Feasible Rate Region of the MIMO Broadcast Channel with Linear Transceivers

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Abstract— **Based on recent results for the vector broadcast channel (BC), the feasibility of the quality of service (QoS) optimization in the multiple-input multiple-output (MIMO) BC with linear precoders and linear equalizers is investigated. Based on the observation that only single data stream transmission must be considered for feasibility, we find that the test for feasibility is the comparison of the sum of the transformed rate requirements with a bound given by the difference of the number of users and the degrees of freedom available at the BC transmitter, as long as the channel realization fulfills some regularity condition.**

I. Introduction

The QoS optimization for the MIMO BC evolves from the idea of satisfying the needs of the receivers with the least effort. Usually the demand is expressed in minimum data rate and the effort is formulated as necessary transmit power. The resulting QoS optimization problem reads as

$$\min P_{\text{tx}} \qquad \text{s.t.:} \quad R_k \geq \rho_k \quad \forall k \in \{1,\ldots,K\} \tag{1}$$

where the transmit power P_{tx} is minimized w.r.t. the linear precoding operation, R_k denotes the rate of user k, and $\rho_k > 0$ is the respective rate requirement. Note that above formulation does not include any upper bound for the transmit power P_{tx}. Therefore, it is possible that the resulting transmit power to meet the rate requirements of the K receivers is impractically large. Additionally, (1) does not necessarily have a solution due to the assumption of linear precoding, i.e., not all requirements can be met even for unbounded transmit power.

Testing feasibility and solving (1) has been researched extensively for the vector BC. As the rate for Gaussian signalling is connected with the signal-to-interference-and-noise ratio (SINR) and also with the minimum mean square error (MMSE) by bijective mappings, (1) is rather formulated with minimum SINR or maximum MMSE constraints. In [1], [2], [3], [4], [5], the following alternative form for (1) was used

$$\min P_{\text{tx}} \qquad \text{s.t.:} \quad \text{SINR}_k \geq \gamma_k \quad \forall k \in \{1,\ldots,K\} \tag{2}$$

whose optimum and optimizer are the same as that of (1), if $\rho_k = \log_2(1+\gamma_k)\ \forall k$ holds. As mentioned in [4], (2) has a solution for any set $\gamma_k > 0\ \forall k$, if the channel vectors describing the propagation from the transmitter to the receivers are linearly independent, because a zero-forcing precoding operation is possible that decouples the SINR's of the different users. This result leads to the necessary condition for the feasibility of arbitrary requirements that the number of transmit antennas N is larger than or equal to the number of receivers K. However, feasibility must be tested for the case of $N < K$ or rank-deficient channels. One possibility is to run an algorithm that solves the QoS optimization and to see whether it converges or not. A more preferable test relies on the balancing optimization, where the ratios of the actual SINR's over the respective targets is made equal for all receivers. Note that the balancing optimization always has a solution. If the noise term in the SINR definition is dropped as in [6], [7], [8], [9], [10], [11], the balancing is done for infinite transmit power, i.e., the signal-to-interference ratios (SIR's) are balanced. Therefore, the test for feasibility based on the balancing optimization is as follows. Run the SIR balancing. In the case that the ratio of the resulting SIRs and the SINR requirement is larger than one, (2) has a solution. Otherwise, the SINR requirements cannot be met.

In [12], [13], it was revealed that feasibility for the QoS requirements in the vector BC can be tested without running any algorithm. Instead, the rate requirements must be transformed to MMSE requirements and only the sum of the MMSE requirements has to be compared to $K - \operatorname{rank}(\boldsymbol{H})$ for regular channels [see (4)], where $\boldsymbol{H}$ denotes the matrix comprising the channel vectors. If the sum of the MMSE requirements lies above $K - \operatorname{rank}(\boldsymbol{H})$, (1) is feasible and otherwise it is not.

In this paper, we generalize the results of [12], [13] to the MIMO BC. It is shown that still such a simple condition suffices to test feasibility. Interestingly, we find that the number of antennas at the receivers has no influence on feasibility.

II. System Model and Assumptions

Relying on the rate duality of the MIMO BC and the MIMO multiple access channel (MAC) with linear transceivers shown in [14], we focus on the dual MIMO MAC. The d_k-dimensional data signal $\boldsymbol{s}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0},\mathbf{I})$ of the k-th user is linearly precoded with $\boldsymbol{T}_k \in \mathbb{C}^{M_k \times d_k}$ and transmitted over the k-th channel $\boldsymbol{H}_k \in \mathbb{C}^{N \times M_k}$ with $\boldsymbol{H}_k \neq \mathbf{0}\ \forall k$. The MIMO MAC receiver observes $\boldsymbol{x} \in \mathbb{C}^N$ which is the superposition of the channel outputs and the additive noise $\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0},\mathbf{I})$. The received signal is passed through the equalizer $\boldsymbol{G}_k \in \mathbb{C}^{d_k \times N}$ to get the estimate $\hat{\boldsymbol{s}}_k \in \mathbb{C}^{d_k}$ for $\boldsymbol{s}_k$.

A practically important setup is when the channels are regular, i.e., they fulfill following condition:

$$\forall \mathcal{I} \subseteq \{1,\ldots,K\}: \quad \operatorname{rank}(\boldsymbol{H}_{\mathcal{I}}) \geq \min(|\mathcal{I}|, N) \tag{3}$$

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where $\boldsymbol{H}_{\mathcal{I}}$ comprises all channels $\boldsymbol{H}_i$ with $i \in \mathcal{I}$ in a block row, e.g., $\boldsymbol{H}_{\mathcal{I}} = [\boldsymbol{H}_1, \boldsymbol{H}_3]$ for $\mathcal{I} = \{1, 3\}$. For vector channels ($M_k = 1\ \forall k$), above regularity condition implies that every user subset must have a full-rank channel matrix [12], [13]:

$$\forall \mathcal{I} \subseteq \{1, \ldots, K\}: \quad \operatorname{rank}(\boldsymbol{H}_{\mathcal{I}}) = \min(|\mathcal{I}|, N). \tag{4}$$

Consequently, the vector channels must be different. Note, however, that (3) allows for identical channels in the MIMO case. For example, if $\boldsymbol{H}_1 = \boldsymbol{H}_2 = \boldsymbol{H}_3 \in \mathbb{C}^{N \times M_1}$ is full rank, (3) is not violated for $M_1 \geq 3$.

The setup with channels that do not fulfill (3) is denoted as singular case in the following.

III. Vector Broadcast Channel

In the dual vector MAC, the channels are vectors $\boldsymbol{h}_i \in \mathbb{C}^N$ and the precoders are scalars $t_k \in \mathbb{R}_+\ \forall k$.[1] Additionally, the inequality of the regularity condition (3) becomes an equality [see (4)]. Define the mean square error (MSE) for user k as

$$\varepsilon_k = \mathrm{E}\left[|s_k - \hat{s}_k|^2\right]$$

and the minimum MSE resulting from the MMSE optimum equalizer as

$$\mathrm{MMSE}_k = \min_{\boldsymbol{g}_k^{\mathrm{H}}} \varepsilon_k.$$

The optimizer $\boldsymbol{g}_{\mathrm{MMSE},k}^{\mathrm{H}}$ is the linear MMSE filter (e.g., [15])

$$\boldsymbol{g}_{\mathrm{MMSE},k}^{\mathrm{H}} = t_k \boldsymbol{h}_k^{\mathrm{H}} \left(\mathbf{I} + \sum_{i=1}^{K} t_i^2 \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}}\right)^{-1}$$

leading to

$$\begin{aligned}\mathrm{MMSE}_k &= 1 - t_k^2 \boldsymbol{h}_k^{\mathrm{H}} \left(\mathbf{I} + \sum_{i=1}^{K} t_i^2 \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}}\right)^{-1} \boldsymbol{h}_k \qquad (5)\\ &= \frac{1}{1 + t_k^2 \boldsymbol{h}_k^{\mathrm{H}} \left(\mathbf{I} + \sum_{i=1, i\neq k}^{K} t_i^2 \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}}\right)^{-1} \boldsymbol{h}_k}.\end{aligned}$$

With this result and assuming complex Gaussian signalling, the data rate for user k can be expressed as

$$R_k = -\log_2(\mathrm{MMSE}_k). \tag{6}$$

Therefore, any rate target ρ_k can be translated into an MMSE target with

$$\mu_k = 2^{-\rho_k}. \tag{7}$$

Employing these definitions, (1) translates to

$$\min_{t_1, \ldots, t_K} P_{\mathrm{tx}} \qquad \text{s.t.:} \quad \mathrm{MMSE}_k \leq \mu_k. \tag{8}$$

Clearly, (1) is feasibiliy, if and only if (8) is feasible. The feasibility test for either (1), (2), or (8) is discussed in the following.

