This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE ICC 2009 proceedings

On Opportunistic Beamforming in Fast Fading Scenarios

Michel T. Ivrlač, Mario Castañeda and Josef A. Nossek
Institute for Circuit Theory and Signal Processing
Technische Universität München
email: {ivrlac, mario.castaneda, josef.a.nossek}@tum.de

***Abstract*—We comment on the famous work »Opportunistic beamforming using dumb antennas« by P. Viswanath, D.N.C. Tse, and R. Laroia, from 2002. In that paper, it is argued that in an independent and *fast fading* environment, the opportunistic beamforming technique provides *no* performance gain. This assessment is based on the argument that the fading distribution of the equivalent channel, which is obtained through opportunistic beamforming, does *not* depend on the number of transmit antennas in independent and fast fading scenarios. In this paper, we show that this argument of the original work is based on a non-physical model of Rayleigh fading, and as a consequence, results in a wrong conclusion. In contrast, we show that there *is* beamforming gain to be obtained by the opportunistic beamforming technique *even for independent and fast fading* environments. For the case of uncoupled antennas, the amount of obtainable beamforming gain approaches – with probability one – the number of transmit antennas from below as the number of users approaches infinity.**

I. Introduction

In a *slowly* Rayleigh fading multi-user downlink scenario with a channel quality feedback, a transmit beamforming gain approaching the number of (uncoupled) transmit antennas can be achieved by random beamforming and scheduling transmission to the user which happens to have the best channel quality [1]. Slow fading means, in this context, that the random beamforming vectors change much more quickly than the channel vectors. In a *fast* Rayleigh fading scenario, on the other hand, the channel vectors change much more quickly than the random beamforming vectors. In this way, one can view the beamforming vector as a constant vector, while the channel vectors quickly jump from one random realization of Rayleigh fading to another. In the original work [1], it is argued that in such a fast and independent (and identically distributed) Rayleigh fading scenario, at any time t, the k-th user's probability density function of the equivalent channel coefficient $h_k(t)$ does *not* depend on the number of transmit antennas. Near the end of the right hand side column on page 1283 of [1], we find the following argument: »It can be seen that in this case, $h_k(t)$ has exactly the same distribution as each of the individual gains $h_{nk}(t)$, and moreover, the overall gains are independent across the users. Thus, in an independent fast Rayleigh-fading environment, the opportunistic beamforming technique does not provide any performance gain.« However, we shall see that this argument is based upon a non-physical model of Rayleigh fading, and as a consequence, results in the non-physical conclusion of having no beamforming gain.

In asymptotic analysis of large opportunistic beamforming systems, there are *two* limit operations involved – one obvious, another one subtly hidden. It is the *order* of application of those two limit operations which turns out to make a big difference. The obvious limit operation is applied when the number of users is growing toward infinity. The second limit operation is hidden inside the Rayleigh distribution itself. A Rayleigh distribution arises when both the real part and the imaginary part of a complex random variable are zero-mean independent and identically Gaussian distributed. Physical phenomena which give rise to be modeled by a Gaussian distributed random variable, always are made up of a superposition of many independent micro-phenomena. The chaotic Brownian motion of electrons, as an example, results – in their superposition – in Gaussian thermal noise. The finiteness of the electric charge quantum leads to statistical fluctuations of electric current, which – again in the superposition of a large number of charge quantums – results in Gaussian current noise (so-called shot-noise). Yet another example is the case where many electro-magnetic wavefronts are scattered from objects and superimpose with different Doppler shifts in frequency at a receive antenna. In the equivalent complex baseband representation, this can be modeled by a complex random channel coefficient, which is zero-mean Gaussian distributed in real and imaginary part, and leads to the famous Rayleigh fading – the darling of all wireless communication engineers.

The reason why so many physical phenomena can be modeled by a Gaussian random variable is founded in the well known central limit theorem (see, for instance, [2] for an in-depth mathematical treatment), which basically states that the distribution of the superposition of – almost arbitrarily distributed – independent random variables approaches the Gaussian distribution as the number of these independent random variables approaches infinity. This constitutes the *second* limit operation, as mentioned beforehand, which is hidden inside *every* Rayleigh distribution that is intended to model a physical phenomenon.

