrho\alpha_1(t)}{p}.$$

*is optimal for* $u$ *and initial endowment* $v \in (0,\infty)$ *with value process* $V(\varphi) = v\mathscr{E}(\eta \bullet X)$. *The corresponding maximal expected utility is finite and given by* $E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p}L_0$ *for the opportunity process*

$$L_t = \exp\left(\vartheta \int_t^T \alpha_1(s)ds + \alpha_1(t)y_t\right). \tag{4.7}$$

PROOF. Conditions 1, 2 and 5 of Theorem 4.20 are satisfied, because $\kappa_0 = \kappa_1 = 0$. Now notice that the denominator in the definition of $\alpha_3$ does not vanish on $[0,T]$ in all three cases. Thus $\alpha_1$ belongs to $C^\infty([0,T],\mathbb{R})$. In particular, $\alpha_0$ is a well defined and in $C^\infty([0,T],\mathbb{R})$ as well. It now follows by insertion that Condition 3 of Theorem 4.20 holds with equality. Moreover, Condition 4 is satisfied, too, because in view of (Bronstein et al., 2001, 21.5.1.2) and $|b| > \sqrt{D}$ for $D > 0$ or by direct calculations, $\alpha_3$ solves the following terminal value problem:

$$\alpha_1'(t) = a\alpha_1^2(t) + b\alpha_1(t) + c, \quad \alpha_1(T) = 0. \tag{4.8}$$

The assertions now follow from Theorem 4.20. □

**Remarks.**

1. For $p \in (0,1)$, the solution to Case 1 is derived by stochastic control methods in Kraft (2005). Case 3 appears on an informal level in Liu (2007). Observe that Corollary 4.26 does not provide a solution beyond some critical time horizon $T_\infty$, which may be finite for $p < 1$ in Cases 2 and 3. A straightforward analysis of (4.7) shows that the maximal expected utility increases to $\infty$ as $T$ tends to $T_\infty$ if the latter is finite. On the other hand, the optimal expected utility is generally an increasing function of the time horizon because one can always stop investing in the risky asset. Consequently, we have

   $$\sup\{E(u(V_T(\varphi))) : \varphi \text{ admissible strategy}\} = \infty$$

   for $T_\infty \leq T < \infty$, which means that no optimal strategy with finite expected utility exists in this case. This complements related discussions in Hobson (2004), Korn & Kraft (2004) and Kim & Omberg (1996).

2. The optimal fraction of wealth invested into stocks is not constant if the correlation $\rho$ differs from 0. We have

   $$\eta = \frac{b^X}{p\tilde{c}^X} + \frac{\alpha_1(t)\tilde{c}^{y,X}}{p\tilde{c}^X},$$

   which shows that the optimal fraction now consists of 2 parts: In addition to the myopic first term from the uncorrelated case, we now have an additional *Merton-Breeden term* that tends to 0 as $t \to T$ (cf. Merton (1973) for an economic interpretation).

3. Since Condition 3 in Theorem 4.20 is always satisfied with equality here, $\frac{LV_T^{-p}(\varphi)}{L_0 v^{-p}}$ is the density process of the $q$-optimal martingale measure for $q := 1 - \frac{1}{p} \in (-\infty, 1)$, if the conditions of Corollary 4.26 are satisfied. As a side remark, the corresponding $q$-optimal measure in Heston's model for $q > 1$ is computed in Hobson (2004).

### 4.4.3 Carr et al. (2003)

If the volatility process $y$ in Theorem 4.20 is chosen to be of finite variation (i.e. $\gamma_1^{22} = 0$ and hence $\gamma_1^{12} = 0$), one is lead to the model of Carr et al. (2003) considered in Section 2.3.3 above. The corresponding differential characteristics of $(y, X)$ are given by

$$b^{(y,X)} = \begin{pmatrix} \lambda b^Z - \lambda y_- \\ b^B y_- \end{pmatrix}, \quad c^{(y,X)} = \begin{pmatrix} 0 & 0 \\ 0 & c^B y_- \end{pmatrix}, \tag{4.9}$$
$$K^{(y,X)}(G) = \int 1_G(z,0) K^Z(dz) + \int 1_G(0,x) \lambda K^B(dx) y_-, \quad \forall G \in \mathscr{B}^2,$$

where $\lambda \neq 0$ is a constant and $(b^B, c^B, K^B)$ as well as $(b^Z, 0, K^Z)$ denote the Lévy-Khintchine triplets of a Lévy process $B$ and a subordinator $Z$, respectively.

In this case, Conditions 1, 2 and 3 of Theorem 4.20 depend neither on $t$ nor on $\alpha_1$. If there exists some $\eta$ satisfying these conditions, it therefore can be chosen to be constant. Moreover, given sufficient exponential integrability of the subordinator $Z$, the ODE for $\alpha_1$ turns out to be linear and hence always admits an explicit solution.

**Corollary 4.27** *Suppose $B$ is neither a.s. increasing nor a.s. decreasing and assume that there exists $\eta \in \mathbb{R}$ such that the following conditions hold.*

1. $K^B(\{x \in \mathbb{R} : 1 + \eta x \leq 0\}) = 0$
2. $\int |x(1+\eta x)^{-p} - h(x)| \, K^B(dx) < \infty$
3. $$b^B - pc^B\eta + \int \left( \frac{x}{(1+\eta x)^p} - h(x) \right) K^B(dx) \geq 0$$
   *if there exists some $\theta < \eta$ such that $K^B(\{x \in \mathbb{R} : 1 + \theta x < 0\}) = 0$ and*
   $$b^B - pc^B\eta + \int \left( \frac{x}{(1+\eta x)^p} - h(x) \right) K^B(dx) \leq 0$$
   *if there exists some $\theta > \eta$ such that $K^B(\{x \in \mathbb{R} : 1 + \theta x < 0\}) = 0$.*
4. *If $p \in (0,1)$, then $\int_0^T \int_1^\infty e^{\alpha_1(t)z} K^Z(dz) < \infty$, where*

$$\alpha_1(t) := \frac{e^{-\lambda(T-t)} - 1}{\lambda} \times \left( (p-1) b^B \eta + \frac{p(1-p)}{2} c^B \eta^2 - \int (1+\eta x)^{1-p} - 1 - (1-p)\eta h(x) K^B(dx) \right). \tag{4.10}$$

*Then* $\varphi_t = \eta v \mathscr{E}(\eta X)_{t-}/S_{t-}$ *is optimal for* $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0,1\}$ *and initial endowment* $v \in (0,\infty)$ *with value process* $V(\varphi) = v\mathscr{E}(\eta X)$. *The corresponding maximal expected utility is finite and given by* $E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p} L_0$ *for the opportunity process*

$$L_t = \exp\left(\int_t^T \psi^Z(\alpha_1(s))\, ds + \alpha_1(t) y_t\right).$$

PROOF. Since $B$ is not monotone, it follows from (Cont & Tankov, 2004, Proposition 9.9), Proposition A.3 and Lemma A.8 that the NFLVR Assumption 4.7 is satisfied.

Conditions 1, 2, 3 imply Conditions 1, 2, 3 of Theorem 4.20, respectively. A second order Taylor expansion yields that $\frac{1+p\eta x}{(1+\eta x)^p} - 1 = O(x^2)$ for $x \to 0$. Together with Condition 2 this implies that $\alpha_1$ is well defined because $K^B$ is a Lévy measure and

$$\begin{aligned} \frac{\lambda \alpha_1(t)}{e^{-\lambda(T-t)} - 1} =& (p-1)\eta \left( b^B - pc^B\eta + \int \left( \frac{x}{(1+\eta x)^p} - h(x)\right) K^B(dx)\right) \\ &+ \left(\frac{p(p-1)}{2} c^B \eta^2 - \int \left(\frac{1+p\eta x}{(1+\eta x)^p} - 1\right) K^B(dx)\right). \end{aligned} \tag{4.11}$$

If $p \in (0,1)$, Condition 4 ensures that Condition 5 of Theorem 4.20 holds as well. This remains true for $p \in (1,\infty)$, since Condition 4 is then automatically satisfied: Indeed, Condition 1 and the Bernoulli inequality imply that $\alpha_1$ is negative in this case, which in turn yields that Condition 4 holds, because $K^Z$ is concentrated on $\mathbb{R}_+$ (cf. (Sato, 1999, Theorem 21.5)).

Now notice that Condition 4 of Theorem 4.20 is satisfied as well, because $\alpha_1$ is a solution of the linear ODE

$$\begin{aligned} \alpha_1'(t) =& \lambda\alpha_1(t) + (p-1)b^B\eta + \frac{p(1-p)}{2} c^B \eta^2 \\ &- \int (1+\eta x)^{1-p} - 1 - (1-p)\eta h(x) K^B(dx), \\ \alpha_1(T) =& 0. \end{aligned}$$

The assertions now follow from Theorem 4.20. □

**Remarks.**

1. Notice that it follows literally as in Proposition 4.22 above that there exists a unique $\eta \in [0,1]$ satisfying Conditions 1-3 above, if the Lévy process $B$ satisfies the Conditions of Proposition 4.22. Moreover, as for exponential Lévy processes, the optimal fraction $\eta$ of stocks is constant over time and the optimal strategy is myopic, since it is completely characterized by Conditions 3, which can equivalently be rewritten in terms of the characteristics $(b^X, c^X, K^X, I)$ of $X$.

2. As for exponential Lévy processes, if $h(x) = x$ can be used as the truncation function and both inequalities in Condition 3 are satisfied with equality, a second-order Taylor expansion yields
$$\eta \approx \frac{b^B}{p(c^B + \int x^2 K^B(dx))} = \frac{b^X}{p\tilde{c}^X}$$

if the driving Lévy process $B$ has predominantly small jumps. Hence the Black-Scholes strategy is once again a good proxy for the optimal strategy in this case (cf. Example 4.28 below for a concrete parametric specification).

3. While Conditions 1-3 do not depend on the parameters of the volatility process $y$, these do appear in Condition 4, which shows that Corollary 4.27 only holds if $y$ does not have too many big jumps. In Section 4.5.3 below, we will show that this condition is not needed to establish optimality of $\varphi$, but is equivalent to the finiteness of the corresponding maximal expected utility. Indeed, as with Heston's model it may happen that Corollary 4.27 does not provide a solution for $p < 1$ beyond some finite time horizon $T_\infty$. In Section 4.5.3 below, it will turn out that the optimal expected utility is infinite for time horizons $T > T_\infty$.

4. As for exponential Lévy processes, $\frac{LV_T(\varphi)^{-p}}{L_0 v^{-p}}$ only is the density process of the $q$-optimal martingale measure, if both inequalities in Condition 3 are satisfied with equality.

5. Consider now the special case where $B_t = \delta t + W_t$ with a standard Wiener process $W$ and triplet $(b^B, c^B, K^B) = (\delta, 1, 0)$, i.e. the BNS model from Section 2.3.2. In this case the asset price process is continuous and the first two conditions of Corollary 4.27 are automatically satisfied. The third then yields that the optimal fraction of wealth in stock is given by
$$\eta = \frac{\delta}{p} = \frac{b^X}{p\tilde{c}^X}.$$
As for the integrability condition on $K^Z$, we have
$$\alpha_1(t) = \frac{1-p}{2p}\delta^2 \frac{1-e^{-\lambda(T-t)}}{\lambda}.$$
Portfolio selection in the BNS model is studied using stochastic control methods by Benth et al. (2003). They allow for an additional constant drift term in the equation for $X$ (cf. Section 4.5.1 for how to deal with this using the present martingale approach). On the other hand, they do not obtain closed-form expressions for the expected utility and for the density process of the corresponding $q$-optimal martingale measure.

6. Remark 4 after Corollary 4.21 remains true here.

We now have a look at how to verify the assumptions of Corollary 4.27 in a concrete parametric example.

**Example 4.28** Consider the discounted NIG-Gamma-OU or NIG-IG-OU specifications of the model of Carr et al. (2003) estimated in Chapter 3. More specifically, assume the stock price is modelled as $S = S_0 \exp(X)$ with $(y, X)$ as in (4.9) and parameters
$$\alpha = 90.1, \quad \beta = -16.0, \quad \vartheta = 85.9, \quad \delta = 15.5$$

for the triplet (4.6) of the NIG process $B$ and

$$\lambda = 2.54, \quad \xi = 0.0485, \quad \omega^2 = 0.00277$$

of the Gamma-OU respectively IG-OU process $y$. By Lemma 2.6, we have $S = S_0\mathscr{E}(\widetilde{X})$ with $\partial(y, \widetilde{X})$ of the form (4.9), if $(b^B, c^B, K^B)$ are replaced with

$$b^{\widetilde{B}} = b^B + \frac{c^B}{2} + \int (h(e^x - 1) - h(x))K^B(dx), \quad c^{\widetilde{B}} = c^B,$$
$$K^{\widetilde{B}} = \int 1_G(e^x - 1)K^B(dx) \quad \forall G \in \mathscr{B}. \tag{4.12}$$

Since the asset price has unbounded jumps, it follows as in Example 4.24 above that all admissible strategies are therefore of the form $\phi = \theta V(\phi)_-/S_-$ with $\theta \in [0, 1]$ and hence automatically satisfy Condition 1 in Corollary 4.27. Since $\int \exp(mx)K^B < \infty$ for $-74.1 \leq m \leq 106.1$, it follows from (4.12) that Condition 2 holds for all admissible strategies, if $p = 2$ or $p = \frac{1}{2}$. For $p = 2$, Condition 4 of Corollary 4.27 is not needed. For $p = \frac{1}{2}$ we have

$$\alpha_1(0) = 0.0797,$$

which in view of (4.12) implies that Condition 4 is satisfied, because our Gamma-OU respectively IG-OU process has finite $m$-th exponential moments for $m \leq 17.5$ respectively $m \leq 17.5/2$ by e.g. (Schoutens, 2003, Sections 5.5.1, 5.5.2). Hence $\eta^{NIG-OU} \in [0, 1]$ is the optimal fraction of stocks if it satisfies Condition 3. For $p = 2$ both inequalities in Condition 3 are satisfied with equality for $\eta^{NIG-OU} = 0.701$. For $p = \frac{1}{2}$, we have

$$\left(b^B + \frac{c^B}{2}\right) - pc^B\alpha_0 + \int \left(\frac{e^x - 1}{(1 + \alpha_0(e^x - 1))^p} - h(x)\right) K^B(dx) > 0 \tag{4.13}$$

for all $\theta \in [0, 1]$, which means that $\eta^{NIG-OU} = 1$ satisfies Condition 3. Analogously, one verifies directly that $\eta^{NIG-OU} = 0.00936 \in (0, 1)$ satisfies the conditions of Corollary 4.27 and therefore represents the optimal fraction of stocks for $p = 150$. Hence we get the same results as for the NIG model in Example 4.24 above. Similarly, the BNS model leads to the same optimal fractions $\eta^{BNS} = 2.808$ (for $p = \frac{1}{2}$), $\eta^{BNS} = 0.702$ (for $p = 2$) and $\eta^{BNS} = 0.00936$ (for $p = 150$) as the Black-Scholes model.

Further specific examples where Theorem 4.20 is applicable include the model of Carr & Wu (2003) as well as generalizations of the Heston model featuring jumps in the asset price (cf. (Kallsen, 2006, Section 4.4) for more details). Other rather straightforward extensions concern a superposition of Lévy-driven Ornstein-Uhlenbeck processes as in Barndorff-Nielsen & Shephard (2001) as well as multivariate versions of the models in Sections 4.4.2, 4.4.3. For more details on these issues we refer the reader to Vierthauer (2009), who considers utility maximization for exponential utility in a general multidimensional affine model.

## 4.5 Solution in models with conditionally independent increments

In Section 4.4 we obtained optimal investment strategies for power utility in affine stochastic volatility models. Moreover, we also determined the corresponding opportunity process (and hence the value function), which turned out to be an exponentially affine function of the volatility process $y$, too. However, the approach of Chapter 4.4 crucially depends on the following two assumptions:

1. The stochastic volatility model under consideration has to be *affine*.
2. The dynamics of the asset price have to be proportional to the volatility of the market, i.e. the differential semimartingale characteristics of $X$ have to be *linear* functions of $y_-$ without additional constant terms.

However, optimal strategies have in some instances been obtained in the literature if one or both of these assumptions are not satisfied (cf. e.g. Benth et al. (2003) and Delong & Klüppelberg (2008)), even though the value function cannot be determined explicitly in these cases. Loosely speaking, this is possible due to the fact that in the setup of Benth et al. (2003) and Delong & Klüppelberg (2008) the driving processes of the asset prices and the stochastic factors are assumed to be independent. Hence the problem can be reduced to dealing with processes with independent increments by conditioning on the whole factor process. Then one can proceed by applying martingale methods almost literally as in the Lévy case considered in Kallsen (2000). In the remainder of this chapter, we will make this statement precise.

Since it does not require additional effort here, we consider the general multidimensional case and assume that the discounted stock prices $S^1, \dots, S^d$ are modelled as positive processes of the form $S^i = S^i_0 \mathscr{E}(X^i)$, $i = 1, \dots, d$ for some semimartingale $X$ with differential characteristics $(b^X, c^X, K^X, I)$.

**Remark 4.29** Notice that in the present very general framework, modelling the stock prices as ordinary exponentials $S^i = S^i_0 \exp(\widetilde{X}^i)$, $i = 1, \dots, d$ for some semimartingale $\widetilde{X}$ leads to the same class of models. Hence, all results can easily be transfered to models of the form $S^i = S^i_0 \exp(X^i)$, $i = 1, \dots, d$ by applying Propositions A.3 and A.4.

In the following, we will show that the optimal portfolio is *myopic*, i.e. only depends on the local dynamics of $S$ respectively $X$, if these are deterministic conditional on some semimartingale $y$. More specifically, let $y$ be some semimartingale and define the augmented $\sigma$-fields

$$\mathscr{G}_t := \bigcap_{s>t} \sigma(\mathscr{F}_s \cup \sigma(y_r, 0 \leq r \leq T)), \quad 0 \leq t \leq T,$$

and the filtration

$$\mathbf{G} := (\mathscr{G}_t)_{t \in [0,T]}.$$

**Assumption 4.30** *$X$ is a semimartingale with differential characteristics $(b^X, c^X, K^X, I)$ relative to the enlarged filtration $\mathbf{G}$ and $(b^X, c^X, K^X)$ are $\mathscr{G}_0$-measurable, i.e. $X$ is a $\mathbf{G}$-semimartingale with $\mathscr{G}_0$-*conditionally independent increments *(cf. (JS, II.6) for more details).*

**Remark 4.31** Assumption 4.30 means that the dynamics $(b^X, c^X, K^X)$ of $X$ are measurable functions of $y$ which can therefore be interpreted as a *stochastic factor process*. In general, a semimartingale $X$ will not remain a semimartingale with respect to an enlarged filtration (cf. e.g. Protter (2004) and the references therein). Even if the semimartingale property is preserved, the characteristics generally do not remain unchanged. Nevertheless, we shall show in Sections 4.5.1 and 4.5.2 below that some fairly general models satisfy this property, if the factor process $y$ is *independent* of the other sources of randomness in the model.

Subject to Assumption 4.30 we now have the following general result in models satisfying the NFLVR Assumption 4.7.

**Theorem 4.32** *Let $u(x) = \frac{x^{1-p}}{1-p}$, $p \in \mathbb{R}_+ \setminus \{0,1\}$ and $v \in (0,\infty)$. Suppose Assumptions 4.7 and 4.30 hold and assume there exists an $\mathbb{R}^d$-valued stochastic process $\eta \in L(X)$ such that the following conditions are satisfied up to a $dP \otimes dt$-null set on $\Omega \times [0,T]$.*

1. $K^X(\{x \in \mathbb{R}^d : 1 + \eta^\top x \leq 0\}) = 0.$
2. $\int \left| x(1+\eta^\top x)^{-p} - h(x) \right| K^X(dx) < \infty.$
3. *For all $\theta \in \mathbb{R}^d$ such that $K^X(\{x \in \mathbb{R}^d : 1 + \theta^\top x < 0\} = 0$ we have*
$$(\theta^\top - \eta^\top)\left(b^X - pc^X\eta + \int \frac{x}{(1+\eta^\top x)^p} - h(x)K^X(dx)\right) \leq 0,$$
4. $\int_0^T |\alpha_s| ds < \infty$, *where*
$$\alpha := (1-p)\eta^\top b^X - \frac{p(1-p)}{2}\eta^\top c^X \eta + \int \left((1+\eta^\top x)^{1-p} - 1 - (1-p)\eta^\top h(x)\right) K^X(dx),$$

*Then there exists a $\mathscr{G}_0$-measurable process $\widetilde{\eta}$ satisfying Conditions 1-4 such that the strategy $\varphi = (\varphi^1, \ldots, \varphi^d)$ defined as*
$$\varphi^i_t := \widetilde{\eta}^i(t)\frac{v\mathscr{E}(\widetilde{\eta} \bullet X)_{t-}}{S^i_{t-}}, \quad i = 1, \ldots, d, \quad t \in [0,T], \tag{4.14}$$
*is optimal for $u$ and initial endowment $v$ with value process $V(\varphi) = v\mathscr{E}(\widetilde{\eta} \bullet X)$. The corresponding maximal expected utility is given by*
$$E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p}E\left(\exp\left(\int_0^T \alpha_s ds\right)\right),$$
*In particular, if $\eta$ is $\mathscr{G}_0$-measurable, this holds for $\varphi^i = \eta^i v\mathscr{E}(\eta \bullet X)_-/S^i_-$.*

PROOF. In view of Conditions 1-4, the measurable selection theorem (Sainte-Beuve, 1974, Theorem 3) and (Jacod, 1979, Proposition 1.1) show the existence of $\widetilde{\eta}$, since $(b^X, c^X, K^X)$ are $\mathscr{G}_0$-measurable by Assumption 4.30. Hence we can assume w.l.o.g. that $\eta$ is $\mathscr{G}_0$-measurable, because we can otherwise pass to $\widetilde{\eta}$ instead.

Since $X$ is a $\mathbf{G}$-semimartingale with $\mathscr{G}_0$-measurable differential characteristics by Assumption 4.30, (JS, 6.6) shows that it has $\mathscr{G}_0$-conditionally independent increments. Relative to the filtration $\mathbf{G}$, the optimality of the strategy $\varphi$ can now be derived almost literally as in the Lévy case discussed in Kallsen (2000). This is done in the following lemma.

**Lemma 4.33** *Suppose the assumptions of Theorem 4.32 are satisfied. Then the strategy $\varphi$ defined in* (4.14) *maximizes $\phi \mapsto E(u(V_T(\phi))|\mathscr{G}_0)$ over all $\phi$ which are admissible w.r.t.* $\mathbf{G}$. *Moreover, the corresponding maximal conditional expected utility is finite and given by*

$$E(u(V_T(\varphi))|\mathscr{G}_0) = \frac{v^{1-p}}{1-p} \exp\left(\int_0^T \alpha_s ds\right).$$

PROOF. *First step*: We begin by showing $\varphi \in L(S)$. Since $\eta$ and hence $\varphi$ is $\mathbf{F}$-predictable by assumption and $\mathscr{F}_t \subset \mathscr{G}_t$, $t \in [0,T]$, we obtain that $\varphi$ is $\mathbf{G}$-predictable as well. In view of Assumption 4.30, the differential characteristics of $X$ relative to $\mathbf{F}$ coincide with those relative to $\mathbf{G}$. Together with (JS, III.6.30) this implies $H \in L(X)$ and hence $\varphi \in L(S)$ with respect to $\mathbf{G}$.

*Second step*: As in Lemma 4.8 it follows that the value process of $\varphi$ is given by $V(\varphi) = v\mathscr{E}(\eta \bullet X)$. By Condition 1 in Theorem 4.32 and (JS, I.4.61) this implies $V(\varphi) > 0$. Hence $\varphi$ is admissible w.r.t $\mathbf{G}$.

*Third step*: Since $\int_0^T |\alpha_s| ds < \infty$ outside some $dP$-null set by Condition 4, the process

$$L_t := \exp\left(\int_t^T \alpha_s ds\right) = L_0 \mathscr{E}\left(\int_\cdot^T \alpha_s ds\right)_t.$$

is indistinguishable from a real-valued càdlàg process of finite variation and hence a $\mathbf{G}$-semimartingale, since $\eta$ and $(b^X, c^X, K^X)$ are $\mathscr{G}_0$ measurable. Let $\phi$ be any admissible strategy w.r.t. $\mathbf{G}$. In view of (Delbaen & Schachermayer, 1998, Theorem 1.1), Lemma A.8 and Assumption 4.30, Assumption 4.7 implies that the market with enlarged filtration $\mathbf{G}$ satisfies NFLVR as well. In view of Lemma 4.8 the corresponding value process can therefore be written as $V(\phi) = v + V_-(\phi)\theta \bullet X$ for some $\mathbf{G}$-predictable process $\theta$. The admissibility of $\phi$ implies $\theta_t^\top \Delta X_t \geq -1$ which in turn yields that outside some $dP \otimes dt$ null set,

$$K^X(\{x \in \mathbb{R}^d : 1 + \theta_t^\top x < 0\}) = 0. \tag{4.15}$$

The characteristics $(b^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}, c^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}, K^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}, I)$ of $\frac{L}{L_0}V(\varphi)^{-p}V(\phi)$ can now be computed similarly as in the proof of Theorem 4.20 using Propositions A.3 and A.4. In particular, we obtain

$$K^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}(G) = \int 1_G\left(\frac{L_-}{L_0}V_-(\varphi)^{-p}V_-(\phi)\left(\frac{1+\theta^\top x}{(1+\eta^\top x)^p} - 1\right)\right) K^X(dx),$$

for all $G \in \mathscr{B}$, which combined with Condition 2 yields

$$\int_{\{|x|>1\}} |x| K^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}(dx) < \infty \tag{4.16}$$

outside some $dP \otimes dt$-null set. Moreover, insertion of the definition of $\alpha$ leads to

$$b^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)} = \int (h(x)-x) K^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}(dx)$$
$$+\frac{L_-}{L_0} V_-(\varphi)^{-p} V_-(\phi)(\theta^\top - \eta^\top)\left(b^X - pc^X\eta + \int \frac{x}{(1+\eta^\top x)^p} - h(x)K^X(dx)\right),$$

and hence

$$b^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)} + \int (x-h(x)) K^{\frac{L}{L_0}V(\varphi)^{-p}V(\phi)}(dx) \leq 0 \tag{4.17}$$

$dP \otimes dt$-almost everywhere on $\Omega \times [0,T]$ by (4.15) and Condition 3. In view of (4.16) and (4.17) the process $\frac{L}{L_0}V(\varphi)^{-p}V(\phi)$ is therefore a supermartingale by Lemma A.8 and Proposition A.9.

If we set $\phi = \varphi$, we obtain from (4.17) and Lemma A.8 that the supermartingale $\frac{L}{L_0}V(\varphi)^{1-p}$ is a positive $\sigma$-martingale with initial value $v^{1-p} < \infty$ and hence a local martingale by (Jacod, 1979, Proposition 2.18) and (Kallsen, 2004, Corollary 3.1). We now show that it is a true martingale.

Since $\frac{L}{L_0}V(\varphi)^{1-p}$ is a supermartingale it is sufficient to show $E(\frac{L_T}{L_0}V_T(\varphi)^{1-p}) = v^{1-p}$. As this property only depends on the distribution of $\frac{L}{L_0}V(\varphi)^{1-p}$ we can assume w.l.o.g. that $(\Omega, \mathscr{F}, \mathbf{G})$ is given by the canonical path space $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ of $\mathbb{R}^d$-valued càdlàg functions endowed with its natural filtration (cf. (JS, Chapter VI)). An application of Proposition A.3 shows that the differential characteristics of $\mathscr{L}(\frac{L}{L_0}V(\varphi)^{1-p})$ are $\mathscr{G}_0$-measurable. Hence $\mathscr{L}(\frac{L}{L_0}V^{1-p})$ is a process with $\mathscr{G}_0$-conditionally independent increments by (JS, II.6.6). In view of (Shiryaev, 1995, Theorem II.7.5) there exists a regular version $R(\omega, d\omega')$ of the conditional probability relative to $\mathscr{G}_0$ on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$. Moreover, it follows from (JS, II.6.13) and (JS, II.6.15) that $\mathscr{L}(\frac{L}{L_0}V(\varphi)^{1-p})$ is a process with independent increments and a local martingale under the measure $R(\omega, \cdot)$ for $P$-almost all $\omega$. Hence Proposition 2.20 yields that it is true martingale under the measure $R(\omega, \cdot)$ for $P$-almost all $\omega$. Together with (Shiryaev, 1995, Theorem II.7.3) this implies

$$E\left(\frac{L_T}{L_0}V_T(\varphi)^{1-p}\right) = E\left(E\left(\frac{L_T}{L_0}V_T(\varphi)^{1-p}\middle|\mathscr{G}_0\right)\right)$$
$$= \int\int \frac{L_T(\widetilde{\omega})}{L_0(\widetilde{\omega})} V_T(\varphi)^{1-p}(\widetilde{\omega}) R(\omega, d\widetilde{\omega}) P(d\omega) = v^{1-p}.$$

*Fourth step*: Now we are ready to show that $\varphi$ is indeed optimal. Since $u$ is concave, we have

$$u(V_T(\phi)) \leq u(V_T(\varphi)) + u'(V_T(\varphi))(V_T(\phi) - V_T(\varphi)) \tag{4.18}$$

for any admissible $\phi$. This implies

$$\begin{aligned} E(u(V_T(\phi))|\mathscr{G}_0) &\leq E(u(V_T(\varphi))|\mathscr{G}_0) + L_0 E\left(\frac{L_T}{L_0}V_T(\varphi)^{-p}V_T(\phi) - \frac{L_T}{L_0}V_T(\varphi)^{1-p}\middle|\mathscr{G}_0\right) \\ &\leq E(u(V_T(\varphi))|\mathscr{G}_0), \end{aligned}$$

because by the third step, $\frac{L}{L_0}V(\varphi)^{-p}V(\phi)$ is a $\mathbf{G}$-supermartingale and $\frac{L}{L_0}V(\varphi)^{1-p}$ is a $\mathbf{G}$-martingale, both starting at $v^{1-p}$. This shows that $\varphi$ is optimal conditional on $\mathscr{G}_0$ as claimed. The formula for the corresponding maximal expected utility follows immediately from $L_T = 1$ and the martingale property of $\frac{L}{L_0}V(\varphi)^{1-p}$. □

We can now complete the proof of Theorem 4.32. First note that the first two steps in the proof of Lemma 4.33 also show that $\varphi$ is admissible w.r.t. the filtration $\mathbf{F}$. As $X$ and hence $S$ have the same differential characteristics with respect to both $\mathbf{G}$ and $\mathbf{F}$ and $\mathbf{F}$-predictability implies $\mathbf{G}$-predictability, (JS, III.6.30) yields that any other admissible strategy in the market with the original filtration $\mathbf{F}$ is admissible with respect to $\mathbf{G}$ as well. Hence optimality of $\varphi$ relative to $\mathbf{G}$ yields optimality of $\varphi$ with respect to $\mathbf{F}$. This completes the proof of Theorem 4.32. □

**Remarks.**

1. Conditions 1-3 are local versions of the corresponding Conditions 1-3 in Corollary 4.21 for Lévy processes. They show that the optimal strategy is characterized completely by the local dynamics of $X$ (or equivalently $S$) in the present setup, i.e. the optimal strategy is myopic. Put differently, the optimal proportions $\eta^i$ of wealth allocated to stock $i$ are the same as in the Lévy case considered in Corollary 4.21 above, if the Lévy-Khintchine triplet is replaced by the random, time-dependent differential characteristics. This is a generalization of an observation from Benth et al. (2003): The investor invests locally as in the Lévy case, but adapts her strategy depending on the local behaviour of the factor process $Y$. This corresponds to well known results for logarithmic utility (cf. e.g Goll & Kallsen (2000, 2003)). However, it is important to note that whereas the optimal strategy is myopic in the general semimartingale case for logarithmic utility, one also needs the (weak) Condition 4 of Theorem 4.32 as well as the crucial Assumption 4.30 for power utility. Nevertheless, we show in Sections 4.5.1 and 4.5.2 below that a wide range of commonly used models fall into this framework, if the drivers of the asset price are independent of the drivers of the stochastic factor process.

2. Condition 4 is needed to ensure that the conditional expected utility of $\varphi$ is finite. However, even if it is satisfied, the unconditional expected utility corresponding to $\varphi$ generally does not have to be finite for $p \in (0,1)$. On the contrary the maximal expected utility is obviously finite for $p > 1$, since the power utility function $u(x) = x^{1-p}/(1-p)$ is bounded from above in this case. Indeed, this can also be derived from Theorem 4.32, because Condition 3 of Theorem 4.32 and the Bernoulli inequality imply that $\alpha$ is negative in this case.

3. The NFLVR Assumption is only needed to apply Lemma 4.8 in the market with enlarged filtration $\mathbf{G}$. It is therefore not needed if

$$b^X - pc^X\eta + \int \left( \frac{x}{(1+\eta^\top x)^p} - h(x) \right) K^X(dx) = 0$$

outside some $dP \otimes dt$-null set, because $\frac{LV(\varphi)^{-p}}{L_0 v^{-p}}$ is the density process of the $q$-optimal equivalent martingale measure for this market in this case.

4. For the constrained problem of maximizing expected utility over all $\phi \in \Theta(v, C)$ the optimal fraction $\eta$ of stocks has to be $C$-valued, but Condition 3 only has to be checked for $\{\theta \in C : K^X(\{1 + \theta^\top x < 0\}) = 0\}$.

5. As for the assumption $\eta \in L(X)$, the crucial point is integrability rather than predictability. This is because if pointwise solutions to Condition 3 exist, the measurable selection theorem (Sainte-Beuve, 1974, Theorem 3) shows that they can be chosen to be both $\mathbf{F}$-predictable and $\mathscr{G}_0$-measurable.

6. In view of Lemma 4.33 one can also interpret Theorem 4.32 as follows. Given Assumption 4.30, complete information about some factor process $y$ does not yield any benefits to the investor, as it remains optimal to invest according to the same strategy. Hence *inside information* in the sense of Di Nunno et al. (2006) about this stochastic factor does not make a difference subject to Assumption 4.30.

We now consider two particular special cases that suffice to cover a wide range of applications.

**Corollary 4.34 (Continuous paths)** *Suppose $X$ is continuous, Assumption 4.30 holds and there exists $\eta \in L(X)$ such that $b^X = pc^X\eta$. Then $\varphi^i = \eta^i v \mathscr{E}(\eta \bullet X)/S$, $i = 1, \dots, d$ is optimal.*

PROOF. Since $X$ is continuous, we have $K^X = 0$, Conditions 1-2 of Theorem 4.32 are satisfied and $\eta \in L(X)$ as well as (JS, III.6.30) yield that Condition 4 holds as well. By Condition 1, both inequalities in Condition 3 of Theorem 4.32 are satisfied. Hence we are in the situation of Remark 3 after Theorem 4.32, i.e. the NFLVR Assumption 4.7 is not needed to apply Theorem 4.32. This proves the assertion. □

**Corollary 4.35 (Arbitrary positive and negative jumps)** *Let $d = 1$. Then Conditions 1-4 of Theorem 4.32 are satisfied for a unique $[0,1]$-valued process $\eta$, if the following holds up to a $dP \otimes dt$-null set.*

1. *$K^X((-1, b)), K^X((a, \infty)) > 0$ for any $b \in (-1, 0)$, $a \in (0, \infty)$,*

2. *$\int_0^T \int_\varepsilon^\infty x K_t^X(dx)dt < \infty$ and $\int_0^t \int_{-1}^{-\varepsilon} \frac{-x}{(1+x)^p} K_t^X(dx)dt < \infty$ for some $\varepsilon \in (0, 1)$.*

PROOF. Literally as in the proof of Proposition 4.22, Conditions 1 and 2 show that Conditions 1-3 of Theorem 4.32 are satisfied for a unique $[0,1]$-valued process $\eta$. Since $\eta$ is bounded, Condition 2 yields that Condition 4 of Theorem 4.32 holds as well. □

**Remark 4.36** If $d=1$ and $S=S_0\exp(\widetilde{X})$, Propositions A.4 and A.3 show that the conditions of Corollary 4.35 hold if Conditions 1 and 2 are replaced with

1'. $K^{\widetilde{X}}((-\infty,-a)), K^{\widetilde{X}}((a,\infty))>0$ for any $a\in(0,\infty)$,

2'. $\int_0^T\int_\varepsilon^\infty e^x K_t^{\widetilde{X}}(dx)dt<\infty$ and $\int_0^T\int_{-\infty}^{-\varepsilon} e^{-px}K_t^{\widetilde{X}}(dx)dt<\infty$ for some $\varepsilon>0$.

### 4.5.1 Integrated Lévy models

In this section, we assume that the discounted stock price $S$ is modelled as a positive process of the form $S=(S^1,\dots,S^d)$, where $S^i=S_0^i\mathscr{E}(X^i)$, $i=1,\dots,d$ and

$$X=y_-\bullet B, \tag{4.19}$$

for an $\mathbb{R}^{d\times n}$-valued semimartingale $y$ and an independent $\mathbb{R}^n$-valued Lévy process $B$ with Lévy triplet $(b^B,c^B,K^B)$. Furthermore, we suppose that the underlying filtration $\mathbf{F}$ is generated by $B$ and $Y$ (or equivalently by $X$ and $y$ if $d=n$ and $y$ takes values in the invertible $\mathbb{R}^{d\times d}$-matrices). The following result shows that Assumption 4.30 is satisfied in this case.

**Lemma 4.37** *Relative to both* $\mathbf{F}$ *and* $\mathbf{G}$*,* $X$ *is a semimartingale with* $\mathscr{G}_0$*-measurable differential characteristics* $(b^X,c^X,K^X,I)$ *given by*

$$b^X=y_-b^B+\int(h(y_-x)-y_-h(x))K^B(dx),\quad c^X=y_-c^By_-^\top,$$
$$K^X(G)=\int 1_G(y_-x)K^B(dx)\quad \forall G\in\mathscr{B}^d.$$

*In particular, Assumption 4.30 is satisfied.*

PROOF. Since $B$ is independent of $y$ and $\mathbf{F}$ is generated by $y$ and $B$, it follows from (Bauer, 2002, Theorem 15.5) that $B$ remains a Lévy process (and in particular a semimartingale), if its natural filtration is replaced with either $\mathbf{F}$ or $\mathbf{G}$. Since the distribution of $B$ does not depend on the underlying filtration, we know from the Lévy-Khintchine formula and Proposition A.2 that $B$ admits the same differential characteristics $(b^B,c^B,K^B,I)$ with respect to its natural filtration and both $\mathbf{F}$ and $\mathbf{G}$. Since $y_-$ is locally bounded and ($\mathbf{F}$-,$\mathbf{G}$-) predictable, the process $X$ is a ($\mathbf{F}$-,$\mathbf{G}$-) semimartingale by (JS, I.4.31). Its differential characteristics can now be derived by applying Proposition A.3. The $\mathscr{G}_0$-measurability is obvious. □

**Remark 4.38** Notice that if the Lévy process $B$ has jumps, one is lead to a different class of models if $S^i=S_0^i\exp(X^i)$, $i=1,\dots,d$ for $X$ as in (4.19) above. However, we have $S^i=S^i\mathscr{E}(\widetilde{X}^i)$, $i=1,\dots,d$ for the processes $\widetilde{X}^i=\mathscr{L}(\exp(X^i))$, $i=1,\dots,d$. Moreover, subject to Assumption 4.30, Lemma 4.37 and Propositions A.4, A.3 show that $\widetilde{X}$ admits the same differential characteristics $(b^{\widetilde{X}},c^{\widetilde{X}},K^{\widetilde{X}},I)$ w.r.t. both $\mathbf{F}$ and $\mathbf{G}$. Since $(b^{\widetilde{X}},c^{\widetilde{X}},K^{\widetilde{X}},I)$ also turn out to be $\mathscr{G}_0$-measurable, Assumption 4.30 holds for $\widetilde{X}$ as well.

To show that NFLVR holds, it suffices to consider the Lévy process $B$.

**Lemma 4.39** *If there exists $Q \sim P$ such that $B$ is a $\sigma$-martingale, Assumption 4.7 is satisfied. For $d = 1$ this holds unless $B$ is either $P$-a.s. increasing or $P$-a.s. decreasing.*

PROOF. By (Kallsen, 2004, Lemma 3.3), $X$ and $S$ are $Q$-$\sigma$-martingales as well. The first part of the assertion now follows from the Fundamental Theorem of Asset Pricing. The second is a consequence of (Cont & Tankov, 2004, Proposition 9.9). □

**Remark 4.40** In view of Lemma 4.39, only degenerate *monotone* Lévy process do not satisfy NFLVR in the univariate case. For multiple stocks this ceases to hold, consider e.g. $B = (1 + W, 2 + W)$ for a Wiener process $W$. In the continuous case, i.e. if $B$ is a multivariate Brownian motion with drift $\mu \in \mathbb{R}^d$ and diffusion matrix $\sigma^\top \sigma \in \mathbb{R}^{d\times d}$, (Karatzas & Shreve, 1998, Theorem 4.2) ascertains that NFLVR holds, if the drift vector $\mu$ lies in the range of $x \mapsto \sigma x$. For Lévy processes with jumps a criterion with similar intuitive appeal does not seem to exist to the best of our knowledge.

### 4.5.2 Time-changed Lévy models

In this section we show that Theorem 4.32 can also be applied to time-changed Lévy models. For Brownian motion, stochastic integration and time changes lead to essentially the same models by the Dambins-Dubins-Schwarz theorem (cf. e.g. (Revuz & Yor, 1999, V.1.6)). For general Lévy processes with jumps, however, the two classes are quite different. More details concerning the theory of time changes can be found in Jacod (1979) and Kallsen & Shiryaev (2002), whereas their use in modelling is dealt with in Kallsen (2006).

Here, we assume that the discounted asset price process is of the form $S = (S^1, \dots, S^d)$, with $S^i = S^i_0 \mathscr{E}(X^i)$, $i = 1, \dots d$ and

$$X = \mu(y_-) \bullet I^d + B_{\int_0^\cdot y_s ds}, \tag{4.20}$$

for the identity process $I^d_t = (t, \dots, t)$ on $\mathbb{R}^d$, a mapping $\mu : \mathbb{R} \to \mathbb{R}^d$ such that $\mu(y_-) \in L(I^d)$ for a strictly positive semimartingale $y$ and an independent $\mathbb{R}^d$-valued Lévy process $B$ with Lévy-Khintchine triplet $(b^B, c^B, K^B)$. Moreover, we suppose that the underlying filtration is generated by $X$ and $y$. We have the following analogue of Lemma 4.37.