[1] The precoders are assumed to be real, since the MMSE, the SINR, and the rate are independent of the phase shift introduced by the precoder.

A. Previous Results on QoS Feasibility

In [16], the feasibility of (2) is discussed for a synchronous code division multiple access (CDMA) system with MMSE receivers. It is proven [16, Theorem 5.1] that some SINR requirements $\gamma_1, \ldots, \gamma_K$ are feasible, iff

$$\sum_{i=1}^{K} \frac{\gamma_i}{1 + \gamma_i} < N. \tag{9}$$

In other words, any CDMA codes of length N lead to SINR's fulfilling (9) and for any SINR requirements obeying (9), codes can be constructed such that (2) has a solution. Translated into the considered setup, it is proven that always a channel exists for which (9) holds. This would mean that the channels $\boldsymbol{h}_i\ \forall i$ have to be appropriately designed to ensure feasibility. Unfortunately, the channels are given and fixed in the BC problem at hand.

The QoS formulation (8) was optimally solved and feasibility conditions were provided for point-to-point MIMO systems with MMSE receivers in [17]. According to [17, Theorem 2], (8) has a solution for point-to-point MIMO systems, iff

$$\sum_{i=1}^{K} \mu_i > K - \operatorname{rank}(\boldsymbol{H}) \tag{10}$$

where $\boldsymbol{H} = [\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K] \in \mathbb{C}^{N \times K}$ comprises the N-dimensional channels of the K users. Due to the relation between the maximum SINR and the minimum MSE, we have that

$$\gamma_i = \frac{1}{\mu_i} - 1 \tag{11}$$

and (9) is equivalent to (10), if $\operatorname{rank}(\boldsymbol{H}) = N$. The point-to-point MIMO precoder $\boldsymbol{T}$ is assumed to have no special structure in [17].[2] In contrast, the precoder of the dual MAC has a particular structure, e.g., $\boldsymbol{T} = \operatorname{diag}(t_1, \ldots, t_K)$ for the vector MAC. Therefore, the result from [17] is not applicable to the vector MAC.

The QoS optimization (2) for the MIMO BC with fixed receivers (not necessarily the MMSE receivers), that is finally equivalent to a vector BC, is investigated in [4]. Besides the reformulation of (2) as a second order cone program (SOCP), it is proven [4, Proposition 1] that there exists a precoder in the vector BC such that $\min_k \mathrm{SINR}_k \geq \gamma$ or equivalently, $\mathrm{SINR}_k \geq \gamma\ \forall k$, is fulfilled, only if

$$\gamma \leq \frac{1}{\frac{K}{\operatorname{rank}(\boldsymbol{H})} - 1}.$$

This necessary condition shows that any precoder in the vector BC leads to a minimum SINR fulfilling above condition. However, no sufficient condition is provided in [4]. So, it is unclear whether a precoder for a QoS requirement γ fulfilling above condition exists or not, as discussed in [4].

[2] Except for the carrier-noncooperative scheme, where the precoder is block-diagonal. However, this comes along with a block-diagonal equalizer. Additionally, the corresponding channel is also block-diagonal. Thus, the carrier-noncooperative problem falls apart into sub-problems having also no structural constraints for the precoder and the equalizer.

The guaranteed MSE region of an orthogonal space-time block coded (OSTBC) BC with an average transmit power constraint was identified in [18]. Including the receiver, aforementioned system is equivalent to a single-input single-output (SISO) BC. Although an average transmit power constraint was included, it was possible to show in [18, Theorem 1] that the MSE's $\mu_1, \ldots, \mu_K$ are guaranteed, iff

$$\sum_{k=1}^{K} \mathrm{E}\left[\frac{1}{\alpha_k}\right](1-\mu_k) \leq \frac{\mathrm{E}[P_{\mathrm{tx}}]}{\sigma_\eta^2}\left(1-\sum_{k=1}^{K}(1-\mu_k)\right)$$

with the SISO channel coefficient α_k for user k and the common noise variance σ_η^2. From this result, it was concluded in [18, Remark 1] that some MMSE QoS requirements are not feasible, if

$$\sum_{k=1}^{K} \mu_k < K-1.$$

Note that $\mathrm{rank}(\boldsymbol{H}) = 1$ holds, since a SISO BC was considered in [18]. Therefore, above non-feasibility condition is very similar to the feasibility condition in (10).

B. Regular Channels

In [12], [13], following theorem for the vector BC with regular channels fulfilling (4) was proven.

Theorem 1: The closure of the feasible MMSE region in the vector broadcast channel with linear transceivers and regular channels satisfying (4) is a polytope $\mathcal{P}$ whose bounding half-spaces are the individual box constraints $0 \leq \mathrm{MMSE}_k \leq 1\ \forall k \in \{1,\ldots,K\}$ and the sum MMSE constraint $\sum_{k=1}^{K} \mathrm{MMSE}_k \geq K-N$. By means of a positive power allocation with finite sum power, any point belonging to the interior of the polytope can be achieved. For MMSEs equal to one, no power is allocated to the respective user.

Note that this result was obtained without any assumptions regarding the channels except that (4) must hold.

Since the MMSE requirements μ_k are found via (7), the box constraints $0 < \mu_k \leq 1$ are always fulfilled for valid rate requirements, i.e., $0 < \rho_k < \infty\ \forall k$. Therefore, the test for feasibility in the vector BC with regular channels is simply [cf. (10)]

$$\sum_{k=1}^{K} \mu_k > K-N.$$

The QoS optimization (1) for the vector BC has a solution, if and only if above condition for the sum of the MMSE requirements is fulfilled.

C. Singular Channels

If the channels do not fulfill (4), the closure of the feasible MMSE region is still a polytope, but the sum MMSE bound is slightly changed and additional bounds apply [13]. In the following, we will give a rigorous prove for this statement.

