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Determination and Valuation of Recovery Risk in Credit-Risk Models

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Abstract

This thesis is concerned with the modelling of recovery rates, their behaviour, and the valuation of recovery risk in credit-risk models. In particular, the characteristics and determinants of facility-level as well as aggregated recovery rates are examined on a unique Pan-European dataset. The empirical insights obtained from these investigations combined with stylized facts like the negative correlation between recovery rates and default rates are emphasized to derive a consistent model for the valuation of single-name credit derivatives under stochastic recovery. Based on the class of intensity-based credit risk models, the model yields analytically tractable pricing formulas and in particular allows for the pricing of single-name credit derivatives with payoffs that are directly linked to the recovery rate at default, like recovery locks. Furthermore, stochastic recovery rates are also considered in the context of portfolio credit-risk modelling. Using nested Archimedean copulas the joint modelling of recovery rates and default rates in a portfolio of credit-risky assets is extended to a non-Gaussian dependence structure. Within this framework an efficient algorithm for sampling the loss process of the portfolio and pricing tranches of CDOs is presented. Finally, the well-known concept of base correlations is adapted to stochastic recoveries in a non-Gaussian setting. This modelling approach yields significantly flatter base correlation curves compared to the current market standard and therefore simplifies the pricing of non-standardized CDO tranches.

Zusammenfassung

Diese Arbeit befasst sich mit der Modellierung von Erlösquoten und ihrem Verhalten sowie der Bewertung von Erlösquotenrisiken in Kreditrisikomodellen. Basierend auf einer einzigartigen paneuropäischen Datenbank werden zunächst die Eigenschaften und bestimmenden Faktoren von Erlösquoten sowohl auf einem Einzelkredit- als auch auf einem aggregierten Level untersucht. Unter Verwendung der so gewonnenen empirischen Erkenntnisse, in Verbindung mit charakteristischen Eigenschaften wie der negativen Korrelation von Ausfallraten und Erlösquoten, wird ein konsistentes Modell zur Bewertung von Kreditderivaten unter Berücksichtigung stochastischer Erlösquoten entwickelt. Aufbauend auf der Klasse der intensitäts-basierten Kreditrisikomodelle liefert das betrachtete Modell analytisch gut handhabbare Bewertungsformeln und erlaubt es insbesondere Kreditderivate zu bewerten, deren Auszahlungsprofil direkt von der Erlösquote bei Ausfall abhängt. Darüber hinaus werden stochastische Erlösquoten auch im Kontext von Portfoliokreditrisikomodellen betrachtet. Mittels hierarchischer archimedischer Kopulas wird die gemeinsame Modellierung von Ausfallraten und Erlösquoten in einem Portfolio ausfallbehafteter Anlagen auf nicht-Gaußsche Abhängigkeitsstrukturen erweitert. Im Rahmen dieses Modells wird ein effizienter Monte Carlo Algorithmus zur Bestimmung des Portfolioverlustprozesses und zur Bewertung der Tranchen eines CDO vorgestellt. Abschließend wird das marktübliche Konzept der Basis-Korrelation auf den Fall stochastischer Erlösquoten in einem nicht-Gaußschen Modellrahmen erweitert, wodurch, verglichen mit dem bisherigen Marktstandard, signifikant flachere Basis-Korrelations-Kurven erzielt werden und somit die Bewertung von nicht-standardisierten CDO Tranchen erleichtert wird.

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Contents

Chapter 1

Introduction

1.1 Motivation

Over the last decades a lot of progress has been made in the field of credit-risk modelling. While there has been a great variety of literature concerning the description, modelling, and prediction of default probabilities (going back to the seminal works of Beaver (1966), Altman (1968), and Merton (1974)), there have been only few studies concerning the description of recovery rates, their behaviour, and their determinants. Accordingly most credit-risk pricing models, both for single-name and portfolio credit derivatives, rather concentrate on default-event risk or default-loss risk and neglect recovery risks. Caused by the rapid growth in the credit-derivatives market at the beginning of this century (see Figure 1.1) and the appearance of contingent claims on recoveries, e.g. fixed-recovery CDS, recovery locks, or recovery swaps, the sound modelling of recovery rates gained in importance lately, both for pricing purposes and portfolio risk management as well as for economic capital requirements.

Particularly, the worldwide financial crisis starting in 2007 gave rise to a new discussion on recovery rates in the academic literature as well as among practitioners. While during the years of continuous economic growth and very low default rates from 2003 until mid 2007, the amount recovered after a potential default event was rather of theoretical interest, things changed when economic and credit conditions deteriorated due to the upcoming US sub-prime crisis. The increasing number of companies in financial distress, the large variation in realized recovery rates, and the high complexity of many credit derivatives led to a need for a better understanding of all variables involved in the loss process of a defaultable security. Table 1.1 shows the 2008 CDS credit event auction results from Creditex Group Inc.™ and Markit

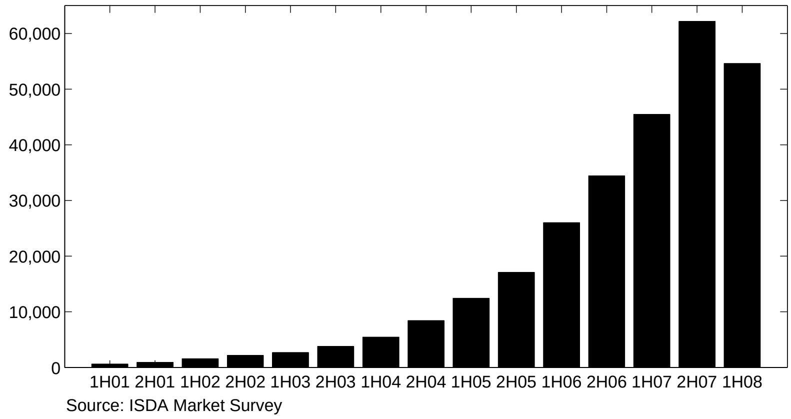

Figure 1.1: CDS notional amount outstanding in billions of US dollar.

Group Limited™ (`www.creditfixings.com`). As can be easily seen recovery rates are not constant and therefore the assumption of a constant recovery rate of 40% typically used in standard CDS and CDO models might be questionable.

Company	Date	Recovery (Senior unsecured)
Quebecor	19/02/2008	41.25%
Tembec	02/10/2008	83.00%
Fannie Mae	06/10/2008	91.51%
Freddie Mac	06/10/2008	94.00%
Lehman Brothers	10/10/2008	8.63%
Washington Mutual	23/10/2008	57.00%
Landsbanki	04/11/2008	1.25%
Glitnir	05/11/2008	3.00%
Kaupthing Bank	06/11/2008	6.63%

Table 1.1: 2008 CDS credit event auction results.

Moreover, it can be observed that recovery rates are lower in a distressed economy than in a healthy economy and default rates and recovery rates are negatively correlated (see Figure 1.2).
These facts as well as increasing regulatory requirements put a stronger emphasis on the development of models including a realistic recovery-rate specification. E.g. within the internal ratings-based (IRB) approach of the new Basel Accord banks are allowed to use internal estimates of default probabil-

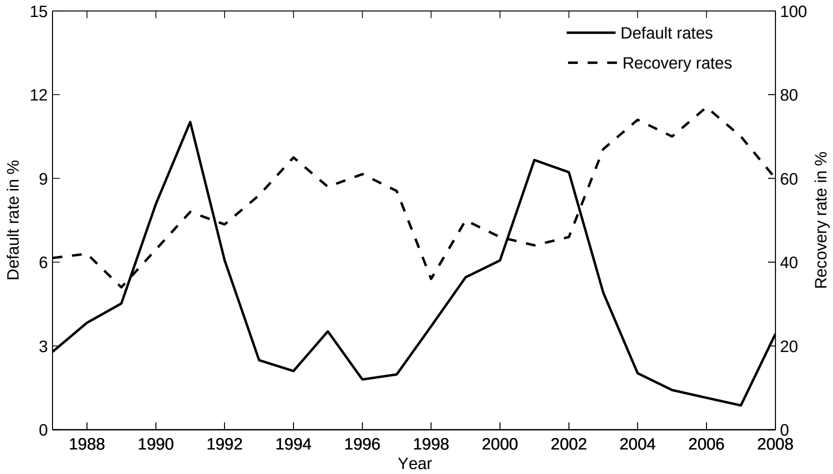

Figure 1.2: S&P speculative-grade default rates and discounted workout recovery rates.

ities as well as of recovery in the event of default (= 1 − loss given default) to calculate their credit-risk capital. In contrast to the probability of default, which is calculated on the obligor level, the recovery rate has to be calculated on the facility level. It is usually defined as the amount recovered as a percentage of the exposure at default.

Although there has been a growing number of studies dealing with the determinants of recovery rates, especially from US rating agencies, the behaviour and prediction of recovery rates is by far not yet fully understood. One of the first and probably most famous articles that examined bond recovery rates is the work by Altman and Kishore (1996), who examined the prices of bonds at the time of default of more than 700 defaulting bonds from 1978 to 1995. As many of these studies try to explain the recovery rates of individual defaulted bonds, they mostly concentrate on facility-level factors like the impact of seniority, collateralisation, or industry affiliation.

In the meantime studies on the determinants of historical recovery rates (see e.g. Altman and Kishore (1996), Schuermann (2004), or Davydenko and Franks (2008)) have been conducted, succeeding those on the determinants of default probabilities (see e.g. Beaver (1966), Ohlson (1980), or Duffie *et al.* (2007)). Moreover, after a long history of scoring models for defaults (see e.g. Altman (1968), Ohlson (1980), or Berg (2007)), recovery-prediction models (see e.g. Friedman *et al.* (2005) or Gupton and Stein (2005)) have been developed recently. Nevertheless, pricing models both for single-name and portfolio credit derivatives with an explicit stochastic modelling of recovery rates are still scarce. In the past, most models have treated the recovery rate as a constant or independent of the default process.

1.2 Objectives and structure

The main objectives of this thesis are the following: After introducing mathematical preliminaries and some basic facts about recovery rates and risks (Part I) the behaviour and determinants of recovery rates are examined (Part II) and new valuation approaches for credit derivatives using this information are developed (Part III). Based on well established concepts for pricing single-name as well as portfolio credit derivatives, the stochastic behaviour of recovery rates and their correlation with other market factors are modelled to achieve more reasonable frameworks for pricing as well as risk management purposes. While the theoretical development of the models is a crucial part of this thesis, the models are applied to real market data whenever possible and implementation issues like parameter estimation and calibration are discussed in detail.
The remainder of this thesis is organized as follows: Chapter 2 provides the basic mathematical concepts which are needed for the determination and valuation of recovery risks. The chapter is also intended to familiarise the reader with the mathematical notation used throughout this thesis. In Chapter 3 recovery rates and risks are defined and it is shown how they can be measured and modelled. In addition to that, this chapter contains a short review on existing recovery-rate models for pricing as well as risk management. Chapters 4 and 5 contain the empirical parts of this thesis. Chapter 4 gives a detailed overview on factors that might influence recoveries including their consideration in the relevant literature and introduces further explanatory variables which have not been considered yet. For the first time (to the author's best knowledge) determinants and behaviour of loan recovery rates on a facility level are described using such a large Pan-European dataset. In Chapter 5 the evolution over time of aggregated recovery rates in dependence of macroeconomic factors is investigated. Furthermore, Markov-switching concepts are applied to the analysis of aggregated recovery rates and an additional factor that tries to explain the credit environment is incorporated into the analysis. These results are used to construct a framework for the joint modelling of default and recovery risk in Chapter 6. The model accounts for the typical characteristics known from empirical studies, e.g. negative correlation between recovery-rate process and default intensity, as well as between default intensity and state of the economy, and a positive dependence of recovery rates on the economic environment. Within this framework analytically

tractable pricing formulas for credit derivatives are derived. The main building blocks for the pricing formulas are presented in Theorems 6.4 and 6.5. Corollaries 6.6 - 6.10 contain the valuation formulas for various defaultable contingent claims like coupon bonds or credit default swaps. The stochastic model for the recovery process allows, in contrast to many other credit-risk models, for the pricing of single-name credit derivatives with payoffs that are directly linked to the recovery rate at default, e.g. recovery locks (Corollary 6.11). Chapter 7 is devoted to the joint modelling of default and recovery risk in a portfolio of credit risky assets. One distinctive feature of this portfolio model is that it especially accounts for the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand (as observed e.g. in Figure 1.2). Nested Archimedean copulas are used to model different dependence structures, namely dependence among default triggers as well as between default triggers and loss triggers. Furthermore, a very flexible continuous recovery-rate distribution with bounded support on $[0, 1]$ is chosen, which allows for an efficient sampling of the loss process. This is especially important, as in most cases the loss process distribution will not be given in closed form. This approach extends the class of copula models for the valuation of CDO tranches to stochastic recovery rates in a non-Gaussian setting. Thereby, some of the "inconsistencies" observed in the credit market since mid 2007 can be resolved. The algorithm for the pricing of CDO tranches via Monte Carlo simulation is presented in Algorithm 7.2. Within this framework, an extensive model calibration case study, sensitivity analyses, and an application on delta hedging are presented. Furthermore, in Section 7.4 the concept of base correlations is extended to a non-Gaussian setting with stochastic recovery rates. Finally, Chapter 8 concludes.

Part I

Fundamentals

Chapter 2

Mathematical preliminaries

In this chapter, the basic mathematical concepts needed for the determination and valuation of recovery risks in Chapters 4 – 7 are introduced. While it is assumed that the basic concepts of probability theory, stochastic processes, and stochastic calculus are known to the reader, some of the main mathematical tools that will be applied later are repeated here. The first section of this chapter repeats the basic principles of discrete-time Markov-switching models. Section 2.2 gives a brief overview of the basic ideas of point processes and intensities and their application to financial modelling. The Cauchy problem and the Feynman-Kac representation needed for the evaluation of conditional expectations in the risk-neutral pricing framework is described in Section 2.3. The Kalman filtering method is presented in Section 2.4. Section 2.5 outlines the general ideas of using copulas for modelling multivariate dependence structures.
While original articles and further literature sources are cited where appropriate, the mathematical notation and presentation of the necessary preliminaries from mathematical finance are mainly based on Bingham and Kiesel (2004), Zagst (2002), Schönbucher (2003), and Schmid (2004). For the basics of intensity-based credit-risk models Schönbucher (2003), Bielecki and Rutkowski (2004), Schmid (2004), or Chapter 22 of Brigo and Mercurio (2001) are good references. Schmid (2004) also deals with the Kalman filtering method. Textbooks covering copula theory in general are Joe (1997) and Nelsen (1998).

2.1 Markov-switching models

In this section, the basic properties of discrete-time Markov-switching models are reviewed. A more general overview on the class of Markov-switching models and its applications to finance can e.g. be found in Frühwirth-Schnatter (2006), Cappé *et al.* (2007), or Mamon and Elliott (2007). A Markov-switching model (MSM) or hidden Markov model (HMM) is given by two stochastic processes. The first stochastic process, $(X_k)_{k\in\mathbb{N}_0}$, is a Markov chain with several states. The sequence of states is often not observable ("hidden"). By contrast, the second stochastic process, $(Y_k)_{k\in\mathbb{N}_0}$, is observable. The probability distribution of Y_k, $k \in \mathbb{N}_0$, depends on the particular state of the Markov chain. Here, the state space Ω of the Markov chain $(X_k)_{k\in\mathbb{N}_0}$ is assumed to be finite, i.e. $\Omega = \{1, \dots, J\}$, $J \in \mathbb{N}$. In the following, the transition matrix of the Markov chain is denoted by $\Pi = (\pi_{jl})_{j,l=1,\dots,J}$ with $\pi_{jl} = \mathbb{P}(X_k = l|X_{k-1} = j)$ and the initial distribution by δ with $\delta_j = \delta(x_j) = \mathbb{P}(X_0 = x_j)$, $j = 1, \dots, J$. The stochastic process $(Y_k)_{k\in\mathbb{N}_0}$ is assumed to be $\mathbb{R}$-valued with realizations $y_0, \dots, y_n$. The probability density of Y_k, $k \in \mathbb{N}_0$, conditioned on the particular state of the Markov Chain is denoted by p, where $p_k(x) = p(x, y_k; \Theta) = \mathbb{P}(Y_k = y_k|X_k = x)$ and Θ denoting the parameter set of the distribution. To determine the unobservable state sequence from the observed realizations of the second stochastic process, knowledge of the transition matrix, the initial distribution, and the parameters of the distribution is required. Unfortunately, this information is often not available. So the unknown parameters must be estimated at first. For that purpose the Baum-Welch algorithm (see Baum *et al.* (1970)) can be used. This algorithm is an expectation-maximization algorithm (EM-algorithm), i.e. it maximizes the conditional expectation of the (logarithmised) joint density given the observations. In principle, an EM-algorithm consists of two steps in every iteration, the E-step and the M-step. In the E-step the expectation functional is evaluated and in the M-step this functional is maximized.

Algorithm 2.1. *Baum-Welch Algorithm*

1. *Initialization:*
   Choose a maximum number of iterations I and a tolerance level ϵ_{tol}.
   For $i = 1, \dots, I$ let

   $$\Theta^i = ((\delta_j^{(i)})_{j=1,\dots,J}, (\pi_{jl}^{(i)})_{j,l=1,\dots,J}, (\mu_j^{(i)})_{j=1,\dots,J}, (\sigma_j^{(i)})_{j=1,\dots,J})$$

   denote the parameter vector after the i^{th} iteration and denote all expressions depending on Θ^i by a superscript (i).
   Set $i = 0$ and choose initial parameters

   $$\Theta^0 = ((\delta_j^{(0)})_{j=1,\dots,J}, (\pi_{jl}^{(0)})_{j,l=1,\dots,J}, (\mu_j^{(0)})_{j=1,\dots,J}, (\sigma_j^{(0)})_{j=1,\dots,J}).$$

2. *E-Step:*

   (a) *Forward recursion:*
   For $j = 1, \ldots, J$ set

$$\alpha_{0,j}^{(i)} = \mathbb{P}^{(i)}(Y_0 = y_0, X_0 = j) = p_0^{(i)}(j)\delta_j^{(i)}.$$

   For $k = 1, \ldots, n$ evaluate

$$\alpha_{k,j}^{(i)} = \mathbb{P}^{(i)}(Y_0 = y_0, \ldots, Y_k = y_k, X_k = j) = \sum_{l=1}^{J} \alpha_{k-1,l}^{(i)} p_k^{(i)}(j) \pi_{lj}^{(i)}$$

   for all $j = 1, \ldots, J$.

   (b) *Backward recursion:*
   For $j = 1, \ldots, J$ set

$$\beta_{n,j}^{(i)} = 1.$$

   For $k = n-1, \ldots, 0$ evaluate

$$\beta_{k,j}^{(i)} = \mathbb{P}^{(i)}(Y_{k+1} = y_{k+1}, \ldots, Y_n = y_n | X_k = j) = \sum_{l=1}^{J} p_{k+1}^{(i)}(l) \beta_{k+1,l}^{(i)} \pi_{jl}^{(i)}$$

   for all $j = 1, \ldots, J$.
   For $k < n$ set

$$\Psi_{k|n}(j; \Theta^i) = \mathbb{P}^{(i)}(X_k = j | Y_0 = y_0, \ldots, Y_n = y_n) = \frac{\alpha_{k,j}^{(i)} \beta_{k,j}^{(i)}}{\sum_{l=1}^{J} \alpha_{k,l}^{(i)} \beta_{k,l}^{(i)}},$$

$$\begin{aligned} \Psi_{k-1:k|n}(j, l; \Theta^i) &= \mathbb{P}^{(i)}(X_{k-1} = j, X_k = l | Y_0 = y_0, \ldots, Y_n = y_n) \\ &= \frac{\alpha_{k-1,j}^{(i)} \pi_{jl}^{(i)} p_k^{(i)}(l) \beta_{k,l}^{(i)}}{\sum_{m=1}^{J} \alpha_{k,m}^{(i)} \beta_{k,m}^{(i)}}. \end{aligned}$$

3. *M-step:*
   Choose $\Theta = ((\delta_j)_{j=1,\ldots,J}, (\pi_{jl})_{j,l=1,\ldots,J}, (\mu_j)_{j=1,\ldots,J}, (\sigma_j)_{j=1,\ldots,J})$, such that

$$
\begin{aligned}
\mathcal{Q}(\Theta;\Theta^i) &= \sum_{j=1}^{J} \Psi_{0|n}(j;\Theta^i)\log\delta_j \\
&\quad -\frac{1}{2}\sum_{k=0}^{n}\sum_{j=1}^{J}\Psi_{k|n}(j;\Theta^i)\left[\log 2\pi\sigma_j^2 + \frac{(y_k-\mu_j)^2}{\sigma_j^2}\right] \\
&\quad +\sum_{k=1}^{n}\sum_{j=1}^{J}\sum_{l=1}^{J}\Psi_{k-1:k|n}(j,l;\Theta^i)\log\pi_{jl}
\end{aligned}
$$

is maximal. The solution of this optimization problem can be evaluated by using the Lagrange multiplicator method under the constraints $\sum_{j=1}^{J}\delta_j = 1$ *and* $\sum_{l=1}^{J}\pi_{jl} = 1$. *The solution is given by*

$$
\begin{aligned}
\delta_j &= \Psi_{0|n}(j), \\
\mu_j &= \frac{\sum_{k=0}^{n}\Psi_{k|n}(j)y_k}{\sum_{k=0}^{n}\Psi_{k|n}(j)}, \\
\sigma_j &= \sqrt{\frac{\sum_{k=0}^{n}\Psi_{k|n}(j)(y_k-\mu_j)^2}{\sum_{k=0}^{n}\Psi_{k|n}(j)}}, \\
\pi_{jl} &= \frac{\sum_{k=1}^{n}\Psi_{k-1:k|n}(j,l)}{\sum_{k=1}^{n}\sum_{l=1}^{J}\Psi_{k-1:k|n}(j,l)},
\end{aligned}
$$

for $j,l = 1,\ldots,J$.

4. *Termination:*
   If $|\mathcal{Q}(\Theta;\Theta^i) - \mathcal{Q}(\Theta^i;\Theta^{i-1})| < \epsilon_{tol}$ *or* $i+1 = I$, *then stop and return* Θ, *else set* $\Theta^{i+1} = \Theta$ *and* $i = i+1$ *and go back to 2.*

If the transition matrix, the initial distribution, and the parameters of the distribution of a Markov-switching model are given, it is possible to estimate the "most-likely" state sequence according to Viterbi's algorithm (see Viterbi (1967)).

Algorithm 2.2. *Viterbi Algorithm*

1. *Initialization:*
   For $i = 1,\ldots,J$ *set*

$$m_0(i) = \log\mathbb{P}(X_0 = i, Y_0 = y_0) = \log(\delta_i p_0(i)).$$

2. *Forward recursion:*
   For $k = 0, \dots, n-1$ *evaluate*

$$\begin{aligned} m_{k+1}(j) &= \max_{\{x_0,\dots,x_k\}\in\Omega_1^k} \log \mathbb{P}(X_0 = x_0, \dots, X_k = x_k, X_{k+1} = j, \\ &\qquad\qquad Y_0 = y_0, \dots, Y_{k+1} = y_{k+1}) \\ &= \max_{i\in\{1,\dots,J\}} [m_k(i) + \log(\pi_{ij})] + \log(p_{k+1}(j)), \\ b_{k+1}(j) &= \arg\max m_{k+1}(j) \end{aligned}$$

   for all $j = 1, \dots, J$.

3. *Backward recursion:*
   Let $\hat{x}_n$ *be the state* j *for which* $m_n(j)$ *is maximal.*
   For $k = n-1, \dots, 0$ *set*

$$\hat{x}_k = b_{k+1}(\hat{x}_{k+1}).$$

2.2 Intensity-based models

Since the valuation of defaultable contingent claims in Part III of this thesis will be based on the concepts of point processes and intensities, a brief overview of the basic ideas and theorems is given in this section. In applications to financial market problems, intensity-based models are sometimes also referred to as reduced-form models and have become very popular, in particular in the context of credit-risk modelling. Historically, the limited success of structural-form models, which describe a defaultable security as a contingent claim on the firm's (unobservable) asset-value process, in explaining credit spreads have led to the invention of intensity-based credit-risk models. These models are often the preferred methodology for pricing and hedging purposes. Intensity-based models don't consider an explicit relation between default and asset value, but rather model default as a stopping time of some given hazard-rate process. Therefore, default is specified exogenously in this model class. This leads to a more realistic behaviour of short-term credit spreads and avoids the usage of an unobservable asset-value process at the cost of less interpretability. The family of reduced-form models goes back to Jarrow and Turnbull (1992) and Jarrow and Turnbull (1995). Since then many articles following this approach have been published, e.g. Madan and Unal (1998), Lando (1998), Schönbucher (1998), Duffie and Singleton (1999), and Duffie and Singleton (2003) just to name a few.
For a mathematical formulation of intensity-based models, let $(\Omega, \mathcal{G}, \mathbb{P})$ de-

note a given probability space. A counting process, i.e. a non-decreasing, integer-valued process which starts in 0, can be defined as follows.

Definition 2.3. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of random variables in $[0,\infty]$ such that $\tau_n(\omega) < \infty$ yields $\tau_n(\omega) < \tau_{n+1}(\omega)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. The process $(N(t))_{t\geq 0}$ defined by*

$$N(t) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \leq t\}}$$

is called point or counting process. In the following the process $(N(t))_{t\geq 0}$ often will be abbreviated by $N(t)$.

The process $N(t)$ can be considered as a stochastic process, counting the number of events associated with the sequence τ_n. Throughout this thesis, it will be assumed that $\tau_i \neq \tau_j$ for $i \neq j$ (i.e. $\tau_n < \tau_{n+1}$ for all $n \in \mathbb{N}$) and that the point process is non-explosive, i.e. $\lim_{n\to\infty} \tau_n = \infty$. In credit-risk modelling the default time τ of an obligor is often associated with the first jump of $N(t)$, i.e. $\tau = \inf\{t > 0 : N(t) > 0\}$.
In the following it will be assumed that $(\Omega, \mathcal{G}, \mathbb{P})$ is equipped with three filtrations $\mathbb{G}$, $\mathbb{F}$, and $\mathbb{F}^N$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ denote the filtration generated by all occurring stochastic processes other than the counting process $N(t)$. For the sake of simplicity $\mathcal{F}_0$ is assumed to be trivial. As $\mathbb{F}$ is assumed to be generated by a background process it is sometimes called the background filtration (see e.g. Schönbucher (2003)). Furthermore, let $\mathbb{F}^N := \{\mathcal{F}_t^N\}_{t\geq 0}$ be the filtration generated by the counting process $N(t)$ and $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ be the enlarged filtration $\mathbb{G} = \mathbb{F}^N \vee \mathbb{F}$, i.e. the smallest filtration generated by $\mathbb{F}^N$ and $\mathbb{F}$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. It should be emphasized that τ is not necessarily a stopping time with respect to the filtration $\mathbb{F}$, but of course with respect to the filtration $\mathbb{G}$. It is also assumed that for any $t \in (0, T^*]$ the σ-fields $\mathcal{F}_{T^*}$ and $\mathcal{F}_t^N$ are conditionally independent given $\mathcal{F}_t$, where T^* denotes a fixed time horizon. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale (see p.167 of Bielecki and Rutkowski (2004)). In some cases it is more suitable to use another condition, which is equivalent to the martingale invariance property (see p.242 of Bielecki and Rutkowski (2004)): For any $t \in (0, T^*]$ and any $\mathbb{P}$-integrable $\mathcal{F}_{T^*}$-measurable random variable X it holds that $\mathbb{E}_{\mathbb{P}}[X|\, \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[X|\, \mathcal{F}_t]$. The following definition introduces the concept of the intensity of a point process.

Definition 2.4. *Let $N(t)$ be a point process as in Definition 2.3, adapted to the filtration $\{\mathcal{F}_t^N\}_{t\geq 0}$ and let $\lambda(t)$ be a non-negative $\mathcal{F}_t$-progressively measurable process*[1] *such that for all $t \geq 0$ it holds that*

$$\int_0^t \lambda(s)ds < \infty \ \mathbb{P} - a.s.$$

If for all non-negative $\mathcal{F}_t$-predictable[2] *processes $C(t)$ the equality*

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^\infty C(s)dN(s)\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^\infty C(s)\lambda(s)ds\right]$$

holds, the point process $N(t)$ is said to admit the $(\mathbb{P}, \mathcal{F}_t)$-intensity $\lambda(t)$.

The following two theorems are concerned with fundamental properties, existence, and uniqueness of intensities (see e.g. p.28ff in Brémaud (1981) or p.60 in Schmid (2004)).

Theorem 2.5. *If $N(t)$ admits the $(\mathbb{P}, \mathcal{F}_t)$-intensity $\lambda(t)$, then $N(t)$ is non-explosive and $M = (M(t))_{t\geq 0}$ with*

$$M(t) = N(t) - \int_0^t \lambda(s)ds \tag{2.1}$$

is a $\mathcal{G}_t$-local martingale. Conversely, let $N(t)$ be a non-explosive point process adapted to $\mathcal{F}_t^N$, and suppose that for some non-negative $\mathcal{F}_t$-progressively measurable process $\lambda(t)$ and for all $n \geq 1$,

$$N(t \wedge \tau_n) - \int_0^{t\wedge\tau_n} \lambda(s)ds$$

is a $(\mathbb{P}, \mathcal{G}_t)$-martingale. Then, $\lambda(t)$ is the $(\mathbb{P}, \mathcal{F}_t)$-intensity of $N(t)$.

Proof. See p.27f in Brémaud (1981). □

[1] Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and $X(t)$ an E-valued process endowed with a σ-algebra $\mathcal{E}$. The process $X(t)$ is $\mathcal{F}_t$-progressively measurable if for all $t \geq 0$ the mapping $(t, \omega) \to X(t, \omega)$ from $[0, t] \times \Omega \to E$ is $\mathcal{B}([0, t] \otimes \mathcal{F}_t) - \mathcal{E}$-measurable (see e.g. p.281 of Brémaud (1981) or p.59 of Schmid (2004)).

[2] Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and define $\mathcal{P}(\mathcal{F}_t)$ to be the σ-algebra over $(0, \infty) \times \Omega$ generated by rectangles of the form $(s, t] \times A$ with $0 \leq s \leq t$ and $A \in \mathcal{F}_s$. $\mathcal{P}(\mathcal{F}_t)$ is referred to as the $\mathcal{F}_t$-predictable σ-algebra over $(0, \infty) \times \Omega$. A real-valued process $X(t)$ where $X(0)$ is $\mathcal{F}_0$-measurable and the mapping $(t, \omega) \to X(t, \omega)$ is $\mathcal{P}(\mathcal{F}_t)$-measurable is called $\mathcal{F}_t$-predictable (see e.g. p.8 of Brémaud (1981) or p.59 of Schmid (2004)).

Theorem 2.6. *Let $N(t)$ be a point process with a $(\mathbb{P}, \mathcal{F}_t)$-intensity $\lambda(t)$. Then one can find a $(\mathbb{P}, \mathcal{F}_t)$-intensity $\tilde{\lambda}(t)$ which is $\mathcal{F}_t$-predictable. Now, let $\tilde{\lambda}(t)$ and $\bar{\lambda}(t)$ be two $(\mathbb{P}, \mathcal{F}_t)$-intensities of $N(t)$ which are $\mathcal{F}_t$-predictable. Then, $\tilde{\lambda}(t) = \bar{\lambda}(t)$ $\mathbb{P}(d\omega)dN(t,\omega)$-almost everywhere.*

Proof. See p.31f in Brémaud (1981). □

Consider the aforementioned special case of only one random time τ and let $H(t)$ denote the indicator function $H(t) = \mathbb{1}_{\{\tau \leq t\}}$ associated e.g. with the occurrence of a credit event. Let $\lambda(t)$ denote the $(\mathbb{P}, \mathcal{F}_t)$-intensity of $H(t)$. Then, recalling Theorem 2.5,

$$M(t) := H(t) - \int_0^t \lambda(s) \mathbb{1}_{\{\tau \geq s\}} ds$$

is a martingale and therefore it is straightforward to see that for small $\Delta t > 0$

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[H(t+\Delta t) - H(t)|\mathcal{G}_t] =& 0 \cdot \mathbb{P}(\tau \leq t|\mathcal{G}_t) + 1 \cdot \mathbb{P}(t < \tau \leq t + \Delta t|\mathcal{G}_t) \\ &+ 0 \cdot \mathbb{P}(\tau > t + \Delta t|\mathcal{G}_t) \\ =& \mathbb{P}(t < \tau \leq t + \Delta t|\mathcal{G}_t) \end{aligned} \tag{2.2}$$

as well as

$$\mathbb{E}_{\mathbb{P}}[H(t+\Delta t) - H(t)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}\left[\int_t^{t+\Delta t} \lambda(s) \mathbb{1}_{\{\tau \geq s\}} ds \,\middle|\, \mathcal{G}_t \right]. \tag{2.3}$$

Furthermore, it can be shown (see e.g. p.61 in Schmid (2004)) that

$$\lim_{\Delta t \to 0} \frac{\mathbb{E}_{\mathbb{P}}[H(t+\Delta t) - H(t)|\mathcal{G}_t]}{\Delta t} = \lambda(t) \mathbb{1}_{\{\tau \geq t\}}, \quad \mathbb{P}\text{-a.s.}$$

This yields

$$\lambda(t) \mathbb{1}_{\{\tau \geq t\}} = \lim_{\Delta t \to 0} \frac{\mathbb{P}(t < \tau \leq t + \Delta t|\mathcal{G}_t)}{\Delta t},$$

i.e. the intensity $\lambda(t)$ can be interpreted as the (instantaneous) arrival rate of a default event associated with τ, given all information at time t. Hence, it can be concluded that the probability of default over the next infinitesimal time interval of length Δt is approximately given by $\lambda(t)\Delta t$. Furthermore, combining Equations (2.2) and (2.3) the conditional default probability over the interval $(t, T]$ is given by

$$\bar{p}\,(t, T|\mathcal{G}_t) = \mathbb{P}\,(t < \tau \leq T|\mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}\left[\int_t^T \lambda(s)\mathbb{1}_{\{\tau\geq s\}}ds\Big|\mathcal{G}_t\right].$$

Under some integrability conditions it can be shown further that (see e.g. Duffie (1998b))

$$\bar{p}\,(t, T|\mathcal{G}_t) = 1 - \mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T \lambda(s)ds}\Big|\mathcal{G}_t\right].$$

Using the martingale-invariance property, it follows that

$$\bar{p}\,(t, T|\mathcal{G}_t) = 1 - \mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T \lambda(s)ds}\Big|\mathcal{F}_t\right] = \bar{p}\,(t, T|\mathcal{F}_t).$$

The conditional survival probability $p\,(t, T|\mathcal{G}_t) = \mathbb{P}\,(N(T) = 0|\mathcal{G}_t)$ is given by

$$p\,(t, T|\mathcal{G}_t) = p\,(t, T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T \lambda(s)ds}\Big|\mathcal{F}_t\right]\mathbb{1}_{\{\tau\geq t\}}.$$

The survival and default probability $p\,(t, T|\mathcal{F}_t)$ and $\bar{p}\,(t, T|\mathcal{F}_t)$ respectively, will be abbreviated by $p(t, T)$ and $\bar{p}(t, T)$ in what follows.
The literature on intensity-based models can be divided in three categories: models with constant intensities, models with time-varying deterministic intensities, and models with stochastic intensities. If the intensity λ is constant, $N(t)$ is usually assumed to follow a Poisson process, i.e. $(\tau_{i+1} - \tau_i) \sim Exp(\lambda)$. If the intensity $\lambda(t)$ is a (non-constant) deterministic function, the process is usually a time-inhomogeneous Poisson process. The incorporation of stochastic intensities yields a doubly stochastic Poisson process, also called Cox process, which can be defined as follows (see also p.121 of Schönbucher (2003)):

Definition 2.7. *A point process $N(t)$ with intensity process $\lambda(t)$ is a Cox process if, conditional on the background filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $N(t)$ is a time-inhomogeneous Poisson process with intensity $\lambda(t)$.*

Note that this definition ensures that the Cox process can not be measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Thus, knowledge of the intensity process does not reveal any information about the realisation of $N(t)$. In what follows, some well known examples for intensity specifications in credit-risk models will be given.

Example 2.8.

1. *Constant intensity:*
   If $\lambda(t) = \lambda > 0$ the survival and default probability are given by

$p(t,T) = e^{-\lambda(T-t)}\mathbb{1}_{\{\tau\geq t\}}$ and $\bar{p}(t,T) = 1 - e^{-\lambda(T-t)}$, respectively. In this model the time to default is exponentially distributed with expected time to default equal to λ^{-1}.

2. *Deterministic intensity:*
   If $\lambda(t)$ is a time-varying deterministic function, one obtains $p(t,T) = e^{-\int_t^T \lambda(s)ds}\mathbb{1}_{\{\tau\geq t\}}$ and $\bar{p}(t,T) = 1 - e^{-\int_t^T \lambda(s)ds}$.

3. *Vasicek model (Vasicek (1977)):*
   The intensity $\lambda(t)$ is assumed to follow an Ornstein-Uhlenbeck process, i.e.

$$d\lambda(t) = (\theta - a\lambda(t))\, dt + \sigma dW(t), \quad \lambda(0) > 0,$$

with constant parameters $\theta \geq 0$, $a, \sigma > 0$, and a one-dimensional Wiener process W. In this case, $\lambda(t)$ is normally distributed with mean

$$\mathbb{E}_{\mathbb{P}}\left[\lambda(t)|\mathcal{F}_0\right] = e^{-at}\lambda(0) + \frac{\theta}{a}\left(1 - e^{-at}\right)$$

and variance

$$\mathbb{V}ar_{\mathbb{P}}\left[\lambda(t)|\mathcal{F}_0\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

The survival probability is given by

$$p\,(t,T) = \mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T \lambda(s)ds}\middle|\mathcal{F}_t\right]\mathbb{1}_{\{\tau\geq t\}} = e^{A(t,T)-B(t,T)\lambda(t)}\mathbb{1}_{\{\tau\geq t\}},$$

with

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \\ A(t,T) &= \left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right)\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4a}B^2(t,T). \end{aligned}$$

The disadvantage of Vasicek's model is that the normal distribution of $\lambda(t)$ implies $\mathbb{P}(\lambda(t) < 0) > 0$ and hence positivity of the intensity is not guaranteed. This drawback is often accepted in applications due to the tractability advantages of the model (see also the discussion in Section 6.1).

4. *Cox-Ingersoll-Ross model (Cox* et al. *(1985)):*
   A process that guarantees (under certain parameter restrictions) posi-

tivity of the intensity is the square-root diffusion process given by

$$d\lambda(t) = (\theta - a\lambda(t))\,dt + \sigma\sqrt{\lambda(t)}dW(t), \quad \lambda(0) > 0,$$

with constant parameters $\theta \geq 0$, $a, \sigma > 0$, where $2\theta > \sigma^2$ is assumed, and a one-dimensional Wiener process W. In this case, $\lambda(t)$ follows a non-central chi-squared distribution with mean

$$\mathbb{E}_{\mathbb{P}}\left[\lambda(t)|\mathcal{F}_0\right] = e^{-at}\lambda(0) + \frac{\theta}{a}\left(1 - e^{-at}\right)$$

and variance

$$\mathbb{V}ar_{\mathbb{P}}\left[\lambda(t)|\mathcal{F}_0\right] = \lambda(0)\frac{\sigma^2}{a}\left(e^{-at} - e^{-2at}\right) + \theta\frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2.$$

The survival probability is again an exponentially affine function in $\lambda(t)$. For more details see e.g. p.76f of Schmid (2004).

Further examples of intensity-based models and their application to credit-risk modelling can e.g. be found in Schönbucher (2003), Schmid (2004), or Chapter 22 of Brigo and Mercurio (2001). Examples of intensity specifications including jump processes can e.g. be found in Gaspar and Schmidt (2007) or Cariboni and Schoutens (2009).

2.3 Cauchy problem and Feynman-Kac representation

The aim of this section is to recall the Cauchy problem and its Feynman-Kac representation. Besides the version of the Feynman-Kac theorem which is often cited in the literature, another version is derived which is tailored to the needs of the model from Chapter 6.
For the first version of the theorem, let $X(t)$ be an n-dimensional Itô process on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ defined by

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) \tag{2.4}$$

with $\mathbb{F} = \mathbb{F}(W) = \{\mathcal{F}_t\}_{t\geq 0}$, $W(t) = (W_1(t), ..., W_m(t))^T$ denoting an m-dimensional Wiener process, $X(0)$ a $\mathcal{F}_0$-measurable random variable, and μ

and σ two progressively measurable stochastic processes with

$$\int_0^t |\mu_i(s)|ds < \infty \text{ and } \int_0^t \sigma_{ij}^2(s)ds < \infty \tag{2.5}$$

$\mathbb{P}$-a.s. for all $t \geq 0$, $i = 1, ..., n$, and $j = 1, ..., m$. As usual, Equation (2.4) is symbolically abbreviated by

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) = \mu(t)dt + \sum_{j=1}^{m} \sigma_j(t)dW_j(t).$$

If there exists an n-dimensional stochastic process $X = X(t) = (X^{0,x_0}(t))_{t\geq 0}$ of the form (2.4) with $\mu(t) = \mu(X(t), t)$ and $\sigma(t) = \sigma(X(t), t)$ satisfying Equation (2.5), this process $X(t)$ is called the strong solution of the stochastic differential equation (SDE)

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t),\ X(0) = x_0. \tag{2.6}$$

The existence and uniqueness of such a strong solution is discussed in the following theorem (see e.g. p.36 of Zagst (2002)).

Theorem 2.9. *Let μ and σ in Equation (2.6) be continuous functions such that for all $t \geq 0$, $x, y \in \mathbb{R}$, and for some constant $K > 0$ the following conditions hold:*

1. *Lipschitz condition:*

$$||\mu(x, t) - \mu(y, t)|| + ||\sigma(x, t) - \sigma(y, t)|| \leq K||x - y||$$

2. *Growth condition:*

$$||\mu(x, t)||^2 + ||\sigma(x, t)||^2 \leq K^2 \left(1 + ||x||^2\right)$$

Then there exists a unique, continuous strong solution $X(t)$ of the stochastic differential equation from Equation (2.6) and a constant C, depending only on K and $T > 0$, such that

$$\mathbb{E}_{\mathbb{P}}\left[||X(t)||^2\right] \leq C\left(1 + ||x||^2\right)e^{Ct}$$

for all $t \in [0, T]$. Moreover,

$$\mathbb{E}_{\mathbb{P}}\left[\sup_{0\leq t\leq T} ||X(t)||^2\right] < \infty.$$

Proof. A detailed proof can e.g. be found on p.127-133 in Korn and Korn (1999). □

The following theorems and corollaries deal with the Markov property of Itô processes.

Theorem 2.10. *Under the assumptions of Theorem 2.9, let $\mu(x,t)$ and $\sigma(x,t)$ be constant in t. Further, let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function. Then, for the solution X of the SDE from Equation (2.6) and $t, h \geq 0$ it holds that*

$$\mathbb{E}_{\mathbb{P}}^{0,x}\left[f(X(t+h))|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f(X(t+h))\right] = \mathbb{E}_{\mathbb{P}}^{0,X^{0,x}(t)}\left[f(X(h))\right], \quad (2.7)$$

where $\mathbb{E}_{\mathbb{P}}^{0,x}[X(t)] := \mathbb{E}_{\mathbb{P}}[X^{0,x}(t)]$.

Proof. See e.g. Section 7.1 of Øksendal (1998). □

While the second equation in (2.7) is only valid if $\mu(x,t)$ and $\sigma(x,t)$ are constant in t, the first equation can be shown in a more general setting.

Corollary 2.11. *Under the assumptions of Theorem 2.9, let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function. For $t, h \geq 0$ it holds that*

$$\mathbb{E}_{\mathbb{P}}^{0,x}\left[f(X(t+h))|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f(X(t+h))\right],$$

where X denotes the solution of the SDE from Equation (2.6).

Proof. For the solution of the $(n+1)$-dimensional differential equation

$$\begin{aligned} d\bar{X}(s) = &\begin{pmatrix} \mu\left((\bar{X}_1(s),...,\bar{X}_n(s))^T, \bar{X}_{n+1}(s)\right) \\ 1 \end{pmatrix} ds \\ &+ \begin{pmatrix} \sigma\left((\bar{X}_1(s),...,\bar{X}_n(s))^T, \bar{X}_{n+1}(s)\right) \\ 0 \end{pmatrix} dW(s) \end{aligned}$$

with initial condition $\bar{X}(0) = (x,0)^T$ it holds that $(\bar{X}_1(s),...,\bar{X}_n(s))^T = X(s)$ for all $s \geq 0$. Defining $\bar{f} : \mathbb{R}^{n+1} \to \mathbb{R}$, $\bar{f}(\bar{x}) := f\left((\bar{x}_1,...,\bar{x}_n)^T\right)$, Theorem 2.10 yields

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}^{0,x}\left[f(X(t+h))|\mathcal{F}_t\right] = E_{\mathbb{P}}^{0,(x,0)}\left[\bar{f}(\bar{X}(t+h))|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{P}}^{t,(X^{0,x}(t),t)}\left[\bar{f}(\bar{X}(t+h))\right] = \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f(X(t+h))\right]. \end{aligned}$$

□

For the following corollary, f is no longer required to be bounded.

Corollary 2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Under the assumptions of Theorem 2.9, the following holds for the solution of the SDE from Equation (2.6) and $t, h \geq 0$:*
If for all $x \in \mathbb{R}^n$ both $\mathbb{E}_{\mathbb{P}}^{t,x}\left[|f(X(t+h))|\right] < \infty$ and $\mathbb{E}_{\mathbb{P}}^{0,x}\left[|f(X(t+h))|\right] < \infty$ hold, then

$$\mathbb{E}_{\mathbb{P}}^{0,x}\left[f(X(t+h))|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f(X(t+h))\right].$$

Proof. Define $f_n : \mathbb{R}^n \to \mathbb{R}$, $f_n(x) := f(x)\mathbb{1}_{\{|f(x)|\leq n\}}$. The functions f_n are bounded. Hence, from Lebesgue's theorem for conditional expectations and Corollary 2.11 it follows that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}^{0,x}\left[f(X(t+h))|\mathcal{F}_t\right] &= \lim_{n\to\infty} E_{\mathbb{P}}^{0,x}\left[f_n(X(t+h))|\mathcal{F}_t\right] \\
&= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f_n(X(t+h))\right] \\
&= \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[f(X(t+h))\right].
\end{aligned}$$

□

Corollary 2.13. *Let $T \geq t \geq 0$, $g : \mathbb{R}^n \to \mathbb{R}$ affine linear, and $f : \mathbb{R}^n \to \mathbb{R}$ continuous. Under the assumptions of Theorem 2.9, the following holds for the solution of the SDE from Equation (2.6):*
If $\mathbb{E}_{\mathbb{P}}^{0,x}\left[\left|e^{-\int_t^T g(X(l))dl} f(X(T))\right|\right] < \infty$ and $\mathbb{E}_{\mathbb{P}}^{t,x}\left[\left|e^{-\int_t^T g(X(l))dl} f(X(T))\right|\right] < \infty$ for all $x \in \mathbb{R}^n$, then

$$\mathbb{E}_{\mathbb{P}}^{0,x}\left[e^{-\int_t^T g(X(l))dl} f(X(T))|\mathcal{F}_t\right] = E_{\mathbb{P}}^{t,X^{0,x}(t)}\left[e^{-\int_t^T g(X(l))dl} f(X(T))\right].$$

Proof. Let $T \geq t \geq 0$ be arbitrary but fixed. For the solution of the $(n+1)$-dimensional SDE

$$\begin{aligned}
d\bar{X}(s) = &\begin{pmatrix} \mu\left((\bar{X}_1(s), ..., \bar{X}_n(s))^T, s\right) \\ -g(\bar{X}_1(s), ..., \bar{X}_n(s))\mathbb{1}_{\{s\geq t\}} \end{pmatrix} ds \\
&+ \begin{pmatrix} \sigma\left((\bar{X}_1(s), ..., \bar{X}_n(s))^T, s\right) \\ 0 \end{pmatrix} dW(s)
\end{aligned}$$

with initial condition $\bar{X}(0) = (x, 0)'$ it obviously holds that $(\bar{X}_1(s), ..., \bar{X}_n(s))^T = X(s)$ and

$$\bar{X}_{n+1}(s) = -\int_t^s g(X(l))dl\mathbb{1}_{\{s\geq t\}} \text{ for all } s \geq 0.$$

This $(n+1)$-dimensional SDE fulfils the conditions of Theorem 2.9. Hence, one obtains from Corollary 2.12 that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}^{0,x}\left[e^{-\int_t^T g(X(l))dl} f(X(T))|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{P}}^{0,(x,0)}\left[e^{\bar{X}_{n+1}(T)} f\left((\bar{X}_1(T),...,\bar{X}_n(T))^T\right)|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{P}}^{t,(X^{0,x}(t),0)}\left[e^{\bar{X}_{n+1}(T)} f\left((\bar{X}_1(T),...,\bar{X}_n(T))^T\right)\right] \\ &= \mathbb{E}_{\mathbb{P}}^{t,X^{0,x}(t)}\left[e^{-\int_t^T g(X(l))dl} f(X(T))\right].\end{aligned}$$

□

Next, the Cauchy problem, a partial differential equation with a certain boundary condition, is defined.

Definition 2.14. *Under the assumptions of Theorem 2.9, let $D : \mathbb{R}^n \to \mathbb{R}$, $k : \mathbb{R}^n \times [0,T] \to [0,\infty)$ be continuous functions and $T > 0$ arbitrary but fixed. The problem to find a function $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ which is continuously differentiable in t, twice continuously differentiable in x, and solves the partial differential equation*

$$\begin{aligned}\mathcal{D}v(x,t) := v_t(x,t) &+ \sum_{i=1}^n \mu_i(x,t) v_{x_i}(x,t) \\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t) v_{x_i x_j}(x,t) = k(x,t)v(x,t) \\ &\qquad v(x,T) = D(x)\end{aligned}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $a_{ij}(x,t) := \sum_{k=1}^m \sigma_{ik}(x,t)\sigma_{jk}(x,t)$ and X is the unique strong solution of the stochastic differential equation from Theorem 2.9 with initial condition $X(t) = x$, is called the Cauchy problem. The operator $\mathcal{D}$ is called the characteristic operator for X.

Under certain regularity conditions it can be shown that there exists a unique solution of the Cauchy problem (see e.g. p.366 of Karatzas and Shreve (1991)).

Theorem 2.15. *Let v be a solution of the Cauchy problem from Definition 2.14. Furthermore, assume that for all $x \in \mathbb{R}^n$*

$$|D(x)| \le L\left(1 + \|x\|^{2\lambda}\right) \quad \text{or } D(x) \ge 0$$

and

$$\max_{0\le t\le T} |v(t,x)| \le M\left(1+\|x\|^{2\nu}\right)$$

with appropriate constants $L, M > 0$ *and* $\lambda, \nu \ge 1$. *Then,* v *can be represented by*

$$v(x,t) = \mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T k(X(s),s)ds} D\left(X(T)\right)|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}^{t,x}\left[e^{-\int_t^T k(X(s),s)ds} D\left(X(T)\right)\right]. \tag{2.8}$$

In particular, such a solution is unique.

Proof. See e.g. p.366f of Karatzas and Shreve (1991). □

The representation (2.8) is called the Feynman-Kac representation of the Cauchy problem. In a more general setting, which is in particular not restricted to the case of $k(x,t) \ge 0$, the uniqueness of a solution of the Cauchy problem can be shown.

Theorem 2.16. *Under the assumptions of Theorem 2.9, let the operator* $\mathcal{D}$ *be given by*

$$\mathcal{D}v(x,t) := v_t(x,t) + \sum_{i=1}^n \mu_i(x,t) v_{x_i}(x,t) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x,t) v_{x_i,x_j}(x,t)$$

with $a_{ij}(x,t) := \sum_{k=1}^m \sigma_{ik}(x,t)\sigma_{jk}(x,t)$ *and* $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ *a function which is continuously differentiable in* t *and twice continuously differentiable in* x. *For* $k : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ *and* $D : \mathbb{R}^n \to \mathbb{R}$ *consider the Cauchy problem*

$$\begin{aligned} \mathcal{D}v(x,t) &= k(x,t)v(x,t) \\ v(x,T) &= D(x) \end{aligned}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$. *If there is a constant* $K > 0$ *such that*

$$|a_{ij}(x,t)| \le K,\ |\mu_i(x,t)| \le K\left(1+\|x\|_2\right),\ -k(x,t) \le K\left(1+\|x\|_2^2\right)$$

and the matrices $\left(a_{ij}(x,t)\right)_{1\le i,j\le n}$ *are positive semidefinite, then there exists at most one solution* v *of the Cauchy problem which fulfils*

$$|v(x,t)| \le K_1 e^{K_2\|x\|_2^2}$$

with positive constants K_1 *and* K_2.

Proof. See e.g. Corollary 4.2 in Friedman (1975). □

In the following, another version of the Feynman-Kac theorem will be derived, which allows the function k to take values in $\mathbb{R}$. Unfortunately, this is not possible in general. Nevertheless, in the special setting of the model from Chapter 6, an analogous theorem can be proven. As this theorem is very important for the valuation of credit derivatives in Chapter 6, a detailed derivation is given in what follows. The following discussion will be restricted to a special class of stochastic differential equations.