**Lemma 4.41** *Relative to both* $\mathbf{F}$ *and* $\mathbf{G}$*, $X$ is a semimartingale with $\mathscr{G}_0$-measurable differential characteristics $(b^X, c^X, K^X, I)$ given by*

$$b^X = \mu(y_-) + b^B y_-, \quad c^X = c^B y_-, \quad K^X(G) = K^B(G) y_- \quad \forall G \in \mathscr{B}^d.$$

*In particular, Assumption 4.30 is satisfied.*

PROOF. Let $Y = \int_0^\cdot y_s ds$ and $U_r := \inf\{q \in \mathbb{R}_+ : Y_q \geq r\}$ and define the $\sigma$-fields

$$\mathscr{H}_t := \bigcap_{s>t} \sigma(B_q, 0 \leq q \leq s, U_r, 0 \leq r < \infty).$$

Since $B$ is independent of $y$ and hence $Y$, it remains a Lévy process relative to the filtration $\mathbf{H} := (\mathscr{H}_t)_{t\in\mathbb{R}_+}$. Its distribution does not depend on the underlying filtration, hence we know from the Lévy-Khintchine formula and Proposition A.2 that it is a semimartingale with differential characteristics $(b^B, c^B, K^B, I)$ relative to $\mathbf{H}$.

By Proposition A.6 the time-changed process $(\widetilde{B}_\vartheta)_{\vartheta\in[0,T]} := (B_{Y_\vartheta})_{\vartheta\in[0,T]}$ is a semimartingale on $[0,T]$ relative to the time-changed filtration $(\widetilde{\mathscr{H}_\vartheta})_{\vartheta\in[0,T]} := (\mathscr{H}_{Y_\vartheta})_{\vartheta\in[0,T]}$ with differential characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ given by

$$\widetilde{b}_\vartheta = b^B y_{\vartheta-}, \quad \widetilde{c}_\vartheta = c^B y_{\vartheta-}, \quad \widetilde{K}_\vartheta(G) = K^B(G) y_{\vartheta-} \quad \forall G \in \mathscr{B}^d.$$

Furthermore, it follows from the proof of (Pauwels, 2007, Proposition 4.3) that $\widetilde{\mathscr{H}_t} = \mathscr{G}_t$ for all $t \in [0,T]$. The assertion now follows by applying Propositions A.3 and A.4 to compute the characteristics of $X$. □

**Remarks.**

1. For the proof of Lemma 4.41 we had to assume that the given filtration is generated by the process $(y, X)$ or equivalently $(Y, X)$. In reality, though, the integrated volatility $Y$ and the volatility $y$ typically cannot be observed directly. Therefore the canonical filtration of the logarithmized asset prize $X$ would be a more natural choice. Fortunately, $Y$ and $y$ are typically adapted to the latter if $B$ is an infinite activity process (cf. e.g. Winkel (2001)).

2. A natural generalization of (4.20) is given by models of the form
$$X = \mu(y_-^{(1)}, \dots, y_-^{(n)}) \bullet I + \sum_{i=1}^{n} B^{(i)}_{Y^{(i)}},$$
for $\mu : (0,\infty)^n \to \mathbb{R}^d$, strictly positive semimartingales $y^{(i)}$, $Y^{(i)} = \int_0^\cdot y_s ds$ and independent Lévy processes $B^{(i)}$, $i = 1, \dots, n$. If one allows for the use of the even larger filtration generated by all $y^{(i)}$, $B^{(i)}_{Y^{(i)}}$, $i = 1, \dots, n$ the proof of Lemma 4.41 remains valid. If $Y^{(i)}$ is interpreted as business time in some market $i$, this class of models allows assets to be influenced by the changing activity in different markets.

3. Unlike in the previous section, models of the form $S^i = S_0^i \exp(X^i)$ and $S^i = S_0^i \mathscr{E}(X^i)$ lead to the same class of processes for time-changed Lévy processes: If $S^i = S_0^i \exp(\mu^i(y_-) + B_Y^i)$, $i = 1, \dots, d$, we have $S^i = S_0^i \mathscr{E}(\mu^i(y_-) \bullet I + \widetilde{B}_Y^i)$ for some other Lévy process $\widetilde{B}$ by Propositions A.3 and A.4.

The NFLVR Assumption 4.7 is rather difficult to check here, since it no longer suffices to consider structure-preserving measure changes as e.g. in Section 2.5. We leave more general results to future research and only consider the univariate case where $(y, X)$ is given by the model of Carr et al. from Section 2.3.3 with $\mu$ not necessarily equal to $0$. Since $y$ is bounded from below in this model, the results of Cheridito et al. (2005) allow us to show that similarly as for Lévy processes, NFLVR holds if $B$ and hence the asset price $X$ has either a Gaussian component or both positive and negative jumps.

**Lemma 4.42** *Let $d = 1$ and suppose $(y, X)$ is given by the model of Carr et al. (2003) from Section 2.3.3. Then the NFLVR Assumption 4.7 is satisfied if the following holds.*

$$c^B > 0 \text{ or } K^B((a, 0)), K^B((0, b)) > 0 \text{ for some } a \in (-1, 0) \text{ and } b \in (0, \infty).$$

PROOF. First notice that by the proof of (Cont & Tankov, 2004, Proposition 9.9), we can assume w.l.o.g. that $b^B = 0$ and $K^B$ has moments of all orders. In particular, $h(x_1, x_2) := (\chi(x_1), x_2)$ can be used as the truncation function. Denote by $(b, c, K, I)$ the differential characteristics

$$b = \begin{pmatrix} \lambda b^Z - \lambda y_- \\ \mu \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ 0 & c^B y_- \end{pmatrix},$$

$$K(G) = \int 1_G(z, 0)\lambda K^Z(dz) + \int 1_G(0, x) K^B(dx) y_- \quad \forall G \in \mathscr{B}^2,$$

of the affine semimartingale $(y, X)$ w.r.t. $h$ and set

$$E := [\underline{y}/2, \infty) \times \mathbb{R}, \quad U := (\underline{y}/2, \infty) \times \mathbb{R}, \quad U^n := (\underline{y}/2, n) \times (-n, n),$$

where $\underline{y} := e^{-\lambda T} y_0$ denotes the lower bound of $y$ (cf. e.g. Barndorff-Nielsen & Shephard (2001)). First consider the case $c^B > 0$ and let

$$b^* := (b^1, 0)^\top, \quad c^* := c, \quad K^* := K.$$

It then follows from Theorem 2.4 that there exists a unique probability measure $Q$ on the canonical path space $(\mathbb{D}^2, \mathscr{D}^2, \mathbf{D}^2)$ such that the canonical process has $Q$-differential characteristics $(b^*, c^*, K^*, I)$. For the mappings

$$\phi_1 : U \to \mathbb{R}^2,\ \xi \mapsto \left(0, -\frac{\mu}{c^B \xi_1}\right)^\top, \quad \phi_2 : U \times \mathbb{R}^2 \to (0, \infty),\ (\xi, x) \mapsto 1,$$

we have $c^* = c$ and

$$b^* = b + c\phi_1(y_-, X_-) + \int (\phi_2((y_-, X_-), x) - 1) h(x) K(dx), \quad \frac{dK^*}{dK} = \phi_2((y_-, X_-), x).$$

Since $\phi_1$ is obviously bounded on $U^n$, the conditions of (Cheridito et al., 2005, Remark 2.5) are satisfied and by Theorem 2.4 the canonical process is a $U$-valued affine semimartingale under both $P$ and $Q$. Hence it follows from (Duffie et al., 2003, Theorem 2.12) and (Cheridito et al., 2005, Theorem 2.4) that $Q \sim P$ with some density process $Z$. Now define the positive local martingale

$$Z^* := \mathscr{E}(\phi_1(y_-, X_-) \bullet X^c),$$

which is a supermartingale by Proposition A.9. Since $Z^*$ is a continuous semimartingale, the strict stopping times

$$T_n = \inf\{t > 0 : |(y_{t-}, X_{t-})| \geq n \text{ or } |(y_t, X_t)| \geq n\}, \quad n \in \mathbb{N}$$

form a localizing sequence for $Z^*$. As in the proofs of Lemma 2.16 and Theorem 2.9 this allows us to obtain that $E(Z^*_T) = E(Z_T) = 1$. Consequently, $Z^*$ is a martingale and we can use it to define a measure $Q^* \sim P$ on $(\Omega, \mathscr{F})$. By (Kallsen, 2006, Proposition 4), the differential $Q^*$-characteristics of $(y, X)$ coincide with $(b^*, c^*, K^*, I)$. In view of Lemma A.8 this shows that $X$ and hence $S$ is a local $Q^*$-martingale. In particular, Assumption 4.7 holds.

Now consider the case $K^B((a,0)), K^B((0,b)) > 0$ for some $a \in (-1,0)$ and $b \in (0,\infty)$. If $\mu \geq 0$, define

$$b^* := (b^1, 0)^\top, \quad c^* := c, \quad K^*(G) := \int 1_G(x)\left(-\frac{\mu 1_{(a,0)}(x_2)x_2}{\int_a^0 y^2 K^B(dy)} + 1\right)K(dx),$$

for $B \in \mathscr{B}^2$ and relative to $h(x_1,x_2) = (\chi(x_1), x_2)$. Set $\phi_1 : U \to \mathbb{R}^2,\ (\xi_1,\xi_2) \mapsto (0,0)^\top$ as well as

$$\phi_2 : U \times \mathbb{R}^2 \to (0,\infty),\ (\xi, x) \mapsto -\frac{\mu 1_{(a,0)}(x_2)x_2}{\xi_1 \int_a^0 y^2 K^B(dy)} + 1.$$

Then we have $c^* = c$ and

$$b^* = b + c\phi_1(y_-, X_-) + \int (\phi_2((y_-,X_-),x) - 1)xK(dx), \qquad \frac{dK^*}{dK} = \phi_2((y_-,X_-),x).$$

Since $K^B$ is a Lévy measure and $\phi_2$ is bounded on $U^n$, the conditions of (Cheridito et al., 2005, Remark 2.5) are satisfied. Hence it follows as in the first case by applying Theorem 2.4 and (Cheridito et al., 2007, Theorem 2.4) that there exists a unique probability measure $Q \sim P$ with density process $Z$ on the canonical path space $(\mathbb{D}^2, \mathscr{D}^2, \mathbf{D}^2)$, such that the canonical process $Y$ has $Q$-characteristics $(b^*, c^*, K^*, I)$. Since $K^B$ has moments of all orders, it follows as in the proof of Lemma 2.14 that the sequence $(T_n)_{n\in\mathbb{N}}$ from above is a localizing sequence for the positive local martingale

$$Z^* := \mathscr{E}((\phi_2((y_-,X_-),\cdot) - 1) * (\mu^{(y,X)} - \nu^{(y,X)})).$$

As in the first case, this yields $E(Z^*_T) = E(Z_T) = 1$, which in turn shows that $Z^*$ is a true martingale. Once more applying (Kallsen, 2006, Proposition 4), we get that $X$ and hence $S$ is a local $Q^*$-martingale.

If $\mu < 0$, the same result follows analogously by replacing $(a,0)$ with $(0,b)$ in the definitions of $b^*, c^*, K^*$ and $\phi_2$ above. □

### 4.5.3 Examples

We now consider some concrete models where the results of the previous three sections can be applied. For ease of notation, we consider only a single risky asset (i.e. $d = 1$), but the extension to multivariate versions of the corresponding models is straightforward.

## Generalized Black-Scholes models

Let $B$ be a standard Brownian motion and $y$ an independent semimartingale. Consider measurable mappings $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ such that $\mu(y_-) \in L(I)$ and $\sigma(y_-) \in L(B)$ and suppose the discounted stock price $S$ is given by

$$S = S_0 \mathscr{E}(\mu(y_-) \bullet I + \sigma(y_-) \bullet B).$$

For $X := \mu(y_-) \bullet I + \sigma(y_-) \bullet B$, Propositions A.2 and A.4 yield $b^X = \mu(y_-)$ as well as $c^X = \sigma^2(y_-)$ and $K^X = 0$. In view of Lemma 4.37, Assumption 4.30 is satisfied. Let

$$\eta_t := \frac{\mu(y_{t-})}{p\sigma^2(y_{t-})}, \quad t \in [0, T].$$

By Corollary 4.34, the strategy $\varphi := \eta v \mathscr{E}(\eta \bullet X)/S$ is optimal for $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0, 1\}$ and initial endowment $v \in (0, \infty)$, if $\eta \in L(X)$. If $y_-$ is $E$-valued for $E \subset \mathbb{R}$, this holds true e.g. if the mapping $x \mapsto \mu(x)/\sigma^2(x)$ is bounded on compact subsets of $E$.

**Remark 4.43** If one works with the set $\Theta(v, [0, 1])$ of strategies without shortselling or leverage, the content of this section generalizes results of Delong & Klüppelberg (2008) by allowing for an arbitrary semimartingale factor process.

Notice however, that unlike Delong & Klüppelberg (2008) we only consider utility from terminal wealth and do not obtain a solution to more general consumption problems.

Finiteness of the maximal expected utility is ensured in the case $p > 1$ in our setup, which complements the results of Delong & Klüppelberg (2008). They consider the case $p \in (0, 1)$ and prove that for a OU process $y$ driven by a subordinator $Z$, the maximal expected utility is finite subject to suitable linear growth conditions on the coefficient functions $\mu(\cdot)$ and $\sigma(\cdot)$ as well as certain exponential moment conditions on the Lévy measure $K^Z$ of $Z$.

## Barndorff-Nielsen and Shephard (2001)

If we set $\mu(x) := \widetilde{\mu} + \delta x$ for constants $\widetilde{\mu}, \delta \in \mathbb{R}$, $\sigma(x) := \sqrt{x}$ and choose

$$dy_t = -\lambda y_{t-} + dZ_{\lambda t}, \quad y_0 > 0$$

for a constant $\lambda > 0$ and some subordinator $Z$ in the generalized Black-Scholes model above, we obtain the BNS model introduced in Section 2.3.2. By e.g. Barndorff-Nielsen & Shephard (2001), we have $y_t \geq y_0 e^{-\lambda T} > 0$ in this case. This shows that

$$\eta := \frac{\mu(y_-)}{p\sigma^2(y_-)} = \frac{\widetilde{\mu}}{p y_-} + \frac{\delta}{p}$$

is bounded and hence belongs to $L(X)$. Consequently, $\varphi_t = \eta V(\varphi)/S$ is optimal for $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0, 1\}$ and initial endowment $v \in (0, \infty)$.

**Remark 4.44** If one works with the set $\Theta(v,[0,1])$ of strategies without shortselling or leverage, this recovers the optimal strategy obtained by Benth et al. (2003). Similarly as in Delong & Klüppelberg (2008), Benth et al. (2003) consider the case $p \in (0,1)$ and prove that the maximal expected utility is finite subject to an exponential moment condition on the Lévy measure $K^Z$ of $Z$. Our results complement this by ascertaining that the same strategy is always optimal (with not necessarily finite expected utility), as well as optimal with finite expected utility in the case $p > 1$.

The next example applies the results of this section to one of the parametric versions of the BNS model estimated in Chapter 3.

**Example 4.45** Consider the parameters of the discounted BNS model estimated from a DAX time series in Chapter 3 above (cf. Remark 3.18 and Examples 3.21, 3.22), i.e. let $S = S_0 \exp(\widetilde{X})$ for a BNS-IG-OU model $(y, X)$ with $\widetilde{\mu} = 0.0438$, $\delta = 0$, mean reversion $\lambda = 2.54$ and stationary IG$(0.203, 4.1835)$-distribution of $y$.

By Lemma 2.6, this means that $S = S_0 \mathscr{E}(X)$ for a BNS-IG-OU process with $\widetilde{\mu} = 0.0438$, $\delta = \frac{1}{2}$, $\lambda = 2.54$ and stationary IG$(0.203, 4.1835)$-distribution. A simulated trajectory of of the optimal fractions of stocks for $p = 2$ is shown in Figure 4.1 below. There we also plot the corresponding optimal fraction of stocks if the constrained set $\Theta(v,[0,1])$ of strategies without shortselling and leverage is used.

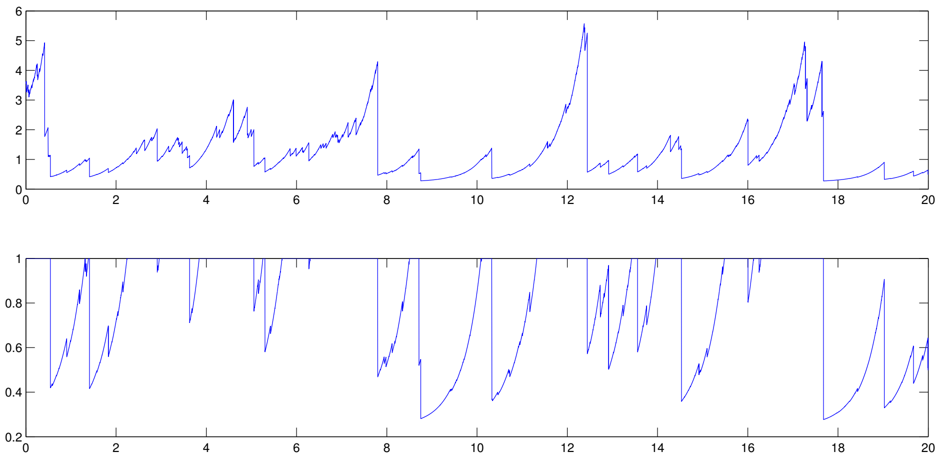


Figure 4.1: Sample paths of the unconstrained (above) and constrained (below) optimal fractions of stocks for $p = 2$ in a BNS-IG-OU model

Note that the optimal fraction now fluctuates according to the stochastic volatility around the constant fraction $0.702$ obtained in Example 4.28 above, leading to severe leverage in the unconstrained case.

## Carr et. al (2003)

In this section we consider the time-changed Lévy generalizations of the BNS model introduced in Section 2.3.3. More specifically, we assume that $S = S_0\mathscr{E}(X) > 0$ for

$$X_t = \mu t + B_{Y_t}, \quad \mu \in \mathbb{R}, \tag{4.21}$$

as in (4.20) above and let the time change $Y$ be given by $Y_t = \int_0^t y_s ds$ with

$$dy_t = -\lambda y_{t-}dt + dZ_{\lambda t}, \quad y_0 > 0, \tag{4.22}$$

where $\lambda > 0$ and $Z$ denotes a subordinator. By Lemmas 4.41 and 4.42, Theorem 4.32 is applicable if $B$ has either a Brownian component or both positive and negative jumps. Here we confine ourselves to the case of arbitrary positive and negative jumps of the asset price and leave the straightforward extension to more general setups to the interested reader. Note that unlike in Section 4.4, Theorem 4.32 now allows us to deal with the case $\mu \neq 0$.

**Corollary 4.46** *Let $u(x) = \frac{x^{1-p}}{1-p}$ for some $p \in \mathbb{R}_+\backslash\{0,1\}$ and assume the Lévy process $B$ satisfies the following conditions:*

1. $K^B((-1,-b))), K^B((a,\infty)) > 0$ *for any* $b \in (-1,0)$ *and* $a \in (0,\infty)$.
2. $\int_\varepsilon^\infty xK^B(dx) < \infty$ *and* $\int_{-\infty}^{-\varepsilon} \frac{-x}{(1+x)^p}K^B(dx) < \infty$ *for some* $\varepsilon > 0$.

*Then there exists a unique $[0,1]$-valued process $\eta \in L(X)$ such that outside some $dP \otimes dt$-null set,*

$$\left(\frac{\mu}{y_-} + b^B\right) - pc^B\eta + \int \left(\frac{x}{(1+\eta x)^p} - h(x)\right) K^B(dx) \geq 0,$$

*if* $0 < \eta$,

$$\left(\frac{\mu}{y_-} + b^B\right) - pc^B\eta + \int \left(\frac{x}{(1+\eta x)^p} - h(x)\right) K^B(dx) \leq 0,$$

*if $\eta < 1$ and $\varphi = \eta v\mathscr{E}(\eta \bullet X)_-/S_-$ is optimal for $u$ and initial endowment $v \in (0,\infty)$.*

PROOF. Follows immediately from Lemma 4.41, Lemma 4.42 and Corollary 4.35, since the predictable process $y_-$ is locally bounded and hence $P$-a.s. bounded on $[0,T]$. □

**Example 4.47** Consider the parameters of the NIG-IG-OU model estimated in Section 3.3. More specifically, let $S = S_0\exp(X)$ for a time-changed Lévy process as in (4.21), (4.22) and suppose $B$ is given by an NIG process with parameters $\beta = -13.9$, $\alpha = 88.3$, $\vartheta = 85.0$, $\delta = 13.6$ and $y$ follows an IG-OU process with mean reversion $\lambda = 2.54$ and stationary IG$(0.203, 4.18)$-distribution.

By Lemma 2.6, this implies that $S = S_0\mathscr{E}(\mu I + \widetilde{B}_Y)$ for the Lévy process $\widetilde{B}$ with triplet

$$b^{\widetilde{B}} = \frac{1}{2}, \quad c^{\widetilde{B}} = c^B, \quad K^{\widetilde{B}}(G) = \int \left(1_G(e^x-1)\frac{\alpha\vartheta}{\pi}e^{\beta x}\frac{K_1(\alpha|x|)}{|x|}\right) dx, \quad \forall G \in \mathscr{B},$$

for the modified Bessel function $K_1$ of the third kind with index 1 and relative to the truncation function $h(x) = x$ which can be used, since $\widetilde{B}$ is a special semimartingale. It now follows by insertion that the conditions of Corollary 4.46 are satisfied. The optimal fractions to be held in stocks can now be obtained by computing pointwise solutions to the inequalities in Corollary 4.46.

A trajectory of the optimal fractions of stock for $p = 2$ and the same simulated path of $y$ as in Figure 4.1 is shown in Figure 4.2 below.

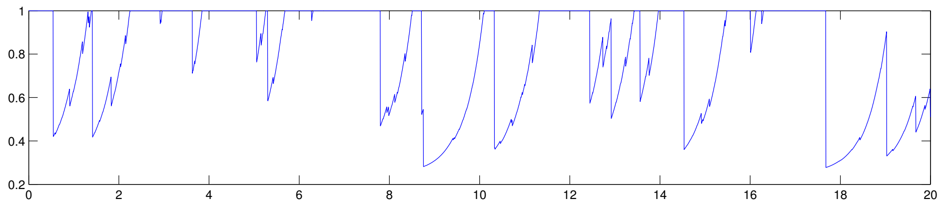


Figure 4.2: Sample path of the optimal fraction of stocks for $p = 2$ in a NIG-IG-OU model

Note that similarly as in Example 4.28 above, we obtain almost the same result as for the NIG-OU models and the BNS model without shortselling and leverage.

As for the generalized Black-Scholes models above, it is important to emphasize that the optimal strategy $\varphi$ is only ensured to lead to finite expected utility in the case $p > 1$.

However, the results provided here allow us to complete the study of the case $p \in (0, 1)$ for $\mu = 0$ started in Section 4.4. Using Theorem 4.32, we can show that the exponential moment Condition 4 in Corollary 4.27 is actually necessary and sufficient for the maximal expected utility to be finite. The key insight is that the process $\int_0^\cdot \alpha_s ds$ from Theorem 4.32 turns out to be an infinitely divisible random variable for $\mu = 0$.

**Corollary 4.48** *Let $v > 0$ and $u(x) = \frac{x^{1-p}}{1-p}$ for some $p \in \mathbb{R}_+\backslash\{0,1\}$. Assume $\mu = 0$ and suppose there exists $\eta \in \mathbb{R}$ satisfying Conditions 1-3 of Corollary 4.27. Then $\varphi := \eta v \mathscr{E}(\eta X)_-/S_-$ is optimal for $u$ and initial endowment $v$. The corresponding maximal expected utility $E(u(V_T(\varphi)))$ is always finite for $p > 1$, whereas for $p \in (0, 1)$ it is finite if and only if*

$$\int_0^T \int_1^\infty \exp\left(\frac{e^{-\lambda t} - 1}{\lambda} Cz\right) K^Z(dz)dt < \infty \tag{4.23}$$

*for*

$$C := (p-1)b^B\eta + \frac{p(1-p)}{2}c^B\eta^2 - \int \left((1+\eta x)^{1-p} - 1 - \eta h(x)\right) K^B(dx).$$

*If the maximal expected utility is finite, it is given by the formula in Corollary 4.27 above.*

PROOF. Conditions 1-3 of Corollary 4.27 yield that Conditions 1-3 of Theorem 4.32 are satisfied for $\eta$. Since $\eta$ is constant and $y_-$ is predictable and locally bounded, Condition 2 of Corollary 4.27 implies that Condition 4 of Theorem 4.32 holds, too. Therefore $\varphi$ is optimal.

For $p \in (1, \infty)$, the corresponding maximal expected utility is finite by Corollary 4.27. Let $p \in (0, 1)$. After inserting the characteristics of $X$ from Section 2.3.3, Theorem 4.32 shows that the maximal expected utility is given by

$$E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p} E\left(\exp\left(-C \int_0^T y_{t-} dt\right)\right) = \frac{v^{1-p}}{1-p} E\left(\exp\left(-C \int_0^T y_t dt\right)\right). \tag{4.24}$$

Since $(y, \int_0^\cdot y_s ds)$ is an affine semimartingale by Proposition A.3, (Kallsen, 2006, Corollary 3.2) implies that the characteristic function of the random variable $\int_0^T y_s ds$ is given by

$$E\left(\exp\left(iu \int_0^T y_s ds\right)\right) = \exp\left(ibu + \int \left(e^{iux} - 1 - iuh(x)\right) K(dx)\right), \quad \forall u \in \mathbb{R},$$

with

$$K(G) := \int_0^T \int 1_G \left(\frac{1 - e^{-\lambda t}}{\lambda} z\right) \lambda K^Z(dz) dt, \quad \forall G \in \mathscr{B}$$

and

$$\begin{aligned} b := {} & b^Z \left(\frac{e^{-\lambda T} - 1 + \lambda T}{\lambda}\right) + y_0 \left(\frac{1 - e^{-\lambda T}}{\lambda}\right) \\ & + \int_0^T \int \left(h\left(\frac{1 - e^{-\lambda t}}{\lambda} z\right) - \frac{1 - e^{-\lambda t}}{\lambda} h(z)\right) \lambda K^Z(dz) dt. \end{aligned}$$

Since $K^Z$ is a Lévy measure, i.e. satisfies $K^Z(\{0\}) = 0$ and integrates $1 \wedge |x|^2$, one easily verifies that $b$ is finite and $K$ is a Lévy measure, too. By the Lévy-Khintchine formula (cf. e.g. (Sato, 1999, Theorem 8.1)), the distribution of $\int_0^T y_s ds$ is therefore infinitely divisible. Consequently (4.24) and (Sato, 1999, Theorem 7.10, Theorem 25.17) yield that $E(u(V_T(\varphi)))$ is finite if and only if

$$\int_{\{|x|>1\}} e^{-Cx} K(dx) = \int_0^T \int_{\{|(1-e^{-\lambda t})z/\lambda|>1\}} \exp\left(\frac{e^{-\lambda t} - 1}{\lambda} Cz\right) \lambda K^Z(dz) dt < \infty.$$

Since $\lambda > 0$ and the Lévy measure $K^Z$ is concentrated on $\mathbb{R}_+$, the assertion follows. □

Since the exponential moment condition in Corollary 4.48 depends on the time horizon $T$, it is potentially only satisfied if the planning horizon is sufficiently small. This resembles the situation in the Heston model, where the maximal expected utility also turned out to be infinite for some parameters and sufficiently large $T$, if $p \in (0, 1)$.

However, a qualitatively different phenomenon arises here. Whereas expected utility could only increase towards infinity in a continuous way in the Heston model, it can suddenly jump to infinity here. More specifically, there possibly exists $T_\infty < \infty$ such that the maximal expected utility is bounded from above for all $T \leq T_\infty$ but infinite for $T > T_\infty$.

Moreover, the following example using the BNS model shows that this effect is not a consequence of a discontinuous asset price $X$, but is much rather induced by jumps of the volatility process $y$.

**Example 4.49 (Sudden explosion of maximal expected utility)** In the setup of Corollary 4.48 consider $p \in (0,1)$, $K^B = 0$, $b^B \neq 0$, $c^B = 1$ and hence $C = \frac{p-1}{2p}(b^B)^2 < 0$. Define the Lévy measure

$$K^Z(dz) := 1_{(1,\infty)}(z) \exp\left(\frac{C}{2\lambda} z\right) \frac{dz}{z^2},$$

and let $b^Z = 0$ relative to the truncation function $h(z) := \chi(z)$ on $\mathbb{R}$. Setting $T_\infty := \log(2)/\lambda$, we obtain

$$\int_1^\infty \exp\left(\frac{e^{-\lambda t} - 1}{\lambda} Cz\right) K^Z(dz) \begin{cases} \leq 1, & \text{for } t \leq T_\infty, \\ = \infty, & \text{for } t > T_\infty. \end{cases}$$

Consequently, by Corollary 4.48, the maximal expected utility that can be obtained by trading on $[0,T]$ is finite for $T \leq T_\infty$. Moreover, by inserting into Corollary 4.27 we obtain that, for $T \leq T_\infty$,

$$E(u(V_T(\varphi))) \leq \exp(1 + |C| y_0) < \infty.$$

Hence the maximal expected utility is bounded from above for $T \leq T_\infty$. For $T > T_\infty$, however, is is infinite by Corollary 4.48.

Since $u(V_T(\varphi)) = V_T(\varphi)^{1-p}/(1-p)$ is an exponentially affine process for $\mu = 0$, the finiteness of the maximal expected utility is intimately linked to *moment explosions* of affine processes. These are studied in Lions & Musiela (2007) and Andersen & Piterbarg (2007) in a diffusion setting, as well as in Keller-Ressel (2008) for possibly discontinuous affine processes.

In line with Corollary 4.26 and Korn & Kraft (2004), Example 4.49 again exemplifies that one has to be careful when dealing with utility maximization in stochastic volatility models. Even subject to NFLVR the maximal expected utility does not have to be finite for all parameter constellations and all time horizons.

In general, infinite expected utility can lead to economically dubious phenomena (cf. Remark 4.10). However, in the special setup considered here, Lemma 4.33 shows that the optimal strategy $\varphi$ obtained via Theorem 4.32 is also optimal for an insider who knows the entire evolution of the stochastic factor process $y$. Since the corresponding conditional expected utility is finite, the respective optimal value process $V(\varphi)$ is unique by e.g. (Kallsen, 2000, Lemma 2.5) in the sense that its conditional expected utility strictly dominates all other value processes. Thus even if other fundamentally different investment strategies also lead to infinite unconditional utility, it still makes sense economically to invest into the strategy $\varphi$ obtained here.

# Chapter 5

# Asymptotic power utility-based pricing and hedging

## 5.1 Introduction

As in Chapter 4 we consider an investor with initial endowment $v$, whose goal is to maximize her expected utility from terminal wealth. However, we now also consider how to price and hedge nontraded contingent claims.

More specifically, suppose the investor is approached by another economic agent who offers her a premium $q\pi^q$ in exchange for $q$ units of some nontraded contingent claim $H$. At this point, the investor has two choices: If she rejects the offer, her utility from terminal wealth will be

$$U(v) := \sup_{\phi \in \Theta(v)} E(u(v + \phi \bullet S_T)), \tag{5.1}$$

where $\phi$ ranges over some suitable set $\Theta(v)$ of self-financing strategies *admissible* for initial endowment $v$. If she accepts the offer, her utility from terminal wealth will instead be given by

$$U^q(v + q\pi^q) := \sup_{\phi \in \Theta^q(v + q\pi^q)} E(u(v + q\pi^q + \phi \bullet S_T - qH)), \tag{5.2}$$

maximizing over some set $\Theta^q(v + q\pi^q)$ of self-financing strategies admissible for initial capital $v + q\pi^q$ as well as an initial position of $-q$ units of the contingent claim $H$. Of course a sensible investor will only accept the deal if it raises her expected utility, i.e. if $U^q(v + q\pi^q) \geq U^0(v)$. The minimal price $\pi^q$ per unit of $H$ with this property is called *utility indifference price*.

If the investor declines the offer, her optimal trading strategy is given by the optimal trading strategy $\varphi$ in (5.1), whereas it is optimal to trade according to the optimal strategy $\varphi^q$ in (5.2) if the offer is accepted. Therefore the difference $\varphi^q - \varphi$ is called *utility-based hedging strategy*, because it describes the action the investor needs to take in order to compensate for the risk resulting from the addition of $-q$ contingent claims to her portfolio.

This utility-based approach is appealing from an economic point of view and therefore has been studied extensively in the literature (cf. e.g. Hodges & Neuberger (1989), Duffie et al. (1997), Rouge & El Karoui (2000), Cvitanić et al. (2001), Delbaen et al. (2002), Karatzas & Žitković (2003), Hugonnier & Kramkov (2004), Hugonnier et al. (2005), Žitković (2005), Ilhan & Sircar (2006)). However, it is generally very difficult to determine the relevant quantities explicitly even for standard utility functions and in simple concrete models.

A feasible alternative is to consider a first-order approximation for a small number $q$ of contingent claims, which has recently been considered by Mania & Schweizer (2005), Becherer (2006), Kallsen & Rheinländer (2008) for exponential utility as well as Henderson (2002), Henderson & Hobson (2002) and Kramkov & Sîrbu (2006, 2007) for utility functions defined on $\mathbb{R}_+$.

As in Chapter 4, we consider here power utility functions $u(x) = x^{1-p}/(1-p)$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$. In Section 5.2 we provide an heuristic account of how to tackle the computation of first order approximations of utility-based prices and hedging strategies. In the remainder of the chapter we then go on to show how to make these arguments precise. We first state existence results mainly due to Hugonnier & Kramkov (2004) concerning utility-based pricing and hedging in Section 5.3. Afterwards, we introduce the asymptotic results of Kramkov & Sîrbu (2006, 2007) in Section 5.4. In a nutshell, these state that after a suitable change of numeraire, first order approximations can be computed by solving a quadratic hedging problem under a certain equivalent martingale measure. Due to the change of numeraire, the dimensionality of the problem is increased by one in this approach. In order to facilitate computations in concrete models in Chapter 6, we therefore put forward an alternative approach in Section 5.5. Here, the first-order approximations are again represented as the solution to a quadratic hedging problem, but in terms of the original numeraire and subject to a different equivalent probability measure.

## 5.2 Heuristic derivation of the solution

In this section we give a heuristic account of how to determine asymptotic expansions of the utility indifference price and the corresponding utility-based hedging strategy as the number of contingent claims tends to zero. In doing so, we first proceed along the lines of Kallsen (2008) and explain how to obtain the results of Kramkov & Sîrbu (2006, 2007) in a heuristic way (cf. Section 5.4 below for a mathematically precise statement of the corresponding results). We then go on to provide the heuristic derivation of the alternative representation that is worked out in Section 5.5.

As already alluded to above, we want to determine first-order approximations of the utility indifference price $\pi^q$ per unit of $H$ and the utility-based hedging strategy $\varphi^q$ for $q$ units of $H$ for small $q$. Therefore we assume a smooth dependency

$$\pi^q = \pi(0) + q\pi' + o(q) \tag{5.3}$$

with constants $\pi(0), \pi'$. The *marginal utility-based price* $\pi(0)$ can be interpreted as a limiting price for very small $q$. It is studied in Davis (1997) and Karatzas & Kou (1996). The *risk premium per option* $\pi'$ represents a kind of sensitivity of the option price relative to the number $q$ of options sold. Analogously, we suppose

$$\varphi^q = \varphi + q\varphi' + o(q), \tag{5.4}$$

where $\varphi$ denotes the optimal strategy in the pure investment problem (5.1) and $\varphi'$ represents the *marginal utility-based hedging strategy* per unit of $H$. In the following we show how to determine $\pi^0, \pi', \varphi$ and $\varphi'$.

We have considered the pure investment problem (5.1) in Chapter 4. In particular, if one forgets about technical details, a strategy $\varphi$ is optimal w.r.t. the power utility function $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+\backslash\{0,1\}$ if and only if $S$ is a martingale under the $q$-optimal measure $Q_0 \sim P$ with density process

$$\frac{dQ_0}{dP} := \frac{u'(V_T(\varphi))}{C_1} = \frac{(V_T(\varphi)/v)^{-p}}{C_1}, \tag{5.5}$$

where $C_1 := E((V_T(\varphi)/v)^{-p}))$ denotes the normalizing constant (cf. Corollary 4.13 for more details). Notice that here and in the following we work in terms of $V(\varphi)/v$ instead of $V(\varphi)$, because the former does not depend on the intial endowment $v$ for power utility by Corollary 4.13. Using this criterion, we showed how to compute the optimal strategy $\varphi$, its value process $V(\varphi)$ and the corresponding maximal expected utility for power utility functions in affine stochastic volatility models (see Section 4.4). We now turn to the optimization problem (5.2) including $q$ options sold for $\pi^q$ each. This amounts to maximizing

$$\begin{aligned} g(\varphi') &:= E\big(u(v + q\pi^q + \varphi^q \bullet S_T - qH)\big) \\ &= E\big(u(V_T(\varphi) + q(\pi(0) + q\pi' + (\varphi' + o(1)) \bullet S_T - H) + o(q^2))\big) \\ &= E\big(u(V_T(\varphi))\big) + qE\big(u'(V_T(\varphi))(\pi(0) + q\pi' + (\varphi' + o(1)) \bullet S_T - H)\big) \\ &\quad + \frac{q^2}{2}E\big(u''(V_T(\varphi))(\pi(0) + \varphi' \bullet S_T - H)^2\big) + o(q^2). \end{aligned}$$

For power utility functions we have $u'(x) = x^{-p}$ and $u''(x) = -px^{-1-p}$. Hence

$$\begin{aligned} g(\varphi') &= C_0 v^{1-p}/(1-p) + qC_1 v^{-p} E_{Q_0}\big(\pi(0) + q\pi' + (\varphi' + o(1)) \bullet S_T - H\big) \\ &\quad - q^2\frac{p}{2}E\left(V_T(\varphi)^{-1-p}\left(\pi(0) + \varphi' \bullet S_T - H\right)^2\right) + o(q^2) \qquad (5.6) \\ &= C_0 v^{1-p}/(1-p) + qC_1 v^{-p} E_{Q_0}\big(\pi(0) + q\pi' + (\varphi' + o(1)) \bullet S_T - H\big) \\ &\quad - q^2 C_1 v^{-p}\,\frac{p}{2}E_{Q_0}\left(\frac{V_T(\varphi)}{v^2}\left(\frac{\pi(0) + \varphi' \bullet S_T - H}{V_T(\varphi)/v}\right)^2\right) + o(q^2) \qquad (5.7) \end{aligned}$$

with $C_0 := E((V_T(\varphi)/v)^{1-p})$. Since $Q_0$ is an equivalent martingale measure (EMM) for $S$, it follows that

$$E_{Q_0}((\varphi' + o(1)) \bullet S_T) = 0. \tag{5.8}$$

Now define a probability measure $Q^\$ \sim Q_0$ through the Radon-Nikodym density

$$\frac{dQ^\$}{dQ_0} := \frac{V_T(\varphi)}{v}. \tag{5.9}$$

Since $Q_0$ is an EMM for $S$, it follows that $Q^\$$ is an EMM relative to the numeraire $V(\varphi)/v$, i.e. $S^\$ := Sv/V(\varphi)$ is a $Q^\$$-martingale. Write

$$\pi^\$(0) := \frac{\pi(0)v}{V(\varphi)}, \quad H^\$ := \frac{Hv}{V(\varphi)} \tag{5.10}$$

for the discounted values relative to the new numeraire $V(\varphi)/v$. Since $\varphi'$ is assumed to be self-financing, (Goll & Kallsen, 2000, Proposition 2.1) yields

$$\frac{\pi(0) + \varphi' \bullet S_T - H}{V_T(\varphi)/v} = \pi^\$(0) + \varphi' \bullet S_T^\$ - H^\$.$$

By (5.8), (5.9), (5.10) we have to maximize

$$g(\varphi') = \frac{C_0 v^{1-p}}{1-p} + q\frac{C_1}{v^p}\left(\pi(0) - E_{Q_0}(H)\right) + q^2\frac{C_1}{v^p}\left(\pi' - \frac{p}{2v}\varepsilon_\$^2(\varphi')\right) + o(q^2),$$

with

$$\varepsilon_\$^2(\varphi') := E_{Q^\$}\left(\left(\pi^\$(0) + \varphi' \bullet S_T^\$ - H^\$\right)^2\right).$$

If we disregard the $o(q^2)$-term, this means that $\varphi'$ has to minimize $\varepsilon_\$^2$ and hence represents the *variance-optimal hedging strategy* of the claim $H^\$$ under the measure $Q^\$$ and relative to the numeraire $V(\varphi)/v$. Moreover, $\varepsilon_\$^2$ is given by the corresponding *minimal expected squared hedging error*. Since $S^\$$ is a $Q^\$$-martingale, $\varphi'$ is given as the integrand in the Galtchouk-Kunita-Watanabe decomposition

$$V_t^\$ = V_0^\$ + \varphi' \bullet S_t^\$ + N_t^\$ \tag{5.11}$$

of the $Q^\$$-martingale $V^\$ := E_{Q^\$}(H^\$|\mathscr{F}_t)$ relative to $S^\$$, where $N^\$$ denotes a martingale which is orthogonal to $S^\$$ (cf. Föllmer & Sondermann (1986)). Consequently, $\varepsilon_\$^2 = E((N_T^\$)^2)$ by orthogonality of $S^\$, N^\$$. Moreover, it follows from the indifference criterion $U(v) = U^q(v + q\pi^q)$ that

$$\pi' = \frac{p}{2v}\varepsilon_\$^2(\varphi') \tag{5.12}$$

as well as

$$\pi(0) = E_{Q_0}(H) \tag{5.13}$$

or equivalently $\pi^\$(0) = E_{Q^\$}(H^\$)$. This shows that $\pi^\$(0)$ coincides with the *variance-optimal initial endowment* of the claim $H^\$$ hedged with $S^\$$ under the measure $Q^\$$. Hence one has to proceed as follows in order to obtain first-order expansions for the utility indifference price $\pi^q$ and the utility-based hedging strategy $\varphi^q$:

1. Solve the pure investment problem (5.1) without any contingent claims, i.e. determine the optimal strategy $\varphi$ and its value process $V(\varphi)$.

2. Compute the density process of $Q^\$$ w.r.t. $P$ and the dynamics of $S^\$$ under $Q^\$$.

3. Solve the quadratic hedging problem for the claim $H^\$$ under the martingale measure $Q^\$$ and relative to the numeraire $V(\varphi)/v$.