In many situations we do not care about this limit operation, but use the Gaussian distribution »out of the box«. And many times it just works out fine. However, in cases when there are additional limit operations involved, we need to be careful with this approach, since the exchange of the order of limit operations does not always preserve equality. In cases when, indeed, such a change of order leads to different results, it is natural to ask which result is »true«, that is, which result represents a *physical truth*, that – in principle, at least – is verifiable by measurement.

The analysis of the gain due to multiple transmit antennas obtainable by opportunistic beamforming in a *fast* Rayleigh fading scenario turns out to be an example of such a case, where the order of limit operations leads to drastically different results. The order chosen – implicitly, though – by the authors of [1], is to let the number of independent random variables, which make up the channel fading – we will call them *fading-constituent random variables* – approach infinity *before* the number of users starts to approach infinity. This leads to the result of [1], that there is *no* gain in having multiple transmit antennas. On the other hand, if one lets the number of fading-constituent random variables approach infinity *after* the number of users has approached infinity, the result is completely different. There *is* a beamforming gain achievable by opportunistic beamforming in a *fast* Rayleigh fading scenario, that approaches the *number of transmit antennas* from below, as the number of users approaches infinity. Now, which of these results possesses physical truth?

The order in which the number of fading-constituent random variables is allowed to approach infinity first, leads to non-physical results, like the receive signal power of the scheduled user approaching infinity as the number of users approaches infinity, even though the transmit power is finite. By reversing the order, and therefore keeping the number of fading-constituent random variables constant and finite while the number of users approaches infinity, such a non-physical behavior is avoided. The receive power of the scheduled user remains finite as the number of user approaches infinity. Indeed, we show in this paper that the obtainable beamforming gain does *not* depend on the number of the fading-constituent random variables, as long as this number is greater than zero. As a consequence, any conflicts with basic principles of physics – including the principle of conservation of energy – can be completely avoided. Because the beamforming gain does not depend on the number of fading-constituent random variables, we can also let this number approach infinity without worries. Consequently, the resulting beamforming gain holds true also for the case of fast Rayleigh fading, which we obtain in the limit of the number of fading-constituent random variables approaching infinity.

In this paper, we assume that there is *no* mutual near-field coupling of the antennas. This assumption is all common in the signal processing and information theory literature – even though it is seldom stated as clearly as this! This assumption is however consistent with physics only when the antennas are separated a positive integer multiple of half the wavelength [3]. When mutual antenna coupling *is* considered, exciting things can happen, like a *super-linear* growth of beamforming gain with the number of antennas, as is shown by two of the authors in [4] for transmit beamforming, and in [5] for receive beamforming. In this paper, however, we do not want to complicate matters, and just stick with the common assumption of uncoupled antennas.

II. Modeling Rayleigh Fading

It is a well known fact that if the real-part and the imaginary-part of a complex random variable are drawn independently from a Gaussian distribution with zero mean, the magnitude of the resulting complex number is Rayleigh distributed. In case where this random number describes the value of an instantaneous channel coefficient, it is common to speak of Rayleigh fading. Let $h_{k,i}^{(N)} \in \mathbb{C}$ be such a complex random channel coefficient defined by:

$$h_{k,i}^{(N)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{j}\Phi_{k,i,n}}, \tag{1}$$

where the $\mathrm{e}^{\mathrm{j}\Phi_{k,i,n}}/\sqrt{N}$, with $n \in \{1,2,\ldots,N\}$, are $N \geq 1$ fading-constituent random variables for the channel coefficient of the k-th user and the i-th transmit antenna, while the constants $\mathrm{j}^2 = -1$, and $\mathrm{e} = \exp(1)$. The random variables $\Phi_{k,i,n}$ are *uniformly distributed* in the interval $[-\pi;\pi]$. Therefore, it follows that

$$\mathrm{E}\left[h_{k,i}^{(N)}\right] = 0, \quad \mathrm{E}\left[\left|h_{k,i}^{(N)}\right|^2\right] = 1, \tag{2}$$