If (4) is violated, it is possible that the composite channel $\boldsymbol{H} = [\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K] \in \mathbb{C}^{N\times K}$ is rank deficient, i.e., $\mathrm{rank}(\boldsymbol{H}) < \min(K,N)$. Let us define $r = \mathrm{rank}(\boldsymbol{H})$ for notational brevity. Then, some sub-unitary basis $\boldsymbol{U} \in \mathbb{C}^{N\times r}$ for $\mathrm{range}(\boldsymbol{H})$ can be easily obtained, e.g., with the QR factorization with column pivoting [19, Subsection 5.4.1]. Since the dual vector MAC is constructed such that the noise is white, i.e., $\boldsymbol{\eta} \sim \mathcal{N}_\mathbb{C}(\boldsymbol{0}, \mathbf{I})$, the front-end $\boldsymbol{U}^\mathrm{H}$ delivers a sufficient statistic for $\boldsymbol{s} = [s_1, \ldots, s_K]^\mathrm{T} \in \mathbb{C}^K$ and its output is

$$\boldsymbol{\xi} = \boldsymbol{U}^\mathrm{H}\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{T}\boldsymbol{s} + \boldsymbol{n} \in \mathbb{C}^r$$

with the equivalent channel $\boldsymbol{\Phi} = \boldsymbol{U}^\mathrm{H}\boldsymbol{H} \in \mathbb{C}^{r\times K}$, the noise $\boldsymbol{n} \sim \mathcal{N}_\mathbb{C}(\boldsymbol{0}, \mathbf{I})$, and $\boldsymbol{T} = \mathrm{diag}(t_1, \ldots, t_K) \in \mathbb{R}^{K\times K}$ comprises the precoders of the K vector MAC transmitters. It is possible that the columns $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_K \in \mathbb{C}^r$ of the equivalent channel $\boldsymbol{\Phi}$ fulfill (4), e.g., for a two user system with colinear channels. In this case, Theorem 1 can be applied and (1) has a solution, iff $\sum_{k=1}^{K} \mu_k > K-r$.

However, the equivalent channels might again violate (4). Also in this case, $\sum_{k=1}^{K} \mu_k > K-r$ must hold. But as we will see in the following, additional conditions for the MMSE requirements of the user sets $\mathcal{I} \subset \{1,\ldots,K\}$ violating $\mathrm{rank}(\boldsymbol{H}_\mathcal{I}) = \min(|\mathcal{I}|, r)$ [cf. (4)] arise.

With (5) and replacing $\boldsymbol{H}$ by $\boldsymbol{\Phi}$, the sum MMSE for any power allocation $t_1^2, \ldots, t_K^2$ in the dual MAC can be written as

$$\sum_{k=1}^{K} \mathrm{MMSE}_k = K - \mathrm{tr}\left(\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^\mathrm{H}\left(\mathbf{I}+\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^\mathrm{H}\right)^{-1}\right).$$

Since the identity matrix does not change the eigenbasis, we can infer that all eigenvalues of $\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^\mathrm{H}(\mathbf{I}+\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^\mathrm{H})^{-1}$ are non-negative and smaller than one. Therefore, the supremum of the trace is r and we have for the sum MMSE that

$$\sum_{k=1}^{K} \mathrm{MMSE}_k > K-r. \tag{12}$$

The bound can only be reached for infinite power, e.g., $\sum_{k=1}^{K} \mathrm{MMSE}_k = K-r$, when all diagonal elements of $\boldsymbol{T}^2$ tend to infinity.

Let $\mathcal{I} \subset \{1,\ldots,K\}$. Additionally, the columns of $\boldsymbol{\Phi}$ are reordered with the permutation matrix $\boldsymbol{\Pi}_\mathcal{I} \in \{0,1\}^{K\times K}$ such that it can be partitioned as $\boldsymbol{\Phi}\boldsymbol{\Pi}_\mathcal{I} = [\boldsymbol{\Phi}_\mathcal{I}, \boldsymbol{\Phi}_{\bar{\mathcal{I}}}]$, where $\boldsymbol{\Phi}_\mathcal{I} \in \mathbb{C}^{r\times|\mathcal{I}|}$ comprises the columns of $\boldsymbol{\Phi}$ with an index out of $\mathcal{I}$ and the rest is collected in $\boldsymbol{\Phi}_{\bar{\mathcal{I}}} \in \mathbb{C}^{r\times K-|\mathcal{I}|}$. Similarly, the symmetric permutation $\boldsymbol{\Pi}_\mathcal{I}^\mathrm{T}\boldsymbol{T}\boldsymbol{\Pi}_\mathcal{I}$ of the diagonal precoder matrix $\boldsymbol{T}$ gives the upper left block $\boldsymbol{T}_\mathcal{I} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{I}|}$ and the lower right block $\boldsymbol{T}_{\bar{\mathcal{I}}} \in \mathbb{R}^{K-|\mathcal{I}|\times K-|\mathcal{I}|}$. With the canonical unit vector $\boldsymbol{e}_i \in \{0,1\}^K$, whose i-th element is one, applying the matrix inversion lemma to (5), and substituting $\boldsymbol{\Phi}$ for $\boldsymbol{H}$ yields

$$\begin{aligned}
\mathrm{MMSE}_k &= 1 - \boldsymbol{e}_k^\mathrm{T}\boldsymbol{T}\boldsymbol{\Phi}^\mathrm{H}\left(\mathbf{I}+\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^\mathrm{H}\right)^{-1}\boldsymbol{\Phi}\boldsymbol{T}\boldsymbol{e}_k \\
&= \boldsymbol{e}_k^\mathrm{T}\left(\mathbf{I}+\boldsymbol{T}\boldsymbol{\Phi}^\mathrm{H}\boldsymbol{\Phi}\boldsymbol{T}\right)^{-1}\boldsymbol{e}_k \\
&= \boldsymbol{e}_{\pi(k)}^\mathrm{T}\left(\mathbf{I}+[\boldsymbol{\Phi}_\mathcal{I}\boldsymbol{T}_\mathcal{I}, \boldsymbol{\Phi}_{\bar{\mathcal{I}}}\boldsymbol{T}_{\bar{\mathcal{I}}}]^\mathrm{H}[\boldsymbol{\Phi}_\mathcal{I}\boldsymbol{T}_\mathcal{I}, \boldsymbol{\Phi}_{\bar{\mathcal{I}}}\boldsymbol{T}_{\bar{\mathcal{I}}}]\right)^{-1}\boldsymbol{e}_{\pi(k)}
\end{aligned}$$

with $\boldsymbol{e}_{\pi(k)} = \boldsymbol{\Pi}_\mathcal{I}^\mathrm{T}\boldsymbol{e}_k$. Note that the $|\mathcal{I}|$ upper left diagonal elements of the inverse in the last line are the MMSE's of the users with indices out of $\mathcal{I}$. The $|\mathcal{I}|\times|\mathcal{I}|$ MMSE matrix of these users can be found with the matrix inversion lemma for partitioned matrices

$$\boldsymbol{M}_{\mathcal{I}} = \left(\mathbf{I} + \boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\left(\mathbf{I} - \boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\left(\mathbf{I} + \boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\right)^{-1}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\right)\boldsymbol{\Upsilon}_{\mathcal{I}}\right)^{-1}$$

where $\boldsymbol{\Upsilon}_{\mathcal{I}} = \boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}} \in \mathbb{C}^{r\times|\mathcal{I}|}$ and $\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}} = \boldsymbol{\Phi}_{\bar{\mathcal{I}}}\boldsymbol{T}_{\bar{\mathcal{I}}} \in \mathbb{C}^{r\times K-|\mathcal{I}|}$. To find the minimum MMSE's, we have to investigate the case, where the powers of the users go to infinity. To this end, we introduce ν which grows with increasing power. However, we let the powers of the users in $\mathcal{I}$ have a different growth rate than the powers of the other users, i.e., $\boldsymbol{T}_{\bar{\mathcal{I}}}$ is weighted with $\nu^{\omega/2}$ and $\boldsymbol{T}_{\mathcal{I}}$ with $\nu^{\omega_{\mathcal{I}}/2}$, where $\omega_{\mathcal{I}}, \omega > 0$:

$$\boldsymbol{M}_{\mathcal{I}} = \left(\mathbf{I} + \nu^{\omega_{\mathcal{I}}}\boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\left(\mathbf{I} - \nu^{\omega}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\left(\mathbf{I} + \nu^{\omega}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\right)^{-1}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\right)\boldsymbol{\Upsilon}_{\mathcal{I}}\right)^{-1}.$$