Problems 1 and 2 can be solved explicitly in a number of *affine stochastic volatility models* (cf. Chapters 4 and 6). Problem 3 has been dealt with in the univariate case ($d = 1$) for Lévy processes by Hubalek et al. (2006) and for affine stochastic volatility models by Pauwels (2007). In principle, a similar approach could also be used to tackle Problem 3 above in affine stochastic volatility models. However, after changing the numeraire to $V(\varphi)/v$, one has to deal with two non-trivial assets even in the simplest case of a market consisting of just one bond and one stock. This is the motivation for deriving an alternative representation for $\pi(0), \pi', \varphi, \varphi'$ in terms of a quadratic hedging problem relative to the original numeraire. Instead of introducing the measure $Q^{\$}$, rewrite (5.6) in terms of the measure $P^{€} \sim P$ for

$$\frac{dP^{€}}{dP} = \frac{(V_T(\varphi)/v)^{-1-p}}{C_2}, \tag{5.14}$$

with normalizing constant $C_2 := E((V_T(\varphi)/v)^{-1-p})$. This yields

$$g(\varphi') = \frac{C_0 v^{1-p}}{1-p} + q\frac{C_1}{v^p}\left(\pi(0) - E_{Q_0}(H)\right) + q^2\frac{C_1}{v^p}\left(\pi' - \frac{pC_2}{2vC_1}\varepsilon^2_{€}(\varphi')\right) + o(q^2),$$

with

$$\varepsilon^2_{€}(\varphi') := E_{P^{€}}\left((\pi(0) + \varphi' \bullet S_T - H)^2\right).$$

Again using the indifference criterion $U(v) = U^q(v + q\pi^q)$, we obtain that $\pi(0), \varphi'$ and $\varepsilon^2_{€}$ represent the variance-optimal initial endowment resp. hedging strategy and the corresponding minimal expected squared hedging error for the original claim $H$ hedged with $S$ under the measure $P^{€}$ relative to the original numeraire. This means that one can now proceed in the following way to obtain $\pi^0, \pi', \varphi^0, \varphi'$:

1. Solve the pure investment problem (5.1) without any contingent claims, i.e. determine the optimal strategy $\varphi$ and its value process $V(\varphi)$.

2. Compute the density process of $P^{€}$ w.r.t. $P$ and the dynamics of $S$ under $P^{€}$.

3. Solve the quadratic hedging problem for the claim $H$ relative to the original numeraire and under $P^{€}$, which is typically not an EMM.

Comparing the two approaches, we find that the situation resembles quadratic hedging in the case where the underlying asset is not necessarily a martingale. One can either use the approach of Gourieroux et al. (1998), Rheinländer & Schweizer (1997), Arai (2005) and solve a mean-variance hedging problem relative to an EMM and a new numeraire, or one can turn to the methodology of ČK to solve the hedging problem by different means relative to the original numeraire.

As already mentioned above, working relative to the original numeraire leads to simpler formulas in the actual computations, because the dimensionality of the problem is not increased by one (cf. Chapter 6 for more details).

## 5.3 Utility-based pricing and hedging

We work in the same setup as in Chapter 4. In addition to the traded securities, we now also consider a nontraded European contingent claim with maturity $T$ and payment function $H$, which is a $\mathscr{F}_T$-measurable random variable. Following Kramkov & Sîrbu (2006, 2007), we assume that $H$ is dominated by the terminal value of the value process corresponding to some admissible strategy.

**Assumption 5.1** *$|H| \leq w + \psi \bullet S_T$ for some $w \in (0, \infty)$ and $\psi \in \Theta(w)$.*

**Remark 5.2** Put differently, Assumption 5.1 means that a *superhedging strategy* exists for $|H|$ given some initial capital $w$. It is trivially satisfied for all bounded contingent claims, as e.g. European put options and also holds for European call options if $S$ is positive.

If the investor sells $q$ units of $H$ at time 0, her terminal wealth should be sufficiently large to cover the payment $-qH$ due at time $T$. Consequently, one has to consider the following slightly different set $\Theta_v^q$ of admissible strategies in this case (cf. e.g. Hugonnier & Kramkov (2004) and Delbaen & Schachermayer (1997) for more details).

**Definition 5.3** A trading strategy $\phi \in \Theta(v)$ is called *maximal*, if the terminal value $V_T(\phi)$ of its wealth process cannot be dominated by that of any other strategy in $\Theta(v)$. An arbitrary strategy $\phi$ is called *acceptable*, if its wealth process can be written as

$$V(\phi) = v' + \phi' \bullet S - (v'' + \phi'' \bullet S), \quad v', v'' \in \mathbb{R}_+,$$

where $\phi' \in \Theta(v')$, $\phi'' \in \Theta(v'')$ and, in addition, $\phi''$ is maximal. For $v \in (0, \infty)$ and $q \in \mathbb{R}$ we denote by

$$\Theta^q(v) := \{\phi : \phi \text{ is acceptable}, v + \phi \bullet S_T - qH \geq 0\},$$

the set of acceptable strategies whose terminal value dominates $qH$.

**Remark 5.4** Subject to the NFLVR Assumption 4.7, $\Theta(v)$ coincides with $\Theta^q(v)$ for $q = 0$.

If in addition to an initial endowment of $v \in (0, \infty)$, a number of $q$ units of $H$ is sold for a price of $x \in \mathbb{R}$ each, the investor's initial position consists of $v + qx$ in cash as well as $-q$ units of the contingent claim $H$. Hence $\Theta^q(v + qx)$ represents the natural set of admissible trading strategies for utility functions defined on $\mathbb{R}_+$. The maximal expected utility the investor can achieve by dynamic trading in the market is then given by

$$U^q(v + qx) := \sup_{\phi \in \Theta^q(v+qx)} E(u(v + qx + \phi \bullet S_T - qH)).$$

**Definition 5.5** Fix $q \in \mathbb{R}$. A number $\pi^q \in \mathbb{R}$ is called *utility indifference price* (or *reservation price*) of $H$, if

$$U^q(v + q\pi^q) = U(v), \tag{5.15}$$

The following example shows that indifference prices do not exist in general, even if Assumption 5.1 is satisfied and NFLVR holds in the given financial market.

**Example 5.6** Let $T = 1$, $d = 1$ and consider $S = \exp(X)$ for a symmetric NIG process $X$ with characteristic function

$$E(\exp(iuX_1) = \exp\left(i(\sqrt{3}-2)u + (2-\sqrt{4+u^2})\right).$$

By (Sato, 1999, Theorem 25.17) the process $S$ is a martingale. In view of (Kallsen, 2000, Lemma 4.2, Theorem 3.2) this implies that $\varphi = 0$ is optimal for $u(x) = 2\sqrt{x}$ and initial endowment $v = \frac{1}{4}$. The corresponding maximal expected utility is obviously given by

$$U(v) = 2\sqrt{v} = 1. \tag{5.16}$$

In particular, Assumption 4.9 is satisfied and since $S$ is a $P$-martingale Assumption 4.7 holds, too. Now consider a European call option with payment function $H = (S_1 - K)^+$, $K > 0$. Then Assumption 5.1 is satisfied as well with $w = S_0 = 1$ and $\psi = 1$.

For $x < \frac{3}{4}$, it follows from (Eberlein & Jacod, 1997, Theorem 2) that $\Theta^1(v + x) = \varnothing$, because no superhedging strategy with initial capital $v + x < 1$ exists in this case. Hence

$$U^1(v+x) = -\infty < 1, \quad x < \frac{3}{4}. \tag{5.17}$$

For $x \geq \frac{3}{4}$ we have $1 \in \Theta^1(v + x)$. Together with monotone convergence this yields

$$\begin{aligned} U^1(v+x) \geq 2E\left(\sqrt{S_1 - (S_1-100)^+}\right) &\geq 2E(\sqrt{S_1}1_{\{S_1\leq K\}}) \\ &\overset{K\to\infty}{\longrightarrow} 2E(\sqrt{S_1}) \\ &= 2\exp(1+\sqrt{3}/2-\sqrt{15}/2) = 1.864, \end{aligned}$$

hence $U^1(v + x) > 1$ for $x \geq \frac{3}{4}$ and sufficiently large $K$. Combining this with (5.16) and (5.17) we obtain that no utility indifference price exists in this case.

**Remark 5.7** Similarly as in Example 5.6 one can show that utility indifference prices do not exist in general, even if the investor receives contingent claims with a positive payoff. This structurally differs from the setup of Henderson (2002) and Henderson & Hobson (2002), where the utility indifference price always exists, if the investor receives positive random endowments, but never exists if the investor sells endowments of the same kind.

However, a unique indifference price $\pi^q$ always exists if the number $q$ of contingent claims sold is sufficiently small or conversely, if the initial endowment $v$ is sufficiently large.

**Lemma 5.8** *Suppose Assumptions 4.7, 4.9 and 5.1 hold. Then a unique indifference price exists for sufficiently small $q$. More specifically,* (5.15) *has a unique solution $\pi^q$ if $q < \frac{v}{2w}$, respectively if $q < \frac{v}{w}$ and $H \geq 0$.*

PROOF. First notice that $g_v^q : x \mapsto U^q(v+qx)$ is concave and strictly increasing on its effective domain. Denote by $w$ and $\psi$ the initial endowment and the superhedging strategy of $H$ from Assumption 5.1. Then $g_v^q(x) \leq U(v+qx+qw) < \infty$ for all $x \in \mathbb{R}$ by (Kramkov & Schachermayer, 1999, Theorem 2.1). For $H \geq 0$ and $q < \frac{v}{w}$ we have $g_v^q(x) > -\infty$ for $x > w - \frac{v}{q}$. In particular, $g_v^q$ is continuous and strictly increasing on $(w-\frac{v}{q}, \infty)$ by (Rockafellar, 1970, Theorem 10.1). By $H \geq 0$ we have $g_v^q(0) \leq U(v)$. Moreover, Assumption 5.1 implies $g_v^q(w) \geq U(v)$. Hence there exists a unique solution $\pi^q \in [0,w]$ to $g_v^q(x) = U(v)$. Similarly, for general $H$ and $q < \frac{v}{2w}$, the function $g_v^q$ is continuous and strictly increasing on an open set containing $[-w,w]$. Moreover, $g(-w) \leq U(v)$ and $g(w) \geq U(v)$. Hence there exists a unique $\pi^q \in (-w,w)$ such that $g(\pi^q) = U(v)$. This proves the assertion. □

We now turn to utility-based hedging strategies. Their existence has been established by Hugonnier & Kramkov (2004) under the following additional assumption, which is slightly stronger than the NFLVR Assumption 4.9 by (Ansel & Stricker, 1994, Corollaire 3.5).

**Assumption 5.9** *There exists an* equivalent local martingale measure*, i.e. a probability measure $Q \sim P$ such that $S$ is a local $Q$-martingale.*

Notice that Assumption 5.9 is equivalent to Assumption 4.9 and hence to NFLVR, if the asset price process $S$ is positive.

**Theorem 5.10** *Fix $q \in \mathbb{R}$ satisfying the conditions of Lemma 5.8 and suppose Assumptions 4.9, 5.1 and 5.9 are satisfied. Then there exists $\varphi^q \in \Theta^q(v+q\pi^q)$ such that*

$$E(u(v+q\pi^q+\varphi^q \bullet S_T - qH)) = U^q(v+q\pi^q).$$

*Moreover, the corresponding optimal value process $v+q\pi^q+\varphi^q \bullet S$ is unique.*

PROOF. This follows from (Hugonnier & Kramkov, 2004, Theorem 2), as the proof of Lemma 5.8 shows that $(v+q\pi^q, q)$ belongs to the interior of $\{(x,r) \in \mathbb{R}^2 : \Theta^r(x) \neq \varnothing\}$. □

Without contingent claims, the investor will trade according to the strategy $\varphi$, whereas she will invest into $\varphi^q$ if she sells $q$ units of $H$ for $\pi^q$ each. Hence the difference between both strategies represents the action the investors needs to take in order to compensate for the risk of selling $q$ units of $H$. This motivates the following notion.

**Definition 5.11** The trading strategy $\varphi^q - \varphi$ from Theorem 5.10 is called *utility-based hedging strategy*.

## 5.4 The asymptotic results of Kramkov and Sîrbu applied to power utility

We now give a brief exposition of some of the results of Kramkov & Sîrbu (2006, 2007) concerning the existence and computation of first-order approximations of utility-based prices and hedging strategies.

Throughout, we suppose that Assumptions 4.9 and 5.9 are satisfied. Moreover, we now restrict our attention to power utility functions $u(x) = x^{1-p}/(1-p)$, $p \in \mathbb{R}_+ \backslash \{0,1\}$ and denote by $\varphi$, $V(\varphi) = v\mathscr{E}(-\widetilde{a} \bullet S)$ and $L$ the optimal strategy as well as the corresponding value and opportunity processes for $u$ and initial endowment $v$ from Proposition 4.15.

We begin by providing formal definitions of the first-order approximations of utility-based prices and hedging strategies introduced heuristically in Section 5.2

**Definition 5.12** Real numbers $\pi(0)$ and $\pi'$ are called *marginal utility-based price* resp. *risk premium* per option sold if

$$\pi^q = \pi(0) + q\pi' + o(q^2).$$

for $q \to 0$, where $\pi^q$ is well-defined for sufficiently small $q$ by Lemma 5.8. Notice that $\pi(0)$ and $\pi'$ are necessarily unique, if they exist.

We now turn to the approximation of utility-based hedging strategies in the sense of Kramkov & Sîrbu (2007). Unlike for exponential utility (cf. e.g. Mania & Schweizer (2005), Becherer (2006) and Kallsen & Rheinländer (2008)) convergence of the approximation only refers to the terminal values of the corresponding value processes here.

**Definition 5.13** A trading strategy $\varphi' \in L(S)$ is called *marginal utility-based hedging strategy*, if there exists $v' \in \mathbb{R}$ such that

$$\lim_{q \to 0} \frac{(v + q\pi^q + \varphi^q \bullet S_T) - (v + \varphi \bullet S_T) - q(v' + \varphi' \bullet S_T)}{q} = 0$$

in $P$-probability and $(v' + \varphi' \bullet S)L\mathscr{E}(-\widetilde{a} \bullet S)^{-p}$ is a martingale.

Notice that in contrast to the corresponding wealth process, marginal utility-based hedging strategies are not necessarily unique.

**Remark 5.14** In addition to utility indifference prices, Kramkov & Sîrbu (2007) also consider the *(dynamic) certainty equivalence value* $c^q$, which is defined as the solution to the equation

$$U^q(v) = U(v - qc^q).$$

For power utility, (Kramkov & Sîrbu, 2007, Theorem A.1, Theorem 8) show that $c^q$ and $\pi^q$ admit the same first-order approximations. Moreover, $\varphi' \in L(S)$ is a marginal utility-based hedging strategy in the sense of Definition 5.13 if and only if its wealth process is the wealth process of a marginal utility-based hedging strategy in the sense of (Kramkov & Sîrbu, 2007, Definition 2). Together with (Kramkov & Sîrbu, 2007, Theorem A.1, Theorem 8), this follows from the observation that the corresponding value process does not depend on the initial endowment for power utility by (Kramkov & Sîrbu, 2007, Theorem 2) and Corollary 4.13. In particular, marginal utility-based hedging strategies are independent of the current level of wealth for power utility.

The results of Kramkov and Sîrbu are derived subject to two technical assumptions.

**Assumption 5.15** *The following process is* $\sigma$-bounded*:*

$$S^{\$} := \left( \frac{1}{\mathscr{E}(-\widetilde{a} \bullet S)}, \frac{S}{\mathscr{E}(-\widetilde{a} \bullet S)} \right).$$

The reader is referred to Kramkov & Sîrbu (2006) for more details on $\sigma$-bounded processes as well as for sufficient conditions that ensure the validity of this assumption.

Since $L\mathscr{E}(-\widetilde{a} \bullet S)^{1-p}$ is a martingale with terminal value $\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}$ by Proposition 4.15, we can define an equivalent probability measure $Q^{\$} \sim P$ through

$$\frac{dQ^{\$}}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}}{C_0}, \quad C_0 := L_0 = E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}). \tag{5.18}$$

Let $\mathscr{H}_0^2(Q^{\$})$ be the space of square-integrable $Q^{\$}$-martingales starting at 0 and set

$$\mathscr{M}_{\$}^2 := \left\{ M \in \mathscr{H}_0^2(Q^{\$}) : M = \phi \bullet S^{\$} \text{ with } \phi \in L(S^{\$}) \right\}.$$

**Assumption 5.16** *There exists a constant* $w^{\$} \geq 0$ *and a process* $M^{\$} \in \mathscr{M}_{\$}^2$*, such that*

$$|H^{\$}| := \frac{|H|}{\mathscr{E}(-\widetilde{a} \bullet S)_T} \leq w^{\$} + M_T^{\$}.$$

**Remark 5.17** By (Kramkov & Sîrbu, 2006, Remark 1), Assumption 5.16 implies that Assumption 5.1 holds. In particular, indifference prices and utility-based hedging strategies exist for sufficiently small $q$ if Assumptions 4.9, 5.9 and 5.16 are satisfied.

In the proof of (Kramkov & Sîrbu, 2007, Lemma 1) it is shown that

$$V_t^{\$} := E_{Q^{\$}}\left(H^{\$} | \mathscr{F}_t\right), \quad t \in [0, T]$$

is a square-integrable $Q^{\$}$-martingale. Hence it admits a decomposition

$$V_t^{\$} = E_{Q^{\$}}\left(H^{\$}\right) + \xi \bullet S^{\$} + N^{\$} = \frac{1}{C_0} E\left(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H\right) + \xi \bullet S^{\$} + N^{\$}, \tag{5.19}$$

where $\xi \bullet S^{\$} \in \mathscr{M}_{\$}^2$ and $N^{\$}$ is an element of the orthogonal complement of $\mathscr{M}_{\$}^2$ in $\mathscr{H}_0^2(Q^{\$})$. Note that this decomposition coincides with the classical *Galtchouk-Kunita-Watanabe* decomposition, if $S^{\$}$ is a locally square integrable martingale. The following Theorem summarizes the results of Kramkov & Sîrbu (2006, 2007) applied to power utility.

**Theorem 5.18** *Suppose Assumptions 4.9, 5.9, 5.15 and 5.16 hold. Then the marginal utility-based price* $\pi(0)$ *and the risk premium* $\pi'$ *exist and are given by*

$$\pi(0) := \frac{1}{C_0} E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H),$$
$$\pi' := \frac{p}{2v} E_{Q^{\$}}((N_T^{\$})^2).$$

*A marginal-utility-based hedging strategy* $\phi'$ *is given in feedback form as*

$$\phi' := (\widetilde{a}, E_d + \widetilde{a} S_-^{\top})\xi - \left(\pi^0 + \phi' \bullet S\right)\widetilde{a},$$

*with* $\xi$ *from* (5.19).

PROOF. The first two assertions follow immediately from (Kramkov & Sîrbu, 2006, Theorem A.1, Theorem 8, Theorem 4) adapted to the present notation. For the second, (Kramkov & Sîrbu, 2007, Theorem 2) and (Kramkov & Sîrbu, 2006, Theorem A.1, Theorem 8) yield

$$\lim_{q\to 0} \frac{(v+q\pi^q+\varphi^q \bullet S_T)-(v+\varphi \bullet S_T)-q\mathscr{E}(-\widetilde{a}\bullet S)_T(\pi(0)+\xi\bullet S_T^\$)}{q}=0. \quad (5.20)$$

because the process $X'_T(x)$ from (Kramkov & Sîrbu, 2007, Equation 23) coincides with $\mathscr{E}(-\widetilde{a} \bullet S)$ for power utility by Corollary 4.13. Now notice that for $\xi^0 := \pi(0) + \xi \bullet S^\$ - \xi^\top S^\$ = \pi(0) + \xi \bullet S^\$_- - \xi^\top S^\$_-$, Remark 4.5 gives

$$\begin{aligned}
&\mathscr{E}(-\widetilde{a}\bullet S)(\pi(0)+\xi\bullet S^\$)\\
&\qquad = \pi(0)+\xi^0\bullet\mathscr{E}(-\widetilde{a}\bullet S)+(\xi^2,\dots,\xi^{d+1})\bullet S\\
&\qquad = \pi(0)+((\xi^2,\dots,\xi^{d+1})-\mathscr{E}(-\widetilde{a}\bullet S)_-\xi^0\widetilde{a})\bullet S\\
&\qquad = \pi(0)+((\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi)\bullet S-(\mathscr{E}(-\widetilde{a}\bullet S)_-(\pi(0)+\xi\bullet S_-^\$))\bullet(\widetilde{a}\bullet S).
\end{aligned}$$

Hence $\mathscr{E}(-\widetilde{a} \bullet S)(\pi^0 + \xi \bullet S^\$)$ solves the stochastic differential equation

$$G = \pi^0 + ((\widetilde{a}, E_d + \widetilde{a}S_-^\top)\xi) \bullet S - G_- \bullet (\widetilde{a} \bullet S). \quad (5.21)$$

By (Jacod, 1979, (6.8)) this solution is unique. Since $\phi'$ is well-defined by (ČK, Lemma 4.9) and $\pi^0 + \phi' \bullet S$ also solves (5.21), we therefore have

$$\mathscr{E}(-\widetilde{a}\bullet S)(\pi^0+\xi\bullet\widetilde{S}) = \pi^0+\phi'\bullet S,$$

which combined with (5.20) yields the third assertion. □

**Remark 5.19** If the dual minimizer $v^{-p}L\mathscr{E}(-\widetilde{a} \bullet S)$ is a martingale and hence the density process of the $q$-optimal martingale measure $Q_0$, we have $C_0 = C_1 := E(\mathscr{E}(-\widetilde{a} \bullet S)^{-p})$ and therefore $\pi(0) = E_{Q_0}(H)$ as in Section 5.2.

## 5.5 An alternative representation

We now consider how to compute the quantities $\pi(0), \pi', \varphi, \varphi'$ from the asymptotic expansions in Theorem 5.18. $\varphi$ has been computed in a wide range of affine stochastic volatility models in Chapter 4, where we also obtained conditions for the existence of the corresponding $q$-optimal martingale measure $Q_0$. If $Q_0$ exists, the marginal utility-based price $\pi(0)$ can be computed by calculating $E_{Q_0}(H)$. Since the $Q_0$-characteristics of $S$ can be computed with Proposition A.5, this can be dealt with in affine models using Laplace transform methods similarly as in Raible (2000), Hubalek et al. (2006) and Pauwels (2007). More generally, a similar approach still can also be used if $Q_0$ does not exist (cf. Vierthauer (2009) for more details). Consequently, we suppose in the remainder of this section that $\varphi$ and $\pi(0)$ are known and proceed to discuss how to obtain $\pi', \varphi'$.

By Kramkov & Sîrbu (2006, 2007), $\varphi'$ and $\pi'$ can be obtained by calculating the generalized Galtchouk-Kunita-Watanabe decomposition (5.19). Since $S^\$$ generally only is a $Q^\$$-supermartingale rather than a martingale, this is typically very difficult. If however, $S^\$$ happens to be a locally square-integrable $Q^\$$-martingale, (5.19) coincides with the classical Galtchouk-Kunita-Watanabe decomposition. By Föllmer & Sondermann (1986) this shows that $\xi$ represents the variance-optimal hedging strategy of $H$ hedged with $S^\$$ under the measure $Q^\$$ and $E((N_T^\$)^2)$ is given by the corresponding minimal expected squared hedging error in this case. Moreover, $\xi$ and $E((N_T^\$)^2)$ can then be characterized in terms of semimartingale characteristics.

**Assumption 5.20** *$S^\$$ is a locally square-integrable $Q^\$$-martingale.*

**Lemma 5.21** *Suppose Assumptions 4.9, 5.9, 5.15, 5.16, 5.20 hold and denote by $\tilde{c}^{(S^\$,V^\$)\$}$ the modified second $Q^\$$-characteristic of $(S^\$, V^\$)$ w.r.t. some $A \in \mathscr{A}_{\text{loc}}^+$. Then we have*

$$\xi = (\tilde{c}^{S^\$\$})^{-1}\tilde{c}^{S^\$,V^\$\$}, \tag{5.22}$$

$$E((N_T^\$)^2) = E_{Q^\$}\left((\tilde{c}^{V^\$\$} - (\tilde{c}^{S^\$,V^\$\$})^\top(\tilde{c}^{S^\$\$})^{-1}\tilde{c}^{S^\$,V^\$\$}) \bullet A_T\right). \tag{5.23}$$

PROOF. Since $S^\$$ is a locally square integrable $Q^\$$-martingale by Assumption 5.20, the claim follows from (ČK, Theorem 4.10, Theorem 4.12) applied to the (local-) martingale case. □

The key to applying Lemma 5.21 in concrete models is the computation of the joint characteristics of $S^\$$ and $V^\$$. In principle, this problem can be tackled using semimartingale calculus and Laplace transform inversion techniques similarly as in Pauwels (2007). However, this direct approach requires the solution of a $d+1$-dimensional quadratic hedging problem. Instead we pursue a different approach here that represents the relevant quantities as the solution to a $d$-dimensional quadratic hedging problem in terms of the original numeraire. For our approach to work, we need the following integrability condition, which is satisfied e.g. if $S^\$$ is a square-integrable $Q^\$$-martingale.

**Assumption 5.22**

$$C_2 := E((\mathscr{E}(-\widetilde{a} \bullet S)_T)^{-1-p}) < \infty.$$

Subject to Assumption 5.22 we can define a probability measure $P^€ \sim P$ via

$$\frac{dP^€}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}}{C_2}. \tag{5.24}$$

**Lemma 5.23** *Suppose Assumptions 4.9, 5.9 and 5.22 hold. Then the process*

$$L_t^\$ := E_{P^€}\left(\frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^2}{\mathscr{E}(-\widetilde{a} \bullet S)_t^2}\middle|\mathscr{F}_t\right), \quad 0 \leq t \leq T,$$

*satisfies $L_T^\$ = 1$. Moreover, we have*

$$E_{P^€}\left(\frac{dQ^\$}{dP^€}\middle|\mathscr{F}_t\right) = \frac{C_2}{C_0}L_t^\$\mathscr{E}(-\widetilde{a} \bullet S)_t^2 = \frac{L_t^\$\mathscr{E}(-\widetilde{a} \bullet S)_t^2}{L_0^\$}.$$

*and $L^\$, L_-^\$ > 0$. Hence the stochastic logarithm $K := \mathscr{L}(L^\$) = \frac{1}{L_-^\$} \bullet L^\$$ is well-defined.*

PROOF. The first part of the assertion is obvious, whereas the second follows from $\frac{dQ^\$}{dP^€} = \frac{C_2}{C_0}\mathscr{E}(-\widetilde{a}\bullet S)_T^2$. By Remark 4.12 and Lemma 4.8 we have $\mathscr{E}(-\widetilde{a}\bullet S), \mathscr{E}(-\widetilde{a}\bullet S)_- > 0$. Combined with $Q^\$ \sim P^€$ and (JS, I.2.27) this implies $L^\$, L^\$_- > 0$ and hence the third part of the assertion by (JS, II.8.3). □

**Remark 5.24** $L^\$$ is linked to the opportunity process $L$ of the pure investment problem via

$$L_t^\$ = \frac{\mathscr{E}(-\widetilde{a}\bullet S)_t^{-1-p}}{E(\mathscr{E}(-\widetilde{a}\bullet S)_t^{-1-p}|\mathscr{F}_t)} L_t,$$

by the generalized Bayes' formula, $L_T = 1$ and since $L\mathscr{E}(-\widetilde{a}\bullet S)^{1-p}$ is a martingale.

Set

$$V_t := \mathscr{E}(-\widetilde{a}\bullet S)_t V_t^\$ = E\left(\frac{\mathscr{E}(-\widetilde{a}\bullet S)_T^{-p}}{\mathscr{E}(-\widetilde{a}\bullet S)_t^{-p}} H \middle| \mathscr{F}_t\right), \quad 0 \le t \le T, \tag{5.25}$$

denote by

$$\left(\begin{pmatrix} b^{S€} \\ b^{V€} \\ b^{K€} \end{pmatrix}, \begin{pmatrix} c^{S€} & c^{S,V€} & c^{S,K€} \\ c^{V,S€} & c^{V€} & c^{V,K€} \\ c^{K,S€} & c^{K,V€} & c^{K€} \end{pmatrix}, K^{(S,V,K)€}, A\right)$$

$P^€$-differential characteristics of the semimartingale $(S, V, K)$ and define

$$\tilde{c}^{S\star} := \frac{1}{1+\Delta A^K}\left(c^{S€} + \int (1+x_3)x_1 x_1^\top K^{(S,V,K)€}(dx)\right),$$
$$\tilde{c}^{S,V\star} := \frac{1}{1+\Delta A^K}\left(c^{S,V€} + \int (1+x_3)x_1 x_2 K^{(S,V,K)€}(dx)\right),$$
$$\tilde{c}^{V\star} := \frac{1}{1+\Delta A^K}\left(c^{V€} + \int (1+x_3)x_2^2 K^{(S,V,K)€}(dx)\right),$$

where $K = K_0 + A^K + M^K$ denotes an *arbitrary* semimartingale decomposition of $K$. We then have the following representation, which is the main result of this chapter. Note that it coincides with the heuristic in Section 5.2 if the $q$-optimal martingale measure $Q_0$ exists, since this implies $C_0 = C_1$.

**Theorem 5.25** *Suppose Assumptions 4.9, 5.9, 5.15, 5.16, 5.20 and 5.22 are satisfied and* $R := E_d + S_-\widetilde{a}^\top$ *is invertible* $P \otimes A$*-almost everywhere. Then* $\tilde{c}^{S*}, \tilde{c}^{S,V*}, \tilde{c}^{V*}$ *are well defined,* $\varphi'$ *given in feedback form as*

$$\varphi' = (\tilde{c}^{S*})^{-1}\tilde{c}^{S,V*} - \left(\frac{1}{C_0}E(\mathscr{E}(-\widetilde{a}\bullet S)_T)^{-p}H) + \varphi'\bullet S_- - V_-\right)\widetilde{a} \tag{5.26}$$

*is a marginal utility-based hedging strategy and the corresponding risk premium is*

$$\pi' = \frac{pC_2}{2vC_0}E_{P€}\Big(\left(\left(\tilde{c}^{V*} - (\tilde{c}^{S,V*})^\top(\tilde{c}^{S*})^{-1}\tilde{c}^{S,V*}\right)L^\$\right)\bullet A_T\Big). \tag{5.27}$$

PROOF. An application of Propositions A.3, A.4 yields the $P^€$-differential characteristics of the process $(S, V, \mathscr{E}(-\widetilde{a} \bullet S), \mathscr{L}(\frac{C_2}{C_0} L^\$ \mathscr{E}(-\widetilde{a} \bullet S)^2))$. Since $\frac{C_2}{C_0} L^\$ \mathscr{E}(-\widetilde{a} \bullet S)^2$ is the density process of $Q^\$$ with respect to $P^€$, the $Q^\$$-characteristics of $(S, V, \mathscr{E}(-\widetilde{a} \bullet S))$ can now be obtained with Proposition A.5. Another application of Proposition A.4 then allows to compute the $Q^\$$-characteristics of $(S^\$, V^\$)$. Since $S^\$ \in \mathscr{H}^2_{\mathrm{loc}}(Q^\$)$ by Assumption 5.20 and $V^\$ \in \mathscr{H}^2(Q^\$)$ by the proof of (Kramkov & Sîrbu, 2007, Lemma 1), the modified second characteristics $\tilde{c}^{V^\$\$}$, $\tilde{c}^{S^\$,V^\$\$}$ and $\tilde{c}^{S^\$\$}$ exist and are given by

$$\tilde{c}^{V^\$\$} = \frac{1+\Delta A^K}{\mathscr{E}(-\widetilde{a} \bullet S)^2_-} \left(\tilde{c}^{V\star} + 2V_- \widetilde{a}^\top \tilde{c}^{S,V\star} + V_-^2 \widetilde{a}^\top \tilde{c}^{S\star} \widetilde{a}\right), \tag{5.28}$$

$$\tilde{c}^{S^\$,V^\$\$} = \frac{1+\Delta A^K}{\mathscr{E}(-\widetilde{a} \bullet S)^2_-} \begin{pmatrix} \widetilde{a}^\top \\ E_d + S_- \widetilde{a}^\top \end{pmatrix} \left(\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \widetilde{a} V_-\right), \tag{5.29}$$

$$\tilde{c}^{S^\$} = \frac{1+\Delta A^K}{\mathscr{E}(-\widetilde{a} \bullet S)^2_-} \begin{pmatrix} \widetilde{a}^\top \tilde{c}^{S\star} \widetilde{a} & \widetilde{a}^\top \tilde{c}^{S\star}(E_d + \widetilde{a} S^\top) \\ (E_d + S\widetilde{a}^\top)\tilde{c}^{S\star}\widetilde{a} & (E_d + S\widetilde{a}^\top)\tilde{c}^{S\star}(E_d + \widetilde{a} S^\top) \end{pmatrix}. \tag{5.30}$$

In particular it follows that $\tilde{c}^{V\star}$, $\tilde{c}^{S,V\star}$ and $\tilde{c}^{S\star}$ are well defined. Let $\phi'$ be the marginal utility-based hedging strategy from Theorem 5.18. Combining the definition of $\phi'$, Lemma 5.21, (5.29) and (Albert, 1972, Theorem 3.9, Theorem 9.1.6), we obtain

$$\phi' = lA^{-1} r \left((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} + (\tilde{c}^{S\star})^{-1} (\tilde{c}^{S\star}) V_- \widetilde{a}\right) - (\pi(0) + \phi' \bullet S_-)\widetilde{a}, \tag{5.31}$$

with

$$l = (\widetilde{a}, R^\top), \quad r = \begin{pmatrix} \widetilde{a}^\top \tilde{c}^{S\star} \\ R\tilde{c}^{S\star} \end{pmatrix}, \quad A = \begin{pmatrix} d & b^\top \\ b & C \end{pmatrix},$$

for $R = E_d + S\widetilde{a}^\top$, $d = \widetilde{a}^\top \tilde{c}^{S\star} \widetilde{a}$, $b = R\tilde{c}^{S\star}\widetilde{a}$, $C = R\tilde{c}^{S\star} R^\top$. In view of Lemma B.2, (5.31) implies

$$R\tilde{c}^{S\star}\phi' = R\tilde{c}^{S\star} \left((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - (\pi(0) + \phi' \bullet S_- - V_-)\widetilde{a}\right). \tag{5.32}$$

Since $R$ is invertible by assumption, this leads to $\tilde{c}^{S\star}\psi' = 0$ for

$$\psi' := \phi' - ((\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - (\pi(0) + \phi' \bullet S - V_-)\widetilde{a}). \tag{5.33}$$

Hence $(\psi')^\top \tilde{c}^{S\star} \psi' = 0$ and it follows from the definition of $\tilde{c}^{S\star}$ that $c^{\psi' \bullet S€} = 0$ and $K^{\psi' \bullet S€} = 0$. By Proposition A.5, Assumption 5.20, (JS, III.3.8) and Lemma A.8, this implies $b^{\psi' \bullet S€} = 0$ and hence $\psi' \bullet S = 0$. By the definition of $\psi'$, this shows that the value process $\phi' \bullet S$ solves the feedback equation

$$G = (\tilde{c}^{S\star} \tilde{c}^{S,V\star} - (\pi(0) - V)\widetilde{a}) \bullet S - G_- \bullet (a \bullet S).$$

Since $\varphi' \bullet S$ also solves this equation and the solution is unique by (Jacod, 1979, (6.8)), we get $\varphi' \bullet S = \phi' \bullet S$. Consequently, $\varphi'$ is a marginal utility-based hedging strategy.

We now turn to the risk premium $\pi'$. First notice that by (Albert, 1972, Theorem 9.1.6),

$$C^\$ := \tilde{c}^{V^\$\$} - (\tilde{c}^{S^\$,V^\$\$})^\top \xi = \tilde{c}^{V^\$\$} - (\tilde{c}^{S^\$,V^\$\$})^\top (\tilde{c}^{S^\$\$})^{-1} \tilde{c}^{S^\$,V^\$\$} \geq 0,$$
$$C^€ := \tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} \geq 0.$$

Hence $C^\$ \bullet A$ is an increasing predictable process and by Lemmas 5.21 and A.12,

$$\begin{aligned} E_{Q^\$}((N_T^\$)^2) &= E_{Q^\$}(C^\$ \bullet A) \\ &= \frac{C_2}{C_0} E_{P^€}\left(L_-^\$ \mathscr{E}(-\widetilde{a} \bullet S)_-^2 C^\$ \bullet A\right) \\ &= \frac{C_2}{C_0} E_{P^€}\left(L_-^\$ \mathscr{E}(-\widetilde{a} \bullet S)_-^2 \bullet (\langle V^\$, V^\$\rangle^{Q^\$} - \langle V^\$, \xi \bullet S^\$\rangle^{Q^\$})\right). \end{aligned}$$

Since we have shown $\phi' \bullet S = \varphi' \bullet S$ above, (Goll & Kallsen, 2000, Proposition 2.1) and the proof of Theorem 5.18 yield $\xi \bullet S^\$ = (\varphi'^0, \varphi') \bullet S^\$$ for $\varphi'^0 := \pi(0) + \varphi' \bullet S - \varphi' S^\$$. Hence

$$\begin{aligned} E_{Q^\$}((N_T^\$)^2) &= \frac{C_2}{C_0} E_{P^€}\left(L_-^\$ \mathscr{E}(-\widetilde{a} \bullet S)_-^2 \bullet \left(\langle V^\$, V^\$\rangle^{Q^\$} - \langle V^\$, (\varphi'^0, \varphi') \bullet S^\$\rangle^{Q^\$}\right)\right) \\ &= \frac{C_2}{C_0} E_{P^€}\left(L_-^\$ \mathscr{E}(-\widetilde{a} \bullet S)_-^2 (\tilde{c}^{V^\$\$} - (\tilde{c}^{S^\$, V^\$\$})^\top (\varphi'^0, \varphi')) \bullet A\right). \end{aligned}$$

After inserting $\tilde{c}^{V^\$\$}$, $\tilde{c}^{S^\$, V^\$\$}$ from (5.28) resp. (5.29) and the definition of $(\varphi'^0, \varphi')$, this leads to

$$E_{Q^\$}((N_T^\$)^2) = \frac{C_2}{C_0} E_{P^€}\left((1 + \Delta A^K) L_-^\$ C^€ \bullet A_T\right). \tag{5.34}$$

Now notice that

$$L^\$ = L_-^\$(1 + \Delta A^K + \Delta M^K).$$

By (JS, I.4.49, I.4.34) the process $\Delta M^K \bullet (L_-^\$ C^€ \bullet A)$ is a local martingale. If $(T_n)_{n \in \mathbb{N}}$ denotes a localizing sequence, this yields

$$\begin{aligned} E_{P^€}(L^\$ C^€ \bullet A_{T \wedge T_n}) &= E_{P^€}((1 + \Delta A^K + \Delta M^K) L_-^\$ C^€ \bullet A_{T \wedge T_n}) \\ &= E_{P^€}((1 + \Delta A^K) L_-^\$ C^€ \bullet A_{T \wedge T_n}), \end{aligned}$$

and hence

$$E_{P^€}(L^\$ C^€ \bullet A_T) = E_{P^€}((1 + \Delta A^K) L_-^\$ C^€ \bullet A_T)$$

by monotone convergence. Combining this with (5.34), we obtain

$$E((N_T^\$)^2) = \frac{C_2}{C_0} E_{P^€}\left(\left(\tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star}\right) L^\$ \bullet A_T\right).$$

In view of Theorem 5.18 this completes the proof. □

**Remarks.**

1. Note that an inspection of the proof of Theorem 5.25 shows that the formulas for $\varphi'$ and $\pi'$ are independent of the specific semimartingale decomposition of $K$ that is used.

2. One easily verifies that if $R$ is not invertible, $S_-$ is an eigenvector of $R$ for the eigenvalue 0 and hence $\widetilde{a}^\top S_- = -1$. By Lemma 4.8 this implies that $V(\varphi)_- = \varphi^\top S_-$, i.e. all funds are invested into stocks in the pure investment problem.

3. We are convinced that $R$ does not have to be invertible for Theorem 5.25 to hold. For $d = 1$, this follows from direct computations, which show that $\varphi' = \phi'$ regardless of whether $R$ is invertible or not. Moreover, if the process $S$ is continuous and the $q$-optimal martingale measure $Q_0$ exists, $\varphi'$ is an utility-based hedging strategy even if $R$ is not always invertible. This is a consequence of (Kramkov & Sîrbu, 2007, Theorem 3) and the fact that the modified second characteristics of $S$ and $S, V$ are invariant w.r.t. equivalent measure changes for continuous $S$. By the proof of Theorem 5.25, the formula for the risk premium $\pi'$ remains unchanged in this case.

In view of (ČK, Theorems 4.10, 4.12), Theorem 5.25 states that the first order approximations for $\varphi^q$ and $\pi^q$ can *essentially* be computed by solving the quadratic hedging problem for the contingent claim $H$ under the (non-martingale) measure $P^€$ relative to the original numeraire. However, this assertion only holds true *literally* if the optimal strategy $\varphi$ in the pure investment problem is *admissible* in the sense of (ČK, Definition 2.2, Corollary 2.5), i.e. if $\varphi \bullet S_T \in L^2(P^€)$ and $(\varphi \bullet S)Z^Q$ is a $P^€$-martingale for any absolutely continuous signed $\sigma$-martingale measure $Q$ with density process $Z^Q$ and $\frac{dQ}{dP^€} \in L^2(P^€)$. More precisely, one easily verifies that in this case $\widetilde{a}$ is the *adjustement process* in the sense of (ČK, Definition 3.8) and the strategy $-\widetilde{a}1_{]\!]\tau,T]\!]}\mathscr{E}(-\widetilde{a} \bullet S)_-$ is *efficient* on the stochastic interval $]\!]\tau, T]\!]$ in the sense of (ČK, Section 3.1). By (ČK, Corollary 3.4) this in turn implies that $L^\$$ is the *opportunity process* in the sense of (ČK, Definition 3.3). Hence it follows along the lines of (ČK, Lemma 3.15) that the *opportunity neutral measure* $P^\star$ with density process

$$Z^{P\star} := \frac{L^\$}{L_0^\$ \mathscr{E}(A^K)}$$

exists. By (ČK, Lemma 3.17, Theorem 4.10), $\tilde{c}^{S\star}, \tilde{c}^{V\star}, \tilde{c}^{S,V\star}$ indeed coincide with the corresponding modified second characteristics of $(S, V, K)$ under $P^\star$. Hence (ČK, Theorems 4.10, 4.12) yield that subject to the probability measure $P^€$, $\varphi'$ represents the variance-optimal hedging strategy for $H$ whereas the minimal expected squared hedging error of $H$ is given by the $\frac{2C_0 v}{pC_2}$-fold of $\pi'$.

Admissibility of a given candidate strategy is typically hard to verify even in concrete models (cf. e.g. Černý & Kallsen (2008a,b) for more details). Nevertheless, Theorem 5.25 ascertains that even if $\varphi$ is not admissible, the corresponding formulas typically still admit an interpretation in the context of asymptotic expansions for utility-based prices and hedging strategies.