Theorem 2.17. *Let $J : [0,\infty) \to \mathbb{R}^n$ be a continuous function, $H \in \mathbb{R}^{n\times n}$, and $V \in \mathbb{R}^{n\times m}$. Then, the unique strong solution of the linear stochastic differential equation*

$$dX(t) = (HX(t) + J(t))dt + VdW(t) \tag{2.9}$$

with initial condition $X(0) = x$ is given by

$$X(t) = e^{Ht}x + \int_0^t e^{H(t-l)}J(l)dl + \int_0^t e^{H(t-l)}VdW(l). \tag{2.10}$$

In particular, $X(t)$ is normally distributed for all $t > 0$.

Proof. See e.g. p.354 of Karatzas and Shreve (1991). □

In the next step, some properties of linear stochastic differential equations will be derived, which will be used to prove the second version of the Feynman-Kac theorem.

Theorem 2.18. *Let $0 \leq s \leq T$, $X(t)$ a solution of the linear SDE from Equation (2.9) with initial condition $X(s) = x$ and $\|\cdot\|$ an arbitrary norm on $\mathbb{R}^n$. Then, for $q \geq 1$ it holds that*

$$\mathbb{E}_{\mathbb{P}}\left[\left(\sup_{s\leq t\leq T} e^{\|X(t)\|}\right)^q\right] < \infty.$$

Proof. For the proof, $q > 1$ will be assumed w.l.o.g. (the square integrability of $\sup_{s\leq t\leq T} e^{\|X(t)\|}$ also yields the integrability). Let K_1, K_2, and K_3 denote adequate positive constants. Then,[3]

[3]For $x \in \mathbb{R}^n$, $\|x\|_1$ denotes the l_1-norm of x given by $\|x\|_1 = \sum_{j=1}^n |x_j|$.

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\left(\sup_{s\le t\le T} e^{\|X(t)\|}\right)^q\right] \\
&\le \mathbb{E}_{\mathbb{P}}\left[\sup_{s\le t\le T}\left(\left(\sup_{s\le t\le T} e^{\|e^{H(t-s)}x+\int_s^t e^{H(t-l)}J(l)dl\|}\right)\right.\right. \\
&\qquad\qquad\left.\left.\cdot\, e^{\left(\sup_{s\le t\le T}\|e^{Ht}\|\right)\|\int_s^t e^{H(-l)}VdW(l)\|}\right)^q\right] \\
&\le \mathbb{E}_{\mathbb{P}}\left[\sup_{s\le t\le T}\left(K_1 e^{K_2\|\int_s^t e^{H(-l)}VdW(l)\|}\right)^q\right] \\
&\le \mathbb{E}_{\mathbb{P}}\left[\sup_{s\le t\le T}\left(K_1 e^{\|\int_s^t K_3 e^{H(-l)}VdW(l)\|_1}\right)^q\right] \\
&\le \left(\frac{q}{q-1}\right)^q \mathbb{E}_{\mathbb{P}}\left[\left(K_1 e^{\|\int_s^T K_3 e^{H(-l)}VdW(l)\|_1}\right)^q\right]
\end{aligned}
$$

In the second to last inequality the equivalence of norms on $\mathbb{R}^n$ was used. For the last inequality Doob's inequality (see e.g. Theorem 1.3.8 in Karatzas and Shreve (1991)) was used. This is possible because $K_1 e^{\|\int_s^t K_3 e^{H(-l)}VdW(l)\|_1}$ is a convex function of the stochastic integral and hence a (non-negative, continuous) submartingal (see e.g. Proposition 1.3.6 in Karatzas and Shreve (1991)). It only remains to be proven that for all $t\in[s,T]$ it holds that

$$\mathbb{E}_{\mathbb{P}}\left[\left(K_1 e^{\|\int_s^t K_3 e^{H(-l)}VdW(l)\|_1}\right)^q\right] < \infty.$$

To show this, it is sufficient to prove that for a normally distributed random variable $Y\sim\mathcal{N}_n(0,\Sigma)$ it holds that $\mathbb{E}_{\mathbb{P}}\left[e^{q\|Y\|_1}\right]<\infty$. As Y can be represented by $Y=AZ$ with $Z\sim\mathcal{N}_n(0,I_n)$, $A\in\mathbb{R}^{n\times n}$, and $A^TA=\Sigma$ it follows that[4]

$$\mathbb{E}_{\mathbb{P}}\left[e^{q\|Y\|_1}\right] \le \mathbb{E}_{\mathbb{P}}\left[e^{q\|A\|_1\|Z\|_1}\right] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{K_4\|z\|_1}\phi_n(z)dz_1\cdots dz_n$$

with $K_4 := q\|A\|_1$ and ϕ_n denoting the density of Z. As the integrand is a product of symmetric functions, the last term can be approximated from

[4]For $A\in\mathbb{R}^{m\times n}$, the matrix norm $\|A\|$ induced by an arbitrary vector norm $\|x\|$, $x\in\mathbb{R}^n$, is given by $\|A\| := \max_{x\ne 0}\frac{\|Ax\|}{\|x\|}$. In the special case of the l_1-norm, it holds that $\|A\|_1 = \max_{j=1,\ldots,n}\sum_{i=1}^m |a_{ij}|$.

above by

$$
\begin{aligned}
&2^n \int_0^\infty \cdots \int_0^\infty e^{K_4 \sum_{i=1}^n z_i} \phi_n(z) dz_1 \cdots dz_n \\
&\leq 2^n \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{K_4 \sum_{i=1}^n z_i} \phi_n(z) dz_1 \cdots dz_n.
\end{aligned}
$$

The last integral is the moment-generating function of the normally distributed random variable Z evaluated at $K_4(1,1,...,1)^T$ and therefore exists. $\square$

Corollary 2.19. *Let $X(t)$ be the solution of the linear SDE from Equation (2.9), $f, g : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := e^{f_1(x)} f_2(x) = e^{A^T x + b}\left(F^T x + d\right)$, $g(x) := G^T x + c$, $A, F, G \in \mathbb{R}^n$, $b, c, d \in \mathbb{R}$, $q \geq 1$, and $0 \leq s \leq t \leq T$. Then, it holds for all $x \in \mathbb{R}^n$ that*

$$
\mathbb{E}_{\mathbb{P}}^{s,x}\left[\left(e^{-\int_t^T g(X(l))dl} f(X(T))\right)^q\right] < \infty.
$$

Proof. Let ξ denote the solution of the SDE $d\xi(l) = (dX(l), g(X(l))dl)$ with initial condition $\xi(s) = (x, 0)$. Then,[5]

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}^{s,x}\left[\left(e^{-\int_t^T g(X(l))dl} f(X(T))\right)^q\right] \\
&\leq \mathbb{E}_{\mathbb{P}}^{s,x}\left[\left(e^{\left|\int_t^T g(X(l))dl\right|} e^{|f_1(X(T))|} \sup_{s \leq l \leq T} e^{|f_2(X(l))|}\right)^q\right] \\
&\leq \mathbb{E}_{\mathbb{P}}^{s,x}\left[\left(e^{\left|\int_s^T g(X(l))dl\right| + \left|\int_s^t g(X(l))dl\right| + |f_1(X(T))|} \sup_{s \leq l \leq T} e^{|f_2(X(l))|}\right)^q\right] \\
&= \mathbb{E}_{\mathbb{P}}^{s,(x,0)}\left[\left(e^{|\xi_{n+1}(T)| + |\xi_{n+1}(t)| + \left|f_1\left((\xi_1(T),\ldots,\xi_n(T))^T\right)\right|} \sup_{s \leq l \leq T} e^{\left|f_2\left((\xi_1(l),\ldots,\xi_n(l))^T\right)\right|}\right)^q\right] \\
&\leq \mathbb{E}_{\mathbb{P}}^{s,(x,0)}\left[\left(e^{\|\xi(T)\|_\infty + \|\xi(t)\|_\infty + \|A\|_1 \|\xi(T)\|_\infty + |b|} \sup_{s \leq l \leq T} e^{\|F\|_1 \|\xi(l)\|_\infty + |d|}\right)^q\right] \\
&\leq e^{q(|b|+|d|)} \mathbb{E}_{\mathbb{P}}^{s,(x,0)}\left[\left(\sup_{s \leq l \leq T} e^{\|\xi(l)\|_\infty}\right)^{(2+\|A\|_1+\|F\|_1)q}\right] < \infty.
\end{aligned}
$$

The finiteness of the last expectation follows from Theorem 2.18. $\square$

Using the aforementioned theorems and corollaries, a version of the Feynman-Kac theorem will be proven, which is suitable for the modelling framework in Chapter 6.

[5] For $x \in \mathbb{R}^n$, $\|x\|_\infty$ denotes the l_∞-norm of x given by $\|x\|_\infty = \max_{j=1,\ldots,n} |x_j|$.

Theorem 2.20. *Let $T \geq 0$, $X(t)$ the solution of the linear SDE from Equation (2.9), and VV^T positive definite. Further, let $f, g : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := e^{f_1(x)} f_2(x) = e^{A^T x + b} \left(F^T x + d\right)$, $g(x) := G^T x + c$, $A, F, G \in \mathbb{R}^n$, $b, c, d \in \mathbb{R}$, $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ with*

$$B(\tilde{x}, t) := \mathbb{E}_{\mathbb{P}}^{t,\tilde{x}} \left[e^{-\int_t^T g(X(l))dl} f(X(T)) \right],$$

and the operator $\mathcal{D}$ be defined by

$$\mathcal{D}B(\tilde{x}, t) := B_t(\tilde{x}, t) + \sum_{i=1}^{n} \mu_i(\tilde{x}, t) B_{\tilde{x}_i}(\tilde{x}, t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\tilde{x}, t) B_{\tilde{x}_i, \tilde{x}_j}$$

with $\mu(\tilde{x}, t) := H\tilde{x} + J(t)$ and $a_{ij}(\tilde{x}, t) := \sum_{k=1}^{m} V_{ik} V_{jk} = \left(VV^T\right)_{ij}$. Then,

$$B(X^{0,x}(t), t) = \mathbb{E}_{\mathbb{P}}^{0,x} \left[e^{-\int_t^T g(X(l))dl} f(X(T)) | \mathcal{F}_t \right] \tag{2.11}$$

and $B(\tilde{x}, t)$ is the only solution of the Cauchy problem

$$\begin{aligned} \mathcal{D}B(\tilde{x}, t) &= g(\tilde{x}) B(\tilde{x}, t) \\ B(\tilde{x}, T) &= f(\tilde{x}) \end{aligned} \tag{2.12}$$

for all $(\tilde{x}, t) \in \mathbb{R}^n \times [0, T]$, fulfilling the growth condition

$$|B(\tilde{x}, t)| \leq K_1 e^{K_2 \|\tilde{x}\|_2^2}$$

with positive constants K_1 and K_2.

Proof. Equation (2.11) follows directly from Corollaries 2.13 and 2.19. In particular, $(M(t))_{0 \leq t \leq T}$ defined by

$$M(t) := e^{-\int_0^t g(X(l))dl} B(X^{0,x}(t), t) = \mathbb{E}_{\mathbb{P}}^{0,x} \left[e^{-\int_0^T g(X(l))dl} f(X(T)) | \mathcal{F}_t \right]$$

is a martingale.
If $B(\tilde{x}, t)$ is twice continuously differentiable in $\tilde{x}$ and continuously differentiable in t, Itô's formula can be applied to the $(n+1)$-dimensional Itô process ξ defined by $d\xi(t) = (dX(t), g(X(t))dt)$ with initial condition $\xi(0) = (x, 0)$. As $M(t) = e^{-\xi_{n+1}(t)} B((\xi_1, ..., \xi_n)(t), t)$, it follows that

$$
\begin{aligned}
dM(t) = e^{-\int_0^t g(X^{0,x}(l))dl} \Bigg(& B_t(X^{0,x}(t),t) + \sum_{i=1}^n B_{x_i}(X^{0,x}(t),t)(HX^{0,x}(t)+J(t))_i \\
& -g(X^{0,x}(t))B(X^{0,x}(t),t) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n B_{x_ix_j}(X^{0,x}(t),t)(VV^T)_{ij} \Bigg) dt \\
& + \sum_{i=1}^n \left(B_{x_i}(X^{0,x}(t),t) \sum_{j=1}^m V_{ij} dW_j \right).
\end{aligned}
$$

Since M is a martingale, its drift is zero $\mathbb{P}$-a.s. (see e.g. Theorem 2.42 in Zagst (2002)). Using the fact that VV^T is positive definite and that the drift of M is continuous, one obtains Equation (2.12).
It remains to be shown that $B(\tilde{x},t)$ is twice continuously differentiable in $\tilde{x}$ and continuously differentiable in t. Let ξ denote the solution of the SDE $d\xi(s) = (dX(s), g(X(s))ds)$ with initial condition $\xi(t) = \xi_0 := (\tilde{x}, 0)$. Applying Theorem 2.17 on ξ yields for all $s \geq t$

$$
\xi^{t,\xi_0}(s) = e^{H_\xi s}\left(e^{H_\xi(-t)}\xi_0 + \int_t^s e^{H_\xi(-l)} J_\xi(l)dl + \int_t^s e^{H_\xi(-l)} V_\xi dW(l) \right)
$$

with

$$
H_\xi = \begin{pmatrix} H & 0 \\ G^T & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},
$$

$V_\xi = (V,0)^T \in \mathbb{R}^{(n+1)\times m}$, and $J_\xi(l) = (J(l),c)^T \in \mathbb{R}^{(n+1)}$. Let $h(t,T)$ denote the last line of $e^{H_\xi(T-t)}$. Then,

$$
\begin{aligned}
\xi^{t,\xi_0}_{n+1}(T) &= h(t,T)\xi_0 + \left(e^{H_\xi(T)} \left(\int_t^T e^{H_\xi(-l)} J_\xi(l)dl + \int_t^T e^{H_\xi(-l)} V_\xi dW(l) \right)\right)_{n+1} \\
&= h(t,T)\xi_0 + \xi^{t,0}_{n+1}(T).
\end{aligned}
$$

The differentiability in $\tilde{x}$ can be directly seen from

$$\begin{aligned}B(\tilde{x},t) =& \mathbb{E}_{\mathbb{P}}\left[e^{-\xi_{n+1}^{t,\xi_0}(T)} f(\xi_1^{t,\xi_0}(T),...,\xi_n^{t,\xi_0}(T))\right]\\ =& \mathbb{E}_{\mathbb{P}}\left[e^{-\xi_{n+1}^{t,\xi_0}(T)} e^{f_1(\xi_1^{t,\xi_0}(T),...,\xi_n^{t,\xi_0}(T))} f_2(\xi_1^{t,\xi_0}(T),...,\xi_n^{t,\xi_0}(T))\right]\\ =& e^{-h(t,T)(\tilde{x}^T,0)^T} e^{(A^T,0)e^{H_\xi(T-t)}(\tilde{x}^T,0)^T}\\ &\cdot\left\{\mathbb{E}_{\mathbb{P}}\left[e^{-\xi_{n+1}^{t,0}(T)} e^{f_1(\xi_1^{t,0}(T),...,\xi_n^{t,0}(T))}\right](F^T,0)e^{H_\xi(T-t)}(\tilde{x}^T,0)^T\right.\\ &\left.+\mathbb{E}_{\mathbb{P}}\left[e^{-\xi_{n+1}^{t,0}(T)} f(\xi_1^{t,0}(T),...,\xi_n^{t,0}(T))\right]\right\}.\end{aligned}$$

Furthermore, Theorem 2.17 yields that $\xi^{t,\xi_0}(T)$ is normally distributed with mean vector and covariance matrix given by (see e.g. p.355 of Karatzas and Shreve (1991))

$$m(t,T) := \mathbb{E}_{\mathbb{P}}\left[\xi^{t,\xi_0}(T)\right] = e^{H(T-t)}\xi_0 + \int_t^T e^{H_\xi(T-l)} J_\xi(l)dl$$

and

$$\begin{aligned}V(t,T) &:= \mathbb{E}_{\mathbb{P}}\left[\left(\xi^{t,\xi_0}(T)-m(t,T)\right)\left(\xi^{t,\xi_0}(T)-m(t,T)\right)^T\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[\int_t^T e^{H_\xi(T-l)}V_\xi dW(l)\int_t^T e^{H_\xi(T-l)}V_\xi dW(l)^T\right]\\ &= \int_t^T e^{H_\xi(T-l)}V_\xi V_\xi^T e^{H_\xi^T(T-l)}dl.\end{aligned}$$

Since e^{Ht} is continuously differentiable in t and J is continuous, it follows that $m(t,T)$ and $V(t,T)$ are continuously differentiable in t. The vector

$$Y := (Y_1,Y_2) = \begin{pmatrix} A^T & -1 \\ F^T & 0 \end{pmatrix}\xi^{t,\xi_0}(T) + \begin{pmatrix} b \\ d \end{pmatrix}$$

is two-dimensional normally distributed with mean vector

$$\begin{pmatrix} A^T & -1 \\ F^T & 0 \end{pmatrix} m(t,T) + \begin{pmatrix} b \\ d \end{pmatrix}$$

and covariance matrix

$$\begin{pmatrix} A^T & -1 \\ F^T & 0 \end{pmatrix} V(t,T)\begin{pmatrix} A^T & -1 \\ F^T & 0 \end{pmatrix}^T.$$

This yields[6]

$$
\begin{aligned}
B(\tilde{x},t) = & e^{(A^T,-1)m(t,T)+b+\frac{1}{2}(A^T,-1)V(t,T)(A^T,-1)^T} \\
& \cdot \left((F^T,0)m(t,T)+d+(F^T,0)V(t,T)(A^T,-1)^T\right)
\end{aligned}
$$

This shows the differentiability in t. Furthermore, it holds that

$$
|B(\tilde{x},t)| \leq e^{\|(A^T,-1)e^{H(T-t)}\|_2\|\xi_0\|_2+|a(t,T)|} \left(\|(F^T,0)e^{H(T-t)}\|_2\|\xi_0\|_2+|b(t,T)|\right)
$$

with $a(t,T)$ and $b(t,T)$ denoting continuous functions in t. Hence, there exist constants

$$
K_2 := \sup_{0\leq t\leq T} \left(\|(A^T,-1)e^{H(T-t)}\|_2 + \|(F^T,0)e^{H(T-t)}\|_2\right)
$$

and

$$
K_1 := e^{\sup_{0\leq t\leq T}(|a(t,T)|+|b(t,T)|)}
$$

such that with $\|\xi_0\|_2 = \|\tilde{x}\|_2 \leq 1+\|\tilde{x}\|_2^2$ it holds that

$$
|B(\tilde{x},t)| \leq K_1 e^{K_2\|\tilde{x}\|_2^2}.
$$

The uniqueness of B follow from Theorem 2.16. □

This result will be used in the valuation of credit derivatives in Chapter 6, where conditional expectations of the form (2.11) appear as the main building blocks of the pricing formulas.

2.4 Kalman filter

This section is concerned with the discrete-time Kalman filter and maximum likelihood estimation for state-space models based on the work of Kalman (1960). A more extensive and detailed discussion of Kalman filtering techniques can e.g. be found in Harvey (1989), Chapter 12 of Brockwell and Davis (1991) or Koopman *et al.* (1999). The latter presents efficient algorithms for prediction, filtering, and smoothing in state-space models. Applications of

[6]For a two-dimensional random variable $Y=(Y_1,Y_2)^T \sim \mathcal{N}_2(\mu,\Sigma)$ with $\mu=(\mu_1,\mu_2)$ and $\Sigma=(\sigma_{ij})_{i,j=1,2}$ it holds that

$$
\mathbb{E}_{\mathbb{P}}\left[Y_2 e^{Y_1}\right] = e^{\mu_1+\frac{1}{2}\sigma_{11}}\left(\mu_2+\sigma_{12}\right).
$$

Kalman filtering to problems in finance can e.g. be found in Schmid (2004) or Kolbe and Zagst (2008).
In general, Kalman filtering can be used whenever the state of a stochastic process given by a linear stochastic differential equation can only be observed from a series of noisy measurements. The Kalman filter provides a numerically efficient way to estimate the state of the process based on the current information. Using these estimates, unknown model parameters can be estimated via maximum likelihood.
The standard Kalman filter is based on a linear Gaussian state-space model consisting of a transition equation and a measurement equation. The transition equation describes the dynamics of an unobservable state vector, while the measurement equation relates an observable variable to the state vector.

Definition 2.21. *A discrete-time linear Gaussian state-space model is defined by two stochastic processes $(\alpha(t))_{t=1,\dots,T}$ and $(Y(t))_{t=1,\dots,T}$, where $\alpha(t)$ fulfils the transition equation*

$$\alpha(t) = c(t) + W\alpha(t-1) + H\epsilon(t), \; t = 1, ..., T \tag{2.13}$$

and $Y(t)$ the measurement equation

$$Y(t) = d(t) + Z\alpha(t) + G\epsilon(t), \; t = 1, ..., T. \tag{2.14}$$

Here, $\alpha(t)$ denotes the unobservable $m \times 1$ state vector at time t, $Y(t)$ the $N \times 1$ observation vector at time t, $c(t)$ and $d(t)$ are unknown fixed effects at time t with dimension $m \times 1$ and $N \times 1$ respectively, $\epsilon(t)$ is the $r \times 1$ disturbance vector, where usually $r = m + N$, and $W, Z, G,$ and H are the deterministic system matrices with dimensions $m \times m$, $N \times m$, $N \times r$, and $m \times r$. Furthermore, the disturbance vectors $(\epsilon(t))_{t=1,\dots,T}$ are i.i.d. multivariate-normal random vectors with expectation $\mathbf{0}$ and the r-dimensional identity matrix $\mathbf{I_r}$ as covariance, i.e.

$$\epsilon_t \sim \mathcal{N}_r(\mathbf{0}, \mathbf{I_r}).$$

The initial state vector is drawn from a normal distribution with expectation a_0 and covariance P_0, i.e.

$$\alpha(0) \sim \mathcal{N}_m(a_0, P_0).$$

Note that if the initial conditions are not explicitly defined, one can assume that the initial state vector is fully diffuse, i.e. $a_0 = \mathbf{0}$ and $P_0 = \kappa\mathbf{I_m}$ and hence $\alpha(0) \sim \mathcal{N}_m(\mathbf{0}, \kappa\mathbf{I_m})$, where κ is some large scalar, e.g. $\kappa = 10^6$ (see p.111 of Koopman *et al.* (1999)). The following algorithm shows how to obtain an estimate $a(t)$ for the state $\alpha(t)$ based on the current information up to time t. This algorithm is also the basis of the maximum likelihood estimation of the parameters in the state-space model.

Algorithm 2.22. *Kalman filter*

1. *Initialization:*
   Set $t = 0$ *and choose initial parameters* a_0, P_0.

2. *Prediction:*
   Set $t = t + 1$.
   Evaluate the prediction equations

$$\begin{aligned} a(t|t-1) &= Wa(t-1) + c(t) \\ P(t|t-1) &= WP(t-1)W^T + HH^T. \end{aligned}$$

3. *Update:*
   Evaluate the update equations

$$\begin{aligned} a(t) &= a(t|t-1) + P(t|t-1)Z^T F(t)(y(t) - Za(t|t-1) - d(t)) \\ P(t) &= P(t|t-1) - P(t|t-1)Z^T F(t)^{-1} ZP(t|t-1) \end{aligned}$$

   with $F(t) := ZP(t|t-1)Z^T + GG^T$.

4. *Termination:*
   If $t = T$ *stop, else go back to 2.*

The following theorem states the distributional properties of the quantities in the Kalman filter.

Theorem 2.23. *For* $t = 1, ..., T$ *it holds that*

$$\left.\begin{pmatrix} \alpha(t) \\ Y(t) \end{pmatrix}\right| y_1, ..., y_{t-1} \sim$$

$$\mathcal{N}_{m+N}\left(\begin{pmatrix} a(t|t-1) \\ Za(t|t-1) + d(t) \end{pmatrix}, \begin{pmatrix} P(t|t-1) & P(t|t-1)Z^T \\ ZP(t|t-1) & F(t) \end{pmatrix}\right)$$

and

$$\alpha(t)|y_1, ..., y_t \sim \mathcal{N}_m(a(t), P(t)).$$

In particular, $a(t)$ *is the minimum mean square estimate of the unobservable state* $\alpha(t)$*, given the observed data* $y_1, ..., y_t$.

Proof. A proof can e.g. be found on p.109f of Harvey (1989). □

Knowing the distributional properties from Theorem 2.23 the likelihood function of the state-space model can be derived as follows. Let $y_1, y_2, ..., y_T$ denote the observations and Θ the model parameter vector. Then, the log-likelihood is, up to some constants, given by (for more details see e.g. Chapter 3.4 in Harvey (1989))

$$\begin{aligned}\log l(y_1, ..., y_T; \Theta) &= \sum_{t=1}^{T} \log p(y_t | y_1, ..., y_{t-1}; \Theta) \\ &\propto -\sum_{t=1}^{T} (\log |F(t)| + v(t)^T F(t)^{-1} v(t))\end{aligned}$$

with $v(t)$ denoting the innovations $y_t - (d(t) + Za(t|t-1))$ from the Kalman filter. Hence, maximum likelihood estimates of the parameter vector Θ can be obtained by maximising the expression

$$f(\Theta | y_1, ..., y_T) = -\sum_{t=1}^{T} (\log |F(t)| + v(t)^T F(t)^{-1} v(t)).$$

Unfortunately, in many situations the functional relations in Equations (2.13) and (2.14) are non-linear, i.e.

$$\begin{aligned}\alpha(t) &= g(c(t), \alpha(t-1)) + H\epsilon(t), \ t = 1, ..., T, \\ Y(t) &= h(d(t), \alpha(t)) + G\epsilon(t), \ t = 1, ..., T\end{aligned}$$

with g and h denoting some (sufficiently smooth) non-linear functions. In this case an extended Kalman filter which linearises the non-linear functions g and h in a Taylor-series expansion around the current estimates $a(t)$ and $a(t|t-1)$ can be applied. Although the extended Kalman filter is in contrast to the standard Kalman filter not an optimal estimator in general, it often shows a good performance in practical applications. Further discussion on the extended Kalman filter can e.g. be found in Section 3.7.2 of Harvey (1989).

2.5 Copulas

This section describes the basic principles of copula theory in general and especially for exchangeable and nested Archimedean copulas. A detailed introduction on copulas is e.g. given by Joe (1997) or Nelsen (1998).

2.5.1 General Properties

The distribution of any I-dimensional random vector $X = (X_1, \ldots, X_I)$ can be described by its distribution function

$$F(x_1, \ldots, x_I) := \mathbb{P}(X_1 \leq x_1, \ldots, X_I \leq x_I).$$

If the one-dimensional marginals of X are known, i.e.

$$F_i(x_i) := \mathbb{P}(X_i \leq x_i) = F(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty),\ x_i \in \mathbb{R},\ i = 1, \ldots, I,$$

they have to be coupled to determine the I-dimensional distribution function F. This is achieved by means of the copula of $(X_1, \ldots, X_I)$ which is defined as follows.

Definition 2.24. *An I-dimensional copula C is a distribution function $C : [0,1]^I \mapsto [0,1]$ on the I-dimensional unit cube with uniformly distributed marginals.*

In other words, knowing the marginals and the copula is equivalent to knowing the multi-dimensional distribution. Any copula C fulfils the following three properties:

1. $C(u_1, \ldots, u_I)$ is increasing in each component u_i, $i = 1, \ldots, I$.
2. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i = 1, \ldots, I$ and $u_i \in [0,1]$.
3. For all $a_1, \ldots, a_I$, $b_1, \ldots, b_I \in [0,1]$ with $a_i \leq b_i$ it holds that

$$\sum_{i_1=1}^{2} \cdots \sum_{i_I=1}^{2} (-1)^{i_1+\ldots+i_I} C(u_{1i_1}, \ldots, u_{Ii_I}) \geq 0$$

with $u_{i1} = a_i$ and $u_{i2} = b_i$ for all $i = 1, \ldots, I$.

Furthermore, for each $k \in \{2, \ldots, I-1\}$ the k-dimensional marginal of the I-dimensional copula is a copula itself.

Example 2.25. *Two simple examples of copulas are the independence copula Π and the copula of complete comonotonicity M defined by*

$$\Pi(u_1, \ldots, u_I) := \prod_{i=1}^{I} u_i,$$
$$M(u_1, \ldots, u_I) := \min\{u_1, \ldots, u_I\}.$$

Note that random variables with continuous distribution functions are independent if and only if their dependence structure is induced by an independence copula Π. *In contrast, for a random vector* $(U_1, \ldots, U_I)$ *with joint distribution function* M *the components are perfectly positively dependent, i.e. for each* $i = 2, \ldots, I$ *the random variable* U_i *is almost surely a strictly increasing function of* U_1.

The most important theorem in copula theory is the popular Theorem of Sklar (see Sklar (1959)) which shows that copulas can be used in conjunction with univariate marginals to construct multivariate distribution functions.

Theorem 2.26. *Let* F *be an* I*-dimensional distribution function with marginals* $F_1, \ldots, F_I$. *Then there exists an* I*-dimensional copula* $C : [0,1]^I \mapsto [0,1]$ *such that for all* $(x_1, \ldots, x_I) \in \mathbb{R}^I$ *it holds that*

$$F(x_1, \ldots, x_I) = C\big(F_1(x_1), \ldots, F_I(x_I)\big). \tag{2.15}$$

If the marginals $F_1, \ldots, F_I$ *are continuous, then* C *is unique. Otherwise,* C *is uniquely determined on* $Ran(F_1) \times \ldots \times Ran(F_I)$, *where* $Ran(F_i)$ *denotes the range of* F_i, $i = 1, \ldots, I$.
Conversely, if C *is an* I*-dimensional copula and* $F_1, \ldots, F_I$ *are univariate distribution functions, then the function* F *defined via Equation (2.15) is an* I*-dimensional distribution function.*

Proof. A complete proof can be found e.g. in Nelsen (1998). □

Sklar's Theorem allows to construct multivariate distribution functions in two steps. In a first step one may choose the univariate marginals and in a second step a copula. This construction principle will be used in Chapter 7 by first fitting the parameters of the marginals to portfolio CDS and subsequently the copula parameters to CDO tranches. This separation of marginals and dependence structure is the main reason for the popularity of copulas in statistical and financial applications.
A very important class of copulas are elliptical copulas (see e.g. Embrechts *et al.* (2003)), i.e. the copulas of elliptical distributions, and especially Gaussian copulas.

Example 2.27. *Let* $(X_1, \ldots, X_I)$ *be a normally distributed random vector with joint distribution function*

$$\begin{aligned}&\Phi_I(x_1, \ldots, x_I; \mu, \Sigma) \\ &:= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_I} \frac{1}{(2\,\pi)^{\frac{I}{2}}\, det(\Sigma)} \exp\Big(-\frac{1}{2} (\mathbf{s} - \mu)^T\, \Sigma^{-1}\, (\mathbf{s} - \mu)\Big)\, ds_I \ldots ds_1\end{aligned}$$

with Σ denoting a positive-definite $I \times I$-dimensional matrix, $\mu \in \mathbb{R}^I$, and $\mathbf{s} := (s_1, \ldots, s_I)^T$. Let $\sigma_1^2, \ldots, \sigma_I^2 > 0$ denote the diagonal entries of Σ. Then, $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, \ldots, I$, and the copula C of $(X_1, \ldots, X_I)$ is called a Gaussian copula given by

$$C(u_1, \ldots, u_I) = \Phi_I\big(\Phi_1^{-1}(u_1; \mu_1, \sigma_1), \ldots, \Phi_1^{-1}(u_I; \mu_I, \sigma_I); \mu, \Sigma\big). \tag{2.16}$$

Note that the copula of an I-dimensional multivariate distribution F with strictly increasing continuous marginals $F_1, \ldots, F_I$ is always given in a similar way like in Equation (2.16), but in many cases this expression can be computed explicitly. In the Gaussian case however, this is not possible due to the fact that no closed-form antiderivatives of normal densities are known.

While the dependence structure of a copula in two or three dimensions can be visualized by a scatter plot (see e.g. Figure 2.1), this is no longer possible for higher dimensions. Therefore, it is sometimes more convenient to express the dependence structure of a copula in probabilistic terms. Popular examples of such dependence measures are e.g. Spearman's Rho, Kendall's Tau, and Blomqvist's Beta (see e.g. Schmid and Schmidt (2006) or Schmid and Schmidt (2007)). In the following, the upper-tail dependence is defined, which will be used later in Chapter 7.

Definition 2.28. *For a random vector $X = (X_1, X_2)$ with marginals F_1 and F_2 the coefficient of upper-tail dependence is defined as*

$$\lambda_U := \lim_{q \to 1} \mathbb{P}(X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q)),$$

provided the limit exists. Here, $F^{\leftarrow}(x) = \inf\{z : F(z) \geq x\}$ denotes the quantile function of F.

Hence, the coefficient of upper-tail dependence gives the probability that X_2 is large given X_1 is large. For a random vector $U = (U_1, U_2)$ whose joint distribution function is the copula C it holds that

$$\begin{aligned}
\lambda_U &= \lim_{u \to 1} \mathbb{P}(U_2 > u \,|\, U_1 > u) \\
&= \lim_{u \to 1} \frac{\mathbb{P}(U_1 > u, U_2 > u)}{\mathbb{P}(U_1 > u)} \\
&= \lim_{u \to 1} \frac{\mathbb{P}(U_1 \leq u, U_2 \leq u) + \mathbb{P}(U_1 > u) + \mathbb{P}(U_2 > u) - 1}{\mathbb{P}(U_1 > u)} \\
&= \lim_{u \to 1} \frac{C(u, u) - 2u + 1}{1 - u}.
\end{aligned} \tag{2.17}$$

The analogous concept of lower-tail dependence is defined by the probability that U_2 is small given U_1 is small. Positive upper- or lower-tail dependence is desirable whenever extreme scenarios shall be modelled. The bivariate normal distribution is a popular example for a distribution whose tail dependences are both zero. Therefore, models which are based on normality assumptions are often criticised as they don't support extreme events. A family of copulas with lower and/or upper tail-dependence is introduced in the next subsection.

2.5.2 Archimedean Copulas

In this section, the basic concepts of exchangeable and nested Archimedean copulas are repeated, which will be used in Chapter 7 to create dependence between defaults of different firms as well as between default rates and recovery rates. One of the main properties of Archimedean copulas is that they are fully specified by some generator function. Furthermore, Archimedean copulas are flexible to capture various dependence structures, e.g. the aforementioned tail dependence. Applications of Archimedean copulas in financial modelling can e.g. be found in Schönbucher (2003) or Cherubini *et al.* (2004). In the last years, nested Archimedean copulas, which extend the concept of exchangeable Archimedean copulas by allowing for some asymmetries, have become quite popular in financial market applications (see e.g. Savu and Trede (2006) or Hofert and Scherer (2009)). Exchangeable Archimedean copulas are defined as follows.

Definition 2.29. *An I-dimensional exchangeable (i.e. distributionally invariant under permutations) Archimedean copula is given by*

$$C(u) = C(u_1, \ldots, u_I; \varphi_0) = \varphi_0^{[-1]} \left[\varphi_0(u_1) + \ldots + \varphi_0(u_I)\right], \ u \in [0,1]^I, \tag{2.18}$$

where the generator $\varphi_0 : [0,1] \mapsto [0,\infty]$ is a continuous and strictly decreasing function, which satisfies $\varphi_0(1) = 0$ and

$$\varphi_0^{[-1]}(t) = \begin{cases} \varphi_0^{-1}(t), & 0 \le t \le \varphi_0(0) \\ 0, & \varphi_0(0) < t < \infty \end{cases}$$

denoting the pseudo-inverse of φ_0.

The function φ_0 is called copula generator. However, for a given generator φ_0 the function C from Equation (2.18) is not always a copula. A necessary and sufficient condition on the inverse of the generator $\varphi_0^{[-1]}$ such that C is a copula uses the following definition.

Definition 2.30. *A function $\varphi_0^{[-1]} : [0,\infty) \to \mathbb{R}$ is called completely monotonic, if it is continuous on $[0,\infty)$, has derivatives of all orders on $(0,\infty)$, and*

$$(-1)^k \frac{d^k}{dt^k}\varphi_0^{[-1]}(t) \geq 0, \; k \in \mathbb{N}, \; t > 0.$$

Using this definition, Kimberling (1974) proved one of the main theorems for Archimedean copulas.

Theorem 2.31. *Let φ_0 be an Archimedean generator. Then Equation (2.18) defines a copula for all $I \geq 2$ if and only if $\varphi_0^{[-1]}$ is completely monotonic.*

Proof. See Kimberling (1974). □

Hence, C from Equation (2.18) defines a proper copula if and only if the inverse of the generator $\varphi_0^{[-1]}$ is a completely monotonic function. In this case, the pseudo-inverse is equal to the inverse function, i.e. $\varphi_0^{[-1]} = \varphi_0^{-1}$ (see e.g. p.122 of Nelsen (1998)).
Completely monotonic functions are known from probability theory as Laplace transforms of non-negative random variables. This relationship is given in Bernstein's theorem as follows.

Theorem 2.32. *A function $\varphi_0^{-1} : [0,\infty) \to \mathbb{R}$ is completely monotonic with $\varphi_0^{-1}(0) = 1$ if and only if there exists a non-negative random variable $V \geq 0$ such that $\varphi_0^{-1}(t) = \mathbb{E}[\exp(-tV)]$, i.e. φ_0^{-1} is the Laplace transform of V.*

Proof. See p.439 of Feller (1971). □

Hence, the class of completely monotonic functions φ_0^{-1} on $[0,\infty)$ with $\varphi_0^{-1}(0) = 1$ coincides with the class of Laplace-Stieltjes transforms of distribution functions G on $[0,\infty)$, i.e. $\varphi_0^{-1}(t) = \int_0^\infty e^{-tv} dG(v)$ for $t \geq 0$. This can be used to construct random vectors according to a multivariate Archimedean copula as given in the following theorem (see e.g. p.223 of McNeil *et al.* (2005)).

Theorem 2.33. *Let G be a distribution function on $[0,\infty)$ satisfying $G(0) = 0$ with Laplace-Stieltjes transform φ_0^{-1}. Further, let V denote a random variable with $V \sim G$ and $U_1,\dots,U_I$ a sequence of random variables conditionally independent given V, where the distribution function of U_i conditioned on V is given by $F_{U_i|V=v}(u) = e^{-v\varphi_0(u)}$ for $u \in [0,1]$. Then, the joint distribution of the random vector $U = (U_1,\dots,U_I)$ is an Archimedean copula with generator φ_0, i.e.*

$$\mathbb{P}(U_1 \leq u_1,\dots,U_I \leq u_I) = \varphi_0^{-1}\left[\varphi_0(u_1) + \ldots + \varphi_0(u_I)\right].$$

Proof.

$$\begin{aligned}\mathbb{P}(U_1 \le u_1, \dots, U_I \le u_I) &= \int_0^\infty \mathbb{P}(U_1 \le u_1, \dots, U_I \le u_I | V = v) dG(v) \\ &= \int_0^\infty \prod_{i=1}^{I} F_{U_i|V=v}(u_i) dG(v) \\ &= \int_0^\infty e^{-v(\varphi_0(u_1)+\dots+\varphi_0(u_I))} dG(v) \\ &= \varphi_0^{-1}\left[\varphi_0(u_1) + \dots + \varphi_0(u_I)\right].\end{aligned}$$

□

Using this representation, Marshall and Olkin (1988) presented an efficient algorithm for sampling from multi-dimensional exchangeable Archimedean copulas under the assumption that G is known.

Algorithm 2.34. *Marshall and Olkin*

1. *Sample from a random variable* $V \sim G$.
2. *Sample from i.i.d. random variables* $X_i \sim Unif[0,1]$, $i \in \{1, \dots, I\}$.
3. *Return the random vector* $(U_1, \dots, U_I)$ *with* $U_i = \varphi_0^{-1}(-\log(X_i)/V)$.

An example of a one-parametric family of Archimedean copulas is given in what follows (see also McNeil (2008)).

Example 2.35. *The Gumbel copula is given by its generator* $\varphi_0(t) = (-\log(t))^\theta$ *with* $\theta \in [1, \infty)$. *The random variable* V *from Algorithm 2.34 follows a positive stable distribution, i.e.* $V \sim St(1/\theta, 1, \gamma, 0)$ *with* $\gamma = (cos(\pi/(2\theta)))^\theta$ *and characteristic function given by*

$$\mathbb{E}_{\mathbb{P}}\left[e^{itV}\right] = \begin{cases} \exp\left(-\gamma^{1/\theta}|t|^{1/\theta}\left(1 - i\,\mathrm{sign}(t)\tan(\frac{\pi}{2\theta})\right)\right), & \theta \neq 1 \\ \exp\left(-\gamma|t|\left(1 + i\,\mathrm{sign}(t)\frac{2}{\pi}\ln|t|\right)\right), & \theta = 1 \end{cases}.$$

The coefficient of upper tail dependence is given by $\lambda_U = 2 - 2^{1/\theta}$. *Figure 2.1 shows* 1000 *realizations of a bivariate Gumbel copula with* $\theta = 1.5$.

A more flexible multivariate Archimedean copula can be constructed by the nesting of generators. These copulas allow for different degrees of positive dependence in different margins.

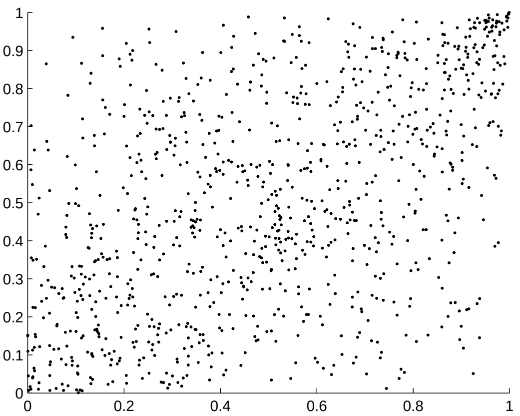

Figure 2.1: Scatterplot of 1000 samples of a bivariate Gumbel copula with $\theta = 1.5$.

Definition 2.36. *An I-dimensional partially nested Archimedean copula is given by*

$$\begin{aligned} C(u) &= C\left(C\left(u_{11},\ldots,u_{1d_1};\varphi_1\right),\ldots,C\left(u_{H1},\ldots,u_{Hd_H};\varphi_H\right);\varphi_0\right) \\ &= \varphi_0^{[-1]}\left[\varphi_0\left(\varphi_1^{[-1]}\left[\varphi_1(u_{11})+\ldots+\varphi_1(u_{1d_1})\right]\right)+\ldots\right. \\ &\quad \left.+\varphi_0\left(\varphi_H^{[-1]}\left[\varphi_H(u_{H1})+\ldots+\varphi_H(u_{Hd_H})\right]\right)\right] \\ &= \varphi_0^{[-1]}\left[\sum_{h=1}^{H}\varphi_0\left(\varphi_h^{[-1]}\left[\sum_{k=1}^{d_h}\varphi_h\left(u_{hk}\right)\right]\right)\right] \end{aligned}$$

with $u_{hk} \in [0,1]$, $h \in \{1,\ldots,H\}$, $k \in \{1,\ldots,d_h\}$, and $\sum_{h=1}^{H} d_h = I$.

Here, d_h denotes the dimension of the h-th subgroup. This is a copula if all $\varphi_0 \circ \varphi_h^{[-1]}$ have completely monotonic derivatives (see McNeil (2008)). In the special case that the generators belong to the same one-parameter family of Archimedean copulas, i.e. $\varphi_h = \varphi_0(\cdot;\theta_h)$, it is often sufficient to claim that $\theta_0 \le \theta_h$, $h \in \{1,\ldots,H\}$, with θ_h denoting the parameter corresponding to φ_h, $h \in \{0,\ldots,H\}$, (see Hofert (2008)). This assumption will be used throughout the remainder of this chapter and in the application of nested Archimedean copulas for CDO pricing in Chapter 7.
Based on Algorithm 2.34, McNeil (2008) suggested an algorithm for sampling from partially nested Archimedean copulas. This algorithm applies Algorithm 2.34 iteratively by sampling from distribution functions associated with Laplace-Stieltjes transforms which are the inverses of the generators

denoted by $\varphi_0(\cdot;\theta_h)$. Here, G again denotes the distribution function with Laplace-Stieltjes transform $\varphi_0^{[-1]}(\cdot;\theta_0)$.

Algorithm 2.37. *McNeil*

1. *Sample from a random variable $V \sim G$.*
2. *For $h \in \{1,\dots,H\}$ sample from a random vector*
$$(X_{h1},\dots,X_{hd_h}) \sim C\left(u_{h1},\dots,u_{hd_h};\varphi_0(\cdot;\theta_h)\right)$$
according to Algorithm 2.34.
3. *Return the random vector $(U_{11},\dots,U_{Hd_H})$ with*
$$U_{hk} = \varphi_0^{[-1]}\left(-\log(X_{hk})/V\right),\ h \in \{1,\dots,H\},\ k \in \{1,\dots,d_h\}\,.$$

Example 2.35 can be easily extended to the case of nested Archimedean copulas.

Example 2.38. *Let C denote a four-dimensional nested Gumbel copula with two subgroups given by*

$$C(u) = \varphi_0^{-1}\left[\varphi_0\left(\varphi_1^{-1}\left[\varphi_1(u_{11})+\varphi_1(u_{12})\right]\right)+\varphi_0\left(\varphi_2^{-1}\left[\varphi_2(u_{21})+\varphi_2(u_{22})\right]\right)\right],$$

where $\varphi_i(t) = (-\log(t))^{\theta_i}$. For $(U_1,\dots,U_4) \sim C$ with $\theta = (1.2, 2.2, 1.5)$, Figure 2.2 contains a two-dimensional scatterplot with 1000 realizations of (U_i,U_j), $i,j \in \{1,2,3,4\}$ and $i \neq j$, in each off-diagonal subplot. The diagonal subplots show the histograms of the univariate marginals.

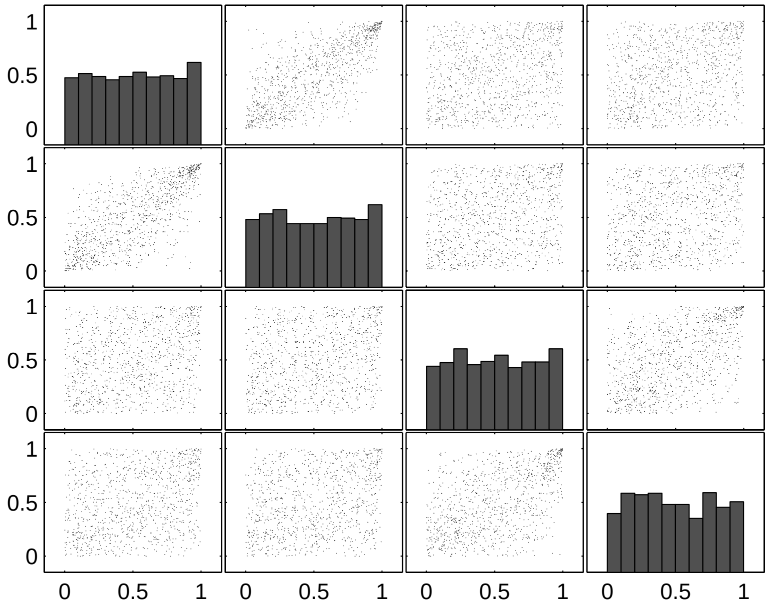

Figure 2.2: Two-dimensional scatterplots of 1000 samples of a four-dimensional Gumbel copula with $\theta = (1.2, 2.2, 1.5)$.

Chapter 3

Recovery rates in credit-risk modelling

This chapter is devoted to the measurement and modelling of recovery rates and risks in defaultable assets. In the first part of this chapter a definition of recovery rates and risks and examples of their occurrence in pricing and risk management will be given. The second part of this chapter is concerned with the different types of measuring recovery rates. Finally, a literature review on various modelling approaches for different credit-risk applications is presented.

3.1 Recovery rates and risks

In contrast to the probability of default, which is calculated on the obligor level, the recovery rate, or loss given default (LGD) respectively, has to be calculated on the facility level. It is usually defined as the amount recovered from a defaulted facility expressed as a percentage of the exposure at default (EAD). Recovery risk measures not only the risk that recovery payments upon default are higher or lower than expected but also changes in mark-to-market values of defaultable assets due to changes in expected market recovery. For a creditor the recovery rate is as important as the probability of default. Nevertheless, for many years practitioners, regulators, as well as academics have concentrated on modelling and predicting default probabilities and ignored the stochastic nature of recovery rates. Due to the rapid growth in the credit derivatives market at the beginning of this century and the appearance of contingent claims on recoveries since 2003, e.g. fixed-recovery CDS or recovery locks (see e.g. Berd (2005) or Liu *et al.* (2005)), the sound modelling of recovery rates gained in importance lately, both for

pricing purposes and portfolio-risk management as well as for economic capital requirements. E.g. within the internal ratings-based (IRB) approach of the new Basel Accord (see Basel Committee on Banking Supervision (2004)) banks are allowed to use internal estimates of default probabilities as well as of recovery in the event of default to calculate credit-risk capital. There are two different versions within the IRB approach available for banking institutions: the foundation approach on the one hand and the advanced approach on the other hand. In the foundation approach a bank is only allowed to estimate the probability of default internally, whereas the loss given default is a constant, e.g. 75% for subordinated debt and 45% for senior unsecured debt. The advanced approach allows the bank to determine all parameters, i.e. probability of default, loss given default and exposure at default, internally subject to supervisory review. As the advanced approach is, due to its flexibility, the more interesting method for banks with large enough portfolios, the methods referring to the IRB approach in the following all bear on this approach.
Not only supervisory institutions, but also rating agencies turned their attention to recovery rates lately. Besides the classical credit ratings, which date back to the middle of the 19^{th} century and measure the credit worthiness of a corporation or a country, major rating agencies started to publish so-called recovery ratings in 2003. Based on cash-flow stress tests as well as debt and equity valuation, recovery ratings measure the expected recovery at emergence from the bankruptcy process. E.g. Fitch Ratings assigns six different recovery ratings with recovery expectation bands $0-10\%$, $11-30\%$, $31-50\%$, $51-70\%$, $71-90\%$, and $91-100\%$.
Besides such risk management approaches, several credit derivatives have emerged in the last years that enable investors to trade recovery risk separately from pure default-event risk. While in a standard credit-default swap (CDS) there is uncertainty about the height of the payment made in case of a default event to cover the resulting loss, a fixed-recovery CDS eliminates this uncertainty. In case of a default event, the fixed-recovery CDS is cash settled with the contract's fixed recovery rate. In addition to standard and fixed-recovery CDS, recovery locks, sometimes also called recovery swaps, which are forward contracts on the recovery rate in case of a default event, started trading in 2003. Within such a recovery lock, there are no upfront or running payments. The only payment stream that occurs is the exchange of realized and contractual recovery rate in case a default has happened. If no default event occurs during the lifetime of the contract, it expires unused. The relation between the different products can be expressed as follows: a long position in a recovery lock can be separated in a long position in a fixed-recovery CDS and a short position in a standard

CDS (see also Section 6.2). While recovery locks have been traded only very rarely in their first years, they became more interesting with the increasing number of companies in financial distress since mid 2007. Meanwhile, recovery locks are traded on the debt of more than 70 companies with a total volume of about 10 billion US dollar. The International Swaps and Derivatives Association (ISDA) published a template for such products (available at `www.isda.org/publications/docs/Recovery-Lock-Template.doc`) in 2006. Nevertheless, recovery locks are only traded over the counter and bid-ask spreads are relatively high. A good example for the functionality and usefulness of recovery locks is the bankruptcy of Lehman Brothers Holdings Inc. According to `www.creditfixings.com` the auction on bankrupt Lehman Brothers' standard CDS held on October 10$^{\text{th}}$ 2008 set a value of 8.625 cents per dollar for the investment bank's debt. Only three days before the bankruptcy on September 15$^{\text{th}}$ 2008, recovery locks ensuring a 20 percent recovery rate on Lehman Brothers debt were still traded. Therefore, a protection seller in a Lehman Brothers CDS who entered such a recovery lock (at no cost at initiation) had two different payment streams in his portfolio after the default of Lehman Brothers. On the one hand he received 8.625 cents per dollar from the CDS contract (in exchange for the default compensation) and on the other hand he received the fixed 20 cents per dollar in exchange for 8.625 cents per dollar from the recovery lock. Hence, such an investor would have gained $20 - 8.625 = 11.375$ cents per dollar from this recovery lock contract.

3.2 Measuring recovery rates

Before recovery rates can be incorporated into the credit-risk modelling process, one has to clarify how they are measured. There are different ways to accomplish this. The methods described in this section are mainly based on the suggestions of the IRB approach. To define a measure for the recovery rate or loss given default respectively, first of all a coherent definition of default has to be found. A formal definition is given in §§452ff of Basel Committee on Banking Supervision (2004). Schuermann (2004) summarizes the default definitions and gives four indicators for a default: a loan is placed on non-accrual, a charge-off has already occurred, the obligor is more than 90 days past due, or the obligor has filed bankruptcy.