Due to the matrix inversion lemma, we have that

$$\mathbf{I} - \nu^{\omega}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\left(\mathbf{I} + \nu^{\omega}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\right)^{-1}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}} = \left(\mathbf{I} + \nu^{\omega}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}\right)^{-1}$$

leading to

$$\boldsymbol{M}_{\mathcal{I}} = \mathbf{I} - \boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\left(\nu^{-\omega_{\mathcal{I}}}\mathbf{I} + \nu^{\omega-\omega_{\mathcal{I}}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}} + \boldsymbol{\Upsilon}_{\mathcal{I}}\boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\right)^{-1}\boldsymbol{\Upsilon}_{\mathcal{I}}.$$

This result enables to investigate the behavior of the MMSE matrix $\boldsymbol{M}_{\mathcal{I}}$ of the users in $\mathcal{I}$ for infinite power ($\nu \to \infty$):

$$\begin{aligned}
\omega_{\mathcal{I}} > \omega:&\quad \boldsymbol{M}_{\mathcal{I}} \to \mathbf{I} - \boldsymbol{P}_{\mathcal{I}}\\
\omega_{\mathcal{I}} = \omega:&\quad \boldsymbol{M}_{\mathcal{I}} \to \mathbf{I} - \boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\left(\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^{\mathrm{H}}\right)^{-1}\boldsymbol{\Upsilon}_{\mathcal{I}}\\
\omega_{\mathcal{I}} < \omega:&\quad \boldsymbol{M}_{\mathcal{I}} \to \mathbf{I} - \boldsymbol{P}_{\boldsymbol{P}_{\bar{\mathcal{I}}}(\mathcal{I})}
\end{aligned} \tag{13}$$

where $\boldsymbol{P}_{\mathcal{I}}$ denotes the projector onto the span of $\boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}$. For the case $\omega_{\mathcal{I}} = \omega$, we restrict to the setup with full rank $\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}$, i.e., $\operatorname{rank}(\boldsymbol{\Phi}\boldsymbol{T}) = r$. For $\omega_{\mathcal{I}} < \omega$, we used the substitute $\boldsymbol{B} = \nu^{-\omega_{\mathcal{I}}}\mathbf{I} + \nu^{\omega-\omega_{\mathcal{I}}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}$ together with the matrix inversion lemma to find

$$\boldsymbol{M}_{\mathcal{I}} = \left(\mathbf{I} + \boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\boldsymbol{B}^{-1}\boldsymbol{\Upsilon}_{\mathcal{I}}\right)^{-1}.$$

For infinitely large power and $\omega_{\mathcal{I}} < \omega$, the inverse of the substitute becomes the weighted projector onto the nullspace of $\boldsymbol{\Upsilon}_{\bar{\mathcal{I}}}^{\mathrm{H}}$, i.e., $\boldsymbol{B}^{-1} \to \nu^{\omega_{\mathcal{I}}}\boldsymbol{P}_{\bar{\mathcal{I}}}^{\perp}$ for $\nu \to \infty$. Similarly, the MMSE matrix $\boldsymbol{M}_{\mathcal{I}}$ converges to the projector $\mathbf{I} - \boldsymbol{P}_{\boldsymbol{P}_{\bar{\mathcal{I}}}(\mathcal{I})}$, where $\boldsymbol{P}_{\boldsymbol{P}_{\bar{\mathcal{I}}}(\mathcal{I})}$ projects onto the span of $\boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\boldsymbol{P}_{\bar{\mathcal{I}}}^{\perp}$.

The result for the MMSE matrix in (13) can be generalized as follows. Assume that there are F user groups $\mathcal{I}_1, \ldots, \mathcal{I}_F$ with different exponents $\omega_1 > \cdots > \omega_F > 0$ of ν which grows with increasing transmit power. For the users with the largest exponent ω_1, i.e., the users in $\mathcal{I}_1$, the first line in (13) is valid ($\boldsymbol{M}_{\mathcal{I}_1} \to \mathbf{I} - \boldsymbol{P}_{\mathcal{I}_1}$ for $\nu \to \infty$) and the MMSE matrix of the users with the smallest exponent ω_F has the form of the last line in (13), that is, $\boldsymbol{M}_{\mathcal{I}_F} \to \boldsymbol{P}_{\boldsymbol{P}_{\overline{\mathcal{I}_F}}(\mathcal{I}_F)}$ for $\nu \to \infty$. For the other user groups, users with smaller exponents can be neglected [first line of (13)] and the users with larger exponents lead to a projection of the channels [last line of (13)]. Thus, $\boldsymbol{M}_{\mathcal{I}_\ell} \to \mathbf{I} - \boldsymbol{P}_{\boldsymbol{P}_{\overline{\mathcal{I}_\ell \cup \cdots \cup \mathcal{I}_F}}(\mathcal{I}_\ell)}$ for $\nu \to \infty$ and $\ell > 1$.

Note that

$$\operatorname{rank}(\boldsymbol{P}_{\mathcal{I}}) = \operatorname{rank}(\boldsymbol{\Upsilon}_{\mathcal{I}}^{\mathrm{H}}\left(\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^{\mathrm{H}}\right)^{-1}\boldsymbol{\Upsilon}_{\mathcal{I}}) = \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}).$$

and

$$\operatorname{rank}(\boldsymbol{P}_{\boldsymbol{P}_{\bar{\mathcal{I}}}(\mathcal{I})}) = \operatorname{rank}(\boldsymbol{P}_{\bar{\mathcal{I}}}^{\perp}\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}) \le \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}).$$

Therefore, the sum MMSE $\sum_{k\in\mathcal{I}} \mathrm{MMSE}_k = \operatorname{tr}(\boldsymbol{M}_{\mathcal{I}})$ of the users out of $\mathcal{I}$ converges to [cf. (13)]

$$\operatorname{tr}(\boldsymbol{M}_{\mathcal{I}}) \to \begin{cases} |\mathcal{I}| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}) & \omega_{\mathcal{I}} > \omega\\ |\mathcal{I}| - \operatorname{tr}\left(\left(\boldsymbol{\Phi}\boldsymbol{T}^2\boldsymbol{\Phi}^{\mathrm{H}}\right)^{-1}\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}^2\boldsymbol{\Phi}_{\mathcal{I}}^{\mathrm{H}}\right) & \omega_{\mathcal{I}} = \omega\\ |\mathcal{I}| - \operatorname{rank}(\boldsymbol{P}_{\bar{\mathcal{I}}}^{\perp}\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}) & \omega_{\mathcal{I}} < \omega \end{cases}$$

for $\nu \to \infty$ and can be bounded for any finite power:

$$\sum_{k\in\mathcal{I}} \mathrm{MMSE}_k > |\mathcal{I}| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}) \tag{14}$$

since $\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}) \ge \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}}) \ge \operatorname{rank}(\boldsymbol{P}_{\bar{\mathcal{I}}}^{\perp}\boldsymbol{\Phi}_{\mathcal{I}}\boldsymbol{T}_{\mathcal{I}})$. Note that (14) is trivial for $\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}) = |\mathcal{I}|$, since the MMSE's are non-negative by definition. Additionally, (14) follows from (12), if $\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}}) = r$, because the MMSE of any other user not included in $\mathcal{I}$ is upper bounded by one. Sorting these redundant bounds out leads to the condition used in (4). Any user set $\mathcal{I} \subset \{1, \ldots, K\}$ with $\operatorname{rank}(\boldsymbol{H}_{\mathcal{I}}) < \min(|\mathcal{I}|, N)$ has a sum MMSE larger than $|\mathcal{I}| - \operatorname{rank}(\boldsymbol{H}_{\mathcal{I}})$.