# Chapter 6

# Asymptotic utility-based pricing and hedging in affine volatility models

## 6.1 Introduction

In this chapter we turn to the computation of asymptotic utility-based prices and hedging strategies in affine stochastic volatility models. We first provide easy-to-check conditions in Section 6.2 that ensure the existence of the first-order approximations of utility-based prices and hedging strategies considered in Chapter 5.

Afterwards, in Section 6.3 we then show how to tackle the computation of these quantities. As already noted in Chapter 5, this *essentially* amounts to solving a quadratic hedging problem under the measure $P^{€}$. Affine stochastic volatility models satisfying the structure condition 4.19 turn out to be (time-inhomogeneously) affine under this measure as well. Quadratic hedging with no martingale assumption on the underlying is dealt with in affine stochastic volatility models in the upcoming Ph.D. thesis Vierthauer (2009). These results can be applied directly here, if the optimal strategy $\varphi$ from the pure investment problem is *admissible* in the sense of ČK. However, this is often difficult to show and, as remarked above, not needed for our purposes. In Section 6.3, we therefore show that for affine models the approach of Vierthauer (2009) still leads to $\pi(0)$, $\pi'$ and $\varphi'$, even if admissibility of $\varphi$ is not guaranteed.

For Lévy processes we show in Section 6.4 that $\varphi$ is always admissible. Hence $\pi(0)$, $\pi'$ and $\varphi'$ indeed solve a quadratic hedging problem in this case. Since the Lévy property is preserved under the change of measure to $P^{€}$, the results of Hubalek et al. (2006) lead directly to semi-explicit formulas in terms of complex integrals.

For more general affine models with stochastic volatility we use the corresponding results of Vierthauer (2009). Here, we only provide formulas and a numerical example for the BNS model in Section 6.5 and refer the interested reader to Vierthauer (2009) for the general affine case. Note that in this case it is generally unclear whether or not $\varphi$ is admissible, consequently $\pi(0)$, $\pi'$ and $\varphi'$ do not necessarily solve a quadratic hedging problem.

Summing up, the purpose of this chapter comprises the following. First, we establish that the first-order approximations considered in Chapter 5 actually exist in a wide range of models allowing for jumps and stochastic volatility. Secondly, we show how to compute the relevant quantities, by linking our results to quadratic hedging.

## 6.2 Existence of the first-order approximations

Throughout this chapter, let $d = 1$ and $S = S_0 \exp(X)$ for a stochastic volatility model $(y, X)$ satisfying the structure condition 4.19. We need the following crucial

**Assumption 6.1** *There exist $\alpha_1 \in C^1([0,T],\mathbb{R})$ and $\eta \in C^1([0,T],(0,1))$ satisfying Conditions 1-5 of Theorem 4.20.*

The interval $(0,1)$ is chosen here, because it allows to accomodate arbitrarily large positive and negative jumps of the asset price. If the jump measure of $X$ has bounded support, the interval can be chosen to be larger, whereas one may allow for $\eta \in C^1([0,T],\mathbb{R})$ is $X$ is continuous. The key point is to ensure the existence of an optimal fraction $\eta$ of stocks in the *interior* of the set of admissible allocations.

**Example 6.2** By Examples 4.24 and 4.28, Assumption 6.1 is satisfied in the NIG resp. NIG-OU models estimated in Chapter 3 for $p = 2$ and $p = 150$, but not for $p = \frac{1}{2}$. For the BNS models it is not needed, since the asset price process is continuous in this case.

Given Assumption 6.1, the strategy $\varphi = \eta v \mathscr{E}(-\widetilde{a} \bullet S)_-/S_-$ is optimal with value process $v\mathscr{E}(-\widetilde{a} \bullet S)$ for $\widetilde{a} = -\eta/S_-$ by Theorem 4.20. Moreover, the corresponding opportunity process is given by $L = \exp(\int_\cdot^T \psi_0^{(y,X)}(\alpha_1(s), 0)ds + \alpha_1 y)$.

**Lemma 6.3** *Suppose Assumption 6.1 is satisfied. Then Assumptions 4.9, 5.9 and 5.15 hold.*

PROOF. Since $\eta$ is $(0,1)$-valued by Assumption 6.1, Condition 3 of Theorem 4.20 is satisfied with equality. Hence $\mathscr{E}(-\widetilde{a} \bullet S)^{-p} L/L_0$ is the density process of the $q$-optimal martingale measure by Remark 2 after Theorem 4.20. In particular, Assumption 5.9 is satisfied. Again by Theorem 4.20, Assumption 4.9 holds as well. We now turn to Assumption 5.15, which can easily be verified using (Kramkov & Sîrbu, 2006, Lemma 8) for $(0,1)$-valued $\eta$. Indeed, since $\varphi S > 0$, we have

$$\left|\frac{1}{\mathscr{E}(-\widetilde{a} \bullet S)}\right| = \frac{v}{V(\varphi)} = \frac{v}{(1-\eta)V_-(\varphi) + \varphi S} \leq \frac{v}{(1-\eta)V_-(\varphi)},$$

as well as

$$\left|\frac{S}{\mathscr{E}(-\widetilde{a} \bullet S)}\right| = \frac{S}{(1-\eta)V_-(\varphi) + \varphi S} \leq \frac{1}{\varphi},$$

because $(1-\eta)V_-(\varphi) > 0$. In view of (Kramkov & Sîrbu, 2006, Lemma 8) we are done. □

Note that if $S$ is continuous, $\eta \in (0,1)$ is not needed, since $S$ and $V(\varphi)$ are predictable and Condition 3 of Theorem 3 is always satisfied with equality in this case. The following example shows that in general, $S^\$$ does not have to be $\sigma$-bounded if $\eta$ is not $(0,1)$-valued.

**Example 6.4** Let $p = \frac{1}{2}$ and consider the NIG process $X$ from Example 4.24. Then the optimal fraction of wealth is $\eta = 1$ and we have $\mathscr{E}(-\widetilde{a} \bullet S)^{-1} = S_0/S = \exp(-X)$. Since $X$ has arbitrarily large negative jumps with positive probability, one can show that this process is not $\sigma$-bounded.

Next, we consider Assumptions 5.20 and 5.22. As a first step, we provide necessary conditions for the existence of the measure $P^{€}$ and calculate the $P^{€}$-dynamics of $(y, X)$. In order to compute the density process of $P^{€}$ w.r.t. $P$, we make a similar ansatz as in Chapter 4. Since we are looking for a strictly positive martingale with terminal value $\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}$, we have to find $L^{€}$ with $L^{€}_T = 1$ such that $L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$ is a martingale. Because $\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$ is exponentially affine, this problem can be tackled by making an affine ansatz

$$L^{€}_t := \exp\left(\alpha^{€}_0(t) + \alpha^{€}_1(t) y_t\right)$$

with deterministic functions $\alpha^{€}_0, \alpha^{€}_1$ satisfying $\alpha^{€}_0(T) = \alpha^{€}_1(T) = 0$. This leads to

**Lemma 6.5** *Suppose Assumption 6.1 holds and there exists a $C^1([0,T], \mathbb{R})$-function $\alpha^{€}_1$ such that the following conditions are satisfied up to a $dt$-null set on $[0,T]$.*

1. $\int_1^\infty e^{\alpha^{€}_1(t) z_1} \kappa_0(dz) < \infty$.
2. $\alpha^{€}_1(T) = 0$ *and*

$$0 = \alpha^{€}_1(t)' + \psi_1^{(y,X)}(\alpha^{€}_1(t), 0) - (1+p)\eta(t)\left(\beta_1^2 + \frac{\gamma_1^{22}}{2}\right) + \frac{(p+1)(p+2)}{2}\eta^2(t)\gamma_1^{22}$$
$$- (1+p)\alpha^{€}_1(t)\eta(t)\gamma_1^{12} + \int (1 + \eta(t)(e^{x_2} - 1))^{-1-p} - 1 + (1+p)\eta(t) h_2(x) \kappa_1(dx).$$

*Then $L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$ is a martingale for*

$$L^{€} = \exp\left(\int_t^T \psi_0^{(y,X)}(\alpha^{€}_1(s), 0) ds + \alpha^{€}_1(t) y_t\right).$$

*Moreover, under $P^{€} \sim P$ with density process $\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p} L^{€}/L^{€}_0$, the stochastic volatility model $(y, X)$ is a (time-inhomogeneous) affine process relative to triplets $(\beta^{€}_i, \gamma^{€}_i, \kappa^{€}_i)$, $i = 0, 1, 2$ given by*

$$(\beta^{€}_0, \gamma^{€}_0, \kappa^{€}_0(G)) = \left(\begin{pmatrix} \beta_0^1 + \int h(z_1)(e^{\alpha^{€}_1 z_1} - 1)\kappa_0(dz) \\ 0 \end{pmatrix}, 0, \int e^{\alpha^{€}_1 z_1} 1_G(z_1, 0) \kappa_0(dz)\right),$$

$$\beta^{€}_1 = \begin{pmatrix} \beta_1^1 + \gamma_1^{11}\alpha^{€}_1 - (1+p)\eta\gamma_1^{12} \\ \beta_1^2 + \alpha^{€}_1\gamma_1^{12} - (1+p)\eta\gamma_1^{22} + \int (h_2(x)(1 + \eta(e^{x_2} - 1))^{-1-p} - 1)\kappa_1(dx) \end{pmatrix},$$

$$\gamma^{€}_1 = \begin{pmatrix} \gamma_1^{11} & \gamma_1^{12} \\ \gamma_1^{12} & \gamma_1^{22} \end{pmatrix},$$

$$\kappa^{€}_1(G) = \int (1 + \eta(e^{x_2} - 1))^{-1-p} 1_G(0, x_2) \kappa_1(dx),$$

$$(\beta^{€}_2, \gamma^{€}_2, \kappa^{€}_2) = (0, 0, 0).$$

*for $G \in \mathscr{B}^2$.*

PROOF. The differential characteristics of $\mathscr{L}(L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p})$ can be computed with Propositions A.3 and A.4. By Condition 1 and since $\eta \in C^1([0,T],(0,1))$ we have

$$\int_{\{|x|>1\}} |x| K^{\mathscr{L}(L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p})}(dx) < \infty.$$

Moreover, we obtain

$$b^{\mathscr{L}(L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p})} + \int (x-h(x)) K^{\mathscr{L}(L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p})}(dx) = 0$$

due to Condition 2. Hence $\mathscr{L}(L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p})$ is a $\sigma$-martingale by Lemma A.8. Since it is also the second component of the affine process $\left(y, \mathscr{L}\left(L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}\right)\right)$ it then follows from Theorem 2.9 that $L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$ is a true martingale. Since $L^{€}_T = 1$, this shows that $\mathscr{E}(-\widetilde{a} \bullet S)L^{€}/L_0$ coincides with the density process of $P^{€}$ w.r.t. $P$. The $P^{€}$-characteristics of $(y, X)$ can now be derived by applying Proposition A.5. □

Again notice that $\eta$ does not have to $(0,1)$-value for continuous $S$, because we have $\kappa_1 = 0$ in this case.

**Example 6.6** Consider the model of Carr et al. (2003) from Section 2.3.3. Then conditions 1 and 2 of Lemma 6.5 are satisfied if

$$\int_1^\infty \exp\left(M_p\left(\frac{e^{-\lambda(T-t)}-1}{\lambda}\right)z\right) K^Z(dz) < \infty \tag{6.1}$$

up to a $dt$-null set on [0,T], for the constant

$$\begin{aligned} M_p :=& (1+p)\eta\big(b^B + \frac{c^B}{2}\big) - \frac{(p+1)(p+2)}{2}\eta^2 c^B \\ &- \int \big((1+\eta(e^x-1))^{-1-p} - 1 + (1+p)\eta h(x)\big) K^B(dx) \\ =& \frac{(p-2)(p+1)}{2}\eta^2 c^B \\ &+ \int \frac{(1+\eta(e^x-1))^{1+p} - 1 - (1+p)\eta(e^x-1)(1+\eta(e^x-1))}{(1+\eta(e^x-1))^{1+p}} K^B(dx), \end{aligned}$$

where we have used Condition 3 of Theorem 4.20 for the second equality. Furthermore, we have $\alpha_1^{€}(t) = M_p(e^{-\lambda(T-t)}-1)/\lambda$ in this case. For the BNS model with $K^B = 0$, $c^B = 1$ and $\eta = (\delta + \frac{1}{2})/p$, this simplifies to

$$M_p = \frac{(p-2)(p+1)}{2p^2}\big(\delta + \frac{1}{2}\big)^2.$$

Notice that for $p \geq 2$, we have $M_p \geq 0$. This is obvious for the BNS model, for the models of Carr et al. (2003) it follows from differentiation and the Bernoulli inequality. Since $K^Z$ is concentrated on the positive real line, this implies that (6.1) is always satisfied for $p \geq 2$. Combining this with Example 4.28, we find that Lemma 6.5 is applicable in particular for

$p = 2$ and $p = 150$ and for the NIG-Gamma-OU, NIG-IG-OU, BNS-Gamma-OU and BNS-IG-OU models with parameters as estimated in Chapter 3.

If $X$ is a Lévy process satisfying Assumption 6.1 (e.g. the NIG process from Example 4.24 for $p = 2$ or $p = 150$), the conditions of Lemma 6.5 are trivially satisfied because $\kappa_0 = 0$ in this case (cf. Section 4.4.1). Moreover $\alpha_1^{€}$ is given by $\alpha_1^{€}(t) = (t-T)M_p$ if $(b^B, c^B, K^B)$ in the definition of $M_p$ are replaced with the Lévy-Khintchine triplet $(b^X, c^X, K^X)$ of $X$.

We can now provide conditions on the model parameters that ensure the validity of Assumptions 5.20 and 5.22.

**Lemma 6.7** *Let Assumption 6.1 and the conditions of Lemma 6.5 be satisfied and suppose there exists $\Phi^1 \in C^1([0,T], \mathbb{R})$ such that the following conditions hold.*

1. $\int_1^\infty e^{\Phi^1(t)z_1} \kappa_0(dz) < \infty$ *for* $t \in [0,T]$.
2. $\int_{\{|x|>1\}} e^{2x_2} \kappa_1(dx) < \infty$.
3. $\Phi^1(T) = 0$ *and* $\frac{d}{dt}\Phi^1(t) = -\psi_1^{(y,\log(S^2\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p}))€}(t, \Phi^1(t), 1)$ *for* $t \in [0,T]$.

*Then Assumptions 5.20 and 5.22 hold.*

PROOF. By Assumption 6.1 and Remark 2 after Theorem 4.20 the $q$-optimal martingale measure $Q_0$ exists and $(1, S)$ is a $Q_0$-martingale. The density process of $Q^\$$ w.r.t. $Q_0$ is given by $\mathscr{E}(-\widetilde{a} \bullet S)$, hence $S^\$ = (1, S)/\mathscr{E}(-\widetilde{a} \bullet S)$ is a $Q^\$$-martingale by (JS, III.3.8). Consequently, it suffices to show $S_T^\$ \in L^2(Q^\$)$ for Assumptions 5.20 and 5.22.

Since $\frac{dQ^\$}{dP} = \frac{1}{L_0}\mathscr{E}(-\widetilde{a} \bullet S)^{1-p}$, this is equivalent to showing $\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p} \in L^1(P)$ and $S^2\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p} \in L^1(P)$. The first assertion has already been established as part of Lemma 6.5. The second follows from Conditions 1-3 and Theorem 2.24, since $\eta$ is bounded away from zero by Assumption 6.1. □

**Example 6.8** If $X$ is a Lévy process with Lévy measure $K^X$, Conditions 1-3 of Lemma 6.7 simplify to

$$\int_{\{|x|>1\}} e^{2x} K^X(dx) < \infty,$$

i.e. $E(S_T^2) < \infty$, because $\kappa_0 = 0$ and the mapping $\Phi$ is always given as a the solution to a constant ODE in this case. Since Example 6.6 shows that the assumptions of Lemma 6.5 are fulfilled too, Lemma 6.7 is applicable in the NIG model from Example 4.24 for $p = 2$ or $p = 150$.

Now consider the model of Carr et al. (2003) with parameters as estimated in Chapter 3. Condition 2 is then equivalent to the existence of the second exponential moment of the driving Lévy process $B$, which holds e.g. for the NIG-OU and BNS-OU models estimated in Chapter 3 (cf. Example 4.28). In view of Example 6.6 the prerequisites of Lemma 6.5 are satisfied for $p \geq 2$ as well. Since the ODE for $\Phi^1$ turns out to be linear here, Conditions 1 and 3 of Lemma 6.7 hold too, if

$$\int_1^\infty \exp\left(N_p\left(\frac{e^{-\lambda(T-t)}-1}{\lambda}\right)z\right)K^Z(dz) < \infty$$

holds up to a $dt$-null set on $[0,T]$ for the constant

$$\begin{aligned}
N_p :=& \big((1+p)\eta - 2\big)\big(b^B + \frac{c^B}{2}\big) - \left(1 - 2(1+p)\eta + \frac{(p+1)(p+2)}{2}\eta^2\right)c^B \\
&- \int\left(\frac{e^{2x}}{(1+\eta(e^x-1))^{1+p}} - 1 + ((1+p)\eta-2)h(x)\right)K^B(dx) \\
=& \left(\frac{(p-2)(p+1)}{2}\eta^2 + 2\eta - 1\right)c^B \\
&+ \int \frac{(1+\eta(e^x-1))^{1+p} - e^{2x} - ((1+p)\eta-2)(e^x-1)(1+\eta(e^x-1))}{(1+\eta(e^x-1))^{1+p}}K^B(dx),
\end{aligned}$$

where we have again used Condition 3 of Theorem 4.20 for the second equality. By insertion (for the BNS models) resp. numerical quadrature (for the NIG models), we obtain that this holds for the parameters from Examples 3.30, 3.31, 3.32 and $p = 2$ or $p = 150$. Consequently, Assumptions 5.20 and 5.22 are satisfied for $p = 2$ or $p = 150$ in the discounted NIG-OU and BNS-OU models with parameters as estimated in Chapter 3.

The validity of Assumption 5.16 depends on the contingent claim under consideration. For example, it is trivially satisfied for European calls and puts if $S^\$$ is a square-integrable $Q^\$$-martingale.

**Lemma 6.9** *Suppose the assumptions of Lemma 6.7 hold. Then Assumption 5.16 is satisfied for Europeran call- and put-options with payoff functions* $(S_T - K)^+$, $K > 0$ *respectively* $(K - S_T)^+$, $K > 0$.

PROOF. Since $S^\$ \in \mathscr{H}^2(Q^\$)$ by the proof of Lemma 6.7, this follows immediately from $(S_T - K)^+/\mathscr{E}(-\widetilde{a}\bullet S) \leq S/\mathscr{E}(-\widetilde{a}\bullet S)$ resp. $(K - S_T)^+ \leq K/\mathscr{E}(-\widetilde{a}\bullet S)$. □

**Example 6.10** Piecing together Examples 4.28, 6.6 , 6.8 and Lemma 6.9 we obtain that first-order approximations of utility-based prices and hedging strategies of European call- and put-options exist for $p = 2$ and $p = 150$ in the NIG model, the BNS-Gamma-OU (resp. BNS-IG-OU) model and in the NIG-Gamma-OU (resp. NIG-IG-OU) model for the parameters estimated in Chapter 3.

## 6.3 Computation of the first-order approximations

Having ensured the existence of the first-order approximations in Section 6.2 above, we now turn to the computation of $\pi(0), \pi', \varphi'$. The following result reduces the computation of $\pi'$ to the calculation of $\varepsilon^2_{€}$.

**Corollary 6.11** *Suppose the assumptions of Lemma 6.5 are satisfied. Then we have*

$$\frac{pC_2}{2vC_0} = \frac{p}{2v} \exp\left( \int_0^T \big(\psi_0^{(y,X)}(\alpha_1^{€}(s),0) - \psi_0^{(y,X)}(\alpha_1(s),0)\big) ds + (\alpha_1^{€}(0) - \alpha_1(0)) y_0 \right),$$

*for $C_0$ and $C_2$ defined as in* (5.18) *and Assumption 5.22, respectively.*

PROOF. Follows immediately from $L_T^{€} = L_T = 1$ and the martingale property of $L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$ and $L\mathscr{E}(-\widetilde{a} \bullet S)^{1-p}$ established in Lemma 6.5 and Theorem 4.20, respectively. □

The next step is to calculate the process $L^{\$}$ from Lemma 5.23, which is characterized by $L_T^{\$} = 1$ and the martingale property of $L^{\$}\mathscr{E}(-\widetilde{a} \bullet S)^2$ under $P^{€}$. Therefore its computation could in principle be tackled just as in Lemma 6.5 above. However, since $(y, X)$ is a time-inhomogeneous affine process under $P^{€}$, it is in many instances no longer possible to solve the resulting time-inhomogeneous generalized Riccati equations explicitly. For instance, this occurs for the Heston model, since even classical Riccati equations typically do not admit closed-form solutions for time-dependent coefficients. Fortunately, one can sidestep this problem by using the link to the opportunity process $L = \exp(\int_\cdot^T \psi_0^{(y,X)}(\alpha_1(s))ds + \alpha_1 y)$ of the pure investment problem provided in Remark 5.24. This is done in the following Lemma.

**Lemma 6.12** *Suppose the conditions of Lemma 6.5 are satisfied. Then we have*

$$\begin{aligned} L^{\$} &= \exp\left( \int_\cdot^T (\psi_0^{(y,X)}(\alpha_1(s),0) - \psi_0^{(y,X)}(\alpha_1^{€}(s),0)) ds + (\alpha_1 - \alpha_1^{€}) y \right) \\ &= \exp\left( \int_\cdot^T \psi_0^{(y,X)€}(s, \alpha_1^{\$}(s), 0) ds + \alpha_1^{\$} y \right) \end{aligned}$$

*for $\alpha_1^{\$} := \alpha_1 - \alpha_1^{€}$. Moreover,*

$$\widetilde{a}_t = \frac{1}{S_{t-}} \frac{\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),1) - \psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),0)}{\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),2) - 2\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),1) + \psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),0)}$$

*as well as $\alpha_1^{\$}(T) = 0$ and*

$$\begin{aligned} 0 = {} & \alpha_1^{\$}(t)' + \psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),0) \\ & - \frac{(\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),1) - \psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),0))^2}{\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),2) - 2\psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),1) + \psi_1^{(y,X)€}(t,\alpha_1^{\$}(t),0)} \end{aligned}$$

*for $t \in [0, T]$.*

PROOF. By definition, the process $L^{\$}$ is uniquely determined by $L_T^{\$} = 1$ and $L^{\$}\mathscr{E}(-\widetilde{a} \bullet S)^2$ being a $P^{€}$-martingale. By (JS, III.3.8) and Lemma 6.5 the latter property is equivalent to $L^{\$}L^{€}\mathscr{E}(-\widetilde{a} \bullet S)^{1-p}$ being a $P$-martingale. Since this also holds for $L$ instead of $L^{\$}L^{€}$ and we have $L_T^{\$}L_T^{€} = 1 = L_T$, we obtain $L^{\$} = L/L^{€}$ which combined with Theorem 4.20

and Lemma 6.5 proves the first equality of the first assertion. The second now follows from Lemma 6.5 by insertion. The second assertion again follows by insertion by making use of $\alpha_1^{\$} + \alpha_1^{€} = \alpha_1$ and exploiting that since $\eta$ is $(0,1)$-valued by Assumption 6.1, Condition 3 of Theorem 4.20 is satisfied with equality. We now turn to the third assertion. $\alpha_1^{\$}(T) = 0$ is a consequence of $\alpha_1(T) = \alpha_1^{€}(T) = 0$. Finally, the ODE for $\alpha_1^{\$}$ follows from the ODEs for $\alpha_1$ and $\alpha_1^{€}$ (cf. Theorem 4.20 and Lemma 6.5, respectively), by using Condition 3 of Theorem 4.20 once again. □

In view of Lemma 6.12, $L^{\$}$ and $\tilde{a}$ coincide with the candidates obtained in Vierthauer (2009) for the *opportunity* resp. *adjustment process* of the quadratic hedging problem for the claim $H$ under the measure $P^{€}$. The remainder of the calculation of the joint characteristics of $S$, $K = \mathscr{L}(L^{\$})$ and $V = E_{Q_0}(H|\mathscr{F}_\cdot)$ then proceeds literally as in Vierthauer (2009), because the subsequent Lemma shows that for affine models the quantities $\tilde{c}^{S\star}, \tilde{c}^{S,V\star}, \dots$ indeed coincide with the modified second characteristics of the respective processes under an equivalent probability measure $P^\star$.

**Lemma 6.13** *Suppose the prerequisites of Lemma 6.5 are satisfied. Then $K := \mathscr{L}(L^{\$})$ is a special semimartingale and*

$$Z^\star := \frac{L^{\$}}{L_0^{\$}\mathscr{E}(A^{K€})}$$

*is the density process of a probability measure $P^\star \sim P^{€}$. Moreover, the modified second $P^\star$-characteristics of $(S, V)$ are given by the formulas preceding Theorem 5.25.*

PROOF. After applying Lemma 6.5 and Propositions A.3, A.4, we obtain

$$\int_{\{|z|>1\}} |z| K^{K€}(dz) \leq \int_{\{|z|>1\}} (e^{\alpha_1 z} + e^{\alpha_1^{€} z}) \kappa_0(dz) < \infty,$$

by Assumption 6.1 and Condition 1 of Lemma 6.5. Together with the continuity of $\alpha_1$ and $\alpha_1^{€}$ as well as dominated convergence, this implies that $K^{€}$ is a $P^{€}$-special semimartingale by (Kallsen, 2004, Lemma 3.2). Consequently, its $P^{€}$-compensator $A^{K€}$ is unique and given by

$$A^{K€} = \int_0^{\cdot} \left( b_t^{K€} + \int (z - h(z)) K_t^{K€}(dz) \right) dt.$$

Since $A^{K€}$ is continuous and of finite variation, Yor's formula yields $Z^\star = \mathscr{E}(K - A^{K€})$. Hence $Z^\star$ is a positive $P^{€}$-$\sigma$-martingale. From Lemmas 6.5 and 6.12 as well as Propositions A.3, A.4 we infer that it is also the stochastic exponential of the second component of the affine process $(y, \mathscr{L}(Z^\star))$. It then follows from Theorem 2.9 that it is a true $P^{€}$-martingale, which yields the second assertion. The third then is a consequence of Proposition A.5. □

## 6.4 Exponential Lévy models

We now consider exponential Lévy models which can be embedded in the affine framework as in Section 4.4.1 above. In this case without stochastic volatility, Lemma 6.5 shows that

$X$ also is a Lévy process under $P^€$ with corresponding Lévy exponent

$$\psi^{X€}(u) = \left( b^X - (1+p)\eta c^X + \int h(x)\big((1+\eta(e^x-1))^{-1-p} - 1\big) K^X(dx) \right) u$$
$$+ \frac{1}{2} c^X u^2 + \int \big(1+\eta(e^x-1))^{-1-p}\big)(e^{ux} - 1 - uh(x)) K^X(dx),$$

which can be evaluated using numerical quadrature if the Lévy measure $K^X$ of $X$ is known in closed form. Piecing together results from Schweizer (1994), Hubalek et al. (2006) and ČK we show in Theorem 6.17 below that for Lévy processes $\pi(0)$, $\pi'$ and $\varphi'$ are indeed given as the solution to a quadratic hedging problem. Solutions to this problem have been obtained in Hubalek et al. (2006) using the *Laplace transform approach* put forward in Raible (2000). The key assumption for this approach is the existence of an integral representation of the payoff function in the following sense.

**Assumption 6.14** *Suppose $H = f(S_T)$ for a function $f : (0,\infty) \mapsto \mathbb{R}$ such that*

$$f(s) = \int_{R-i\infty}^{R+i\infty} l(z) s^z dz, \quad s \in (0,\infty),$$

*for $l : \mathbb{C} \to \mathbb{C}$ s.t. the integral exists for all $s \in (0,\infty)$ and $R \in \mathbb{R}$ s.t. $E(e^{RX_T}) < \infty$.*

**Remark 6.15** If the *(bilateral) Laplace transform* $\widetilde{f_e}$ of $f_e : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(e^x)$ exists for some $R \in \mathbb{R}$ and $v \mapsto \widetilde{f_e}(R+iv)$ is integrable, Assumption 6.14 holds with $l = \frac{1}{2\pi i}\widetilde{f_e}$ by (Hubalek et al., 2006, Theorem A.1).

Most European options admit a representation of this kind, see e.g. Hubalek et al. (2006).

**Example 6.16** For a European call option with strike $K > 0$ we have $H = (S_T - K)^+$ and, for $s > 0$ and $R > 1$,

$$(s-K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \frac{K^{1-z}}{z(z-1)} s^z dz.$$

**Theorem 6.17** *Suppose $X$ is a Lévy process s.t. $E(e^{2X_T}) < \infty$, $\psi^{X€}(2) - 2\psi^{X€}(1) \neq 0$ and Assumption 6.1 holds. For a contingent claim $H$ satisfying Assumptions 5.16 and 6.14 the marginal utility-based price and a marginal utility-based hedging strategy are then given by*

$$\pi(0) = V_0,$$
$$\varphi'_t = \xi_t - (V_0 + \varphi' \bullet S_{t-} - V_{t-})\,\widetilde{a},$$

*with*

$$\Psi(z) := \psi^{X€}(z) - \psi^{X€}(1) \frac{\psi^{X€}(z+1) - \psi^{X€}(z) - \psi^{X€}(1)}{\psi^{X€}(2) - 2\psi^{X€}(1)},$$
$$\widetilde{a} := \frac{1}{S_{t-}} \frac{\psi^{X€}(1)}{\psi^{X€}(2) - 2\psi^{X€}(1)},$$
$$V_t := \int_{R-i\infty}^{R+i\infty} S_t^z e^{\Psi(z)(T-t)} l(z) dz,$$
$$\xi_t := \int_{R-i\infty}^{R+i\infty} S_{t-}^{z-1} \frac{\psi^{X€}(z+1) - \psi^{X€}(z) - \psi^{X€}(1)}{\psi^{X€}(2) - 2\psi^{X€}(1)} e^{\Psi(z)(T-t)} l(z) dz.$$

*Moreover, we have*

$$\varepsilon_{€}^2 = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} J(z_1, z_1) l(z_1) l(z_2) dz_1 dz_2,$$

*where*

$$\begin{aligned} k(z_1, z_2) &:= \Psi(z_1) + \Psi(z_2) - \frac{\psi^{X€}(1)^2}{\psi^{X€}(2) - 2\psi^{X€}(1)}, \\ j(z_1, z_2) &:= \psi^{X€}(z_1 + z_2) - \psi^{X€}(z_1) - \psi^{X€}(z_2) \\ &\quad - \frac{(\psi^{X€}(z_1+1) - \psi^{X€}(z_1) - \psi^{X€}(1))(\psi^{X€}(z_2+1) - \psi^{X€}(z_2) - \psi^{X€}(1))}{\psi^{X€}(2) - 2\psi^{X€}(1)}, \\ J(z_1, z_2) &:= \begin{cases} S_0^{z_1+z_2} j(z_1, z_2) \dfrac{e^{k(z_1,z_2)T} - e^{\psi^{X€}(z_1+z_2)T}}{k(z_1, z_2) - \psi^{X€}(z_1+z_2)} & \textit{if } k(z_1, z_2) \neq \psi^{X€}(z_1 + z_2), \\ S_0^{z_1+z_2} j(z_1, z_2) T e^{\psi^{X€}(z_1,z_2)T} & \textit{if } k(z_1, z_2) = \psi^{X€}(z_1, z_2). \end{cases} \end{aligned}$$

PROOF. First notice that Assumption 5.16 implies $H \in L^2(P^{€})$. It then follows from Lemma 6.5, (Hubalek et al., 2006, Proposition 3.1) and the proof of (Hubalek et al., 2006, Lemma 3.1) that (Schweizer, 1994, Theorem 3) is applicable under $P^{€}$. By (Schweizer, 1994, Proposition 13), the proof of (Hubalek et al., 2006, Lemma 3.1) and Lemma 6.12 this yields that the optimal strategy $\varphi$ from the pure investment problem represents the variance-optimal hedging strategy of the constant payoff $1$ for initial endowment $0$ under $P^{€}$. In particular, $\varphi$ is admissible in the sense of Schweizer (1994) and hence in the sense of ČK as well by (ČK, Corollary 2.9). As remarked at the end of Chapter 5 above, this shows that the marginal utility-based price $\pi(0)$ and a marginal utility-based hedging strategy are given by the variance-optimal initial capital resp. hedging strategy for the claim $H$ hedged with $S$ under $P^{€}$. Moreover, $\varepsilon_{€}^2$ coincides with the corresponding minimal expected squared hedging error in this case. Notice that this refers to the solution of the quadratic hedging problem w.r.t. the set of admissible strategies from ČK. However, (ČK, Corollary 2.9) shows that the terminal portfolio values of admissible strategies in the sense of Schweizer (1994) are $L^2$-dense in the set of terminal portfolio values of admissible strategies in the sense of ČK. Hence it follows that the minimal expected squared hedging error $\varepsilon_{€}^2$ coincides with the one obtained in (Hubalek et al., 2006, Theorem 3.2) using Schweizer's notion of admissibility. By ČK, Lemma 2.11 the value process of the corresponding hedging strategy is unique, which shows that $\pi(0)$ and a marginal utility-based hedging strategy are given by the initial endowment resp. optimal hedging strategy from (Hubalek et al., 2006, Theorem 3.1). This proves the assertion. □

## 6.5 Barndorff-Nielsen & Shephard (2001)

For ease of exposition, we only consider here the BNS model as a first example and refer the reader to Vierthauer (2009) for the general affine case as well as the corresponding proofs. If

Lemmas 6.5 is applicable, as e.g. for $p \geq 2$ and the BNS-Gamma-OU (resp. BNS-IG-OU) model with parameters as estimated in Chapter 3 (cf. Examples 4.28, 6.10), Example 6.6 shows that

$$\alpha_1^{€}(t) = \frac{(p-2)(p+1)}{2p^2}\frac{(\delta+1/2)^2}{\lambda}\left(e^{-\lambda(T-t)}-1\right), \quad t \in [0,T].$$

Lemma 6.5 then yields that the characteristics of $(y,X)$ under $P^{€}$ are affine relative to

$$(\beta_0^{€}, \gamma_0^{€}, \kappa_0^{€}(G)) = \left(\begin{pmatrix} b^Z + \int h(z)(e^{\alpha_1^{€} z}-1)K^Z(dz) \\ 0 \end{pmatrix}, 0, \int e^{\alpha_1^{€} z} 1_G(z,0)K^Z(dz)\right),$$

$$(\beta_1^{€}, \gamma_1^{€}, \kappa_1^{€}) = \left(\begin{pmatrix} -\lambda \\ -(2\delta+1+p)/2p \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0\right),$$

$$(\beta_2^{€}, \gamma_2^{€}, \kappa_2^{€}) = (0,0,0),$$

for $t \in [0,T]$ and $G \in \mathscr{B}^2$, since $\eta = \frac{\delta+1/2}{p}$. In particular,

$$\psi^{Z€}(t,u_1) = \psi_0^{(y,X)€}(t,u_1,u_2) = \psi^Z(u_1+\alpha_1^{€}(t)) - \psi^Z(\alpha_1^{€}(t)),$$

$$\psi_1^{(y,X)€}(u_1,u_2) = -\lambda u_1 - u_2(2\delta+1+p)/2p + \frac{1}{2}u_2^2.$$

Moreover, by the formula for $\alpha_1^{€}$ and Remark 5 after Corollary 4.27,

$$\alpha_1^{\$}(t) = \alpha_1(t) - \alpha_1^{€}(t) = \frac{(\delta+1/2)^2}{p^2\lambda}(e^{-\lambda(T-t)}-1), \quad t \in [0,T].$$

$\pi(0)$, $\varphi'$ and $\varepsilon_{€}^2$ are now given by the formulas derived in Vierthauer (2009) using the Laplace transform approach. Suppose Assumptions 5.16, 6.14 hold and Lemmas 6.5, 6.7 are applicable. One can then prove the following result subject to further technical regularity conditions. For more details we refer to Pauwels (2007) and Vierthauer (2009).

**Theorem 6.18** *The marginal utility-based price and hedging strategy are given by*

$$\pi(0) = V_0,$$

$$\varphi_t' = \xi_t - (V_0 + \varphi' \bullet S_{t-} - V_{t-})\widetilde{a}_t,$$

*with*

$$\Psi^1(t,T,z) = \frac{(1-z)z}{2\lambda}(e^{-\lambda(T-t)}-1),$$

$$\Psi^0(t,T,z) = \int_t^T \left(\psi^Z(\alpha_1(s)+\Psi^1(s,T,z)) - \psi^Z(\alpha_1(s))\right) ds$$

$$V_t = \int_{R-i\infty}^{R+i\infty} S_t^z \exp\left(\Psi^0(t,T,z)+\Psi^1(t,T,z)y_t\right) l(z)dz,$$

$$\xi_t = \int_{R-i\infty}^{R+i\infty} zS_t^{z-1} \exp\left(\Psi^0(t,T,z)+\Psi^1(t,T,z)y_{t-}\right) l(z)dz.$$

*Moreover, we have*

$$\varepsilon_{€}^2 = \int_0^T \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} J(t,z_1,z_2) l(z_1) l(z_2) dz_1 dz_2 dt,$$

*for*

$$\begin{aligned}
j(t,z_1,z_2) =& \psi^{Z€}(t, \alpha_1^{\$}(t) + \Psi^1(t,T,z_1) + \Psi^1(t,T,z_2)) + \psi^{Z€}(t,\alpha_1^{\$}(t)) \\
& - \psi^{Z€}(t, \alpha_1^{\$}(t) + \Psi^1(t,T,z_1)) - \psi^{Z€}(t, \alpha_1^{\$}(t) + \Psi^1(t,T,z_2)), \\
g(z_1,z_1) =& \frac{2\delta+1+p}{2p}(z_1+z_2) - \frac{1}{2}(z_1+z_2)^2, \\
\Upsilon^1(s,t,T,z_1,z_2) =& (\alpha_1^{\$}(t) + \Psi^1(t,T,z_1) + \Psi^1(t,T,z_2)) e^{\lambda(s-t)} + g(z_1,z_2)\frac{e^{\lambda(s-t)}-1}{\lambda}, \\
\Upsilon^0(s,t,T,z_1,z_2) =& \int_s^t \psi^{Z€}(r, \Upsilon^1(r,t,T,z_1,z_2)) dr, \\
J(t,z_1,z_2) =& S_0^{z_1+z_2} j(t,z_1,z_2) \exp\big(\Upsilon^0(0,t,T,z_1,z_2) + \Upsilon^1(0,t,T,z_1,z_2) y_0\big) \\
& \times \exp\Big(\int_t^T \psi^{Z€}(s, \alpha_1^{\$}(s)) ds + \Psi^0(t,T,z_1) + \Psi^0(t,T,z_2)\Big).
\end{aligned}$$

PROOF. Vierthauer (2009). □

**Remarks.**

1. Notice that if one can swap the order of differentiation and integration, the *pure hedge coefficient* $\xi$ is given by the derivative w.r.t. $S$ of the marginal utility-based *option price* $V$ for the BNS model. Consequently, the initial value of the marginal utility-based hedging strategy $\varphi'$ is given by a kind of *Delta-hedge*. This ceases to hold for processes with jumps (cf. e.g. Section 6.4).

2. Theorem 6.18 can be generalized to other affine stochastic volatility models satisfying Assumption 4.19. More details on this will be provided in Vierthauer (2009), where the present results are also compared to other hedging approaches.

If $y$ is chosen to be a $\Gamma$-OU process, all expressions involving the characteristic exponent $\psi^{Z€}(t,u) = \psi^Z(u+\alpha_1^{€}(t)) - \psi^Z(\alpha_1^{€}(t))$ can be computed in closed form as well

**Lemma 6.19** *Let $y$ be a $\Gamma$-OU process with mean reversion $\lambda$ and stationary $\Gamma(a,b)$-distribution and*

$$m(s) := c_1\left(e^{-\lambda(\widetilde{t}-s)} - 1\right) + c_2 e^{-\lambda(\widetilde{t}-s)} + c_3, \quad \widetilde{t} \in [0,T],$$

*for constants $c_1, c_2, c_3 \in \mathbb{C}$. Then if $m(s) \neq b$, $s \in [t,T]$ we have, for $0 \leq t_1 \leq t_2 \leq T$,*

$$\int_{t_1}^{t_2} \psi^Z(m(s)) ds = \frac{-a}{b+c_1-c_3}\left(\lambda(t_2-t_1)(c_1-c_3) - b\log\left(\frac{-b+m(t_1)}{-b+m(t_2)}\right)\right),$$

*where* log *denotes the* distinguished logarithm *in the sense of (Sato, 1999, Lemma 7.6).*

PROOF. Follows by inserting $\psi^Z(u) = \frac{\lambda a u}{b-u}$, which is analytic on $\mathbb{C}\backslash\{b\}$, and integration using decomposition into partial fractions. □

For the BNS-Gamma-OU model one therefore has to evaluate a complex integral each for the marginal utility-based price and hedging strategy as well as a complex triple integral for the corresponding risk premium. Notice that all integrands are known in closed form here, whereas for Lévy processes the Lévy exponent $\psi^{X€}$ of $X$ under $P^{€}$ typically has to be evaluated numerically. We now present a numerical example.

**Example 6.20** Consider the discounted BNS-Gamma-OU model with parameters as estimated in Chapter 3, i.e. $\delta = 0.904$, $\lambda = 2.54$, $a = 0.847$ and $b = 17.5$. We let $y_0 = 0.0485$, $S_0 = 100$ and put $v = 241$, which implies that indifference prices and utility-based hedging strategies exist for $S_0 \in [80, 120]$ and $q \in [-2, 2]$. By Example 6.10, first-order approximations of the utility-indifference price and the utility-based hedging strategy exist for $p = 2$ and $p = 150$ by Lemma 5.8 resp. Theorem 5.10. Moreover, Assumptions Assumption 5.16 and 6.14 hold for European call-options by Example 6.16. Let $R = 1.2$. Then one can verify that all expressions in Theorem 6.17 are well-defined. The formulas of Theorem 6.18 can now be evaluated using numerical quadrature. The resulting first-order approximations for $p = 2$ resp. $p = 150$ and $q = -2, -1, 0, 1, 2$ are shown below.

The initial hedges for $p = 2$ and $p = 150$ in Figure 6.1 below cannot be distinguished by eye. Indeed, the maximal relative difference between the two strategies is $0.4\%$ for $80 \leq S_0 \leq 120$, which implies that the utility-based hedging strategy is very robust w.r.t. the investor's risk aversion. Moreover, both strategies are quite close to the Black-Scholes hedging strategy, the maximal relative difference being about $8.7\%$.