for all k, i and $N \geq 1$. Here E[.] denotes the expectation operation. Note that

$$\left|h_{k,i}^{(N)}\right| \leq \sqrt{N}, \tag{3}$$

where equality holds iff $\Phi_{k,i,n} = \Phi_{k,i}$ for $n \in \{1,2,\ldots,N\}$. From the central limit theorem, it follows that

$$\left(\lim_{N\to\infty} h_{k,i}^{(N)}\right) \sim \mathcal{N}_{\mathbb{C}}\left(0,1\right), \tag{4}$$

that is, as the number N of fading-constituent random variables approaches infinity, the distribution of the channel fading coefficient $h_{k,i}^{(N)}$ converges to the complex, circularly symmetric, Gaussian distribution with zero mean, and unity variance. The random channel vector for user k is defined as:

$$\boldsymbol{h}_k^{(N)} = \left[\begin{array}{cccc} h_{k,1}^{(N)} & h_{k,2}^{(N)} & \cdots & h_{k,M}^{(N)} \end{array}\right]^{\mathrm{T}} \in \mathbb{C}^{M\times 1}, \tag{5}$$

where $M \geq 1$ denotes the number of transmit antennas. Since all the fading-constituent random variables are independent, it follows

$$\left(\lim_{N\to\infty} \boldsymbol{h}_k^{(N)}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_M, \mathbf{I}_M\right), \tag{6}$$

where $\mathbf{0}_M$ is the M-dimensional all-zero vector, and $\mathbf{I}_M$ is the $M \times M$ identity matrix. Rayleigh fading is therefore obtained in the limit as $N \to \infty$.

III. Opportunistic Beamforming in Fast Rayleigh Fading

In fast Rayleigh fading scenarios, the channel vectors change much quicker than the beamforming vectors. Hence, we regard the beamforming vector as constant, while the random channel vectors jump from one fading realization to another. Let us fix the beamforming vector to:

$$\boldsymbol{w}_M = \frac{1}{\sqrt{M}} \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]^{\mathrm{T}} \in \mathbb{C}^{M\times 1}, \tag{7}$$

which has unity norm for all $M \geq 1$. When we assume feedback of channel-quality, the transmitter is aware of the channel power gains $\left|\boldsymbol{w}_M^{\mathrm{T}}\boldsymbol{h}_k^{(N)}\right|^2$, for each user $k \in \{1,2,\ldots,K\}$, and chooses to schedule that user for transmission which happens to have the largest channel power gain:

$$\gamma\left(\boldsymbol{w}_M, K, N\right) = \max_{k\in\{1,2,\ldots,K\}} \left|\boldsymbol{w}_M^{\mathrm{T}}\boldsymbol{h}_k^{(N)}\right|^2. \tag{8}$$

Since the $\boldsymbol{h}_k^{(N)}$ are random, the channel power gain defined in (8) is random, too. We define the average channel power gain of the scheduled user as:

$$\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right) \;=\; \mathrm{E}\left[\gamma\left(\boldsymbol{w}_M, K, N\right)\right]. \qquad (9)$$

IV. Average Obtainable Beamforming Gain in Fast Rayleigh Fading

We are interested in the *two* average beamforming gains:

$$\Gamma\left(M\right) \;=\; \lim_{N\to\infty}\left(\lim_{K\to\infty}\left(\frac{\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right)}{\overline{\gamma}\left(\boldsymbol{w}_1, K, N\right)}\right)\right), \qquad (10)$$

$$\widetilde{\Gamma}\left(M\right) \;=\; \lim_{K\to\infty}\left(\lim_{N\to\infty}\left(\frac{\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right)}{\overline{\gamma}\left(\boldsymbol{w}_1, K, N\right)}\right)\right), \qquad (10a)$$

which differ in the *order* of the application of the two limiting operations. In the following, we show that:

$$\Gamma\left(M\right) \;=\; M, \quad \text{with probability } 1, \qquad (11)$$

$$\widetilde{\Gamma}\left(M\right) \;=\; 1. \qquad (11a)$$

Furthermore, we show that only for the definition (10), conflict with basic principles of physics can be avoided. In this way, it is the result (11) which constitutes the true result from a physics point of view: *in a fast Rayleigh fading scenario a beamforming gain approaching the number of (uncoupled) transmit antennas can be achieved with a probability of one, by the technique of opportunistic beamforming, as the number of users approaches infinity.*