As mentioned above the loss given default is, in contrast to the probability of default, which is calculated on the obligor level, calculated on the facility level. It is usually defined as the loss expressed as a percentage of the exposure at default. This definition can be improved by differentiating between defaulted and non-defaulted facilities. For the first it is the ex-post ratio of loss to EAD at the time of default and for the latter the ex-ante estimate of the loss given default as a random variable (see e.g. Basel Committee on Banking Supervision (2005b)). Sometimes the term loss given default is used only for the ex-ante estimates and 1 minus the observed losses are called recovery rates. In general there are four different methods for the estimation of loss given default (see, e.g., Basel Committee on Banking Supervision (2005b)):

1. *Workout* or *ultimate LGD* based on discounted cash flows after default: here, the most important things to concern are the correct timing of the cash flows and the discount rate which is applied. It is by far not clear which discount rate is the correct one. In addition to that, one has to take care to incorporate the right amount of direct and indirect costs which arise during the workout procedure.

2. *Market LGD* based on prices of traded defaulted bonds and loans shortly after the default has occurred (typically 30 days): as the prices of traded bonds and loans are based on par = 100, the recovery rate can be derived from the actual prices. Advantages of this method are that the values are obtainable instantaneously after default and that market LGD expresses the market sentiment. Unfortunately, market LGD is not available for all loans, as many of them, especially those to small and medium-sized enterprises (SMEs), are not traded on the market.

3. *Implied market LGD* derived from risky but not yet defaulted bond prices by means of a theoretical asset pricing model: here, LGD is estimated via credit spreads of non-defaulted, risky bonds.

4. *Implied historical LGD* based on the experience of total losses and PD estimates (only allowed for retail portfolios).

As the estimates of LGD must be based on historical recovery rates (see §470 of Basel Committee on Banking Supervision (2004)), it is sometimes questioned if the implied methods should be excluded from this list because they are based on information of non-defaulted debt.

As mentioned above the correct discount rate for workout LGD can be a very important issue, especially when the workout time is long, as it is often the case for loans to large corporates. The theoretical correct discount rate would be the risk-appropriate discount rate, but, as facilities considered for workout LGD are often not traded on the market, the problem is to infer this risk-appropriate rate. The discount rate used for the workout LGD should reflect the costs of holding the defaulted assets during the workout period including an appropriate risk premium (see Basel Committee on Banking Supervision (2005a)). There are broadly two different ways in which the discount rates for modelling recovery rates, or loss given default respectively, can be chosen. Either the discount rate is in some way connected to the borrower interest rate at default or current comparable market rates are considered (see Davis (2004)). While the first measures how well the institution collects on the defaulted facility, the latter also takes the opportunity costs of funds into account. There are different discount rates that have been proposed in empirical as well as theoretical literature (see Basel Committee on Banking Supervision (2005b) or Maclachlan (2005)): Carty *et al.* (1998) for example use the contractual rate fixed at the date of origination, while Asarnow and Edwards (1995) include a penalty term to the contractual rate. Friedman and Sandow (2003b) use the coupon rate as a discount rate. Other possible discount rates that are proposed in Basel Committee on Banking Supervision (2005b) are the risk-free rate plus a spread at the default date for the average recovery, the zero-coupon yields plus a spread at the default date, the average risk-free rate plus a spread during the last business cycle at the date of transaction, the average rate of an asset of similar risk over the last business cycle at the date of transaction, and the spot rate plus spread existing at the date of transaction. Finally, in FSA (2003) the usage of a suitable rate for an asset of similar rate at the date of default is recommended.

3.3 Modelling recovery rates

While the previous section dealt with measuring recovery rates, this section is concerned with modelling aspects of recovery rates in different types of credit-risk models.

3.3.1 Econometric recovery rate prediction models

There are two main industry models for the prediction of recovery rates based on econometric models, namely Moody's LossCalc™ v2 (see Gupton and Stein (2005)) and S&P's LossStats® (see Friedman and Sandow (2003b)). Both models work on a set of explanatory variables and predict recovery rates, or loss given defaults respectively, for a given time horizon.

3.3.1.1 Moody's LossCalc™

Moody's LossCalc™ v2 predicts recovery rates for defaults which occur immediately or for defaults that may occur in one year. As required by the Basel

Committee the methodology includes time-varying factors and a history that is longer than seven years (see §468 and §472 of Basel Committee on Banking Supervision (2004)). LossCalc$^{\text{TM}}$ uses different explanatory factors from five different groups (collateral, instrument, firm, industry, and macroeconomy and geography). In detail the explanatory variables are:

- Collateral: The proportion of coverage of the exposure by cash, "all assets", property, plant and equipment, and "unknown" as well as yes or no for support from subsidiaries and unsecured.
- Debt type (loan, bond, and preferred stock) and seniority class (senior, junior, secured, unsecured, subordinate, etc.).
- Firm-specific distance-to-default (for publicly traded firms only): The distance to default is defined by (see e.g. Crosbie and Bohn (2003))

$$\text{Distance-to-default} = \frac{\text{Market Value of Assets} - \text{Default Point}}{\text{Market Value of Assets} \times \text{Asset Volatility}},$$

  where the default point is defined as the market value of assets where the firm will default. Usually the default point is assumed to be short-term debt plus half the long-term debt.
- Relative seniority: Seniority standing within the firm's overall capital structure.
- Firm leverage: How much asset value is available to cover the liabilities. Not used for secured debt or financial industries.
- The industry's historical recovery rates.
- The aggregated distance-to-default across all firms in that industry and region.
- Trailing 12-month all corporate default rate (for a definition see e.g. p.20 of Hamilton *et al.* (2001)) published monthly by Moody's Investors Service.

In LossCalc$^{\text{TM}}$ recovery is defined as the observed debt price approximately one month after default. The observed market values seem to be approximately beta distributed. Before modelling a multivariate model some of the explanatory variables are transformed or adjusted. This step is called "mini-modelling" in the original technical document. The main modelling process can be divided into three steps:

1. Transform the raw recovery observations z_i to an approximately normally distributed dependent variable y_i, i.e.

$$y_i = \Phi^{-1}\left(F_{Beta}\left(z_i, a, b\right)\right),$$

where a and b are the parameters of the beta distribution, Φ the cumulative distribution function of a standard normal distribution and F_{Beta} the cumulative distribution function of the beta distribution.

2. Run a multivariate linear regression on the normalized recovery rates, i.e.

$$y = \beta_1 x_1 + \ldots + \beta_K x_K + \epsilon,$$

where the x_k, $k = 1, \ldots, K$ are the (transformed) explanatory variables and $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

3. Retransform the normalized recovery predictions

$$\widehat{y_i} = \widehat{\beta}_1 x_{1,i} + \ldots + \widehat{\beta}_K x_{K,i}$$

via

$$\widehat{z_i} = F_{Beta}^{-1}\left(\Phi\left(\widehat{y_i}\right), a, b\right)$$

with $x_{k,i}$ denoting the i-th observation of the k-th explanatory variable and $\widehat{\beta}_k$, $k = 1, \ldots, K$, the estimated coefficients from step 2.

3.3.1.2 S&P's LossStats®

Standard & Poor's LossStats® is mainly based on a general machine learning approach which is explained in detail in Friedman and Sandow (2003a). This approach is not only suitable for LGD modelling, but also for the modelling of probabilities of default (see e.g. Friedman and Huang (2003)). This very general concept is based on different aspects from statistics, optimization, and utility theory. Therefore, only the main features are sketched here. A more detailed overview of this approach can be found e.g. in Friedman and Sandow (2003a).
In contrast to LossCalc™, where market prices shortly after default are used as a proxy for recoveries, discounted recovery rates at the time the obligor emerges from bankruptcy (workout or ultimate recoveries) are used in LossStats®. These ultimate recoveries are allowed to take values in $[0, z_{\max}]$, where $z_{\max} = 1.2$ is suggested in the original technical document. The explanatory variables used for estimating the conditional probability distribution of the recovery rates are the following:

- Collateral quality classified into 16 categories ranging from "all assets" to "unsecured".

- Debt below class, defined as the percentage of debt on the balance sheet that is inferior to the class of debt instrument considered.

- Debt above class, defined as the percentage of debt on the balance sheet that is superior to the class of debt instrument considered.

- Aggregate default rate expressed by the percentage of S&P-rated US-bonds that defaulted within a time horizon of 12 months prior to the default date.

In addition to that, industry default rate, regional default rate, and quarterly GDP growth rate are used as explanatory variables for bonds.
The modelling approach can be summarized as follows: Let x be the vector of the observed explanatory variables and z describe the ultimate recovery rate. Further, a continuous probability density function on $[0, z_{\max}]$ is assumed for the recovery rates conditioned on x with positive point masses in 0 and 1, i.e.

$$p(z|x) = p_{[0,z_{\max}]}(z|x) + p_0(x)\delta(z) + p_1(x)\delta(z-1)$$

with $\delta(z) = \mathbb{1}_{\{z=0\}}$, $p_{[0,z_{\max}]}(z|x)$ denoting the recovery rate density on $[0, z_{\max}]$ given the explanatory variable x, and $p_0(x)$ and $p_1(x)$ the point masses in 0 and 1 respectively, conditioned on the explanatory variable x. Following Friedman and Sandow (2003a) the optimal probability measure is found by maximizing the out-of-sample expected utility of an investor who chooses his strategy so as to maximize his expected utility under the model he beliefs in. This leads to a procedure which chooses the measure from an efficient frontier of pareto optimal measures defined in terms of consistency with a prior distribution $p^0(z|x)$ and consistency with the data. The latter is expressed as the difference between the theoretical and the sample expectations of a set of features, where each feature f_j is a mapping from the pair (z, x) to $\mathbb{R}$. Here, a logarithmic utility function and a prior of the form $p^0(z|x) = p^0_{[0,z_{\max}]}(z|x) + p^0_0(x)\delta(z) + p^0_1(x)\delta(z-1)$ are used. This leads to an optimization problem of the form:

$$\begin{aligned}
\text{Minimize} \qquad & \sum_x \widetilde{p}(x) \Bigg\{ \int_0^{z_{max}} p_{[0,z_{\max}]}(z|x) \log \frac{p_{[0,z_{\max}]}(z|x)}{p^0_{[0,z_{\max}]}(z|x)} dz \\
& \qquad + \sum_{k=0,1} p_k(x) \log \frac{p_k(x)}{p^0_k(x)} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\text{s.t. } Nc^T\Sigma^{-1}c &\leq \alpha \\
\text{with } c &= E_p[f] - E_{\widetilde{p}}[f] \\
E_p[f] &= \sum_x \widetilde{p}(x) \int_{[0,z_{\max}]} p(z|x) f(z,x) dz \\
\text{and } E_{\widetilde{p}}[f] &= \sum_x \widetilde{p}(x) \int_{[0,z_{\max}]} \widetilde{p}(z|x) f(z,x) dz.
\end{aligned}$$

Here, $f(z,x) = (f_1(z,x), \ldots, f_J(z,x))^T$ is the feature vector, $\widetilde{p}$ the empirical distribution function, N the number of observations, and Σ the empirical covariance matrix of the features. The corresponding dual problem is given by:

$$\begin{aligned}
\text{Find } \beta^* &= \arg\max_\beta h(\beta) \\
\text{with } h(\beta) &= \frac{1}{N} \sum_{k=1}^N \log p^{(\beta)}(z_k|x_k) - \sqrt{\frac{\alpha}{N} \beta^T \Sigma \beta} \\
\text{with } p^{(\beta)}(z|x) &= \frac{1}{\Upsilon_x(\beta)} e^{\beta^T f(z,x)} \\
\text{and } \Upsilon_x(\beta) &= \int_0^{z_{max}} p^0_{[0,z_{\max}]}(z|x) e^{\beta^T f(z,x)} dz + \sum_{k=0,1} p^0_k(x) e^{\beta^T f(k,x)},
\end{aligned}$$

where (x_k, z_k) are the observed pairs of explanatory variables and recovery rates. The optimal probability measure is then given by

$$p(z|x) = p^{(\beta^*)}(z|x) p^0(z|x).$$

As the explicit choice of the prior does not seem to have a great impact on the results, the authors chose

$$p^0(z|x) = \frac{1}{2 + z_{max}} \left[1 + \delta(z) + \delta(z-1)\right]$$

as a prior. The features used in the technical document are given by $f_j(z,x) = z^n x_i^m$ for $z \in [0, z_{\max}]$ and $f_j(z,x) = x_i^m$ for $z \in \{0,1\}$ with x_i denoting the i-th component of x, $n = 1,2,3$, and $m = 0,1$.

3.3.2 Risk management using stochastic recoveries

In the late Nineties some industry models were developed to measure the potential loss within a given confidence level on a fixed time horizon. These value-at-risk models include CreditMetrics™ (see Gupton *et al.* (1997)), CreditRisk+™ (see Credit Suisse Financial Products (1997)) as well as CreditPortfolioView™ (see Wilson (1998)). One thing that all these credit value-at-risk models have in common is that they assume independence between probabilities of default and recovery rates. In CreditMetrics™ the recovery rate is a beta distributed random variable, in CreditRisk+™ it is treated as a constant parameter that has to be identified for each credit exposure, and in CreditPortfolioView™ the recovery rate is modelled as a random variable independent of the default process.

In recent years, research in modelling recovery rates for risk management purposes, especially management of tail risks like the calculation of economic capital, has focused on models that reflect the possible relationship between probabilities of default and recovery rates. Therefore, recovery rate and probability of default are both modelled depending on a single systematic risk factor X (e.g. representing the state of the economy). This approach is mainly based on the assumption that the same economic conditions that cause defaults to rise might cause recoveries to go down. One feature that all these models have in common is that the asset-value process of firm j is assumed to be of the form $A_j = \sqrt{\rho}X + \sqrt{1-\rho}\widetilde{X}_j$ with a systematic risk factor $X \sim \mathcal{N}(0,1)$ and an idiosyncratic risk factor $\widetilde{X}_j \sim \mathcal{N}(0,1)$ and that default occurs if A_j falls below a given threshold. The parameter ρ is sometimes referred to as the asset correlation between two firms. The higher ρ the higher is the influence of fluctuations of the business cycle on the asset value. For the modelling of recovery rates different distributional assumptions have been suggested.

Frye (2000b) models the recovery rate as a normally distributed (and hence unbounded) random variable via

$$z_j = \mu + \sigma\sqrt{q}X + \sigma\sqrt{1-q}X_j,$$

with an idiosyncratic (recovery-) risk factor $X_j \sim \mathcal{N}(0,1)$.

In Frye (2000a) as well as in Miu and Ozdemir (2006) the recovery rate is modelled as a beta distributed random variable given by

$$z_j = F_{Beta}^{-1}\left(\Phi(Y_j), a, b\right),$$

where $Y_j = \sqrt{q}X + \sqrt{1-q}X_j$, Φ the cdf of a standard normal distribution, and F_{Beta}^{-1} the inverse cdf of the beta distribution with parameters a and b.
Following the approach of Schönbucher (2003), the recovery rate in Duellmann and Trapp (2004) is modelled as a logit transformation of a normally distributed random variable and hence lies between 0 and 1, i.e.

$$z_j = \frac{e^{Y_j}}{1+e^{Y_j}},$$

where $Y_j = \mu + \sigma\sqrt{q}X + \sigma\sqrt{1-q}X_j$. A quite similar model is to be found in Rösch and Scheule (2005), where an extension to a multifactor model with more than one idiosyncratic factor is allowed.
There are also two approaches in this area that do not model recovery rates directly but rather the value of the collateral. In Frye (2000a) the recovery rate is given by

$$z_j = \min(1, Coll_j)$$

with $Coll_j = \mu(1+\sigma C_j)$ and $C_j = \sqrt{q}X + \sqrt{1-q}X_j$, whereas in Pykhtin (2003) the collateral is modelled by a lognormally distributed random variable $Coll_j = \exp\left(\mu + \sigma C_j\right)$ with $C_j = \sqrt{\beta}X + \sqrt{\gamma}X_j + \sqrt{1-\beta-\gamma}\nu_j$ and $\nu_j \sim \mathcal{N}(0,1)$ i.i.d.
Another interesting approach with an explicit relationship between the probability of default and the recovery rate is introduced in Bruche and González-Aguado (2008). This paper proposes a model in which the two variables depend on an unobservable credit cycle, modelled by a two-state Markov chain. Conditional on the credit cycle, probability of default and recovery rate are independent, where the marginal distribution of the recovery rates is a beta distribution. The parameters of the beta distribution are allowed to vary across seniority and industry. The only information used to model the (unobservable) Markov chain is default and recovery data.

3.3.3 Recovery rates in pricing models

Also pricing models for both single-name and portfolio credit derivatives relied on deterministic recovery assumptions for a long time. Only in recent years, some models have been developed to incorporate a stochastic behaviour of recovery rates in pricing credit derivatives. Before these models are discussed, the different modelling concepts that are available are repeated.

3.3.3.1 Modelling concepts

There are basically three different concepts of expressing and modelling recovery rates in credit-risk pricing models.

1. *Recovery of face value*, i.e. the recovery payment in case of a default event is a fraction of the face value of the defaulted asset (applied e.g. in Duffie (1998a)).

2. *Recovery of treasury*, i.e. in case of a default event there is compensation in terms of a fraction of an equivalent non-defaultable asset, where equivalent means the same maturity, face value, and payoff structure if no default happens (applied e.g. in Jarrow and Turnbull (1995) and Madan and Unal (1998)).

3. *Recovery of market value*, i.e. the recovery payment in case of a default event is assumed to be a fraction of the market value of the defaulted instrument instantaneously before default (applied e.g. in Duffie and Singleton (1999)). This approach is, in terms of pricing relationships, equivalent to the multiple defaults model as described in Section 6.1.3 of Schönbucher (2003).

All these concepts have in common that they don't model the realized recovery in a bankruptcy process, but rather the value of the settlement. The different concepts can all be transformed into each other and are hence equivalent in a mathematical sense. Nevertheless, in different situations one model might be preferred over the other due to tractability reasons or a higher degree of interpretability. The general pricing problems for defaultable contingent claims within the different concepts can e.g. be found in Section 6.1 of Schönbucher (2003).

Comparing the three approaches, the recovery of treasury concept is the weakest. It leads to unrealistic spread curve shapes and the exclusion of recoveries above 100% requires some severe restrictions. The recovery of market value assumption leads to analytically tractable pricing formulas, e.g. for bonds. Unfortunately, it does not allow for a separation of default and recovery risk, i.e. only the product of intensity and recovery is observable. This is problematic whenever credit derivatives with payoffs solely depending on intensity or recovery are considered. Also for risk management and hedging purposes, the distributions of both recovery and intensity are required. By contrast the recovery of face value approach allows for a separation of default and recovery risk. Although this is at the cost of slightly more complicated pricing formulas, the recovery of face value assumption is applied in most credit-derivatives pricing models.

3.3.3.2 Single-name pricing models

During the last thirty years different approaches to modelling recovery rates in single-name credit-pricing models have been developed by academics as well as by practitioners. This subsection describes the usage of recovery rates in the most important classes of these models. A more detailed description of the different models is to be found in Altman *et al.* (2004).
In the first generation of structural-form models based on the seminal work of Merton (see Merton (1974)), where defaults can only be observed at maturity, the recovery at default is, as well as the probability of default, a function of the firm's asset volatility and the firm's leverage expressed by the firm's debt. Hence, the recovery rate is a model-endogenous variable.
The second generation of structural-form models overcomes the drawback that default can only occur at maturity at the cost of a recovery rate that is exogenously specified and independent of the firm's asset value as well as of the probability of default (see e.g. Longstaff and Schwartz (1995)). Here, a default occurs when the asset value falls below a given threshold level and the recovery rate in case of default is assumed to be a fixed ratio of the outstanding debt value. Exceptions are models based on discontinuous processes (see e.g. Zhou (2001)).
One of the main drawbacks of the structural-form models is that default depends on the unobservable asset-value process. A class of models which can be implemented without estimating the parameters of the asset-value process are the reduced-form models described in Section 2.2 (see e.g. Jarrow and Turnbull (1995), Madan and Unal (1998), or Duffie and Singleton (1999)). In general, these models assume an exogenous recovery rate independent from the probability of default. This recovery rate can either be deterministic or stochastic and different recovery rates for different issuers or seniority are possible. Exceptions with correlated default and recovery rates are described in more detail in the following.
Over the last years different approaches have been made to incorporate the stochastic behaviour of recovery rates and their correlation with the default process in intensity-based pricing models.
One of the first attempts to a joint modelling of recovery and default risk was proposed by Bakshi *et al.* (2001), who assume that the recovery rate is related to the underlying hazard rate. More precisely, the authors assume that the recovery rate $z(t)$ is related to the underlying hazard rate $\lambda(t)$ via

$$z(t) = a_z + b_z e^{-\lambda(t)}$$

with $a_z \geq 0$, $b_z \geq 0$, and $0 \leq a_z + b_z \leq 1$. The hazard-rate process itself is assumed to be linear in the short-term interest rate which is driven by a CIR-process, i.e.

$$\begin{aligned} \lambda(t) &= \Lambda_0 + \Lambda_1 r(t), \\ dr(t) &= (\theta_r - a_r r(t))dt + \sigma_r \sqrt{r(t)} dW_r(t),\ r(0) = r_0, \end{aligned}$$

with $\Lambda_0 > 0$ and $r_0 > 0$. The main drawback of this modelling approach is that there is only one factor, the short rate, that explains the whole variation. In the appendix of Bakshi *et al.* (2006) an illustrative multi-factor defaultable bond valuation model is presented. However, this model leads to complex valuation formulas and is hence difficult to implement. In this model, the same relation among recovery rates and intensity is valid as in Bakshi *et al.* (2001), but the intensity is given by a three-factor model. The model can formally be described by the following set of stochastic differential equations:

$$\begin{aligned} dr(t) &= (\theta_r w(t) - a_r r(t))dt + \sigma_r dW_r(t),\ r(0) = r_0, \\ dw(t) &= (\theta_w - a_w w(t))dt + \sigma_w dW_w(t),\ w(0) = w_0, \\ du(t) &= (\theta_u - a_u u(t))dt + \sigma_u dW_u(t),\ u(0) = u_0, \\ \lambda(t) &= \Lambda_0 + \Lambda_1 r(t) + \Lambda_2 w(t) + \Lambda_3 u(t), \\ z(t) &= a_z + b_z e^{-\lambda(t)}, \end{aligned}$$

with $r_0,\ w_0,\ u_0 \in \mathbb{R}$.
In Christensen (2005) a three-factor model for the joint evolution of interest rates, intensity, and recovery is introduced, where both intensity and recovery rate are affine functions of a common risk factor (short-term interest rate) and an idiosyncratic risk factor, i.e.

$$\begin{aligned} dr(t) &= (\theta_r - a_r r(t))dt + \sigma_r dW_r(t),\ r(0) = r_0, \\ \lambda(t) &= \Lambda_0 + \Lambda_r (a_r r(t) - \theta_r) + \Lambda_1 \left(a_\lambda X_\lambda(t) - \theta_\lambda\right), \\ z(t) &= z_0 + z_r (a_r r(t) - \theta_r) + z_1 \left(a_z X_z(t) - \theta_z\right), \\ dX_\lambda(t) &= (\theta_\lambda - a_\lambda X_\lambda(t))dt + \sigma_\lambda dW_\lambda(t),\ X_\lambda(0) = X_{\lambda,0}, \\ dX_z(t) &= (\theta_z - a_z X_z(t))dt + \sigma_z dW_z(t),\ X_z(0) = X_{z,0}. \end{aligned}$$

In a simulation-based study with arbitrarily chosen parameters the default intensity and recovery risk components are, under certain conditions (recovery of face value assumption, limited noise in the bond yields), estimated simultaneously from bond yields. Still, there is a fundamental identification problem inherent in the corporate bond yields as soon as measurement noise is added to the true yields.

An extension to this model is developed in Christensen (2007). In this four-factor affine model, the short-term interest rate is an affine function of two interest-rate factors given by

$$\begin{aligned} r(t) &= \delta_0 + \delta_1 X_1(t) + \delta_2 X_2(t), \\ dX_1(t) &= (\theta_1 - a_1 X_1(t))dt + \sqrt{X_1(t)}dW_1(t),\ X_1(0) = X_{1,0}, \\ dX_2(t) &= (\theta_{12} - a_{12} X_1(t) - a_2 X_2(t))dt + \sqrt{1 + \beta_{12} X_1(t)}dW_2(t), \\ &\quad X_2(0) = X_{2,0}. \end{aligned}$$

The default-intensity risk factor is assumed to be a CIR-process and the recovery-rate risk factor is assumed to be Gaussian. Both intensity and recovery rate are driven by the two interest-rate risk factors and their corresponding idiosyncratic risk factor, i.e. the default intensity is given by

$$\begin{aligned} \lambda(t) &= \Lambda_0 + \Lambda_1(a_1 X_1(t) - \theta_1) + \Lambda_2 X_2(t) + X_\lambda(t), \\ dX_\lambda(t) &= (\theta_\lambda - a_\lambda X_\lambda(t))dt + \sigma_\lambda \sqrt{X_\lambda(t)}dW_\lambda(t), X_\lambda(0) = X_{\lambda,0}. \end{aligned}$$

The recovery-rate process is given by

$$\begin{aligned} z(t) &= z_0 + z_1(a_1 X_1(t) - \theta_1) + z_2 X_2(t) + X_z(t), \\ dX_z(t) &= (\theta_z - a_z X_z(t))dt + \sigma_z dW_z(t), X_z(0) = X_{z,0}. \end{aligned}$$

The author's aim is the separation of default and recovery risk in an affine reduced-form setting. A numerical example is exercised with CDS quotes of Ford Motor Co.

Karoui (2007) proposes a discrete-time framework for modelling defaultable instruments under stochastic recovery as the pricing formulas are easier to handle than in a corresponding continuous-time setting. Gaspar and Slinko (2006) present a model based on the dynamics of a market index which determines the default intensity as well as the distribution of the loss quota. To be more precise, the loss quota is assumed to be a beta distributed random variable with one constant parameter and the other driven by the market index. This leads to intractable formulas and requires a simulation-based approach. A completely different approach is used in Das and Hanouna (2007). Here, a reduced-form calibration method for the joint derivation of market-implied forward hazard rates and forward recovery rates is presented, but without a dynamic representation of default and recovery risk components.

3.3.3.3 Multi-name pricing models

While most CDO pricing models assume a constant recovery rate of 40%, only very few CDO models with a stochastic recovery specification exist. The first one was introduced by Andersen and Sidenius (2004). In this article an extension to the Gaussian copula model (see e.g. Li (2000)) is presented. Thereby, a stochastic recovery related to the systematic factor driving the default events is assumed, explicitly allowing for an inverse correlation between recovery rates and default rates. To be more precise, the recovery rate of an obligor in case of a default is given by an application of the normal cumulative distribution function on a normally distributed random variable. This random variable is correlated with the default triggering variable through a common systematic factor, i.e.

$$\begin{aligned} A_i(t) &= \sqrt{\rho}X(t) + \sqrt{1-\rho}\widetilde{X}_i(t) \\ z_i(t) &= \Phi(\mu(t) + b(t)X(t) + X_i(t)) \end{aligned}$$

with $X(t)$, $\widetilde{X}_i(t)$, and $X_i(t)$ *i.i.d.* $\mathcal{N}(0,1)$, $i = 1,\ldots,N$ denoting firm i, Φ denoting the normal cdf, and t the point in time. Firm i defaults with recovery rate $z_i(t)$ if $A_i(t) \leq c_i(t)$. In a numerical investigation the authors noted that the base correlation skew effect of random recovery is quite minor and hence the random recovery approach was not further investigated.

Due to the credit market crisis, recently some articles on using stochastic recovery rates in CDO pricing have been published. Krekel (2008) uses a discrete stochastic recovery rate in a Gaussian base correlation setting to overcome the problem that super senior tranches in a standard Gaussian base correlation model have zero fair spread. In this model the discrete recovery rates are defined as constants on buckets of the default triggering factors, i.e. the recovery rate is a step function of the default triggering variable and hence

$$\begin{aligned} A_i(t) &= \sqrt{\rho}X(t) + \sqrt{1-\rho}\widetilde{X}_i(t) \\ z_i(t) &= z_i(t, A_i(t)) \end{aligned}$$

with $X(t)$ and $\widetilde{X}_i(t)$ *i.i.d.* $\mathcal{N}(0,1)$, $i = 1,\ldots,N$ denoting firm i, and t the point in time. Again, firm i defaults with recovery $z_i(t, A_i(t))$ if $A_i(t) \leq c_i(t)$. The function $z_i(t,x)$ is defined by

$$z_i(t,x) = \begin{cases} z_{i,j} & \text{if } c_{i,j}(t) < x \leq c_{i,j-1}(t) \quad j = 1,\ldots,J \\ 0 & \text{else} \end{cases},$$

where $c_{i,0}(t) = c_i(t)$ and $c_{i,J}(t) = -\infty$. E.g., in the empirical part of the article a recovery-rate distribution with only four possible realizations (60%, 40%, 20%, and 0%) is used. Amraoui and Hitier (2008) extend the approach of Krekel (2008) by modelling the recovery rate as some deterministic function of the systematic risk factor of the default triggering variable. Ech-Chatbi (2008) uses a multiple default approach (similar to Section 6.1.3 in Schönbucher (2003)), where the recovery is lowered by a random factor each time a default event occurs. Hence, the recovery-rate process is some geometric compound Poisson process, where the current recovery rate is multiplied by a random variable, e.g. beta or log-gamma distributed, each time a default event occurs. One thing that all these models have in common is that they rely on the assumption of a Gaussian dependence structure, which might not be appropriate, especially in distressed market situations. This Gaussian assumption will be relaxed in the modelling approach presented in Chapter 7.

Part II

Empirical analysis of recovery rates

Chapter 4

Loan recovery determinants

Although there has been a growing number of studies dealing with the determinants of recovery rates, especially from US rating agencies, the behaviour and prediction of recovery rates is by far not yet fully understood. Most of the studies on recovery rates are based on data from the US bond market rather than on loan recoveries. One of the first and probably most famous articles that examined bond recovery rates is the work by Altman and Kishore (1996), who examined the prices of bonds at the time of default of more than 700 defaulting bonds from 1978 to 1995. There are also some studies that concentrate on recoveries from bank loans. Again most of them with focus on the US (see e.g. Asarnow and Edwards (1995) or Gupton *et al.* (2000)). Studies from outside the US are e.g. from Latin America (see Hurt and Felsovalyi (1998) and La Porta *et al.* (2003)) or Australia (see Eales and Bosworth (1998)). In recent years some studies on bank loan recovery rates on the European market came up (see Grippa *et al.* (2005), Grunert and Weber (2005), Dermine and Neto de Carvalho (2006), Davydenko and Franks (2008), and Bastos (2009)), most of them relying on data from only one source or from one to three countries. Table 4.1 gives an overview of some studies on loan recoveries supplemented by some of the most important studies on bond recoveries. Note that the average recovery rates reported in Table 4.1 might be misleading as the standard deviations for the recovery rates are quite high (up to 40%).

Authors	Instruments	Measurement	Number	Time	Region	Mean
Acharya *et al.* (2003)	Bonds & Loans	Market	1758	82-99	US	41.9
Altman and Kishore (1996)	Bonds	Market	728	78-95	US	41.7
Araten *et al.* (2004)	Commercial Loans	Workout	3761	82-99	US	61.2
Asarnow and Edwards (1995)	Bank Loans	Workout	831	70-93	US	65.2
Bastos (2009)	Bank Loans (SME)	Workout	374	95-00	POR	71.0
Carey and Gordy (2004)	Firm Level	Workout	443	87-02	US	53.0
Carty and Lieberman (1996)	Bank Loans	Market	58	89-96	US	71.0
Carty and Lieberman (1996)	Bank Loans (SME)	Workout	229	90-96	US	79.0
Carty *et al.* (1998)	Bank Loans	Workout	200	86-97	US	86.0
Covitz and Han (2004)	Bonds	Market	1350	83-02	US	40.0
Davydenko and Franks (2008)	Bank Loans	Workout	276	84-03	GER	61.4
Davydenko and Franks (2008)	Bank Loans	Workout	1418	84-03	UK	75.0
Davydenko and Franks (2008)	Bank Loans	Workout	586	84-03	F	52.9
de Laurentis and Riani (2005)	Financial Leases	Workout	1118	80-00	IT	NA
Dermine and Neto de Carvalho (2006)	Bank Loans (SME)	Workout	374	95-00	POR	71.0
Eales and Bosworth (1998)	Loans	Workout	5782	92-95	AU	69.0
Emery *et al.* (2004)	Bank Loans	Market	370	89-03	NA	37.0
Grippa *et al.* (2005)	Loans	Workout	NA	-99	IT	37.0
Grossman *et al.* (2001)	Bonds & Loans	Market	35	97-00	US	NA
Grunert and Weber (2005)	Bank Loans	Workout	120	92-03	GER	72.4
Gupton *et al.* (2000)	Loans	Workout	181	89-00	US	69.5
Hamilton and Carty (1999)	Bonds & Loans	Workout	829	82-97	US	NA
Hamilton *et al.* (2005)	Bonds	Market	NA	82-04	US	42.2
Hu and Perraudin (2002)	Bonds	Market	958	71-00	US	41.0

Table 4.1: Overview of empirical studies on loan and bond recoveries – continued on next page.

Authors	Instruments	Measurement	Number	Time	Region	Mean
Hurt and Felsovalyi (1998)	Loans	Workout	1149	70-96	Lat. Am.	68.2
Izvorski (1997)	Bonds	Market	153	83-93	US	35.3
Keisman *et al.* (2004)	Bonds & Loans	Workout	852	88-03	US	78.3
La Porta *et al.* (2003)	Bank Loans	Workout	1500	-95	MEX	65.6
Roche *et al.* (1998)	Bank Loans	Market	60	91-97	US	NA
Schuermann (2004)	Bonds	Market	2025	70-03	US	39.9
Thorburn (2000)	Firm Level	Workout	263	88-91	SWE	35
van de Castle and Keisman (1999)	Bonds & Loans	Workout	829	87-97	US	54.0
van de Castle and Keisman (1999)	Bank Debt	Workout	258	87-97	US	84.5
van de Castle *et al.* (2000)	Bonds & Loans	Workout	954	87-96	US	51.1
van de Castle *et al.* (2000)	Bank Debt	Workout	264	87-96	US	83.5
Varma (2005)	Bonds	Market	50	90-03	Asia-Pac.	33.4
Varma and Cantor (2005)	Bonds & Loans	Market	1084	83-03	US	39.5

Table 4.1: Overview of empirical studies on loan and bond recoveries.

In this chapter, a detailed overview on factors that might influence recoveries and their consideration in the literature is given. Further explanatory variables which have not been considered yet in the literature (e.g. the different asset classes as proposed in § 215ff of the Basel Committee on Banking Supervision (2004) or the utilization rate) are introduced. A large Pan-European database is employed to describe the determinants and behaviour of loan recovery rates on a facility level. One of the great advantages of the application of this broad database is that it uses a consistent definition of default and recovery rate over different jurisdictions and hence allows for an empirical comparison over different countries, industry sectors, and asset classes.

4.1 The data

The LGD database used in this chapter consists of 42632 individual resolved defaulted loans and 34350 entities (24491 acting as a borrower, 9722 as a guarantor, and 137 as both). As the considered default events took place between April 1983 and February 2007, the data pool spans more than one full economic cycle as postulated in § 472 of Basel Committee on Banking Supervision (2004). The data pool contains information on defaulted loans from 37 European and 45 Non-European countries and from 10 different asset classes (SME (34018 facilities), Large Corporates (6222), Banks (381), Shipping Finance (207), Aircraft Finance (577), Real Estate Finance (724), Project Finance (249), Commodities Finance (162), Public Services (3), and Private Banking (89)). The economic recovery rate reported is the present value of all post-default cash flows as a percentage of the default amount, where the cash flows are discounted by the Euro Libor risk-free rate as at the loan default date.
All facilities with default amount 0 (1921 observations) have been removed from the database since only real physical losses are of interest. Furthermore, all facilities where the total sum of all reported cash flows (including charge-offs and waivers which are not present in the calculation of economic recovery rates) divided by the outstanding amount at default is greater than 110% (another 1494 observations) or smaller than 90% (6930 observations) of the outstanding amount at default have been excluded. This is accomplished to exclude all facilities that are not yet fully resolved or exhibit cash flows that are not reasonable. As some of the facilities in the database have abnormally high or low recoveries, outliers with a recovery rate lower than -0.5 (another 372 observations) or higher than 1.5 (50 observations) have been removed to mitigate the impact such observations may have. Additionally, a subsample of the dataset which only contains facilities with recoveries in $[0, 1]$ is under

examination. Some basic statistics of the recovery rates are to be found in Table 4.2.

	RR in $[-0.5, 1.5]$		RR in $[0, 1]$	
	Simple	Weighted	Simple	Weighted
Mean	0.556	0.712	0.607	0.693
St.dev.	0.443	0.323	0.401	0.308
Median	0.756	0.858	0.803	0.822
25%-Quantile	0.000	0.487	0.142	0.481
75%-Quantile	0.982	0.975	0.976	0.961
Number	31865		25232	

Table 4.2: Basic statistics of recovery rates.

Here, "weighted" means weighted with the size of the issue as at the date of default (this type of weighting was amongst others used by Duellmann and Trapp (2004), Varma (2005), Dermine and Neto de Carvalho (2006), and Bruche and González-Aguado (2008)). Simple means all facilities are equally weighted.

Figure 4.1 shows the distribution of the recovery rates. As one can clearly see the distribution of the recovery rates is bimodal (U-shaped for the case of recovery rates in $[0, 1]$). This holds true for the recovery-rate distributions of almost all subcategories, e.g. recovery-rate distributions for different industries, facility types, asset classes, or geographical jurisdictions. This supports the findings of many other studies on loan recovery rates that also observe a bimodal or U-shaped distribution (see e.g. Asarnow and Edwards (1995), Hurt and Felsovalyi (1998), Araten *et al.* (2004), Schuermann (2004) or Dermine and Neto de Carvalho (2006)), whereas some studies especially for bond recovery rates report a unimodal and more or less skewed distribution (see e.g. Carty and Lieberman (1996), Carty *et al.* (1998), Hu and Perraudin (2002), or La Porta *et al.* (2003)).

All following results refer to recovery rates in $[0, 1]$. As the results for recoveries in $[-0.5, 1.5]$ are very similar, they are omitted here.

As mentioned above, the discount rate used in the calculation of workout recoveries is quite important, especially if the time to resolution is long. A question one might ask is how big the influence of the discount rate on recovery rates actually is and how much the recovery rates change if the risk-free rate is substituted by a risk-adjusted rate. Maclachlan (2005) found that the mean discount rate for defaulted SME bank loans is on average similar to the contract rate at the time of default. Hence, the recovery rates discounted with the risk-free rate (Euro Libor risk-free rate) are compared with recovery

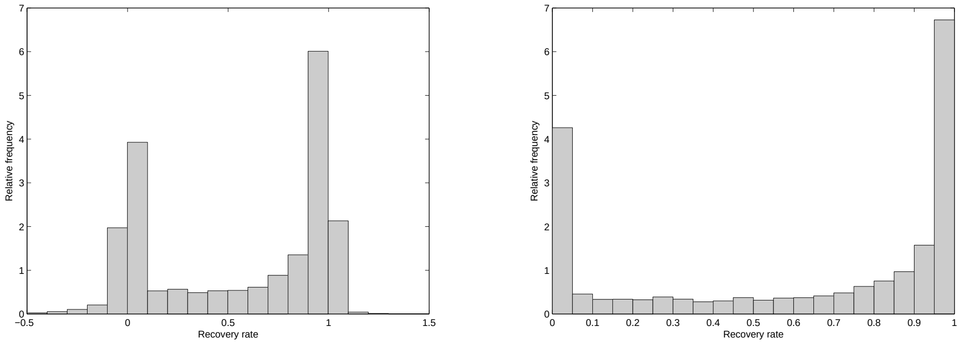

Figure 4.1: Recoveries in $[-0.5, 1.5]$ (left) and in $[0, 1]$ (right).

rates that were computed using the contract's base rate as at the date of default plus the contractual spread (available for 7008 facilities). Furthermore, two extreme cases are tested on the whole dataset: a fixed discount rate of 1% and a fixed discount rate of 15% (this is the discount rate used in Araten *et al.* (2004)).
The overall distribution of the recovery rates discounted with the risk-free discount rate (Euro Libor risk-free rate) and the risk-adjusted rate (contract's base rate plus the contractual spread) are quite similar. For the first the average recovery rate is 57.3% with a standard deviation of 40.7%, for the latter 54.7% (39.3%). The average difference in recovery rates is 2.6% with a standard deviation of 3.5%. This difference seems to be negligible, but if the facilities are grouped according to their time to resolution the differences can become quite high. Figure 4.2 shows the distributions of recovery rates with risk-free and risk-adjusted discount rates for facilities which were resolved within the first two months after default or for which the workout period was longer than 5 years. While the distributions for the facilities resolved in the first two months after default are almost equal, the recovery distributions for the longer workout period are completely different. The recovery rates (for the long workout period) with risk-adjusted discount rate (44.2% on average with a standard deviation of 45.0%) are a lot smaller than those discounted with the risk-free rate (54.6% on average with a standard deviation of 28.6%).

The average difference in recovery rates discounted with 1% and 15% is 8.8%. While the difference is rather small in most cases and not significant for short workout periods (e.g. for workout periods smaller than 2 months the average recovery rates are 25.4% and 25.1%), for those facilities with a long workout period the recovery rates differ substantially (see Figure 4.3).

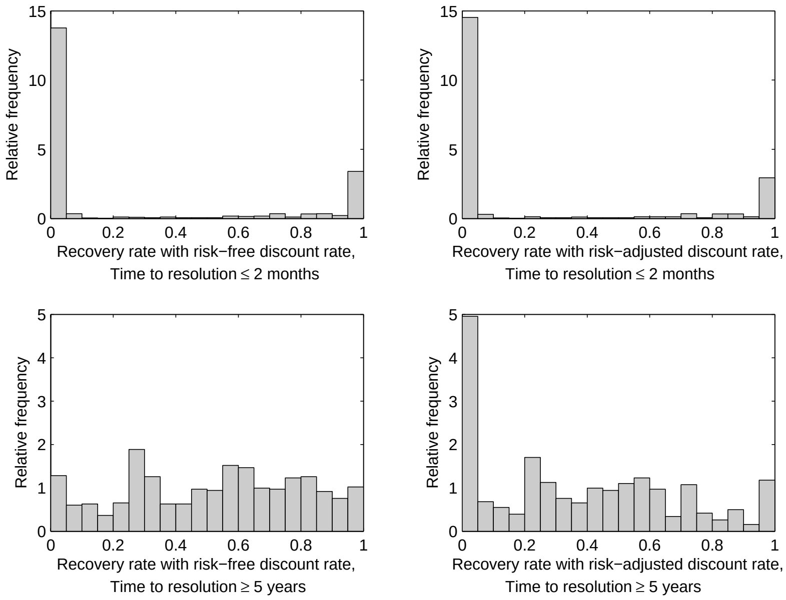

Figure 4.2: Recovery rates with risk-free and risk-adjusted discount rates.

As expected, the recovery rates (for the long workout period) with discount rate 15% (43.1% on average with a standard deviation of 27.8%) are a lot smaller than those discounted with 1% (65.5% on average with a standard deviation of 32.8%).
Concerning the influence factors for recovery rates, the chosen discount rate is rather unimportant. The explanatory variables (see Section 4.2) show the same behaviour regardless what discount rate is chosen. Only the default amount becomes less important for very high discount rates. This is due to the fact that the default amount discriminates the set of facilities with very high recoveries from those with low recoveries (see Subsection 4.2.1). As a high discount rate leads to a shift of all recovery rates towards the low end, this separation effect is no longer possible. Therefore, the recovery rates discounted with the Euro Libor risk-free rate as at the date of default will be used in the next sections.

4.2 Univariate analysis

Before the multivariate dependences in the data are analysed some univariate analyses are carried out. Here, also explanatory variables which are only available for a small subsample of the population and will therefore no longer

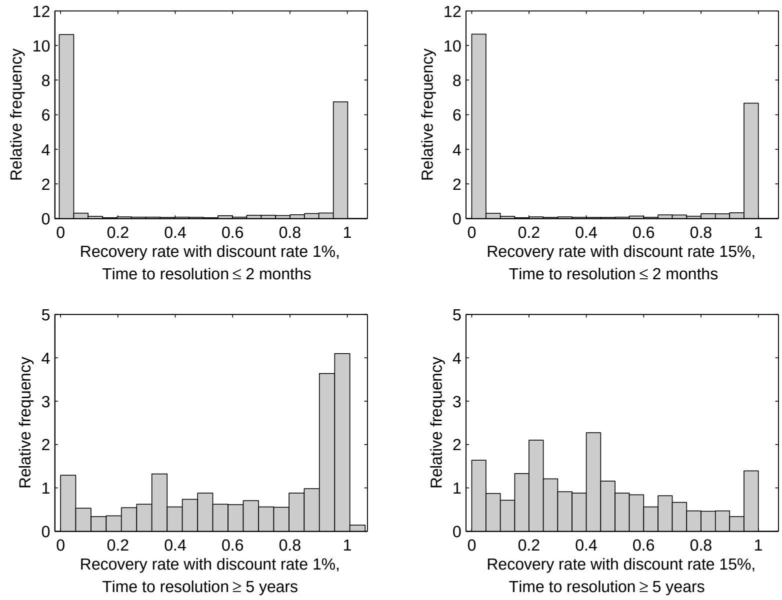

Figure 4.3: Recovery rates with discount rates 1% and 15%.

be under consideration when it comes to multivariate analyses (see Section 4.3) are taken into consideration. The univariate analyses described below contain linear regressions, correlation tests, and tests for the difference in means like t-Test, Kruskal-Wallis-Test or Wilcoxon-Test (see, e.g., Lehmann (1975) or Draper and Smith (1998)). As simple linear regression models are based on the assumption of normality and recovery rates are far from being normal (see Figure 4.1), the recovery rates are transformed before a regression analysis is applied. This is accomplished similar to Gupton and Stein (2005) via

$$\text{Transformed Recovery Rate} = \Phi^{-1}\left(F_{Beta}(\text{Recovery Rate}, a, b)\right),$$

where $F_{Beta}(x, a, b)$ is the distribution function of the beta distribution with parameters $a = 0.292$ and $b = 0.189$ and Φ the cdf of the standard normal distribution. The parameters a and b have been estimated from the recovery rates from Figure 4.1. Consequently, the transformed recovery rates are approximately normal. The choice of this transformation seems to be reasonable as the observed distribution of facility-level recovery rates is often bimodal or U-shaped in the interval $[0, 1]$ (see, e.g., Asarnow and Edwards (1995) or Schuermann (2004)) and the beta distribution is a very flexible, non-symmetric distribution on a bounded interval. The explanatory factors investigated are divided into five different classes: default process, facility, entity, collateralisation, and macroeconomic factors. Here, facility-level factors are all factors directly linked to one single loan, whereas entity-level factors are factors that describe the characteristics of the borrower of one or more loans. The independent variable to be described is in all subsections the facility-level economic recovery rate introduced in Section 4.1.

4.2.1 Default process

First of all, explanatory variables regarding the default process are considered, i.e. factors describing the time from initiation of the contract to default or from default to resolution as well as factors describing the reason that caused the default event and the exposure at default.

Workout period. The average time to resolution is 1.86 years with a standard deviation of 2.13 years. If a defaulted loan is resolved within the first two years after default, the time to resolution is positively correlated with the recovery rate. Otherwise, if the workout process takes longer than two years, the recovery rate decreases with an increasing time to resolution (see Figure 4.4). One possible explanation for the rather low recovery rates for very short workout periods, is the fact that the percentage of charge-offs is much higher for facilities with a workout period shorter than half a year (78.7%) than for those with longer workout periods (56.8%).

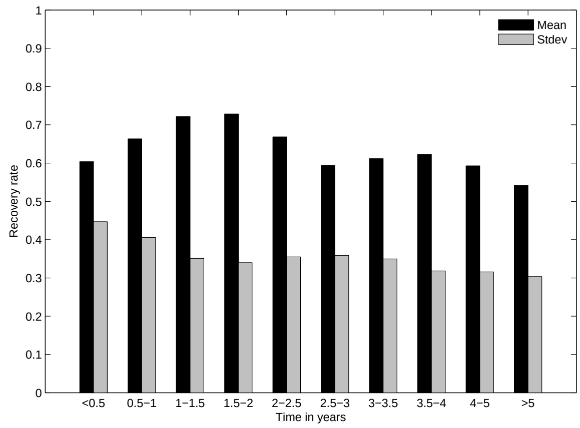

Figure 4.4: Recovery rates by time to resolution.

Other studies on workout recoveries report that the average time to resolution ranges from 1.25 years (see Carty *et al.* (1998)) to 4.5 years (see Grippa *et al.* (2005)) with high standard deviations. While Grippa *et al.* (2005) note an inverse relation between recovery and time to resolution, Carey and Gordy (2004) find little evidence that it affects firm-level recovery.

Workout costs. Almost 85% of the facilities in the database report zero workout costs resulting in average workout costs of 0.67% of the default amount. If only those facilities with positive workout costs are considered, the average workout costs are 4.34% with a standard deviation of 10.1%. The distribution of the workout costs is in both cases heavily right-skewed. One thing that is worth mentioning is the fact that secured loans have on average higher workout costs than unsecured loans, which might be explained by the costs resulting from the liquidation of collateral positions. Grippa *et al.* (2005) report average workout costs of 1.2% of the default amount. In a study from the leasing industry de Laurentis and Riani (2005) find average workout costs of up to 7.6% of the exposure at default.

Time to default. Furthermore, the time to default (available for 2598 observations) seems to have a positive effect on the recovery rate, i.e. the longer the time between origination and default the higher the recovery rate will be on average. Facilities that default within the first year after origination have on average a recovery rate of 44.2%, while facilities which survive at least one year recover 75.9% on average. The results from the literature for the time between origination and default are ambiguous. While Altman and Kishore (1996) can't find a relation between time to default and amount recovered, Emery *et al.* (2004) show a significant positive effect and support the aforementioned findings.

Reason for default. The reason for default is another important factor for the determination of workout recovery rates. Seven different types of default are observed: "90 days past due", "unlikely to pay", "bankruptcy", "charge-off or specific provision", "sold at material credit loss", "distressed restructuring", and "non-accrual" (see also the definition of default in § 452f of Basel Committee on Banking Supervision (2004)). All these different default types are highly significant with the highest recovery rates observed for "unlikely to pay" and the lowest recoveries for "90 days past due" and "sold at material credit loss". The type of default as a potential predictor for recoveries has only been discussed in very few studies so far. While Hamilton and Carty (1999) state that recoveries vary not significantly by bankruptcy type, Carty *et al.* (1998) find that prepackaged Chapter 11s have higher recoveries than regular Chapter 11s.

Default amount (EAD). Another interesting fact is that the default amount has a significant positive effect on the recovery rate (see Figure 4.5). This confirms the findings of Eales and Bosworth (1998). Many other empirical research articles find a negative impact (see e.g. Hurt and Felsovalyi (1998)) or no significant impact at all (see e.g. Asarnow and Edwards (1995), Grunert and Weber (2005), and Davydenko and Franks (2008)) of the EAD on the observed recovery rate.

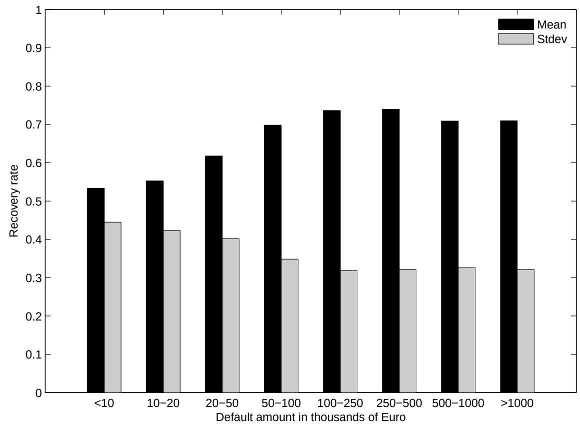

Figure 4.5: Recovery rates by default amount (EAD).

One possible reason for the aformentioned findings is the frequent occurrence of zero recovery for facilities with a smaller EAD. More than 85% (92%) of all facilities with zero recovery and 75% (83%) of all facilities with positve recovery smaller than 5% have a default amount smaller than 50000€ (100000€). In contrast, recoveries greater than 95% and fully recovered facilities are almost evenly distributed over the whole range of EADs. As can be seen from Figure 4.6, by removing loans with a small EAD the percentage of facilities with zero recovery decreases. The differences in average recovery rates for different default amount cut-offs are significant. If only facilities with an EAD greater than 100000€ are considered, the overall degree of explanation decreases. Some of the explanatory variables found to be significant in this section are less important or even no longer significant for facilities with high EADs, e.g. the size of the issue, the facility asset class, or the rank of security. Hence, it can be stated that these factors help to separate low recoveries from high recoveries, but have only a small impact on the actual value of the recovery.

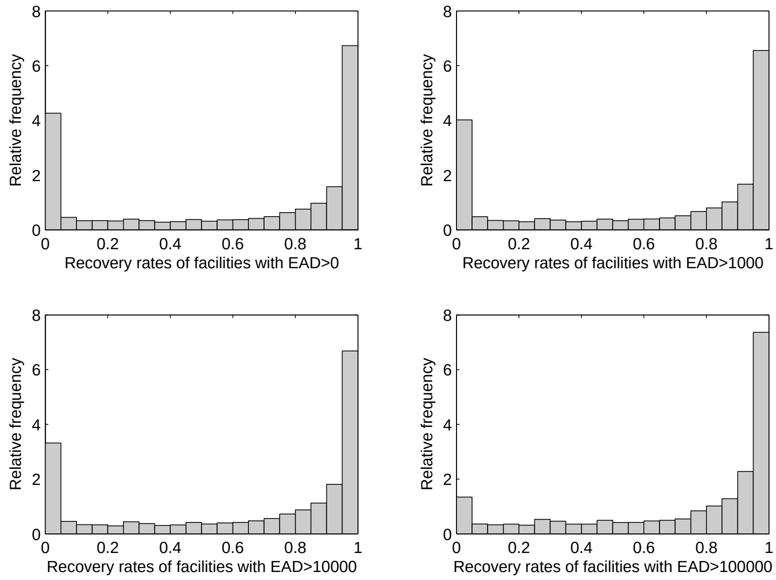

Figure 4.6: Recovery rates by minimum EAD.