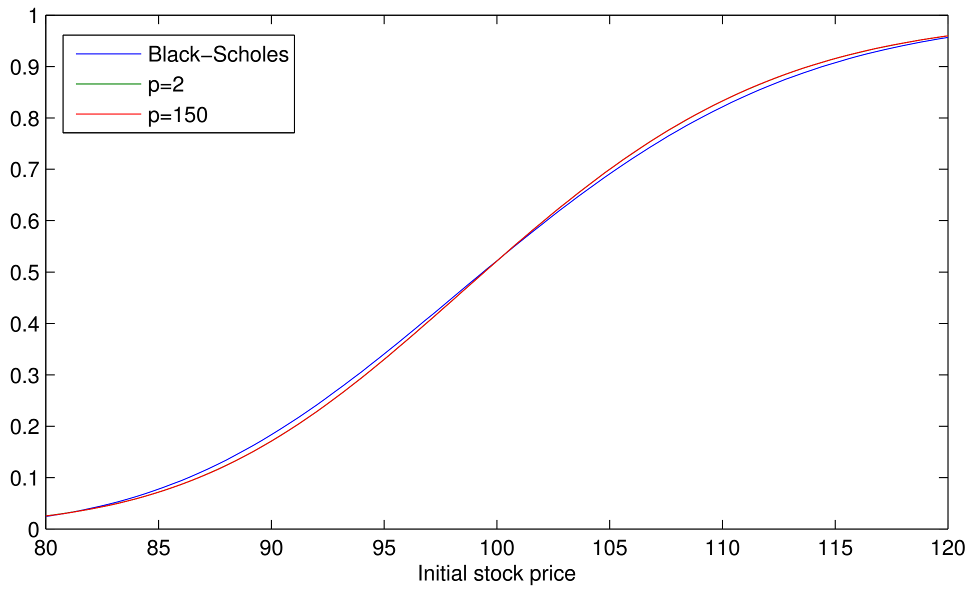


Figure 6.1: Initial Black-Scholes hedge and initial utility-based BNS-hedges for $p = 2$, $p = 150$ and a European call with strike $K = 100$ and maturity $T = 0.25$

For $p = 2$, the effect of the first-order risk adjustment is rather small (cf. Figures 6.2, 6.3 below). This resembles similar findings of Henderson (2002) and Henderson & Hobson (2002) on utility-based pricing and hedging in the presence of basis risk.

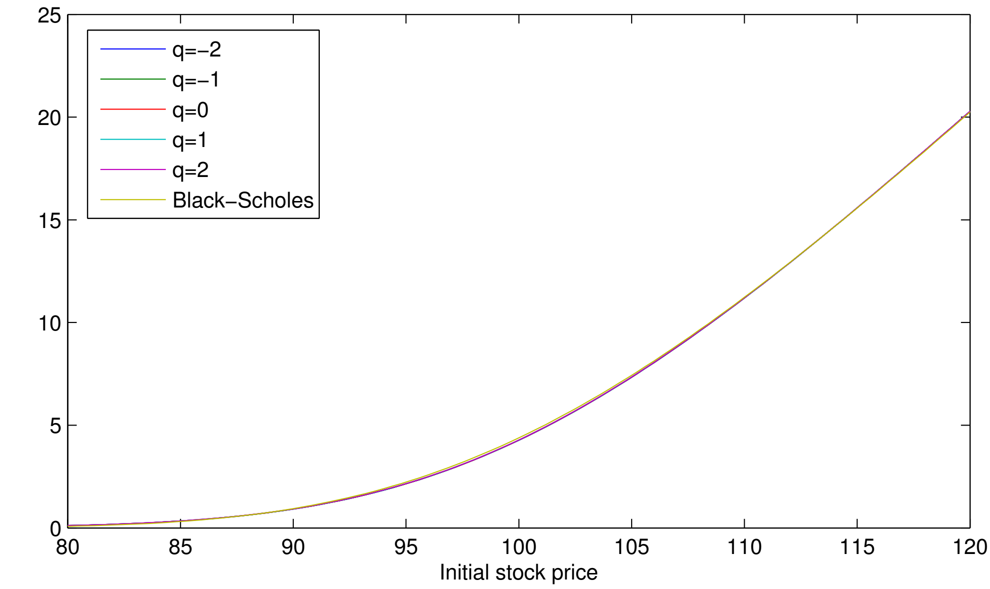


Figure 6.2: Black-Schoples price and approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 2$ and a European call with strike $K = 100$ and maturity $T = 0.25$

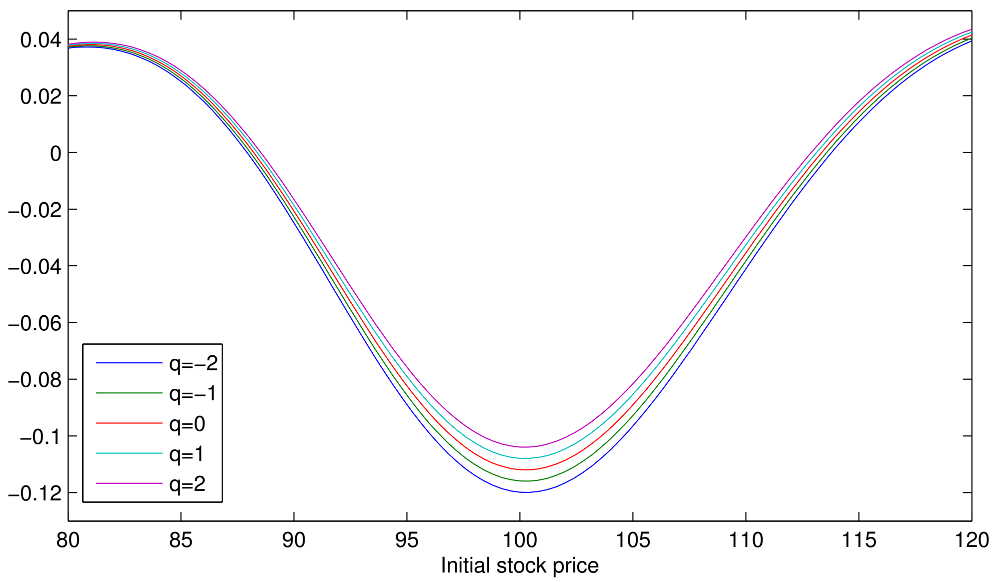


Figure 6.3: Difference between the approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 2$ and the Black-Scholes price for a European call with strike $K = 100$ and maturity $T = 0.25$

On the contrary, for higher risk aversions as e.g. $p = 150$ in Figures 6.4, 6.5 below, the first-order risk adjustment leads to a *bid-price* below and an *ask-price* above the Black-Scholes price for one option.

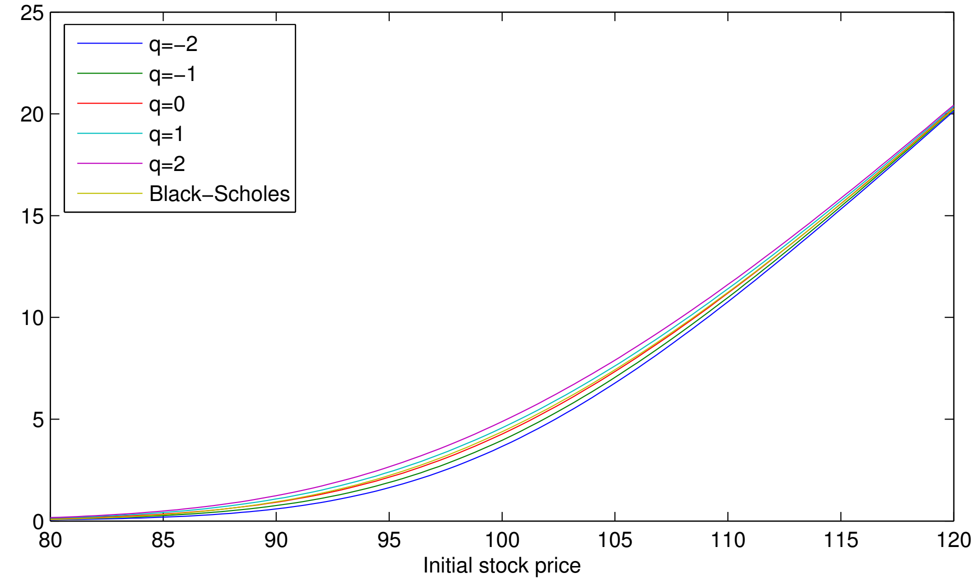


Figure 6.4: Black-Schoples price and approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 150$ and a European call with strike $K = 100$ and maturity $T = 0.25$

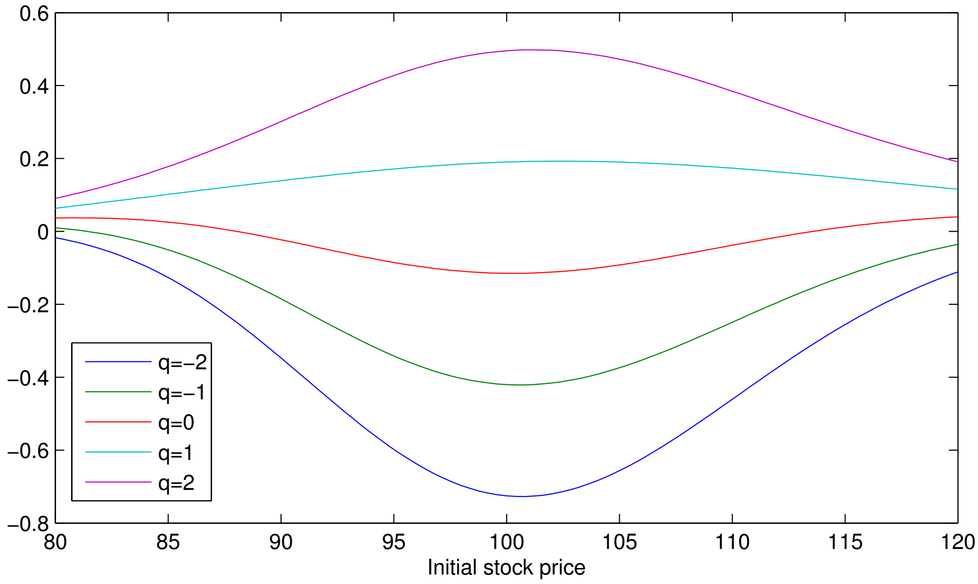


Figure 6.5: Difference between the approximate indifference price $\pi(0) + q\pi'$ in the BNS model for $p = 150$ and the Black-Scholes price for a European call with strike $K = 100$ and maturity $T = 0.25$

# Part II

# Models with transaction costs

# Chapter 7

# On the existence of shadow prices in finite discrete time

## 7.1 Introduction

When considering problems in Mathematical Finance, one classically works with a *frictionless* market, i.e. one assumes that securities can be purchased and sold for the same price $S$. This is clearly a strong modeling assumption, since in reality one usually has to pay a higher *ask price* when purchasing securities, whereas one only receives a lower *bid price* when selling them. In addition, the introduction of even miniscule transaction costs often fundamentally changes the structure of the problem at hand (cf. e.g. Magill & Constantinides (1976), Davis & Norman (1990) and Shreve & Soner (1994) for portfolio optimization, Cvitanić et al. (1995), Levental & Skorohod (1997), Cvitanić et al. (1999), Kabanov (1999) and Jakubėnas et al. (2003) for super-replication, Hodges & Neuberger (1989), Davis et al. (1993), Whalley & Wilmott (1997), Barles & Soner (1998) and Zakamouline (2006) for utility based option pricing and hedging as well as Jouini & Kallal (1995), Kabanov et al. (2002), Schachermayer (2004), Guasoni (2006) and Guasoni et al. (2008a,b) for no-arbitrage). Therefore models with transaction costs have been extensively studied in the literature.

Problems involving transaction costs are usually tackled by one of two different approaches. The first employs methods from stochastic control theory (cf. e.g. Davis & Norman (1990), Shreve & Soner (1994)), whereas the second reformulates the task at hand as a similar problem in a frictionless market. This second approach goes back to the pioneering paper of Jouini & Kallal (1995). They showed that under suitable conditions, a market with bid/ask prices $\underline{S}, \overline{S}$ is arbitrage free if and only if there exists a *shadow price* $\widetilde{S}$ lying within the bid/ask bounds, such that the frictionless market with price process $\widetilde{S}$ is arbitrage free. The same idea has since been employed extensively leading to various other versions of the fundamental theorem of asset pricing in the presence of transaction costs (cf. e.g. Guasoni et al. (2008b) and the references therein). It has also found its way into other branches of Mathematical Finance. For example, Lamberton et al. (1998) have shown that bid/ask prices can be replaced by a shadow price in the context of local risk-minimization, whereas Cvitanić & Karatzas (1996), Cvitanić & Wang (2001) and Loewenstein (2002) prove that the same is true for certain Itô process settings when dealing with optimal portfolios.

In this chapter we establish that in finite discrete time, this *general principle* holds true literally for investment/consumption problems. After introducing our finite market model with proportional transaction costs in Section 7.2, the main result concerning the existence of shadow prices is then stated and proved in Section 7.3.

## 7.2 Utility maximization with transaction costs in finite discrete time

We study the problem of maximizing expected utility from consumption in a finite market model with proportional transaction costs. Our general framework is as follows. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\{0,1,\dots,T\}}, P)$ be a filtered probability space, where $\Omega = \{\omega_1, \dots, \omega_K\}$ and the time set $\{0, 1, \dots, T\}$ are finite. In order to avoid lengthy notation, we let $\mathscr{F} = \mathscr{P}(\Omega)$, $\mathscr{F}_0 = \{\varnothing, \Omega\}$, and assume that $P(\{\omega_k\}) > 0$ for all $k \in \{1, \dots, K\}$. However, one can show that all following statements remain true without these restrictions.

The financial market we consider consists of a risk-free asset $0$ (also called *bank account*) with price process $S^0$ normalized to $S^0_t = 1$, $t = 0, \dots, T$, and risky assets $1, \dots, d$ whose prices are expressed in multiples of $S^0$. More specifically, they are modelled by their (discounted) *bid price process* $\underline{S} = (\underline{S}^1, \dots, \underline{S}^d)$ and their (discounted) *ask price process* $\overline{S} = (\overline{S}^1, \dots, \overline{S}^d)$, where we naturally assume $\overline{S} \geq \underline{S} > 0$. Their meaning should be obvious: if one wants to purchase security $i$ at time $t$, one must pay the higher price $\overline{S}^i_t$ whereas one receives only $\underline{S}^i_t$ for selling it.

**Remark 7.1** This setup amounts to assuming that the risk-free asset can be purchased and sold without incurring any transaction costs. This assumption is commonly made in the literature dealing with optimal portfolios in the presence of transaction costs (cf. Davis & Norman (1990), Shreve & Soner (1994)), and seems reasonable when thinking of security $0$ as a bank account.

For foreign exchange markets where it appears less plausible, a numeraire free approach has been introduced by Kabanov (1999). This approach would, however, require the use of multidimensional utility functions as in Deelstra et al. (2001) in our context.

**Definition 7.2** A *trading strategy* is an $\mathbb{R}^{d+1}$-valued predictable stochastic process $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$, where $\varphi^i_t$, $t = 0, \dots, T$ denotes the number of shares held in security $i$ until time $t$ after rearranging the portfolio at time $t-1$. A (discounted) *consumption process* is an $\mathbb{R}$-valued, adapted stochastic process $c$, where $c_t$, $t = 0, \dots, T$ represents the amount consumed at time $t$. A pair $(\varphi, c)$ of a trading strategy $\varphi$ and a consumption process $c$ is called *portfolio/consumption pair*.

To capture the notion of a self-financing strategy, we use the intuition that no funds are added or withdrawn. More specifically, this means that the proceeds of selling stock must be added to the bank account while the expenses from consumption and the purchase of stock have to be deducted from the bank account whenever the portfolio is readjusted from $\varphi_t$ to $\varphi_{t+1}$ and an amount $c_t$ is consumed at time $t \in \{0, \dots, T-1\}$. Defining purchases and sales at $t$ as

$$\Delta\varphi^{\uparrow,i} := (\Delta\varphi^i)^+, \quad \Delta\varphi^{\downarrow,i} := (\Delta\varphi^i)^-, \quad i = 1, \dots, d, \tag{7.1}$$

this leads to the following

**Definition 7.3** A portfolio/consumption pair $(\varphi, c)$ is called *self-financing* (or $\varphi$ *c-financing*) if

$$\Delta\varphi^0_{t+1} = \sum_{i=1}^{d} \left( \underline{S}^i_t \Delta\varphi^{\downarrow,i}_{t+1} - \overline{S}^i_t \Delta\varphi^{\uparrow,i}_{t+1} \right) - c_t, \quad t = 0, \dots, T-1. \tag{7.2}$$

**Remark 7.4** For $i = 1, \dots, d$, define the total purchases $\varphi^{\uparrow,i}$ and sales $\varphi^{\downarrow,i}$ as

$$\varphi^{\uparrow,i} := (\varphi^i_0)^+ + \sum_{t=1}^{\cdot} \Delta\varphi^{\uparrow,i}_t, \quad \varphi^{\downarrow,i} := (\varphi^i_0)^- + \sum_{t=1}^{\cdot} \Delta\varphi^{\downarrow,i}_t.$$

Then the self-financing condition (7.2) can equivalently be represented as

$$\varphi^0 = \varphi^0_0 + \int_0^\cdot \underline{S}_{t-} d\varphi^{\downarrow}_t - \int_0^\cdot \overline{S}_{t-} d\varphi^{\uparrow}_t - \int_0^\cdot c_t dA_t,$$

or, using integration by parts in the sense of (JS, I.4.45, I.4.49b), as

$$\varphi^0 + \overline{S}^\top \varphi^\uparrow - \underline{S}^\top \varphi^\downarrow = \varphi^0_0 + \left( \overline{S}^\top_0 \varphi^\uparrow_0 - \underline{S}^\top_0 \varphi^\downarrow_0 \right) + \left( \int_0^\cdot \varphi^\uparrow_t d\overline{S}_t - \int_0^\cdot \varphi^\downarrow_t d\underline{S}_t \right) - \int_0^\cdot c_t dA_t$$

for $A_t = \sum_{s \leq t} 1_{\mathbb{N}}(s)$. This means that the pair $\left((\varphi^0, \varphi^\uparrow, -\varphi^\downarrow), c\right)$ is self-financing in the usual sense for a frictionless market with $2d+1$ securities $(1, \overline{S}, \underline{S})$. Note that for $\underline{S} = \overline{S}$, we recover the usual self-financing condition for frictionless markets (cf. e.g. (Karatzas & Shreve, 1988, Section 5.8)). Moreover, this alternative formulation also makes sense in continuous time, where it can be used to define self-financing strategies (cf. Chapter 8).

We consider an investor who disposes of an *initial endowment* $(\zeta_0, \dots, \zeta_d) \in \mathbb{R}^{d+1}_+$, referring to the initial number of securities of type $i$, $i = 0, \dots, d$, respectively. To rule out infinite consumption at time $T$, we require that the investor is able to cover her consumption by liquidating her portfolio at the terminal time $T$. Again this corresponds to admissibility without transaction costs if $\overline{S} = \underline{S}$ (c.f. e.g. Pliska (1997)).

**Definition 7.5** The *(liquidation) value process* of a self-financing portfolio/consumption pair $(\varphi, c)$ is defined as

$$V(\varphi) := \varphi^0 + \sum_{i=1}^{d} \left( (\varphi^i)^+ \underline{S}^i - (\varphi^i)^- \overline{S}_i \right).$$

A self-financing portfolio/consumption pair $(\varphi, c)$ is called *admissible* if $(\varphi_0^0, \varphi_0^1, \dots, \varphi_0^d) = (\zeta_0, \zeta_1, \dots, \zeta_d)$ and $c_T = V_T(\varphi)$. An admissible portfolio/consumption pair $(\varphi, c)$ is called *optimal* if it maximizes

$$\kappa \mapsto E\left(\sum_{t=0}^{T} u_t(\kappa_t)\right) \tag{7.3}$$

over all admissible portfolio/consumption pairs $(\psi, \kappa)$, where the *utility process* $u$ is a mapping $u : \Omega \times \{0, \dots, T\} \times \mathbb{R} \to [-\infty, \infty)$, such that $(\omega, t) \mapsto u_t(\omega, x)$ is predictable for any $x \in \mathbb{R}$ and $x \mapsto u_t(\omega, x)$ is a proper, upper-semicontinuous, concave function for any $(\omega, t) \in \Omega \times \{0, \dots, T\}$, which is strictly increasing on its effective domain $\{x \in \mathbb{R} : u_t(\omega, x) > -\infty\}$.

**Remark 7.6** Since we allow the utility process to be random, assuming $S_t^0 = 1$, $t = 0, \dots, T$ does not entail a loss of generality in the present setup. More specifically, let $S^0$ be an arbitrary strictly positive, predictable process. In this undiscounted case a portfolio/consumption pair $(\varphi, c)$ should be called *self-financing* if

$$\Delta\varphi_{t+1}^0 S_t^0 = \sum_{i=1}^{d} \left(\underline{S}_t^i \Delta\varphi_{t+1}^{\downarrow,i} - \overline{S}_t^i \Delta\varphi_{t+1}^{\uparrow,i}\right) - c_t,$$

for $t = 0, \dots, T-1$. *Admissibility* is defined as before but in terms of the *liquidation value process*

$$V(\varphi) := \varphi^0 S^0 + \sum_{i=1}^{d} \left((\varphi^i)^+ \underline{S}^i - (\varphi^i)^- \overline{S}^i\right).$$

By direct calculations, one easily verifies that $(\varphi, c)$ is self-financing resp. admissible if and only if $(\varphi, \hat{c}) = (\varphi, c/S^0)$ is self-financing resp. admissible relative to the discounted processes $\hat{S}^0 := S^0/S^0 = 1$, $\hat{\overline{S}} := \overline{S}/S^0$ and $\hat{\underline{S}} := \underline{S}/S^0$. In view of

$$E\left(\sum_{t=0}^{T} u_t(c_t)\right) = E\left(\sum_{t=0}^{T} \hat{u}_t(\hat{c}_t)\right)$$

for the utility process $\hat{u}_t(x) = u_t(S^0 x)$, the problem of maximizing undiscounted utility with respect to $u$ is equivalent to maximizing discounted expected utility with respect to $\hat{u}$.

## 7.3 Existence of shadow prices

We now introduce the central concept of this part of the thesis.

**Definition 7.7** We call an adapted process $\widetilde{S}$ *shadow price process* if

$$\underline{S} \leq \widetilde{S} \leq \overline{S}$$

and if the maximal expected utilities in the market with bid/ask-prices $\underline{S}, \overline{S}$ and in the market with price process $\widetilde{S}$ *without* transaction costs coincide.

The following theorem shows that in our finite market model, shadow price processes exist if there is an optimal portfolio with finite expected utility.

**Theorem 7.8** *Suppose an optimal portfolio/consumption pair $(\varphi, c)$ exists for the market with bid/ask prices $\underline{S}, \overline{S}$. Then if $E(\sum_{t=0}^{T} u_t(c_t)) > -\infty$, a shadow price process $\widetilde{S}$ exists.*

PROOF. *First step*: Demanding the consumption of the liquidation value of the portfolio at time $T$ is equivalent to requiring the portfolio to be liquidated at $T$ in a self-financing way. More specifically, it follows by insertion that we can identify the set of admissible portfolio/consumption pairs $((\phi_t)_{t=0,\dots,T}, (\kappa_t)_{t=0,\dots,T})$ with the set of all $((\widetilde{\phi}_{t=0,\dots,T+1}), (\widetilde{\kappa})_{t=0,\dots T})$, where $(\widetilde{\phi}_t)_{t=0,\dots,T+1}$ is an $\mathbb{R}^{d+1}$-valued predictable process with $\widetilde{\phi}^i_0 = \zeta_i$, $\widetilde{\phi}^i_{T+1} = 0$ for $i = 0, \dots, d$ and $(\widetilde{\kappa}_t)_{t=0,\dots,T}$ is a consumption process such that (7.2) holds for $t = 0, \dots, T$.

*Second step*: Next, notice that since the given utility process is increasing, no utility can be gained by allowing sales and purchases of the same asset at the same time. Formally, by the first step and since $x \mapsto u_t(x)$ is increasing for fixed $t$, maximizing (7.3) over all admissible portfolio/consumption pairs yields the same maximal expected utility as maximizing (7.3) over all $((\phi^0, \phi^\uparrow, \phi^\downarrow), \kappa)$, where $(\phi^0(t))_{t=0,\dots,T+1}$ is an $\mathbb{R}$-valued predictable process with $\phi^0_0 = \zeta_0$ and $\phi^0_{T+1} = 0$, the increasing, $\mathbb{R}^d$-valued predictable processes $(\phi^\uparrow_t)_{t=0,\dots,T+1}$ and $(\phi^\downarrow_t)_{t=0,\dots,T+1}$ satisfy $\phi^{\uparrow,i}_0 = (\zeta_i)^+$, $\phi^{\downarrow,i}_0 = (\zeta_i)^-$, $\phi^{\uparrow,i}_{T+1} - \phi^{\downarrow,i}_{T+1} = 0$ for $i = 1, \dots, d$ and $(\kappa_t)_{t=0,\dots,T}$ is a consumption process such that (7.2) holds for $t = 0, \dots, T$. Moreover, if we define $\Delta\varphi^\uparrow$ and $\Delta\varphi^\downarrow$ as in (7.1) above and set

$$\varphi^\uparrow := ((\zeta^1)^+, \dots, (\zeta^d)^+) + \sum_{t=1}^{\cdot} (\Delta\varphi^{\uparrow,1}_t, \dots, \Delta\varphi^{\uparrow,d}_t),$$

$$\varphi^\downarrow := ((\zeta^1)^-, \dots, (\zeta^d)^-) + \sum_{t=1}^{\cdot} (\Delta\varphi^{\downarrow,1}_t, \dots, \Delta\varphi^{\downarrow,d}_t),$$

then $((\varphi^0, \varphi^\uparrow, \varphi^\downarrow), c)$ is an optimal strategy in this set.

*Third step*: Denote by $F_{t,1}, \dots, F_{t,m_t}$ the partition of $\Omega$ that generates $\mathscr{F}_t$, $t \in \{0, \dots, T\}$. Since a mapping is $\mathscr{F}_t$-measurable if and only if it is constant on the sets $F_{t,j}$, $j = 1, \dots, m_t$, we can identify the set of all $((\phi^0, \phi^\uparrow, \phi^\downarrow), \kappa)$, where $(\phi^0_t)_{t=0,\dots,T+1}$ is $\mathbb{R}$-valued and predictable with $\phi^0_0 = \zeta_0$, $(\phi^\uparrow_t)_{t=0,\dots,T+1}$ and $(\phi^\downarrow_t)_{t=0,\dots,T+1}$ are increasing, $\mathbb{R}^d$-valued and predictable with $\phi^{\uparrow,i}_0 = (\zeta_i)^+$, $\phi^{\downarrow,i}_0 = (\zeta_i)^-$ and $(\kappa_t)_{t=0,\dots,T}$ is a consumption process such that (7.2) holds for $t = 0, \dots, T$ with

$$\mathbb{R}^{2dn}_+ \times \mathbb{R}^n := (\mathbb{R}^{m_0 d}_+ \times \dots \times \mathbb{R}^{m_T d}_+) \times (\mathbb{R}^{m_0 d}_+ \times \dots \times \mathbb{R}^{m_T d}_+) \times (\mathbb{R}^{m_0} \times \dots \times \mathbb{R}^{m_T}),$$

and vice versa, namely with

$$(\Delta\phi^\uparrow, \Delta\phi^\downarrow, c) := (\Delta\phi^{\uparrow,1,1}_1, \dots, \Delta\phi^{\uparrow,d,m_T}_{T+1}, \Delta\phi^{\downarrow,1,1}_1, \dots, \Delta\phi^{\downarrow,d,m_T}_{T+1}, c^1_0, \dots, c^{m_T}_T),$$

where we use the notation $\Delta\phi^{\uparrow,i,j}_t := \Delta\varphi^{\uparrow,i}_t(\omega)$ for $i = 1, \dots, d$, $t = 0, \dots, T$, $j = 1, \dots, m_t$ and $\omega \in F_{t,j}$ (and analogously for $\Delta\phi^\downarrow$, $c$, $\underline{S}$, $\overline{S}$). Using this identification, we can then define mappings $f : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $k^j : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R}$ and $h^{i,j} : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R}$ (for $i = 1, \dots, d$, $j = 1, \dots, m_T$) by

$$\begin{aligned} f(\Delta\underline{\phi}, \Delta\overline{\phi}, c) &:= -E\left(\sum_{t=1}^T u_t(c_t)\right), \\ k^j(\Delta\underline{\phi}, \Delta\overline{\phi}, c) &:= \zeta_0 + \sum_{t=1}^T \left(\sum_{i=1}^d \underline{S}^{i,j}_{t-1}\Delta\phi^{\downarrow,i,j}_t - \overline{S}^{i,j}_{t-1}\Delta\phi^{\uparrow,i,j}_t\right) - \sum_{t=0}^T c^j_t, \\ h^{i,j}(\Delta\phi^{\uparrow}, \Delta\phi^{\downarrow}, c) &:= \zeta_j + \sum_{t=1}^{T+1}\left(\Delta\phi^{\uparrow,i,j}_t - \Delta\phi^{\downarrow,i,j}_t\right). \end{aligned}$$

With this notion, $(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$ is optimal if and only if it minimizes $f$ over $\mathbb{R}^{2dn}_+ \times \mathbb{R}^n$ subject to the constraints $k^j = 0$ and $h^{i,j} = 0$ (for $i = 1, \dots, d, j = 1, \dots, m_T$). Since all mappings are actually convex functions on $\mathbb{R}^{(2d+1)n}$, this is equivalent to $(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$ minimizing $f$ over $\mathbb{R}^{(2d+1)n}$ subject to the constraints $k^j = 0$, $h^{i,j} = 0$ (for $i = 1, \dots, d$ and $j = 1, \dots, m_T$) and $g^{\uparrow,i,j}_t, g^{\downarrow,i,j}_t \le 0$ (for $t = 0, \dots, T$, $i = 1, \dots, d$ and $j = 1, \dots, m_t$), where the convex mappings $g^{\uparrow,i,j}_t, g^{\downarrow,i,j}_t : \mathbb{R}^{(2d+1)n} \to \mathbb{R}$ are given by

$$g^{\uparrow,i,j}_t(\Delta\phi^{\uparrow}, \Delta\phi^{\downarrow}, c) := -\Delta\phi^{\uparrow,i,j}_{t+1}, \quad g^{\downarrow,i,j}_t(\Delta\phi^{\uparrow}, \Delta\phi^{\downarrow}, c) := -\Delta\phi^{\downarrow,i,j}_{t+1}.$$

In view of (Rockafellar, 1970, Theorems 28.2 and 28.3) $(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$ is therefore optimal if and only if there exists a Lagrange multiplier, i.e. real numbers $\nu^j$, $\mu^{i,j}$ (for $i = 1, \dots, d$ and $j = 1, \dots, m_T$) and $\lambda^{\uparrow,i,j}_t, \lambda^{\downarrow,i,j}_t$ (for $t = 0, \dots, T$, $i = 1, \dots, d$ and $j = 1, \dots, m_t$) such that the following holds.

1. $\lambda^{\uparrow,i,j}_t, \lambda^{\downarrow,i,j}_t \ge 0$, $\lambda^{\uparrow,i,j}_t g^{\uparrow,i,j}_t(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $\lambda^{\downarrow,i,j}_t g^{\downarrow,i,j}_t(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $t = 0, \dots, T$, $i = 1, \dots, d$ and $j = 1, \dots, m_t$.

2. $k^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $h^{i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $i = 1, \dots, d$, $j = 1, \dots, m_T$.

3. $$\begin{aligned} 0 \in &\partial f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{j=1}^{m_T} \nu^j \partial k^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{i=1}^d\sum_{j=1}^{m_T} \mu^{i,j}\partial h^{i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \\ &+ \sum_{t=0}^T\sum_{i=1}^d\sum_{j=1}^{m_t} \lambda^{\uparrow,i,j}_t \partial g^{\uparrow,i,j}_t(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{t=0}^T\sum_{i=1}^d\sum_{j=1}^{m_t} \lambda^{\downarrow,i,j}_t \partial g^{\downarrow,i,j}_t(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c). \end{aligned}$$

Here, $\partial$ denotes the subdifferential of a convex mapping $\mathbb{R}^{(2d+1)n} \to \mathbb{R}$ (cf. Rockafellar (1970) for more details).

*Fourth step*: By (Rockafellar & Wets, 1998, Proposition 10.5) we can split Statement 3 into many similar statements where the subdifferentials on the right-hand side are replaced with partial subdifferentials relative to $\Delta\varphi^{\uparrow,1,1}_1, \dots, \Delta\varphi^{\uparrow,d,m_T}_{T+1}, \Delta\varphi^{\downarrow,1,1}_1, \dots, \Delta\varphi^{\downarrow,d,m_T}_{T+1}$, $c^1_t, \dots, c^{m_T}_T$, respectively. In particular, for $c^j_T$, $j \in \{1, \dots, m_T\}$, we obtain

$$0 \in \partial_{c^j_T} f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) - \nu^j, \tag{7.1}$$

where $\partial_x$ denotes the partial subdifferential of a convex function relative to a vector $x$. Hence $\nu^j < 0$, $j = 1, \ldots, m_T$, because $f$ is strictly decreasing in $c_T^j$. Furthermore, since the mappings $g_t^{\uparrow,i,j}, g_t^{\downarrow,i,j}$ (for $t = 0, \ldots, T$, $i = 1, \ldots, d$ and $j = 1, \ldots, m_t$) and $k^j, h^{i,j}$ (for $i = 0, \ldots, d$ and $j = 1, \ldots, m_T$) are differentiable, their partial subdifferentials coincide with the respective partial derivatives by (Rockafellar, 1970, Theorem 25.1). Hence, taking partial derivatives with respect to $\Delta\varphi_{t+1}^{\uparrow,i,j}$ resp. $\Delta\varphi_{t+1}^{\downarrow,i,j}$, $i \in \{0, \ldots, d\}$, $t \in \{0, \ldots, T\}$, $j \in \{1, \ldots, m_t\}$, Statement 3 above implies that

$$\begin{aligned} 0 &= \sum_{k:\omega_k \in F_{t,j}} \mu^{i,j} - \left( \sum_{k:\omega_k \in F_{t,j}} \nu^k \right) \overline{S}_t^{i,j} - \underline{\lambda}_t^{\uparrow,i,j} \\ &= \sum_{k:\omega_k \in F_{t,j}} \mu^{i,j} - \sum_{k:\omega_k \in F_{t,j}} \nu^k \left( 1 + \frac{\lambda_t^{\uparrow,i,j}}{\overline{S}_t^{i,j} \sum_{k:\omega_k \in F_{t,j}} \nu^k} \right) \overline{S}_t^{i,j}, \end{aligned} \tag{7.2}$$

and likewise

$$0 = \sum_{k:\omega_k \in F_{t,j}} \mu^{i,j} - \sum_{k:\omega_k \in F_{t,j}} \nu^k \left( 1 - \frac{\lambda_t^{\downarrow,i,j}}{\underline{S}_t^{i,j} \sum_{k:\omega_k \in F_{t,j}} \nu^k} \right) \underline{S}_t^{i,j}. \tag{7.3}$$

In particular we have, for $i = 1, \ldots, d$, $t = 0, \ldots, T$, $j = 1, \ldots, m_t$,

$$\left( 1 + \frac{\lambda_t^{\uparrow,i,j}}{\overline{S}_t^{i,j} \sum_{k:\omega_k \in F_{t,j}} \nu^k} \right) \overline{S}_t^{i,j} = \left( 1 - \frac{\lambda_t^{\downarrow,i,j}}{\underline{S}_t^{i,j} \sum_{k:\omega_k \in F_{t,j}} \nu^k} \right) \underline{S}_t^{i,j} =: \widetilde{S}_t^{i,j}.$$

Since $\widetilde{S}$ is constant on $F_{t,j}$ by definition, this defines an adapted process. Furthermore, we have $\underline{S} \leq \widetilde{S} \leq \overline{S}$, by Statement 1 above and because $\nu^k < 0$ for $k = 1, \ldots, m_T$. Moreover, Statement 1 above also implies that

$$\widetilde{S} = \overline{S} \text{ on } \{\Delta\varphi^{\uparrow} > 0\}, \quad \widetilde{S} = \underline{S} \text{ on } \{\Delta\varphi^{\downarrow} > 0\}. \tag{7.4}$$

Set $\widetilde{\mu}^{i,j} := \mu^{i,j}$ (for $i = 1, \ldots, d$, $j = 1, \ldots, m_T$), $\widetilde{\nu}^j := \nu^j$ (for $j = 1, \ldots, m_T$) and $\widetilde{\lambda}_t^{\uparrow,i,j}, \widetilde{\lambda}_t^{\downarrow,i,j} := 0$ (for $i = 1, \ldots, d$, $t = 0, \ldots, T$, $j = 1, \ldots, m_t$). It then follows from Statements 1,2,3 above, Equations (7.2), (7.3), (7.4) and the definition of $\widetilde{S}$ that

1. $\tilde{\lambda}_t^{\uparrow,i,j}, \tilde{\lambda}_t^{\downarrow,i,j} \geq 0$ and $\tilde{\lambda}_t^{\uparrow,i,j} \tilde{g}_t^{\uparrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c), \widetilde{\lambda}_t^{\downarrow,j,i} \widetilde{g}_t^{\downarrow,i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $t = 0, \ldots, T$, $i = 1, \ldots, d$ and $j = 1, \ldots, m_{t-1}$,
2. $\widetilde{h}^{i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $\widetilde{k}^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $i = 1, \ldots, d$, $j = 1, \ldots, m_T$,
3. $$\begin{aligned} 0 \in &\partial f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{j=1}^{m_T} \widetilde{\nu}^j \partial \widetilde{k}^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{i=1}^{d} \sum_{j=1}^{m_T} \widetilde{\mu}^{i,j} \partial h^{i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \\ &- \sum_{t=0}^{T} \sum_{i=1}^{d} \sum_{j=1}^{m_t} \widetilde{\lambda}_t^{\uparrow,i,j} \partial g_t^{\uparrow,i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) - \sum_{t=0}^{T} \sum_{i=1}^{d} \sum_{j=1}^{m_t} \widetilde{\lambda}_t^{\downarrow,i,j} \partial g_t^{\downarrow,i,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c), \end{aligned}$$

where the mappings $\widetilde{f}, \widetilde{k}^j, \widetilde{h}^{i,j}, \widetilde{g}_t^{\uparrow,i,j}, \widetilde{g}_t^{\downarrow,i,j}$ are defined by setting $\underline{S} = \overline{S} = \widetilde{S}$ in the definition of the mappings $f, k^j, h^{i,j}, g_t^{\uparrow,i,j}, g_t^{\downarrow,i,j}$ above. In view of (Rockafellar, 1970, Theorem 28.3) and Steps 1-3 above, $(\varphi, c)$ is therefore not only optimal in the market with bid/ask prices $\underline{S}, \overline{S}$, but in the market with bid-ask prices $\widetilde{S}, \widetilde{S}$ (i.e. in the frictionless market with price process $\widetilde{S}$) as well. Hence $\widetilde{S}$ is a shadow price process and we are done. □

**Remark 7.9** An analogue of Theorem 7.8 for utility from terminal wealth can be obtained by considering the objective function $f((\Delta\underline{\varphi}, \Delta\overline{\varphi}), c)) := -E(u_T(c_T))$ subject to the additional constraints $c_1 = \ldots = c_{T-1} = 0$.

**Corollary 7.10 (Fundamental Theorem of Utility Maximization with transaction costs)** *Let $(\varphi, c)$ be an admissible portfolio consumption pair for the market with bid/ask prices $\underline{S}, \overline{S}$ satisfying $E(\sum_{t=0}^T u_t(c_t)) > -\infty$. Then we have equivalence between:*

1. *$(\varphi, c)$ is optimal in the market with bid/ask prices $\underline{S}, \overline{S}$.*
2. *There exists an adapted process $\widetilde{S}$ with $\underline{S} \leq \widetilde{S} \leq \overline{S}$, a number $\kappa \in (0, \infty)$ and an probability measure $Q_0 \sim P$ such that $\widetilde{S}$ is a $Q$-martingale and*

$$E\left(\frac{dQ}{dP}\middle|\mathscr{F}_t\right) \in \frac{1}{\kappa}\partial u_t(c_t), \quad t = 0, \ldots, T.$$

PROOF. $1 \Rightarrow 2$: This follows from Theorem 7.8 combined with Kallsen (1998), Theorem 3.5, Remark 3 after Theorem 3.7 and Definition 2.3.

$2 \Rightarrow 1$: By (Kallsen, 1998, Theorem 3.5), Statement 2 above is equivalent to $(\varphi, c)$ being optimal in the frictionless market with price process $\widetilde{S}$. Let $(\phi, k)$ be any admissible portfolio consumption pair in the market with bid/ask prices $\underline{S}, \overline{S}$. Define $\Delta\phi_t^{\uparrow,i} := (\Delta\phi_t^i)^+$, $\Delta\phi_t^{\downarrow,i} := (\Delta\phi_t^i)^-$, $t = 1, \ldots, T$ as above and let

$$\widetilde{k}(t) := k(t) + \sum_{i=1}^d \left(\Delta\phi_t^{\uparrow,i}(\overline{S}_t^i - \widetilde{S}_t^i) + \Delta\phi_t^{\downarrow,i}(\widetilde{S}_t^i - \underline{S}_t^i)\right).$$

Then $\widetilde{k} \geq k$ since $\underline{S} \leq \widetilde{S} \leq \overline{S}$ and $(\phi, \widetilde{k})$ is a self-financing portfolio/consumption pair in the frictionless market with price process $\widetilde{S}$, i.e. with bid/ask-prices $\widetilde{S}, \widetilde{S}$. Since $(\varphi, c)$ is optimal in this market, we have

$$E\left(\sum_{t=0}^T u_t(k_t)\right) \leq E\left(\sum_{t=0}^T u_t(\widetilde{k}_t)\right) \leq E\left(\sum_{t=0}^T u_t(c_t)\right).$$

Therefore $((\underline{\varphi}, \overline{\varphi}), c)$ is optimal in the market with bid/ask prices $\underline{S}, \overline{S}$. □

**Remarks.**

1. If, for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$, $x \mapsto u_t(\omega, x)$ is differentiable on its effective domain with derivative $u'$, $E(\frac{dQ}{dP}|\mathscr{F}_t) \in \frac{1}{\kappa}\partial u_t(c_t)$ reduces to $E(\frac{dQ}{dP}|\mathscr{F}_t) = \frac{1}{\kappa}u_t'(c_t)$.
2. The pair $(\widetilde{S}, Q)$ consisting of the shadow price process $\widetilde{S}$ and the corresponding dual martingale measure $Q$ is called a *consistent price system* by Guasoni et al. (2008b).