After providing the necessary conditions (12) and (14) for the case that (4) is not fulfilled, we show that they are also sufficient. The proof for the singular case is similar to the proof of sufficiency for the regular case given in [13].

Assume that some power allocation $\check{\tau}_1, \ldots, \check{\tau}_K$ with $\check{\tau}_i = \check{t}_i^2$, $\forall i$, achieves some targets $\mu_1, \ldots, \mu_K$. Since the case is trivial, where some target MMSE is one, the powers of the respective users are set to zero and these users are dropped in the following to simplify the derivation. Setting $\mathrm{MMSE}_k = \mu_k < 1$, using the equivalent channels $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_K$, and with the second line of (5), following fixed point equation can be derived

$$\check{\tau}_k = f_k(\check{\boldsymbol{\tau}}; \boldsymbol{\mu}) = \frac{\mu_k^{-1} - 1}{\boldsymbol{\phi}_k^{\mathrm{H}}\left(\mathbf{I} + \sum_{\ell\neq k}\boldsymbol{\phi}_\ell\boldsymbol{\phi}_\ell^{\mathrm{H}}\check{\tau}_\ell\right)^{-1}\boldsymbol{\phi}_k} \quad \forall k \tag{15}$$

where $\check{\boldsymbol{\tau}} = [\check{\tau}_1, \ldots, \check{\tau}_K]^{\mathrm{T}} \in \mathbb{R}^K$ and $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_K]^{\mathrm{T}} \in \mathbb{R}^K$ with $\mathbf{0} < \boldsymbol{\mu} < \mathbf{1}$ were defined for notational brevity. Note that the inequality $\boldsymbol{\mu} < \boldsymbol{a}$ means that every entry of $\boldsymbol{\mu}$ is smaller than the respective entry of $\boldsymbol{a}$ and $\mathbf{1}$ denotes the all-ones vector. Combining above functions to one vector leads to the function $\boldsymbol{f}(\boldsymbol{\tau}; \boldsymbol{\mu}) = [f_1(\boldsymbol{\tau}; \boldsymbol{\mu}), \ldots, f_K(\boldsymbol{\tau}; \boldsymbol{\mu})]^{\mathrm{T}}$ that maps from $\mathbb{R}^K$ to $\mathbb{R}^K$. Note that $\boldsymbol{f}(\boldsymbol{\tau}; \boldsymbol{\mu})$ is an interference function as defined in [20]:

$$\begin{aligned}
\text{Positivity:}&\quad \boldsymbol{f}(\boldsymbol{\tau}; \boldsymbol{\mu}) > \mathbf{0} \quad \forall \boldsymbol{\tau} \ge \mathbf{0}\\
\text{Monotonicity:}&\quad \boldsymbol{f}(\boldsymbol{\tau}; \boldsymbol{\mu}) \ge \boldsymbol{f}(\boldsymbol{\tau}'; \boldsymbol{\mu}) \quad \forall \boldsymbol{\tau} \ge \boldsymbol{\tau}' \ge \mathbf{0}\\
\text{Scalability:}&\quad \alpha\boldsymbol{f}(\boldsymbol{\tau}; \boldsymbol{\mu}) > \boldsymbol{f}(\alpha\boldsymbol{\tau}; \boldsymbol{\mu}) \quad \forall \alpha > 1 \text{ and } \boldsymbol{\tau} \ge \mathbf{0}.
\end{aligned} \tag{16}$$

The scalability property of interference functions implies radial quasi-concavity of $\boldsymbol{g}(\boldsymbol{\tau}) = \boldsymbol{f}(\boldsymbol{\tau}, \boldsymbol{\mu}) - \boldsymbol{\tau}$ (see [21, Definition 2.1]), since for $0 < \lambda < 1$,

$$\begin{aligned}
\boldsymbol{g}(\boldsymbol{\tau}^*) = \mathbf{0} \;\Rightarrow\; \boldsymbol{g}(\lambda\boldsymbol{\tau}^*) &= \boldsymbol{f}(\lambda\boldsymbol{\tau}^*; \boldsymbol{\mu}) - \lambda\boldsymbol{\tau}^*\\
&> \lambda\boldsymbol{f}(\boldsymbol{\tau}^*; \boldsymbol{\mu}) - \lambda\boldsymbol{\tau}^* = \lambda\boldsymbol{g}(\boldsymbol{\tau}^*) = \mathbf{0}.
\end{aligned}$$

So, the fixed point iteration (15) has at most one positive solution, since $\boldsymbol{g}(\boldsymbol{\tau})$ is radially quasi-concave [21, Theorem 3.1] or equivalently, since $\boldsymbol{f}(\boldsymbol{\tau};\boldsymbol{\mu})$ is an interference function [20, Theorem 1]. To prove the existence of a solution for the fixed point iteration (15), $\boldsymbol{f}(\boldsymbol{\tau};\boldsymbol{\mu})$ must be increasing in $\boldsymbol{\tau}$ and the following conditions must be fulfilled (e.g., [21, Theorem 3.2])

$$\exists \boldsymbol{a} > \boldsymbol{0}: \quad \boldsymbol{f}(\boldsymbol{a};\boldsymbol{\mu}) > \boldsymbol{a} \tag{17}$$

$$\exists \boldsymbol{b} > \boldsymbol{a}: \quad \boldsymbol{f}(\boldsymbol{b};\boldsymbol{\mu}) < \boldsymbol{b}. \tag{18}$$

For (17), the choice $\boldsymbol{a} = \alpha\mathbf{1}$ is appropriate. To see this, drop the sum in the inverse of (15) to find the lower bound for $\mathbf{0} < \boldsymbol{\mu} < \mathbf{1}$

$$f_k(\boldsymbol{\tau};\boldsymbol{\mu}) \geq \frac{\mu_k^{-1}-1}{\|\boldsymbol{\phi}_k\|_2^2} > 0 \quad \forall \boldsymbol{\tau} > \mathbf{0}.$$

Thus, any positive $\alpha < \min_{k\in\{1,\ldots,K\}}(\mu_k^{-1}-1)/\|\boldsymbol{\phi}_k\|_2^2$ leads to an $\boldsymbol{a} = \alpha\mathbf{1}$ that fulfills (17).

If the equivalent channels $\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_K \in \mathbb{C}^r$ fulfill (4), the proof for (18) in [13] is valid. So, we concentrate on the case, where (4) is violated. Hence, $r < K$ holds in the following.

First, we prove that a vector $\boldsymbol{b} > \alpha\mathbf{1}$ exists in the limit with $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\mu}) \leq \boldsymbol{b}$, if the MMSE targets $\mu_1,\ldots,\mu_K$ with $\boldsymbol{\mu} < \mathbf{1}$ are chosen such that some of the conditions (12) and (14) are fulfilled with equality. In a second step, the equalities are relaxed to inequalities which proves (18).