4.2.2 Facility level

In the next step, the relation between recovery rates and factors that characterise the defaulted instrument is analysed.

Facility type. Another factor of interest is the facility type. A significant impact on the 5%-level can be found for the facility types "bridge loan", "revolver", "overdraft", "demand loan", "uncommitted line", and "other derivative or security claim". All other facility types are summarised in the category "other/unkown". The highest recovery rates are observed for "uncommitted line" (92.0%), the lowest for "demand loan" (35.2%). According to Siddiqi and Zhang (2004) some of the facility type dummies in his study seem to be significant for determining recovery rates.

Facility asset class. For the facility asset classes, significant differences between the following categories can be found: "SME", "large corporates", "banks" and "specialized lending". A summary of the basic statistics of the recovery rates for the different facility asset classes is to be found in Table 4.3.

Size of the issue. Similar to the default amount, which is defined as the lender

	SME	Large Corporates	Banks	Specialized Lending
Mean	0.588	0.722	0.729	0.737
St.dev.	0.409	0.326	0.350	0.314
Median	0.768	0.892	0.905	0.910
Number	21655	2489	207	807

Table 4.3: Basic recovery statistics for different facility asset classes.

outstanding amount at default, the lender outstanding amount at origination and 1 year prior to default have a clear positive effect on the recovery rate. This confirms the findings of Acharya *et al.* (2003). Many other empirical research articles find a negative impact (see e.g. Grippa *et al.* (2005) and Dermine and Neto de Carvalho (2006)) or no significant impact at all (see e.g. Altman and Kishore (1996) and Schuermann (2004)) of the size of the issue on the recovery rates.

Creditworthiness. Regarding the creditworthiness, recovery rates show a strong negative correlation (correlation coefficient = -24.0%) with the contractual spread (available for 9720 facilities), i.e. facilities with a lower creditworthiness (measured by a higher contractual spread) have lower recoveries on average and vice versa. This finding is similar to the results in Grunert and Weber (2005), who state that the creditworthiness has a positive impact on the recovery rate for bank loans. One factor that is only of interest for publicly traded debt is the original rating of the issue. According to Altman and Kishore (1996) it has no significant impact on the recovery rate once seniority is taken into consideration. More important than the original rating is the rating at time of default. Gupton *et al.* (2000) and Hamilton *et al.* (2001) find that rating at default acts as a predictor for recoveries.

Utilization rate. A quite interesting result was achieved when looking at the utilization rate which is defined as the percentage of the available commitment amount on a loan that is drawn. The height of the utilization rate has a statistical significant effect on the amount recovered if one distinguishes between utilization rates smaller than 100% and greater than 100%. For utilization rates smaller than 100% this effect is positive, i.e. the more of the commitment amount is drawn at the default date the higher is the recovery rate on average. By contrast utilization rates and recovery rates are negatively correlated if the utilization rate is greater than 100%. Nevertheless, this effect is rather small and for more than 85% of the data the utilization rate equals 100%.

Syndication. Facilities which are part of a syndication show recovery rates (on average 73.3%) that are a little bit higher than those of facilities which are not part of a syndication (71.5%), but this difference is not significant.

4.2.3 Entity level

In this subsection, the relation between (facility-level) recovery rates and explanatory variables connected with the borrowing entity as a whole is explained.

Entity asset class. For the entity asset classes ("corporate", "corporate specialized lending", "corporate mixture", and "banks") "banks" showed the highest average recovery rate (74.7%) and "corporate mixture" the lowest (49.7%). To the author's best knowledge no other studies concerning the impact of the entity asset class on facility-level recovery rates exist yet.

Size of borrower. The size of the borrower measured by the entity sales has a significant positive effect on the recovery rate, similar to the results obtained for the size of the issue (correlation coefficient 27.1%). As for the size of the issue, one can find studies that observe a negative relationship between the size of the borrowing company and the recovery rate (see e.g. Asarnow and Edwards (1995), Eales and Bosworth (1998), Hurt and Felsovalyi (1998), or Grunert and Weber (2005)), studies that find a positive relationship (see e.g. Acharya *et al.* (2003)) and studies that don't find any relationship at all (see e.g. Carty and Lieberman (1996), Roche *et al.* (1998), Thorburn (2000), Carey and Gordy (2004), or Davydenko and Franks (2008)).

Geography. In addition to that, the influence of the geographical jurisdiction on recovery rates was tested. It can be stated that there is a significant difference in the recovery rates for different regions. This is in line with Hu and Perraudin (2002) and Grippa *et al.* (2005), who show a significant influence of geographical dummies on recovery rates. By contrast, Araten *et al.* (2004) and Grunert and Weber (2005) find wide spread recovery rates over different geographic domiciles but no statistical significance. One interesting fact is that on average facilities from France recover more than those from Germany and both recover more than facilities from Great Britain. This contrasts the results that Davydenko and Franks (2008) found in their study on different bankruptcy codes in France (average recovery rate of 54%), Germany (61%), and the UK (74%). Table 4.4 gives an overview on recovery rates in the regions in the database from Section 4.1 (Africa, Asia, Austria & Germany, Benelux, France, Great Britain & Ireland, North America, Northern Europe, South America, and Southern Europe).

	AFRICA	ASIA	AT&GE	BENEL	FR
Mean	0.451	0.608	0.613	0.637	0.831
St.dev.	0.351	0.365	0.404	0.390	0.300
Median	0.389	0.797	0.807	0.831	0.964
Number	521	132	8747	189	548
	GB&IE	NAMER	NEURO	SAMER	SEURO
Mean	0.569	0.724	0.757	0.679	0.808
St.dev.	0.411	0.335	0.350	0.349	0.313
Median	0.726	0.910	0.962	0.913	0.974
Number	12102	308	103	79	72

Table 4.4: Basic recovery statistics for different regions.

Industry. Another factor that is often mentioned in literature is the industry of the borrower. Gupton *et al.* (2000) and Grunert and Weber (2005) find no evidence that different industries have different recoveries. Araten *et al.* (2004) state that recovery rates in dependence of industry dummies are widespread but not statistically significant, whereas Roche *et al.* (1998), Hu and Perraudin (2002), Acharya *et al.* (2003), Varma (2005), and Dermine and Neto de Carvalho (2006) find a significant and robust effect of industry on recovery rates. Altman and Kishore (1996) report great differences in average recovery rates for different industry sectors with a maximum of 70.5% for public utility and a minimum of 26.5% for lodging, hospitals, and nursing facilities. These results are confirmed by the works of Izvorski (1997) and Carey and Gordy (2004). Different levels of aggregation have been tested on the dataset introduced in Section 4.1 but no significant influence of industry dummies could be found, although there are quite high differences in the average recovery rates for different industries (between 51.9% for "education" and 83.1% for "public administration and defence").

Number of loans. The number of loans per borrower has a significant positive impact on the recoveries, i.e. single loans (56.0%) have on average a lower recovery rate than multiple-loan defaulters (64.3%). When the facilities are divided according to whether they are secured or not, one can see that for unsecured facilities single-loans have again a significant lower recovery rate (41.0% on average) than facilities of multiple-loan borrowers (51.0%

on average), which is in contrast to the results of Gupton *et al.* (2000). The authors of this study report an average recovery rate of 63.4% for single-loan defaulters for unsecured loans, while for multiple-loan defaulters only 36.8% are recovered for unsececured loans on average. For secured facilities no significant difference in recovery rates can be found between single-loan (65.2%) and multiple-loan defaulters (65.5%).

Operating firm indicator / Public-private indicator. The operating firm indicator (available for 5114 facilities) shows that on average recovery rates of facilities belonging to non-operating companies (80.8%) are higher than for loans of operating companies (72.2%). The difference in mean is significant on a 5%-level, but the impact of the operating firm indicator on recovery rates in a univariate linear model is rather small. The results obtained for the public-private indicator (7250 observations) also show only little explanatory power. Private (public) firms exhibit an average recovery rate of 70.5% (67.4%) with a standard deviation of 33.5% (36.4%). To the author's best knowledge no other studies concerning the impact of these indicators on facility-level recovery rates exist yet.

4.2.4 Collateralisation

This subsection deals with the impact of collaterals on the recovery rate of defaulted loans.

Collateral. Probably one of the most important factors influencing loan recovery rates is the presence, quality (liquidity), and quota of collateral. Most studies on loan recoveries report a significant difference between secured and unsecured loans (see e.g. Acharya *et al.* (2003), Emery *et al.* (2004), or Dermine and Neto de Carvalho (2006)). Carty *et al.* (1998) report an average recovery rate of 87% for secured loans (with a standard deviation of 23%) and an average of 79% for unsecured loans (27%). Gupton *et al.* (2000) find an average recovery rate of 69.5% for secured loans (22.5%) and an average of 52.1% for unsecured loans (28.6%). Araten *et al.* (2004) report an average recovery rate of 59.1% for secured and 49.5% for unsecured facilities. In the work of Grippa *et al.* (2005) fully collateralised loans have an average recovery rate of 70%, whereas non-collateralised loans recover only 32%. In the database presented in Section 4.1 it can be found that secured loans (average recovery rate of 65.3% with a standard deviation of 37.0%) lead to a higher mean recovery rate with a lower standard deviation than unsecured loans (mean recovery rate of 45.5% with a standard deviation of 44.2%).
For the quota of collateral, defined as the ratio of collateral value to default

amount, capped to the interval $[0, 1]$, a strong positive correlation to the amount recovered can be detected. This is also consistent to other studies which report a positive impact of the quota of collateral on the amount recovered (see e.g. Carty *et al.* (1998), Gupton *et al.* (2000), Araten *et al.* (2004), Emery *et al.* (2004), or Grunert and Weber (2005)).
Furthermore, the types of collateral were divided into six different classes ("cash" (loans with collateral "cash" recover 77.7% on average), "accounts receivable" (73.3%), "fixed assets" (63.8%), "real estate" (77.1%), "commodities" (67.6%), and "others" (71.6%)), which are all significant at a 5%-level. The relatively high average recovery rate of facilities collateralised with real estate can be explained by the high quota of collateral those facilities exhibit on average (more than 90%) in comparison to the other collateral classes (around 60%). Regarding the type of collateral, Carty *et al.* (1998) propose the difference in quality of collateral as a driver for recovery rates. They observed a recovery rate of 89.8% for loans collateralised with accounts receivable/cash/inventory but only 73.6% for loans collateralised with stocks of subsidiaries. Keisman *et al.* (2004) state that instruments with higher quality collateral achieve on average higher recovery rates with a lower standard deviation.
In addition to that, facilities with more than one piece of collateral (75.2%) recover on average more than those with only one piece of collateral (73.3%), but the difference is neglible.
Nevertheless, there have to be other sources of recovery, as the average recovery rate is in general much higher than the average quota of collateral. In the investigated database, the average recovery rate is 60.7% while the average quota of collateral is only 35.6%. Grunert and Weber (2005) report an average recovery rate of 72.5% and an average quota of collateral of 30.6%. The rank of security represented by the five different categories "secured by first and non-shared lien on assets" (average recovery rate of 68.4%), "secured by first and pari-passu lien on assets" (74.9%), "secured by second lien on assets" (76.6%), "secured by other means" (51.8%), and "unsecured" (45.5%) are also statistically significant, whereas "secured by other means" and "unsecured" show clearly the lowest recovery rates with the highest standard deviations.

Seniority. The seniority classes ("senior", "pari-passu", "subordinated", "equity", and "not known") play only a minor role in determining the recovery in the considered data set as almost 85% of the data are either "senior" or "pari-passu"[7] and only 11 observations are "equity". By contrast, for bond

[7] "Senior" is assumed to mean everyone else is subordinated/junior to this obligation. If

recoveries the seniority or the place in the capital structure is one of the most important influence factors. Altman and Kishore (1996) for example found average recovery rates for senior secured debt of 57.9% (with a standard deviation of 23.0%), for senior unsecured debt of 47.6% (26.7%), for senior subordinated debt of 34.4% (25.1%), and for subordinated debt of 31.3% (25.1%). This result is confirmed by many other studies, e.g. Acharya *et al.* (2003) or Schuermann (2004). In a study on Swedish small business bankruptcies, Thorburn (2000) finds that senior claims recover 69% on average while junior claims receive only 2%.
In contrast to these absolute seniority dummies, the relative seniority, debt cushion, or debt subordinated percentage, defined as the percentage of debt that is subordinated to the obligation in question, seems to have a significant positive impact on the recovery rates in the investigated database. Other studies, e.g. van de Castle *et al.* (2000), Keisman *et al.* (2004), Emery *et al.* (2004), or Grippa *et al.* (2005), also find that the higher the debt cushion the higher the average recovery rate and the lower the standard deviation.

Guarantees. The presence and quality of guarantees was also tested as a possible explanatory variable. Facilities with a guarantee (average recovery of 63.0% with standard deviation 38.2%) have on average higher recovery rates than those facilities without (59.6% with standard deviation 40.9%). Similar results were found by Grippa *et al.* (2005) with different magnitudes depending on the quality of guarantees. Dermine and Neto de Carvalho (2006) report a negative but not significant effect of personal guarantees on the amount recovered.

4.2.5 Macroeconomic factors

Finally, the impact of the macroeconomic environment on the (facility-level) recovery rates was tested as well. Different kinds of macroeconomic explanatory variables like GDP, TED spread, the S&P 500 total return index, the Dow Jones Euro Stoxx 50, the growth rate in industrial production, the volatility index from CBOE as well as the VSTOXX and the VDAX, the 5-year treasury constant maturity rate, the 3-month Euribor, the Consumer and Producer Price Index, as well as S&P's annual global corporate default rate (see e.g. Vazza *et al.* (2006)) as a proxy for the credit environment have been examined. For the macroeconomic variables monthly, quarterly, as well

the obligation is senior but it isn't known if this is the only senior lender than "pari-passu" is used.

as yearly absolute values and changes at default and one year prior to default were tested. The impact of all of those factors on the recovery rates at the facility level is rather small. They become more important when aggregated recovery rates are considered (see Chapter 5).
The macroeconomic environment is a factor that has been often and controversially discussed in empirical research, mostly for aggregated recovery rates. Frye (2000b) states that recovery rates are about one-third lower in recessions, Roche *et al.* (1998) find a positive correlation between recoveries and stock prices measured by the Dow Jones Industrial Average, and Emery *et al.* (2004) note a positive effect of the growth rate in industrial production on recoveries. Covitz and Han (2004) claim that recovery rates increase as economic conditions improve from low levels but decrease as economic conditions become robust. Araten *et al.* (2004) only find a significant correlation between recovery rates and the economic cycle for unsecured credits. Altman *et al.* (2001), Grunert and Weber (2005), and Dermine and Neto de Carvalho (2006) can't find a significant relationship at all. Hu and Perraudin (2002) find a negative correlation (-19%) between the quarterly default rates and the quartely recovery rates and Frye (2003) rejects the hypothesis that recovery is independent of high default years. Keisman *et al.* (2004) propose the aggregate default rate to be one of the five main factors influencing recovery rates in their modeling framework and Altman *et al.* (2001) find a significant negative correlation between aggregated default rates and bond recoveries. In contrast to this, Carey and Gordy (2004) state that the correlation between simple default rates and recoveries is close to zero.

4.2.6 Overview

Table 4.5 gives a short summary of the empirical findings in literature on the impact of the factors described above on recovery rates and compares them to the results obtained from the database introduced in Section 4.1.[8] For some of the variables no information was found in literature. For the aggregate default rates and the economic cycle different indicators were tested at different times in the default process. Some of them had a positive influence on recovery rates, some of them a had negative impact, and some were not significant at all.

[8]Here, "+" means positive influence, "o" no statistical significant influence, "-" negative influence and "$\diamond$" statistical significant influence (with no statement about sign possible) on recovery rates. No statement about sign possible especially applies to factors with different categories where some of the possible outcomes have a positive impact and some a negative impact.

Influence factor	+	o	-	◇	This study
Seniority	×				o
Debt cushion or rel. seniority [9]	×				+
Presence of collateral	×				+
Liquidity of collateral	×				+
Quota of collateral	×				+
Presence of guarantee	×				+
Industry of borrower		×		×	o
Size of issue	×	×	×		+
Size of borrower	×	×	×		+
Number of loans per borrower			×		+
Facility type		×		×	◇
Default type		×		×	◇
Time between origination and default		×			+
Time between default and resolution		×	×		◇
Geographical dummies		×		×	◇
Rating / creditworthiness	×	×			+
Aggregate default rate		×	×		o/-
Macroeconomics	×	×			+/o/-
Rank of security					+
Facility asset class					◇
Guarantee type					o
Syndication indicator					◇
Operating company indicator					◇
Public-private indicator					◇
Utilization rate					◇

Table 4.5: Influence factors and their impact on recovery rates.

Levels of recoveries. In addition to these analyses, the reliability of influence factors on different levels of recovery rates was tested. Therefore, the database was divided into different classes, e.g. the 25% lowest observed recovery rates, the 25% highest observed recovery rates or the 50% highest observed recovery rates, and the procedures described above were repeated on the subsamples. It turned out that some factors which are good at predicting whether a defaulted facility leads to a high or a low recovery rate, are not very predictive when only facilities with low or high recovery rates are considered. For the facilities with low recovery rates (the bottom 25% of the data with recoveries ranging from 0% to 14.2%) a much higher degree of explanation can be found than for high recoveries. The most important influence factors for low levels of recovery are presence, quality, and quota of collateral. As the top 25% of recoveries only range from 97.6% to 100% the top 50% of the data with recoveries from 80.3% to 100% was investigated as well. In the sample containing these high recoveries collateralisation only plays a minor role. The most important factor is the default type. In addition to that, many of the industry dummies are significant. Nevertheless, the degree of explanation is rather small. Another interesting result obtained from the high recovery dataset is that in contrast to the previous findings now the size of the issue is negatively correlated with the recovery rate (at a rather low level).
Grunert and Weber (2005) also show that there are different influence factors for different levels of recoveries, but in their case study high recovery rates are mostly influenced by the exposure at default (high EAD leads to a high recovery rate) and low recovery rates by the risk premium and by the economic cycle.

4.3 Multivariate analysis

In addition to the univariate analyses from Section 4.2, various combinations of explanatory variables were tested for a predictive multivariate regression model. Again, the independent variable in this multivariate regression is the (facility-level) economic recovery rate introduced in Section 4.1. The set of explanatory variables used in this section contains all explanatory variables described in Section 4.2. Furthermore, the set of explanatory variables was divided into two subsets, one containing all variables which are available for all defaulted facilities and the other including all those facilities for which spread observations are available. Analogously to Section 4.2, the

[9] Here, "Debt Cushion or Rel. Seniority" is defined as the percentage of debt that is subordinated to the obligation in question.

transformed recovery rates are used for the multivariate linear model. Using Mallow's C_p-statistic[10] (see e.g. Draper and Smith (1998)) as variable selection criteria in a backward-forward selection procedure leads to the results in Tables 4.6 and 4.7[11]. In these (in terms of Mallow's C_p-statistic) "best" multivariate models, which have adjusted R^2 of 18.8% (with spread information) and 21.7% (without spread information), all the variables are significant on a 5%−level. In many other studies the multivariate R^2 is of similar height, e.g. 15%-21% in Grunert and Weber (2005) or 30.2% in Grippa *et al.* (2005). In the multivariate model with spread information much less factors are significant than in the model without spread information, while both models have a similar degree of explanation. Besides the spread, which has a negative impact on the recovery rates, the most important factors in the model with spread information refer to the degree and quality of collateralisation and the type of default. The higher the quantity of collateral and rank of security, the higher is the recovery rate. The default amount (EAD) is also poitively correlated with the recovery rate. Considering the type of default, "unlikely to pay" has the highest positive impact on (facility-level) recovery rates. In the model without spreads the facility asset class, the facility type, and the industry group as well as the size of the issue are significant besides the factors describing collateralisation and the type of default.

4.4 Recovery rate of collaterals

As seen in Subsection 4.2.4, the most important explanatory variable for the recovery rate of secured loans is the quality and quantity of collateral. Therefore, a closer look is taken at the recovery rates of collaterals and their relationship to overall recovery rates in this section. Hence, in contrast to Sections 4.2 and 4.3, where the independent variable under consideration was the workout recovery rate of the defaulted facilities, now the payment streams occurring from the liquidation of collateral positions for each defaulted facility are investigated. For this, all loans for which no collateral value information is available (neither book nor market value) in the database are removed, leaving a sample of 2973 different pieces of collateral distributed among 3202 facilities (one piece of collateral can be used as a security for more than one loan and one loan can be secured by more than one piece of collateral). Again, all facilities with default amount 0 are removed from the

[10] Mallow's C_p-statistic is defined by $C_p = \frac{RSS_p}{\widetilde{\sigma}^2} - n + 2p$, where RSS_p is the residual sum of squares for the model with p regressors, $\widetilde{\sigma}^2$ is the residual mean square after regression on the complete set of regressors, and n is the sample size.

[11] *** indicates significance at a 1%-level, ** at a 5%-level, and * at a 10%-level.

	Coefficient	(t-statistic)	
Intercept	-1.630	(-18.365)	***
Quota of collateral	0.267	(4.102)	***
Collateral acc. receivable	-0.226	(-3.119)	***
Collateral real estate	0.174	(3.025)	***
Guarantee indicator	0.137	(3.553)	***
Syndicated indicator	0.335	(4.402)	***
Sec. by first lien (non shared)	0.583	(13.081)	***
Sec. by first lien (pari-passu)	0.437	(5.512)	***
Sec. by other means	0.381	(7.773)	***
Default - 90 days past due	0.263	(3.471)	***
Default - Unlikely to pay	0.606	(14.753)	***
Default - Bankruptcy	0.096	(3.125)	***
Default - Distr. restructuring	0.368	(4.721)	***
Log default amount	0.061	(8.622)	***
Spread in bps	-1e-04	(-8.059)	***

Table 4.6: Multivariate regression model for transformed recovery rates with spread information.

sample (another 19 observations).

In the following, two major questions concerning the liquidation of collateral after default are addressed. First, how much recovery is gained from collateral positions. Second, how much of the collateral value is liquidated during the workout process. The first question is answered by looking at the recovery rate gained from collateral, i.e. the ratio of the present value of all net cash flows either from liquidation of collateral or realized book value of collateral to the default amount ("recovery from collateral"), and the ratio of recovery rate from collateral to overall recovery rate ("ratio of recovery from collateral to overall recovery"). The ratio of collateral that is liquidated or collateral with book value that is realized to the collateral value prior to default ("proportion of liquidated collateral") is used as a measure for the proportion of provided collateral used during the workout process. To avoid the impact of abnormally high or low ratios they have been capped to the interval $[-0.5, 1.5]$. The ratio of recovery rate from collateral to overall recovery rate is only computed when both quantities have the same sign. Facilities where the recovery rate from collateral is negative due to liquidation expenses and the overall recovery rate is positive or vice versa (103 observations) are excluded from the analysis as such negative ratios are hard to interpret. Table 4.8 gives the summary statistics of the three ratios and compares them

	Coefficient	(t-statistic)	
Intercept	-1.854	(-14.578)	***
Collateral indicator	-0.062	(-2.067)	**
Collateral acc. receivable	-0.074	(-2.336)	**
Collateral fixed assets	-0.243	(-6.934)	***
Collateral real estate	0.172	(6.779)	***
Collateral commodities	-0.742	(-3.818)	***
Facil. class - SME	0.624	(5.346)	***
Facil. class - Large corp.	1.058	(8.919)	***
Facil. class - Banks	0.736	(5.302)	***
Facil. class - Spec. lending	0.694	(5.592)	***
Guarantee indicator	0.253	(15.574)	***
Syndicated indicator	0.139	(2.594)	***
Private firm	0.327	(18.081)	***
Facil. type - Demand loan	-0.631	(-20.799)	***
Facil. type - Uncommitted line	0.532	(2.202)	**
Facil. type - Lease	-0.551	(-3.885)	***
Facil. type - Other deriv.	0.521	(17.610)	***
Senior	0.092	(2.603)	***
Pari-passu	0.426	(12.035)	***
Sec. by first/non-shared lien	0.466	(17.364)	***
Sec. by first/pari-passu lien	0.214	(5.267)	***
Sec. by second lien	0.750	(11.837)	***
Default - 90 days past due	0.253	(10.428)	***
Default - Unlikely to pay	0.341	(12.176)	***
Default - Bankruptcy	0.149	(6.737)	***
Default - Charge-off	0.102	(4.158)	***
Default - Non accrual	0.225	(6.078)	***
Ind. - Aggriculture	0.156	(2.793)	***
Ind. - Public adm. & Defence	0.344	(2.378)	**
Log borrower default amount	0.018	(6.059)	***
Quota of collateral	0.187	(4.982)	***

Table 4.7: Multivariate regression model for transformed recovery rates without spread information.

to the overall recovery rates of the facilities in this sample ("overall recovery rate").

	Recovery from collateral	Ratio of recovery from collateral to overall recovery	Proportion of liquidated collateral	Overall recovery rate
Mean	0.651	0.861	0.619	0.756
St.dev.	0.394	0.349	0.186	0.333
Median	0.698	0.997	0.579	0.904
Number	3183	3080	3183	3183

Table 4.8: Summary statistics for collateral recovery.

The distribution of recovery rates gained from collateral is U-shaped with a high amount of recoveries higher than 90% (more than 35% of the observations) or lower than 5% (8% of the observations). The remainder is almost uniformly distributed in the interval $[5\%, 90\%]$. Not surpisingly the recovery rates gained from collateral are highly correlated with the overall recovery rates and the quota of collateral, i.e. the ratio of collateral value to default amount. The ratio of recovery rates from collateral to overall recovery rates is in most cases rather high, with a ratio of more than 90% in 60% of the observations and more than 100% in 42% of the observations. The remainder is almost evenly distributed. It has a positive correlation with the number of collaterals as well as with the quota of collateral and decreases on average when the number of loans that are secured with this piece of collateral increases. All in all one can say that for secured facilities the recovery payments from collateral liquidation or realisation of book value constitute the majority of all recovery payments for the secured facilities in this sample and depend on the number and quota of collateral.

The proportion of liquidated collateral is in most cases either very small (smaller than 5% in 14% of the observations) or very high (greater than 95% in 27% of the observations). This ratio has a negative correlation with the value and quota of the collateral as well as with the number of loans that are secured with this collateral. A possible explanation for this might be the fact that in the case of high collateral values and overcollateralisation most of the collateral is not needed to pay the creditors. In the other case, when there are many defaulted facilities that share one piece of collateral, the share of each facility is rather small. To sum it up, in most cases the collateral is either liquidated (almost) completely or (nearly) not at all.

Chapter 5

Explaining aggregated recovery rates

The determinants of aggregated recovery rates have only been examined by very few studies. Altman *et al.* (2001) for example, who investigate annual weighted average recovery rates of defaulted US corporate bonds, only find secondary effects of macroeconomic variables on recovery rates. Covitz and Han (2004) find a weak positive correlation between GDP and annual aggregated recovery rates of US bonds. Nevertheless, there has also been a growing amount of empirical research articles showing that aggregated recovery rates vary over time and are lower in a distressed economy than in a healthy economy (see e.g. Frye (2000b) or Schuermann (2004)). Emery *et al.* (2004) find a positive effect of the industrial production growth on recovery rates and Roche *et al.* (1998) note that there is a positive correlation between recovery rates of syndicated bank loans and stock prices.

In this chapter, the above mentioned relationships, that have been proposed in different studies, are investigated on a unique dataset. Such an examination is accomplished, to the author's best knowledge, for the first time on such a broad European dataset. Furthermore, different Markov-switching concepts are applied to the analysis of aggregated recovery rates. An additional factor that tries to explain the credit environment is incorporated into the analysis as well. Finally, the empirical results give an indication of how the knowledge gained from the analyses can be used in the modelling of recovery rates, e.g. for pricing purposes (see Chapter 6).

Recent studies with similar scope but different focus on other aspects of credit-risk modelling are e.g. Bystroem (2008), Alexander and Kaeck (2008), and Hofert *et al.* (2008). The former two investigate the relation between CDS spreads on the one hand and stock prices, stock return volatilities, and interest rate movements on the other hand. The latter describes compound and base correlations of CDO tranches in terms of linear regressions. The concept of Markov-switching models for the description of recovery rates is e.g. used in Bruche and González-Aguado (2008) and Chourdakis (2008).

5.1 The data

The data set used in this chapter consists of monthly average recovery rates derived from the database described in Section 4.1 and various macroeconomic indicators acting as explanatory variables. Again, outliers with recovery rates smaller than 0 or greater than 1 were removed from the database just as those facilities with default amount equal to 0. A time horizon between January 1998 and January 2007 is considered in what follows. Monthly average recovery rates were computed for the defaulted facilities within this time horizon. Here, the results achieved with the unweighted arithmetic mean are presented. A weighted average with the default amount as weighting factor, which was used e.g. in Duellmann and Trapp (2004), Varma (2005), Dermine and Neto de Carvalho (2006), and Bruche and González-Aguado (2008) was also tested but these results were not very promising. Figure 5.1 shows the monthly recovery rates in the considered time horizon.

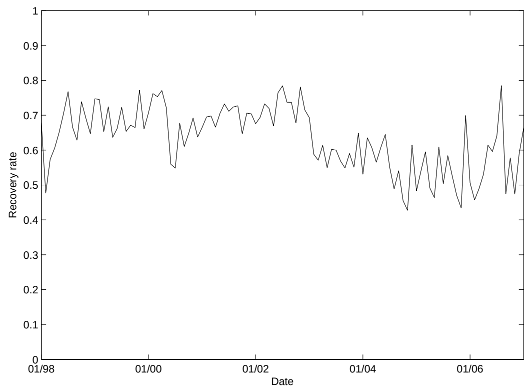

Figure 5.1: Monthly average recovery rates.

In the following, six different explanatory variables will be regarded. The first two factors describe the macroeconomic environment, the third and fourth describe interest rate movements, and the last two serve as proxies for the stock market behaviour. The growth rate in industrial production (GIP) includes the industrial production of 13 Euro countries.[12] GGDP

[12]At the end of 2007 the 13 Euro countries were Belgium, Germany, Ireland, Spain,

denotes the growth rate of the gross domestic product (GDP). The GDP which is calculated from the GDP of 13 Euro countries is the seasonally adjusted GDP (i.e. after removing seasonal effects) at market price.[13] As the GDP is not observable on a monthly basis, monthly data was generated from quarterly data by interpolation. In addition to that, the GDP with a delay of 3 months due to publication issues was tested but as the results were very similar to the case without delay, the following results will be restricted to GDP without delay. The Euro Interbank Offered Rate (EURIBOR) serves as a proxy for the risk-free interest rates in the Euro zone. Furthermore, the 5-year Euro area Government Benchmark Bond yield (GY) calculated as the weighted mean of government bond yields with maturities between 4.5 and 5.5 years is used. GIP, GDP, EURIBOR, and GY are available at the website of the European central bank (`http://sdw.ecb.europa.eu/`). As a proxy for equity markets the return of the Dow Jones Euro STOXX 50 (DJES) is employed. Finally, the VSTOXX volatility index based on the Dow Jones Euro STOXX 50 serves as a proxy for equity volatility. It is calculated from the implicit volatilities of Dow Jones Euro STOXX 50 options. As the VSTOXX data only date back to 1999, it is replaced by the VDAX-NEW index (VDAX) whenever time series that date before 1999 are used. The equity index DJES as well as VSTOXX and VDAX were downloaded from Reuters. The evolution of the explanatory variables in the considered time period is shown in Figure 5.2.

In the following, an exponential relationship between the recovery rates and the macroeconomic factors is assumed. This assumption prevents recovery rates from becoming negative. Although recovery rates are not bounded below 1 in this case (recovery rates greater than 1 do also appear in some cases in reality, see e.g. p.13 of Schuermann (2004)), this is a common approach used e.g. in Altman *et al.* (2001). Therefore, the recovery rates are transformed with the ln-function. The connection between the logarithmised recovery rates and the macroeconomic factors is assumed to be linear. Other possible transformations include e.g. logistic transformation (see, e.g., Schönbucher (2003) or Duellmann and Trapp (2004)) or beta transformations (see e.g. Gupton and Stein (2005)), which lead to distributions bounded on $[0, 1]$ at the cost of loosing analytical tractability in the modelling framework.

France, Italy, Luxembourg, the Netherlands, Austria, Portugal, Finland, Greece, and Slovenia.

[13]For further information on seasonally adjustment see e.g. `http://www.ecb.int/stats/money/aggregates/season/html/index.en.html`

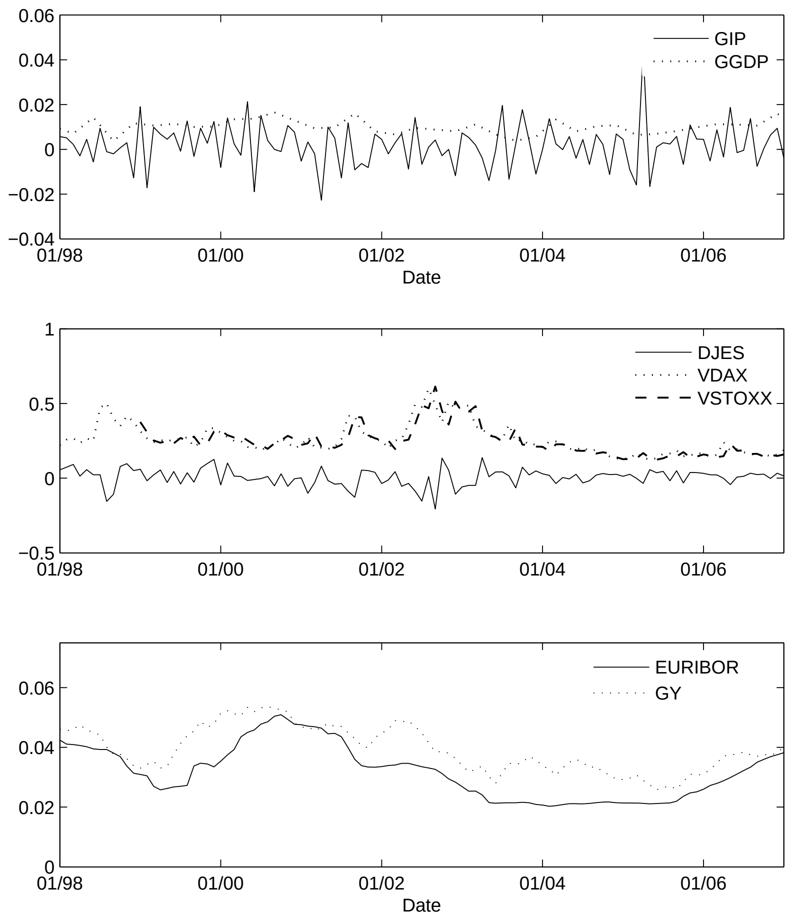

Figure 5.2: Explanatory variables.

5.2 Linear regression analysis

In this section, the linear regression model

$$\begin{aligned} \ln(z(t)) &= \beta_0 + \beta_1 GIP(t) + \beta_2 GGDP(t) + \beta_3 DJES(t) + \beta_4 VDAX(t) \\ &\quad + \beta_5 EURIBOR(t) + \beta_6 GY(t) + \epsilon(t), \end{aligned} \tag{5.1}$$

with $z(t)$ denoting the recovery rate and $\epsilon(t) \sim \mathcal{N}(0, \sigma^2)$ *i.i.d.* is estimated. The coefficients and t-statistics are given in Table 5.1.[14]

Variable	**Coefficient (t-statistic)**
Intercept	-1.004 *** (-14.641)
GIP	0.942 (0.848)
GGDP	0.941 (0.199)
DJES	0.111 (0.580)
VDAX	0.704 *** (6.168)
EURIBOR	-0.075 (-0.030)
GY	8.718 *** (3.075)
R_a^2 in %	46.23

Table 5.1: Coefficients (t-statistics) and significance codes of the linear regression (5.1) for response $\ln(z(t))$.

The coefficient of determination (R^2) of this model is 49.22%, the adjusted R-squared (R_a^2) equals 46.23% which is in line with other studies on recovery rates. Partial t-tests show that the variables VDAX and GY are significant on a 1%-level. All other variables are not significant on a 10%-level and hence have only a small impact on the response variable.
Note that the sign and the significance code of the coefficients in such a multivariate model like in (5.1) may differ from the sign and significance code of the coefficient in a univariate model (see also Table A.1 in Appendix A). Especially the results for EURIBOR and GY (correlation coefficient: 86.38%)

[14]*** indicates significance at a 1%-level, ** at a 5%-level, and * at a 10%-level.

as well as for EURIBOR and GGDP (correlation coefficient: 51.39%) might be misleading as they are highly correlated. E.g. in a univariate model both EURIBOR and GY have a significant positive impact on the logarithmised recovery rates. In addition to that in a univariate setting the variable GGDP is positively correlated with the logarithmised recovery rates. To avoid the impact of a possible multi-collinearity a variable selection procedure for each multivariate model is conducted in the following.
Using Mallow's C_p-statistic as variable selection criterion in a backward-forward selection procedure leads to the results presented in Table 5.2. In this (in terms of Mallow's C_p-statistic) "best" model with an adjusted R^2 of 47.70% the variables VDAX and GY are significant on a $1\%-$level. The higher the variables VDAX and GY are, the higher are the recovery rates.

Variable	**Coefficient (t-statistic)**
Intercept	-0.991 *** (-17.209)
VDAX	0.676 *** (6.417)
GY	8.806 *** (6.242)
R_a^2 in %	47.70

Table 5.2: Coefficients (t-statistics) and significance codes in "best" model of the linear regression (5.1) for response $\ln(z(t))$.

As can be seen from Figure 5.1 there seems to be a break in the data at some time in the year 2002. Such a break might lead to changes in the behaviour of the regression coefficients and a lower degree of explanation. Figure 5.3 contains the evolution of the coefficients of the (multivariate) linear regression (5.1) rolled over the last three years. The dashed lines correspond to the coefficients from Table 5.1.
To verify this observation and test the stability of the regression parameters a Chow breakpoint test (see Chow (1960)) is applied to the model from Table 5.2. As a potential breakpoint all observation dates except for the first and last four observation dates are used. The hypothesis that the coefficients of the regression models before and after the breakpoint are equal is rejected on a 5%-level for all possible breakpoints between January 2000 and April 2002 with the test statistic reaching its maximum in April 2002. According to these results, two different ways in the examination of the determinants of aggregated recovery rates will be conducted in what follows: a Markov-

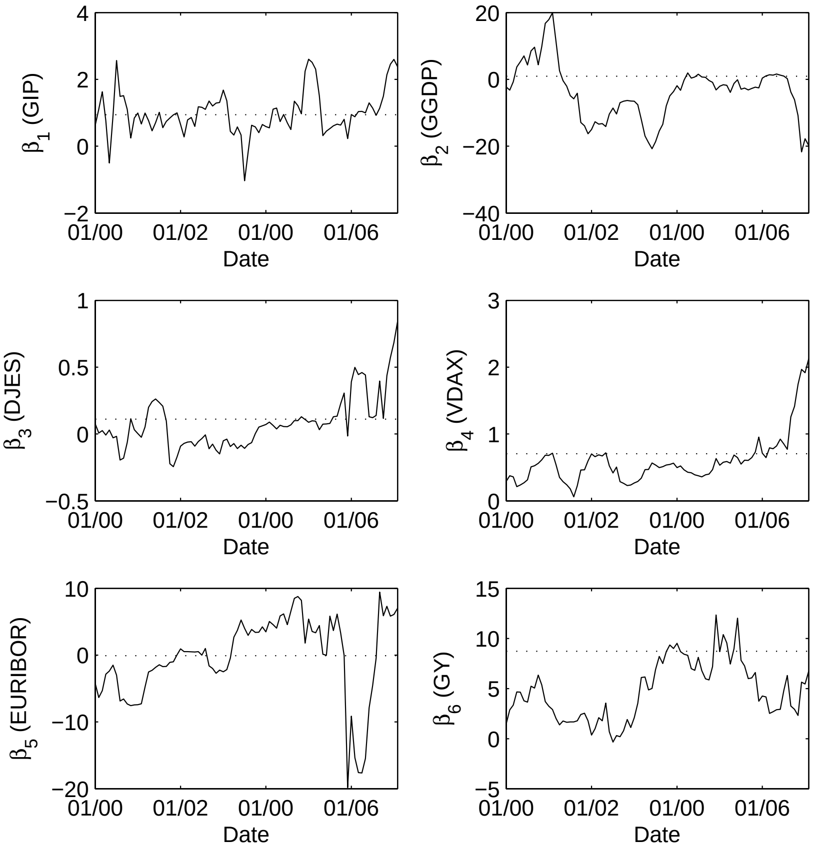

Figure 5.3: Evolution of regression parameters

switching model which allows the regression coefficients to switch according to different regimes and a more detailed regression analysis on a sub-sample of the data that contains no structural breaks.

5.3 Markov-switching analysis

In this section, the procedures described in Section 2.1 will be applied to the aggregated (logarithmised) recovery rates, i.e. the observable process is given by $Y(t) = \ln(z(t))$. Furthermore, it is assumed that p is the density of the normal distribution, i.e.

$$p(x, y; \Phi) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left\{ -\frac{(y-\mu_x)^2}{2\sigma_x^2} \right\},$$

where μ_x, σ_x denote the state-dependent distribution parameters with $x \in \{1, \dots, J\}$.
Before the Markov-switching analysis is conducted, the number of states J has to be determined. For this, the parameters of the HMM for $J \in \{2, 3, 4\}$ ($J > 4$ is not realistic as the number of parameters becomes to high) are estimated and the decision which model fits best is made according to Akaike's information criterion (see Akaike (1974)). I.e. we calculate

$$AIC = 2n(J) - 2l(J, \Theta^*),$$

where $n(J)$ denotes the number of parameters, Θ^* the set of estimated parameters, and $l(J, \Theta^*)$ the maximal log-likelihood of the model with J states. The lowest AIC-value is obtained for a model with $J = 2$ states. Hence, the following analyses concentrate on a HMM with two different states, $S1$ and $S2$, applied to the logarithmised recovery rates. The parameter estimates obtained from the Baum-Welch algorithm (see Algorithm 2.1 in Section 2.1) can be found in Table 5.3.

	State $S1$	**State $S2$**
δ_j	2.1E-06	0.999
π_{j1}	0.960	0.040
π_{j2}	0.041	0.959
μ_j	-0.355	-0.579
σ_j	0.061	0.128

Table 5.3: Parameter estimates of HMM with two states.

The parameters for μ_j and σ_j in Table 5.3 correspond to an expected value for the recovery rates in state S1 of 0.702 with standard deviation 0.043 and an expected value of 0.565 with standard deviation 0.073 in state S2. In the first state the expectation is higher and the standard deviation is lower than in the second state. Applying Viterbi's algorithm (see Algorithm 2.2 in Section 2.1) leads to the state sequence shown in Figure 5.4.

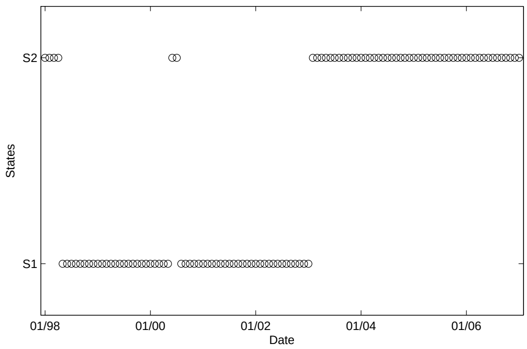

Figure 5.4: States of the Markov chain.

It can be seen that the Markov Chain stays mostly in state $S1$ at the beginning of the considered time period. After about 60 months (end of 2002) the Markov Chain changes to state $S2$ and remains in this state. This change is in line with the results from the breakpoint tests in Section 5.2.
Since the Markov-switching analysis has shown that the distribution of the recovery rates depends on the state of the Markov chain, the impact of each explanatory variable in Equation (5.1) is also state-dependent. Therefore, the impact of the different states on the regression coefficients is analysed in what follows, i.e. the estimated sequence of states is introduced as a further explanatory variable. The new linear model is then given by

$$\begin{aligned} \ln(z(t)) &= \beta_0 + \beta_1 GIP(t) + \beta_2 GGDP(t) + \beta_3 DJES(t) \\ &\quad + \beta_4 VDAX(t) + \beta_5 EURIBOR(t) + \beta_6 GY(t) + \sigma_1 \epsilon(t) \\ &\quad + S(t)\left[\beta_7 + \beta_8 GIP(t) + \beta_9 GGDP(t)\right. \\ &\quad + \beta_{10} DJES(t) + \beta_{11} VDAX(t) \\ &\quad \left. + \beta_{12} EURIBOR(t) + \beta_{13} GY(t) + \sigma_2 \epsilon(t)\right], \end{aligned} \tag{5.2}$$

with $\epsilon(t) \sim \mathcal{N}(0,1)$ *i.i.d.*, and $\sigma_1, \sigma_2 > 0$. Let $\widehat{\beta}_0, \ldots, \widehat{\beta}_{13}, \widehat{\sigma}_1, \widehat{\sigma}_2$ be the least-squares estimates of $\beta_0, \ldots, \beta_{13}, \sigma_1, \sigma_2$. Assuming $S(t) \in \{0,1\}$, Equation (5.2) can be reformulated in two linear models, one for each state. Let $\ln(z^{S1}(t))$ and $\ln(z^{S2}(t))$ denote the logarithmised recovery rates in state $S1$

and $S2$ respectively. This leads to

$$\begin{aligned}
\ln(z^{S1}(t)) &= \widehat{\beta}_0^{S1} + \widehat{\beta}_1^{S1} GIP(t) + \widehat{\beta}_2^{S1} GGDP(t) + \widehat{\beta}_3^{S1} DJES(t) \\
&\quad + \widehat{\beta}_4^{S1} VDAX(t) + \widehat{\beta}_5^{S1} EURIBOR(t) + \widehat{\beta}_6^{S1} GY(t) \\
&\quad + \widehat{\sigma}^{S1}\epsilon(t), \qquad (5.3) \\
\ln(z^{S2}(t)) &= \widehat{\beta}_0^{S2} + \widehat{\beta}_1^{S2} GIP(t) + \widehat{\beta}_2^{S2} GGDP(t) + \widehat{\beta}_3^{S2} DJES(t) \\
&\quad + \widehat{\beta}_4^{S2} VDAX(t) + \widehat{\beta}_5^{S2} EURIBOR(t) + \widehat{\beta}_6^{S2} GY(t) \\
&\quad + \widehat{\sigma}^{S2}\epsilon(t), \qquad (5.4)
\end{aligned}$$

where $\epsilon(t) \sim \mathcal{N}(0,1)$ *i.i.d.*, $\widehat{\beta}_0^{S1} = \widehat{\beta}_0$, $\widehat{\beta}_0^{S2} = \widehat{\beta}_0 + \widehat{\beta}_7$, $\widehat{\beta}_i^{S1} = \widehat{\beta}_i$, $\widehat{\beta}_i^{S2} = \widehat{\beta}_i + \widehat{\beta}_{i+7}$ for $i = 1, \ldots, 6$, $\widehat{\sigma}^{S1} = \widehat{\sigma}_1$, and $\widehat{\sigma}^{S2} = \widehat{\sigma}_1 + \widehat{\sigma}_2$.
The coefficients of the models are given in Table 5.4 (the corresponding univariate results can be found in Appendix A).

Variable	**Coefficient (t-stat) in State** **S1**		**S2**	
Intercept	-0.508 (-5.927)	***	-0.867 (-6.507)	***
GIP	0.408 (0.432)		1.608 (0.973)	
GGDP	-0.325 (-0.093)		3.458 (0.411)	
DJES	0.015 (0.121)		0.369 (0.729)	
VDAX	0.238 (2.530)	**	0.4840 (1.765)	*
EURIBOR	-2.176 (-1.244)		-3.213 (-0.592)	
GY	3.627 (1.790)	*	6.585 (1.057)	
R_a^2 in %	58.05			

Table 5.4: Coefficients (t-statistics) and significance codes of Equations (5.3) - (5.4) with responses $\ln(z^S(t))$.

As one can easily see, the differences between $\widehat{\beta}_i^{S1}$ and $\widehat{\beta}_i^{S2}$ are quite high. While in state $S1$ GY, VDAX, and EURIBOR are the most significant explanatory variables, the variables VDAX, GIP, and GY have the greatest impact in state $S2$. Using an F-test, the hypothesis that the models from Tables 5.1 and 5.4 are equal can be rejected on a 1%-level. In comparison to

the regression from Equation (5.1) ($R_a^2 = 0.4623$), the model from Equation (5.2) ($R_a^2 = 0.5805$) has a much higher degree of explanation.
Again, Mallow's C_p-statistic is applied in a backward-forward selection procedure to get the "best" model given in Table 5.5 consisting of the variables VDAX, EURIBOR, and GY. The higher the variables VDAX and GY are, the higher are the recovery rates in each state. For the EURIBOR the reverse is true.

Variable	**Coefficient (t-stat) in State** **S1**		**S2**	
Intercept	-0.510	***	-0.826	***
	(-6.537)		(-7.377)	
VDAX	0.234	***	0.448	*
	(2.773)		(1.687)	
EURIBOR	-2.392		-1.712	
	(-1.496)		(-0.365)	
GY	3.811	*	5.653	
	(1.995)		(0.349)	
R_a^2 in %	59.56			

Table 5.5: Coefficients (t-statistics) and significance codes in "best" model of Equations (5.3) - (5.4) with responses $\ln(z^S(t))$.

Until now, it can be stated that recovery rates can change between two states according to a Markov-switching model with state $S2$ having a lower mean and higher standard deviation. Furthermore, the sensitivity of the recovery rates to the explanatory variables varies between the states in this case. Another open question is what drives changes in the regimes. This question is addressed by using a logistic regression model given by

$$\mathbb{P}(S(t) = S2) = \frac{1}{1 + e^{-\beta^T x(t)}} \tag{5.5}$$

with

$$x(t) = (1, GIP(t), GGDP(t), DJES(t), VDAX(t), EURIBOR(t), GY(t)),$$

$\beta = (\beta_0, \ldots, \beta_6)$, and $S(t)$ denoting the state at time t. In this multivariate logistic regression model only the two variables VDAX and GY are significant. The coefficients and z-statistics of this reduced logistic regression model, which corresponds to the (according to Mallow's C_p-statistic) optimal model, are given in Table 5.6. As one can easily see, the higher the variables

VDAX and GY, the lower is the probability of being in state S2.

Variable	**Coefficient (z-statistic)**
Intercept	14.249 *** (5.537)
VDAX	-18.661 *** (-3.952)
GY	-239.036 *** (-4.929)
R_a^2 in %	52.40

Table 5.6: Coefficients (z-statistics) and significance codes of the logistic regression from Equation (5.5).

For logistic regression models it is often more convenient to measure the quality of the model not in terms of R_a^2 but in terms of other performance measures like accuracy ratio or percentage of right predictions (see e.g. Engelmann *et al.* (2003) or Höcht and Zagst (2007) for applications of logistic regression models and different performance measures in the field of credit risk modelling). Both the accuracy ratio (85.49%) and the percentage of right predictions (87.14%) indicate a very good degree of explanation. Figure 5.5 contains the estimated probabilities from Equation (5.5) with explanatory variables VDAX and GY and the inferred sequence of states from Viterbi's algorithm.

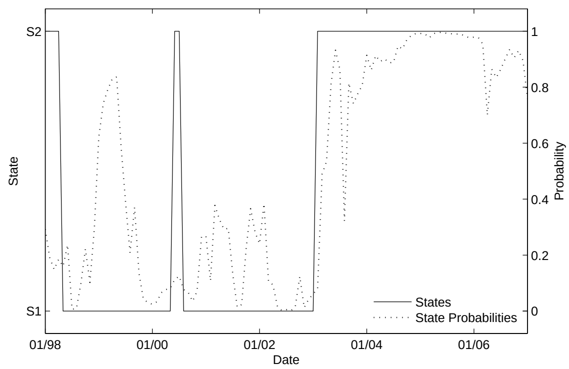

Figure 5.5: State probabilities from Equation (5.5) and states of the Markov chain from Viterbi's algorithm.

The estimated coefficients, significance codes, and performance measures are almost the same when replacing $x(t)$ by $x(t-1)$ in Equation (5.5), i.e. this model is not only good in explaining regime changes, it provides also a suitable state prediction for the next time step.