V. Inconsistency with Physics

Let us first investigate the beamforming gain according to the definition (10a). First notice that:

$$\widetilde{\Gamma}\left(M\right) \;=\; \lim_{K\to\infty}\left(\lim_{N\to\infty}\left(\frac{\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right)}{\overline{\gamma}\left(\boldsymbol{w}_1, K, N\right)}\right)\right) \qquad (12)$$

$$\;=\; \lim_{K\to\infty}\left(\frac{\lim_{N\to\infty}\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right)}{\lim_{N\to\infty}\overline{\gamma}\left(\boldsymbol{w}_1, K, N\right)}\right) \qquad (12a)$$

$$\;=\; \lim_{K\to\infty}\left(\frac{\mathrm{E}\left[\lim_{N\to\infty}\gamma\left(\boldsymbol{w}_M, K, N\right)\right]}{\mathrm{E}\left[\lim_{N\to\infty}\gamma\left(\boldsymbol{w}_1, K, N\right)\right]}\right). \qquad (12b)$$

The equality of (12a) and (12b) follows from the fact, that the maximum channel power gain $\gamma\left(\boldsymbol{w}_M, K, N\right)$ for a given finite K remains positive finite for any channel realization, with probability 1, when $N \to \infty$. That is why we can safely pull the limit operation inside the expectation operation, as is done in the transition from (12a) to (12b). Since $\gamma\left(\boldsymbol{w}_M, K, N\right)$ remains finite and positive (with probability 1) as $N \to \infty$, it follows that also its expected value $\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right)$ remains positive finite as $N \to \infty$. From this follows the equality between (12) and (12a). Defining the random variable:

$$\beta_k \;=\; \lim_{N\to\infty} \boldsymbol{w}_M^{\mathrm{T}}\boldsymbol{h}_k^{(N)} \qquad (13)$$

$$\;=\; \frac{1}{\sqrt{M}}\sum_{i=1}^{M}\lim_{N\to\infty} h_{k,i}^{(N)}, \qquad (13a)$$

it follows from (4) that

$$\beta_k \;\sim\; \mathcal{N}_{\mathbb{C}}\left(0, 1\right), \qquad (14)$$

hence, β_k is also a complex Gaussian distributed random variable with zero mean and unity variance. Especially, its statistical properties do *not* depend on the number M of transmit antennas. From (8), (9), (13) and (14), then follows:

$$\lim_{N\to\infty}\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right) = \mathrm{E}\left[\max_{k\in\{1,2,\ldots,K\}} |\beta_k|^2\right],$$

and therefore, we obtain:

$$\lim_{N\to\infty}\overline{\gamma}\left(\boldsymbol{w}_M, K, N\right) \;=\; \lim_{N\to\infty}\overline{\gamma}\left(\boldsymbol{w}_1, K, N\right), \qquad (15)$$

for any M and K. With (12a) then follows

$$\widetilde{\Gamma}\left(M\right) \;=\; 1. \qquad (16)$$

That is, *no* beamforming gain can be obtained. This is exactly the same result obtained on exactly the same line of argument as in the original work [1]. However, this result is *wrong* from a physics viewpoint. To see why, notice that the channel power gain of the scheduled user, that is

$$\lim_{N\to\infty}\gamma\left(\boldsymbol{w}_M, K, N\right) \;=\; \max_{k\in\{1,2,\ldots,K\}} |\beta_k|^2,$$

approaches infinity as the number of users $K \to \infty$, even though the transmit power (squared norm of $\boldsymbol{w}_M$ when there is no mutual antenna coupling) is finite. This shows that it is not sensible to allow the number N of fading-constituent random variables to approach infinity *before* computing the user's channel power gains $|\beta_k|^2$. By letting the number N of fading-constituent random variables approach infinity at the very beginning of the calculation, the remaining calculations are based on a non-physical channel model. This is the reason why the beamforming gain from (16), is *not* the right answer from a physics point of view.