Assume that two user subsets $\mathcal{I}_1$ and $\mathcal{I}_2$ have MMSE targets with [cf. (14)]

$$i \in \{1,2\}: \quad \sum_{k\in\mathcal{I}_i} \mu_k = |\mathcal{I}_i| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_i}) \tag{19}$$

and that the other users $\{1,\ldots,K\}\setminus\mathcal{I}_{1,2}$ are not member of user subsets that fulfill (14) with equality. From the proof of the necessary condition (14), we know that such a setup can only be reached, if the exponents of ν for the users in $\mathcal{I}_1$ and $\mathcal{I}_2$ are larger than the exponents for the other users $\{1,\ldots,K\}\setminus\mathcal{I}_{1,2}$ with the union set $\mathcal{I}_{1,2} = \mathcal{I}_1\cup\mathcal{I}_2$. Therefore, we have that [cf. first line of (13)]

$$\sum_{k\in\mathcal{I}_{1,2}} \mu_k = |\mathcal{I}_{1,2}| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_{1,2}}). \tag{20}$$

In a first step, we prove that such an equality also holds for the intersection set $\mathcal{I}_3 = \mathcal{I}_1\cap\mathcal{I}_2 \neq \emptyset$, i.e.,

$$\sum_{k\in\mathcal{I}_3} \mu_k = |\mathcal{I}_3| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3}). \tag{21}$$

As this result is obvious for $\mathcal{I}_3 = \mathcal{I}_1$ or $\mathcal{I}_3 = \mathcal{I}_2$, we concentrate on $\mathcal{I}_3 \neq \mathcal{I}_1$ and $\mathcal{I}_3 \neq \mathcal{I}_2$. Summing the two equations in (19) yields

$$\begin{aligned}\sum_{k\in\mathcal{I}_1}\mu_k + \sum_{k\in\mathcal{I}_2}\mu_k &= \sum_{k\in\mathcal{I}_{1,2}}\mu_k + \sum_{k\in\mathcal{I}_3}\mu_k \\ &= |\mathcal{I}_1| + |\mathcal{I}_2| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_1}) - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_2}) \\ &= |\mathcal{I}_{1,2}| + |\mathcal{I}_3| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_1}) - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_2}).\end{aligned}$$

Note that

$$\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_{1,2}}) \leq \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_1}) + \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_2}) - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3}).$$

To understand this inequality, let $\boldsymbol{Q}_3$ be some orthogonal basis for the range of $\boldsymbol{\Phi}_{\mathcal{I}_3}$. Moreover, $[\boldsymbol{Q}_3,\boldsymbol{Q}_1]$ and $[\boldsymbol{Q}_3,\boldsymbol{Q}_2]$ are orthogonal bases for $\operatorname{range}(\boldsymbol{\Phi}_{\mathcal{I}_1})$ and $\operatorname{range}(\boldsymbol{\Phi}_{\mathcal{I}_2})$, respectively. Accordingly, $\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_j}) = \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3}) + \operatorname{rank}(\boldsymbol{Q}_j)$ for $j\in\{1,2\}$. Clearly, $[\boldsymbol{Q}_1,\boldsymbol{Q}_2,\boldsymbol{Q}_3]$ is a basis for $\operatorname{range}(\boldsymbol{\Phi}_{\mathcal{I}_{1,2}})$ and above inequality becomes obvious. Iff $[\boldsymbol{Q}_1,\boldsymbol{Q}_2]$ is full rank, equality holds. Combining the two last results and (20) leads to

$$\sum_{k\in\mathcal{I}_3}\mu_k \leq |\mathcal{I}_3| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3}).$$

Comparing this result to the necessary condition (14) shows that equality must hold, i.e., (21) is valid. Additionally, we can infer that the initial assumption (19) is only possible, if $[\boldsymbol{Q}_1,\boldsymbol{Q}_2]$ is full rank.

Based on the assumption (19) and the consequences (20) and (21), we can prove the existence of a $\boldsymbol{b} > \boldsymbol{a}$ that fulfills (18). To simplify notation, the users are relabelled such that $\mathcal{I}_3 = \{1,\ldots,|\mathcal{I}_3|\}$, $\mathcal{I}_1\setminus\mathcal{I}_3 = \{|\mathcal{I}_3|+1,\ldots,|\mathcal{I}_1|\}$ and $\mathcal{I}_2\setminus\mathcal{I}_3 = \{|\mathcal{I}_1|+1,\ldots,|\mathcal{I}_1|+|\mathcal{I}_2|-|\mathcal{I}_3|\}$. Let

$$\boldsymbol{b} = [\nu^{\omega_3}\boldsymbol{b}_3^{\mathrm{T}}, \nu^{\omega}\boldsymbol{b}_1^{\mathrm{T}}, \nu^{\omega}\boldsymbol{b}_2^{\mathrm{T}}, \nu\boldsymbol{b}_4^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^K \tag{22}$$

with $\omega_3 > \omega > 1$ and $\nu\to\infty$. The vectors $\boldsymbol{b}_3 \in \mathbb{R}^{|\mathcal{I}_3|}$, $\boldsymbol{b}_1 \in \mathbb{R}^{|\mathcal{I}_1\setminus\mathcal{I}_3|}$, $\boldsymbol{b}_2 \in \mathbb{R}^{|\mathcal{I}_2\setminus\mathcal{I}_3|}$, and $\boldsymbol{b}_4 \in \mathbb{R}^{K-|\mathcal{I}_{1,2}|}$ have positive entries. Thus, $\boldsymbol{b} > \boldsymbol{a}$ holds. Due to the highest exponent of ν, the users in $\mathcal{I}_3$ can be considered separately. For the following paragraph, we set $\mathcal{I} = \mathcal{I}_3$ and $\boldsymbol{\Phi}_{\mathcal{I}} = \boldsymbol{\Phi}_{\mathcal{I}_3}$.

Let $\boldsymbol{V} \in \mathbb{C}^{r\times\operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}})}$ be an orthonormal basis for $\operatorname{range}(\boldsymbol{\Phi}_{\mathcal{I}})$ and $\mathcal{I}\subset\{1,\ldots,K\}$. If the equivalent channel $\boldsymbol{\Theta}_{\mathcal{I}} = \boldsymbol{V}^{\mathrm{H}}\boldsymbol{\Phi}_{\mathcal{I}} \in \mathbb{C}^{r_{\mathcal{I}}\times|\mathcal{I}|}$ is full rank, i.e., $r_{\mathcal{I}} = |\mathcal{I}|$ with $r_{\mathcal{I}} = \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}})$, any targets $\mu_k\ \forall k\in\mathcal{I}$ can be achieved by zero-forcing and power loading. So, we concentrate on $r_{\mathcal{I}} < |\mathcal{I}|$. The limit $\nu\to\infty$ leads to the setup of negligible noise that is the main assumption for SIR balancing (e.g., [8]) which can be formulated as

$$\max_{r,\boldsymbol{b}_{\mathcal{I}}} r \quad \text{ s.t.: } \quad \mathrm{SIR}_k = r\gamma_k \quad \forall k\in\mathcal{I} \tag{23}$$

with the SIR of user k (the indices of the elements of $\boldsymbol{b}_{\mathcal{I}}$ and the columns of $\boldsymbol{\Theta}_{\mathcal{I}}$ are not chosen from 1 to $|\mathcal{I}|$ but are taken from $\mathcal{I}$ to match the user indices)

$$\mathrm{SIR}_k = b_k\boldsymbol{\theta}_k^{\mathrm{H}}\left(\sum_{\ell\neq k:\ell\in\mathcal{I}}\boldsymbol{\theta}_\ell\boldsymbol{\theta}_\ell^{\mathrm{H}}b_\ell\right)^{-1}\boldsymbol{\theta}_k.$$

Note that (23) always has a solution (e.g., [8]) contrary to (2). Due to (11), the constraints can be expressed in terms of the MMSE's:

$$\mathrm{MMSE}_k = \frac{1}{r(\mu_k^{-1}-1)+1}.$$

Obviously, the right-hand side is monotonically decreasing in r for $r>0$ and so is $\sum_{k\in\mathcal{I}}(r(\mu_k^{-1}-1)+1)^{-1}$. Additionally, we know from the derivation of (14) that for $\nu\to\infty$, $\sum_{k\in\mathcal{I}}\mathrm{MMSE}_k = |\mathcal{I}| - \operatorname{rank}(\boldsymbol{\Theta}_{\mathcal{I}})$. Consequently, the equation $\sum_{k\in\mathcal{I}}(r(\mu_k^{-1}-1)+1)^{-1} = |\mathcal{I}| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}})$ has a unique solution $r>0$. If $\sum_{k\in\mathcal{I}}\mu_k = |\mathcal{I}| - \operatorname{rank}(\boldsymbol{\Phi}_{\mathcal{I}})$ (as assumed), we obtain $r=1$.