5.4 More detailed linear regression analysis

According to the results from Sections 5.2 and 5.3, it can be concluded that there is no structural change in the recovery rates data before January 2000 and especially after mid 2002. Therefore, different states and different behaviour of explanatory variables can be neglected in these time periods. As the time period from January 1998 until January 2000 is rather short, the following analyses will mainly concentrate on the recovery rates starting in mid 2002. By taking a closer look at the data sample before January 2000, it can be found that the GGDP has a significant positive impact on the recovery rates while the EURIBOR a significant negative impact. The adjusted coefficient of determination is only about 20% in this case.
The recovery rates starting in mid 2002 are analysed in what follows. In addition to the analysis subject to Equation (5.1), the recovery rates are divided according to the securisation of the underlying credit instruments, i.e. recovery rates from secured and unsecured facilities are differentiated. A further classification of secured facilities, e.g. according to rank of securities, is not done because the resulting data samples in the different subclasses would be too small. The corresponding linear regression is given by

$$\begin{aligned} \ln(z_i(t)) &= \beta_0 + \beta_1 GIP(t) + \beta_2 GGDP(t) + \beta_3 DJES(t) \\ &\quad + \beta_4 VSTOXX(t) + \beta_5 EURIBOR(t) + \beta_6 GY(t) + \epsilon(t) \qquad (5.6) \end{aligned}$$

with $i \in \{T, U, S\}$ and $\epsilon(t) \sim \mathcal{N}(0, \sigma^2)$ *i.i.d.* Here z_T denotes the total recovery rates, z_U the recovery rates of unsecured facilities, and z_S the recovery rates of secured facilities.
In Table 5.7, the estimated coefficients and t-statistics for the three response variables are given (the corresponding univariate results can be found in Appendix A).
For the total recovery rates the coefficient of determination R^2 is 48.35%, the adjusted coefficient of determination R_a^2 is 42.27%. The null hypothesis, that all coefficients are zero, can be rejected at a 1%-level. The p-values of the partial t-tests show that the explanatory variables VSTOXX and GY are significant.
In the case of the recovery rates of unsecured facilities R^2 is 64.21% and R_a^2 is 60.00%. The null hypothesis, that all coefficients are zero, can be rejected at a 1%-level. The p-values of the partial t-tests show that the explanatory

Variable	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**
Response	$\ln(z_T(t))$	$\ln(z_S(t))$	$\ln(z_U(t))$
Intercept	-1.091 *** (-8.482)	-0.429 *** (-2.716)	-2.104 *** (-9.367)
GIP	1.486 (0.942)	1.9381 (1.211)	1.368 (0.496)
GGDP	-5.837 (-0.682)	3.402 (0.324)	-32.218 ** (-2.155)
DJES	0.256 (0.760)	-0.275 (-0.666)	1.098 * (1.866)
VSTOXX	0.578 *** (3.410)	-0.153 (-0.735)	1.024 *** (3.458)
EURIBOR	1.377 (0.260)	-1.358 (-0.209)	32.463 *** (3.506)
GY	12.464 ** (2.429)	1.771 (0.282)	13.754 (1.535)
R_a^2 in %	42.27	-6.00	60.00

Table 5.7: Coefficients (t-statistics) and significance codes of Equation (5.6) with responses $\ln(z_i(t))$.

variables VSTOXX and EURIBOR are significant at a 1%-level, the variable GGDP at a 5%-level, and the variable DJES at a 10%-level.
For the response $\ln(z_S)$, i.e. for the recovery rates of secured facilities, R^2 is only 5.15% and R_a^2 is even negative. This means that the variance of recovery rates steming from secured facilities can't be explained by the macroeconomic factors. The null hypothesis, that all coefficients are zero, can not be rejected at a 5%-level and partial t-tests show that none of the variables is significant at a 10%-level.
In the next step, the "best" model according to Mallow's C_p-statistic for the three response variables is determined. The achieved coefficients and t-statistics are given in Table 5.8.

Variable	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**
Response	$\ln(z_T(t))$	$\ln(z_S(t))$	$\ln(z_U(t))$
Intercept	-1.095 *** (-10.470)	-0.405 *** (-20.432)	-2.098 *** (-9.422)
GGDP	-	-	-32.047 ** (-2.160)
DJES	-	-	1.066 * (1.836)
VSTOXX	0.541 *** (3.788)	-	1.011 *** (3.452)
EURIBOR	-	-	32.561 *** (3.543)
GY	12.438 *** (3.820)	-	13.632 (1.533)
R_a^2 in %	44.50	0.00	60.58

Table 5.8: Coefficients (t-statistics) and significance codes of "best" model from Equation (5.6) with responses $\ln(z_i(t))$.

The results show that the variables GY and VSTOXX form the "best" model for the total recovery rates as in Section 5.2. Both variables have a positive impact on the recovery rates. The adjusted coefficient of determination R_a^2 is 44.50% and hence about 2%-points higher in comparison to the full model from Table 5.7 and 3%-points lower than in the model with the data from Section 5.2. The p-values of the partial t-tests show that both variables, GY and VSTOXX, are significant at a 1%-level.
The "best" model for recovery rates of unsecured facilities includes GGDP, VSTOXX, EURIBOR, DJES, and GY as explanatory variables. With these

variables an adjusted R-squared R_a^2 of 60.58% is achieved. The p-values of the partial t-test show that four of the five variables are significant. The variables EURIBOR and VSTOXX are significant at a 1%-level, the variable GGDP is significant at a 5%-level, and the variable DJES is significant at a 10%-level.

For the recovery rates of secured facilities none of the explanatory variables is chosen in the "best" model.

To sum it up, the preceding regression analysis shows that securisation effects the relation between recovery rates and macroeconomic factors. The best adjustment can be achieved for recovery rates of unsecured facilities. 60% of the total variance of recovery rates of unsecured facilities can be explained by macroeconomic factors. In contrast, less than 50% of the total variance of total recovery rates can be explained by macroeconomic factors. As expected, the worst adjustment is obtained for recovery rates of secured facilities. The recovery rates of secured facilities depend more on the rank and type of the security, especially the value of the underlying collateral, than on macroeconomic indicators. Hence, the following analyses will concentrate on total recovery rates and recovery rates from unsecured facilities.

Regarding the explanatory variables that form the "best" model for the different response variables, it can be seen that the variables VSTOXX and GY belong to the best model for the response $\ln(z_T)$ and $\ln(z_U)$ respectively. The variables GGDP, DJES, and EURIBOR are part of the "best" model for the response $\ln(z_U)$. This means that these variables have a lot of influence on the recovery rates of unsecured facilities but only little influence on total recovery rates and on recovery rates of secured facilities.

In the next step, the preceding analysis shall be further enlarged by including a measure for the credit environment. Therefore, a measure for the uncertainty in the prices of defaultable securities is used. This is accomplished by using the so-called uncertainty index u from the extended Schmid/Zagst model as proposed in Antes *et al.* (2008) (a short review on this model is given in Appendix B). The higher the uncertainty of the obligor, the higher is the factor u. The unobservable process u used in the following analysis, has been filtered from corporate-composite yields Euro area for rating class A.

A linear regression with both the uncertainty index and the macroeconomic factors as explanatory variables is conducted in what follows. The regression is now given by

$$\begin{aligned}\ln(z_i(t)) &= \beta_0 + \beta_1 GIP(t) + \beta_2 GDP(t) + \beta_3 DJES(t) \\ &\quad + \beta_4 VSTOXX(t) + \beta_5 EURIBOR(t) + \beta_6 GY(t) \\ &\quad + \beta_7 u(t) + \epsilon(t), \end{aligned} \tag{5.7}$$

where $i \in \{T, U\}$ and $\epsilon(t) \sim \mathcal{N}(0, \sigma^2)$ *i.i.d.* Again, z_T denotes total recovery rates, z_U recovery rates of unsecured facilities, and $u(t)$ the uncertainty index. In Table 5.9 the estimated coefficients and t-statistics are given for the two response variables $\ln(z_T(t))$ and $\ln(z_U(t))$.

Variable	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**
Response	$\ln(z_T(t))$	$\ln(z_U(t))$
Intercept	-1.021 *** (-6.142)	-1.959 *** (-6.763)
GIP	1.514 (0.954)	1.427 (0.516)
GGDP	-6.398 (-0.740)	-33.376 ** (-2.214)
DJES	0.085 (0.200)	0.744 (1.007)
VSTOXX	0.242 (0.457)	0.330 (0.358)
EURIBOR	-0.016 (-0.003)	29.588 *** (2.967)
GY	11.911 ** (2.280)	12.615 (1.385)
u	15.610 (0.671)	32.204 (0.794)
R_a^2 in %	41.64	59.71

Table 5.9: Coefficients (t-statistics) and significance codes of Equation (5.7) with responses $\ln(z_i(t))$.

In the case of total recovery rates, the adjusted coefficient of determination R_a^2 is 41.64%. The null hypothesis that all coefficients are zero can be rejected at a 1%-level. The p-values of the partial t-tests show that only the explanatory variable GY is significant at a 5%-level.
Regarding recovery rates of unsecured facilities an adjusted R-squared R_a^2 of 59.71% can be achieved. About 60% of the total variation of the recovery

rates of unsecured facilities can be explained by the macroeconomic factors and the variable u. The null hypothesis that all coefficients are zero can be rejected at a 1% level. The p-values of the partial t-tests show that the explanatory variable GGDP is significant at a 5%-level, while the variable EURIBOR is even significant at a 1%-level.
For both response variables the achieved results are similar to the results of the regression model without the uncertainty index as additional explanatory variable.
Finally, the explanatory variables are chosen such that Mallow's C_p-statistic of the corresponding linear model is optimal. The uncertainty index is contained in both "best" models. The corresponding regression coefficients and t-statistics in the reduced models are given in Table 5.10.

Variable	**Coefficient (t-statistic)**	**Coefficient (t-statistic)**
Response	$\ln(z_T(t))$	$\ln(z_U(t))$
Intercept	-1.015 *** (-9.310)	-1.636 *** (-21.570)
GGDP	-	-39.666 *** (-2.830)
EURIBOR	-	36.872 *** (5.470)
GY	10.405 *** (2.963)	-
u	26.075 *** (3.766)	46.075 *** (3.758)
R_a^2 in %	44.37	60.41

Table 5.10: Coefficients (t-statistics) and significance codes of "best" model of Equation (5.7) with responses $\ln(z_i(t))$.

In the case of total recovery rates the variables GY and u form the best model. With these explanatory variables an adjusted coefficient of determination R_a^2 of almost 45% is achieved, where both variables are significant on a 1% level. For recovery rates of unsecured facilities the adjusted coefficient of determination is even higher. Using the explanatory variables u, GGDP, and EURIBOR, which are all significant at a 1%-level, an adjusted coefficient of determination R_a^2 of 60% can be achieved. In both "best" models the variable VSTOXX is replaced by the variable u. This can be interpreted as follows: both the uncertainty/volatility in credit markets and the uncertainty/volatility in equity markets are suitable explanatory variables for recovery rates. If both variables are available, it is more reasonable to use the credit market uncertainty.

To sum it up, it can be concluded that a suitable model for recovery rates should distinguish between secured and unsecured facilities. For recovery rates of secured facilities a closer look at the type and value of collateral might be more important than any macroeconomic indicator. In contrast to this, recovery rates of unsecured facilities can be modelled by macroeconomic variables, e.g. the short rate, an economic indicator, and a factor that describes the uncertainty in credit markets as shown in Table 5.10. These results motivate the pricing approach introduced in the following chapter.

Part III

Pricing credit derivatives under stochastic recovery

Chapter 6

A hybrid model for pricing single-name credit derivatives under stochastic recovery

The aim of this chapter is to present a joint modelling of default and recovery risk accounting for negative correlation between default rates and recovery rates as well as the dependence of recovery rates on the economic environment. Within this framework analytically tractable pricing formulas are derived for different credit derivatives including recovery products.
The modelling approach presented here is based on the framework of the extended Schmid-Zagst defaultable term-structure model (see Antes *et al.* (2008)), which is an extension of the three-factor Schmid-Zagst model (see Schmid and Zagst (2000)). This hybrid model models directly the short-rate credit spread in dependence of some unobservable, firm-specific uncertainty index. Under the assumption of fractional recovery of market value, i.e. the recovery payment in case of a default event is assumed to be a fraction of the market value instantaneously before default, closed-form solutions for defaultable bond prices are available without specifying a recovery-rate process (see e.g. Antes *et al.* (2008)). Within the same framework the pricing of credit derivatives under constant recovery is developed in Schmid *et al.* (2009). Unlike Antes *et al.* (2008), Schmid and Zagst (2000), and Schmid *et al.* (2009) the model presented in this chapter rather models the default intensity instead of the short-rate credit spread and uses a recovery of face value instead of a recovery of market value assumption. Under this recovery of face value assumption the recovery payment in case of a default event at time t is a fraction $z(t)$, called the recovery rate, of the face value.

6.1 Modelling framework

In the following a fixed terminal time horizon T^* is assumed. Uncertainty in the financial market is modelled on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. All random variables and stochastic processes introduced below are defined on this probability space. It is assumed throughout that $(\Omega, \mathcal{G}, \mathbb{P})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing and right-continuous families of sub-σ-fields of $\mathcal{G}$. The default time τ of an obligor is an arbitrary random time on $(\Omega, \mathcal{G}, \mathbb{P})$. For the sake of convenience it is assumed that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for every $t \in (0, T^*]$. For a given default time τ, consider the associated default indicator or hazard function $H(t) = \mathbb{1}_{\{\tau \leq t\}}$ and the survival indicator function $L(t) = 1 - H(t), t \in (0, T^*]$. Let $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be the filtration generated by the process H. Note that while $\mathbb{F}^N$ from Section 2.2 is the filtration generated by a counting process N, $\mathbb{H}$ is the filtration generated by the first jump of the counting process. In addition, let the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ be defined as the filtration generated by the multi-dimensional standard Brownian motion $W(t)^T$ containing all one-dimensional Brownian motions appearing in the modelled processes. Additionally, let $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ denote the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. for every t set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. This subfiltration structure is very common in intensity-based models (see e.g. Antes *et al.* (2008) or Bielecki and Rutkowski (2004)). Furthermore, it is assumed throughout that for any $t \in (0, T^*]$ the $\sigma-$fields $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}-$martingale is also a $\mathbb{G}-$martingale (see p.167 of Bielecki and Rutkowski (2004)).

6.1.1 Short-rate model

The short-rate model is specified by a two factor Hull-White model, with stochastic processes r and w describing the non-defaultable short rate and a market factor. The dynamics of the non-defaultable short rate are given by the stochastic differential equation

$$dr(t) = [\theta_r(t) + b_{rw}w(t) - a_r r(t)]\, dt + \sigma_r dW_r(t), \; r(0) = r_0, \; 0 \leq t \leq T^*, \tag{6.1}$$

where r_0, a_r, b_{rw}, $\sigma_r > 0$ are positive constants and $\theta_r(t)$ is a non-negative valued deterministic function.

The dynamics of the market factor are given by the following SDE:

$$dw\,(t) = [\theta_w - a_w w\,(t)]\,dt + \sigma_w dW_w\,(t)\,,\ w(0) = w_0,\ 0 \le t \le T^*, \qquad (6.2)$$

where a_w, $\sigma_w > 0$ are positive constants, θ_w is a non-negative constant, and $w_0 \in \mathbb{R}$.

6.1.2 Recovery and intensity model

The recovery rate $z(t)$, or equivalently $\widetilde{z}(t) := z(t) - a_z$ with $a_z \ge 0$, is given by

$$\widetilde{z}(t) = b_z e^{-c_z u(t) + d_z w(t)} \qquad (6.3)$$

with $b_z \ge 0$, $a_z + b_z < 1$, and u denoting an (unobservable) idiosyncratic risk factor given by the SDE

$$du\,(t) = [\theta_u - a_u u\,(t)]\,dt + \sigma_u dW_u\,(t)\,,\ u(0) = u_0,\ 0 \le t \le T^*, \qquad (6.4)$$

where a_u, $\sigma_u > 0$ are positive constants, θ_u is a non-negative constant, and $u_0 \in \mathbb{R}$. Hence, the dynamics of the recovery-rate process are given by

$$\begin{aligned} d\widetilde{z}(t) &= dz(t),\ \widetilde{z}(0) = \widetilde{z}_0, \\ &= -\widetilde{z}(t)\Big\{\Big[c_z\left(\theta_u - a_u u(t)\right) - d_z\left(\theta_w - a_w w(t)\right) \\ &\qquad -\frac{1}{2}\left(c_z^2\sigma_u^2 + d_z^2\sigma_w^2\right)\Big]\,dt \\ &\qquad + c_z\sigma_u dW_u(t) - d_z\sigma_w dW_w(t)\Big\} \end{aligned}$$

with $\widetilde{z}_0 \ge 0$.

The dynamics of the default intensity are given by the SDE

$$d\lambda\,(t) = [\theta_\lambda + b_{\lambda u} u\,(t) - b_{\lambda w} w(t) - a_\lambda \lambda\,(t)]\,dt + \sigma_\lambda dW_\lambda\,(t)\,,\ \lambda(0) = \lambda_0,\ (6.5)$$

where λ_0, a_λ, $b_{\lambda u}$, $b_{\lambda w}$, $\sigma_\lambda > 0$ are positive constants, θ_λ is a non-negative constant, and $0 \le t \le T^*$.

In the following, the Wiener processes W_r, W_w, W_u, and W_λ are assumed to be uncorrelated.

As mentioned above it is general consent that default risk and recovery risk are correlated and that recovery rates depend on the state of the economy. The first observation is accounted for in the modelling framework presented above by the impact of u on z and λ and the latter by the positive dependence of z on w. This modelling approach is also in line with the empirical insights presented in Chapter 5, where it was shown that the recovery rates of unsecured facilities can be described best by a macroeconomic index, a short-term interest rate, and an index describing the uncertainty in credit markets. Since the market factor w also drives the short rate r in the presented modelling framework, the recovery rate z is assumed to depend only on w and u. This also avoids potential identification problems. While the index describing the uncertainty in credit markets used in Chapter 5 was filtered from bond prices, in this chapter the factor u will be estimated directly from historical aggregated recovery rates and afterwards be used to model the dependence between default rates and recovery rates. Alternatively, the filtered time series of the uncertainty index as in Chapter 5 could be used as an input for the model. Then, an additional source of randomness had to be introduced to Equation (6.3), e.g. by adding a white noise term. The dependence structure in the modelling framework is illustrated in Figure 6.1.[15]

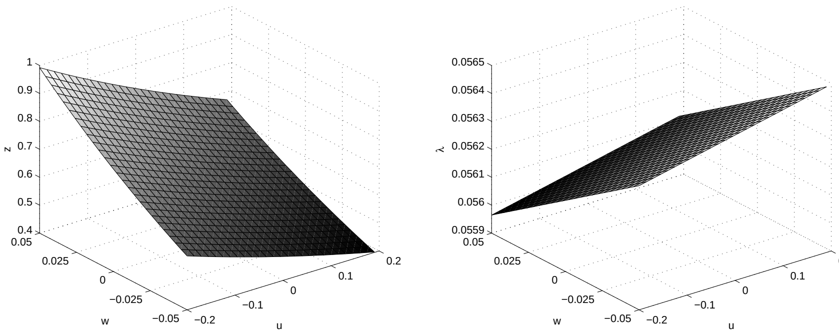

Figure 6.1: Recovery rate z and default intensity λ as functions of w and u.

Note that in this modelling framework the recovery-rate process can take values greater than 1. Recoveries of more than 100% can indeed be observed in certain situations (see e.g. p.13 of Schuermann (2004)). Also, short rates as well as default intensities can become negative in this framework. Here, we follow Brigo and Mercurio (2001) (see p.74), Duffie and Singleton (2003) (see p.108), and Schönbucher (2003) (see p.166) stating that the computational advantages are worth the approximation error and that small probabilities of negative short rates or default intensities are accepted in practical applications. Using the parameter set from Table 6.1, the (real-world as well as risk-neutral) one-year probability that the default intensity λ is negative

[15]For this illustration the parameter set from Table 6.1 in Section 6.3 estimated from market data was used.

is 0.033 and the (real-world as well as risk-neutral) one-year probabilities that the short rate r is negative and that the recovery rate z is greater 1 are both below 10^{-10}. One way to overcome the problem of possibly negative short rates and default intensities while preserving the aforementioned dependences would be to assume Cox-Ingersoll-Ross processes (CIR) with correlated Brownian motions for r, w, u, and λ instead of the dynamics assumed in Equations (6.1), (6.2), (6.4), and (6.5). However, this would lead to a significant loss of computational tractability of the pricing formulas presented in Section 6.2 (see also p.140 of Brigo and Mercurio (2001) or p.255 of Schmid (2004)). In such models with correlated CIR processes tree- (see e.g. Hull and White (1994)) or simulation-based (see e.g. Brigo and Alfonsi (2005)) methods are required.

6.1.3 Change of measure

So far, the modelling has taken place under the real-world measure $\mathbb{P}$. For pricing purposes we need a characterization of all processes of Subsections 6.1.1 and 6.1.2 under an equivalent martingale measure $\mathbb{Q}$, i.e. all discounted security price processes have to be $\mathbb{Q}-$martingales with respect to a suitable numéraire. As numéraire the money-market account $B\left(t\right) = e^{\int_0^t r(l)dl}$ is chosen, where $r\left(t\right)$ is the non-defaultable short rate from Equation (6.1).
It is well known that each martingale measure $\mathbb{Q}$ is given by the Radon-Nikodym-derivative

$$L(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|\mathcal{F}_t = \exp\left(-\int_0^t \gamma(s)^T dW(s) - \frac{1}{2}\int_0^t \left\|\gamma(s)\right\|^2 ds\right),$$

where $\gamma(s)^T = (\gamma_r(s), \gamma_w(s), \gamma_u(s), \gamma_\lambda(s))$ is an adapted, measurable four-dimensional process satisfying

$$\int_0^{T^*} \gamma_i(s)^2 ds < \infty \quad \mathbb{P}\text{-a.s. for } i \in \{r, w, u, \lambda\}.$$

Following Chen (1996) and Schmid (2004), the change of measure is assumed to have a parametric form given by

$$\gamma_i(t) = \eta_i \sigma_i i(t)$$

with $t \in [0, T^*]$ and $\eta_i \in \mathbb{R}$, $i \in \{r, w, u, \lambda\}$, such that Novikov's condition

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{T^*} \left\|\gamma(s)\right\|^2 ds\right)\right] < \infty$$

holds. This assumption is made in order to preserve the structure of the SDEs (6.1), (6.2), (6.4), and (6.5) under $\mathbb{Q}$. From Girsanov's theorem (see e.g. p.159 of Bingham and Kiesel (2004)) it is known that

$$\widehat{W}(t)^T = \left(\widehat{W}_r(t), \widehat{W}_w(t), \widehat{W}_u(t), \widehat{W}_\lambda(t)\right)$$

with

$$\widehat{W}_i(t) = W_i(t) + \int_0^t \gamma_i(s)ds,\ i \in \{r, w, u, \lambda\},\ 0 \le t \le T^*$$

is a four-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Under $\mathbb{Q}$, the dynamics of r, w, u, and λ are given by

$$\begin{aligned}
dr\,(t) &= [\theta_r\,(t) + b_{rw}w(t) - \widehat{a}_r r\,(t)]\,dt + \sigma_r d\widehat{W}_r\,(t)\,,\ r(0) = r_0, \\
dw\,(t) &= [\theta_w - \widehat{a}_w w\,(t)]\,dt + \sigma_w d\widehat{W}_w\,(t)\,,\ w(0) = w_0, \\
du\,(t) &= [\theta_u - \widehat{a}_u u\,(t)]\,dt + \sigma_u d\widehat{W}_u\,(t)\,,\ u(0) = u_0, \\
d\lambda\,(t) &= [\theta_\lambda + b_{\lambda u} u\,(t) - b_{\lambda w} w(t) - \widehat{a}_\lambda \lambda\,(t)]\,dt + \sigma_\lambda d\widehat{W}_\lambda\,(t)\,,\ \lambda(0) = \lambda_0,
\end{aligned}$$

with $\widehat{a}_i = a_i + \eta_i \sigma_i^2$, $i \in \{r, w, u, \lambda\}$, and $0 \le t \le T^*$.

6.1.4 Valuation of defaultable claims

Before the valuation of defaultable claims is addressed, an important result for the pricing of non-defaultable zero-coupon bonds is recalled in this subsection. This result will be used later for the calibration of the non-defaultable short-rate process $r(t)$ from Equation (6.1).

Theorem 6.1. *The time t price of a non-defaultable zero-coupon bond with maturity T is given by*

$$P^{nd}(t, T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right] = e^{A^{nd}(t,T) - B^{nd}(t,T) r(t) - E^{nd}(t,T) w(t)}$$

with

$$
\begin{aligned}
B^{nd}(t,T) &= \frac{1}{\widehat{a}_r}\left(1 - e^{-\widehat{a}_r(T-t)}\right),\\
E^{nd}(t,T) &= \frac{b_{rw}}{\widehat{a}_r}\left(\frac{1 - e^{-\widehat{a}_w(T-t)}}{\widehat{a}_w} + \frac{e^{-\widehat{a}_w(T-t)} - e^{-\widehat{a}_r(T-t)}}{\widehat{a}_w - \widehat{a}_r}\right),\\
A^{nd}(t,T) &= \int_t^T \Bigg[\frac{1}{2}\sigma_r^2 B^{nd}(s,T)^2 + \frac{1}{2}\sigma_w^2 E^{nd}(s,T)^2\\
&\qquad -\theta_r(s)B^{nd}(s,T) - \theta_w E^{nd}(s,T)\Bigg]ds.
\end{aligned}
$$

Proof. This theorem corresponds to a special case of the two-factor Hull-White model (see Hull and White (1994)). □

A defaultable contingent claim is defined as a triplet $DCC = (X, Z, \tau)$ with X denoting the promised payoff at maturity T if no default has taken place up to T, $Z = (Z(t))_{t\in[0,T]}$ the process describing the recovery payoff at the time of default, and τ the default time. If Z is a $\mathbb{G}$-predictable process and X is $\mathcal{G}_T$-measurable, the value process $V(t)$ of the defaultable contingent claim is given by (see e.g. p.180 of Bielecki and Rutkowski (2004))

$$
V(t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(l)dl} Z(s)dH(s) + e^{-\int_t^T r(l)dl} X \mathbb{1}_{\{\tau>T\}} \middle| \mathcal{G}_t\right].
$$

Under certain assumptions, the value of such a defaultable contingent claim can be expressed by the conditional expectation of the claim's payoffs discounted with a default-risk-adjusted short rate (see e.g. Duffie *et al.* (1996) or Bielecki and Rutkowski (2004)).

Theorem 6.2. *Assume that the martingale invariance property assumption is fulfilled, Z is an $\mathbb{F}$-predictable process and X is an $\mathcal{F}_T$-measurable random variable. Then, for every $t \in [0, T^*]$, it holds that*

$$
V(t) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r(l)+\lambda(l))dl}\lambda(s)Z(s)ds + e^{-\int_t^T (r(l)+\lambda(l))dl} X \middle| \mathcal{F}_t\right].
$$

Proof. Section 8.3 of Bielecki and Rutkowski (2004). □

While the recovery of market value assumption is suitable for bond-pricing purposes as it leads to analytically tractable formulas, it contains the problem that intensity and recovery risk are not separable. Hence, the recovery of face value assumption will be used in the following. This assumption is generally preferred when contingent claims on recoveries are considered (see e.g. Bakshi *et al.* (2006)).

Corollary 6.3. *In a model with recovery of face value assumption, i.e.* $Z(t) := z(t)X$ *with* $z(t)$ *denoting the recovery-rate process, the price of a defaultable contingent claim under the assumption of no default up to time* t *is given by*

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r(l)+\lambda(l))dl}\lambda(s)z(s)X ds + e^{-\int_t^T (r(l)+\lambda(l))dl}X \middle| \mathcal{F}_t\right].$$

6.2 Pricing recovery dependent credit derivatives

In this section, pricing equations for credit derivatives under the dynamics assumed in Equations (6.1) - (6.5) are derived.

6.2.1 Building blocks

The main building blocks of the pricing formulas are conditional expectations of the form

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} z\left(T\right) \middle| \mathcal{F}_t\right] \tag{6.6}$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} \lambda\left(T\right) z\left(T\right) \middle| \mathcal{F}_t\right]. \tag{6.7}$$

The following theorems show how to calculate the expected values in Equations (6.6) and (6.7) under the assumptions from Equations (6.1) - (6.5).

Theorem 6.4.

$$\begin{aligned} g(r,\lambda,u,w,t,T) : &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} e^{-c_z u(T)+d_z w(T)} \middle| \mathcal{F}_t\right] \\ &= e^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)} \end{aligned}$$

with

$$
\begin{aligned}
B(t,T) &= \frac{1}{\widehat{a}_r}\left(1-e^{-\widehat{a}_r(T-t)}\right), \\
C(t,T) &= \frac{1}{\widehat{a}_\lambda}\left(1-e^{-\widehat{a}_\lambda(T-t)}\right), \\
D(t,T) &= \frac{b_{\lambda u}}{\widehat{a}_\lambda}\left(\frac{1-e^{-\widehat{a}_u(T-t)}}{\widehat{a}_u}+\frac{e^{-\widehat{a}_u(T-t)}-e^{-\widehat{a}_\lambda(T-t)}}{\widehat{a}_u-\widehat{a}_\lambda}\right)+c_z e^{-\widehat{a}_u(T-t)}, \\
E(t,T) &= -\frac{b_{\lambda w}}{\widehat{a}_\lambda}\left(\frac{1-e^{-\widehat{a}_w(T-t)}}{\widehat{a}_w}+\frac{e^{-\widehat{a}_w(T-t)}-e^{-\widehat{a}_\lambda(T-t)}}{\widehat{a}_w-\widehat{a}_\lambda}\right) \\
&\quad +b_{rw}\left(\frac{1-e^{-\widehat{a}_w(T-t)}}{\widehat{a}_w\widehat{a}_r}+\frac{e^{-\widehat{a}_w(T-t)}-e^{-\widehat{a}_r(T-t)}}{\widehat{a}_w-\widehat{a}_r}\frac{1}{\widehat{a}_r}\right) \\
&\quad -d_z e^{-\widehat{a}_w(T-t)},
\end{aligned}
$$

and

$$
\begin{aligned}
A(t,T) &= \int_t^T \left[\frac{1}{2}\sigma_r^2 B^2(s,T)+\frac{1}{2}\sigma_\lambda^2 C^2(s,T)+\frac{1}{2}\sigma_u^2 D^2(s,T)\right. \\
&\quad +\frac{1}{2}\sigma_w^2 E^2(s,T)-\theta_r(s)B(s,T)-\theta_\lambda C(s,T) \\
&\quad \left.-\theta_u D(s,T)-\theta_w E(s,T)\right]ds.
\end{aligned}
$$

Proof. According to the theorem of Feynman-Kac (see Theorem 2.20 in Section 2.3), g is the solution of the PDE

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_r^2 g_{rr}+\sigma_\lambda^2 g_{\lambda\lambda}+\sigma_u^2 g_{uu}+\sigma_w^2 g_{ww}\right) \\
&\quad +\left(\theta_r(t)+b_{rw}w-\widehat{a}_r r\right)g_r+\left(\theta_w-\widehat{a}_w w\right)g_w+\left(\theta_u-\widehat{a}_u u\right)g_u \\
&\quad +\left(\theta_\lambda+b_{\lambda u}u-b_{\lambda w}w-\widehat{a}_\lambda\lambda\right)g_\lambda-(r+\lambda)g+g_t
\end{aligned}
$$

under the condition $g(r,\lambda,u,w,T,T)=e^{-c_z u(T)+d_z w(T)}$. If

$$
g(r,\lambda,u,w,t,T)=e^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)},
$$

then

$$\begin{aligned} 0 &= \frac{1}{2}\left(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2\right) g \\ &\quad - \left(\theta_r(t) + b_{rw} w - \widehat{a}_r r\right) Bg - \left(\theta_w - \widehat{a}_w w\right) Eg - \left(\theta_u - \widehat{a}_u u\right) Dg \\ &\quad - \left(\theta_\lambda + b_{\lambda u} u - b_{\lambda w} w - \widehat{a}_\lambda \lambda\right) Cg - (r+\lambda) g \\ &\quad + \left(A_t - B_t r - C_t \lambda - D_t u - E_t w\right) g. \end{aligned}$$

This is equivalent to

$$\begin{aligned} 0 &= \frac{1}{2}\left(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2\right) \\ &\quad + r(\widehat{a}_r B - 1 - B_t) + \lambda(\widehat{a}_\lambda C - 1 - C_t) + u(\widehat{a}_u D - b_{\lambda u} C - D_t) \\ &\quad + w(-b_{rw} B + \widehat{a}_w E + b_{\lambda w} C - E_t) \\ &\quad + A_t - \theta_r(t) B - \theta_\lambda C - \theta_u D - \theta_w E. \end{aligned}$$

Therefore, the following system of linear equations has to be solved:

$$\begin{aligned} B_t &= \widehat{a}_r B - 1,\ C_t = \widehat{a}_\lambda C - 1,\ D_t = \widehat{a}_u D - b_{\lambda u} C, \\ E_t &= \widehat{a}_w E - b_{rw} B + b_{\lambda w} C, \\ A_t &= \theta_r(t) B + \theta_\lambda C + \theta_u D + \theta_w E \\ &\quad - \frac{1}{2}\left(\sigma_r^2 B^2 + \sigma_\lambda^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 E^2\right), \end{aligned}$$

with boundary conditions

$$\begin{aligned} &A(T,T) = 0, \quad B(T,T) = 0, \quad C(T,T) = 0, \\ &D(T,T) = c_z, \quad E(T,T) = -d_z. \end{aligned}$$

Using the transformation $s = T - t$ leads to the proposed solutions for $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$. □

Theorem 6.5.

$$\begin{aligned} &\widetilde{g}(r,\lambda,u,w,t,T) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} \lambda(T) e^{-c_z u(T) + d_z w(T)} \middle| \mathcal{F}_t \right] \\ &= g(r,\lambda,u,w,t,T)\left(G(t,T) + I(t,T)\lambda(t) + J(t,T)u(t) + K(t,T)w(t)\right) \\ &= e^{A(t,T) - B(t,T)r(t) - C(t,T)\lambda(t) - D(t,T)u(t) - E(t,T)w(t)} \\ &\quad \cdot \left(G(t,T) + I(t,T)\lambda(t) + J(t,T)u(t) + K(t,T)w(t)\right) \end{aligned}$$

with $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$ from Theorem 6.4,

$$
\begin{aligned}
I(t,T) &= e^{-\widehat{a}_\lambda (T-t)}, \\
J(t,T) &= \frac{b_{\lambda u}}{\widehat{a}_u - \widehat{a}_\lambda} \left(e^{-\widehat{a}_\lambda (T-t)} - e^{-\widehat{a}_u (T-t)} \right), \\
K(t,T) &= \frac{b_{\lambda w}}{\widehat{a}_\lambda - \widehat{a}_w} \left(e^{-\widehat{a}_\lambda (T-t)} - e^{-\widehat{a}_w (T-t)} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,T) &= \int_t^T \Big[\theta_\lambda I(s,T) + \theta_u J(s,T) + \theta_w K(s,T) - \sigma_\lambda^2 C(s,T) I(s,T) \\
&\quad -\sigma_u^2 D(s,T) J(s,T) - \sigma_w^2 E(s,T) K(s,T) \Big] ds.
\end{aligned}
$$

Proof. According to the theorem of Feynman-Kac (see Theorem 2.20 in Section 2.3), $\widetilde{g}$ is the solution of the PDE

$$
\begin{aligned}
0 &= \frac{1}{2} \left(\sigma_r^2 \widetilde{g}_{rr} + \sigma_\lambda^2 \widetilde{g}_{\lambda\lambda} + \sigma_u^2 \widetilde{g}_{uu} + \sigma_w^2 \widetilde{g}_{ww} \right) \\
&\quad + (\theta_r(t) + b_{rw} w - \widehat{a}_r r)\, \widetilde{g}_r + (\theta_w - \widehat{a}_w w)\, \widetilde{g}_w + (\theta_u - \widehat{a}_u u)\, \widetilde{g}_u \\
&\quad + (\theta_\lambda + b_{\lambda u} u - b_{\lambda w} w - \widehat{a}_\lambda \lambda)\, \widetilde{g}_\lambda - (r+\lambda)\widetilde{g} + \widetilde{g}_t
\end{aligned}
$$

under the condition $\widetilde{g}(r,\lambda,u,w,T,T) = \lambda(T) e^{-c_z u(T) + d_z w(T)}$. If

$$
\begin{aligned}
\widetilde{g}(r,\lambda,u,w,t,T) = {} & e^{A(t,T) - B(t,T) r(t) - C(t,T)\lambda(t) - D(t,T) u(t) - E(t,T) w(t)} \\
& \cdot (G(t,T) + H(t,T) r(t) + I(t,T)\lambda(t) \\
& + J(t,T) u(t) + K(t,T) w(t))
\end{aligned}
$$

then

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\left(\sigma_r^2 B^2+\sigma_\lambda^2 C^2+\sigma_u^2 D^2+\sigma_w^2 E^2\right) \\
& \cdot (G+Hr+I\lambda+Ju+Kw) \\
& +\left(-\sigma_r^2 BH-\sigma_\lambda^2 CI-\sigma_u^2 DJ-\sigma_w^2 EK\right) \\
& +\left(\theta_r(t)+b_{rw}w-\widehat{a}_r r\right)\left(-B\left(G+Hr+I\lambda+Ju+Kw\right)+H\right) \\
& +\left(\theta_w-\widehat{a}_w w\right)\left(-E\left(G+Hr+I\lambda+Ju+Kw\right)+K\right) \\
& +\left(\theta_u-\widehat{a}_u u\right)\left(-D\left(G+Hr+I\lambda+Ju+Kw\right)+J\right) \\
& +\left(\theta_\lambda+b_{\lambda u}u-b_{\lambda w}w-\widehat{a}_\lambda\lambda\right) \\
& \cdot\left(-C\left(G+Hr+I\lambda+Ju+Kw\right)+I\right) \\
& -(r+\lambda)\left(G+Hr+I\lambda+Ju+Kw\right) \\
& +\left(G+Hr+I\lambda+Ju+Kw\right) \\
& \cdot\left(A_t-B_t r-C_t\lambda-D_t u-E_t w\right) \\
& +G_t+H_t r+I_t\lambda+J_t u+K_t w.
\end{aligned}
$$

Using the proof of Theorem 6.4, this reduces to

$$
\begin{aligned}
0 \;=\; & -\sigma_r^2 BH-\sigma_\lambda^2 CI-\sigma_u^2 DJ-\sigma_w^2 EK \\
& +H\theta_r(t)+K\theta_w+J\theta_u+I\theta_\lambda+G_t \\
& +r\left(-\widehat{a}_r H+H_t\right)+w\left(b_{rw}H-\widehat{a}_w K-b_{\lambda w}I+K_t\right) \\
& +u\left(-\widehat{a}_u J+b_{\lambda u}I+J_t\right)+\lambda\left(-\widehat{a}_\lambda I+I_t\right).
\end{aligned}
$$

Therefore, the following system of linear equations has to be solved:

$$
\begin{aligned}
H_t &= \widehat{a}_r H,\; I_t=\widehat{a}_\lambda I,\; J_t=\widehat{a}_u J-b_{\lambda u}I, \\
K_t &= \widehat{a}_w K-b_{rw}H+b_{\lambda w}I, \\
G_t &= \sigma_r^2 BH+\sigma_\lambda^2 CI+\sigma_u^2 DJ+\sigma_w^2 EK \\
& \quad -H\theta_r(t)-K\theta_w-J\theta_u-I\theta_\lambda,
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
& G(T,T)=0, \quad H(T,T)=0, \quad I(T,T)=1, \\
& J(T,T)=0, \quad K(T,T)=0.
\end{aligned}
$$

Using the transformation $s=T-t$ leads to $H(t,T)\equiv 0$ and the proposed solutions for $G(t,T)$, $I(t,T)$, $J(t,T)$, and $K(t,T)$. □

As an immediate consequence of Theorems 6.4 and 6.5 the following corollaries can be stated by using Corollary 6.3.

Corollary 6.6. *The time t price of a defaultable zero-coupon bond with maturity T and unit notional under the assumption of zero recovery and no default up to time t is given by*

$$\begin{aligned} P^{d,zero}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} \middle| \mathcal{F}_t \right] \\ &= e^{A(t,T)-B(t,T)r(t)-C(t,T)\lambda(t)-D(t,T)u(t)-E(t,T)w(t)} \end{aligned}$$

with $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, and $E(t,T)$ from Theorem 6.4 and $c_z = d_z = 0$.

Corollary 6.7. *The time t price of a defaultable zero-coupon bond with maturity T and unit notional under the assumption of no default up to time t is given by*

$$\begin{aligned} P^{d}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+\lambda(l))dl} \middle| \mathcal{F}_t \right] \\ &\quad + \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda(s) z(s) \middle| \mathcal{F}_t \right] ds \\ &= P^{d,zero}(t,T) + b_z \int_t^T \widetilde{g}(r,\lambda,u,w,t,s)ds \\ &\quad + a_z \int_t^T \widetilde{g}^{zero}(r,\lambda,u,w,t,s)ds \end{aligned}$$

with $\widetilde{g}(r,\lambda,u,w,t,s)$ from Theorem 6.5 and $\widetilde{g}^{zero}(r,\lambda,u,w,t,s)$ denoting $\widetilde{g}(r,\lambda,u,w,t,s)$ under the assumption $c_z = d_z = 0$.

Corollary 6.8. *A default digital put option on a defaultable zero-coupon bond with maturity T pays one unit of currency in the case of a default before or at maturity and nothing else. Assuming no default up to time t and that the payoff takes place at default, the time t price of the default digital put is given by*

$$\begin{aligned} V^{ddp}(t) &= \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda(s) ds \middle| \mathcal{F}_t \right] \\ &= \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda(s) \middle| \mathcal{F}_t \right] ds \\ &= \int_t^T \widetilde{g}^{zero}(r,\lambda,u,w,t,s)ds, \end{aligned}$$

with $\widetilde{g}^{zero}(r,\lambda,u,w,t,s)$ denoting $\widetilde{g}(r,\lambda,u,w,t,s)$ from Theorem 6.5 under the assumption $c_z = d_z = 0$.

In the following section pricing formulas for credit derivatives based on Equations (6.6) and (6.7) are established.

6.2.2 Credit default swaps

A credit default swap (CDS) is a swap under which one party (the beneficiary) pays the other party (the guarantor) regular fees, called the credit default swap spread or the credit default swap rate. This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default to cover the loss resulting from default. As common for swap products, two payment streams have to be considered, the default and the premium leg. According to the recovery of face value assumption, it is assumed that in case of a default event the payment on the default leg is one minus the recovery rate times the notional. For ease of notation a unit notional is assumed in the following.
The pricing of a credit default swap consists of two problems. At origination ($t = t_0$) there is no exchange of cash flows and the credit default swap spread $S^{CDS}(t_0, T)$ has to be determined such that the market value of the credit default swap is zero. After origination ($t \in (t_0, T]$), the market value of the credit default swap will change due to changes in the underlying variable. Therefore, given the credit default swap spread $S^{CDS}(t_0, T)$, the current market value of the credit default swap has to be computed.
It is assumed throughout that the CDS counterparties (beneficiary and guarantor) are default-free. Furthermore, it is assumed that the underlying reference credit asset has no coupon payments up to the maturity T^* and that there has been no credit event until time t_0. The scheduled payment dates of the credit swap spread are denoted by $\mathcal{T} = \{t_1, \ldots, t_m\}$ with $t_0 \leq t_1 \leq \ldots \leq t_m = T$. The value of the default leg at origination must be the same as paying $S^{CDS}(t_0, T)$ at some predefined times t_i, $i = 1, ..., m$, with $t_0 \leq t_1 \leq \cdots \leq t_m = T$ until a default happens. Finally, for ease of notation we assume that in case of a default event the beneficiary receives the compensation at the next premium date rather than right upon default. Under these assumptions the CDS premium is given as follows:

Corollary 6.9. *The swap premium of a credit default swap is (under the above mentioned assumptions) given by*

$$S^{CDS}(t_0, T) = \frac{V^{ddp}(t_0) - P^d(t_0, T) + P^{d,zero}(t_0, T)}{\sum_{i=1}^{m} \Delta t_i P^{d,zero}(t_0, t_i)}, \tag{6.8}$$

with $\Delta t_i = t_i - t_{i-1}$.

Proof. Under the above mentioned assumptions the time t value of the default leg of a CDS is given by (see e.g. Duffie and Singleton (2003))

$$\begin{aligned} V_{def}^{CDS}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r(l)+\lambda(l))dl}\lambda(s)\left(1-z(s)\right)ds\middle|\mathcal{F}_t\right] \\ &= V^{ddp}(t) - P^d(t,T) + P^{d,zero}(t,T). \end{aligned} \tag{6.9}$$

The time t value of the premium leg is given by

$$\begin{aligned} V_{prem}^{CDS}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^m S^{CDS}(t_0,T)\Delta t_i e^{-\int_t^{t_i}(r(l)+\lambda(l))dl}\middle|\mathcal{F}_t\right] \qquad (6.10) \\ &= S^{CDS}(t_0,T)\sum_{i=1}^m \Delta t_i \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{t_i}(r(l)+\lambda(l))dl}\middle|\mathcal{F}_t\right] \\ &= S^{CDS}(t_0,T)\sum_{i=1}^m \Delta t_i P^{d,zero}(t,t_i), \end{aligned}$$

where $S^{CDS}(t_0,T)$ is the swap premium of the credit default swap and $\Delta t_i = t_i - t_{i-1}$, $i = 1,\ldots,m$. To give the contract a value of zero at origination, the relation

$$V_{def}^{CDS}(t_0,T) = V_{prem}^{CDS}(t_0,T)$$

must hold and hence the swap premium is given by Equation (6.8). □

Figure 6.2 shows the impact of different values of λ and z (all other parameters fixed) on CDS spreads in the modelling framework from Section 6.1.[16]

While the value of the contract at origination is zero, it changes during the lifetime of the contract. The value of the CDS is then given by the difference between the value of the default leg $V_{def}^{CDS}(t,T)$ and the value of the premium leg $V_{prem}^{CDS}(t,T)$.

6.2.3 Fixed-recovery CDS

A fixed-recovery CDS or default digital swap is a credit default swap with a contractually fixed recovery payment in case of default. Hence, the swap premium for the fixed-recovery CDS can be calculated similarly to the swap premium of a standard CDS and is given in the following corollary.

[16]For this illustration the parameter set from Table 6.1 in Section 6.3 estimated from market data was used.

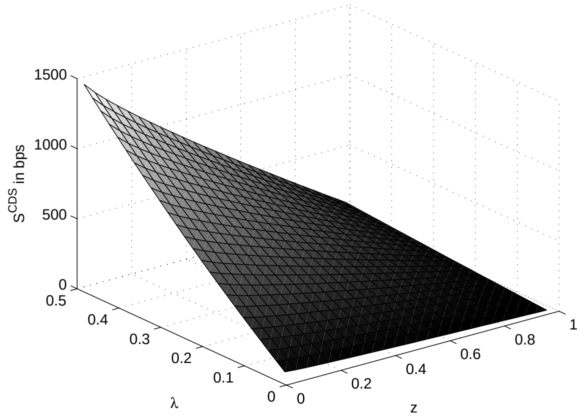

Figure 6.2: CDS spread in dependence of recovery z and intensity λ.

Corollary 6.10. *The swap premium of a fixed-recovery credit default swap with a contractually fixed recovery rate z_{Fix} is given by*

$$S^{FRCDS}\left(t_0, T, z_{Fix}\right) = \frac{\left(1 - z_{Fix}\right) V^{ddp}(t_0)}{\sum_{i=1}^{m} \Delta t_i P^{d,zero}(t_0, t_i)}. \tag{6.11}$$

Proof. The time t value of the default leg of a fixed-recovery CDS is given by

$$\begin{aligned} V_{def}^{FRCDS}(t,T) &= \left(1 - z_{Fix}\right) \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda\left(s\right) \middle| \mathcal{F}_t \right] ds \\ &= \left(1 - z_{Fix}\right) V^{ddp}(t). \end{aligned}$$

The time t value of the premium leg of such a fixed-recovery CDS is given by

$$\begin{aligned} V_{prem}^{FRCDS}(t,T) &= S^{FRCDS}(t_0, T, z_{Fix}) \sum_{i=1}^{m} \Delta t_i \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{t_i} (r(l)+\lambda(l))dl} \middle| \mathcal{F}_t \right] \\ &= S^{FRCDS}(t_0, T, z_{Fix}) \sum_{i=1}^{m} \Delta t_i P^{d,zero}(t, t_i), \end{aligned}$$

where $S^{FRCDS}(t_0, T, z_{Fix})$ is the swap premium of the fixed-recovery credit default swap with contractually fixed recovery rate z_{Fix}. By equating the values of default and premium leg at $t = t_0$ the assertion follows immediately.

□

Similar to Figure 6.2, Figure 6.3 shows the impact of different values of λ and z (all other parameters fixed) on the fixed-recovery CDS spread in the modelling framework of Section 6.1.[17] By construction the fixed-recovery CDS spread is independent of the dynamics of the recovery-rate process and is therefore a measure of pure default-event risk.

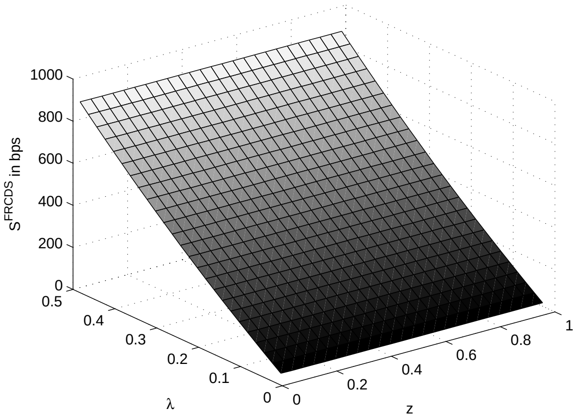

Figure 6.3: Fixed-recovery CDS spread in dependence of recovery z and intensity λ.

As for standard CDS, the value of a fixed-recovery CDS is given by the difference between the value of the default leg $V_{def}^{FRCDS}(t,T)$ and the value of the premium leg $V_{prem}^{FRCDS}(t,T)$.

6.2.4 Recovery lock

While standard CDS give protection against default-loss risk and fixed-recovery CDS against default-event risk, recovery locks give protection against pure recovery risk. Recovery locks, sometimes also called recovery swaps or recovery forwards, allow to purchase or sell the underlying credit instrument at a predetermined price $Z_{Lock}(t_0,T)$ if a credit event occurs. A recovery lock has no upfront or running payments. The only payment stream is the exchange of realized and predetermined recovery in case of a default event. Its payoff can be represented either as a single recovery lock trade or through a recovery swap representation that separates the trade in two legs,

[17]For this illustration the parameter set from Table 6.1 in Section 6.3 estimated from market data was used.

a short protection in a standard CDS and a long protection in fixed-recovery CDS (see e.g. Berd (2005) or Liu *et al.* (2005)). The price of such a recovery lock is given in the following corollary.

Corollary 6.11. *The price of a recovery lock is given by*

$$Z_{Lock}(t_0, T) = 1 - (1 - z_{Fix})\,\frac{S^{CDS}(t_0, T)}{S^{FRCDS}\,(t_0, T, z_{Fix})}$$

with $S^{CDS}(t_0, T)$ denoting the swap premium of a (standard) CDS from Equation (6.8) and $S^{FRCDS}\,(t_0, T, z_{Fix})$ the premium of a fixed-recovery CDS with contractually fixed recovery rate z_{Fix} from Equation (6.11).

Proof. Assume the payoff of a long position in a recovery lock shall be replicated by buying φ^{FRCDS} fixed-recovery CDS and selling φ^{CDS} standard CDS. To circumvent arbitrage opportunities the net cash flows of the two representations have to equal zero in all scenarios. As the recovery lock has no running payments, the running payments of the combined position given by

$$\varphi^{CDS} S^{CDS}(t_0, T) - \varphi^{FRCDS} S^{FRCDS}(t_0, T, z_{Fix})$$

have to equal zero. This leads to a ratio of fixed-recovery CDS to standard CDS of

$$\frac{\varphi^{FRCDS}}{\varphi^{CDS}} = \frac{S^{CDS}(t_0, T)}{S^{FRCDS}\,(t_0, T, z_{Fix})}.$$

In case of a default event the payment of the recovery lock equals

$$Z_{Lock}(t_0, T) - z(t)$$

with $z(t)$ denoting the (actual) recovery rate in case of a default event at time t, while the payoff of the combined position is given by

$$\begin{aligned}&\varphi^{CDS}(1 - z(t)) - \varphi^{FRCDS}(1 - z_{Fix})\\ &= \varphi^{CDS}(1 - z(t)) - \varphi^{CDS}\frac{S^{CDS}(t_0, T)}{S^{FRCDS}\,(t_0, T, z_{Fix})}(1 - z_{Fix}).\end{aligned}$$

To avoid arbitrage opportunities both payoffs have to be equal. Setting $\varphi^{CDS} = 1$ leads to the proposed recovery price. □

Figure 6.4 shows the impact of different values of λ and z (all other parameters fixed) on the recovery lock price in the modelling framework from Section 6.1.[18]

[18] Again, the parameter set from Table 6.1 in Section 6.3 estimated from market data

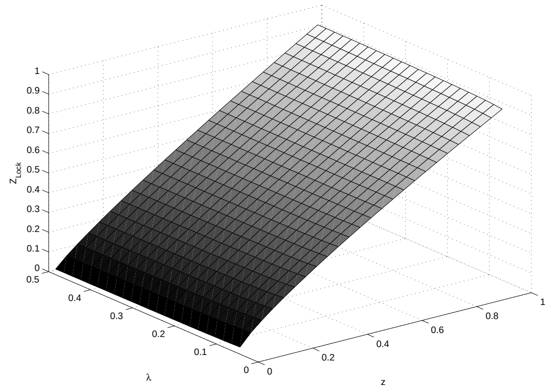

Figure 6.4: Recovery lock price in dependence of recovery z and intensity λ.