VI. The Actual Beamforming Gain

It is not too difficult to avoid performing calculations with a non-physical channel model. All we have to do is, make sure that the number N of the fading-constituent random variables is kept *finite* all the time. In the following, we compute the beamforming gain for a *finite* number N, and consider the limit of $N \to \infty$ only in the *final* step of our calculation. We can interpret this procedure as follows: by holding N finite, we work with a *non*-Rayleigh fading distribution of channel power gain, which has finite support and therefore conflict with the principle of conservation of energy can easily be avoided. Only in the final step, we let this fading distribution *approach* Rayleigh fading when we let $N \to \infty$.

For finite N, the maximum power gain $\gamma\big(\boldsymbol{w}_M, K, N\big)$ remains finite and positive for *any* channel realizations, when $K \to \infty$. Therefore, also its expected value remains finite, such that we can re-write (10) for a given finite N as:

$$\Gamma\big(M,N\big) = \left(\frac{\lim\limits_{K\to\infty} \overline{\gamma}\big(\boldsymbol{w}_M, K, N\big)}{\lim\limits_{K\to\infty} \overline{\gamma}\big(\boldsymbol{w}_1, K, N\big)}\right) \tag{17}$$

$$= \left(\frac{\mathrm{E}\left[\lim\limits_{K\to\infty} \gamma\big(\boldsymbol{w}_M, K, N\big)\right]}{\mathrm{E}\left[\lim\limits_{K\to\infty} \gamma\big(\boldsymbol{w}_1, K, N\big)\right]}\right). \tag{17a}$$

Consider now a set $\mathcal{H}^{(N)}_{\epsilon',\epsilon''}$ of channel vectors:

$$\mathcal{H}^{(N)}_{\epsilon',\epsilon''} = \left\{ \boldsymbol{h}_k^{(N)} \;\middle|\; \boldsymbol{h}_k^{(N)} = \sqrt{N} \begin{pmatrix} l_1 \mathrm{e}^{\mathrm{j}\Psi_1} \\ l_2 \mathrm{e}^{\mathrm{j}\Psi_2} \\ \vdots \\ l_M \mathrm{e}^{\mathrm{j}\Psi_M} \end{pmatrix} \right\}, \tag{18}$$

with

$$1-\epsilon' \le l_i \le 1, \quad \forall i, \tag{19}$$

and

$$-\epsilon'' \le \Psi_i \le \epsilon'', \quad \forall i, \tag{20}$$

for given $\epsilon' > 0$, $\epsilon'' > 0$, and $N \ge 1$. Let us call P_0 the probability that, by chance, a random realization of $\boldsymbol{h}_k^{(N)}$ fulfills the properties of the set $\mathcal{H}^{(N)}_{\epsilon',\epsilon''}$, that is:

$$P_0\left(\epsilon',\epsilon'',N\right) = \mathrm{Prob}\left\{\boldsymbol{h}_k^{(N)} \in \mathcal{H}^{(N)}_{\epsilon',\epsilon''}\right\}. \tag{21}$$

From (3), and the fact that the angles of the complex components of the channel vectors are uniformly distributed within the interval $[-\pi;\pi]$, it follows that for any pair of any positive ϵ' and any positive ϵ'', there is a positive probability to find such a vector, that is:

$$\forall\left(\epsilon' > 0, \epsilon'' > 0, N \ge 1\right): \quad P_0\left(\epsilon',\epsilon'',N\right) > 0. \tag{22}$$

The probability $P(K,N,\epsilon',\epsilon'')$ with which we can find such a vector within K independent realizations of the channel vectors, is then given by:

$$P\left(K,N,\epsilon',\epsilon''\right) = 1 - \Big(\underbrace{1 - P_0\left(\epsilon',\epsilon'',N\right)}_{<1}\Big)^K. \tag{23}$$