So, we have shown via the SIR balancing formulation (23) that an asymptotic power allocation $\nu^{\omega_3}\boldsymbol{b}_3$ exists for any target MMSE's of the users in $\mathcal{I}_3$ fulfilling $\sum_{k\in\mathcal{I}_3}\mu_k = |\mathcal{I}_3| - \mathrm{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3})$. For the fixed point iteration, we conclude that $f_k(\boldsymbol{b};\boldsymbol{\mu}) = b_k$ for $k \in \mathcal{I}_3$.

The proof for the users in $\mathcal{I}_1\setminus\mathcal{I}_3$ and $\mathcal{I}_2\setminus\mathcal{I}_3$ is very similar. We set $\mathcal{I} = \mathcal{I}_{1,2}\setminus\mathcal{I}_3$ and choose $\boldsymbol{\Phi}_\mathcal{I} = \boldsymbol{P}^\perp_{\mathcal{I}_3}[\boldsymbol{\Phi}_{\mathcal{I}_1},\boldsymbol{\Phi}_{\mathcal{I}_2}]$. The projection with $\boldsymbol{P}^\perp_{\mathcal{I}_3}$ onto the orthogonal complement of the span of $\boldsymbol{\Phi}_{\mathcal{I}_3}$ is necessary due to $\omega_3 > \omega$ [cf. third line of (13)]. Because of the conditions (20) and (21), $\sum_{k\in\mathcal{I}}\mu_k = |\mathcal{I}| - \mathrm{rank}(\boldsymbol{\Phi}_\mathcal{I})$ holds for $\nu\to\infty$.[3] Hence, the argumentation based on the balancing formulation (23) also shows that asymptotic power allocations $\nu^\omega\boldsymbol{b}_1$ and $\nu^\omega\boldsymbol{b}_2$ exist for any target MMSE's of the users in $\mathcal{I}_j$ fulfilling $\sum_{k\in\mathcal{I}_j}\mu_k = |\mathcal{I}_j| - \mathrm{rank}(\boldsymbol{\Phi}_j)$ with $j\in\{1,2\}$. For the fixed point iteration, we conclude that $f_k(\boldsymbol{b};\boldsymbol{\mu}) = b_k$ for $k\in\mathcal{I}_{1,2}$.

For the remaining users $\mathcal{I}_{\overline{1,2}} = \{1,\ldots,K\}\setminus\mathcal{I}_{1,2}$, we set $\mathcal{I} = \mathcal{I}_{\overline{1,2}}$ and $\boldsymbol{\Phi}_\mathcal{I} = \boldsymbol{P}^\perp_{\mathcal{I}_{1,2}}\boldsymbol{\Phi}_{\mathcal{I}_{\overline{1,2}}}$. For this user set, the target MMSE's are larger than the bound, i.e., $\sum_{k\in\mathcal{I}}\mu_k > |\mathcal{I}| - \mathrm{rank}(\boldsymbol{\Phi}_\mathcal{I})$. Due to the monotonicity of $\sum_{k\in\mathcal{I}}(r(\mu_k^{-1}-1)+1)^{-1}$ in r and the fact that $\sum_{k\in\mathcal{I}}\mathrm{MMSE}_k = |\mathcal{I}| - \mathrm{rank}(\boldsymbol{\Phi}_\mathcal{I})$ for $\nu\to\infty$, the optimal factor for the balancing of the MMSE's of the user set $\mathcal{I}_{\overline{1,2}}$ is $r > 1$. In other words, an asymptotic power allocation $\nu\boldsymbol{b}_4$ exists for which the target MMSE's can be over-fulfilled. Note that scaling the targets of the user set $\mathcal{I}_{\overline{1,2}}$ with $0<\beta<1$ such that $\sum_{k\in\mathcal{I}}\beta\mu_k = |\mathcal{I}| - \mathrm{rank}(\boldsymbol{\Phi}_\mathcal{I})$ would lead to $r = 1$. For the fixed point iteration, this would result in $f_k(\boldsymbol{b};\boldsymbol{\mu}') = b_k$ for $k\in\mathcal{I}_{\overline{1,2}}$ with $\mu'_k = \beta\mu_k$ for $k\in\mathcal{I}_{\overline{1,2}}$ and $\mu'_k = \mu_k$ otherwise. As $f_k(\boldsymbol{\tau};\boldsymbol{\mu})$ is strictly decreasing with μ_k and independent of $\mu_\ell\ \forall\ell\neq k$, we get that $f_k(\boldsymbol{b};\boldsymbol{\mu}) < b_k$ for $k\in\mathcal{I}_{\overline{1,2}}$.

We have proven so far that the allocation $\boldsymbol{b} > \boldsymbol{a} = \alpha\mathbf{1}$ given in (22) fulfills $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\mu}) \leq \boldsymbol{b}$. To reach the strict inequality, we relax the equalities (19) to inequalities by a factor $\delta > 1$, i.e., $\boldsymbol{\mu}' = \delta\boldsymbol{\mu}$, but δ must be small enough that still $\boldsymbol{\mu}' < \mathbf{1}$. Due to the monotonicity of $\boldsymbol{f}(\boldsymbol{\tau};\boldsymbol{\mu})$ in $\boldsymbol{\mu}$, we get the desired result that $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\mu}') < \boldsymbol{b}$.

The presented proof for two user subsets with active bounds for the target MMSE's can be easily generalized to more active bounds. The asymptotic power allocation must be constructed similar to (22). The users contained in most intersection sets must have the highest exponent ω_{most} of ν and the users that are in no subset with active bound have the smallest exponent 1. The other users have exponents between ω_{most} and 1 depending on the number of intersection sets they are member of.

This leads to following theorem [13].

Theorem 2: The closure of the feasible MMSE region in the vector broadcast channel with linear transceivers is a polytope $\mathcal{P}$ whose bounding half-spaces are the individual box constraints $0 \leq \mathrm{MMSE}_k \leq 1\ \forall k\in\{1,\ldots,K\}$, the sum MMSE constraint $\sum_{k=1}^K \mathrm{MMSE}_k \geq K - \mathrm{rank}(\boldsymbol{H})$, and additional constraints for the sum MMSE of user subsets whose channels are not full rank. Strictly speaking, a half-space constraint $\sum_{k\in\mathcal{I}}\mathrm{MMSE}_k \geq |\mathcal{I}| - \mathrm{rank}(\boldsymbol{H}_\mathcal{I})$ is imposed for every subset $\mathcal{I}\subset\{1,\ldots,K\}$ for which $\mathrm{rank}(\boldsymbol{H}_\mathcal{I}) < \min(N,|\mathcal{I}|)$.

[3] This equality is asymptotically achievable, because $[\boldsymbol{Q}_1,\boldsymbol{Q}_2]$ is full-rank with the orthogonal basis $\boldsymbol{Q}_j$ of the span of $\boldsymbol{P}^\perp_{\mathcal{I}_3}\boldsymbol{\Phi}_{\mathcal{I}_j}$, $j\in\{1,2\}$. To see this, subtract (21) from (20) and exploit the fact that $\mathrm{rank}(\boldsymbol{\Phi}_{\mathcal{I}_{1,2}}) - \mathrm{rank}(\boldsymbol{\Phi}_{\mathcal{I}_3}) = \mathrm{rank}(\boldsymbol{P}^\perp_{\mathcal{I}_3}[\boldsymbol{\Phi}_{\mathcal{I}_1},\boldsymbol{\Phi}_{\mathcal{I}_2}])$, since $[\boldsymbol{Q}_1,\boldsymbol{Q}_2]$ is full-rank.