The following corollary shows how to determine the value of a recovery lock during its lifetime.

Corollary 6.12. *The value of a recovery lock during its lifetime is given by*

$$\begin{aligned} V^{RL}(t,T) &= V^{RL}_{def}(t,T) \\ &= Z_{Lock}(t_0,T)V^{ddp}(t) - P^{d}(t,T) + P^{d,zero}(t,T). \end{aligned}$$

Proof. To obtain the value of a recovery lock during its lifetime, the values of the two different legs have to be computed. Owing to its construction the value of the premium leg always equals zero. Hence, the value of the recovery lock is equal to the value of the default leg and consequently given by

was used.

$$\begin{aligned}
V^{RL}(t,T) &= V_{def}^{RL}(t,T) \\
&= V_{def}^{CDS}(t,T) - \frac{S^{CDS}(t_0,T)}{S^{FRCDS}\left(t_0,T,z_{Fix}\right)} V_{def}^{FRCDS}(t,T) \qquad (6.12)\\
&= \left(1 - \frac{S^{CDS}(t_0,T)}{S^{FRCDS}\left(t_0,T,z_{Fix}\right)}\left(1-z_{Fix}\right)\right) \\
&\quad \cdot \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda\left(s\right) \middle| \mathcal{F}_t \right] ds \\
&\quad - \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda\left(s\right) z\left(s\right) \middle| \mathcal{F}_t \right] ds \\
&= Z_{Lock}(t_0,T) \cdot \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda\left(s\right) \middle| \mathcal{F}_t \right] ds \\
&\quad - \int_t^T \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^s (r(l)+\lambda(l))dl} \lambda\left(s\right) z\left(s\right) \middle| \mathcal{F}_t \right] ds \qquad (6.13)\\
&= Z_{Lock}(t_0,T) V^{ddp}(t) - P^d(t,T) + P^{d,zero}(t,T).
\end{aligned}$$

□

The equivalence of (6.12) and (6.13) corresponds to the two different representations of a recovery lock as a short protection in a standard CDS and a long protection in fixed recovery CDS or as a single recovery lock trade.

6.3 Parameter estimation, model calibration, and empirical results

6.3.1 Parameter estimation

In this section, it is shown how to determine the parameter values for the model introduced in Section 6.1 from market data by using Kalman filter techniques (see Section 2.4). As the number of parameters is quite high, the estimation procedure is divided into three steps.

6.3.1.1 Estimation of the risk-free interest rate

First, the parameters of the short rate r and the market factor w are estimated. Estimating the parameters for w is done by means of maximum likelihood from the market factor, represented e.g. by GDP growth rates. Then, the Kalman filter is applied to time series of non-defaultable zero rates for different maturities to obtain the parameters of the short rate r. Let t_k, $k = 1, \ldots, n$, denote the observation dates of the zero rates and $\Delta t_{k+1} := t_{k+1} - t_k$ for $k = 1, \ldots, n-1$. As the frequency of market data (e.g. weekly zero rates) is in most cases much higher than the frequency of the macroeconomic data (e.g. quarterly GDP growth rates), we obtain $w(t_k)$, $k = 1, \ldots, n$ by a cubic spline interpolation of the macroeconomic data.

Parameter estimation for the market factor (e.g. GDP growth rates).
The parameters of the market factor $w(t)$ from Equation (6.2) are estimated from an observed time series like GDP growth rates via a maximum likelihood estimation. It is well known that Equation (6.2) has the solution

$$w(t) = e^{-a_w t} w(0) + \frac{\theta_w}{a_w}\left(1 - e^{-a_w t}\right) + \int_0^t e^{-a_w(t-s)} \sigma_w dW_w(s).$$

Hence, one obtains for $k = 1, \ldots, n-1$

$$\begin{aligned} w(t_{k+1}) &= e^{-a_w \Delta t_{k+1}} w(t_k) + \frac{\theta_w}{a_w}\left(1 - e^{-a_w \Delta t_{k+1}}\right) \\ &\quad + \int_{t_k}^{t_{k+1}} e^{-a_w(t_{k+1}-s)} \sigma_w dW_w(s) \end{aligned}$$

and

$$w(t_{k+1}) | w(t_k) \sim \mathcal{N}(p_1, p_2^2)$$

with parameters

$$p_1 = e^{-a_w \Delta t_{k+1}} w(t_k) + \frac{\theta_w}{a_w}\left(1 - e^{-a_w \Delta t_{k+1}}\right)$$

and

$$p_2^2 = \frac{\sigma_w^2}{2a_w}\left(1 - e^{-2a_w \Delta t_{k+1}}\right).$$

The maximum likelihood estimates of the parameter vector $\mathbf{\Theta_w} := (\theta_w, a_w, \sigma_w)$ can now be obtained by maximising the likelihood function

$$L(\theta_w, a_w, \sigma_w) = \prod_{k=1}^{n-1} \varphi_{w(t_{k+1})|w(t_k)},$$

where $\varphi_{w(t_{k+1})|w(t_k)}$ denotes the density of a normal distribution with parameters p_1 and p_2^2 as defined above.

Parameter estimation for the short-rate process.
First, the time-dependent function $\theta_r(t)$ is fitted to the initial term structure (see e.g. p.73 of Brigo and Mercurio (2001)). Second, the parameter vector $\mathbf{\Theta_r} := (a_r, \sigma_r, b_{rw}, \eta_r, \eta_w)$ of the short-rate model is estimated with a Kalman filter for state space models with measurement and transition equation as given below.
From Theorem 6.1 we know that in the setting of Section 6.1 the time t price of a zero-coupon bond with maturity T is given by

$$P^{nd}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right] = e^{A^{nd}(t,T) - B^{nd}(t,T)r(t) - E^{nd}(t,T)w(t)},$$

with $A^{nd}(t,T)$, $B^{nd}(t,T)$, and $E^{nd}(t,T)$ as in Theorem 6.1.
The measurement equation can now be derived from

$$R(t,T) = -\frac{1}{T-t} \ln P^{nd}(t,T) = a(t,T) + b(t,T)r(t)$$

with

$$a(t,T) = -\frac{A^{nd}(t,T)}{T-t} + \frac{E^{nd}(t,T)}{T-t} w(t)$$

and

$$b(t,T) = \frac{B^{nd}(t,T)}{T-t}.$$

Let $R(t_k, t_k + T_i)$, $i = 1, \dots, m_r$, denote the observed zero rates at time t_k. Hence, the measurement equation is given by

$$\begin{pmatrix} R(t_k, t_k + T_1) \\ \vdots \\ R(t_k, t_k + T_{m_r}) \end{pmatrix} = \begin{pmatrix} a(t_k, t_k + T_1) \\ \vdots \\ a(t_k, t_k + T_{m_r}) \end{pmatrix} + \begin{pmatrix} b(0, T_1) \\ \vdots \\ b(0, T_{m_r}) \end{pmatrix} \cdot r(t_k) + \epsilon_r(t_k),$$

where the measurement error $\epsilon_r(t_k)$ is assumed to follow an m_r-dimensional normal distribution with expectation vector $\mathbf{0}$ and covariance matrix $h_r^2 \cdot \mathbf{I}_{m_r}$, i.e. $\epsilon_r(t_k) \sim \mathcal{N}_{m_r}(\mathbf{0}, h_r^2 \cdot \mathbf{I}_{m_r})$.
The transition equation is obtained by using the fact that the SDE of the short-rate process from Equation (6.1) has the following solution:

$$r(t) = e^{-a_r t} r(0) + \int_0^t e^{-a_r(t-s)} \left(\theta_r(s) + b_{rw} w(s)\right) ds + \int_0^t e^{-a_r(t-s)} \sigma_r dW_r(s).$$

Approximately, it holds that

$$\begin{aligned} r(t_{k+1}) &= e^{-a_r \Delta t_{k+1}} r(t_k) + \int_0^{\Delta t_{k+1}} e^{-a_r s} \left(\theta_r(t_k) + b_{rw} w(t_k)\right) ds + \nu_r(t_{k+1}) \\ &= e^{-a_r \Delta t_{k+1}} r(t_k) + \frac{1}{a_r} \left(1 - e^{-a_r \Delta t_{k+1}}\right) \left(\theta_r(t_k) + b_{rw} w(t_k)\right) \\ &\quad + \nu_r(t_{k+1}) \end{aligned}$$

with

$$\nu_r(t_{k+1}) := \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-s)} \sigma_r dW_r(s) \sim \mathcal{N}\left(0, \frac{\sigma_r^2}{2a_r}\left(1 - e^{-2a_r \Delta t_{k+1}}\right)\right)$$

and $\Delta t_{k+1} = t_{k+1} - t_k$.

6.3.1.2 Estimation of the recovery-rate process

As the markets for digital default swaps and recovery swaps are not very liquid and a reliable joint estimation of default and recovery risk components from CDS quotes is, because of identifiability problems, only possible under some restrictive assumptions, the estimation of default and recovery risk is seperated. Therefore, in a second step the obtained estimates from the first step are used to estimate the parameters for the recovery-rate process z and the risk factor u from historical time series of average recovery rates by means of the Kalman filter. As the minimum possible recovery rate is given by a_z and the case of zero recovery should be included, a_z is set equal to zero, i.e. $z(t) = \widetilde{z}(t)$. Hence, the parameter vector $\mathbf{\Theta_z} = (b_z, c_z, d_z, \theta_u, a_u, \sigma_u)$ has to be estimated. Defining $\xi(t) := \log(z(t))$ and $b_\xi := \log(b_z)$ the measurement equation can be written as

$$\xi(t_k) = b_\xi - c_z u(t_k) + d_z w(t_k) + \epsilon_u(t_k)$$

with $\epsilon_u(t_k) \sim \mathcal{N}(0, h_z^2)$. The transition equation can be obtained similarly to Subsection 6.3.1.1 and is given by

$$u(t_{k+1}) = e^{-a_u \Delta t_{k+1}} u(t_k) + \frac{\theta_u}{a_u}\left(1 - e^{-a_u \Delta t_{k+1}}\right) + \nu_u(t_{k+1})$$

with $\nu_u(t_{k+1}) \sim \mathcal{N}\left(0, \frac{\sigma_u^2}{2a_u}\left(1 - e^{-2a_u \Delta t_{k+1}}\right)\right)$.

6.3.1.3 Estimation of the default intensity

In the third step, the parameters of the default intensity λ are estimated from market quotes of CDS spreads by using the estimates from the first two steps.
The approach of using empirical time series and parameters in a risk-neutral valuation framework is commonly used e.g. in the prepayment modelling for the valuation of mortgage-backed securities (see e.g. Kolbe and Zagst (2008)). This is similar to the setting presented here, where there is little if anything in liquid markets which can be used for a suitable calibration of the recovery-rate process.
Using the obtained parameter estimates from Subsections 6.3.1.1 and 6.3.1.2 as well as the market quotes of CDS, the parameter vector $\mathbf{\Theta}_\lambda = (\theta_\lambda, a_\lambda, \sigma_\lambda, b_{\lambda w}, b_{\lambda u}, \eta_\lambda, \eta_u)$ is estimated. The transition equation can be obtained similarly to Subsection 6.3.1.1 and is given by

$$\begin{aligned}\lambda(t_{k+1}) &= e^{-a_\lambda \Delta t_{k+1}} \lambda(t_k) \\ &\quad + \frac{\theta_\lambda + b_{\lambda u} u(t_k) - b_{\lambda w} w(t_k)}{a_\lambda} \left(1 - e^{-a_\lambda \Delta t_{k+1}}\right) + \nu_\lambda(t_{k+1})\end{aligned}$$

with $\nu_\lambda(t_{k+1}) \sim \mathcal{N}\left(0, \frac{\sigma_\lambda^2}{2a_\lambda}\left(1 - e^{-2a_\lambda \Delta t_{k+1}}\right)\right)$.
As the CDS swap premium $S^{CDS}(t, T_i)$, $i = 1, \ldots, m_s$, is a non-linear function in $\lambda(t)$, an extended Kalman filter has to be applied. Therefore, the model implied swap premium is linearised by means of a first-order Taylor series expansion around the best prediction $\widehat{\lambda}_{t|t-1}$ in the prediction step of the Kalman filter, i.e.

$$\begin{aligned}& S^{CDS}(t, T_i, \lambda(t)) \doteq \\ & S^{CDS}(t, T_i, \widehat{\lambda}_{t|t-1}) + \frac{\partial}{\partial \lambda(t)} S^{CDS}(t, T_i, \lambda(t))|_{\lambda(t)=\widehat{\lambda}_{t|t-1}} \cdot \left(\lambda(t) - \widehat{\lambda}_{t|t-1}\right).\end{aligned}$$

By defining

$$a^{CDS}(t, T_i) = S^{CDS}(t, T_i, \widehat{\lambda}_{t|t-1}) - \frac{\partial}{\partial \lambda(t)} S^{CDS}(t, T_i, \lambda(t))|_{\lambda(t)=\widehat{\lambda}_{t|t-1}} \cdot \widehat{\lambda}_{t|t-1}$$

and

$$b^{CDS}(t, T_i) = \frac{\partial}{\partial \lambda(t)} S^{CDS}(t, T_i, \lambda(t))|_{\lambda(t)=\widehat{\lambda}_{t|t-1}}$$

the measurement equation of the extended Kalman filter can be written as follows:

$$\begin{pmatrix} S^{CDS}(t_k, t_k + T_1, \lambda(t_k)) \\ \vdots \\ S^{CDS}(t_k, t_k + T_{m_s}, \lambda(t_k)) \end{pmatrix} =$$
$$\begin{pmatrix} a^{CDS}(t_k, t_k + T_1) \\ \vdots \\ a^{CDS}(t_k, t_k + T_{m_s}) \end{pmatrix} + \begin{pmatrix} b^{CDS}(t_k, t_k + T_1) \\ \vdots \\ b^{CDS}(t_k, t_k + T_{m_s}) \end{pmatrix} \cdot \lambda(t_k) + \epsilon_\lambda(t_k)$$

with $\epsilon_\lambda(t_k) \sim \mathcal{N}_{m_s}(\mathbf{0}, h_\lambda^2 \cdot \mathbf{I}_{m_s})$.

6.3.2 An application to market data

Next, the estimation procedure is applied to a sample of market data between September 2004 and March 2007. The market data used in this study are European GDP growth rates, German sovereign yields as a proxy for risk-free interest rates, and iTraxx Europe CDS spreads. In addition to that, aggregated recovery rates of European small and medium-sized enterprises (SMEs) and large corporates are used.
Weekly German sovereign yields with maturities from 3 months to 10 years and quarterly GDP growth rates from Euro countries are used to estimate the parameters of the processes r and w. As the frequency of zero rates is higher than the frequency of the GDP growth rates, a cubic spline interpolation is applied to the quarterly GDP data to obtain a time series of the same length as the zero rates. Furthermore, average recovery rates of European SMEs and large corporates are used to estimate the parameters of the recovery-rate process as described in Subsection 6.3.1.2. Finally, iTraxx Europe CDS spreads with a maturity of 5 years (as these are the most liquid ones) are used to estimate the parameters of the default intensity according to Subsection 6.3.1.3. The parameter estimates of the short-rate, recovery, and intensity model are given in Table 6.1. As the process u is unobservable in this example, c_z is set equal to one. The estimated standard errors of the parameter estimates are obtained by a moving block bootstrapping procedure (see e.g. Lahiri (2003)). A block length of 26 weeks was chosen and then the blocks were randomly concatenated to obtain series with approximatively the same length as the respective original sample series. The standard error estimates given in Table 6.1 are the empirical standard deviations of the respective estimators in a total of 50 bootstrap replications.

	Parameter	Estimate	Std. Error
Short-rate process	a_r	0.0636	(0.0029)
	$\widehat{a}_r$	0.0635	(0.0027)
	σ_r	0.0053	(7.3e-05)
	b_{rw}	0.1397	(0.0273)
Market-factor process	θ_w	0.0093	(0.0033)
	a_w	0.6146	(0.0802)
	$\widehat{a}_w$	0.6140	(0.0801)
	σ_w	0.0017	(0.0026)
Recovery-rate process	b_z	0.6281	(0.0901)
	d_z	5.1494	(5.6202)
	θ_u	0.0135	(0.0258)
	a_u	0.1318	(0.2025)
	$\widehat{a}_u$	0.1472	(0.2137)
	σ_u	0.0554	(0.0099)
Intensity process	θ_λ	0.0076	(0.0008)
	a_λ	0.8601	(0.1825)
	$\widehat{a}_\lambda$	0.8596	(0.1824)
	σ_λ	0.0127	(0.0030)
	$b_{\lambda u}$	0.0001	(0.0001)
	$b_{\lambda w}$	0.1997	(0.0812)

Table 6.1: Estimates of short-rate, recovery, and intensity model.

6.3.3 Model validation

Before the performance of the model is evaluated, it is checked whether the model assumptions from Subsections 6.1.1 and 6.1.2 are fulfilled. Using the filtered time series and estimated parameter values,

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}) \text{ for } i \in \{r, w, u, \lambda\} \text{ and } k = 2, \ldots, n$$

are computed. These are supposed to be realisations of independent normally distributed random variables. Hence, each $(\Delta W_i(t_k))_{k=2,\ldots,n}$ is tested for autocorrelation and normal distribution.
To test for autocorrelation, a Ljung-Box test (see e.g. Box and Ljung (1978)) is performed. The null hypothesis of no autocorrelation up to lag $22 \approx 2\sqrt{n-1}$ is not rejected on a 5%-level for ΔW_r and ΔW_λ and rejected for ΔW_w and ΔW_u.
The null hypothesis that ΔW_i for $i \in \{r, w, u, \lambda\}$ are not realisations of normally distributed random variables is tested according to the test proposed by Bera and Jarque (1980). The test indicates that ΔW_w and ΔW_r are normally distributed. If the (in terms of absolute values) highest 5% of ΔW_u and ΔW_λ are removed, the normal distribution assumption can not be rejected anymore. Therefore, one can conclude that the assumption of a normal distribution is justified for ΔW_w and ΔW_r and adequate at least at the center of the distribution for ΔW_u and ΔW_λ (see also the QQ-plots and histograms in Figures 6.5 and 6.6).
Furthermore, in Section 6.1 the Wiener processes were assumed to be uncorrelated. To verify this assumption the empirical correlations of the processes $(\Delta W_i(t_k))_{k=2,\ldots,n}$ for $i \in \{r, w, u, \lambda\}$ are computed and a t-test for no correlation is performed. The test indicates that the processes ΔW_w and ΔW_r, ΔW_w and ΔW_λ, as well as ΔW_r and ΔW_u are uncorrelated, but also the other correlations are on a rather low level. Table 6.2 contains the empirical correlations of the processes ΔW_w, ΔW_r, ΔW_u, and ΔW_λ and the corresponding test statistics of a t-test with null hypothesis of no correlation. The test statistic T given by

$$T = \frac{\rho\sqrt{n-2}}{\sqrt{1-\rho^2}},$$

where ρ denotes the correlation, follows under the null hypothesis Student's t-distribution with $n-2$ degrees of freedom. Hence, the null hypothesis is rejected on a 5%-level if $|T| > 1.9782$.
Finally, it is also tested if the assumption of log-normally distributed recovery rates (Equation (6.3) with $a_z = 0$) is justified. Although the Jarque-Bera test

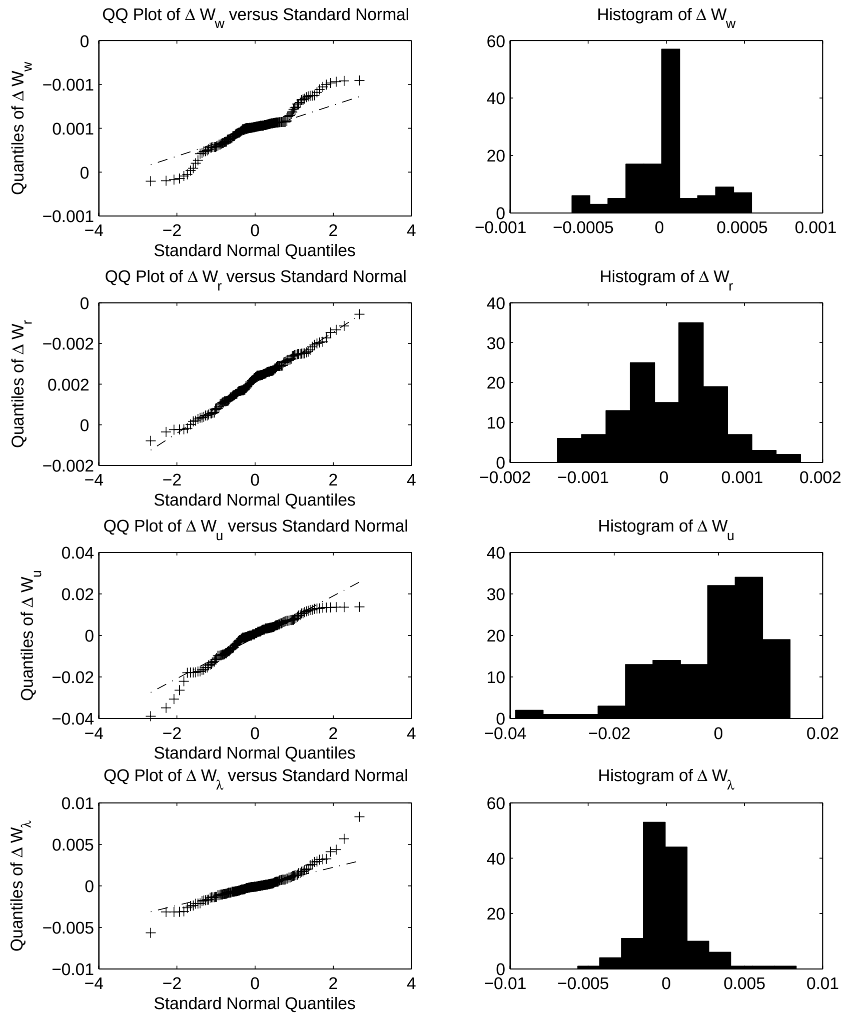

Figure 6.5: QQ-Plots and histograms of ΔW_i.

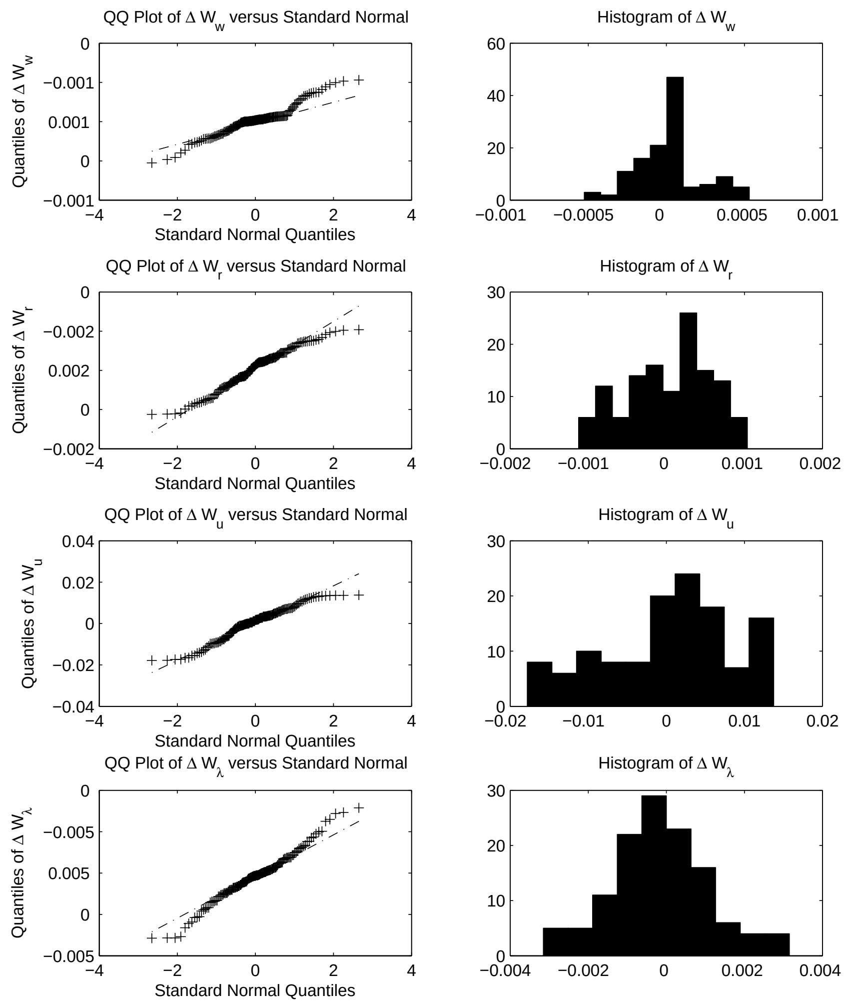

Figure 6.6: QQ-Plots and histograms of ΔW_i with outliers removed.

	ΔW_w	ΔW_r	ΔW_u	ΔW_λ
ΔW_w	1	0.0319 (0.3642)	0.2038 (2.3740)	-0.0487 (-0.5566)
ΔW_r	0.0319 (0.3642)	1	-0.0444 (-0.5069)	-0.2406 (-2.8259)
ΔW_u	0.2038 (2.3740)	-0.0444 (-0.5069)	1	-0.2271 (-2.6586)
ΔW_λ	-0.0487 (-0.5566)	-0.2406 (-2.8259)	-0.2271 (-2.6585)	1

Table 6.2: Empirical correlations and test statistics.

rejects the hypothesis of normal distribution for the logarithm of the recovery rates, the QQ-plot in Figure 6.7 indicates that the distributional assumption fits quite well in most parts of the distribution and that larger deviations appear only in the upper tail.

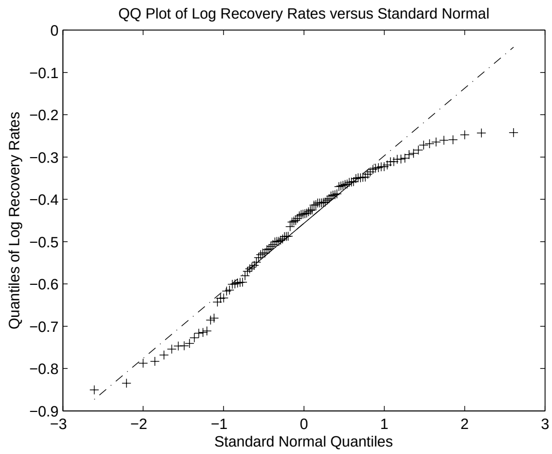

Figure 6.7: QQ-Plot for logarithm of aggregated recovery rates.

6.3.4 Model performance

After having validated the model assumptions, the model performance is investigated. For this, model and market prices/spreads as well as model and market price/spread movements are compared. The first is done by calculating the mean absolute and relative pricing error for each maturity, the latter by regressing the model price/spread movements on the market price/spread movements similar to Titman and Torous (1989).

The average absolute and relative deviations of the model prices of zero-coupon bonds and CDS spreads from the corresponding market prices are given in Table 6.3.

	Risk-free ZCB	CDS
Mean absolute error	0.01052	3.68e-05
Mean relative error	0.01374	0.01031

Table 6.3: Average pricing errors for risk-free zero-coupon bonds and CDS spreads.

Additionally, Figures 6.8 and 6.9 show a comparison of market and model zero rates and CDS spreads for a maturity of 5 years.

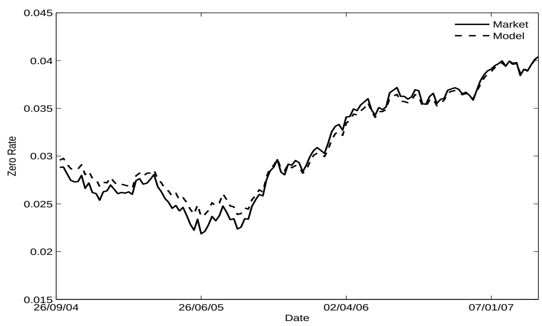

Figure 6.8: Market and model zero rates with 5 year maturity.

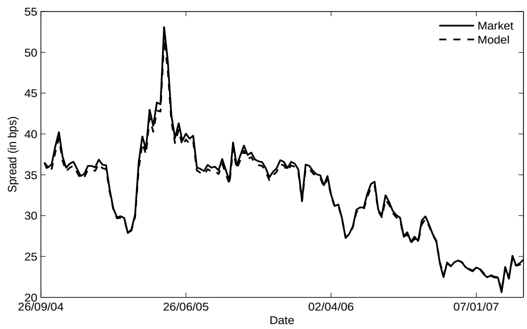

Figure 6.9: Market and model spread for 5-year iTraxx CDS spreads.

For further examination of the model performance, it is tested how well changes in market quotes can be explained by changes in model rates/spreads.

For this, let

$$\Delta R_k(T) := R(t_k, t_k + T) - R(t_{k-1}, t_{k-1} + T)$$

with $R(t,T) = -\frac{1}{T-t} \ln P^{nd}(t,T)$ and

$$\Delta S_k^{CDS}(T) := S^{CDS}(t_k, t_k + T) - S^{CDS}(t_{k-1}, t_{k-1} + T)$$

denote the changes in the zero rate and CDS spread with time to maturity T between t_{k-1} and t_k. The following regressions are performed:

$$\Delta R_k^{market}(T) = a_{R,T} + b_{R,T} \Delta R_k^{model}(T) + \epsilon_{R,T}$$

with $\epsilon_{R,T} \sim \mathcal{N}\left(0, h_{R,T}^2\right)$ and

$$\Delta S_k^{CDS,market}(T) = a_{S^{CDS},T} + b_{S^{CDS},T} \Delta S_k^{CDS,model}(T) + \epsilon_{S^{CDS},T}$$

with $\epsilon_{S^{CDS},T} \sim \mathcal{N}\left(0, h_{S^{CDS},T}^2\right)$.
For a good model one would expect $a_{\cdot,T}$ to be around 0, $b_{\cdot,T}$ around 1, and the coefficient of determination R^2 close to 1. For the interest-rate model, the hypothesis $a_{R,T} = 0$ is only rejected for very short maturities, and the hypothesis $a_{R,T} = 0$ and $b_{R,T} = 1$ is rejected for very short and very long maturities. The R^2 for maturities between 1 year and 10 years lies between 0.76 and 0.98 with an average R^2 of 0.91. Replacing the zero rate changes by absolute zero rates, even higher degrees of explanation for all maturities (between 0.87 and 0.99) can be achieved. For the CDS spreads the hypothesis $a_{S^{CDS},T} = 0$ can not be rejected on a 5%-level. The hypothesis $a_{S^{CDS},T} = 0$ and $b_{S^{CDS},T} = 1$ is rejected but the value for $b_{S^{CDS},T}$ is only slightly higher than 1 and the R^2 is over 0.98.
Finally, the empirical correlations of the historical and filtered time series of the processes w, r, u, and λ are computed (see Table 6.4). The signs of the correlations correspond to what would be expected according to many empirical studies, see e.g. Altman *et al.* (2004) or Schuermann (2004) who both report a negative correlation between recovery rates and default rates or Driessen (2005) who states that default-free interest rates and default intensities are negatively correlated.

6.4 Further applications

The modelling approach introduced in this chapter is not only restricted to the valuation of the above presented credit derivatives, but can also be used

	w	r	z	λ
w	1	0.7374	0.6384	-0.6708
r	0.7374	1	0.4989	-0.4143
z	0.6384	0.4989	1	-0.0962
λ	-0.6708	-0.4143	-0.0962	1

Table 6.4: Empirical correlations of historical (w, z) and filtered (r, λ) processes.

in further applications. As most of them rely on Monte Carlo techniques and are often individually customized, only the main ideas are sketched here.
In Chapters 4 and 5, it was pointed out that the value and type of a collateral position plays a major role in determining the amount recovered from a defaulted secured facility. Yet, the modelling approach presented in Section 6.1 aims at the valuation of unsecured debt positions and credit derivatives written on these positions, which is the case for most credit-risk pricing models. Exceptions with a stochastic collateral modelling in a structural-form model include the approaches presented in Cossin and Hricko (2003) and Jokivuolle and Peura (2003). To include the impact a collateral position might have in the framework presented above, the recovery specification from Equation (6.3) is only applied on that part of the debt which is not covered by the value of the collateral. Then, the recovery payment from a defaulted contingent claim with underlying collateral is given by

$$Z^{Coll}(t) = \min(C(t), F) + \max(F - C(t), 0)\, z(t),$$

where $C(t)$ denotes the value of the collateral at time t. Hence, the recovery-rate process in presence of collateral can be described by

$$\begin{aligned} z^{Coll}(t) &= \frac{\min(C(t), F) + \max(F - C(t), 0)\, z(t)}{F} \\ &= 1 - \frac{1 - z(t)}{F} \max(F - C(t), 0)\,, \end{aligned} \tag{6.14}$$

where the last factor in Equation (6.14) is the payoff of a put option on the collateral with strike equal to the face value F. In the most simple case of a constant or deterministic collateral value with $C(t) < F$, the pricing formulas developed in Section 6.2 remain the same, when the parameters a_z and b_z are replaced by

$$a_z^{Coll} = \frac{C(t)}{F} + \left(1 - \frac{C(t)}{F}\right) a_z$$

and

$$b_z^{Coll} = \left(1 - \frac{C(t)}{F}\right) b_z$$

respectively. In a model with stochastic collateral, $C(t)$ has to be specified in dependence of the type of collateral, for example by a geometric Brownian motion or any other positive stochastic process. Pricing within such an approach will in most cases rely on Monte Carlo or tree methods. The parameters of the collateral-value process can be estimated using information from the specific piece of collateral or a respective index, e.g. the S&P/Case-Shiller Home Price Index for real estate or the Dow Jones - AIG Commodity Index for commodity. Figure 6.10 shows an example of how the distribution of the recovery rates after 1 year changes if different types of collateral are taken into consideration. The upper plot contains a histogram of 10000 simulated recovery rates according to Equation (6.3) with parameter values as in Table 6.1. The other two plots show histograms of the recovery rates based on Equation (6.14), where $C(t)$ is in both cases assumed to be lognormally distributed. For the plot in the middle of Figure 6.10, the parameters of $C(t)$ have been fitted to a time series of the Dow Jones - AIG Commodity Index between 1991 and 2008 and an initial quota of collateral, i.e. $C(0)/F$, of 60% was assumed, while for the bottom plot the parameters have been estimated using a time series of the seasonally adjusted Composite 10 S&P/Case-Shiller Home Price Index between 1987 and 2008 and the initial quota of collateral was set equal to 90%. The quotas of collateral in this example have been chosen according to the results from Subsection 4.2.4. It can be easily seen that changing the assumptions about the collateral-value process influences the location of the recovery-rate distribution as well as its shape. Another interesting question from the field of risk management, which might be addressed in a such model with stochastic recovery and collateral, is the determination of collateral haircut, i.e. the amount of collateral required at initiation for a given face value to receive a desired degree of credit risk.

The presented framework can also be used to price more exotic credit derivatives like constant maturity CDS or credit default swaptions. Unlike in a standard CDS, the premium in a constant maturity CDS (CMCDS) is reset periodically, i.e. the spread is floating and not fixed. This makes the value of the CMCDS less sensitive to changes in the credit-spread level. The spread of a CMCDS is contingent on the spread of a reference entity with a constant maturity, e.g. the current 5-year CDS of a reference entity or the current 5-year CDS index. In many cases the premium of a CMCDS is expressed as a percentage of the reference spread, sometimes called gearing factor or participation rate. While the value of the default leg of a CMCDS is equiva-

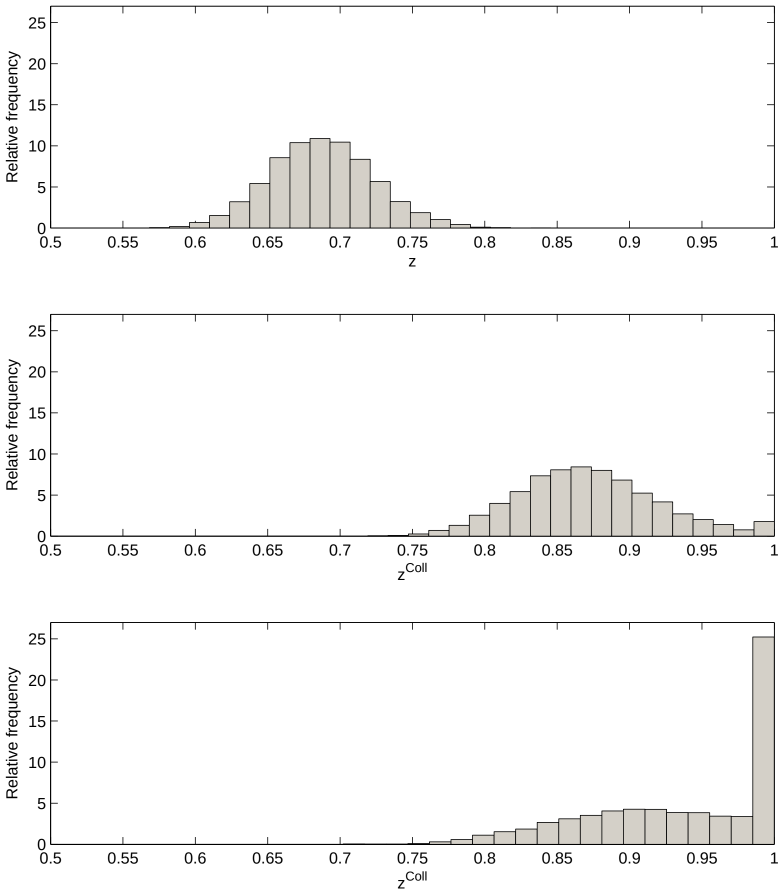

Figure 6.10: Histograms of simulated recovery rates after 1 year without collateral (top), with commodity as collateral (middle), and with real estate as collateral (bottom).

lent to the case of a standard CDS as in Equation (6.9), for the value of the premium leg Equation (6.10) changes to

$$V_{prem}^{CMCDS}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{m} gS^{CDS,ref}(t_i, T+t_i)\Delta t_i e^{-\int_t^{t_i}(r(l)+\lambda(l))dl}\middle|\mathcal{F}_t\right]$$

with g denoting the gearing factor and $S^{CDS,ref}(t_i, T+t_i)$ the time t_i CDS spread of the reference entity with time to maturity T, e.g. the current 5-year CDS spread at time t_i. More on CMCDS pricing can e.g. be found in Brigo (2006) or Jönsson and Schoutens (2009).
A credit default swaption is a contract which gives the holder the opportunity to enter a CDS with a fixed premium K at time t_0 which is usually equal to the initiation date of the CDS. Let $\mathcal{T} = \{t_1, \ldots, t_m\}$ denote the payment schedule of the CDS with $t_0 < t_1 < \ldots < t_m = T$, then the time t ($t \le t_0$) value of a payer default swaption is given by

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{t_0} r(l)dl}\max\left(V^{CDS}(t_0,T,K),0\right)\middle|\mathcal{F}_t\right]$$

with $V^{CDS}(t_0,T,K)$ denoting the time t_0 value of a CDS with maturity T and premium K given by

$$V^{CDS}(t_0,T,K) = 1_{\{\tau>t_0\}}\left[V_{def}^{CDS}(t_0,T) - K\sum_{i=1}^{m}\Delta t_i P^{d,zero}(t_0,t_i)\right].$$

Here, $V_{def}^{CDS}(t_0,T)$ is the value of the default leg from Equation (6.9) and $P^{d,zero}(t_0,t_i)$ the time t_0 price of a defaultable zero-coupon bond with maturity t_i under the assumption of zero recovery from Corollary 6.6. Good references for pricing credit default swaptions numerically are e.g. Brigo and Alfonsi (2005) and Jönsson and Schoutens (2008).

Chapter 7

Pricing distressed CDOs with stochastic recovery

Standard copula models for pricing collateralized debt obligations (CDOs) assume a constant recovery rate of 40%. While this assumption might work quite well in normal market situations, in distressed markets, as observed since the 2^{nd} half of 2007, this assumption is not justified anymore: First, standard copula models, like the Gaussian copula model introduced by Li (2000), often show a bad performance in times of high tranche spreads. Second, in 2008 it was temporarily not possible to calibrate the standard Gaussian base correlation model to the complete set of CDX and iTraxx tranche quotes. And finally, non-standardized super senior tranches ($60\% - 100\%$) have a fair spread of zero in standard market models while being traded on the market with a positive spread of up to 25 bps during distressed market situations.

Nevertheless, only very few CDO models with stochastic recovery exist. The first one was introduced in Andersen and Sidenius (2004). In this article an extension to the Gaussian copula model is presented by assuming a stochastic recovery related to the systematic factor driving the default events, explicitly allowing for an inverse correlation between recovery rates and default rates. To be more precise, the recovery rate of an obligor in case of a default in this model is given by an application of the normal cumulative distribution function on a normally distributed random variable which is correlated with the default triggering variable through a common systematic factor. In a numerical examination the authors noted that the base correlation skew effect of random recovery is quite minor and hence the random recovery approach was not further investigated. Due to the credit market crisis, recently some articles on using stochastic recovery rates in CDO pricing have been published. Krekel (2008) uses a discrete stochastic recovery rate in a Gaussian

base correlation setting to overcome the problem that super senior tranches in a standard Gaussian base correlation model have zero fair spread. In this model the discrete recovery rates are defined as constants on buckets of the default triggering factors, i.e. the recovery rate is a step function of the default triggering variable. In the empirical part a recovery rate distribution with only four possible realizations (60%, 40%, 20%, and 0%) is used. Amraoui and Hitier (2008) extend the approach of Krekel (2008) by modelling the recovery rate as a deterministic function of the systematic risk factor of the default triggering variable. Ech-Chatbi (2008) uses a multiple default approach (similar to Section 6.1.3 in Schönbucher (2003)), where the recovery is lowered by a random factor each time a default event occurs. Hence, the recovery rate process is some geometric compound Poisson process where the current recovery rate is multiplied by a random variable, e.g. beta distributed or log-gamma distributed, each time a default event occurs. One feature that all these models have in common is that they rely on the assumption of a Gaussian copula, which might not be appropriate, especially in distressed market situations, as Gaussian copulas don't support tail dependences.
The aim of this chapter is a joint modelling of default and recovery risk in a portfolio of credit risky assets, especially accounting for the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand. Nested Archimedean copulas as proposed in Hofert and Scherer (2009) are used to model different dependence structures. However, this concept is not applied to model different default dependences for firms in the same sector and firms from different sectors as in Hofert and Scherer (2009), but rather to model dependences between default triggers (inner dependence) as well as between default triggers and loss triggers (outer dependence). Furthermore, a very flexible continuous recovery-rate distribution with bounded support on $[0, 1]$ is chosen, which allows for an efficient sampling of the loss process. This is especially important as in most cases the loss process distribution will not be given in closed form.

7.1 Portfolio credit derivatives

7.1.1 Modelling framework

In the following a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is assumed, where $\mathbb{Q}$ denotes some given pricing measure and $\mathcal{F} \subset \mathcal{G}$ with $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ similar to Chapter 6. Consider a portfolio of I credit risky assets with payment streams depending both on the default status and on the recovery rate in case of default.

The default times τ_i, $i = 1, \ldots, I$, are assumed to follow a default intensity model with $\lambda_i(t)$ denoting the $\mathcal{F}_t$-measurable default intensity of firm i at time t. The survival probabilities in this model are given by

$$p_i(t) := \mathbb{Q}(\tau_i > t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int\limits_0^t \lambda_i(s)ds}\right] \tag{7.1}$$

and the default probabilities are denoted by $\bar{p}_i(t) = 1 - p_i(t)$. According to p.183 of Bielecki and Rutkowski (2004) or p.122 of Schönbucher (2003), τ_i can be constructed in the canonical way as follows: Let U_i^D denote a random variable uniformly distributed on $[0,1]$ and independent of $\mathcal{F}$, then

$$\tau_i \stackrel{d}{=} \inf\left\{t > 0 : e^{-\int_0^t \lambda_i(s)ds} \leq U_i^D\right\}. \tag{7.2}$$

The recovery rates z_i in case of a default event are given by $1 - LGD_i$, where the losses given default, LGD_i, are assumed to be identically distributed according to a distribution function F with support $[0,1]$; if no default has taken place, z_i is set equal to 1. This choice is arbitrary as the recovery rate is only of importance for defaulted assets. Hence, the recovery rate of firm i is given by

$$z_i = \begin{cases} 1 - LGD_i & \text{if } \tau_i \leq T \\ 1 & \text{else} \end{cases}.$$

Assuming U_i^L to be uniformly distributed on $[0,1]$ and independent of $\mathcal{F}$, one can set $LGD_i = F^{\leftarrow}(U_i^L)$ and $z_i = h(U_i^L)$ with $F^{\leftarrow} := \inf\{z : F(z) \geq x\}$ denoting the quantile function of F and

$$h(x) = \begin{cases} 1 - F^{\leftarrow}(x) & \text{if } \tau_i \leq T \\ 1 & \text{else} \end{cases}.$$

Setting e.g. $F^{\leftarrow}(x) \equiv 0.6$ for $x \in (0,1)$ leads to the standard case of a constant recovery of 40%. To introduce dependence between different firms on the one hand and between recovery rates and default rates on the other hand, a certain dependence structure between the trigger variables U_i^D, U_i^L, $i = 1, \ldots, I$, is assumed. The joint distribution of the vector $\left(U_1^D, \ldots, U_I^D, U_1^L, \ldots, U_I^L\right)$ is assumed to be given by some copula C, i.e. $\left(U_1^D, \ldots, U_I^D, U_1^L, \ldots, U_I^L\right) \sim C$. In the standard Gauss copula model with constant recovery as introduced in

Li (2000) C would simply be a Gaussian copula with correlation matrix

$$\Sigma = \begin{pmatrix} 1 & & \rho & & & \\ & \ddots & & & 0 & \\ \rho & & 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

correlation coefficient $\rho \geq 0$, and $F^{\leftarrow}(x) \equiv 0.6$ for $x \in (0,1)$.
To allow for different dependence hierarchies, C is chosen to belong to the class of nested Archimedean copulas in what follows. The concept of pricing CDO tranches using exchangeable Archimedean copulas was introduced in Schönbucher and Schubert (2001). Hofert and Scherer (2009) extended the approach by using nested Archimedean copulas to introduce different default correlations for different industry sectors. This concept will be extended to allow for additional dependence between default rates and recovery rates in what follows.
The most important quantity for pricing portfolio-credit derivatives as well as for risk management purposes is the portfolio-loss process $L(t)$ which can be easily derived once the default process, loss given default, and nominal of each asset in the portfolio are known. Unfortunately, the distribution of the loss process is generally not known in closed form. In some cases, e.g. exchangeable Archimedean copulas with constant recovery (see Proposition 10.7 in Schönbucher (2003)), the portfolio-loss distribution can be approximated by a conditional independence approach. Nevertheless, as long as the occurring processes can be sampled efficiently, it is possible to price portfolio-credit derivatives via a Monte Carlo approach.

7.1.2 Portfolio CDS and CDO tranches

In the following, the payment streams and pricing formulas of portfolio CDS and CDOs are presented. To accomplish this, consider a portfolio of I obligors, where each obligor contributes the same amount to the notional. For ease of notation a notional of 1 is assumed in what follows. The time to maturity is denoted by T. For the pricing of CDS and CDOs the payment streams of two different legs, the premium and the default leg, have to be considered. Premium payments are made at predefined dates given by the payment schedule $\mathcal{T} = \{0 < t_1 < \ldots < t_n = T\}$. Note that a default event can happen at any point in time in the interval $[0, T]$. To simplify computations, all default payments between two premium payment dates are deferred to the next scheduled payment date. Furthermore, to account for accrued interest, defaults are assumed to happen in the middle of two scheduled payment dates, i.e. at $(t_{k-1}+t_k)/2$ for a default event in $[t_{k-1}, t_k)$ with $k = 1, \ldots, n$ and $t_0 = 0$. Both portfolio CDS and CDO tranches are credit derivatives with the portfolio-loss process $L(t)$ given by

$$L(t) = \frac{1}{I}\sum_{i=1}^{I} \mathbb{1}_{\{\tau_i \le t\}} LGD_i, \;\; t \in [0, T]$$

as underlying. The basic idea of a CDO is to pool the risky assets and resell them in slices, the so-called CDO tranches. The loss affecting tranche $j \in \{1, \ldots, J\}$ with lower and upper attachment points l_j and u_j, $0 = l_1 < u_1 = l_2 < \ldots < u_{J-1} = l_J < u_J \le 1$, is given by

$$L_j(t) = \min\{\max\{0, L(t) - l_j\}, u_j - l_j\}, \;\; t \in [0, T].$$

With each default the nominal of the portfolio of obligors is reduced by $1/I$, i.e. the remaining nominal of the portfolio CDS is

$$N(t) = 1 - \frac{1}{I}\sum_{i=1}^{I} \mathbb{1}_{\{\tau_i \le t\}}, \;\; t \in [0, T].$$

For tranche $j \in \{1, \ldots, J\}$ the remaining nominal is given by

$$N_j(t) = u_j - l_j - L_j(t), \;\; t \in [0, T].$$

With s_T^{pCDS} denoting the annualized portfolio-CDS spread, $r(t)$ the non-defaultable short rate, and $\Delta t_k = (t_k - t_{k-1})$, $k = 1, \ldots, n$, the time between two subsequent scheduled payment dates, the expected discounted premium and default leg of a portfolio CDS are given by

$$\begin{aligned} & EDPL_T(s_T^{pCDS}) \\ & = \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} s_T^{pCDS} \Delta t_k \left(N(t_k) + \left(N(t_{k-1}) - N(t_k)\right)/2\right)\right] \end{aligned} \tag{7.3}$$

and

$$EDDL_T = \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} \left(L(t_k) - L(t_{k-1})\right)\right] \tag{7.4}$$

respectively. For the j-th tranche of a CDO, the expected discounted premium and default leg are given by

$$
\begin{aligned}
&EDPL_{T,j}(s_{T,j}^{CDO}) \\
&= \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} s_{T,j}^{CDO} \Delta t_k \left(N_j(t_k) + \left(N_j(t_{k-1}) - N_j(t_k)\right)/2\right)\right] \qquad (7.5)
\end{aligned}
$$

with $s_{T,j}^{CDO}$ denoting the annualized spread of tranche j and

$$
EDDL_{T,j} = \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} \left(L_j(t_k) - L_j(t_{k-1})\right)\right] \qquad (7.6)
$$

respectively. It is market standard to quote the equity tranche, i.e. the most subordinated tranche, with an upfront payment (percentage of the nominal) and a fixed running spread of 500 bps, i.e.

$$
\begin{aligned}
&EDPL_{T,1}(s_{T,1}^{CDO}) = s_{T,1}^{CDO}(u^1 - l^1) \\
&+ \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} 0.05 \Delta t_k \left(N_1(t_k) + \left(N_1(t_{k-1}) - N_1(t_k)\right)/2\right)\right],
\end{aligned}
$$

where $s_{T,1}^{CDO}$ denotes the upfront payment.
The fair spreads of both portfolio CDS and CDO tranches can now be computed by equating the expected discounted premium and default leg and solving for the spread.

7.1.3 Homogeneous portfolio approximation

Assuming a homogeneous portfolio with deterministic intensities in the modelling framework from Section 7.1.1 and using the properties of Archimedean copulas, the following results can be stated (for the first result see also Section 10.8 of Schönbucher (2003)).

Theorem 7.1. *Let C denote the Archimedean copula of the random vector $\left(U_1^D, \ldots, U_I^D, U_1^L, \ldots, U_I^L\right)$ and $\lambda_i(t) = \lambda_j(t)$, $i, j \in \{1, \ldots, I\}$, $t \in [0, T]$, the default intensities from Equation (7.1).*

1. *The default correlation between any two firms k and l is given by*

$$
\begin{aligned}
\rho(t) &= Cor(\mathbb{1}_{\{\tau_k \le t\}}, \mathbb{1}_{\{\tau_l \le t\}}) \\
&= \frac{C_{k,l}(p(t), p(t)) - p^2(t)}{p(t)(1 - p(t))} = \frac{\varphi^{-1}\left[2\varphi(p(t))\right] - p^2(t)}{p(t)(1 - p(t))},
\end{aligned}
$$

where $C_{k,l}$ is the (k,l)-th bivariate margin of C with φ denoting the respective generator.[19] *For copulas with existing upper tail dependence parameter λ_U (see Definition 2.28), the default correlation $\rho(t)$ converges to λ_U for $t \to 0$.*

2. *The expected portfolio loss up to time T is given by*

$$\mathbb{E}_{\mathbb{Q}}\left[L(T)\right] = \bar{p}(T)(1 - z_{const})$$

with z_{const} denoting the expected recovery in case of default.

3. *The portfolio-loss process distribution can be approximated by*

$$\begin{aligned}&\mathbb{Q}(L(t) \leq x)\\ &\approx \int\int\int \mathbb{1}_{\{\bar{p}(t,V,V^D)\cdot\mathbb{E}_{\mathbb{Q}}[LGD_1|V,V^L]\leq x\}} dG(V)dG^D(V^D)dG^L(V^L)\end{aligned}$$

with

$$\bar{p}(t, V, V^D) = 1 - e^{-V^D\varphi_h(e^{-V\varphi_0(p(t))})}$$

and $V \sim G$, $V^D \sim G^D$, and $V^L \sim G^L$ denoting the mixing variables from Algorithm 2.37 and the call of Algorithm 2.34 in Algorithm 2.37 respectively.