Therefore, $\forall\left(\epsilon' > 0, \epsilon'' > 0, N \ge 1\right)$:

$$\lim_{K\to\infty} P\left(K,N,\epsilon',\epsilon''\right) = 1, \tag{24}$$

that is, as the number of users approaches infinity, we can find, with a probability of one, a user which has a channel vector that comes arbitrarily close to the all-ones vector weighted by $\sqrt{N}$. As $K \to \infty$, there is, with probability 1, a user which channel power gain $\left|\boldsymbol{w}_M^{\mathrm{T}} \boldsymbol{h}_k^{(N)}\right|^2$ comes arbitrarily close to

$$\left|\boldsymbol{w}_M^{\mathrm{T}} \mathbf{1}_M \sqrt{N}\right|^2 = MN, \tag{25}$$

where $\mathbf{1}_M$ denotes the M-dimensional all-ones vector. Note that, because N is finite, this channel power gain is finite for any finite number M of antennas. In this way, conflict with the principle of conservation of energy can be avoided. It follows from (25) and (8) that:

$$\forall\tilde{\epsilon} > 0: \quad MN - \tilde{\epsilon} \le \lim_{K\to\infty} \gamma\big(\boldsymbol{w}_M, K, N\big) \le MN, \tag{26}$$

holds true with probability 1. Obviously, we also have (with probability 1):

$$\forall\tilde{\epsilon} > 0: \quad MN - \tilde{\epsilon} \le \mathrm{E}\left[\lim_{K\to\infty} \gamma\big(\boldsymbol{w}_M, K, N\big)\right] \le MN. \tag{27}$$

Hence,

$$\forall\tilde{\epsilon} > 0: \quad M - \frac{\tilde{\epsilon}}{N} < \frac{\mathrm{E}\left[\lim\limits_{K\to\infty} \gamma\big(\boldsymbol{w}_M, K, N\big)\right]}{\mathrm{E}\left[\lim\limits_{K\to\infty} \gamma\big(\boldsymbol{w}_1, K, N\big)\right]} \le M. \tag{28}$$

Since $\tilde{\epsilon} > 0$ can be made arbitrarily small, the result does *not* depend on the number $N \ge 1$ of fading-constituent random variables:

$$\forall\epsilon > 0: \quad M - \epsilon \le \Gamma\big(M,N\big) \le M, \tag{29}$$

No matter what value N actually has, the beamforming gain does *not* depend on it. That is why we do not even need to perform the final limit operation on N, because the beamforming gain anyway does not depend on N. If we do it nevertheless, it follows that the result also holds true for Rayleigh fading:

$$\forall\epsilon > 0: \quad M - \epsilon \le \Gamma\big(M\big) \le M. \tag{30}$$

If we neglect the arbitrarily small $\epsilon > 0$, we find that the beamforming gain obtained by opportunistic beamforming in a fast fading scenario approaches the number M of (uncoupled) transmit antennas as the number of users approaches infinity. This result is true even for Rayleigh fading.

VII. Conclusion

Contrary to the statement in [1], there *is* beamforming gain obtainable by opportunistic beamforming *even in a fast* Rayleigh fading scenario, when the number of users is large. This result is developed in this paper, by application of a physically consistent model of Rayleigh fading.

References

[1] P. Viswanath, D.N.C. Tse, and R. Laroia, "Opportunistic Beamforming using Dumb Antennas," *IEEE Trans. Information Theory*, vol. 48, pp. 1277–1294, June 2002.

[2] W. Feller, *An Introduction to Probability Theory and its Applications, 3rd Edition*, John Wiley & Sons, 1968.

[3] H. Yordanov, M. T. Ivrlač, P. Russer, and J. A. Nossek, "Arrays of Isotropic Radiators – A Field-theoretic Justification," in *Proc. ITG/IEEE Workshop on Smart Antennas*, Berlin, Germany, Feb. 2009.

[4] M. T. Ivrlač and J. A. Nossek, "The Maximum Achievable Array Gain under Physical Transmit Power Constraint," in *Proc. IEEE International Symposium on Information Theory and its Applications*, Dec. 2008, pp. 1338–1343.

[5] M. T. Ivrlač and J. A. Nossek, "Receive Antenna Gain of Uniform Linear Arrays of Isotrops," in *Proc. of the IEEE International Conference on Communications (ICC2009)*, June 2009.