# Chapter 8

# On using shadow prices in utility maximization with transaction costs

## 8.1 Introduction

In this chapter we consider a continuous-time version of the *Merton problem* with transaction costs introduced in Chapter 7 for finite probability spaces. More precisely, we deal with maximizing utility from consumption over an infinite horizon in the presence of proportional transaction costs.

As for the related problem of maximizing utility from terminal wealth, this problem was was solved in *frictionless* markets by Merton (1969, 1971) for power and logarithmic utility functions in a Markovian Itô process model. In a market with a riskless bank account and one risky asset following a geometric Brownian motion, the optimal strategy turns out to invest a *constant fraction* $\eta^*$ of wealth in the risky asset and to consume at a rate proportional to current wealth. This means that it is optimal for the investor to keep her portfolio holdings in bank and stock on the so-called *Merton line* with slope $\eta^*/(1-\eta^*)$.

Since then, this problem has been generalized in several ways. One direction has been to consider different market models (cf. Chapter 4 and the references therein). In this case solutions to utility maximization problems are generally obtained by two different methods. One approach is to use stochastic control theory, which leads to Hamilton-Jacobi-Bellman equations. Alternatively, one can turn to martingale methods which appear in different forms, both in actual computations and in general structural results.

A different generalization of the Merton problem is the introduction of proportional transaction costs. In a continuous time setting this was first done by Magill & Constantinides (1976). Their paper contains the fundamental insight that it is optimal to refrain from transacting while the portfolio holdings remain in a wedge-shaped *no-transaction region*, i.e. while the fraction of wealth held in stock lies inside some interval $[\eta_1^*, \eta_2^*]$. However, their solution is derived in a somewhat heuristic way and also did not show how to compute the location of the boundaries $\eta_1^*$, $\eta_2^*$.

Mathematically rigorous results were first obtained in the seminal paper of Davis & Norman (1990). They show that it is indeed optimal to keep the proportion of total wealth held in stock between fractions $\eta_1^*$, $\eta_2^*$ and they also prove that these two numbers can be determined as the solution to a free boundary value problem. The theory of viscosity solutions to Hamilton-Jacobi-Bellman equations was introduced to this problem by Shreve & Soner (1994) who succeeded in removing several assumptions needed in Davis & Norman (1990). Since then, this approach has also been used to compute optimal portfolios in several variants of the Merton problem with proportional transaction costs, e.g. in the finite horizon case (cf. Akian et al. (1995), Liu & Loewenstein (2002), Dai et al. (2009)), the case of multiple stocks (cf. Akian et al. (1995)) and stocks modelled as jump diffusions (cf. Framstad et al. (2001)).

All these articles aiming for the computation of the optimal portfolio employ tools from stochastic control. It seems that martingale methods have so far only been used to obtain structural existence results in the presence of transaction costs. In this context the martingale and duality theory for frictionless markets is often applied to a *shadow price process* $\widetilde{S}$ lying within the bid-ask bounds of the real price process $S$. Economically speaking, the frictionless price process $\widetilde{S}$ and the original price process $S$ with transaction costs lead to identical decisions and gains for the investor under consideration. We refer the reader to Chapter 7 and the references therein for a brief survey of applications of this concept in different areas of Mathematical Finance. For the particular case of utility maximization, the existence of a shadow price has been established in finite discrete time (cf. Theorem 7.8) as well as for certain Itô process settings (cf. Cvitanić & Karatzas (1996), Cvitanić & Wang (2001) and Loewenstein (2002)).

In this chapter, we reconsider Merton's problem for logarithmic utility and under proportional transaction costs as in Davis & Norman (1990). Our goal is threefold. Most importantly, we show that the shadow price approach can be used to come up with a candidate solution to the utility maximization problem under transaction costs. Moreover, the ensuing verification verification procedure appears — at least for the problem at hand — to be surprisingly simple compared to the very impressive and non-trivial reasoning in Davis & Norman (1990) and Shreve & Soner (1994). Finally, we also construct the shadow price as part of the solution.

The more involved case of power utility is treated in Davis & Norman (1990), Shreve & Soner (1994) as well. The application of the present approach to this case is subject of current research. While it is still possible to come up with a candidate for the shadow price, the corresponding free boundary problem appears to be more difficult than its counterpart in Davis & Norman (1990). This stems from the fact that it may be more difficult to determine the shadow price than the optimal strategy for power utility (cf. Remark 8.14 for more details).

This chapter is organized as follows. The setup is introduced in Section 8.2. Subsequently, we heuristically derive the free-boundary problem that characterizes the solution. Verification is done in Section 8.4. Finally, we present some numerical results.

## 8.2 The Merton Problem with transaction costs

We study the problem of maximizing expected logarithmic utility from consumption over an infinite horizon in the presence of proportional transaction costs. Except for a slightly larger class of admissible strategies we work in the setup of Davis & Norman (1990).

The mathematical framework is as follows: Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a fixed complete, filtered probability space in the sense of (JS, I.1.2), supporting a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$. Our market consists of two investment opportunities: a bank account or bond with constant value 1 and a risky asset ("stock") whose discounted price process $S$ is modelled as a geometric Brownian motion, i.e.

$$S_t := S_0\mathscr{E}(\mu I + \sigma W)_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \tag{8.1}$$

with $I_t := t$ and constants $S_0, \sigma > 0$, $\mu \in \mathbb{R}$. We consider an investor who disposes of an *initial endowment* $(\zeta_B, \zeta_S) \in \mathbb{R}^2_+$, referring to the number of bonds and stocks, respectively. Whenever stock is purchased or sold, transaction costs are imposed equal to a constant fraction of the amount transacted, the fractions being $\overline{\lambda} \in [0, \infty)$ on purchase and $\underline{\lambda} \in [0, 1)$ on sale, not both being equal to zero. Since transactions of infinite variation lead to instantaneous ruin, we limit ourselves to the following set of strategies.

**Definition 8.1** A *trading strategy* is an $\mathbb{R}^2$-valued predictable process $\varphi = (\varphi^0, \varphi^1)$ of finite variation, where $\varphi^0_t$ and $\varphi^1_t$ denote the number of shares held in the bank account and in stock at time $t$ respectively. A (discounted) *consumption rate* is an $\mathbb{R}_+$-valued, adapted stochastic process $c$ satisfying $\int_0^t c_s ds < \infty$ a.s. for all $t \geq 0$. A pair $(\varphi, c)$ of a trading strategy $\varphi$ and a consumption rate $c$ is called *portfolio/consumption pair*.

As in Chapter 7, we use the intuition that no funds are added or withdrawn to capture the notion of a self-financing strategy. To this end, we write the second component $\varphi^1$ of any strategy $\varphi$ as difference $\varphi^1 = \varphi^{\uparrow} - \varphi^{\downarrow}$ of two increasing processes $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ which do not grow at the same time. Moreover, we denote by

$$\underline{S} := (1 - \underline{\lambda})S, \quad \overline{S} := (1 + \overline{\lambda})S \tag{8.2}$$

the bid and ask price of the stock, respectively. The proceeds of selling stock must be added to the bank account while the expenses from consumption and the purchase of stock have to be deducted from the bank account in any infinitesimal period $(t - dt, t]$, i.e. we require

$$d\varphi^0_t = \underline{S}_{t-} d\varphi^{\downarrow}_t - \overline{S}_{t-} d\varphi^{\uparrow}_t - c_t dt. \tag{8.3}$$

for *self-financing strategies*. Written in integral terms this amounts to

$$\varphi^0 = \varphi^0_0 + \int_0^{\cdot} \underline{S}_{t-} d\varphi^{\downarrow}_t - \int_0^{\cdot} \overline{S}_{t-} d\varphi^{\uparrow}_t - \int_0^{\cdot} c_t dt. \tag{8.4}$$

In our setup (8.1,8.2) we obviously have $\underline{S}_- = \underline{S}$ and $\overline{S}_- = \overline{S}$ but the above definition makes sense for discontinuous bid and ask price processes $\underline{S}, \overline{S}$ as well. The second and third term on the right-hand side represent the cumulative amount of wealth gained selling respectively spent buying stock, while the last term represents cumulated consumption.

At this stage it may not be entirely obvious how the integrals in (8.4) are defined because $\varphi^\uparrow$, $\varphi^\downarrow$ are generally neither left nor right continuous. Continuing on an intuitive level, we write the second term on the right-hand side of (8.3) as

$$\begin{aligned}\overline{S}_{t-}d\varphi_t^\uparrow &= \overline{S}_{t-dt}(\varphi_t^\uparrow - \varphi_{t-dt}^\uparrow)\\ &= \varphi_t^\uparrow\overline{S}_t - \varphi_{t-dt}^\uparrow\overline{S}_{t-dt} - \varphi_t^\uparrow(\overline{S}_t - \overline{S}_{t-dt})\\ &= d(\varphi^\uparrow\overline{S})_t - \varphi_t^\uparrow d\overline{S}_t,\end{aligned}$$

which means

$$\int_0^\cdot \overline{S}_{t-}d\varphi_t^\uparrow = \varphi^\uparrow\overline{S} - \varphi_0^\uparrow\overline{S}_0 - \int_0^\cdot \varphi_t^\uparrow d\overline{S}_t \tag{8.5}$$

in integral terms. More precisely, we use the integration by parts formula (8.5) as a definition for the integral on the left-hand side. Accordingly, we set

$$\int_0^\cdot \underline{S}_{t-}d\varphi_t^\downarrow := \varphi^\downarrow\underline{S} - \varphi_0^\downarrow\underline{S}_0 - \int_0^\cdot \varphi_t^\downarrow d\underline{S}_t. \tag{8.6}$$

For right-continuous $\varphi^\uparrow$ resp. $\varphi^\downarrow$ the definition in (8.5,8.6) coincides with the usual Stieltjes integral by (JS, I.4.49b). For general strategies we have the following alternative representation.

**Lemma 8.2** *Write*

$$\varphi_t^\uparrow = \varphi_t^{\uparrow,c} + \sum_{0<s\leq t}\Delta^-\varphi_s^\uparrow + \sum_{0\leq s<t}\Delta^+\varphi_s^\uparrow, \quad t\geq 0$$

*with a continuous process* $\varphi^{\uparrow,c}$ *and jumps*

$$\Delta^-\varphi_t^\uparrow := \varphi_t^\uparrow - \lim_{s\uparrow t}\varphi_s^\uparrow, \quad \Delta^+\varphi_t^\uparrow := \lim_{s\downarrow t}\varphi_s^\uparrow - \varphi_t^\uparrow.$$

*Then the right-hand side of* (8.5) *can be written as*

$$\int_0^t \overline{S}_{s-}d\varphi_s^\uparrow = \int_0^t \overline{S}_{s-}d\varphi_s^{\uparrow,c} + \sum_{0<s\leq t}\overline{S}_{s-}\Delta^-\varphi_s^\uparrow + \sum_{0\leq s<t}\overline{S}_s\Delta^+\varphi_s^\uparrow. \tag{8.7}$$

*A parallel statement holds for* (8.6).

PROOF. If we define right-continuous processes

$$J_t^- := \sum_{0<s\leq t}\Delta^-\varphi_s^\uparrow, \quad J_t^+ := \sum_{0\leq s\leq t}\Delta^+\varphi_s^\uparrow,$$

then $\varphi_t^\uparrow = \varphi_t^{\uparrow,c} + J_t^- + J_{t-}^+$. Setting $\psi_t := \varphi_t^{\uparrow,c} + J_t^- + J_t^+ = \varphi_t^\uparrow + \Delta^+\varphi_t^\uparrow$, we have $\psi_{t-} := \varphi_t^\uparrow - \Delta^-\varphi_t^\uparrow$ and the right-hand side of (8.5) can be written as

$$\begin{aligned}&\varphi_t^\uparrow\overline{S}_t - \varphi_0^\uparrow\overline{S}_0 - \int_0^t \varphi_s^\uparrow d\overline{S}_s\\ &= \psi_t\overline{S}_t - \psi_0\overline{S}_0 - \int_0^t \psi_{s-}d\overline{S}_s - (\Delta^+\varphi_t^\uparrow)\overline{S}_t + (\Delta^+\varphi_0^\uparrow)\overline{S}_0 - \int_0^t \Delta^-\varphi_s^\uparrow d\overline{S}_s.\end{aligned}$$

Using integration by parts as in (JS, I.4.49a), the first three terms yield

$$\begin{aligned}\int_0^t \overline{S}_s d\psi_s &= \int_0^t \overline{S}_s d\varphi_s^{\uparrow,c} + \int_0^t \overline{S}_s dJ_s^- + \int_0^t \overline{S}_s dJ_s^+ \\ &= \int_0^t \overline{S}_{s-} d\varphi_s^{\uparrow,c} + \sum_{0<s\leq t} \overline{S}_s \Delta^- \varphi_s^{\uparrow} + \sum_{0<s\leq t} \overline{S}_s \Delta^+ \varphi_s^{\uparrow}. \end{aligned} \tag{8.8}$$

The remaining three terms can be written as

$$-\overline{S}_t \Delta^+ \varphi_t^{\uparrow} + \overline{S}_0 \Delta^+ \varphi_0^{\uparrow} - \sum_{0<s\leq t} (\Delta^- \varphi_s^{\uparrow}) \Delta \overline{S}_s. \tag{8.9}$$

The sum of (8.8) und (8.9) equals the right-hand side of (8.7) as claimed. □

Piecing together (8.4–8.6), we end up with the following

**Definition 8.3** Let $(\varphi, c)$ be a portfolio/consumption pair with $\varphi = (\varphi^0, \varphi^{\uparrow} - \varphi^{\downarrow})$. We call $(\varphi, c)$ *self-financing* (or $\varphi$ *c-financing*) if (8.4) holds in the sense of (8.5,8.6) or, equivalently,

$$\varphi^0 + (\varphi^{\uparrow}\overline{S} - \varphi^{\downarrow}\underline{S}) = \varphi_0^0 + (\varphi_0^{\uparrow}\overline{S}_0 - \varphi_0^{\downarrow}\underline{S}_0) + \left(\int_0^{\cdot} \varphi_t^{\uparrow} d\overline{S}_t - \int_0^{\cdot} \varphi_t^{\downarrow} d\underline{S}_t\right) - \int_0^{\cdot} c_t dt. \tag{8.10}$$

(8.10) means that the pair $((\varphi^0, \varphi^{\uparrow}, -\varphi^{\downarrow}), c)$ is self-financing in the usual sense for a frictionless market with three securities $(1, \overline{S}, \underline{S})$. The validity of (8.10) does not depend on the choice of the initial values $\varphi_0^{\uparrow}, \varphi_0^{\downarrow}$. Note that for $\underline{S} = \overline{S} = S$ we recover the usual self-financing condition for frictionless markets.

The value of a portfolio is not obvious either because securities have no unique price. As is common in the literature, we use the value that would be obtained if the portfolio were to be liquidated immediately.

**Definition 8.4** The *(liquidation) value process* of a trading strategy $\varphi$ is defined as

$$V(\varphi) := \varphi^0 + (\varphi^1)^+\underline{S} - (\varphi^1)^-\overline{S},$$

A self-financing portfolio/consumption pair $(\varphi, c)$ is called *admissible* if $(\varphi_0^0, \varphi_0^1) = (\zeta_B, \zeta_S)$ and $V(\varphi) \geq 0$. An admissible pair is called *optimal* if it maximizes

$$\kappa \mapsto E\left(\int_0^{\infty} e^{-\delta t} \log(\kappa_t) dt\right) \tag{8.11}$$

over all admissible portfolio/consumption pairs $(\psi, \kappa)$, where $\delta > 0$ denotes a fixed given *impatience rate*.

Note that the "true" price process $S$ is irrelevant for the problem as it does not appear in the definitions; only the bid and ask prices $\underline{S}, \overline{S}$ matter. Moreover, since $\delta > 0$, the value function of the Merton problem *without* transaction costs is finite by (Davis & Norman, 1990, Theorem 2.1). Hence it follows that this holds in the present setup with transaction costs as well.

**Remark 8.5** Let us briefly discuss what happens to Merton's problem if the bank account is modelled as $B_t = e^{rt}$ for some $r \in \mathbb{R}_+$. In this undiscounted case a portfolio/consumption pair $(\varphi, c)$ should be called *self-financing* if

$$\begin{aligned}&\varphi^0 B + \varphi^{\uparrow}\overline{S} - \varphi^{\downarrow}\underline{S} \\ &= \varphi^0_0 B_0 + \varphi^{\uparrow}_0\overline{S}_0 - \varphi^{\downarrow}_0\underline{S}_0 + \int_0^{\cdot} \varphi^0_t dB_t + \int_0^{\cdot} \varphi^{\uparrow}_t d\overline{S}_t - \int_0^{\cdot} \varphi^{\downarrow}_t d\underline{S}_t - \int_0^{\cdot} c_t dt.\end{aligned}$$

*Admissibility* is defined as before but in terms of the *liquidation value process*

$$V(\varphi) := \varphi^0 B + (\varphi^1)^+\underline{S} - (\varphi^1)^-\overline{S}.$$

Obviously, these notions reduce to the definitions above for $B = 1$.

Using integration by parts similarly as in (Goll & Kallsen, 2000, Proposition 2.1) or (Goll & Kallsen, 2001, Lemma 2.3), one easily verifies that $(\varphi, c)$ is self-financing resp. admissible if and only if $(\varphi, c/B)$ is self-financing resp. admissible relative to the discounted processes $\hat{B} = B/B = 1$, $\hat{\underline{S}} := \underline{S}/B$, $\hat{\overline{S}} := \overline{S}/B$. In view of

$$E\left(\int_0^\infty e^{-\delta t}\log\Big(\frac{c_t}{B_t}\Big)\,dt\right) = E\left(\int_0^\infty e^{-\delta t}\log(c_t)\,dt\right) - E\left(\int_0^\infty e^{-\delta t}\log(B_t)\,dt\right)$$

it does not really matter whether one considers the investment and consumption problem for logarithmic utility in undiscounted or discounted terms because the expected utilities differ only by a constant.

Our notion of admissible strategies is slightly more general than that in Davis & Norman (1990), Shreve & Soner (1994). However, it will turn out later on that the optimal strategies in both sets coincide.

**Lemma 8.6** *For any admissible policy $(c, L, U)$ in the sense of Davis & Norman (1990) there exists a corresponding trading strategy $\varphi = (\varphi^0, \varphi^1)$ such that $(\varphi, c)$ is an admissible portfolio/consumption pair.*

PROOF. The initial endowment in Davis & Norman (1990) is expressed in terms of wealth as $(x, y) = (\zeta_B, \zeta_S S_0)$. Define $s_0$, $s_1$ as in (Davis & Norman, 1990, Equation (3.1)) and set

$$\varphi^0_t := s_0(t-), \quad \varphi^1_t := \frac{s_1(t-)}{S_t}.$$

The value process of $((\varphi^0, \varphi^1), c)$ is nonnegative by admissibility in the sense of Davis & Norman (1990). Furthermore, by definition of $s_0$, $s_1$ and (Revuz & Yor, 1999, IX.2.3), we have

$$\begin{aligned}\varphi^0_t &= x - \int_0^t c_s ds - (1+\overline{\lambda})L_{t-} + (1-\underline{\lambda})U_{t-}, \\ \varphi^1_t &= \frac{y}{S_0} + \int_0^{t-} \frac{1}{S_s} dL_s - \int_0^{t-} \frac{1}{S_s} dU_s.\end{aligned}$$

Thus $\varphi$ is of finite variation. Since it is left-continuous, it is also predictable. Using Lemma 8.2, a straightforward computation shows that $((\varphi^0, \varphi^1), c)$ satisfies the self-financing condition (8.4) as well. □

## 8.3 Heuristic derivation of the solution

As indicated in the introduction, the martingale approach relies decisively on shadow price processes, which we define as follows.

**Definition 8.7** We call a semimartingale $\widetilde{S}$ *shadow price process* if

$$\underline{S} \leq \widetilde{S} \leq \overline{S} \tag{8.12}$$

and if the maximal expected utilities for $S, \underline{\lambda}, \overline{\lambda}$ and for the price process $\widetilde{S}$ *without* transaction costs coincide.

Obviously, the maximal expected utility for any frictionless price process $\widetilde{S}$ satisfying (8.12) is at least as high as for the original market with transaction costs. Indeed, trading $\widetilde{S}$ an investor is always buying at $\widetilde{S}_t \leq \overline{S}_t$ and selling for $\widetilde{S}_t \geq \underline{S}_t$. A shadow price process can be interpreted as a kind of least favourable frictionless market extension. The corresponding optimal portfolio trades only when the shadow price happens to coincide with the bid or ask price, respectively. Otherwise it would achieve higher profits with $\widetilde{S}$ than with $S$ and transaction costs.

Let us assume that such a shadow price process $\widetilde{S}$ exists. If it were known in the first place, it would be of great help because portfolio selection problems without transaction costs are considerably easier to solve. But it is not known at this stage. Hence we must solve the problems of determining $\widetilde{S}$ and of portfolio optimization relative to $\widetilde{S}$ simultaneously.

To this end, we parametrize the shadow price process in the following form:

$$\widetilde{S} = S \exp(C) \tag{8.13}$$

with some $[\underline{C}, \overline{C}]$-valued process $C$ where

$$\underline{C} := \log(1 - \underline{\lambda}) \quad \text{and} \quad \overline{C} := \log(1 + \overline{\lambda}).$$

Since $S$ is an Itô process, we expect $\widetilde{S}$ and hence $C$ to be Itô processes as well. We even guess that $C$ is an Itô diffusion, i.e.

$$dC_t = \widetilde{\mu}(C_t)dt + \widetilde{\sigma}(C_t)dW_t \tag{8.14}$$

with some deterministic functions $\widetilde{\mu}, \widetilde{\sigma}$. Any admissible portfolio/consumption pair $(\varphi, c)$ is completely determined by $c$ and the fraction of wealth invested in stocks

$$\widetilde{\eta} := \frac{\varphi^1 \widetilde{S}}{\varphi^0 + \varphi^1 \widetilde{S}}, \tag{8.15}$$

where bookkeeping is done here relative to shadow prices $\widetilde{S}$. Hence we must determine four unknown objects, namely the ansatz functions $\widetilde{\mu}$, $\widetilde{\sigma}$ as well as the optimal consumption rate $c$ and the optimal fraction $\widetilde{\eta}$ of wealth in stocks.

Standard results yield the optimal strategy for the frictionless price process $\widetilde{S}$. E.g. by (Goll & Kallsen, 2000, Theorem 3.1) we have

$$\widetilde{\eta} = \frac{\mu - \frac{\sigma^2}{2} + \widetilde{\mu}(C)}{(\sigma + \widetilde{\sigma}(C))^2} + \frac{1}{2}, \quad c = \delta \widetilde{V}(\varphi), \tag{8.16}$$

where

$$\widetilde{V}(\varphi) = \varphi^0 + \varphi^1 \widetilde{S} \tag{8.17}$$

denotes the value process of $\varphi$ in the frictionless market with price process $\widetilde{S}$. This already determines the optimal consumption rate. To simplify the following calculations, we assume $\widetilde{\eta} > 0$ and work with

$$\beta := \log\left(\frac{\widetilde{\eta}}{1 - \widetilde{\eta}}\right)$$

instead of $\widetilde{\eta}$. By (8.15) this implies $\beta := \log(\varphi^1) + \log(\widetilde{S}) - \log(\varphi^0)$. Since the optimal strategy trades the shadow price process only when it coincides with bid or ask price, $\varphi^1$ must be constant on $]\!]0, T[\![$ with

$$T := \inf\left\{t > 0 : C_t \in \{\underline{C}, \overline{C}\}\right\}.$$

By (8.3) and Itô's formula we have

$$d\log(\varphi_t^0) = \frac{-c_t}{\varphi_t^0} dt = \frac{-\delta \widetilde{V}_t(\varphi)}{\widetilde{V}_t(\varphi) - \widetilde{\eta}_t \widetilde{V}_t(\varphi)} dt = \frac{-\delta}{1 - \widetilde{\eta}_t} dt$$

on $]\!]0, T[\![$, hence insertion of (8.16) yields

$$\begin{aligned} d\beta_t &= d\log(\varphi_t^1) + d\log(\widetilde{S}_t) - d\log(\varphi_t^0) \qquad (8.18)\\ &= \left(\mu - \frac{\sigma^2}{2} + \widetilde{\mu}(C_t) + \frac{\delta(\sigma + \widetilde{\sigma}(C_t))^2}{\frac{1}{2}(\sigma + \widetilde{\sigma}(C_t))^2 - (\mu - \frac{\sigma^2}{2} + \widetilde{\mu}(C_t))}\right) dt + (\sigma + \widetilde{\sigma}(C_t)) dW_t. \end{aligned}$$

On the other hand, we know from (8.16) that $\widetilde{\eta}$ is a function of $C$, which in turn yields $\beta = f(C)$ for some function $f$. By Itô's formula this implies

$$d\beta_t = \left(f'(C_t)\widetilde{\mu}(C_t) + f''(C_t)\frac{\widetilde{\sigma}(C_t)^2}{2}\right) dt + f'(C_t)\widetilde{\sigma}(C_t) dW_t. \tag{8.19}$$

From (8.18), (8.19) and (8.16) we now obtain three conditions for the three functions $\widetilde{\mu}, \widetilde{\sigma}, f$:

$$\frac{1}{1 + e^{-f}} = \frac{\mu - \frac{\sigma^2}{2} + \widetilde{\mu}}{(\sigma + \widetilde{\sigma})^2} + \frac{1}{2}, \tag{8.20}$$

$$\mu - \frac{\sigma^2}{2} + \widetilde{\mu} + \frac{\delta(\sigma + \widetilde{\sigma})^2}{\frac{1}{2}(\sigma + \widetilde{\sigma})^2 - (\mu - \frac{\sigma^2}{2} + \widetilde{\mu})} = f'\widetilde{\mu} + f''\frac{\widetilde{\sigma}^2}{2}, \tag{8.21}$$

$$\sigma + \widetilde{\sigma} = f'\widetilde{\sigma}. \tag{8.22}$$

Equations (8.22) and (8.20) yield

$$\widetilde{\sigma} = \frac{\sigma}{f'-1}, \quad \widetilde{\mu} = -\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}\left(\frac{f'}{f'-1}\right)^2 \frac{1-e^{-f}}{1+e^{-f}}. \tag{8.23}$$

By inserting into (8.21) we obtain the following ordinary differential equation (ODE) for $f$:

$$\begin{aligned} f''(x) = \frac{2\delta}{\sigma^2}(1+e^{f(x)}) + \left(\frac{2\mu}{\sigma^2} - 1 - \frac{4\delta}{\sigma^2}(1+e^{f(x)})\right) f'(x) \\ \left(-\frac{4\mu}{\sigma^2} + \frac{2}{1+e^{-f(x)}} + 1 + \frac{2\delta}{\sigma^2}(1+e^{f(x)})\right)(f'(x))^2 + \left(\frac{2\mu}{\sigma^2} - \frac{2}{1+e^{-f(x)}}\right)(f'(x))^3. \end{aligned} \tag{8.24}$$

Because of missing boundary conditions, (8.24) does not yet yield the solution. We obtain such conditions heuristically as follows. In order to lead to finite maximal expected utility, the shadow price process should be arbitrage-free and hence allow for an equivalent martingale measure. This in turn means that $\widetilde{S}$ and hence also $C$ should not have any singular part in their semimartingale decomposition. Put differently, we expect the Itô process representation (8.14) to hold even when $C$ reaches the boundary points $\underline{C}, \overline{C}$.

The number of shares of stock $\varphi^1$, on the other hand, changes only when $C$ hits the boundary. As this is likely to happen only on a Lebesgue-null set of times, $\varphi^1$ must have a singular part in order to move at all. In view of the connection between $\varphi^1$ and $\beta$, this suggests that $\beta$ has a singular part as well. This means that $f$ cannot be a $C^2$ function on the closed interval $[\underline{C}, \overline{C}]$ because otherwise $\beta = f(C)$ would be an Itô process, too. A natural way out is the ansatz $f'(\underline{C}) = -\infty = f'(\overline{C})$ in order for $\beta$ to have a singular part at the boundary. Hence we complement ODE (8.24) by boundary conditions

$$\lim_{x \downarrow \underline{C}} f'(x) = -\infty = \lim_{x \uparrow \overline{C}} f'(x). \tag{8.25}$$

In order to avoid infinite derivatives we consider instead the inverse function $g := f^{-1}$. Equation (8.24) turns into

$$\begin{aligned} g''(y) = \left(-\frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}}\right) + \left(\frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right) g'(y) \\ + \left(-\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 - \frac{2\delta}{\sigma^2}(1+e^y)(g'(y))^3 \end{aligned} \tag{8.26}$$

on the a priori unknown interval $[\underline{\beta}, \overline{\beta}] := [f(\overline{C}), f(\underline{C})]$ and (8.25) translates into free boundary conditions

$$g(\underline{\beta}) = \overline{C}, \quad g(\overline{\beta}) = \underline{C}, \quad g'(\underline{\beta}) = 0, \quad g'(\overline{\beta}) = 0. \tag{8.27}$$

(8.26,8.27) together with (8.13–8.17) and $f = g^{-1}$ constitute our ansatz for the portfolio optimization problem.

In summary, the solution to the free boundary problem (8.26,8.27) — or equivalently (8.24,8.25) — leads to the optimal strategy. The ODE itself is derived based on the optimality of $\widetilde{\eta}$ for $\widetilde{S}$ and the constancy of $\varphi^1$ on $]\!]0, T[\![$. In the next section we show that this ansatz indeed yields the true solution.

Our result resembles Davis & Norman (1990) in that the solution is expressed in terms of a free boundary problem. However, both the ODE and the boundary conditions are different, since the function $g$ refers to the shadow price process from the present dual approach and therefore does not appear explicitly in the framework of Davis & Norman (1990) (but cf. Remark 8.14).

## 8.4 Construction of the shadow price process

We turn now to verification of the candidate solution from the previous section. The idea is rather simple. Using (8.13,8.14) we define a candidate shadow price process $\widetilde{S}$. In order to prove that it is indeed a shadow price process, we show that the optimal portfolio relative to $\widetilde{S}$ trades only at the boundaries $\underline{S}, \overline{S}$ . However, existence of a solution to stochastic differential equation (SDE) (8.14) is not immediately obvious. Therefore we consider instead the corresponding Skorohod SDE for $\beta = f(C)$ with instantaneous reflection at some boundaries $\underline{\beta} < \overline{\beta}$. The process $C = g(\beta)$ is then defined in a second step.

We begin with an existence result for the free boundary value problem derived above. We make the following assumption which guarantees that the fraction of wealth held in stock remains positive and which is needed in Davis & Norman (1990) as well (Equation 5.1 in that paper).

**Standing assumption**:

$$0 < \mu < \sigma^2. \tag{8.28}$$

**Remark 8.8** It is shown in Shreve & Soner (1994) that this condition is not needed to ensure the existence of an optimal strategy characterized by a wedge-shaped no-transaction region. If the transformation $\beta = \log(\widetilde{\eta}/(1-\widetilde{\eta}))$ was not used in our approach, we would still obtain a free boundary problem, but as in Davis & Norman (1990) it is less obvious whether or not it admits a solution.

**Proposition 8.9** *There exist $\underline{\beta} < \overline{\beta}$ and a strictly decreasing mapping $g : [\underline{\beta}, \overline{\beta}] \to [\underline{C}, \overline{C}]$ satisfying the free boundary problem (8.26,8.27).*

PROOF. Since we have assumed $0 < \frac{\mu}{\sigma^2} < 1$, there is a unique solution $y$ to $\frac{2}{1+e^{-y}} - \frac{2\mu}{\sigma^2} = 0$, namely $y_0 = -\log(\frac{\sigma^2}{\mu} - 1)$. For any $\underline{\beta}_\Delta := y_0 - \Delta$ with $\Delta > 0$, there exists a local solution $g_\Delta$ of the initial value problem corresponding to (8.26) and initial values $g_\Delta(\underline{\beta}_\Delta) = \overline{C}$ and $g'_\Delta(\underline{\beta}_\Delta) = 0$. Set

$$M' := \max\left\{ \sqrt[3]{\frac{4(\mu+\sigma^2)}{\delta}}, \sqrt{\frac{8\mu}{\delta}}, 8 + \frac{4\mu + 2\sigma^2}{\delta} \right\}.$$

Then we have $g''_\Delta(y) > 0$ for $g'_\Delta(y) < -M'$ and $g''_\Delta(y) < 0$ for $g'_\Delta > M'$ by (8.26). Therefore $g'_\Delta$ only takes values in $[-M', M']$, which implies that $g_\Delta$ does not explode.

From (8.26) and $\Delta > 0$ it follows that $g''_\Delta(y) < 0$ in a neighbourhood $\mathscr{U}$ of $\underline{\beta}_\Delta$ and hence $g'_\Delta(y) < 0$ in $\mathscr{U}$. For sufficiently large $y$ and $g'_\Delta(y) < 0$, the right-hand side of (8.26) is positive and bounded away from zero by a positive constant. Hence a comparison argument shows that there exist further zeros of $g'_\Delta$, the first of which we denote by $\overline{\beta}_\Delta$. Note that by definition $g_\Delta$ is strictly decreasing on $[\underline{\beta}_\Delta, \overline{\beta}_\Delta]$.

It remains to show that for properly chosen $\Delta$, we can achieve $g(\overline{\beta}_\Delta) = \underline{C}$ for any $\underline{C} < \overline{C}$.

*Step 1:* We first show $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ as $\Delta \to 0$. This can be seen as follows. Observe that for $|y - y_0| < 1$, (8.26) and $g'_\Delta(y) \in [-M', M']$ yield

$$
\begin{aligned}
|g''_\Delta(y)| < M'' \quad := \quad & \frac{2\mu}{\sigma^2} + 2 + \left(\frac{4\mu}{\sigma^2} + 3 + \frac{2\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)\right) M' \\
& + \left(\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)\right)(M')^2 + \frac{2\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)(M')^3.
\end{aligned}
$$

Hence $|g'_\Delta(y)| \le 2M''\Delta$ for $y \in [y_0 - \Delta, y_0 + \Delta]$ and $\Delta < 1$. Combined with (8.26), this yields

$$
\sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''_\Delta(y)| \to 0, \quad \text{for } \Delta \to 0. \tag{8.29}
$$

For $\Delta$ sufficiently small, $y \in [y_0 + \Delta, y_0 + 1]$, and

$$
|g'_\Delta(y)| < m_\Delta := \max\left\{ \frac{\frac{1}{3}\left(-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-(y_0+\Delta)}}\right)}{\frac{4\mu}{\sigma^2} + 3 + \frac{2\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)}, \sqrt{\frac{\frac{1}{3}\left(-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-(y_0+\Delta)}}\right)}{\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)}}, \sqrt[3]{\frac{\frac{1}{3}\left(-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-(y_0+\Delta)}}\right)}{\frac{2\delta}{\sigma^2}\left(1 + e^{y_0+1}\right)}} \right\},
$$

(8.26) and a first order Taylor expansion imply

$$
g''_\Delta(y) > \frac{\mu}{\sigma^2} + \frac{1}{1 + e^{-(y_0+\Delta)}} > \frac{e^{-y_0}}{2(1 + e^{-y_0})^2}\Delta > 0. \tag{8.30}
$$

(8.29) yields

$$
|g'_\Delta(y_0 + \Delta)| \le 2\Delta \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''_\Delta(y)| < m_\Delta
$$

for sufficiently small $\Delta$. By (8.30) we have that if $g'_\Delta$ does not have a zero on $[y_0 - \Delta, y_0 + \Delta]$, i.e. $g'(y_0 + \Delta) < 0$, then $g''_\Delta(y) > \frac{e^{-y_0}}{2(1+e^{-y_0})^2}\Delta$ on $[y_0 + \Delta, \min\{\overline{\beta}_\Delta, y_0 + 1\}]$. Using (8.29) this yields

$$
\overline{\beta}_\Delta - \underline{\beta}_\Delta < 2\Delta + \frac{2\Delta \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''(y)|}{\frac{e^{-y_0}}{2(1+e^{-y_0})^2}\Delta} \to 0
$$

for $\Delta \to 0$. Since $|g'_\Delta(y)| < M'$, an application of the mean value theorem completes the first step.

*Step 2:* We now establish $\overline{\beta}_\Delta \geq y_0$ and $g(\overline{\beta}_\Delta) \to -\infty$ as $\Delta \to \infty$. To this end, let $y^* < y_0$. Then we have $g''_\Delta(y) < -\frac{\mu}{\sigma^2} + \frac{1}{1+e^{y^*}} < 0$ if $y \leq y^*$ and

$$|g'_\Delta(y)| < m' := \max\left\{ \frac{\frac{1}{3}\left|-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-y^*}}\right|}{\frac{4\mu}{\sigma^2} + 3 + \frac{2\delta}{\sigma^2}(1+e^{y^*})}, \sqrt{\frac{\frac{1}{3}\left|-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-y^*}}\right|}{\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^{y^*})}}, \sqrt[3]{\frac{\frac{1}{3}\left|-\frac{\mu}{\sigma^2} + \frac{1}{1+e^{-y^*}}\right|}{\frac{2\delta}{\sigma^2}(1+e^{y^*})}} \right\}.$$

Since $g''_\Delta(\underline{\beta}_\Delta) < 0$, this implies $g'_\Delta(y) < 0$ for $y \leq y^*$ as well as $|g'_\Delta(y)| \geq m'$ for $y \in [y_0 - \Delta + \frac{m'}{\mu/\sigma^2 - (1+e^{-y^*})^{-1}}, y^*]$. By the first statement and since $y^* < y_0$ was chosen arbitrarily, we have $\overline{\beta}_\Delta \geq y_0$. In addition, the second statement and the mean value theorem show that $g_\Delta(\overline{\beta}_\Delta) \to -\infty$ as $\Delta \to \infty$.

*Step 3:* We now establish $\overline{\beta}_\Delta > y_0$. By Step 2 it remains to show that $\overline{\beta}_\Delta \neq y_0$. Suppose that $\overline{\beta}_\Delta = y_0$. Then $g'_\Delta(y_0) = 0 = g''_\Delta(y_0)$ and it follows from a Taylor expansion around $y_0$ that

$$g''_\Delta(y) = \frac{2e^{-y_0}}{(1+e^{-y_0})^2}(y - y_0) + O((y-y_0)^2) < 0$$

for $y \in (y_0 - \varepsilon, y_0)$ and sufficiently small $\varepsilon > 0$, hence $g'(y) > 0$ for some $y < y_0$. By the intermediate value theorem there exists a zero of $g'$ on $(\underline{\beta}_\Delta, y_0)$, in contradiction to the definition of $\overline{\beta}_\Delta$. Therefore we have $\overline{\beta}_\Delta > y_0$ as claimed.

*Step 4:* Next, we prove that $(g_\Delta, g'_\Delta)$ converges toward $(g_{\Delta_0}, g'_{\Delta_0})$ uniformly on compacts as $\Delta \to \Delta_0$. To this end, we consider the solution $f^\Delta : \mathbb{R}_+ \to \mathbb{R}^3$ to the initial value problem

$$\frac{d}{dy}(f_1^\Delta, f_2^\Delta, f_3^\Delta)(y) = \Big(1, f_3^\Delta(y), h\big(f_1^\Delta(y), f_3^\Delta(y)\big)\Big)$$

with

$$\begin{aligned} h(y,z) \quad := \quad & \left(-\frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}}\right) + \left(\frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right) z \\ & + \left(-\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right) z^2 - \frac{2\delta}{\sigma^2}(1+e^y) z^3 \end{aligned}$$

and initial values $(f_1^\Delta, f_2^\Delta, f_3^\Delta)(0) = (y_0 - \Delta, \overline{C}, 0)$. The solution to this problem is

$$(f_1^\Delta, f_2^\Delta, f_3^\Delta)(y) = \big(y + y_0 - \Delta, g_\Delta(y + y_0 - \Delta), g'_\Delta(y + y_0 - \Delta)\big).$$

Note that

$$\begin{aligned} |g_\Delta(y) - g_{\Delta_0}(y)| \quad &= \quad \left|f_2^\Delta(y - y_0 + \Delta) - f_2^{\Delta_0}(y - y_0 + \Delta_0)\right| \\ &\leq \quad \left|f_2^\Delta(y - y_0 + \Delta) - f_2^{\Delta_0}(y - y_0 + \Delta)\right| + M'|\Delta - \Delta_0| \end{aligned}$$

and similarly for $g''$. Hence it suffices to show that $f^\Delta$ depends uniformly on compacts on its initial value $f^\Delta(0)$.

$h$ is locally Lipschitz and hence globally Lipschitz in $z$ on $[-M', M']$ and in $y$ on compacts. The desired uniform convergence follows now from the corollary to (Birkhoff & Rota, 1962, Theorem V.3.2).

*Step 5:* In view of Steps 1 and 2 as well as the intermediate value theorem, it remains to show that $g(\overline{\beta}_\Delta)$ depends continuously on $\Delta$. Fix $\Delta_0 > 0$. Since $\overline{\beta}_{\Delta_0} > y_0$ by Step 3, a Taylor expansion around $\overline{\beta}_{\Delta_0}$ yields that $g'_{\Delta_0}$ is strictly increasing in a sufficiently small neighbourhood $\mathscr{W}$ of $\overline{\beta}_{\Delta_0}$. Now consider $\Delta$ sufficiently close to $\Delta_0$. Recall that $g'_\Delta(y)$ does not vanish for $\underline{\beta}_\Delta < y \leq y_0$. By the uniform convergence from Step 4, the first zero $\overline{\beta}_\Delta$ of $g'_\Delta$ after $\underline{\beta}_\Delta$ is close to the first zero $\overline{\beta}_{\Delta_0}$ of $g'_{\Delta_0}$ after $\underline{\beta}_{\Delta_0}$. In view of

$$\left|g_\Delta(\overline{\beta}_\Delta) - g_{\Delta_0}(\overline{\beta}_{\Delta_0})\right| \leq \left|g_\Delta(\overline{\beta}_\Delta) - g_{\Delta_0}(\overline{\beta}_\Delta)\right| + \left|g_{\Delta_0}(\overline{\beta}_\Delta) - g_{\Delta_0}(\overline{\beta}_{\Delta_0})\right|$$

and Step 4, this completes the proof. □

We now construct the process $\beta$ as the solution to an SDE with instantaneous reflection. The coefficients $a$ and $b$ in (8.31) below are chosen in line with (8.18) and (8.23).