Proof.

1. The default correlation of two obligors k and l is given by

$$\begin{aligned}\rho(t) &= Cor(\mathbb{1}_{\{\tau_k\leq t\}}, \mathbb{1}_{\{\tau_l\leq t\}})\\ &= \frac{\mathbb{Q}\left(\tau_k \leq t, \tau_l \leq t\right) - \bar{p}^2(t)}{\bar{p}(t)(1-\bar{p}(t))}\\ &= \frac{\mathbb{Q}\left(U_k^D \geq p(t), U_l^D \geq p(t)\right) - (1-p(t))^2}{p(t)(1-p(t))}\\ &= \frac{C_{k,l}(p(t),p(t)) + 1 - 2p(t) - (1-p(t))^2}{p(t)(1-p(t))}\\ &= \frac{C_{k,l}(p(t),p(t)) - p^2(t)}{p(t)(1-p(t))}\\ &\overset{\text{Def.2.29}}{=} \frac{\varphi^{-1}\left[2\varphi(p(t))\right] - p^2(t)}{p(t)(1-p(t))}.\end{aligned}$$

[19]For k and l from the same subgroup h it holds that $\varphi = \varphi_h$, while for k and l from different subgroups it holds that $\varphi = \varphi_0$.

Further, using $p(t) \overset{t\to 0}{\to} 1$ yields (see Equation (2.17))

$$\lim_{t\to 0}\rho(t) = \lim_{t\to 0}\underbrace{\frac{C_{k,l}(p(t),p(t))+1-2p(t)}{p(t)(1-p(t))}}_{\to\lambda_U} - \frac{1-p(t)}{p(t)} = \lambda_U.$$

2. Using the fact that the expected recovery in case of a default event is a constant equal for all obligors i, i.e. $\mathbb{E}\left[R_i|\tau_i \le T\right] = R_{const}$, one obtains

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[L(T)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{I}\sum_{i=1}^{I}\mathbb{1}_{\{\tau_i\le T\}}LGD_i\right]\\ &= \frac{1}{I}\sum_{i=1}^{I}\mathbb{Q}\left(\tau_i\le T\right)\underbrace{\mathbb{E}_{\mathbb{Q}}\left[1-z_i|\tau_i\le T\right]}_{1-z_{const}}\\ &= \bar{p}(T)(1-z_{const})\end{aligned}$$

with z_{const} denoting the expected recovery in case of default.

3. Applying the conditional independence approach as used e.g. in Theorem 10.5 in Schönbucher (2003) results in

$$\begin{aligned}&\mathbb{Q}(L(t)\le x)\\ &= \int\int\int \mathbb{Q}\left(L(t)\le x|\, V, V^D, V^L\right) dG(V)dG^D(V^D)dG^L(V^L).\end{aligned}$$

with $L(t) = \frac{1}{I}\sum_{i=1}^{I}\mathbb{1}_{\{\tau_i\le t\}}LGD_i$, $V\sim G$ denoting the mixing variable from step 1 in Algorithm 2.37 and $V^D\sim G^D$ and $V^L\sim G^L$ the mixing variables from the call of Algorithm 2.34 in step 2 in Algorithm 2.37. Conditioned on the mixing variables V, V^D, and V^L the triggering variables U_i^D, U_i^L, $i=1,\ldots,I$, are independent and so are τ_i and LGD_i, $i=1,\ldots,I$. Using the law of large numbers, yields for $I\to\infty$

$$\frac{1}{I}\sum_{i=1}^{I}\mathbb{1}_{\{\tau_i\le t\}}LGD_i \overset{\mathbb{Q}}{\to} \bar{p}(t,V,V^D)\mathbb{E}_{\mathbb{Q}}\left[LGD_1|V,V^L\right]$$

with $\bar{p}(t, V, V^D)$ denoting the conditional default probability. Hence,

$$\begin{aligned}
&\int\int\int \mathbb{Q}\left(L(t) \leq x|\, V, V^D, V^L\right) dG(V)dG^D(V^D)dG^L(V^L) \\
&\approx \int\int\int \mathbb{1}_{\{\bar{p}(t,V,V^D)\cdot\mathbb{E}_{\mathbb{Q}}[LGD_1|V,V^L]\leq x\}} dG(V)dG^D(V^D)dG^L(V^L).
\end{aligned}$$

According to Algorithm 2.37,

$$U_{hk} = \varphi_0^{-1}\left(-\frac{1}{V}\log\left(\varphi_h^{-1}\left(-\frac{1}{V^D}\log\left(X_k\right)\right)\right)\right)$$

with $X_k \sim Unif[0,1]$. Hence,

$$\begin{aligned}
p(t, v, v^D) &= \mathbb{Q}\left[\tau_k > t|V = v, V^D = v^D\right] \\
&= \mathbb{Q}\left[U_{hk} \leq p(t)|V = v, V^D = v^D\right] \\
&= \mathbb{Q}\left[\varphi_0^{-1}\left(-\frac{1}{v}\log\left(\varphi_h^{-1}\left(-\frac{1}{v^D}\log\left(X_k\right)\right)\right)\right) \leq p(t)\right] \\
&= \mathbb{Q}\left[X_k \leq e^{-v^D\varphi_h\left(e^{-v\varphi_0(p(t))}\right)}\right] \\
&= e^{-v^D\varphi_h\left(e^{-v\varphi_0(p(t))}\right)}.
\end{aligned}$$

□

Unfortunately, for most choices of the recovery-rate distribution, there will be no closed-form approximation for the loss distribution. Nevertheless, if efficient sampling strategies from the recovery-rate distribution are known, Monte Carlo techniques can be applied to price derivatives of the loss process. In the following it is assumed that C belongs to a parametric family of (nested) Archimedean copulas with parameter vector $\theta = (\theta_0, \theta_1, \ldots, \theta_H)$, $H \geq 0$, and θ_l, $l = 0, \ldots, H$, denoting the parameter corresponding to the generator φ_l.

7.1.4 Recovery-rate distribution

As long as there are no liquidly traded credit derivatives on pure default-event risk, e.g. digital default swaps, or on pure recovery risk, e.g. recovery swaps, it is not possible to infer default intensities from credit derivatives without making an assumption on recovery rates. Therefore, while choosing the recovery-rate distribution, one has to ensure that the assumptions made in the pricing of correlation-insensitive credit derivatives used for the determination of default intensities are not violated in the model for correlation-sensitive products. For this, the average recovery rate is assumed to equal the constant recovery rate, z_{const}, used for bootstrapping the default intensities from portfolio-CDS quotes, i.e.

$$\mathbb{E}_{\mathbb{Q}}\left[z_i|\tau_i \leq T\right] = z_{const}. \tag{7.7}$$

Using standard assumptions this means that the recovery distribution conditioned on default has to have an expectation of e.g. 40%. Otherwise the model is not consistent with the portfolio-CDS pricing.
The beta distribution, which is often used for recovery rates / loss given defaults (see e.g. Section 6.1 of Schönbucher (2003), Schuermann (2004), or Gupton and Stein (2005)), is numerically too expensive for Monte Carlo pricing (for the cdf and inverse cdf the incomplete beta function has to be evaluated numerically). Therefore, the Kumaraswamy distribution (see Kumaraswamy (1980)), which is a special case of McDonald's generalized beta distribution (see e.g. McDonald (1984) or Johnson *et al.* (1995)) and more suitable for a Monte Carlo applications, is chosen as marginal distribution for the loss given default. Its density is given by

$$f_{Kum}(x) = abx^{a-1}\left(1-x^a\right)^{b-1}, \tag{7.8}$$

where $0 \leq x \leq 1$ and $a, b > 0$. The Kumaraswamy distribution has similar properties as the beta distribution: it is a continuous distribution with bounded support showing a high degree of flexibility by supporting bimodal as well as unimodal and skewed, J-shaped, U-shaped, and uniform densities (see Figure 7.1).

In contrast to the beta distribution it has a closed-form cdf and inverse cdf given by

$$F_{Kum}(x) = 1-\left(1-x^a\right)^b$$

and

$$F^{-1}_{Kum}(x) = \left(1-(1-x)^{1/b}\right)^{1/a}. \tag{7.9}$$

Other possible continuous distributions with bounded support for the loss given default that show a similar degree of flexibility would be the two-sided power distribution as proposed by Kotz and van Dorp (2004) or the Vasicek distribution (see Vasicek (1991)). The two-sided power distribution has similar properties as the Kumaraswamy distribution but a peak at its mode (antimode), whereas the Vasicek distribution has the drawback that for its cdf and inverse cdf the normal cdf and inverse cdf have to be evaluated numerically.

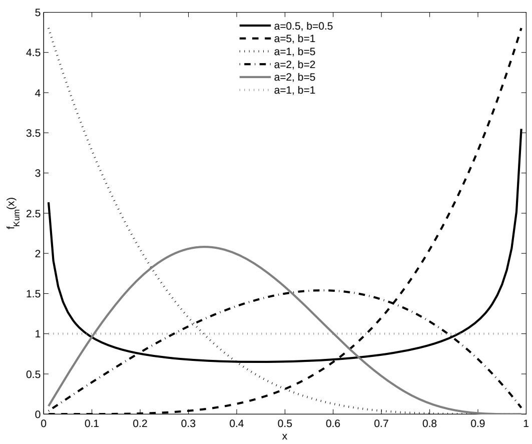

Figure 7.1: Kumaraswamy densities for different parameter constellations.

To ensure that condition (7.7) is fulfilled the parameters a and b have to be chosen such that the expectation of the recovery rate conditioned on default is equal to 40%, i.e. the expectation of the loss given default conditioned on default is equal to 60%. Therefore, set (see McDonald (1984))

$$\mathbb{E}_{\mathbb{Q}}\left[LGD_i | \tau_i \leq T\right] = bB\left(1 + \frac{1}{a}, b\right) = 0.6, \tag{7.10}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denotes the incomplete beta function. The remaining degree of freedom can be resolved e.g. by choosing the second parameter such that the recovery distribution matches the empirically observed standard deviation (e.g. 20% as in Altman and Kishore (1996)), i.e.

$$\sigma_{LGD} := \sqrt{\mathbb{V}ar\left[LGD_i | \tau_i \leq T\right]} = \sqrt{bB\left(1 + \frac{2}{a}, b\right) - b^2 B\left(1 + \frac{1}{a}, b\right)^2} = 0.2. \tag{7.11}$$

This leads to a Kumaraswamy distribution with parameters $a = 2.65$ and $b = 2.13$ for the loss given default with density as given in Figure 7.2.

By choosing other parameter constellations that fulfil Equation (7.10), one can vary the shape of the distribution. Another possibility would be to calibrate the second parameter to the market quotes of CDO tranches.

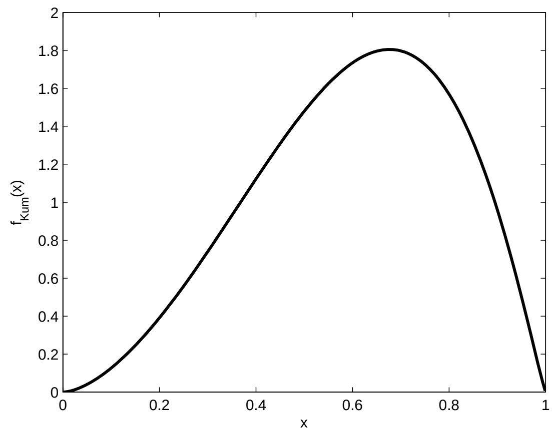

Figure 7.2: Density of Kumaraswamy distribution with parameters $a = 2.65$ and $b = 2.13$.

7.2 Model calibration and empirical results

In this section, a procedure to calibrate the model to market data is presented and a numerical example of the model's fitting capability using iTraxx Europe data is given. First of all, portfolio CDS and CDO tranches are priced using the results from Section 7.1.

7.2.1 Pricing and calibration algorithm

Assuming deterministic discount factors, Equations (7.3) and (7.4) only require the computation of the expected portfolio loss and the expected remaining nominal at time t, which are given by

$$\mathbb{E}_{\mathbb{Q}}\left[L(t)\right] = \frac{1 - z_{const}}{I} \sum_{i=1}^{I} \bar{p}_i(t) \text{ and } \mathbb{E}_{\mathbb{Q}}\left[N(t)\right] = \frac{1}{I} \sum_{i=1}^{I} p_i(t)$$

with z_{const} denoting the constant recovery used for bootstrapping default intensities. Therefore, the model spread of a portfolio CDS can be obtained by equating Equations (7.3) and (7.4) and solving for s_T^{pCDS}, i.e.

$$s_T^{pCDS,model} = \frac{EDDL_T}{EDPL_T(1)}. \tag{7.12}$$

Here, the linearity of the premium and the default leg in the loss process is used. Unfortunately, premium and default leg for a CDO tranche are non-linear functions in the loss process. Therefore, a straightforward calculation of the expected discounted premium and default leg is not possible. Nevertheless, the following algorithm shows how to price CDO tranches in the modelling framework from Section 7.1 via Monte Carlo simulation.

Algorithm 7.2. *Pricing CDO tranches via Monte Carlo simulation.*

1. *Let T denote the maturity, $\mathcal{T} = \{0 < t_1 < \ldots < t_n = T\}$ the payment schedule, t_0 the initiation of the contract, $\lambda_i(t)$ the default intensity of firm i, I the number of firms, z_{const} the constant recovery rate used for bootstrapping default intensities from portfolio-CDS spreads, and $r(t)$ the non-defaultable short rate. Choose the number of simulation runs M, the attachment and detachment point l_j and u_j of tranche j, and a copula C with parameter vector θ.*

2. *Monte Carlo simulation:*

   (a) *Sample $\lambda_i^{(m)}(t_k)$ and compute $\Lambda_i^{(m)}(t_k) = e^{-\sum_{l=0}^{k-1}\lambda_i^{(m)}(t_l)(t_{l+1}-t_l)}$ for $k = 0, \ldots, n$, $i = 1, \ldots, I$, and $m = 1, \ldots, M$.*

   (b) *Sample $U = (U_{i,m})_{i=1,\ldots,2I,m=1,\ldots,M} \in [0,1]^{2I\times M}$, where each column $U_{\cdot,m}$ of U is a sample of the chosen copula C.*

   (c) *Compute the default times $\tau_i^{(m)}$ for each firm $i \in \{1, \ldots, I\}$ and Monte Carlo run $m \in \{1, \ldots, M\}$ via Equation (7.2) using $\Lambda_i^{(m)}(t_k)$ and $U_{i,m}$ from steps 2(a) and (b).*

   (d) *Compute the loss given default for each defaulted firm via*

   $$LGD_i^{(m)}|_{\tau_i^{(m)}\le T} = F_{Kum}^{-1}\left(\widetilde{F}_{U_{I+i,\cdot|\tau_i^{(\cdot)}\le T}}\left(U_{I+i,m|\tau_i^{(m)}\le T}\right)\right) \tag{7.13}$$

   with $\widetilde{F}_{U_{I+i,\cdot|\tau_i^{(\cdot)}\le T}}$ denoting the empirical cdf of $U_{I+i,\cdot|\tau_i^{(\cdot)}\le T}$ and F_{Kum}^{-1} the inverse cdf of the Kumaraswamy distribution from Equation (7.9). Here, $U_{I+i,\cdot|\tau_i^{(\cdot)}\le T}$ denotes the subsample of all $U_{I+i,m}$ for which $\tau_i^{(m)} \le T$.

   (e) *Define the recovery rate of firm i as*

   $$z_i^{(m)} = \begin{cases} 1 - LGD_i^{(m)}|_{\tau_i^{(m)}\le T} & \text{if } \tau_i^{(m)} \le T \\ 1 & \text{else} \end{cases}. \tag{7.14}$$

(f) *Compute the loss process* $L^{(m)}(t_k)$ *for* $t_k \in \mathcal{T}$ *according to*

$$L^{(m)}(t_k) = \frac{1}{I}\sum_{i=1}^{I} \mathbb{1}_{\{\tau_i^{(m)} \leq t_k\}}(1 - z_i^{(m)}).$$

(g) *Compute for each tranche* $j \in \{1, \ldots, J\}$ *and each* $t_k \in \mathcal{T}$ *the loss affecting tranche* j, $L_j^{(m)}(t_k)$, *and the remaining nominal in tranche* j, $N_j^{(m)}(t_k)$, *via*

$$L_j^{(m)}(t_k) = \min\{\max\{0, L^{(m)}(t_k) - l_j\}, u_j - l_j\}$$

and

$$N_j^{(m)}(t_k) = u_j - l_j - L_j^{(m)}(t_k).$$

3. *Compute for each tranche* $j \in \{1, \ldots, J\}$ *and each Monte Carlo run the discounted premium and default legs and estimate their expectations* $EDPL_{T,j}(1)$ *and* $EDDL_{T,j}$ *from Equations (7.5) and (7.6) by their sample means* $\overline{EDPL}_{T,j}(1)$ *and* $\overline{EDDL}_{T,j}$.
Determine the fair spread of tranche $j \in \{2, \ldots, J\}$ *via*

$$\widehat{s}_{T,j}^{CDO,model} = \frac{\overline{EDDL}_{T,j}}{\overline{EDPL}_{T,j}(1)}$$

and $\widehat{s}_{T,1}^{CDO,model}$ *via*

$$\begin{aligned}\widehat{s}_{T,1}^{CDO,model} &= \frac{1}{u_1 - l_1}\Big[\, \overline{EDPL}_{T,1}(1) \\ &- \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} e^{-\int_0^{t_k} r(l)dl} 0.05\Delta t_k \left(N_1(t_k) + \left(N_1(t_{k-1}) - N_1(t_k)\right)/2\right)\right]\Big].\end{aligned}$$

With these pricing routines, the model can now be calibrated to market quotes of portfolio-CDS spreads $s_T^{pCDS,market}$ and CDO tranche spreads $s_{T,j}^{CDO,market}$, $j = 1, \ldots, J$. Since default intensities are specified independently from the dependence structure in the modelling framework from Section 7.1, the default-intensity process can be fitted to portfolio-CDS quotes in a first step and then the copula parameter vector θ can be fitted to the dependence structure induced by market quotes of CDO tranches. The former can be done by adjusting the default-intensity parameters such that the model spread in Equation (7.12) equals the market spread. For the latter the difference in model and market CDO tranche spreads has to be minimized over the Copula parameter vector θ. Since the equity tranche is quoted in terms of an upfront payment and not in terms of a running spread as the other tranches, care has to be taken when comparing pricing errors of different tranches. Accounting for this and the fact that the highest spread is paid for the equity tranche, the model is calibrated by minimizing the deviation of market and model spreads for tranches $j = 2, \dots, J$ provided that the model upfront payment matches the market upfront payment up to a certain accuracy ϵ reflecting bid-ask spreads, e.g. $\epsilon = 10^{-4}$ (see e.g. Hofert and Scherer (2009)). Therefore, the following optimization problem has to be solved

$$\begin{aligned} \min_{\theta} \; D_2 &:= \sum_{j=2}^{J} \left| s_{T,j}^{CDO,model} - s_{T,j}^{CDO,market} \right| \qquad (7.15) \\ \text{s.t. } D_1 &:= \left| s_{T,1}^{CDO,model} - s_{T,1}^{CDO,market} \right| \leq \epsilon, \end{aligned}$$

where the minimization is taken over the involved copula parameter vector θ.

7.2.2 Calibration results

The following numerical example shows that already in a very simple form the model fits market data quite well and leads to significantly smaller pricing errors compared to a standard Gaussian copula model. The following simplifying assumptions are made:

As the main focus of this chapter is on the modelling of recovery rates, the dependence between defaults, and the dependence between default rates and recovery rates, a homogeneous portfolio with constant default intensities $\lambda_i(t) = \lambda$ is assumed. Of course this framework could be easily generalized to the case of stochastic default intensities as in Chapter 6.

Furthermore, three different parametric copula families with parameter vector θ will be compared for modelling dependence between default and loss triggers: Gaussian (Ga), Gumbel (Gu), and outer power Clayton (opC) copula. Each copula is tested in its exchangeable form with constant recoveries and in its nested form with stochastic recoveries. For the latter this means in terms of Section 2.5.2, $H = 2$, $d_1 = d_2 = 125$ (iTraxx standard), $\theta = (\theta_0, \theta_1, \theta_2)$, where the copula generators are given by $\varphi_0(\cdot; \theta_0)$, $\varphi_0(\cdot; \theta_1)$, and $\varphi_0(\cdot; \theta_2)$. Gumbel and outer power Clayton (with additional parameter θ_c) have been chosen from the family of Archimedean copulas, since they have shown consistently the best fitting results in the case of constant recovery rates (see Hofert and Scherer (2009)). These two copulas are compared to the Gaussian copula as a benchmark which is still some kind of market standard. Table 7.1 contains the parameter ranges, generator functions and inverses, and lower and upper tail dependence parameters of the considered copulas (see e.g. Hofert and Scherer (2009)).

Family	θ	$\varphi_0(t)$	$\varphi_0^{-1}(t)$	λ_L	λ_U
Gauss	$[-1,1]$	-	-	0	0
Gumbel	$[1,\infty)$	$(-\log(t))^\theta$	$\exp\left(-t^{\frac{1}{\theta}}\right)$	0	$2-2^{\frac{1}{\theta}}$
opC	$[1,\infty)$	$(t^{-\theta_c}-1)^\theta$	$(1+t^{\frac{1}{\theta}})^{-\frac{1}{\theta_c}}$	$2^{-\frac{1}{\theta\theta_c}}$	$2-2^{\frac{1}{\theta}}$

Table 7.1: Copula properties.

To keep the dimension of the optimization problem small, the inner copula parameters, describing the dependence among default triggers and the dependence among loss triggers, are assumed to be identical, i.e. $\theta_{in} := \theta_1 = \theta_2$. Different correlations for different industry sectors are also not under consideration here. This could be easily done as an extension of the model by adding another hierarchy level to the model. Furthermore, the additional parameter of the outer power Clayton copula is set to $\theta_c = 0.1$ (see Hofert and Scherer (2009)). Hence, the calibration to the market quotes of CDO tranches reduces to a two-dimensional optimization problem over the parameter vector $\theta = (\theta_{out}, \theta_{in})$ with $\theta_{out} := \theta_0$ denoting the outer copula parameter.
The investigated dataset consists of portfolio-CDS and CDO market quotes of the 8^{th} and 9^{th} iTraxx Europe series between February and July 2008. Portfolio-CDS spreads and spreads of the first five CDO tranches with detachment and attachment points given by $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, and $[12\%, 22\%]$ with a maturity of 5 years are used. According to the iTraxx Europe convention a quarter-yearly payment schedule $\mathcal{T}$ is used, i.e. $n = 4T$ and $T = 5$. The portfolio consists of $I = 125$ companies.
Five different, randomly picked trading days were chosen to test the modelling approach (22/02/2008, 31/03/2008, 02/05/2008, 27/06/2008, and 25/07/2008). In the following only the results for 02/05/2008 are discussed in detail. As all other results are quite similar they are deferred to Appendix C.1. The portfolio-CDS spread at this date was 63.74 bps, which leads to a constant default intensity of $\lambda = 0.0106$. This corresponds to a five-year default probability of $\bar{p} = 5.17\%$.
First of all, the model is calibrated to the market spreads. Table 7.2 contains the calibrated parameters $\theta = (\theta_{out}, \theta_{in})$ and the default correlation $\rho(5)$ according to Theorem 7.1. In addition to that, the average correlation between default rates and recovery rates $\widehat{\rho}^{D,R}(5)$ is reported. Of course, in case of deterministic recovery rates $\widehat{\rho}^{D,R}(5)$ is not well-defined.
Table 7.3 contains the market and model upfront payment (in %) for tranche

Model	θ_{in}	θ_{out}	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$
Ga det	0.34		12.93%	-
Ga sto	0.28	0.24	8.40%	-43.35%
Gu det	1.26		26.06%	-
Gu sto	1.19	1.11	20.68%	-29.03%
opC det	1.25		25.77%	-
opC sto	1.16	1.16	18.49%	-48.39%

Table 7.2: Calibrated parameters and correlations for 02/05/2008.

1, the market and model spreads (in bps) for tranches 2 - 5, and the pricing errors D_2 (see Equation (7.15)) and D_2^{rel}, where D_2^{rel} is D_2 divided by the sum of market spreads of tranches 2 - 5, i.e.

$$D_2^{rel} := \frac{\sum_{j=2}^{J} \left| s_{T,j}^{CDO,model} - s_{T,j}^{CDO,market} \right|}{\sum_{j=2}^{J} s_{T,j}^{CDO,market}}.$$

Model	$0-3\%$	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$	D_2	D_2^{rel}
Ga det	29.59%	496.48	250.50	142.08	53.12	411.87	77.67%
Ga sto	29.68%	488.42	241.94	137.92	52.90	390.86	73.70%
Gu det	29.63%	278.80	150.14	104.52	65.00	68.15	12.85%
Gu sto	29.60%	256.67	138.64	97.59	61.27	37.18	7.01%
opC det	29.69%	285.15	152.87	107.70	64.86	80.26	15.13%
opC sto	29.66%	270.51	142.35	97.73	59.98	48.43	9.13%
Market	29.65%	259.09	122.55	101.83	46.84	-	-

Table 7.3: Market and model spreads for 02/05/2008.

By introducing additional dependence between default rates and recovery rates, the default correlation gets lower and the pricing error smaller across all models. For the Gaussian model this gain in performance is the smallest. Both the Gumbel and the outer power Clayton model can generate higher default correlations than the Gaussian model while fitting the spread of the first tranche. This leads to consistently lower pricing errors (across all tranches). A possible explanation for this observation is the positive upper tail dependence of both Gumbel and outer power Clayton copula compared to the zero tail dependence of the Gaussian copula. Nevertheless, the outer power Clayton model needs a much higher correlation of default rates and recovery rates (with only a slightly lower default correlation) compared to the Gumbel model to generate similar fitting results. The absolute calibration error for tranches 2 - 5 in the Gumbel model is only 37.18 bps, which corresponds to a relative error of 7.01%. This result seems to be quite good especially as the sum of bid-ask spreads of tranches 2 -5 on this date is already 34.56 bps (6.16%). Since the Gumbel copula model shows consistently better calibration results compared to the outer power Clayton copula model, the following analyses will be restricted to a comparison of Gaussian and Gumbel copula model, both with and without stochastic recoveries.

7.3 Parameter sensitivity and delta hedging

In the next step, the sensitivities of the five tranches' spreads are investigated in dependence of the copula parameter vector $\theta = (\theta_{out}, \theta_{in})$, while all other parameters are set to the calibrated values from Subsection 7.2.2. For the Gumbel copula model it can be seen from Figure 7.3, that for tranches 1 and 2 the spread or upfront payment respectively decreases with increasing dependence parameters, whereas for tranches 4 and 5 the opposite holds true, i.e. holders of tranche 1 and 2 are short correlation, holders of tranche 4 and 5 are long correlation. In tranche 3 things are a bit different. While the spread decreases with increasing outer dependence parameter θ_{out}, it first increases with increasing inner dependence parameter θ_{in} until it reaches its maximum and decreases afterwards. Across all tranches the inner dependence parameter which drives the default correlation as well as the correlation of loss triggers has the higher impact on the tranche spreads. The outer dependence parameter which drives the dependence of default rates and recovery rates has its highest impact on the equity tranche. This effect can also be observed in the Gaussian model, but with a lower impact compared to the Gumbel case.

For the Gaussian copula model it can be seen from Figure 7.4, that for tranches 1 and 2 the spread or upfront payment respectively again decreases with increasing dependence parameters, i.e. investors in these tranches are short correlation. In contrast, for tranches 3 - 5 the spread increases with increasing θ_{in} until it reaches its maximum and decreases afterwards, i.e. depending on the value of θ_{in} investors in the tranches are short correlation or long correlation. Note that this behaviour was not observed for tranches 4 and 5 in the Gumbel copula model.

Figures 7.5 and 7.6 show the default correlation $\rho(5)$ and the average correlation between default rates and recovery rates $\widehat{\rho}^{D,R}(5)$ in the models with Gumbel and Gauss copula. As expected the default correlation increases with increasing inner dependence parameter θ_{in}, while the main driver of

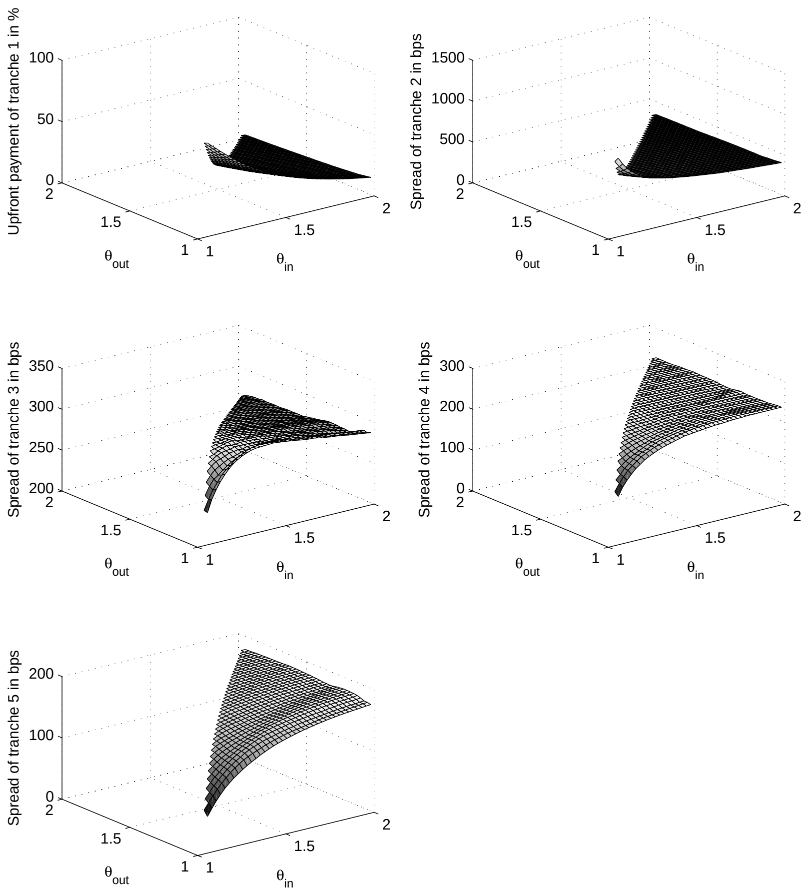

Figure 7.3: Upfront payment and tranche spreads for Gumbel model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (02/05/2008).

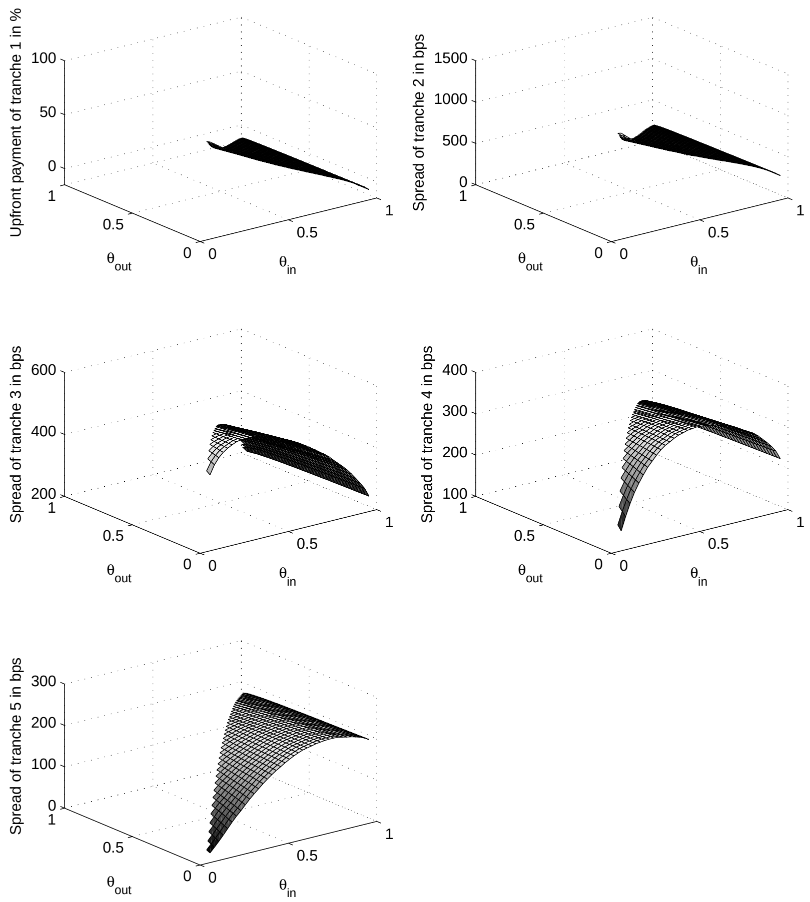

Figure 7.4: Upfront payment and tranche spreads for Gauss model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (02/05/2008).

$\widehat{\rho}^{D,R}(5)$ is the outer dependence parameter θ_{out}.

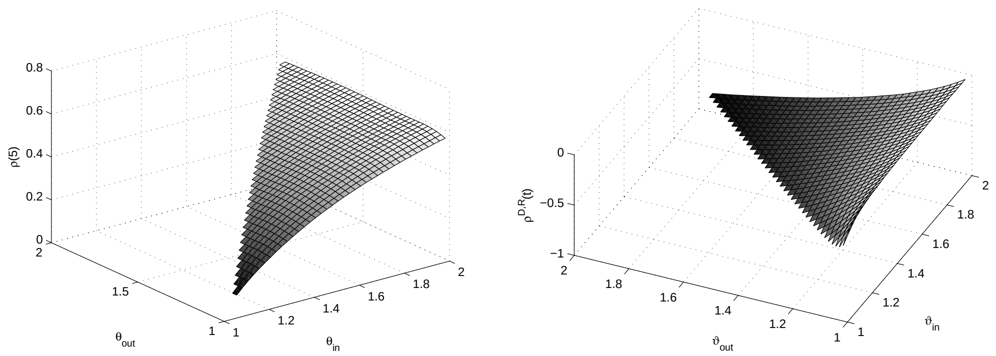

Figure 7.5: $\rho(5)$ (left) and $\widehat{\rho}^{D,R}(5)$ (right) for Gumbel model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (02/05/2008).

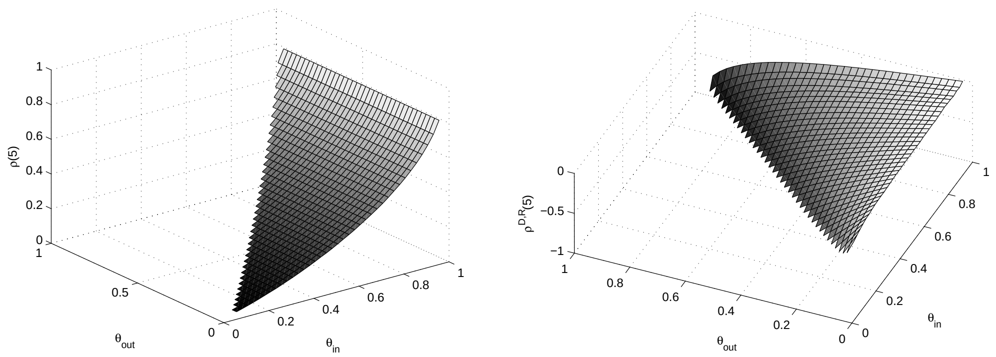

Figure 7.6: $\rho(5)$ (left) and $\widehat{\rho}^{D,R}(5)$ (right) for Gauss model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (02/05/2008).

Besides on the copula parameter vector $\theta = (\theta_{out}, \theta_{in})$, the tranche spreads also depend on the parameters of the recovery distribution a and b. So far they have been chosen such that Equations (7.10) and (7.11) are fulfilled. While one parameter has to be determined by Equation (7.10) to ensure consistency with portfolio-CDS pricing, the other parameter can be chosen arbitrarily. Figure 7.7 shows the tranche spreads for the Gauss and the Gumbel copula model and the loss given default standard deviation σ_{LGD} from Equation (7.11) for different values of a while all other parameters are set according to the calibration at 02/05/2008.
With an increase of the parameter a the parameter b increases as well to ensure the constant loss given default expectation. Furthermore, the dis-

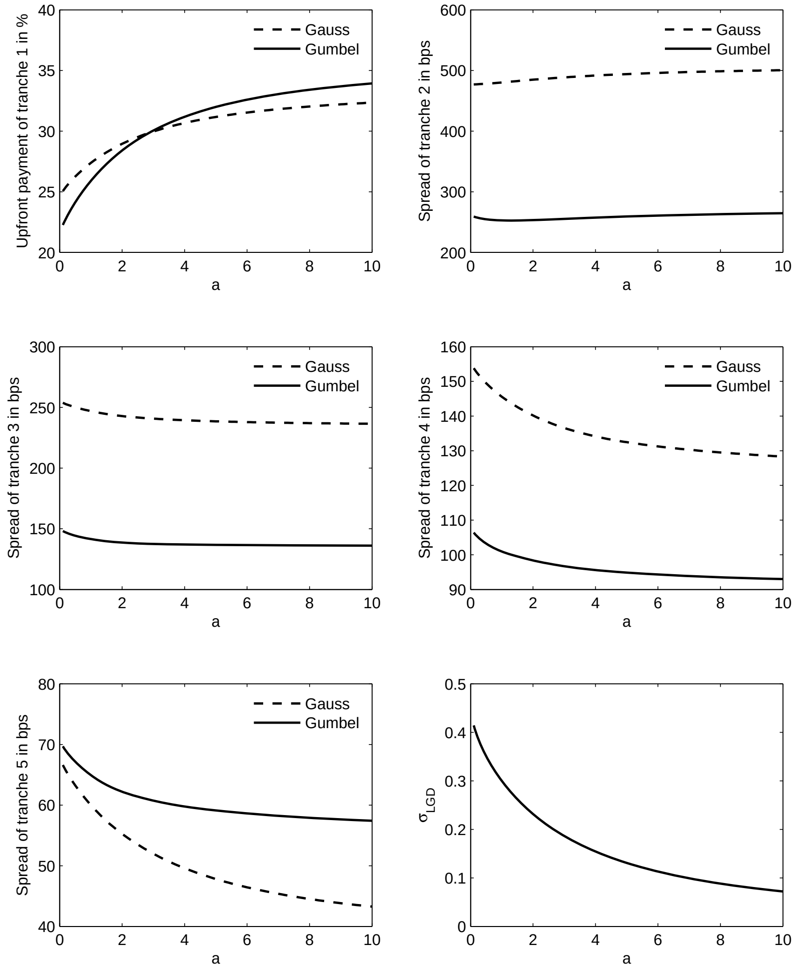

Figure 7.7: Upfront payment and tranche spreads for Gauss and Gumbel model and σ_{LGD} in dependence of a (02/05/2008).

tribution changes from U-shaped to J-shaped to unimodal and finally to a point mass in 0.6 which corresponds to the case of constant recovery (see also Figure 7.1). As expected the spreads of the lower tranches increase with a decreasing recovery-rate variability while the spreads of the higher tranches decrease. For tranches 2 - 4 the behaviour of the Gaussian and the Gumbel model is very similar, i.e. the tranche spreads are almost parallel. Only for the first and the fifth tranche there are significant differences. For the Gumbel model the first tranche reacts more sensitive to changes in the shape of the recovery-rate distribution. In contrast, the fifth tranche in the Gaussian copula model is subject to larger changes if the parameter of the recovery distribution varies.

In addition to the sensitivities of the tranche spreads to changes in the model parameters, one thing that is especially of interest for investors in a tranche is the impact of changes in the spread level of the portfolio CDS on the spread levels of each tranche. Buying protection in the portfolio CDS such that the position in a certain tranche of a CDO is insensitive to changes in the level of the portfolio-CDS spread is called delta hedging. To be more precise, the delta of a tranche with respect to the portfolio CDS is defined as the ratio of the change in the spread of the respective tranche to that of the portfolio CDS. Typically a shift of 1 bp is applied to the portfolio-CDS spread. Assuming constant discount factors, this shift in the portfolio-CDS spread leads to a change in the default intensity, which influences the values of the different CDO tranches. Figure 7.8 shows the deltas for all tranches in the Gauss and Gumbel model at 02/05/2008. The results for all other considered dates are very similar and therefore omitted here.

It can be easily seen that the hedge ratios remain almost the same regardless of a deterministic or stochastic modelling of recovery rates, i.e. from a practical point of view traders don't have to adjust their hedges a lot when switching from a model with deterministic recoveries to a model with stochastic recoveries. Nevertheless, it can be observed that in comparison to the Gaussian model, the Gumbel model features significantly higher deltas for the first tranche and significantly lower deltas for all other tranches. On average, the delta of the first tranche is 25% higher in the Gumbel model than in the Gaussian model while the delta of tranche 2 is 20% lower. For tranches 3 - 5 the Gumbel model exhibits deltas that are on average 35% lower than in the Gaussian model.

Another approach often applied for hedging CDO tranches is to hedge the first with the second tranche, which is also sometimes called mezzanine-equity hedging. The corresponding hedge ratio between the two tranches is simply defined as the ratio of the delta of tranche 1 to the delta of tranche 2, both with respect to the portfolio-CDS spread. On average the hedge ratios in the

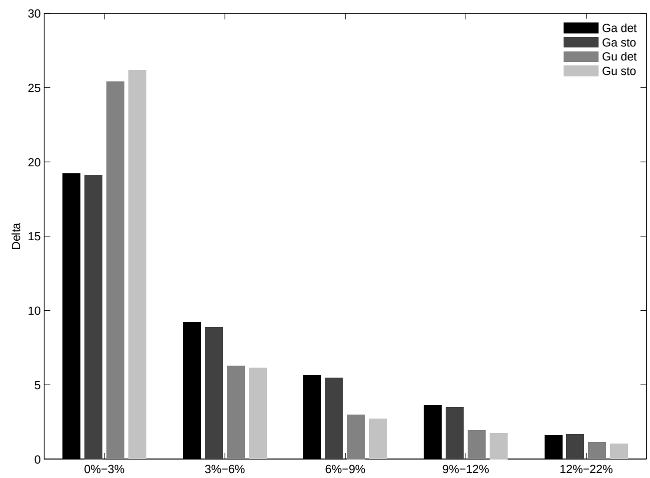

Figure 7.8: Delta w.r.t. portfolio CDS for different tranches in Gauss and Gumbel model (02/05/2008).

Gumbel model are 60% higher than those in the Gaussian model. According to the Gumbel model for both deterministic and stochastic recoveries the notional of the second tranche has to be 3 - 4 times higher than the notional of the first tranche. In the Gaussian model the notional of the second tranche only has to be 1.5 - 2 times the notional of the equity tranche. All the delta hedging results presented here are in line with the results of Masol and Schoutens (2008) who compare hedge ratios of different one-factor Lévy models to those of a Gaussian model.

7.4 An application to base correlations

Similar to the equity market, where it has become standard to quantify the prices of equity options in terms of implied volatility, it has become standard to quantify the spreads of CDO tranches in terms of implied correlation, especially in terms of base correlation as introduced by O'Kane and Livesey (2004). In contrast to the concept of compound correlation, where for each tranche the correlation is chosen such that market and model spread coincide, the concept of base correlation decomposes each tranche into combinations of base tranches, i.e. tranches without subordination covering some interval $[0, u]$. Using the observation that being long a tranche $[l, u]$ coincides with a long position in $[0, u]$ and a short position in $[0, l]$, the base correlation of

tranche j, $j = 2, \ldots, J$, can be calculated in an recursive algorithm from the previously computed base correlations and the market spread of tranche j. The base correlation of the equity trance ($j = 1$) coincides with the compound correlation of this tranche. Though being less intuitive, base correlations are more frequently used in practice than compound correlations. One reason for this is the fact that each base correlation only depends on its detachment point. This facilitates the pricing of non-standard tranches via interpolation. Since the Gaussian base correlation introduced in O'Kane and Livesey (2004) has some drawbacks (correlation skew, sensitivity to interpolation scheme), the concept of base correlation has been extended to other models not relying on Gaussian distributions (see e.g. Hooda (2006) or Garcia *et al.* (2007)).
To define a base correlation curve in the setting from Section 7.1, i.e. $\theta_{in}(u_j)$, $j = 1, \ldots, 5$, the outer copula parameter θ_{out} is chosen to coincide with the estimate from the global calibration in Section 7.2. Alternatively, θ_{out} could also be set to the same fixed value for all days and time horizons, but since $\theta_{out} \leq \theta_{in}$ has to be claimed this would imply a rather low level for θ_{out} and hence a low level of correlation between default rates and recovery rates. Then, one can proceed similar to the Gaussian base correlation case as in O'Kane and Livesey (2004) and subsequently bootstrap the default correlation parameter for each base tranche. In the case of deterministic recovery rates, one can start directly with the bootstrapping.
Note that although θ is in the case of Archimedean copulas not the correlation coefficient, the term base correlation is used here for the sake of simplicity when describing a curve of dependence parameters driving the default correlation for different detachment points. Table 7.4 and Figure 7.9 show the base correlation curves in the Gaussian and Gumbel copula model both with and without stochastic recoveries at 02/05/2008 (the results for all other considered trading days can be found in Appendix C.2).

	3%	6%	9%	12%	22%
Ga det	0.34	0.46	0.54	0.59	0.73
Ga sto	0.28	0.41	0.49	0.54	0.68
Gu det	1.26	1.27	1.29	1.30	1.40
Gu sto	1.19	1.19	1.19	1.20	1.23

Table 7.4: Base correlations $\theta_{in}(u_j)$ for 02/05/2008.

While the difference in the base correlation curves for the Gaussian model is an almost parallel downward shift, the Gumbel base correlation curve with stochastic recovery rates is not only significantly lower but also significantly

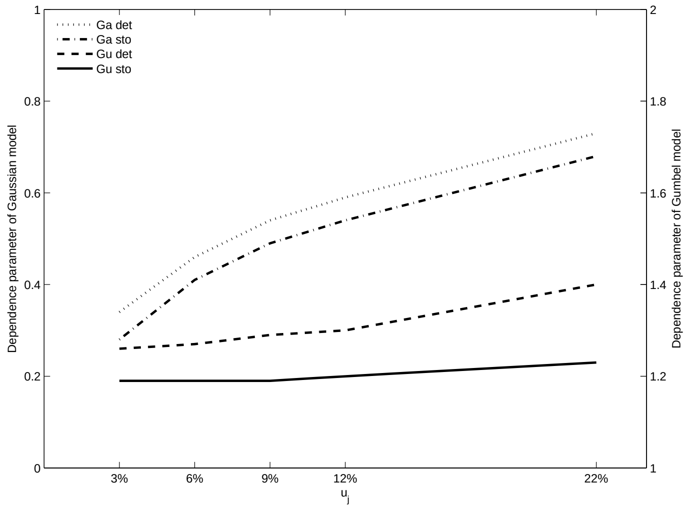

Figure 7.9: Base correlations $\theta_{in}(u_j)$ for different models with deterministic and stochastic recoveries (02/05/2008).

flatter than the curve in the case of deterministic recoveries. As said above, this property is especially useful for the pricing of non-standard tranches using an inter- or extrapolation scheme of the base correlation curve.

7.5 Risk measurement and management case study for a portfolio of credit-risky assets

In most cases, banking institutions are not only interested in the pricing of standardized credit derivatives like in the examples with iTraxx data presented in Sections 6.3 and 7.2 - 7.4, but also in the valuation and risk management of their non-standardized portfolios of credit-risky assets. In this section, an example is given of how this can be accomplished by combining the models presented in Sections 6.1 and 7.1.

One of the crucial problems for such a portfolio model is the estimation of the model parameters. If there are liquidly traded derivatives like standard CDS, fixed-recovery CDS, recovery locks, or CDO tranches on the underlying credits in the portfolio, the parameters of the default-intensity and recovery-rate process as well as of the dependence structure can be easily calibrated. Unfortunately, this is not the case for most credit portfolios. Due to this lack of information other sources of data have to be used to estimate the model parameters. To obtain the parameters of the default intensity, CDS quotes of the obligors in the portfolio or of a respective index can be used as well as historical default rates, if they are available. The parameters of the recovery-rate distribution can be estimated from a time series of aggregated recovery rates. Using the dependence concepts presented in this chapter, the copula parameters will typically be determined using standardized CDO tranches of portfolios with similar characteristics. For this, an additional hierarchy level for different industry sectors or geographical regions might be appropriate. Using Monte Carlo techniques, the following algorithm computes risk measures of the loss-process as well as prices of credit derivatives similar to Algorithm 7.2. It also includes recovery-rate modelling in presence of stochastic collateral.

Algorithm 7.3. *Simulation of the loss process.*

1. *Let T denote the time horizon, $\mathcal{S} = \{0 < t_1 < \ldots < t_m = T\}$ the time grid of the simulation, $\mathcal{T} \subseteq \mathcal{S}$ the payment schedule, $C_i(t) \geq 0$ the collateral-value process of obligor i, E_i the exposure of firm i*[20]*, and I the number of firms. Choose the number of simulation runs M, the attachment and detachment point l_j and u_j of tranche j, the significance level α of the risk measures, and a copula C.*

2. *Monte Carlo simulation:*

   (a) *Sample $r^{(m)}(t_k)$, $w^{(m)}(t_k)$, $u_i^{(m)}(t_k)$, and $\lambda_i^{(m)}(t_k)$ according to Equations (6.1), (6.2), (6.4), and (6.5) and compute $P^{nd,(m)}(0, t_k) = e^{-\sum_{l=0}^{k-1} r_i^{(m)}(t_l)(t_{l+1}-t_l)}$ as well as $\Lambda_i^{(m)}(t_k) = e^{-\sum_{l=0}^{k-1} \lambda_i^{(m)}(t_l)(t_{l+1}-t_l)}$ for $k = 0, \ldots, m$, $i = 1, \ldots, I$, and $m = 1, \ldots, M$.*

   (b) *Sample the collateral-value process $C_i^{(m)}(t_k)$ for $k = 0, \ldots, m$, $i = 1, \ldots, I$, and $m = 1, \ldots, M$ analogously to Section 6.4.*

   (c) *Sample $U = (U_{i,m})_{i=1,\ldots,2I,m=1,\ldots,M} \in [0,1]^{2I \times M}$, where each column $U_{\cdot,m}$ of U is a sample of the chosen copula C.*

   (d) *Compute the default times $\tau_i^{(m)}$ for each firm $i \in \{1, \ldots, I\}$ and Monte Carlo run $m \in \{1, \ldots, M\}$ via Equation (7.2) using $\Lambda_i^{(m)}(t_k)$ and $U_{i,m}$ from steps 2(a) and (c).*

   (e) *Compute the loss given default for each defaulted firm according to Equation (7.13).*

   (f) *Define the recovery rate of firm i as $z^{Coll,(m)}(t)$ according to Equation (6.14) with $z_i^{(m)}$ given by Equation (7.14).*

[20] In the following example, E_i will be assumed to be the same known constant for each firm i. Otherwise, E_i had to be simulated in step 2 as well.

(g) *Compute the loss process* $L^{(m)}(t_k)$ *for* $t_k \in \mathcal{S}$ *according to*

$$L^{(m)}(t_k) = \frac{1}{I}\sum_{i=1}^{I} \mathbb{1}_{\{\tau_i^{(m)} \leq t_k\}}(1 - z_i^{Coll,(m)})E_i.$$

(h) *Compute for each tranche* $j \in \{1, \ldots, J\}$ *and each* $t_k \in \mathcal{T}$ *the loss affecting tranche* j, $L_j^{(m)}(t_k)$, *and the remaining nominal in tranche* j, $N_j^{(m)}(t_k)$ *according to step 2(g) from Algorithm 7.2.*

3. *Determine the fair spreads of portfolio CDS and CDO tranches by equating the sample means of default and premium leg unless they are given in the market. Using these spreads, calculate risk measures and other characteristics of the loss distribution of the whole portfolio or of single tranches by their respective empirical counterparts.*

The following example shows an application of Algorithm 7.3 to a portfolio of credit-risky assets.

Example 7.4. *Consider a portfolio consisting of 100 obligors, where each obligor contributes 1 million Euro to the notional. The portfolio is assumed to be homogeneous in the sense that the default intensity of each obligor is given by Equation (6.5) with parameters from Table 6.1 and the recovery-rate density is for all obligors given by Equation (7.8) with parameters estimated from the time series of aggregated recovery rates as used in Section 6.3.2. For the dependence structure both Gaussian and Gumbel model with parameters from Table 7.2 are employed. Since there is no possibility to estimate the model under the real-world measure, it will be assumed throughout this example that real-world and risk-neutral parameters are equal. Three different situations are distinguished in each model: First, a constant recovery rate is assumed. Second, the recovery rates are stochastic with univariate LGD distribution given in Equation (7.13). Third, stochastic recovery rates are assumed and stochastic collaterals are included. To be more precise, in this case 33 obligors are assumed to have collaterals from the asset class commodity (with 60% initial quota of collateral), another 33 obligors are assumed to have collaterals from the asst class real estate (with 90% initial quota of collateral), and the remaining obligors are assumed to have no collateral at all. The collateral-value processes are simulated analogously to Section 6.4.*
First of all, the fair portfolio-CDS spreads and CDO upfront payments and spreads for the different model specifications are computed (see Table 7.5). Table 7.6 contains the value at risk (VaR) and conditional value at risk (CVaR) with significance levels $\alpha = 0.99$ *and* $\alpha = 0.999$ *for the corresponding*

	s_T^{pCDS}	$s_{T,j}^{CDO}$				
		0-3%	3-6%	6-9%	9-12%	12-22%
Ga det	45.09	23.58%	336.34	142.70	66.67	16.98
Ga sto	45.05	24.60%	325.00	136.45	62.84	16.16
Ga sto, Coll	24.43	10.19%	119.54	30.58	8.56	0.94
Gu det	44.98	20.62%	196.97	108.38	74.40	45.00
Gu sto	45.01	21.64%	179.37	97.36	65.99	39.39
Gu sto, Coll	24.32	5.55%	89.94	49.17	32.90	17.55

Table 7.5: Upfront and spreads of portfolio CDS and CDO tranches.

loss distribution after 5 years ($L(5)$). Histograms of the loss distributions can be found in Figure 7.10. Similar to the results of the previous sections, it can be seen that the introduction of stochastic recovery rates has a higher impact on spread levels as well as on risk measures in the Gumbel model than in the Gaussian model. As losses higher than the expected recovery rate times the notional of the portfolio can occur in the models with stochastic recovery rates, there is more mass in the very high tails of the portfolio distribution in these models. Due to the upper-tail dependence, the Gumbel copula model shows much higher risk measures compared to the Gaussian model regardless of the recovery specification. The introduction of collateral reduces the risk of high losses and hence leads to smaller spreads and risk measures.

	VaR		CVaR	
	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.99$	$\alpha = 0.999$
Ga det	15.43	23.70	19.01	25.92
Ga sto	15.10	24.97	19.59	28.77
Ga sto, Coll	8.37	13.89	10.74	15.74
Gu det	28.66	34.76	34.47	36.74
Gu sto	27.14	52.91	40.61	55.23
Gu sto, Coll	14.93	28.66	22.08	30.08

Table 7.6: VaR and CVaR of $L(5)$ in millions of Euro.

In a second step, the upfront payments and spreads from Table 7.5 are taken as given. Then, simulated profit & loss (P & L) distributions after 5 *years for an investor who acts as a protection seller in one of the CDO tranches or the portfolio CDS respectively are investigated (see Figures C.1 - C.6 in Appendix C.3). Again, the highest differences between Gaussian and Gumbel copula models can be found in the tails, i.e. in the tail of the portfolio-CDS P & L distribution as well as in the P & L distribution of the fifth tranche.*

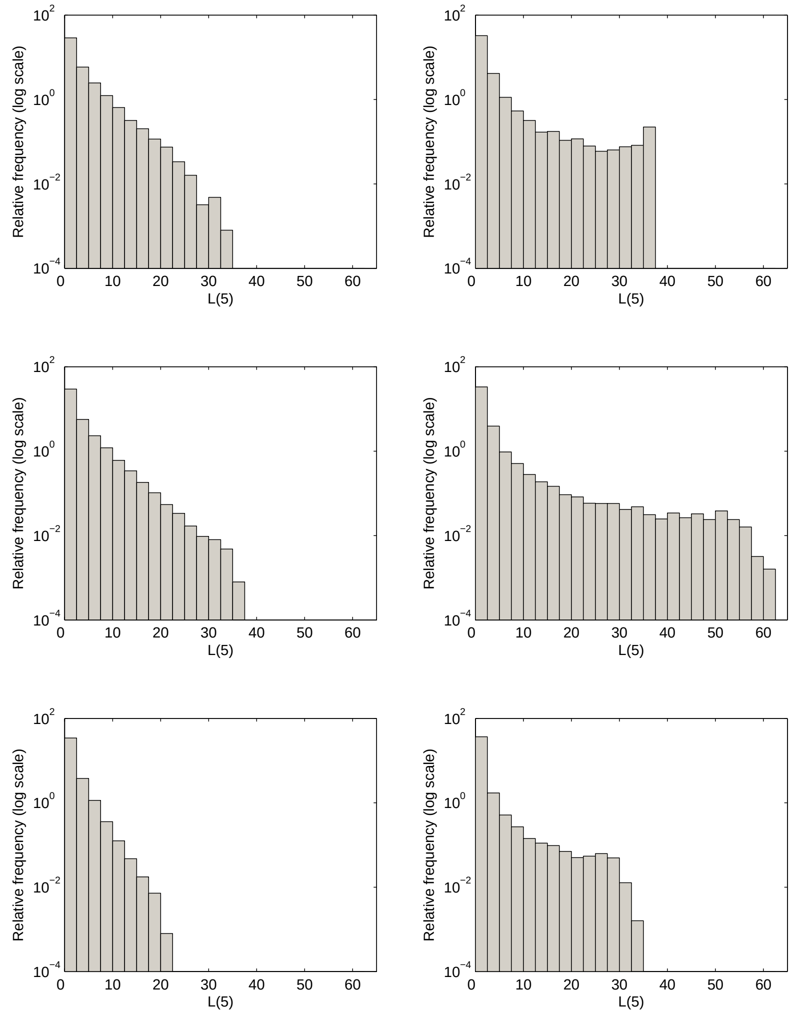

Figure 7.10: Histograms (log scale) of loss distribution (in millions of Euro) after 5 years for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

This is also confirmed by the empirical statistics of the P & L distribution in Tables C.9 - C.14 in Appendix C.3. Note that losses which are higher than the size of the respective tranche can occur in Figures C.2 - C.6 as payments which are made during the considered time period are compounded to the final time horizon.

Finally, Figures 7.11 and 7.12 show the (mean-variance) efficient frontiers and the corresponding (mean-variance) optimal portfolios of CDO tranches. While in the Gumbel model only combinations of the first and fourth tranche are optimal, in the Gaussian model also the fifth tranche is allocated. One possible explanation for this effect is the fact that in the Gaussian model the correlations of the fifth tranche with the other tranches are significantly lower compared to the Gumbel model (see Tables C.15 - C.20 in Appendix C.3). It can be seen that in both models the introduction of stochastic recovery rates leads to an improvement in the efficient frontier, i.e. a shift towards the upper left corner in the mean-variance diagrams in Figure 7.11. This might be explained by the (slightly) higher mean portfolio profits and (slightly) lower portfolio profit standard deviations for most of the allocated tranches in the models with stochastic recoveries compared to those with deterministic recovery rates (see Tables C.10, C.13, and C.14)

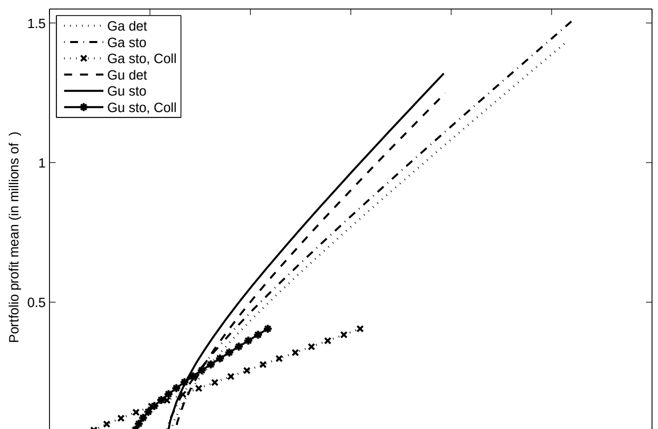

Figure 7.11: Efficient frontiers of mean-variance optimal tranche combinations for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

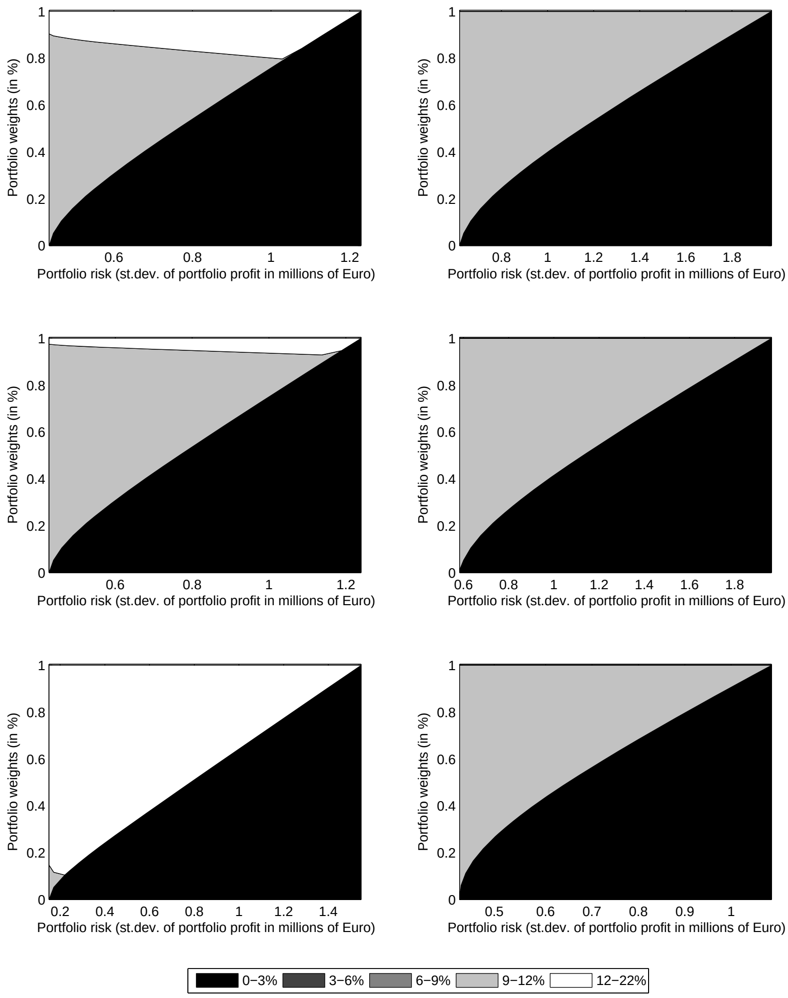

Figure 7.12: Mean-variance optimal tranche combinations for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

Chapter 8

Summary and conclusion

In this thesis, the behaviour and determinants of recovery rates in credit-risk modelling have been examined and new valuation approaches for single-name and portfolio credit derivatives under stochastic recovery have been developed.

In Chapter 4, workout recoveries of bank loans have been investigated on a large Pan-European dataset with regard to their determinants and their behaviour. First of all, it was shown that the discount rate chosen for the calculation of workout recoveries can have a great impact on the recovery value for facilities with a long workout period. For facilities with a moderate workout time the influence of the chosen discount rate is rather small. Furthermore, it was shown that the distribution of recovery rates on a facility level is bimodal or U-shaped in the interval $[0, 1]$. Facility-level and entity-level factors as well as factors that describe the collateralisation or the default proceedings were tested on their impact on workout recoveries. Furthermore, relations between macroeconomic variables and recovery rates were investigated on an individual level. The presence and quality of collateral was found to be the most important component in workout recovery rates on a facility level. In addition to that, the creditworthiness measured by the spread at default and the reason for default play a significant role in determining loan recoveries. In contrast to other studies, a significant positive impact of the size of issue and issuer on recoveries was observed. Industry dummies and macroeconomic variables play only a minor role on the facility level in the considered dataset. All in all, an adjusted degree of explanation of about 20% was obtained in a multivariate model, which lies in the range of other studies.

The dependence of aggregated recovery rates on various explanatory variables like interest rates, equity market indicators, and macroeconomic variables was investigated in Chapter 5. In a Markov-switching setting it was shown that the distribution of the recovery rates and their dependence on the explanatory variables may vary between different states. In addition to that, recovery rates of secured facilities were distinguished from those of unsecured facilities. For the first the explanatory variables have only a minor impact, while for the latter an adjusted degree of explanation up to 60% can be achieved. The small degree of explanation for the recovery rates of secured facilities might be due to the fact that they are mainly driven by the quality and quantity of the underlying collateral. In a further step, the factors describing the equity market were replaced by one single factor describing the uncertainty in credit markets without decreasing the explanatory power. In this final model the recovery rates of unsecured facilities could be explained best by the EURIBOR as a proxy for short-term interest rates, the GDP growth rate as an indicator for the macroeconomic environment, and an additional factor describing the credit environment.

In Chapter 6 a joint modelling framework for recovery and default risk was presented. The model accounts for typical characteristics known from empirical studies like a negative correlation of default rates and recovery rates or the positive impact of a healthy macroeconomic environment on recovery rates. To be more precise, both default intensity and recovery rate are assumed to depend on a common factor describing the credit environment. Furthermore, the recovery rate as well as the short-term interest rate are modelled as functions of a market factor. Despite its realistic features the model is still simple enough to obtain closed-form (at least up to one numerically tractable integral) pricing formulas for many (single-name) defaultable assets, like coupon bonds or credit default swaps. The stochastic nature of the recovery-rate process in this model allows for the pricing of credit derivatives with payoffs directly linked to the recovery rate, e.g. recovery locks. The model parameters are estimated using an (extended) Kalman filter approach. The estimation procedure combines estimation under the real-world measure from historical time series (GDP growth rates and aggregated recovery rates) with calibration to time series of market quotes (zero rates and CDS spreads). In a numerical example the model was applied to a set of European data and tested for its fitting capability.

Chapter 7 is devoted to a joint modelling of default and recovery risk in a portfolio of credit-risky assets. Within this modelling framework, special emphasize was put on modelling the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand. Nested Archimedean copulas were used to introduce different dependence structures for default correlations and the correlation of default rates and recovery rates. The Kumaraswamy distribution, a very flexible continuous distribution with bounded support, was chosen for the recovery rates to allow for an efficient

sampling of the loss process. This is especially important, as in most cases the loss process distribution will not be given in closed form. Due to the relaxation of the constant 40% recovery assumption and the negative correlation of default rates and recovery rates, this modelling framework is especially suited for distressed market situations and the pricing of super senior tranches. In a numerical example the calibration to CDO tranche spreads of the European iTraxx portfolio was performed to demonstrate the fitting capability of the model. Already in a very simplistic setting of the model, the introduction of stochastic recovery rates consistently decreases pricing errors compared to the case of deterministic recovery rates. The best calibration results were achieved when the dependences are modelled by a Gumbel copula. Deltas with repsect to the portfolio-CDS spread were shown to be insensitive to the underlying recovery specification. In an extension to the Gaussian base correlation framework, significantly flatter base correlation curves could be obtained by using a Gumbel copula and stochastic recovery rates, which simplifies the pricing of non-standard tranches.
To summarize, the overall contribution of this thesis is threefold: First, the behaviour and determinants of facility-level as well as aggregated recovery rates are examined on a unique Pan-European dataset. The economic insights gained from these investigations expand the rather small amount of existing recovery-rate studies available for the European market and give hints for a sound modelling of recovery rates in different situations. Second, using these empirical insights a new valuation approach for single-name credit derivatives under stochastic recovery is developed. This approach is based on the classical intensity-based credit risk models, but due to the stochastic modelling of recovery rates it enables to price credit derivatives with payoffs directly linked to the recovery rate, e.g. recovery locks. Third, a tractable framework for pricing distressed CDOs using nested Archimedean copulas is introduced, which resolves some of the "inconsistencies" observed in the credit market since mid 2007.

Appendix A

Univariate results from Chapter 5

The coefficients and t-statistics for the univariate models

$$\ln(z(t)) = \beta_0 + \beta_i x_i(t) \tag{A.1}$$

with

$$x_i(t) \in \{GIP(t), GGDP(t), DJES(t), VDAX(t), EURIBOR(t), GY(t)\}$$

are given in Table A.1. While Table A.2 contains the univariate Markov-switching regression results, Table A.3 presents the univariate regression results corresponding to the multivariate regression from Table 5.7.

Variable	Coefficient (t-statistic)			
	β_0		β_i	
GIP	-0.471	***	0.8226	
	(-31.869)		(0.545)	
GGDP	-0.558	***	8.812	
	(-10.039)		(1.642)	*
DJES	-0.468	***	-0.373	
	(-32.230)		(-1.513)	
VDAX	-0.679	***	0.809	***
	(-20.347)		(6.738)	
EURIBOR	-0.711	***	7.578	***
	(-14.630)		(5.15)	
GY	-0.888	***	10.639	***
	(-13.691)		(6.565)	

Table A.1: Coefficients (t-statistics) and significance codes of the univariate linear regression (A.1) for response $\ln(z(t))$.

Variable	Coefficient (t-stat) in State							
	S1				S2			
	β_0		β_i		β_0		β_i	
GIP	-0.358	***	0.690		-0.587	***	1.222	
	(-42.130)		(0.723)		(-33.065)		(0.732)	
GGDP	-0.332	***	-2.413		-0.593	***	0.913	
	(-9.360)		(-0.743)		(-9.179)		(0.138)	
DJES	-0.358	***	-0.062		-0.588	***	0.229	
	(-42.522)		(-0.537)		(-30.276)		(0.475)	
VDAX	-0.421	***	0.205	**	-0.692	***	0.528	**
	(-16.443)		(2.621)		(-12.891)		(2.117)	
EURIBOR	-0.308	***	-1.345		-0.640	***	2.088	
	(-6.911)		(-1.129)		(-10.116)		(0.917)	
GY	-0.365	***	0.175		-0.741	***	4.513	*
	(-5.819)		(0.123)		(-7.465)		(1.698)	

Table A.2: Coefficients (t-statistics) and significance codes in a univariate model corresponding to Equations (5.3) - (5.4) with responses $\ln(z^S(t))$.

Variable	**Coefficient (t-statistic)**		**Coefficient (t-statistic)**		**Coefficient (t-statistic)**	
Response	$\ln(z_T(t))$		$\ln(z_S(t))$		$\ln(z_U(t))$	
	β_0	β_i	β_0	β_i	β_0	β_i
GIP	-0.543 *** (-25.651)	0.864 (0.417)	-0.409 *** (-21.675)	2.575 (1.392)	-0.839 *** (-18.875)	0.139 (0.003)
GGDP	-0.515 (-6.233)	-2.884 (-0.331)	-0.445 *** (-5.951)	4.339 (0.551)	-0.888 *** (-5.121)	5.320 (0.291)
DJES	-0.538 *** (-26.831)	-0.799 ** (-2.241)	-0.404 *** (-21.345)	-0.164 (-0.488)	-0.835 *** (-19.389)	-1.175 (-1.535)
VSTOXX	-0.718 *** (-18.728)	0.755 *** (5.162)	-0.383 *** (-9.108)	-0.092 (-0.570)	-1.181 *** (-14.188)	1.462 *** (4.601)
EURIBOR	-0.832 *** (-9.140)	11.131 *** (3.270)	-0.420 *** (-4.662)	0.577 (0.170)	-1.815 *** (-11.320)	37.644 *** (6.226)
GY	-1.135 *** (-9.805)	17.291 *** (5.190)	-0.424 *** (-3.319)	0.554 (0.151)	-2.297 *** (-10.413)	42.417 *** (6.346)

Table A.3: Coefficients (t-statistics) and significance codes in a univariate model corresponding to Equation (5.6) with responses $\ln(z_i(t))$.

Appendix B

The four-factor hybrid defaultable bond pricing model of Antes *et al.* (2008)

In the four-factor model of Antes *et al.* (2008), which is an extension of the hybrid defaultable bond price model introduced in Schmid and Zagst (2000), the risk-free short rate r is given by a Hull-White model with the factor w describing the macroeconomic environment. The short-rate spread s is modelled in dependence of w and the so-called uncertainty index u, which is an unobservable process describing an obligor's uncertainty filtered from market prices of defaultable bonds.

To be more precise, let $(\Omega, \mathcal{F}, \mathbb{Q})$ denote a complete probability space and $\mathbb{F}(W) = \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$ the natural filtration. Here, the vector $W(t) = (W_r(t), W_w(t), W_s(t), W_u(t))^T$ is a 4-dimensional standard Brownian motion and T^* is the fixed terminal time horizon. Then, the four factors of the model are given by the following system of stochastic differential equations:

$$
\begin{aligned}
dr(t) &= (\theta_r(t) + b_{rw}w(t) - a_r r(t))dt + \sigma_r dW_r, \ r(0) = r_0 && \text{(B.1)} \\
dw(t) &= (\theta_w - a_w w(t))dt + \sigma_w dW_w, \ w(0) = w_0 && \text{(B.2)} \\
du(t) &= (\theta_u - a_u u(t))dt + \sigma_u dW_u, \ u(0) = u_0 && \text{(B.3)} \\
ds(t) &= (\theta_s + u(t) - b_{sw}w(t) - a_s s(t))dt + \sigma_s dW_s, \ s(0) = s_0 && \text{(B.4)}
\end{aligned}
$$

with $a_r, b_{rw}, \sigma_r, a_w, \sigma_w, a_u, \sigma_u, b_{sw}, a_s,$ and σ_s positive constants, $\theta_w, \theta_u,$ and θ_s non-negative constants, θ_r a continuous, deterministic function in t, and $0 \le t \le T^*$.

In this modelling framework the price of a non-defaultable zero-coupon bond as well as of a defaultable zero-coupon bond can be calculated as an affine-

exponential function in the processes r, w, u, and s (see Theorems 3 and 4 in Antes *et al.* (2008)). The corresponding coefficients are functions of the parameters from Equations (B.1) - (B.4) and the contract's maturity T. The unknown parameters can be estimated from market data by using a Kalman filter technique.
To generate a filtered time series of the process u in the setting of Section 5.4, the model parameters are estimated in the following way: The parameters of the factor w and the short rate r are estimated from German sovereign yields with maturities from 3 months to 10 years and the Euro area GDP growth rates. To estimate the parameters of the uncertainty index and the short-rate spread corporate-composite yields Euro area of rating class A with maturities between 3 months and 10 years are used, i.e. the risky zero rates are not derived from a single company but from an index consisting of bonds of different companies belonging to rating class A. Weekly data between April 2002 and January 2007 for both riskless and risky bonds have been downloaded from Bloomberg (Bloomberg-Ticker: F910, C670). The parameter estimates and the filtered time series of the index u are given in Table B.1 and Figure B.1 respectively.

Interest-rate model		**Credit-spread model**	
Parameter	Estimate	Parameter	Estimate
a_r	0.7655	a_s	0.8358
σ_r	0.0095	σ_s	0.0010
b_{rw}	0.0334	b_{sw}	0.0450
a_w	1.9785	a_u	0.2206
σ_w	0.0056	σ_u	0.0108
θ_w	0.0199	θ_u	0.0004
		θ_s	0.0026

Table B.1: Parameter estimates of the four-factor model of Antes *et al.* (2008).

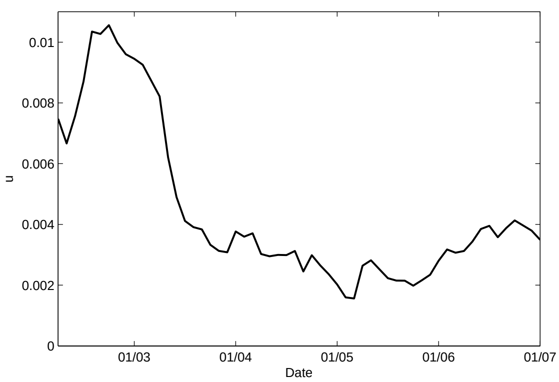

Figure B.1: Filtered time-series of uncertainty index u in the model of Antes *et al.* (2008).

Appendix C

Detailed results from Chapter 7

C.1 Detailed calibration results

In this section, the calibration results for all five trading days (22/02/2008, 31/03/2008, 02/05/2008, 27/06/2008, and 25/07/2008) on which the model was tested are presented. Table C.1 contains the calibration results for the portfolio-CDS model, i.e. default intensity as well as 1-year and 5-year default probabilities. Tables C.2 and C.3 show the absolute and relative calibration errors and the parameters of the CDO calibration. Finally, Table C.4 contains the default correlations and average correlations between default rates and recovery rates for each of the five trading days.

Date	$s^{pCDS,market}$	λ	$\bar{p}(1)$	$\bar{p}(5)$
22/02/2008	124.56	0.0207	2.05%	9.85%
31/03/2008	122.29	0.0204	2.02%	9.68%
02/05/2008	63.74	0.0106	1.06%	5.17%
27/06/2008	106.52	0.0178	1.76%	8.49%
25/07/2008	91.64	0.0153	1.52%	7.35%

Table C.1: Portfolio-CDS calibration.

		Ga		Gu		opC	
		D_2	D_2^{rel}	D_2	D_2^{rel}	D_2	D_2^{rel}
22/02/2008	det	668.96	53.80%	133.51	10.74%	121.24	9.75%
	sto	590.83	47.52%	70.92	5.70%	80.61	6.48%
31/03/2008	det	972.54	86.87%	277.01	24.74%	295.50	26.40%
	sto	903.76	80.73%	189.66	16.94%	191.93	17.14%
02/05/2008	det	411.87	77.67%	68.15	12.85%	80.26	15.13%
	sto	390.86	73.70%	37.18	7.01%	48.43	9.13%
27/06/2008	det	851.59	94.04%	277.37	30.63%	285.39	31.51%
	sto	788.92	87.12%	197.14	21.77%	218.71	24.15%
25/07/2008	det	676.09	80.96%	157.05	18.81%	165.26	19.79%
	sto	624.57	74.80%	86.46	10.35%	124.17	14.87%

Table C.2: Absolute and relative calibration errors.

		Ga		Gu		opC	
		θ_{out}	θ_{in}	θ_{out}	θ_{in}	θ_{out}	θ_{in}
22/02/2008	det	0.62		1.75		1.79	
	sto	0.52	0.54	1.09	1.68	1.49	1.50
31/03/2008	det	0.46		1.49		1.47	
	sto	0.38	0.38	1.17	1.38	1.30	1.30
02/05/2008	det	0.34		1.26		1.25	
	sto	0.24	0.28	1.11	1.19	1.16	1.16
27/06/2008	det	0.50		1.52		1.50	
	sto	0.41	0.42	1.16	1.40	1.33	1.34
25/07/2008	det	0.45		1.42		1.41	
	sto	0.36	0.37	1.17	1.31	1.27	1.28

Table C.3: Calibrated copula parameters.

		Ga		Gu		opC	
		$\rho(5)$	$\widehat{\rho}^{D,R}(5)$	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$
22/02/2008	det	33.91%	-	50.25%	-	51.82%	-
	sto	27.65%	-68.87%	47.49%	-14.90%	40.25%	-72.54%
31/03/2008	det	22.18%	-	39.75%	-	39.26%	-
	sto	17.00%	-68.38%	33.80%	-36.12%	29.43%	-67.82%
02/05/2008	det	12.93%	-	26.06%	-	25.77%	-
	sto	8.46%	-43.35%	20.68%	-29.03%	18.49%	-48.39%
27/06/2008	det	23.75%	-	40.08%	-	39.74%	-
	sto	18.39%	-65.73%	35.18%	-32.13%	31.81%	-65.20%
25/07/2008	det	19.56%	-	36.12%	-	35.81%	-
	sto	14.75%	-61.55%	29.49%	-37.84%	27.94%	-60.33%

Table C.4: Default correlation and average correlation between default rates and recovery rates.

C.2 Detailed base correlation results

In Tables C.5 - C.8 the base correlation curves at 22/02/2008, 31/03/2008, 27/06/2008, and 25/07/2008 are presented. Similar to Section 7.4 the difference in the base correlation curves for the Gaussian model with deterministic and stochastic recovery rates is an almost parallel downward shift. For the Gumbel model the base correlations in the case with stochastic recovery rates are significantly smaller and flatter than in the case with deterministic recovery rates.

	3%	6%	9%	12%	22%
Ga det	0.62	0.71	0.75	0.78	0.86
Ga sto	0.54	0.64	0.68	0.72	0.81
Gu det	1.75	1.80	1.82	1.84	2.06
Gu sto	1.68	1.72	1.72	1.73	1.88

Table C.5: Base correlation $\theta_{in}(u_j)$ for 22/02/2008.

	3%	6%	9%	12%	22%
Ga det	0.46	0.59	0.66	0.71	0.84
Ga sto	0.38	0.52	0.59	0.65	0.80
Gu det	1.49	1.55	1.58	1.62	1.94
Gu sto	1.38	1.42	1.44	1.47	1.66

Table C.6: Base correlation $\theta_{in}(u_j)$ for 31/03/2008.

	3%	6%	9%	12%	22%
Ga det	0.50	0.62	0.69	0.74	0.87
Ga sto	0.42	0.55	0.63	0.69	0.83
Gu det	1.52	1.57	1.62	1.67	2.08
Gu sto	1.40	1.44	1.46	1.50	1.78

Table C.7: Base correlation $\theta_{in}(u_j)$ for 27/06/2008.

	3%	6%	9%	12%	22%
Ga det	0.45	0.56	0.63	0.68	0.81
Ga sto	0.37	0.49	0.56	0.62	0.77
Gu det	1.42	1.45	1.47	1.50	1.72
Gu sto	1.31	1.32	1.33	1.34	1.46

Table C.8: Base correlation $\theta_{in}(u_j)$ for 25/07/2008.

C.3 Detailed results for Example 7.4

Tables C.9 - C.14 contain some empirical statistics of the simulated profit & loss (P & L) distributions after 5 years for an investor who acts as a protection seller in one of the CDO tranches or the portfolio CDS respectively. Figures C.1 - C.6 show the corresponding histograms. The empirical correlations of the P & L distributions after 5 years for the different CDO tranches in the six considered model specifications are given in Tables C.15 - C.20.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	0.147	0.179	0.101	0.159	0.173	0.086
Median	1.569	1.509	0.836	1.485	1.421	0.774
St.dev.	3.699	3.610	1.993	5.060	5.364	2.933
Skewness	-2.775	-2.990	-3.004	-5.079	-6.335	-6.284
Kurtosis	13.120	15.577	15.700	32.608	51.498	51.029
Min	-32.660	-36.026	-19.740	-41.962	-61.839	-35.264
Max	4.147	5.034	2.246	4.195	10.215	4.936
$VaR_{0.99}$	-15.083	-14.809	-7.996	-29.696	-28.144	-15.487
$CVaR_{0.99}$	-19.353	-19.678	-10.892	-35.743	-41.675	-22.766

Table C.9: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the portfolio CDS.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	1.429	1.506	0.405	1.250	1.319	0.180
Median	2.328	2.400	1.033	1.784	1.832	0.267
St.dev.	2.568	2.599	1.546	1.971	1.963	1.087
Skewness	-0.507	-0.498	-1.055	-0.757	-0.763	-1.484
Kurtosis	17.502	17.593	28.153	24.937	25.593	44.628
Min	-2.983	-2.979	-3.517	-2.998	-2.995	-3.868
Max	6.127	6.600	2.690	5.769	5.382	2.226
$VaR_{0.99}$	-2.942	-2.936	-2.995	-2.947	-2.943	-3.082
$CVaR_{0.99}$	-2.953	-2.947	-3.042	-2.960	-2.957	-3.229

Table C.10: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the first tranche.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	0.019	0.022	0.004	0.010	0.012	0.008
Median	0.552	0.535	0.201	0.329	0.300	0.152
St.dev.	1.215	1.188	0.714	0.954	0.901	0.644
Skewness	-1.955	-2.012	-3.787	-2.834	-3.026	-4.539
Kurtosis	5.162	5.418	16.566	9.510	10.723	22.528
Min	-4.704	-5.500	-5.167	-4.902	-5.129	-4.998
Max	1.501	2.321	0.702	1.046	1.970	1.157
$VaR_{0.99}$	-3.360	-3.361	-3.270	-3.374	-3.355	-3.206
$CVaR_{0.99}$	-3.603	-3.605	-3.560	-3.670	-3.643	-3.500

Table C.11: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the second tranche.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	-0.004	0.007	-0.001	0.004	0.011	0.002
Median	0.239	0.230	0.052	0.183	0.165	0.083
St.dev.	0.845	0.805	0.383	0.745	0.690	0.500
Skewness	-3.388	-3.578	-7.972	-4.047	-4.397	-6.255
Kurtosis	13.091	14.559	69.274	18.011	21.004	41.636
Min	-4.881	-5.358	-5.660	-6.128	-5.422	-5.683
Max	1.034	1.581	0.378	1.725	1.437	0.760
$VaR_{0.99}$	-3.443	-3.404	-2.690	-3.369	-3.285	-3.137
$CVaR_{0.99}$	-3.768	-3.745	-3.333	-3.726	-3.651	-3.453

Table C.12: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the third tranche.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	-0.004	-0.001	-0.002	0.005	0.006	0.001
Median	0.113	0.106	0.015	0.126	0.112	0.056
St.dev.	0.590	0.570	0.217	0.618	0.583	0.415
Skewness	-5.206	-5.509	-14.196	-5.020	-5.438	-7.650
Kurtosis	29.514	33.143	213.553	26.973	31.560	61.523
Min	-5.536	-6.323	-4.531	-5.330	-5.801	-5.720
Max	0.704	0.583	0.255	1.310	1.067	0.536
$VaR_{0.99}$	-3.289	-3.283	0.011	-3.256	-3.211	-3.056
$CVaR_{0.99}$	-3.656	-3.684	-1.674	-3.610	-3.601	-3.358

Table C.13: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the fourth tranche.

	Ga det	Ga sto	Ga sto, Coll	Gu det	Gu sto	Gu sto, Coll
Mean	-0.006	-0.004	0.001	0.015	0.005	0.009
Median	0.096	0.092	0.005	0.254	0.223	0.099
St.dev.	0.862	0.847	0.153	1.529	1.455	0.899
Skewness	-10.106	-10.407	-36.009	-6.559	-6.917	-10.671
Kurtosis	114.288	119.034	1521.848	45.672	50.719	120.543
Min	-15.175	-12.878	-8.158	-17.083	-17.576	-14.479
Max	0.310	0.301	0.598	4.300	1.896	0.495
$VaR_{0.99}$	-4.023	-3.638	0.004	-10.273	-10.192	-3.216
$CVaR_{0.99}$	-7.843	-7.783	-0.566	-11.096	-11.011	-8.484

Table C.14: Empirical statistics of P & L distribution (in millions of Euro) after 5 years of a protection seller in the fifth tranche.

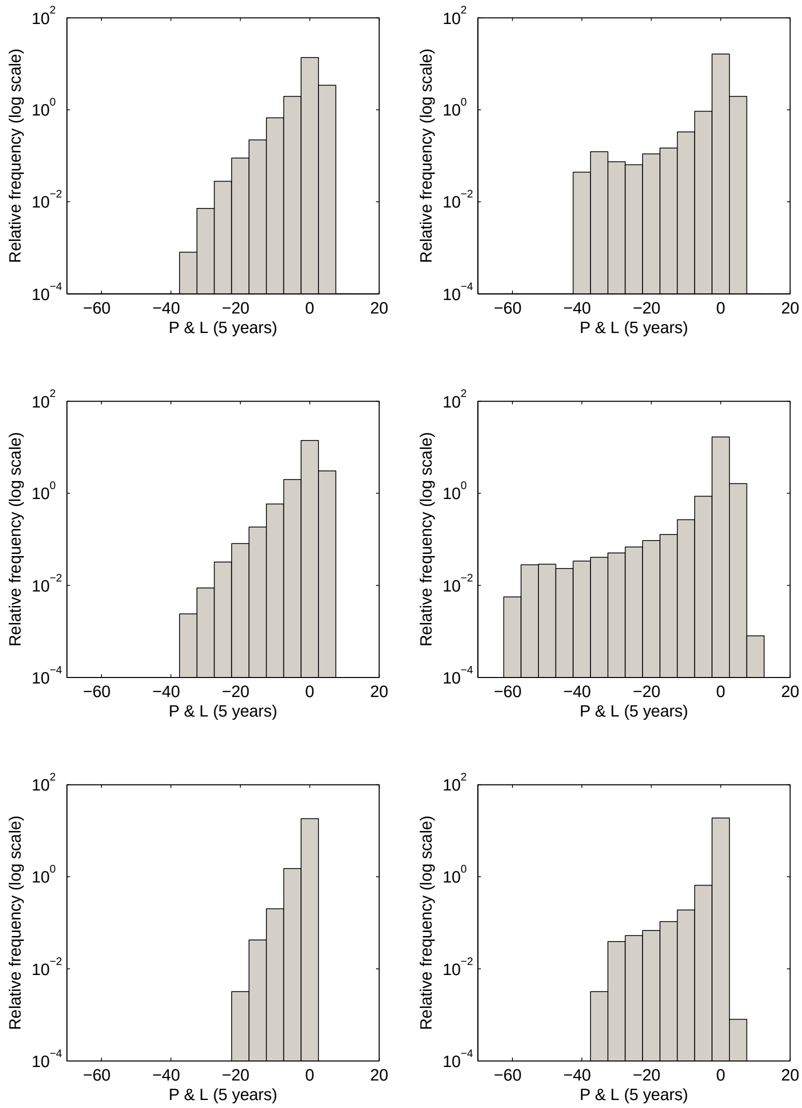

Figure C.1: Histograms (log scale) of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the portfolio CDS for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

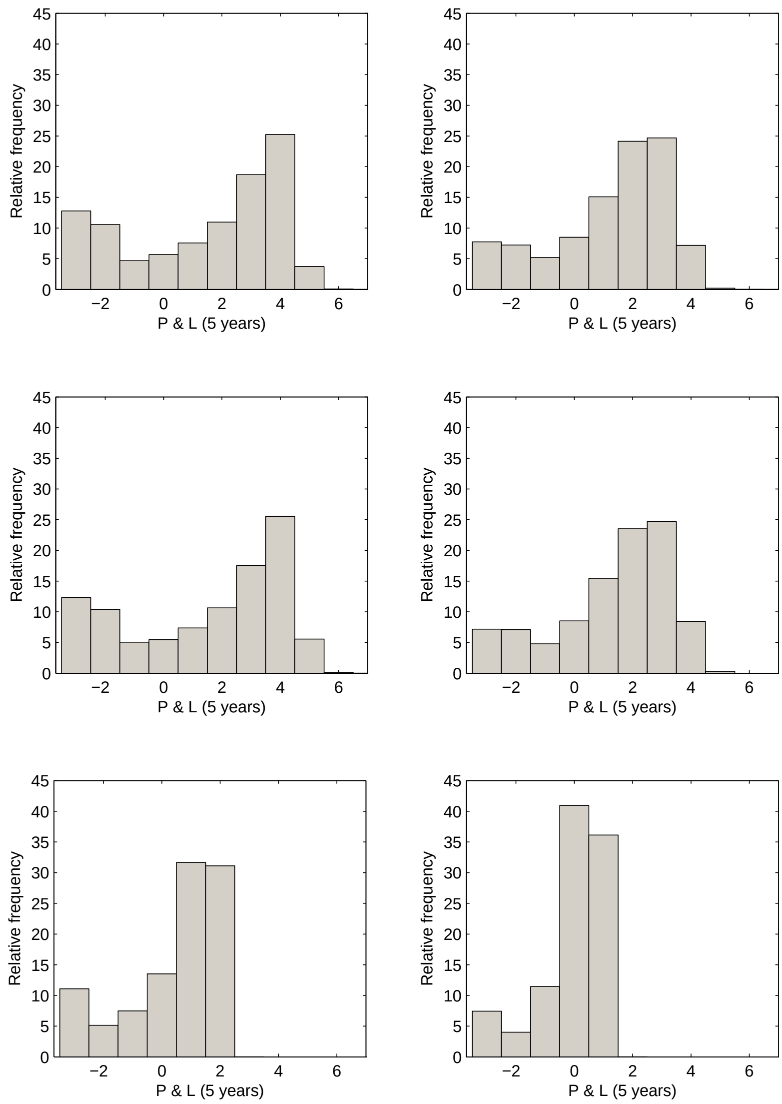

Figure C.2: Histograms of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the first tranche for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

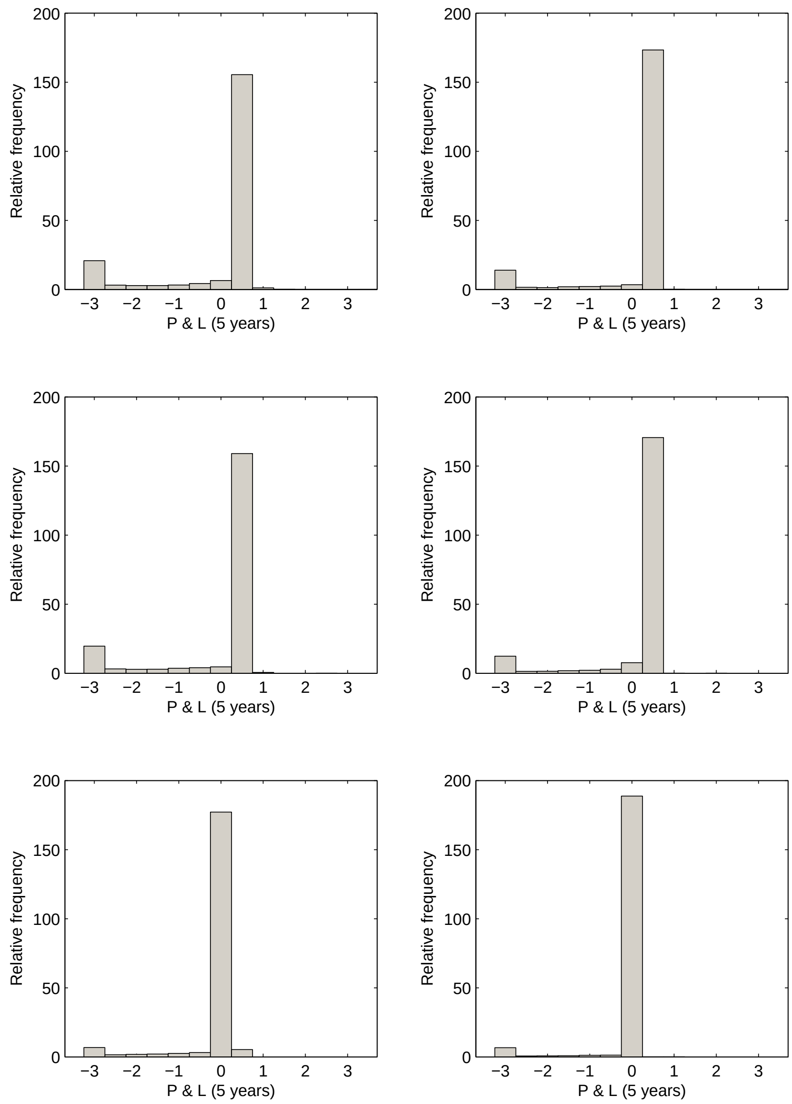

Figure C.3: Histograms of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the second tranche for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

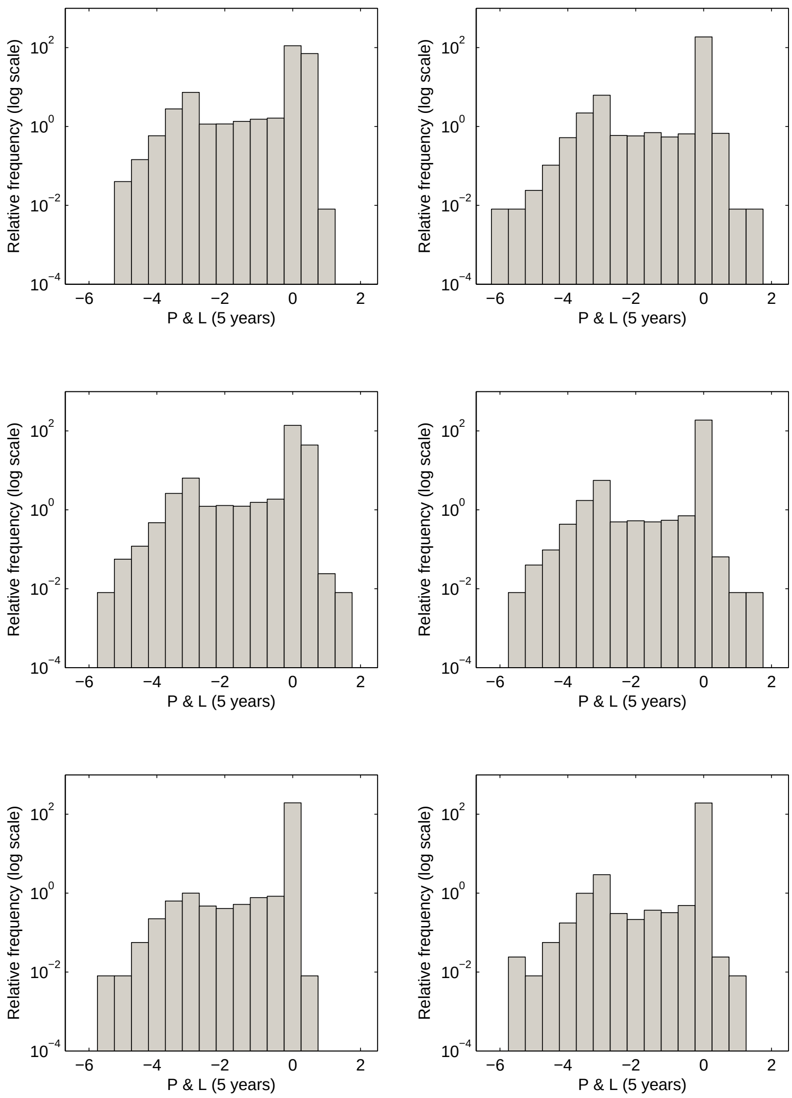

Figure C.4: Histograms (log scale) of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the third tranche for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

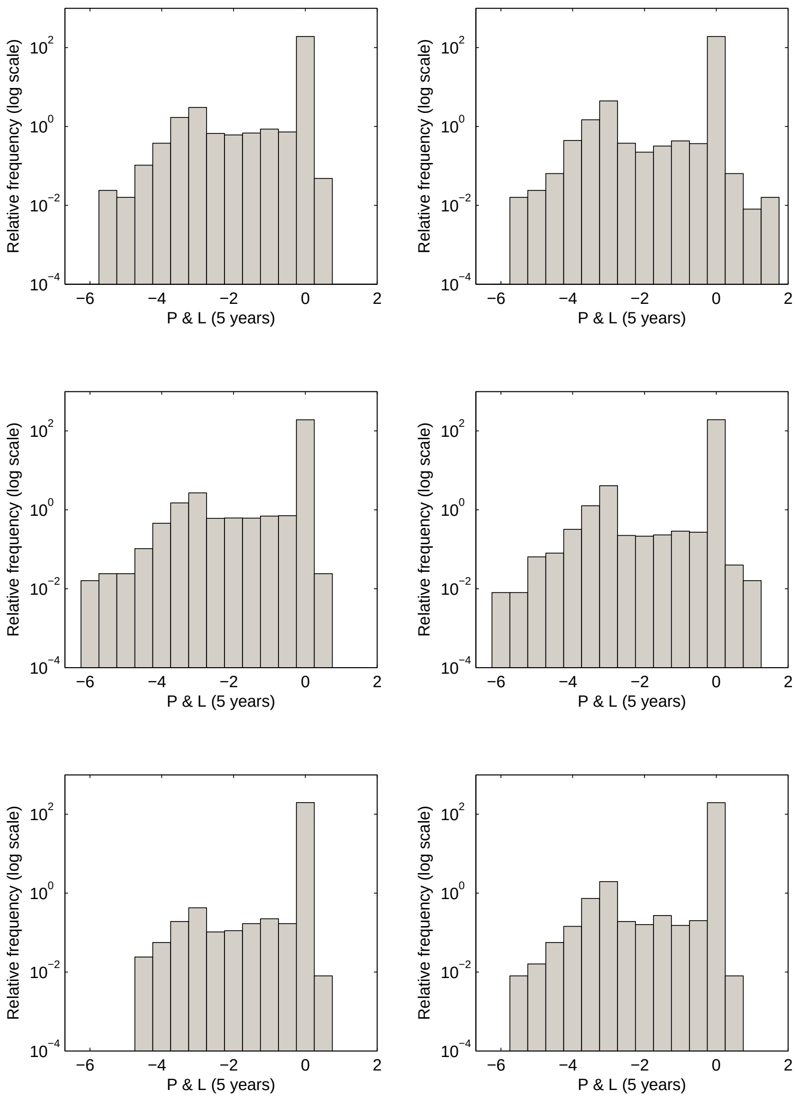

Figure C.5: Histograms (log scale) of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the fourth tranche for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

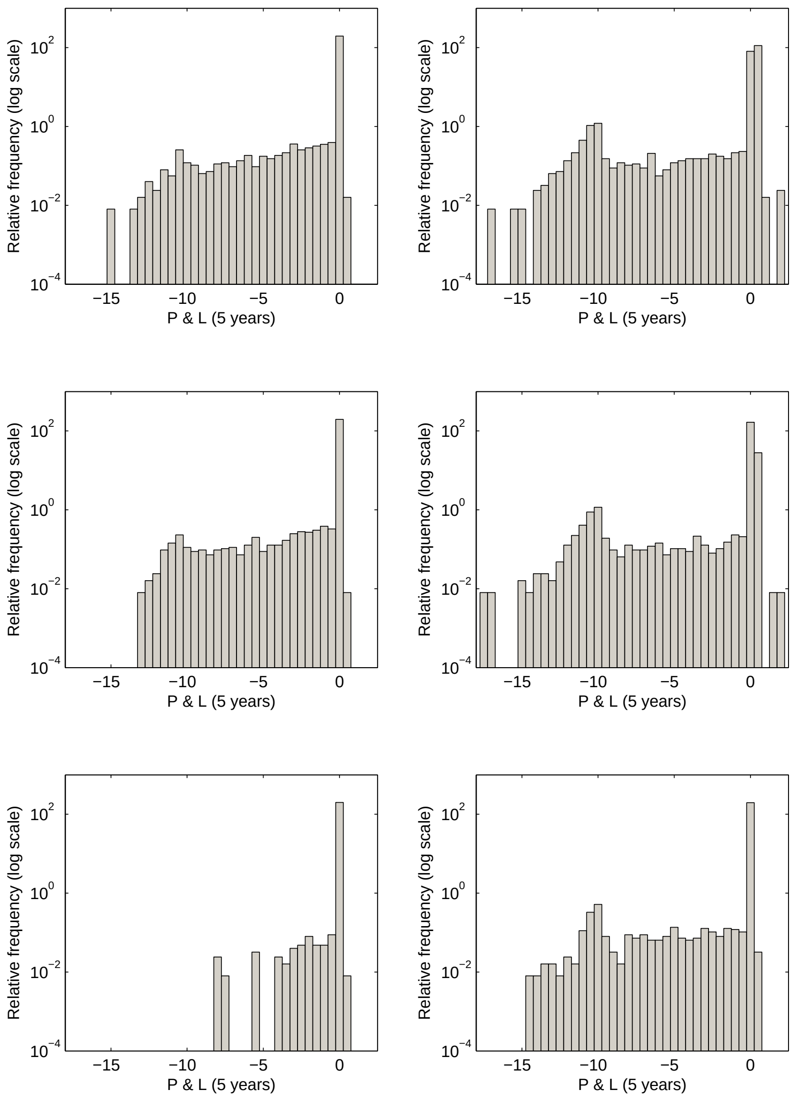

Figure C.6: Histograms (log scale) of the P & L distribution (in millions of Euro) after 5 years of a protection seller in the fifth tranche for Gaussian (left column) and Gumbel (right column) model with deterministic recovery rates (top row), stochastic recovery rates (middle row), and stochastic recovery rates and collateral (bottom row).

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.728	0.482	0.332	0.196
3-6%	0.728	1	0.744	0.497	0.289
6-9%	0.482	0.744	1	0.759	0.427
9-12%	0.332	0.497	0.759	1	0.628
12-22%	0.196	0.289	0.427	0.628	1

Table C.15: Empirical correlations of P & L distributions after 5 years of different tranches in the Gaussian model with deterministic recovery rates.

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.719	0.469	0.322	0.196
3-6%	0.719	1	0.738	0.493	0.295
6-9%	0.469	0.738	1	0.760	0.444
9-12%	0.322	0.493	0.760	1	0.652
12-22%	0.196	0.295	0.444	0.652	1

Table C.16: Empirical correlations of P & L distributions after 5 years of different tranches in the Gaussian model with stochastic recovery rates.

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.605	0.307	0.170	0.082
3-6%	0.605	1	0.627	0.336	0.159
6-9%	0.307	0.627	1	0.667	0.301
9-12%	0.170	0.336	0.667	1	0.554
12-22%	0.082	0.159	0.301	0.554	1

Table C.17: Empirical correlations of P & L distributions after 5 years of different tranches in the Gaussian model with stochastic recovery rates and collateral.

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.682	0.505	0.418	0.337
3-6%	0.682	1	0.809	0.647	0.511
6-9%	0.505	0.809	1	0.856	0.664
9-12%	0.418	0.647	0.856	1	0.818
12-22%	0.337	0.511	0.664	0.818	1

Table C.18: Empirical correlations of P & L distributions after 5 years of different tranches in the Gumbel model with deterministic recovery rates.

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.665	0.480	0.396	0.328
3-6%	0.665	1	0.795	0.637	0.516
6-9%	0.480	0.795	1	0.859	0.685
9-12%	0.396	0.637	0.859	1	0.827
12-22%	0.328	0.516	0.685	0.827	1

Table C.19: Empirical correlations of P & L distributions after 5 years of different tranches in the Gumbel model with stochastic recovery rates.

	0-3%	3-6%	6-9%	9-12%	12-22%
0-3%	1	0.612	0.444	0.363	0.276
3-6%	0.612	1	0.810	0.651	0.487
6-9%	0.444	0.810	1	0.867	0.635
9-12%	0.363	0.651	0.867	1	0.781
12-22%	0.276	0.487	0.635	0.781	1

Table C.20: Empirical correlations of P & L distributions after 5 years of different tranches in the Gumbel model with stochastic recovery rates and collateral.

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