**Lemma 8.10** *Let $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ and*

$$a(y) := \frac{\sigma^2}{2}\left(\frac{1-e^{-y}}{1+e^{-y}}\right)\left(\frac{1}{1-g'(y)}\right)^2 + \delta\left(1+e^y\right), \quad b(y) := \frac{\sigma}{1-g'(y)}$$

*for $\beta \in [\underline{\beta}, \overline{\beta}]$. Then there exists a solution to the Skorohod SDE*

$$d\beta_t = a(\beta_t)dt + b(\beta_t)dW_t$$

*with instantaneous reflection at $\underline{\beta}, \overline{\beta}$, i.e. a continuous, adapted, $[\underline{\beta}, \overline{\beta}]$-valued process $\beta$ and nondecreasing adapted processes $\Phi$, $\Psi$ such that $\Phi$ and $\Psi$ increase only on the sets $\{\beta = \underline{\beta}\}$ and $\{\beta = \overline{\beta}\}$, respectively, and*

$$\beta_t = \beta_0 + \int_0^t a(\beta_s)ds + \int_0^t b(\beta_s)dW_s + \Phi_t - \Psi_t \tag{8.31}$$

*holds for all $t \in \mathbb{R}_+$.*

PROOF. In view of Skorokhod (1961, 1962), it suffices to prove that the coefficients $a(\cdot)$ and $b(\cdot)$ are globally Lipschitz on $[\underline{\beta}, \overline{\beta}]$. By the mean value theorem it is enough to show that their derivatives are bounded on $(\underline{\beta}, \overline{\beta})$. Let $y \in (\underline{\beta}, \overline{\beta})$ be fixed. Then we have

$$b'(y) = \sigma\frac{g''(y)}{(1-g'(y))^2}. \tag{8.32}$$

$g'(y) \leq 0$ implies $|1 - g'(y)| \geq \max\{1, |g'(y)|\}$. Moreover, $g'$ is bounded on $[\underline{\beta}, \overline{\beta}]$ by the proof of Proposition 8.9. Boundedness of $b'$ now follows from (8.26) and (8.32). Boundedness of $a'$ is shown along the same lines. □

We now define $C$ and the shadow price process $\widetilde{S}$ as motivated in Section 8.3.

**Lemma 8.11** *For $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ let $\beta$ be the process from Lemma 8.10. Then $C := g(\beta)$ is a $[\underline{C}, \overline{C}]$-valued Itô process of the form*

$$C_t = g(\beta_0) + \int_0^t \left( -\mu + \frac{\sigma^2}{2} + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\beta_s}}{1 + e^{-\beta_s}} \right) \left( \frac{1}{1 - g'(\beta_s)} \right)^2 \right) ds + \int_0^t \frac{\sigma g'(\beta_s)}{1 - g'(\beta_s)} dW_s$$

*and the Itô process $\widetilde{S} := S \exp(C)$ satisfies*

$$\widetilde{S}_t = S_0 e^{C_0} \exp \left( \int_0^t \frac{\sigma^2}{2} \left( \frac{1 - e^{-\beta_s}}{1 + e^{-\beta_s}} \right) \left( \frac{1}{1 - g'(\beta_s)} \right)^2 ds + \int_0^t \frac{\sigma}{1 - g'(\beta_s)} dW_s \right).$$

PROOF. $g$ can be extended to a $C^2$-function on an open set containing $[\underline{\beta}, \overline{\beta}]$, e.g. by attaching suitable parabolas at $\underline{\beta}, \overline{\beta}$. Since $\Phi$ and $\Psi$ are of finite variation and $g'$ vanishes on the support of the Stieltjes measures corresponding to $\Phi$ and $\Psi$, Itô's formula yields

$$dC_t = \left( g'(\beta_t) a(\beta_t) + \frac{1}{2} g''(\beta_t) b(\beta_t)^2 \right) dt + g'(\beta_t) b(\beta_t) dW_t.$$

The claims follow by inserting the definitions of $a$ and $b$, (8.26), and the definition of $S$. □

Next, we show that $\widetilde{S}$ is indeed a shadow price process, i.e. the same portfolio/consumption pair $(\varphi, c)$ is optimal with the same expected utility both in the frictionless market with price process $\widetilde{S}$ and in the market with price process $S$ and proportional transaction costs $\underline{\lambda}, \overline{\lambda}$. In the frictionless market with price process $\widetilde{S}$, standard results yield the optimal strategy and consumption rate.

**Lemma 8.12** *Set*

$$\beta_0 := \begin{cases} \underline{\beta} & \text{if } \frac{\zeta_S \overline{S}_0}{\zeta_B + \zeta_S \overline{S}_0} < \frac{1}{1+e^{-\underline{\beta}}}, \\ \overline{\beta} & \text{if } \frac{\zeta_S \underline{S}_0}{\zeta_B + \zeta_S \underline{S}_0} > \frac{1}{1+e^{-\overline{\beta}}}. \end{cases} \tag{8.33}$$

*Otherwise, let $\beta_0$ denote the $[\underline{\beta}, \overline{\beta}]$-valued solution $y$ to*

$$\frac{\zeta_S S_0 e^{g(y)}}{\zeta_B + \zeta_S S_0 e^{g(y)}} = \frac{1}{1 + e^{-y}}.$$

*For processes $\beta$ and $\widetilde{S}$ as in Lemma 8.11 define*

$$\widetilde{V}_t := (\zeta_B + \zeta_S \widetilde{S}_0) \mathscr{E} \left( \int_0^\cdot \frac{1}{(1 + e^{-\beta_s}) \widetilde{S}_s} d\widetilde{S}_s - \int_0^\cdot \delta ds \right)_t,$$

$$c_t := -\delta \widetilde{V}_t,$$

$$\varphi_t^1 := \frac{1}{1 + e^{-\beta_t}} \frac{\widetilde{V}_t}{\widetilde{S}_t}, \quad \varphi_t^0 := \widetilde{V}_t - \varphi_t^1 \widetilde{S}_t.$$

*Then*

$$\begin{aligned} \varphi_t^0 &= \varphi_0^0 - \int_0^t c_s ds - \int_0^t \frac{\widetilde{V}_s e^{-\beta_s}}{(1 + e^{-\beta_s})^2} d\Phi_s + \int_0^t \frac{\widetilde{V}_s e^{-\beta_s}}{(1 + e^{-\beta_s})^2} d\Psi_s, \\ \varphi_t^1 &= \varphi_0^1 + \int_0^t \frac{\varphi_s^1 e^{-\beta_s}}{1 + e^{-\beta_s}} d\Phi_s - \int_0^t \frac{\varphi_s^1 e^{-\beta_s}}{1 + e^{-\beta_s}} d\Psi_s, \end{aligned} \tag{8.34}$$

*and $(\varphi, c)$ is an optimal portfolio/consumption pair with value process $\widetilde{V}$ for initial wealth $\zeta_B + \zeta_S \widetilde{S}_0$ in the frictionless market with price process $\widetilde{S}$.*

PROOF. Unless (8.33) holds, $\beta_0$ is the root of the continuous, strictly increasing function

$$h(y) := \frac{\zeta_B e^{-g(y)} + \zeta_S S_0}{1 + e^{-y}} - \zeta_S S_0.$$

If $h$ does not have a root in $[\underline{\beta}, \overline{\beta}]$, then either $h(\underline{\beta}) > 0$ or $h(\overline{\beta}) < 0$, i.e. either

$$\frac{\zeta_S \overline{S}_0}{\zeta_B + \zeta_S \overline{S}_0} < \frac{1}{1 + e^{-\underline{\beta}}} \quad \text{or} \quad \frac{\zeta_S \underline{S}_0}{\zeta_B + \zeta_S \underline{S}_0} > \frac{1}{1 + e^{-\overline{\beta}}}.$$

Hence $\beta_0$ is well defined.

We have

$$\log(\varphi_t^1) = \log(\widetilde{V}_t) - \left(\mu - \frac{\sigma^2}{2}\right) t - \sigma W_t - C_t - \log(1 + e^{-\beta_t}). \tag{8.35}$$

By (JS, I.4.61)

$$\begin{aligned} &d \log(\widetilde{V}_t) \\ &= \left( \frac{\sigma^2}{2(1 + e^{-\beta_t})^2} \left( \frac{1}{1 - g'(\beta_t)} \right)^2 - \delta \right) dt + \frac{\sigma}{1 + e^{-\beta_t}} \left( \frac{1}{1 - g'(\beta_t)} \right) dW_t. \end{aligned}$$

$C$ is given in Lemma 8.11 and for the last term in (8.35), Itô's formula yields

$$-d \log(1 + e^{-\beta_t})) = \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\beta_t - \frac{1}{2} \frac{e^{-\beta_t}}{(1 + e^{-\beta_t})^2} d[\beta, \beta]_t.$$

Summing up terms, we have

$$d \log(\varphi_t^1) = \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\Phi_t - \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\Psi_t.$$

Hence $\log(\varphi^1)$ is of finite variation and another application of Itô's formula yields the claimed representation for $\varphi^1$. Obviously, $\widetilde{V}$ is the value process of $\varphi$ relative to $\widetilde{S}$. By definition we have

$$d\widetilde{V}_t = \varphi_t^1 d\widetilde{S}_t - c_t dt, \tag{8.36}$$

which means that $(\varphi, c)$ is a self-financing portfolio/consumption pair for price process $\widetilde{S}$. The integral representation of $\varphi^0$ now follows from

$$d\varphi_t^0 = d(\widetilde{V}_t - \varphi_t^1 \widetilde{S}_t) = -c_t dt - \widetilde{S}_t d\varphi_t^1,$$

where we used integration by parts in the sense of (JS, I.4.49b). For $t \in \mathbb{R}_+$ set

$$K_t := \int_0^t e^{-\delta s} ds, \quad \kappa_t := e^{\delta t} c_t, \quad \psi_t^0 := \varphi_t^0 + \int_0^t c_s ds, \quad \psi_t^1 := \varphi_t^1.$$

Then $(\varphi, c)$ is optimal in the sense of Definition 8.4 (adapted to frictionless markets where the restriction to strategies of finite variation is dropped) if and only if $(\psi, \kappa)$ is optimal in the sense of (Goll & Kallsen, 2000, Definition 2.2). From Propositions A.3 and A.4 we derive that the differential characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ of $\widetilde{S}$ are given by $\widetilde{F} = 0$ and

$$\widetilde{b}_t = \widetilde{S}_t \sigma^2 \frac{1}{1+e^{-\beta_t}} \left( \frac{1}{1-g'(\beta_t)} \right)^2, \quad \widetilde{c}_t = \widetilde{S}_t^2 \sigma^2 \left( \frac{1}{1-g'(\beta_t)} \right)^2.$$

Hence (Goll & Kallsen, 2000, Theorem 3.1) with $H_t = \widetilde{b}_t / \widetilde{c}_t$, $K_\infty = 1/\delta$ and $K_\infty - K_t = \frac{1}{\delta} e^{-\delta t}$ yields the optimality of $(\varphi, c)$. □

If (8.33) holds, then $(\varphi_0^0, \varphi_0^1) \neq (\zeta_B, \zeta_S)$. In this case we can and do modify the initial portfolio to

$$(\varphi_0^0, \varphi_0^1) := (\zeta_B, \zeta_S) \tag{8.37}$$

without affecting the initial wealth, gains, or optimality. From now on, $\varphi$ refers to this slightly changed strategy. The case (8.33) happens if the initial portfolio is not situated in the no-trade region of the transaction costs model, which makes an initial bulk trade necessary.

(8.34) implies that the optimal strategy $\varphi$ is of finite variation and constant until $\widetilde{S}$ visits the boundary $\{\underline{S}, \overline{S}\}$ the next time. Since sales and purchases take place at the same prices as in the market with transaction costs $\underline{\lambda}$, $\overline{\lambda}$ and price process $S$, the portfolio/consumption pair $(\varphi, c)$ is admissible in this market as well. Conversely, since shares can be bought at least as cheaply and sold at least as expensively, any admissible consumption rate in the market with price process $S$ and transaction costs is admissible in the frictionless market with price process $\widetilde{S}$, too. Hence $(\varphi, c)$ is optimal in the market with transaction costs as well. Made precise, this is stated in the following theorem.

**Theorem 8.13** *The portfolio/consumption pair* $(\varphi, c)$ *defined in Lemma 8.12 and* (8.37) *is also optimal in the market with price process* $S$ *and proportional transaction costs* $\overline{\lambda}$, $\underline{\lambda}$. *In particular,* $\widetilde{S}$ *is a shadow price process in this market.*

PROOF. Let $((\psi^0, \psi^\uparrow - \psi^\downarrow), \kappa)$ be an admissible portfolio/consumption pair in the market with price process $S$ and transaction costs $\underline{\lambda}$, $\overline{\lambda}$. By $\underline{S} \leq \widetilde{S} \leq \overline{S}$ and Lemma 8.2 we have

$$\begin{aligned} \widetilde{\psi}^0 &:= \psi_0^0 + \int_0^\cdot \widetilde{S}_t d\psi_t^\downarrow - \int_0^\cdot \widetilde{S}_t d\psi_t^\uparrow - \int_0^\cdot c_t dt \\ &\geq \psi_0^0 + \int_0^\cdot \underline{S}_t d\psi_t^\downarrow - \int_0^\cdot \overline{S}_t d\psi_t^\uparrow - \int_0^\cdot c_t dt \\ &= \psi^0. \end{aligned}$$

Together with $\underline{S} \leq \widetilde{S} \leq \overline{S}$ it follows that $((\widetilde{\psi}^0, \psi^1), \kappa)$ is an admissible portfolio/consumption pair in the frictionless market with price process $\widetilde{S}$. By optimality of $(\varphi, c)$ defined in Lemma 8.12, this implies

$$E\left( \int_0^\infty e^{-\delta t} \log(c_t) dt \right) \geq E\left( \int_0^\infty e^{-\delta t} \log(\kappa_t) dt \right).$$

Therefore it remains to prove that $(\varphi, c)$ is admissible in the market with price process $S$ and proportional transaction costs $\overline{\lambda}$, $\underline{\lambda}$. Let us begin with $\varphi$ as in Lemma 8.13, i.e. without the modification from (8.37). Since $\Phi$ and $\Psi$ increase only on the sets $\{\widetilde{S} = \overline{S}\}$ and $\{\widetilde{S} = \underline{S}\}$, respectively, the self-financing condition for $(\varphi, c)$ and (8.34) yield

$$\begin{aligned}\varphi^0 &= \varphi^0_0 + \int_0^\cdot \widetilde{S}_t d\varphi^\downarrow_t - \int_0^\cdot \widetilde{S}_t d\varphi^\uparrow_t - \int_0^\cdot c_t dt \\ &= \varphi^0_0 + \int_0^\cdot \underline{S}_t d\varphi^\downarrow_t - \int_0^\cdot \overline{S}_t d\varphi^\uparrow_t - \int_0^\cdot c_t dt.\end{aligned}$$

This shows that $(\varphi, c)$ is self-financing in the market with price process $S$ and transaction costs $\overline{\lambda}$, $\underline{\lambda}$. We now turn back to $\varphi$ as in (8.37). Both sides of (8.10) are unaffected by this modification, at least if the initial values of $\varphi^\uparrow, \varphi^\downarrow$ are chosen accordingly. This implies that the slightly changed $(\varphi, c)$ is self-financing for $S, \overline{\lambda}$, $\underline{\lambda}$ as well. By $\varphi^0$, $\varphi^1 \geq 0$ it is also admissible. This completes the proof. □

In the language of Davis & Norman (1990), the optimal policy is $(c, L, U)$ with

$$\begin{aligned}L_t &= \left(\varphi^1_0 - \zeta_S\right)^+ S_0 + \int_0^t \frac{\varphi^1_s S_s e^{-\beta_s}}{1+e^{-\beta_s}} d\Phi_s, \\ U_t &= \left(\varphi^1_0 - \zeta_S\right)^- S_0 + \int_0^t \frac{\varphi^1_s S_s e^{-\beta_s}}{1+e^{-\beta_s}} d\Psi_s.\end{aligned}$$

In particular, it belongs to the slightly smaller set of admissible controls in Davis & Norman (1990), Shreve & Soner (1994), where the cumulative values $L, U$ of purchases and sales are supposed to be right continuous. Therefore the optimal strategies in our and their setup coincide.

**Remark 8.14** In the case of logarithmic utility, it is possible to recover the shadow price $\widetilde{S}$ from the results of Davis & Norman (1990). General results on logarithmic utility maximization in frictionless markets show that the optimal consumption rate $c$ equals the $1/\delta$-fold of the investor's current wealth measured in terms of the shadow price. Hence the consumption rate calculated in Davis & Norman (1990) determines the shadow value process $\widetilde{V}$, which in turn allows to back out the shadow price $\widetilde{S}$. More precisely, the shadow price can be constructed in a very subtle way using the results of Davis & Norman (1990), as was pointed out to us by the very insightful comments of an anonymous referee: In the proof of (Davis & Norman, 1990, Theorem 5.1) it is shown that the value function is of the form

$$v(x, y) = \frac{1}{\delta} \log\left(p\left(\frac{x}{y}\right)\left(x + q\left(\frac{x}{y}\right) y\right)\right) \tag{8.38}$$

with functions $p, q$ related through the identity

$$p'(x) = -p(x)q'(x)/(x + q(x)). \tag{8.39}$$

Differentiating (8.38) and inserting (8.39) leads to

$$\frac{1}{v_x(x, y)} = \delta\left(x + q\left(\frac{x}{y}\right) y\right).$$

In view of (Davis & Norman, 1990, Theorem 4.3), this shows that the optimal consumption policy is given by $c = \delta(s_0 + q(\frac{s_0}{s_1})s_1)$. By (Goll & Kallsen, 2000, Theorem 3.1) this implies that the optimal value process w.r.t the shadow price is given by

$$\widetilde{V} = s_0 + q\left(\frac{s_0}{s_1}\right)s_1. \tag{8.40}$$

A close look at the construction of the function $q$ in the proof of (Davis & Norman, 1990, Theorem 5.1) reveals that $q$ is increasing with $q(\frac{s_0}{s_1}) = 1-\underline{\lambda}$ when $\frac{s_0}{s_1}$ hits the lower boundary resp. $q(\frac{s_0}{s_1}) = 1+\overline{\lambda}$ for the upper boundary of the no-trade region. Therefore it follows from (Davis & Norman, 1990, (3.1)) that

$$\widetilde{V} = (\varphi^0 + \Delta s_0) + q\left(\frac{s_0}{s_1}\right)(\varphi^1 S + \Delta s_1) = \varphi^0 + \varphi^1 q\left(\frac{s_0}{s_1}\right)S$$

for the optimal trading strategy

$$\varphi^0_t = s_0(t-), \quad \varphi^1_t := \frac{s_1(t-)}{S}$$

corresponding to the optimal policy $(L, U)$ of Davis & Norman (1990). This shows that $q(\frac{s_0}{s_1})S$ coincides with the shadow price process $\widetilde{S}$ constructed above.

However, if one wants to verify that $q(\frac{s_0}{s_1})S$ indeed is a shadow price without using the results provided in this chapter, the ensuing verification procedure turns out to be as involved as our approach of dealing with the utility optimization problem and the computation of the shadow price process simultaneously. More specifically, one knows by construction that $q(\frac{s_0}{s_1})S$ is $[(1-\underline{\lambda})S, (1+\overline{\lambda})S]$-valued and positioned at the respective boundary whenever the strategy $\varphi$ trades. By the proof of Theorem 8.13 it therefore suffices to show that $(\varphi, c)$ is optimal w.r.t. $\widetilde{S} = q(\frac{s_0}{s_1})S$ in order for $q(\frac{s_0}{s_1})S$ to be a shadow price. In view of (Goll & Kallsen, 2000, Theorem 3.1) this amounts to verifying that

$$\frac{\varphi^1 q(s_0/s_1)S}{s_0 + q(s_0/s_1)s_1} = \frac{b}{c} \tag{8.41}$$

for the differential semimartingale characteristics $(b, c, 0)$ of the continuous process $q(\frac{s_0}{s_1})S$. In particular one has to prove that the properties of the function $q$ ensure that $S$ is an Itô process and calculate its Itô decomposition. The optimality condition (8.41) then has to be verified using (Davis & Norman, 1990, (5.7)), which leads to rather tedious computations.

As a side remark, it is interesting to note that this link between the optimal policy and the shadow price is only apparent for logarithmic utility. Therefore it is not possible to extract the shadow price from the results of Davis & Norman (1990) for power utility functions of the form $u(x) = x^{1-p}/(1-p)$. Using the present approach of solving for the optimal strategy and the shadow price simultaneously still leads to equations for the optimal strategy and the shadow price if combined with the notion of an opportunity process from Chapter 4. However, the corresponding free boundary problem appears to be substantially more complicated than its counterpart in Davis & Norman (1990).

At this stage it is not clear whether this additional complexity can be removed through a suitable transformation as in the proof of (Davis & Norman, 1990, Theorem 5.1), or whether the shadow price is indeed more difficult to obtain than the optimal policy for power utility.

## 8.5 Numerical illustration

The key free boundary value problem (8.26) can be readily solved with just a few lines of standard e.g. MATLAB code. For the remainder of this section, we use

$$\underline{\lambda} = \overline{\lambda} = 0.01, \quad \mu = 0.05, \quad \sigma = 0.4, \quad \delta = 0.1.$$

as in (Davis & Norman, 1990, Section 7). For these parameters the functions $g$ and $g'$ are shown in Figure 8.1.

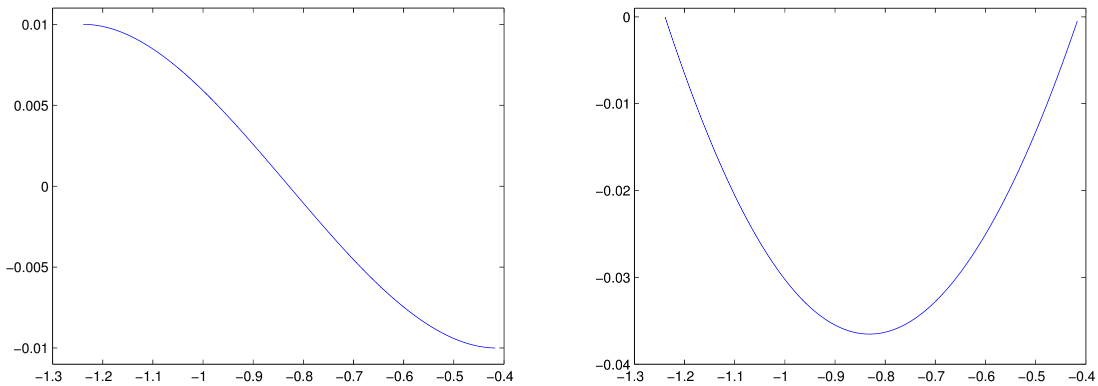


Figure 8.1: The functions $g$ and $g'$

Having determined $g$ and $g'$, the reflected process $\beta$ can be simulated using a simple Euler scheme (cf. e.g. Lépingle (1995)). By applying $g$ this yields $C$ and hence the shadow price process $\widetilde{S}$. We already know the fraction $\widetilde{\eta} = 1/(1+e^{-\beta})$ of wealth held in stocks, computed relative to $\widetilde{S}$. To calculate the liquidation value process $V$ of the optimal portfolio, notice that

$$V = \varphi^0 + \varphi^1 \underline{S} = (1-\widetilde{\eta})\widetilde{V} + \widetilde{\eta}\widetilde{V}e^{-C}(1-\underline{\lambda}) = \left(1 + ((1-\underline{\lambda})e^{-C} - 1)\widetilde{\eta}\right)\widetilde{V}.$$

For the fraction $\eta$ of liquidation wealth held in stocks we obtain

$$\eta := \frac{\varphi^1 \underline{S}}{V} = \frac{1-\underline{\lambda}}{(\widetilde{\eta}^{-1} - 1)e^C + 1 - \underline{\lambda}}.$$

It moves within the limits $0.225$ and $0.397$ in our example. The path of the bivariate process $(V_t, \eta_t V_t)$ in the wedge-shaped no-transaction region is shown in Figure 8.2.

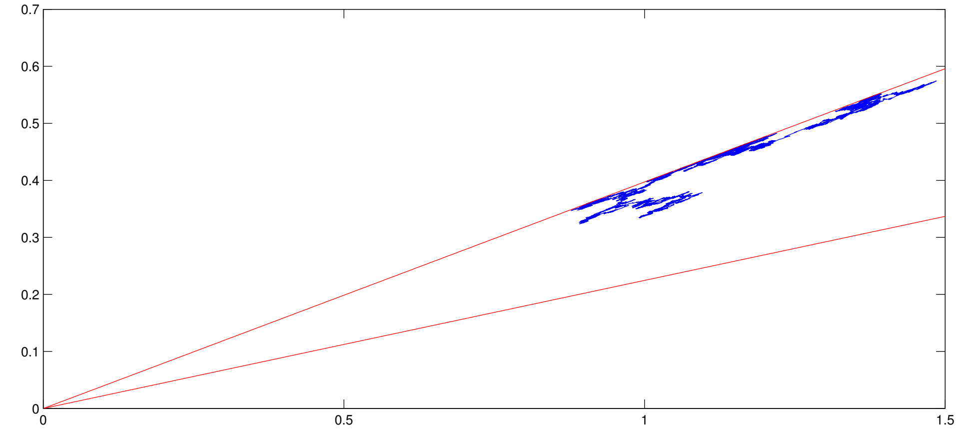


Figure 8.2: (Liquidation wealth, liquidation wealth in stock) with upper boundary, Merton line and lower boundary

Moreover, the ratio $\widetilde{S}/S$ and the optimal fraction $\eta$ are plotted with the corresponding lower respectively upper bounds in Figure 8.3.

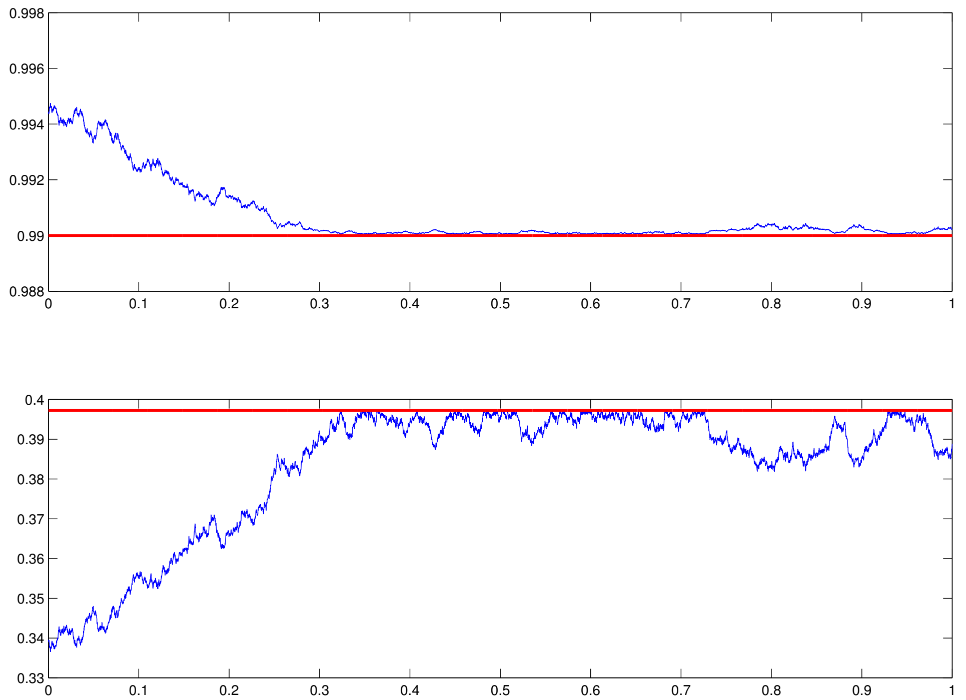


Figure 8.3: Shadow price/real price (above), optimal fraction of stock (below)

One can see that these processes are decreasing functions of one another. Nevertheless, the process in the upper graph is an Itô process whereas the lower one is not because of reflection at the boundary.

# Appendix A

# Tools from Stochastic Calculus

## A.1 Semimartingale calculus

This thesis relies heavily on the calculus of semimartingale characteristics. For the convenience of the reader we summarize here a few basic definitions and properties that are used throughout the thesis. For more details the interested reader is referred to JS.

To any $\mathbb{R}^d$-valued semimartingale $X$ there is associated a triplet $(B, C, \nu)$ of *characteristics*, where $B$ resp. $C$ denote $\mathbb{R}^d$- resp. $\mathbb{R}^{d\times d}$-valued predictable processes and $\nu$ a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ (cf. (JS, II.2.6) for more details). The first characteristic $B$ depends on a *truncation function* $h : \mathbb{R}^d \to \mathbb{R}^d$. In view of Definition 2.2 and Remark 2.3, we assume it to be of the form $h = (h_1, ..., h_d)$ with

$$h_k(x) := \chi(x_k) := \begin{cases} 0 & \text{if } x_k = 0, \\ (1 \wedge |x_k|)\frac{x_k}{|x_k|} & \text{otherwise,} \end{cases}$$

unless $X$ is a *special semimartingale*, in which case it is possible to use $h(x) = x$. Instead of the characteristics themselves, we typically use the following notion.

**Definition A.1** Let $X$ be an $\mathbb{R}^d$-valued semimartingale with characteristics $(B, C, \nu)$ relative to some truncation function $h$ on $\mathbb{R}^d$. In view of (JS, II.2.9), there exist a predictable process $A \in \mathscr{A}^+_{\text{loc}}$, an $\mathbb{R}^d$-valued predictable process $b$, an $\mathbb{R}^{d\times d}$-valued predictable process $c$ and a transition kernel $K$ from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B_t = b \bullet A_t, \quad C_t = c \bullet A_t, \quad \nu([0,t] \times G) = K(G) \bullet A_t \quad \text{for } t \in [0,T],\ G \in \mathscr{B}^d,$$

where we implicitly assume that $(b, c, K)$ is a good version in the sense that the values of $c$ are non-negative symmetric matrices, $K_s(\{0\}) = 0$ and $\int(1 \wedge |x|^2)K_s(dx) < \infty$. We call $(b, c, K, A)$ *differential characteristics* of $X$.

If $(b, c, K, A)$ denote differential characteristics of some semimartingale $X$, we write

$$\tilde{c} := c + \int xx^\top K(dx),$$

provided that the integral exists and call $\tilde{c}$ the *modified second characteristic* of $X$. This notion is motivated by the fact that $\langle X, X\rangle = \tilde{c} \bullet A$ by (JS, I.4.52) if the angle bracket process exists. From now on, we write $(b^X, c^X, K^X, A)$ and $\tilde{c}^X$ for the differential characteristics and the modified second characteristic of a semimartingale $X$. If they refer to some probability measure $P^\star$ rather than $P$, we write instead $(b^{X\star}, c^{X\star}, F^{X\star}, A)$ and $\tilde{c}^{X\star}$. The joint differential characteristics of two semimartingales $X, Y$ are denoted by

$$(b^{(X,Y)}, c^{(X,Y)}, K^{(X,Y)}, A) = \left( \begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}, K^{(X,Y)}, A \right)$$

and likewise

$$\tilde{c}^{(X,Y)} = \begin{pmatrix} \tilde{c}^X & \tilde{c}^{X,Y} \\ \tilde{c}^{Y,X} & \tilde{c}^Y \end{pmatrix},$$

if the modified second characteristic of $(X, Y)$ exists. The decomposition in Definition A.1 is of course not unique, for example $(2b^X, 2c^X, 2K^X, \frac{1}{2}A)$ yields another version. Save for discrete-time processes (where one typically chooses $A_t = \sum_{s\leq t} 1_{\mathbb{N}}(s)$), the characteristics are usually absolutely continuous in time, i.e. one may choose $A_t = t$. In this case the triplet $(b^X, c^X, K^X)$ is unique up to some $dP \otimes dt$-null set and we denote it by $\partial X := (b^X, c^X, K^X)$.

**Proposition A.2 (Lévy process)** *An $\mathbb{R}^d$-valued semimartingale $X$ with $X_0 = 0$ is a Lévy process if and only if there is a version $(b^X, c^X, K^X)$ of $\partial X$ which does not depend on $(\omega, t)$. In this case $(b^X, c^X, K^X)$ is equal to the Lévy-Khintchine triplet.*

PROOF. (JS, II.4.19). □

For $A_t = t$, one can interpret the differential characteristics as a local Lévy-Khintchine triplet. Very loosely speaking, a semimartingale with $\partial X = (b^X, c^X, K^X)$ resembles locally after $t$ a Lévy process with triplet $(b^X, c^X, K^X)(\omega, t)$, i.e. with drift rate $b$, diffusion matrix $c$, and jump measure $K$. Starting from e.g. Lévy processes as building blocks, a number of rules allow to compute the differential characteristics of more complicated processes.

**Proposition A.3 (Stochastic integration)** *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics $(b^X, c^X, K^X, A)$ and H an $\mathbb{R}^{n\times d}$-valued predictable process with $H^{j\cdot} \in L(X), j = 1, \dots, n$. Then differential characteristics of the $\mathbb{R}^n$-valued integral process $H \bullet X := (H^{j\cdot} \bullet X)_{j=1,\dots,n}$ are given by $(b^{H\bullet X}, c^{H\bullet X}, K^{H\bullet X}, A)$, where*

$$\begin{aligned} b_t^{H\bullet X} &= H_t b_t^X + \int (\widetilde{h}(H_t x) - H_t h(x)) K_t^X(dx), \\ c_t^{H\bullet X} &= H_t c_t^X H_t^\top, \\ K_t^{H\bullet X}(G) &= \int 1_G(H_t x) K_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G. \end{aligned}$$

*Here $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the truncation function which is used on $\mathbb{R}^n$.*

PROOF. (Kallsen & Shiryaev, 2002, Lemma 3). □

The combination of Propositions A.2 and A.3 yields that we have

$$b_t^X = \mu_t, \quad c_t^X = \sigma_t^2, \quad K_t^X = 0$$

for the differential characteristics of an Itô process $X$ of the form $dX_t = \mu_t dt + \sigma_t dW_t$. Itô's formula for differential characteristics reads as follows:

**Proposition A.4 ($C^2$-function)** *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics $(b^X, c^X, K^X, A)$. Suppose that $f : U \to \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subset \mathbb{R}^d$ such that $X, X_-$ are $U$-valued. Then the $\mathbb{R}^n$-valued semimartingale $f(X)$ has differential characteristics $(b^{f(X)}, c^{f(X)}, K^{f(X)}, A)$, where*

$$\begin{aligned}
b_t^{f(X),i} &= \sum_{k=1}^{d} D_k f^i(X_{t-}) b_t^{X,k} + \frac{1}{2} \sum_{k,l=1}^{d} D_{kl} f^i(X_{t-}) c_l^{X,kl} \\
&\quad + \int \left( \widetilde{h}^i(f(X_{t-}+x) - f(X_{t-})) - \sum_{k=1}^{d} D_k f^i(X_{t-}) h^k(x) \right) K_t^X(dx), \\
c_t^{f(X),ij} &= \sum_{k,l=1}^{d} D_k f^i(X_{t-}) c_t^{X,kl} D_l f^j(X_{t-}), \\
K_t^{f(X)}(G) &= \int 1_G(f(X_{t-}+x) - f(X_{t-})) K_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G.
\end{aligned}$$

PROOF. Follows immediately from (Goll & Kallsen, 2000, Corollary A.6). □

The Girsanov-Jacod-Memin theorem (cf. (JS, III.3.24)) allows to compute the effect of equivalent measure changes on the characteristics. Here we state a version put forward in Kallsen (2004) that is convenient for applications. Let $P^\star \overset{\text{loc}}{\sim} P$ be a probability measure with density process $Z$. Since $P^\star \overset{\text{loc}}{\sim} P$, $Z$, $Z_-$ are strictly positive by (JS, I.2.27). Hence the *stochastic logarithm* $N := \mathscr{L}(Z) = \frac{1}{Z_-} \bullet Z$ is a well-defined semimartingale. We now have the following result, which relates the differential $P^\star$-characteristics $(b^{(X,N)\star}, c^{(X,N)\star}, K^{(X,N)\star}, A)$ of $(X, N)$ to the characteristics $(b^{(X,N)}, c^{(X,N)}, K^{(X,N)}, A)$ of $(X, N)$ under $P$.

**Proposition A.5 (Equivalent change of measure)** *Differential $P^\star$-characteristics of the process $(X, N)$ are given by*

$$\begin{aligned}
b^{(X,N)\star} &= b^{(X,N)} + c^{(X,N),N} + \int h(x) x_{d+1} K^{(X,N)}(dx), \\
c^{(X,N)\star} &= c^{(X,N)}, \\
K^{(X,N)\star} &= \int 1_G(x)(1 + x_{d+1}) K^{(X,N)}(dx) \quad \forall G \in \mathscr{B}^d.
\end{aligned}$$

PROOF. (Kallsen, 2004, Lemma 5.1). □

The next result considers the effect of absolutely continuous time changes. For ease of exposition we only consider the case of absolutely continuous characteristics (i.e. $A_t = t$), which suffices for our needs here.

**Proposition A.6 (Absolutely continuous time-change)** *Let $X$ be an $\mathbb{R}^d$-valued semimartingale with differential characteristics $(b^X, c^X, K^X, I)$. Suppose that $(T_\vartheta)_{\vartheta\in\mathbb{R}_+}$ is a finite, absolutely continuous time change (i.e. $T_\vartheta$ is a finite stopping time for any $\vartheta$ and $T_\vartheta = \int_0^\vartheta \dot{T}_\varrho d\varrho$ with non-negative derivative $\dot{T}_\vartheta$).*

*Then the time-changed process $(\widetilde{X}_\vartheta)_{\vartheta\in\mathbb{R}_+} := (X_{T_\vartheta})_{\vartheta\in\mathbb{R}_+}$ is a semimartingale relative to the time-changed filtration $(\widetilde{\mathscr{F}_\vartheta})_{\vartheta\in\mathbb{R}_+} := (\mathscr{F}_{T_\vartheta})_{\vartheta\in\mathbb{R}_+}$ with differential characteristics $(b^{\widetilde{X}}, c^{\widetilde{X}}, K^{\widetilde{X}}, I)$ given by*

$$b^{\widetilde{X}}_\vartheta = b^X_{T_\vartheta}\dot{T}_\vartheta, \quad c^{\widetilde{X}}_\vartheta = c^X_{T_\vartheta}\dot{T}_\vartheta, \quad K^{\widetilde{X}}_\vartheta(G) = K^X_{T_\vartheta}(G)\dot{T}_\vartheta, \quad \forall G \in \mathscr{B}^d.$$

PROOF. (Kallsen, 2006, Proposition 5). □

The following Lemma shows how the characteristics are affected by stopping.

**Lemma A.7 (Stopping)** *Let $\tau$ be a stopping time and $X$ an $\mathbb{R}^d$-valued semimartingale with characteristics $(B, C, \nu)$. Then the stopped process $X^\tau$ has characteristics $(B^\tau, C^\tau, \nu^\tau)$, where $\nu^\tau$ here refers to the random measure given by*

$$1_G * \nu^\tau := 1_{\{G\cap [\![0,\tau]\!]\}} * \nu, \quad \forall G \in \mathscr{P}.$$

*If $X$ admits differential characteristics $(b^X, c^X, K^X, A)$, then $X^\tau$ has differential characteristics $(b^X 1_{[\![0,\tau]\!]}, c^X 1_{[\![0,\tau]\!]}, K^X(dx)1_{[\![0,\tau]\!]}, A)$.*

PROOF. By (JS, II.2.42) we have $A(u) \in \mathscr{M}_{\text{loc}}$ for $u \in \mathbb{R}^d$, where

$$A(u) := e^{iu^\top X} - e^{iu^\top X_-} \bullet \left( iu^\top B - \frac{1}{2}u^\top C u + \int_{[0,\cdot]\times\mathbb{R}^d} (e^{iu^\top x} - 1 - iu^\top h(x))\nu(d(t,x)) \right).$$

Since $\mathscr{M}_{\text{loc}}$ is stable under stopping, we have $A^\tau \in \mathscr{M}_{\text{loc}}$. Moreover, (JS, I.4.37) yields

$$\begin{aligned} A^\tau(u) &= e^{iu^\top X^\tau} \\ &\quad - e^{iu^\top X^\tau_-} \bullet \left( iu^\top B^\tau - \frac{1}{2}u^\top C^\tau u + \int_{[0,\cdot]\times\mathbb{R}^d} (e^{iu^\top x} - 1 - iu^\top h(x))\nu^\tau(d(t,x)) \right). \end{aligned}$$

Again by (JS, II.2.42) the characteristics of $X^\tau$ have the desired form. The second part of the claim now follows from $(b1_{[\![0,\tau]\!]}) \bullet A = B^\tau$, $(c1_{[\![0,\tau]\!]}) \bullet A = C^\tau$ and

$$(K(G)1_{[\![0,\tau]\!]}) \bullet A_t = \nu^\tau([0,t] \times G)$$

for all $G \in \mathscr{B}^d$. □

The $\sigma$-martingale property of a semimartingale can be directly read from its characteristics (cf. Kallsen (2004) for further background).

**Lemma A.8 ($\sigma$-(super-)martingales)** *Let $X$ be a semimartingale with differential characteristics $(b, c, K, A)$. Then $X$ is a $\sigma$-martingale (resp. $\sigma$-supermartingale) if and only if $\int_{\{|x|>1\}} |x| K(dx) < \infty$ and*

$$b + \int (x - h(x)) K(dx) = 0 \quad \textit{(resp. } \leq 0\textit{)}$$

*hold outside some $dP \otimes dA$-null set.*

PROOF. (Kallsen & Kühn, 2004, Lemma A.2). □

**Proposition A.9** *Let $X$ be a nonnegative $\sigma$-supermartingale with $E(X_0) < \infty$. Then $X$ is a supermartingale.*

PROOF. (Kallsen, 2004, Proposition 3.1). □

## A.2 Affine processes

In this section we state a time-inhomogeneous version of (Duffie et al., 2003, Lemma 9.2), i.e. a sufficient criterion for a strongly regular affine Markov process to be conservative. Here we use the notation and terminology of Duffie et al. (2003) and Filipović (2005).

**Lemma A.10** *Let $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ be strongly admissible parameters and denote by $X$ the corresponding strongly regular affine Markov process. Suppose $c = 0$, $\gamma = 0$, and*

$$\sup_{t \in [0,T]} \int_{D \setminus \{0\}} \left( |\eta| \wedge |\eta|^2 \right) \mu_i(t, d\xi) < \infty, \quad \forall T \in \mathbb{R}_+, \quad \forall i \in \mathscr{I}. \tag{A.1}$$

*Then $X$ is conservative.*

PROOF. The proof is a modification of Lemma 9.2 and the first part of Lemma 9.1 in Duffie et al. (2003). By definition of conservativeness and (Filipović, 2005, Definition 2.1), $X$ is conservative if $\phi(t, T, 0) = 0$ and $\psi(t, T, 0) = 0$ for $0 \leq t \leq T < \infty$. Since $\gamma = 0$ by assumption, $g = 0$ is an $\mathbb{R}^m_-$-valued solution of the initial value problem

$$\frac{\partial}{\partial t} g(t) = R^{\mathscr{Y}}(T - t, (g(t), 0)), \quad g(0) = 0. \tag{A.2}$$

By (Filipović, 2005, Theorem 2.13), $\psi^{\mathscr{Y}}(T - \cdot, T, 0)$ also solves (A.2) on $[0, T]$. From (Filipović, 2005, Proposition 4.1) it follows that $\psi^{\mathscr{Y}}(T - \cdot, T, (v, 0))$ is $\mathbb{R}^m_{--}$-valued for $v \in \mathbb{R}^m_{--}$. Therefore it is $\mathbb{R}^m_-$-valued for $v \in \mathbb{R}^m_-$ by (Filipović, 2005, Lemma 3.1 and Proposition 4.3). Similarly as in (Duffie et al., 2003, Lemma 5.3), it follows from (A.1) that $R^{\mathscr{Y}}(t, (v, 0))$ is locally Lipschitz continuous in $v \in \mathbb{R}^m_-$ for $t \in \mathbb{R}_+$. Hence we have $\psi^{\mathscr{Y}}(t, T, 0) = 0$ for $t \in [0, T]$ and $\psi(t, T, 0) = 0$ from (Filipović, 2005, (2.26)). Since $c = 0$, inserting into (Filipović, 2005, (2.24)) establishes $\phi(t, T, 0) = 0$, which completes the proof. □

## A.3 Measure changes

In this section, we state two technical results related to measure changes. First we recall a statement on the existence of probability measures on the Skorohod space which are defined in terms of their density process.

**Lemma A.11** *Let* $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d, P)$ *denote the Skorohod space of càdlàg functions endowed with some probability measure* $P$ *and* $Z$ *some nonnegative martingale on that space with* $E(Z_0) = 1$. *Then there exists a probability measure* $Q \overset{\text{loc}}{\ll} P$ *with density process* $Z$.

PROOF. For any $t \in \mathbb{R}_+$ there exists a probability measure $Q_t$ on $\mathscr{D}^d_t$ with density $Z_t$. The family $(Q_t)_{t\in\mathbb{R}_+}$ is consistent in the sense that $Q_t|_{\mathscr{D}^d_s} = Q_s$ for $s \leq t$. The assertion follows now along the same lines as (Revuz & Yor, 1999, Theorem 6.1 in the appendix) by slight modification of the proof in Stroock & Varadhan (1979). □

The next lemma shows that expectations under an equivalent probability measure can sometimes be expressed in terms of a stochastic integral of the density process.

**Lemma A.12** *Let* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ *be a filtered probability space and* $Q \sim P$ *with density process* $Z$. *Then for any increasing, predictable process* $A$ *with* $A_0 = 0$ *we have*

$$E_Q(A_T) = E_P(Z_- \bullet A_T).$$

PROOF. Since $Z$ is a $P$-martingale and $A$ is predictable and of finite variation, $A \bullet Z = \Delta A \bullet Z + A_- \bullet Z$ is a local $P$-martingale by (JS, I.4.49, I.4.34). If $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence for $A \bullet Z$, then $A \bullet Z_{T\wedge T_n}$ is a martingale starting at $0$. By (JS, III.3.4, I.4.49), this implies

$$\begin{aligned} E_Q(A_{T\wedge T_n}) &= E_P(Z_{T\wedge T_n} A_{T\wedge T_n}) \\ &= E_P(Z_- \bullet A_{T\wedge T_n} + A \bullet Z_{T\wedge T_n}) \\ &= E_P(Z_- \bullet A_{T\wedge T_n}). \end{aligned}$$

Hence monotone convergence yields $E_Q(A_T) = E_P(Z_- \bullet A_T)$ as claimed. □

# Appendix B

# Moore-Penrose pseudoinverses

The *Moore-Penrose pseudoinverse* of an arbitrary matrix or matrix valued process $c$ is a particular matrix $c^{-1}$ such that $cc^{-1}c = c$ and $c^{-1}cc^{-1} = c^{-1}$ (cf. Albert (1972) for more details). The proof of Theorem 5.25 uses some technical arguments involving pseudoinverses, due to Richhard Vierthauer, which are provided here.

Throughout, let $S_-, \widetilde{a} \in \mathbb{R}^d$ and denote by $\tilde{c}^{S\star}$ a positive semidefinite $d \times d$-matrix. Set

$$R := E_d + S_- \widetilde{a}^\top, \quad C = R\tilde{c}^{S\star}R^\top, \quad b = R\tilde{c}^{S\star}\widetilde{a}, \quad d = \widetilde{a}^\top \tilde{c}^{S\star}\widetilde{a}$$

and denote by $A$ the positive semidefinite matrix

$$A = \begin{pmatrix} C & b \\ b^\top & d \end{pmatrix}.$$

We start with some identities that are needed later on.

**Proposition B.1** *The following identities hold:*

$$CC^{-1}b = b, \tag{B.1}$$
$$bS_-^\top + R\tilde{c}^{S\star}R^\top C^{-1}R\tilde{c}^{S\star} = R\tilde{c}^{S\star}R^\top,$$
$$R\tilde{c}^{S\star}(E_d - R^\top C^{-1}R\tilde{c}^{S\star}) = 0, \tag{B.2}$$
$$R(E_d - \tilde{c}^{S\star}R^\top C^{-1}R)\tilde{c}^{S\star} = 0,$$
$$\tilde{c}^{S\star}(E_d - R^\top C^{-1}R\tilde{c}^{S\star})R^\top = 0, \tag{B.3}$$
$$(E_d - \tilde{c}^{S\star}R^\top C^{-1}R)\tilde{c}^{S\star}R^\top = 0. \tag{B.4}$$

PROOF. The first identity follows from (Albert, 1972, Theorem 9.1.6) and in turn implies the others by straightforward calculations. □

We can now prove the principal result of this section.

**Lemma B.2** *Let* $l := (R^\top, \widetilde{a})$ *and* $r := (R\tilde{c}^{S\star}, \widetilde{a}^\top \tilde{c}^\star)^\top$. *Then we have*

$$R\tilde{c}^{S\star}lA^{-1}r = R\tilde{c}^{S\star}.$$

PROOF. Define

$$C_m := \begin{pmatrix} C \\ b^\top \end{pmatrix}, \quad u_m := \begin{pmatrix} b \\ d \end{pmatrix}.$$

In order to use Greville's theorem (cf. (Albert, 1972, Theorem 4.3)) for the computation of the pseudoinverse $A^{-1}$, we have to distinguish the following two cases:

1. $(E_{d+1} - C_m C_m^{-1}) u_m = 0$.
2. $(E_{d+1} - C_m C_m^{-1}) u_m \neq 0$.

In Case 1, which is equivalent to $d = b^\top C^{-1} b$, Greville's theorem and (B.1) yield

$$A^{-1} = \begin{pmatrix} (I - w_1 C^{-1} b b^\top C^{-1}) C^{-1} (I - w_1 C^{-1} b b^\top C^{-1}) & (I - w_1 C^{-1} b b^\top C^{-1}) w_1 C^{-1} C^{-1} b \\ w_1 b^\top C^{-1} C^{-1} (I - w_1 C^{-1} b b^\top C^{-1}) & w_1^2 b^\top C^{-1} C^{-1} C^{-1} b \end{pmatrix},$$

with

$$w_1 = \frac{1}{1 + b^\top C^{-1} C^{-1} b}.$$

In Case 2, we get

$$A^{-1} = \begin{pmatrix} (I - w_2 C^{-1} b b^\top) C^{-1} & w_2 C^{-1} b \\ w_2 b^\top C^{-1} & -w_2 \end{pmatrix},$$

where

$$w_2 = \frac{1}{b^\top C^{-1} b - d}.$$

In Case 1, we now obtain

$$l A^{-1} r = R^\top C^{-1} R \tilde{c}^{S\star} + w_1^2 b^\top C^{-1} C^{-1} C^{-1} b A_1 + w_1 A_2,$$

for

$$\begin{aligned}
A_1 &:= (R^\top C^{-1} R \tilde{c}^{S\star} - E_d) \widetilde{a} \widetilde{a}^\top (\tilde{c}^{S\star} R^\top C^{-1} R - E_d) \tilde{c}^{S\star}, \\
A_2 &:= A_2^1 + A_2^2, \\
A_2^1 &:= R^\top C^{-1} C^{-1} R \tilde{c}^{S\star} \widetilde{a} \widetilde{a}^\top (E_d - \tilde{c}^{S\star} R^\top C^{-1} R) \tilde{c}^{S\star}, \\
A_2^2 &:= (E_d - R^\top C^{-1} R \tilde{c}^{S\star}) \widetilde{a} \widetilde{a}^\top \tilde{c}^{S\star} R^\top C^{-1} C^{-1} R \tilde{c}^{S\star}.
\end{aligned}$$

By (B.2) we have $R\tilde{c}^{S\star} A_1 = 0$ and $R\tilde{c}^{S\star} A_2^2 = 0$. Moreover, if the matrix $R$ is invertible, it follows from (B.4) that $A_2^1 = 0$. If $R$ is not invertible one easily verifies that there exists a basis of $\mathbb{R}^d$ consisting of $\widetilde{a}$ and $d-1$ linearly independent vectors orthogonal to $S_-$. Together with $d = b^\top C^{-1} b$, this implies $A_2^1 = 0$. Consequently, we have

$$R\tilde{c}^{S\star} l A^{-1} r = R\tilde{c}^{S\star} R^\top C^{-1} R \tilde{c}^{S\star} = R\tilde{c}^{S\star}$$

by (B.2) in Case 1. We now consider Case 2. For $A_1$ as above, we have

$$l A^{-1} r = R^\top C^{-1} R \tilde{c}^{S\star} - w_2 A_1$$

and therefore

$$R\tilde{c}^{S\star} l A^{-1} r = R\tilde{c}^{S\star}$$

by (B.2) as claimed. □

# Bibliography


Akian, M., Menaldi, J., & Sulem, A. (1995). Multi-asset portfolio selection problem with transaction costs. *Mathematics and Computers in Simulation*, 38, 163–172.

Albert, A. (1972). *Regression and the Moore-Penrose Pseudoinverse*. New York: Academic Press.

Andersen, L. & Piterbarg, V. (2007). Moment explosions in stochastic volatility models. *Finance and Stochastics*, 11, 29–50.

Andersen, T., Benzoni, L., & Lund, J. (2002). An empirical investigation of continuous-time equity return models. *Journal of Finance*, 57, 1239–1284.

Ansel, J. & Stricker, C. (1994). Couverture des actifs contingents et prix maximum. *Annales de l'Institut Henri Poincaré*, 30, 303–315.

Arai, T. (2005). An extension of mean-variance hedging to the discontinuous case. *Finance & Stochastics*, 9, 129–139.

Barles, G. & Soner, M. (1998). Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance and Stochastics*, 2, 369–397.

Barndorff-Nielsen, O. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.*, 24, 1–13.

Barndorff-Nielsen, O. (1998). Processes of normal inverse Gaussian type. *Finance & Stochastics*, 2, 41–68.

Barndorff-Nielsen, O. & Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society, Series B*, 63, 167–241.

Barndorff-Nielsen, O. & Shephard, N. (2006). Impact of jumps on returns and realised variances: Econometric analysis of time-deformed Lévy processes. *Journal of Econometrics*, 131, 217–252.

Bauer, H. (2002). *Wahrscheinlichkeitstheorie*. Berlin: de Gruyter, fifth edition.

Becherer, D. (2001). The numeraire portfolio for unbounded semimartingales. *Finance & Stochastics*, 5, 327–341.

Becherer, D. (2006). Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *The Annals of Applied Probability*, 16, 2027–2054.

Benth, F., Karlsen, K., & Reikvam, K. (2001a). A note on portfolio management under non-Gaussian logreturns. *International Journal of Theoretical and Applied Finance*, 4, 711–731.

Benth, F., Karlsen, K., & Reikvam, K. (2001b). Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach. *Finance & Stochastics*, 5, 275–303.

Benth, F., Karlsen, K., & Reikvam, K. (2003). Merton's portfolio optimization problem in a Black and Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type. *Mathematical Finance*, 13, 215–244.

Birkhoff, G. & Rota, G. (1962). *Ordinary differential equations*. Boston: Ginn and Company.

Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637–659.

Bronstein, I., Semendjajew, K., Musiol, G., & Mühlig, M. (2001). *Taschenbuch der Mathematik*. Thun: Harri Deutsch, expanded edition.

Carr, P., Geman, H., Madan, D., & Yor, M. (2003). Stochastic volatility for Lévy processes. *Mathematical Finance*, 13, 345–382.

Carr, P. & Wu, L. (2003). The finite moment log stable process and option pricing. *The Journal of Finance*, 58, 753–777.

Černý, A. & Kallsen, J. (2007). On the structure of general mean-variance hedging strategies. *The Annals of Probability*, 35, 1479–1531.

Černý, A. & Kallsen, J. (2008a). A counterexample concerning the variance-optimal martingale measure. *Mathematical Finance*, 18, 305–316.

Černý, A. & Kallsen, J. (2008b). Mean-variance hedging and optimal investment in Heston's model with correlation. *Mathematical Finance*, 18, 473–492.

Chen, L. & Filipović, D. (2005). A simple model for credit migration and spread curves. *Finance & Stochastics*, 9, 211–231.

Cheridito, P., Filipović, D., & Kimmel, R. (2007). Market price of risk specifications for affine models: Theory and evidence. *Journal of Financial Economics*, 83, 123–170.

Cheridito, P., Filipović, D., & Yor, M. (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability*, 15, 1713–1732.

Chernov, M., Gallant, R., Ghysels, E., & Tauchen, G. (2003). Alternative models for stock price dynamics. *Journal of Econometrics*, 116, 225–257.

Cont, R. & Tankov, P. (2004). *Financial Modelling with Jump Processes*. Boca Raton: Chapman & Hall/CRC.

Cox, J. & Huang, C.-F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49, 33–83.

Cox, J. & Huang, C.-F. (1991). A variational problem arising in financial economics. *Journal of Mathematical Economics*, 20, 465–487.

Cvitanić, J. & Karatzas, I. (1992). Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2, 767–818.

Cvitanić, J. & Karatzas, I. (1996). Hedging and portfolio optimization under transaction costs: A martingale approach. *Mathematical Finance*, 6, 133–165.

Cvitanić, J., Pham, H., & Touzi, N. (1999). A closed-form solution to the problem of super-replication under transaction costs. *Finance & Stochastics*, 3, 35–54.

Cvitanić, J., Schachermayer, W., & Wang, H. (2001). Utility maximization in incomplete markets with random endowment. *Finance and Stochastics*, 5, 259–272.

Cvitanić, J., Shreve, S., & Soner, H. (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, 5, 327–355.

Cvitanić, J. & Wang, H. (2001). On optimal terminal wealth under transaction costs. *Journal of Mathematical Economics*, 35, 223–231.

Dai, M., Jiang, L., Li, P., & Yi, F. (2009). Finite horizon optimal investment and consumption with transaction costs. *SIAM Journal on Control and Optimization*, 48, 1134–1154.

Davis, M. (1997). Option pricing in incomplete markets. In M. Dempster & S. Pliska (Eds.), *Mathematics of Derivative Securities* (pp. 216–226). Cambridge: Cambridge University Press.

Davis, M. & Norman, A. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15, 676–713.

Davis, M., Panas, V., & Zariphopoulou, T. (1993). European option pricing with transaction costs. *SIAM Journal on Control and Optimization*, 31, 470–493.

Deelstra, G., Pham, H., & Touzi, N. (2001). Dual formulation of the utility maximization problem under transaction costs. *The Annals of Applied Probability*, 11, 1353–1383.

Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M., & Stricker, C. (2002). Exponential hedging and entropic penalties. *Mathematical Finance*, 12, 99–123.

Delbaen, F. & Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300, 463–520.

Delbaen, F. & Schachermayer, W. (1997). The Banach space of workable contingent claims in arbitrage theory. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 33, 113–144.

Delbaen, F. & Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 312, 215–250.

Delong, L. & Klüppelberg, C. (2008). Optimal investment and consumption in a Black-Scholes market with Lévy-driven stochastic coefficients. *Annals of Applied Probability*, 18, 879–908.

Di Nunno, G., Meyer-Brandis, T., Øksendal, B., & Proske, F. (2006). Optimal portfolio for an insider in a market driven by Lévy processes. *Quantitative Finance*, 6, 83–94.

Duffie, D., Filipovic, D., & Schachermayer, W. (2003). Affine processes and applications in finance. *The Annals of Applied Probability*, 13, 984–1053.

Duffie, D., Fleming, V., Soner, M., & Zariphopoulou, T. (1997). Hedging in incomplete markets with HARA utility. *Journal of Economic Dynamics & Control*, 21, 753–782.

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68, 1343–1376.

Duffie, D. & Singleton, K. (1993). Simulated moments estimation of Markov models of asset prices. *Econometrica*, 61, 929–952.

Eberlein, E. & Jacod, J. (1997). On the range of option prices. *Finance & Stochastics*, 1, 131–140.

Eberlein, E., Jacod, J., & Raible, S. (2005). Lévy term structure models: No-arbitrage and completeness. *Finance & Stochastics*, 9, 67–88.

Eberlein, E. & Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli*, 1, 281–299.

Eraker, B., Johannes, M., & Polson, N. (2003). The impact of jumps in volatility and returns. *The Journal of Finance*, 58, 1269–1300.

Filipović, D. (2005). Time-inhomogeneous affine processes. *Stochastic Processes and their Applications*, 115, 639–659.

Foldes, L. (1990). Conditions for optimality in the infinite-horizon portfolio-cum-saving problem with semimartingale investments. *Stochastics and Stochastics Reports*, 29, 133–170.

Foldes, L. (1992). Existence and uniqueness of an optimum in the infinite-horizon portfolio-cum-saving problem with semimartingale investments. *Stochastics and Stochastics Reports*, 41, 241–267.

Föllmer, H. & Sondermann, D. (1986). Hedging of nonredundant contingent claims. In W. Hildenbrand & A. Mas-Colell (Eds.), *Contributions to Mathematical Economics* (pp. 205–223). Amsterdam: North-Holland.

Framstad, N., Øksendal, B., & Sulem, A. (1999). *Optimal Consumption and Portfolio in a Jump Diffusion Market*. Technical report, Norges handelshøyskole. Institutt for foretaksøkonomi.

Framstad, N., Øksendal, B., & Sulem, A. (2001). Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs. *Journal of Mathematical Economics*, 35, 233–257.

Gallant, R. & Tauchen, G. (1996). Which moments to match? *Econometric Theory*, 12, 657–681.

Genon-Catalot, V., Jeantheau, T., & Larédo, C. (2000). Stochastic volatility models as hidden Markov models and statistical applications. *Bernoulli*, 6, 1051–1079.

Goll, T. & Kallsen, J. (2000). Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications*, 89, 31–48.

Goll, T. & Kallsen, J. (2001). *A Note on the Log-Optimal Portfolio Problem*. Technical Report 32/2001, Mathematische Fakultät Universität Freiburg i. Br.

Goll, T. & Kallsen, J. (2003). A complete explicit solution to the log-optimal portfolio problem. *The Annals of Applied Probability,*, 13, 774–799.

Gourieroux, C., Laurent, J., & Pham, H. (1998). Mean-variance hedging and numéraire. *Mathematical Finance*, 8, 179–200.

Guasoni, P. (2006). No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Mathematical Finance*, 16, 569–582.

Guasoni, P., Rásonyi, M., & Schachermayer, W. (2008a). Consistent price systems and face-lifting pricing under transaction costs. *The Annals of Applied Probability*, 18, 491–520.

Guasoni, P., Rásonyi, M., & Schachermayer, W. (2008b). The fundamental theorem of asset pricing for continuous processes under small transaction costs. *The Annals of Finance*. To appear.

Hakansson, N. (1970). Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica*, 38, 587–607.

Hansen, L. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, 50, 1029–1054.

Harvey, A. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge: Cambridge University Press.

Haug, S., Klüppelberg, C., Lindner, A., & Zapp, M. (2007). Method of moment estimation in the COGARCH(1,1) model. *The Econometrics Journal*, 10, 320–341.

He, H. & Pearson, N. (1991a). Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite-dimensional case. *Mathematical Finance*, 1, 1–10.

He, H. & Pearson, N. (1991b). Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite-dimensional case. *Journal of Economic Theory*, 54, 259–304.

Henderson, V. (2002). Valuation of claims on non-traded assets using utility maximization. *Mathematical Finance*, 12, 351–371.

Henderson, V. & Hobson, D. (2002). Real options with constant relative risk aversion. *Journal of Economic Dynamics & Control*, 27, 329–355.

Heston, S. (1993). A closed-form solution for options with stochastic volatilities with applications to bond and currency options. *The Review of Financial Studies*, 6, 327–343.

Hobson, D. (2004). Stochastic volatility models, correlation, and the $q$-optimal martingale measure. *Mathematical Finance*, 14, 537–556.

Hodges, S. & Neuberger, A. (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8, 222–239.

Hubalek, F., Krawczyk, L., & Kallsen, J. (2006). Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability*, 16, 853–885.

Hugonnier, J. & Kramkov, D. (2004). Optimal investment with random endowments in incomplete markets. *The Annals of Applied Probability*, 14, 845–864.

Hugonnier, J., Kramkov, D., & Schachermayer, W. (2005). On utility-based pricing of contingent claims in incomplete markets. *Mathematical Finance*, 15, 203–212.

Hurd, T. (2004). A note on log-optimal portfolios in exponential Lévy markets. *Statistics & Decisions*, 22, 225–233.

Ibragimov, I. & Linnik, Y. (1971). *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing, Groningen.

Ilhan, A. & Sircar, R. (2006). Optimal static-dynamic hedges for barrier options. *Mathematical Finance*, 16, 359–385.

Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*, volume 714 of *Lecture Notes in Mathematics*. Berlin: Springer.

Jacod, J. & Shiryaev, A. (2003). *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Berlin: Springer-Verlag, second edition.

Jakubėnas, P., Levental, S., & Ryznar, M. (2003). The super-replication problem via probabilistic methods. *The Annals of Applied Probability*, 13, 742–773.

Jiang, G. & Knight, J. (2002). Estimation of continuous-time processes via the empirical characteristic function. *Journal of Business & Economic Statistics*, 20, 198–212.

Jouini, E. & Kallal, H. (1995). Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66, 178–197.

Kabanov, Y. (1999). Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, 3, 237–248.

Kabanov, Y., Rásonyi, M., & Stricker, C. (2002). No-arbitrage criteria for financial markets with efficient friction. *Finance and Stochastics*, 6, 371–382.

Kallsen, J. (1998). *Duality Links between Portfolio Optimization and Derivative Pricing*. Technical Report 40/1998, Mathematische Fakultät Universität Freiburg i. Br.

Kallsen, J. (2000). Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, 51, 357–374.

Kallsen, J. (2002). Utility-based derivative pricing in incomplete markets. In H. Geman, D. Madan, S. Pliska, & T. Vorst (Eds.), *Mathematical Finance – Bachelier Congress 2000* (pp. 313–338). Berlin: Springer.

Kallsen, J. (2004). $\sigma$-localization and $\sigma$-martingales. *Theory of Probability and Its Applications*, 48, 152–163.

Kallsen, J. (2006). A didactic note on affine stochastic volatility models. In Y. Kabanov, R. Liptser, & J. Stoyanov (Eds.), *From Stochastic Calculus to Mathematical Finance* (pp. 343–368). Berlin: Springer.

Kallsen, J. (2008). Option pricing. In T. Andersen, R. Davis, J. Kreiß, & T. Mikosch (Eds.), *Handbook of Financial Time Series*. Berlin: Springer. To appear.

Kallsen, J. & Kühn, C. (2004). Pricing derivatives of American and game type in incomplete markets. *Finance & Stochastics*, 8, 261–284.

Kallsen, J. & Muhle-Karbe, J. (2008a). Exponentially affine martingales, affine measure changes and exponential moments of affine processes. Preprint.

Kallsen, J. & Muhle-Karbe, J. (2008b). On using shadow prices in portfolio optimization with transaction costs. Preprint.

Kallsen, J. & Muhle-Karbe, J. (2008c). Utility maximization in affine stochastic volatility models. Preprint.

Kallsen, J. & Rheinländer, T. (2008). Asymptotic utility-based pricing and hedging for exponential utility. Preprint.

Kallsen, J. & Shiryaev, A. (2002). Time change representation of stochastic integrals. *Theory of Probability and Its Applications*, 46, 522–528.

Karatzas, I. & Kou, S. (1996). On the pricing of contingent claims under constraints. *The Annals of Applied Probability*, 6, 321–369.

Karatzas, I., Lehoczky, J., & Shreve, S. (1987). Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization*, 25, 1557–1586.

Karatzas, I., Lehoczky, J., Shreve, S., & Xu, G. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29, 702–730.

Karatzas, I. & Shreve, S. (1988). *Brownian Motion and Stochastic Calculus*. New York: Springer.

Karatzas, I. & Shreve, S. (1998). *Methods of Mathematical Finance*. Berlin: Springer.

Karatzas, I. & Žitković, G. (2003). Optimal consumption from investment and random endowment in incomplete semimartingale markets. *The Annals of Probability*, 31, 1821–1858.

Keller-Ressel, M. (2008). Moment explosions and long-term behavior of affine stochastic volatility models. *Mathematical Finance*. Forthcoming.

Kim, T. & Omberg, E. (1996). Dynamic nonmyopic portfolio behavior. *The Review of Financial Studies*, 9, 141–161.

Klüppelberg, C., Lindner, A., & Maller, R. (2004). A continuous-time GARCH process driven by a Lévy process: Stationarity and second order behaviour. *Journal of Applied Probability*, 41, 601–622.

Klüppelberg, C., Lindner, A., & Maller, R. (2006). Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In Y. Kabanov, R. Liptser, & J. Stoyanov (Eds.), *From Stochastic Calculus to Mathematical Finance* (pp. 393–419). Berlin: Springer.

Korn, R. (1997). *Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management in Continuous Time*. Singapore: World Scientific.

Korn, R. & Kraft, H. (2004). On the stability of continuous-time portfolio problems with stochastic opportunity set. *Mathematical Finance*, 14, 403–414.

Kou, S. (2002). A jump-diffusion model for option pricing. *Management Science*, 48, 1086–1101.

Kraft, H. (2005). Optimal portfolios and Heston's stochastic volatility model: An explicit solution for power utility. *Quantitative Finance*, 5, 303–313.

Kramkov, D. & Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9, 904–950.

Kramkov, D. & Schachermayer, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *The Annals of Applied Probability*, 13, 1504–1516.

Kramkov, D. & Sîrbu, M. (2006). On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. *The Annals of Applied Probability*, 16, 1352–1384.

Kramkov, D. & Sîrbu, M. (2006). The sensitivity analysis of utility based prices and the risk-tolerance wealth processes. *The Annals of Applied Probability*, 16, 2140–2194.

Kramkov, D. & Sîrbu, M. (2007). Asymptotic analysis of utility-based hedging strategies for small number of contingent claims. *Stochastic Processes and their Applications*, 117, 1606–1620.

Lamberton, D., Pham, H., & Schweizer, M. (1998). Local risk-minimization under transaction costs. *Mathematics of Operations Research*, 23, 585–612.

Lépingle, D. (1995). Euler scheme for reflected stochastic differential equations. *Mathematics and Computers in Simulation*, 38, 119–126.

Lépingle, D. & Mémin, J. (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 42, 175–203.

Levental, S. & Skorohod, A. (1997). On the possibility of hedging options in the presence of transaction costs. *The Annals of Applied Probability*, 7, 410–443.

Lions, P. & Musiela, M. (2007). Correlations and bounds for stochastic volatility models. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, 24, 1–16.

Liu, H. & Loewenstein, M. (2002). Optimal portfolio selection with transaction costs and finite horizons. *The Review of Financial Studies*, 15, 805–835.

Liu, J. (2007). Portfolio selection in stochastic environments. *The Review of Financial Studies*, 20, 1–39.

Loewenstein, M. (2002). On optimal portfolio trading strategies for an investor facing transaction costs in a continuous trading market. *Journal of Mathematical Economics*, 33, 209–228.

Madan, D. & Seneta, E. (1990). The VG model for share market returns. *Journal of Business*, 63, 511–524.

Magill, M. & Constantinides, G. (1976). Portfolio selection with transaction costs. *Journal of Economic Theory*, 13, 245–263.

Mania, M. & Schweizer, M. (2005). Dynamic exponential utility indifference valuation. *The Annals of Applied Probability*, 15, 2113–2143.

Masuda, H. (2004). On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process. *Bernoulli*, 10, 97–120.

Merton, R. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51, 247–257.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3, 373–413.

Merton, R. (1973). An intertemporal capital asset pricing model. *Econometrica*, 41, 867–887.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3, 125–144.

Mossin, J. (1968). Optimal multiperiod portfolio policies. *Journal of Business*, 41, 215–229.

Nicolato, E. & Venardos, E. (2003). Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Mathematical Finance*, 13, 445–466.

Øksendal, B. & Sulem, A. (2005). *Applied Stochastic Control of Jump Diffusions*. Berlin: Springer.

Pauwels, A. (2007). *Varianz-optimales Hedging in affinen Volatilitätsmodellen*. Dissertation Technische Universität München.

Pigorsch, C. & Stelzer, R. (2008). A multivariate Ornstein-Uhlenbeck type stochastic volatility model. Preprint.

Pitt, M. & Shephard, N. (1999). Filtering via simulation: Auxiliary particle filters. *Journal of the American Statistical Association*, 94, 590–599.

Pliska, S. (1986). A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operations Research*, 11, 371–384.

Pliska, S. (1997). *Introduction to Mathematical Finance*. Malden, MA: Blackwell.

Protter, P. (2004). *Stochastic Integration and Differential Equations*. Berlin: Springer, second edition.

Raible, S. (2000). *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts*. Dissertation Universität Freiburg i. Br.

Revuz, D. & Yor, M. (1999). *Continuous Martingales and Brownian Motion*. Berlin: Springer, third edition.

Rheinländer, T. & Schweizer, M. (1997). On $L^2$-projections on a space of stochastic integrals. *The Annals of Probability*, 25, 1810–1831.

Rockafellar, T. (1970). *Convex Analysis*. Princeton: Princeton University Press.

Rockafellar, T. & Wets, R. (1998). *Variational Analysis*. Berlin: Springer.

Rouge, R. & El Karoui, N. (2000). Pricing via utility minimization and entropy. *Mathematical Finance*, 10, 259–276.

Rydberg, T. (1997). The normal inverse Gaussian Lévy process: Simulation and approximation. *Communications in Statistics. Stochastic Models*, 13, 887–910.

Sainte-Beuve, M.-F. (1974). On the extension of von Neumann-Aumann's theorem. *Journal of Functional Analysis*, 17, 112–129.

Samuelson, P. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51, 239–246.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Schachermayer, W. (2001). Optimal investment in incomplete markets when wealth may become negative. *The Annals of Applied Probability*, 11, 694–734.

Schachermayer, W. (2004). The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. *Mathematical Finance*, 14, 19–48.

Schoutens, W. (2003). *Lévy Processes in Finance*. New York: Wiley.

Schweizer, M. (1994). Approximating random variables by stochastic integrals. *The Annals of Probability*, 22, 1536–1575.

Shiryaev, A. (1995). *Probability*. New York: Springer, second edition.

Shreve, S. & Soner, M. (1994). Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, 4, 609–692.

Skorokhod, A. (1961). Stochastic equations for diffusion processes in a bounded region. *Theory of Probability and its Applications*, 6, 264–274.

Skorokhod, A. (1962). Stochastic equations for diffusion processes in a bounded region II. *Theory of Probability and its Applications*, 7, 3–23.

Sørensen, M. (2000). Prediction-based estimating functions. *The Econometrics Journal*, 3, 123–147.

Stroock, D. & Varadhan, S. (1979). *Multidimensional Diffusion Processes*. Berlin: Springer.

van der Vaart, A. (1998). *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge: Cambridge University Press.

Vierthauer, R. (2009). *Hedging in Affine Stochastic Volatility Models*. Dissertation Christian-Albrechts-Universität zu Kiel. Forthcoming.

von Neumann, J. & Morgenstern, O. (1947). *Theory of Games and Economic Behavior*. Princeton, N. J.: Princeton University Press. 2d ed.

Whalley, A. & Wilmott, P. (1997). An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7, 307–324.

Winkel, M. (2001). The recovery problem for time-changed Lévy processes. Preprint.

Wong, B. & Heyde, C. (2004). On the martingale property of stochastic exponentials. *Journal of Applied Probability*, 41, 654–664.

Zakamouline, V. (2006). Efficient analytic approximation of the optimal hedging strategy for a European call option with transaction costs. *Quantitative Finance*, 6, 435–445.

Žitković, G. (2005). Utility maximization with a stochastic clock and an unbounded random endowment. *The Annals of Applied Probability*, 15, 748–777.

# General Notation

| | |
|---|---|
| $\mathbb{N}$, $\mathbb{N}^*$ | $\{0,1,2,3,\ldots\}$, $\{1,2,3,\ldots\}$ |
| $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ | $(-\infty,\infty)$, $[0,\infty)$, $(0,\infty)$ |
| $x \wedge y$, $x \vee y$ | $\min\{x,y\}$, $\max\{x,y\}$ for $x$, $y \in \mathbb{R}$ |
| $x^+$, $x^-$ | $x \vee 0$, $-x \vee 0$ |
| $\lfloor x \rfloor$ | $\max\{n \in \mathbb{N} : n \leq x\}$ |
| $\mathbb{R}^d$, $\mathbb{C}^d$ | the Euclidean resp. unitary $d$-dimensional space |
| $\lvert x\rvert$, $\lvert z\rvert$ | the Euclidean resp. unitary norm of $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$ |
| $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ | the real resp. imaginary part of $z \in \mathbb{C}^d$ |
| $\mathbb{R}^d_-$ | $\{x \in \mathbb{R}^d : x_i \leq 0,\ i = 1,\ldots,d\}$ |
| $\mathbb{R}^d_{--}$ | $\{x \in \mathbb{R}^d : x_i < 0,\ i = 1,\ldots,d\}$ |
| $\mathbb{C}^d_-$ | $\{z \in \mathbb{C}^d : \mathrm{Re}(z) \in \mathbb{R}^d_-\}$, |
| $\mathbb{R}^{m\times n}$ | the set of $m \times n$-matrices with real-valued entries |
| $A^\top$ | the transpose of the matrix $A$ |
| $A^{-1}$ | the Moore-Penrose pseudoinverse of the matrix $A$ |
| $e_i$ | the $i$-th unit vector $(0,\ldots,0,1,0,\ldots,0)^\top$ in $\mathbb{R}^d$ |
| $E_d$ | the identity matrix $(e_1,\ldots,e_d)$ in $\mathbb{R}^{d\times d}$ |
| $\mathscr{F}, \mathscr{G}, \mathscr{H}$ | $\sigma$-fields |
| $P, P^{€}, P^\star$ | probability measures |
| $Q, Q_0, Q^\$$ | equivalent martingale measures |
| $Q \overset{\text{loc}}{\ll} P, Q \ll P$ | (local) absolute continuity of $Q$ w.r.t. $P$ |
| $Q \overset{\text{loc}}{\sim} P, Q \sim P$ | (local) equivalence of $Q$ w.r.t. $P$ |
| $\frac{dQ}{dP}$ | the Radon-Nickodym derivative of $Q \ll P$ |
| $P\vert_{\mathscr{G}}$ | the restriction of the measure $P$ to the $\sigma$-field $\mathscr{G}$ |
| $\mathbf{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}, \mathbf{G} = (\mathscr{G}_u)_{u\in\mathbb{R}_+}$ | filtrations |
| $(\Omega, \mathscr{F}, \mathbf{F}, P)$ | filtered probability space |
| $E_P(X)$, $\mathbb{V}\mathrm{ar}_P(X)$ | expectation, variance of the random variable $X$ under $P$ |
| $L^p(P)$ | the random variables s.t. $E_P(\lvert X\rvert^p) < \infty, p \in [1,\infty)$ |
| $\mathbb{C}\mathrm{ov}_P(X)$ | the covariance of random variables $X, Y$ under $P$ |
| $E_P(X\vert\mathscr{G})$ | the conditional expectation of $X$ given $\mathscr{G}$ under $P$ |
| $\mu_i$, $m_i$ | $i$-th centered resp. uncentered moment |
| $\widehat{\mu}_i$, $\widehat{m}_i$ | $i$-th centered resp. uncentered empirical moment |
| $c_i$ | $i$-th cumulant |

| | |
|---|---|
| $P^X$ | the distribution of $X$ under $P$ |
| $P$-a.s. $P$-a.e. | almost surely, almost everywhere w.r.t $P$ |
| $\xrightarrow{\text{a.s.}}$, $\xrightarrow{d}$ | a.s. convergence, convergence in distribution |
| $\otimes$ | the product of $\sigma$-fields or measures |
| $\sigma(X_i : i \in I)$ | the $\sigma$-field generated by $\{X_i : i \in I\}$ |
| $\mathscr{B}$, $\mathscr{B}^d$ | the Borel $\sigma$-field on $\mathbb{R}$ resp. $\mathbb{R}^d$ |
| $\mathscr{P}$ | the predictable $\sigma$-field |
| $\widetilde{\mathscr{P}}$ | $\mathscr{P} \otimes \mathscr{B}^d$ |
| $\mathbb{D}^d$ | the Skorohod space $\{\alpha : \mathbb{R}_+ \to \mathbb{R}^d : \alpha \text{ càdlàg}\}$ |
| $\pi_t$ | the projection $\pi_t(\alpha) := \alpha_t$ for $\alpha \in \mathbb{D}^d$ |
| $\mathscr{D}_t^0(\mathbb{R}^d)$, $\mathscr{D}^d$, $\mathscr{D}_t^d$ | $\sigma(\pi_s : s \le t)$, $\sigma\Big(\bigcup_{t \in \mathbb{R}_+} \mathscr{D}_t^0(\mathbb{R}^d)\Big)$, $\bigcap_{s>t} \mathscr{D}_s^0(\mathbb{R}^d)$ |
| $\varepsilon_x$ | the Dirac-measure in $x \in \mathbb{R}^d$ |
| $\mathrm{IG}(a,b)$ | the inverse Gaussian distribution with parameters $a, b > 0$ |
| $\Gamma(a,b)$ | the gamma distribution with parameters $a, b > 0$ |
| $K_1$ | the modified Bessel function of the third kind with index 1 |
| $A^C$ | the complement of the set $A$ |
| $1_A$ | the indicator function of the set $A$ |
| $A_1 \times A_2$ | the Cartesian product of the sets $A_1$ and $A_2$ |
| $o(\cdot)$, $O(\cdot)$ | the Landau order symbols |
| $C^k([0,T], \mathbb{R}^d)$ | the spaces of $k$-times continuously differentiable functions $f : [0,T] \to \mathbb{R}^d$ |
| $C^\infty([0,T], \mathbb{R}^d)$ | $\bigcap_{k \in \mathbb{N}} C^k([0,T], \mathbb{R}^d)$ |
| $f'$ | the derivative of the real function $f$ |
| $f\|_A$ | the restriction of the function $f$ to the set $A$ |
| $\arg\min_{x \in A} f$ | the point where $f\|_A$ attains its minimum |
| $r_y$ | the autocorrelation function of a stationary process $y$ |
| $\gamma$ | the autocovariance function of a stationary process |
| $(\alpha_y(k))_{k \in \mathbb{N}}$ | the mixing coefficients of an $\alpha$-mixing process $(y_n)_{n \in \mathbb{N}}$ |
| $\mathscr{M}_{\text{loc}}$ | the set of local martingales |
| $\mathscr{H}^2$ | the set of square-integrable martingales |
| $\mathscr{H}_0^2$ | the set of square-integrable martingales starting in $0$ |
| $\mathscr{H}_{\text{loc}}^2$ | the set of locally square-integrable martingales |
| $\mathscr{A}_{\text{loc}}^+$ | the set of cádlág, adapted processes, starting in $0$ that are locally integrable and increasing |
| $L(X)$ | the set of processes integrable w.r.t. the semimartingale $X$ |
| $Y \bullet X.$ | the stochastic integral $\int_0^{\cdot} Y_s \, dX_s$ of $Y \in L(X)$ w.r.t. the semimartingale $X$, |
| $X = X_0 + A^X + M^X$ | the semimartingale decomposition of the semimartingale $X$ |
| $X_{t-}$ | the left limit $\lim_{s \uparrow t} X_s$ of the semimartingale $X$ |
| $\Delta X_t$ | the jump $X_t - X_{t-}$ of the semimartingale $X$ |
| $X_{(n)}$ | the increment $X_{n\Delta} - X_{(n-1)\Delta}$ of the semimartingale $X$ |

| | |
|---|---|
| $\mathscr{E}(X)$ | the stochastic exponential of the semimartingale $X$ |
| $\mathscr{L}(X)$ | the stochastic logarithm of the semimartingale $X$ |
| $X^c$ | the continuous martingale part of the semimartingale $X$ |
| $[X, Y]$ | the quadratic covariation of the semimartingales $X, Y$ |
| $\langle X, Y \rangle$ | the predictable quadratic covariation (angle bracket process) of the semimartingales $X$ and $Y$ |
| $X_Y$ | time-changed process |
| $(\mathscr{G}_{Y_t})_{t\in\mathbb{R}_+}$ | time-changed filtration |
| $X^\tau$ | the process $X$ stopped at $\tau$, i.e. $X^\tau_t = X_{\tau\wedge t}$ |
| $[\![\tau, \tilde{\tau}]\!]$, $[\![\tau, \tilde{\tau}[\![$ etc. | stochastic intervals of $\tau$, $\tilde{\tau}$ |
| $\mu^X$ | the random measure of jumps of the semimartingale $X$ |
| $\nu^X$ | the compensator of $\mu^X$ |
| $W * \mu$ | the integral process of the $\widetilde{\mathscr{P}}$-measurable function $W$ w.r.t. the random measure $\mu$ |
| $W * (\mu^X - \nu^X)$ | the integral process of the $\widetilde{\mathscr{P}}$-measurable function $W$ w.r.t. the compensated random measure $\mu^X - \nu^X$ |
| $h$, $\chi$ | truncation functions |
| $(B^X, C^X, \nu^X)$ | the integral characteristics of the semimartingale $X$ |
| $(b^X, c^X, K^X, A)$ | the differential characteristics of the semimartingale $X$ |
| $I$ | the identity process $I_t = t$ |
| $\partial X$ | the differential characteristics of $X$ w.r.t. $A = I$ or the subdifferential of a convex function in 7.8, 7.10 |
| $(b^L, c^L, K^L)$ | the Lévy-Khintchine triplet of the Lévy process $L$ |
| $\psi^L$ | the Lévy exponent of the Lévy process $L$, i.e. |
| $\psi^L(u)$ | $u^\top b^L + \frac{1}{2}u^\top c^L u + \int_{\mathbb{R}^d} \left(e^{u^\top x} - 1 - u^\top h(x)\right) K^L(dx)$ |
| $\psi^X_i$ | the Lévy exponent corresponding to $(\beta_i, \gamma_i, \kappa_i)$ for a process $X$ affine w.r.t. Lévy-Khintchine triplets $(\beta_i, \gamma_i, \kappa_i)$, $i = 0, \ldots, m$ |

# Index of Terminology