Thus, the general feasibility test in the vector BC has two parts. First, the sum of MMSE targets $\mu_1,\ldots,\mu_K$ must fulfill

$$\sum_{k=1}^K \mu_k > K - \mathrm{rank}(\boldsymbol{H}).$$

Second, the sum of the MMSE targets of any subset violating (4) is bounded:

$$\sum_{k\in\mathcal{I}}\mu_k > |\mathcal{I}| - \mathrm{rank}(\boldsymbol{H}_\mathcal{I})$$
$$\forall\mathcal{I}\subset\{1,\ldots,K\}: \quad \mathrm{rank}(\boldsymbol{H}_\mathcal{I}) < \min(|\mathcal{I}|,N).$$

The QoS optimization (1) has a solution, if and only if above conditions are fulfilled.

IV. MIMO Broadcast Channel

On the first sight, the MIMO BC case seems to be more complicated than the vector BC, since the receivers in the BC (the transmitters in the dual MIMO MAC) are equipped with multiple antennas. Although finding the optimal solution of (1) in fact is much more difficult in the MIMO case, the feasibility test fortunately is as simple as for the vector channel case.

A. Feasibility of Arbitrary Requirements

When discussing feasibility, it is important to note that some requirement of a user can be provided, if at least one data stream can be transmitted to this user without interference. Any additional data stream for this user brings possibly some reduction of the used power but is not necessary for feasibility. These observations lead to the scenario with $N \geq K$. When the transmitter in the MIMO BC (the receiver in the dual MIMO MAC) has at least as many degrees of freedom as there are users in the system, it is possible to allocate one data stream per user and apply a zero-forcing equalizer in the dual MIMO MAC to achieve an interference-free reception of the signals from the different users. With (3) and $N\geq K$, it is possible to find beamformers $\boldsymbol{t}_k\ \forall k$ such that

$$\boldsymbol{H}_{\text{ext}} = [\boldsymbol{H}_1\boldsymbol{t}_1,\ldots,\boldsymbol{H}_K\boldsymbol{t}_K] \in \mathbb{C}^{N\times K}$$

has a left inverse. As a consequence, the K data signals can be separated by the zero-forcing filter $(\boldsymbol{H}_{\text{ext}}^{\mathrm{H}}\boldsymbol{H}_{\text{ext}})^{-1}\boldsymbol{H}_{\text{ext}}^{\mathrm{H}}$. Clearly, any requirements $\rho_k > 0\ \forall k$ can be fulfilled via power allocation, i.e., the choice of $\|\boldsymbol{t}_k\|_2^2\ \forall k$, if the interference is completely suppressed.

B. Non-Zero-Forcing Configuration

When $N < K$, i.e., more users are active than the BC transmitter has degrees of freedom, it is impossible to reach arbitrary requirements, since the system is interference limited even though the BC receivers have multiple antennas. Instead, the result in Theorem 1 for the vector BC must be applied.

Let's assume that only a single data stream is allocated per user. In this case, the rate requirements $\rho_k > 0\ \forall k$ can be translated to MMSE targets $\mu_k = 2^{-\rho_k}$ as in the vector channel case. Due to (3), it is possible to find beamformers $\boldsymbol{t}_k\ \forall k$ such that the total channels

$$\boldsymbol{h}_{\text{tot},k} = \boldsymbol{H}_k \boldsymbol{t}_k \quad \forall k$$

also fulfill (3) or equivalently (4). Based on Theorem 1, we can infer that the MMSE targets and therefore the rate requirements are feasible, if $\sum_{k=1}^{K} \mu_k > K - N$. In other words, the rate requirements must fulfill

$$\sum_{k=1}^{K} 2^{-\rho_k} > K - N.$$

This condition for the rate targets is the same as for the K user N antenna vector BC.

To find out, whether the multiple antennas at the users for the MIMO case can be exploited to relax this condition or not, let's consider following scenario. Assume that the rate requirements $\rho_k\ \forall k$ translate to MMSE requirements $0 < \mu_k \leq 1\ \forall k$ that do not fulfill above feasibility condition, that is, $\sum_{k=1}^{K} \mu_k = K-N-\alpha$ with $\alpha \geq 0$. Remember that this condition resulted from the assumption of allocating a single data stream per user. Now, a second data stream is given to some user j. Without loss of generality, it can be assumed that the two data streams of user j do not interfer with each other. To understand this, construct the covariance matrix $\boldsymbol{Q}_j = \boldsymbol{T}_j \boldsymbol{T}_j^{\text{H}}$ which has rank 2 according to above assumptions. Note that the precoder $\boldsymbol{T}_j$ can be changed without changing the covariance matrix $\boldsymbol{Q}_j$. In particular, consider the alternative precoder $\boldsymbol{T}_j' = \boldsymbol{T}_j \boldsymbol{W}_j$ with the unitary matrix $\boldsymbol{W}_j \in \mathbb{C}^{2\times 2}$. Clearly, $\boldsymbol{Q}_j = \boldsymbol{T}_j' \boldsymbol{T}_j'^{\text{H}}$ holds. Since the MMSE's of the other users are independent of $\boldsymbol{W}_j$, this unitary degree of freedom can be used to diagonalize the combination of the MMSE equalizer $\boldsymbol{G}_j$, the channel $\boldsymbol{H}_j$, and the precoder $\boldsymbol{T}_j$, that is, $\boldsymbol{W}_j$ is chosen such that $\boldsymbol{W}_j^{\text{H}} \boldsymbol{G}_j \boldsymbol{H}_j \boldsymbol{T}_j \boldsymbol{W}_j$ is diagonal. This can be achieved by choosing the columns of $\boldsymbol{W}_j$ to be different eigenvectors of the non-negative definite $\boldsymbol{G}_j \boldsymbol{H}_j \boldsymbol{T}_j$ belonging to non-zero eigenvalues. Therefore,

$$-\log_2\left(\mu_j^{(1)}\right) - \log_2\left(\mu_j^{(2)}\right) = \rho_j$$

must hold to meet the rate requirement, where $\mu_j^{(i)}$ is the MMSE target for the i-th data stream of user j. Alternatively, we must have

$$\mu_j^{(1)} \mu_j^{(2)} = \mu_j.$$

The two data streams of user j can be interpreted as the data streams of two different users in a vector channel scenario. Consequently, we conclude from Theorem 1 that

$$\sum_{k=1,k\neq j}^{K} \mu_k + \mu_j^{(1)} + \mu_j^{(2)} > K - N + 1$$

must be fulfilled for feasibility. Substituting the assumption that $\sum_{k=1}^{K} \mu_k = K - N - \alpha$ leads to

$$\mu_j^{(1)} + \mu_j^{(2)} - \mu_j > \alpha + 1$$

and with $\mu_j^{(1)} \mu_j^{(2)} = \mu_j$,

$$\alpha + \left(1 - \mu_j^{(1)}\right)\left(1 - \mu_j^{(2)}\right) < 0.$$

To be meaningful MMSE targets, $0 < \mu_j^{(i)} \leq 1$ must hold. Thus, the second term on the left-hand side of above inequality is always non-negative. Since $\alpha \geq 0$, the inequality can never be fulfilled. So, when the rate requirements cannot be met by allocating a single data stream per user, the rate requirements are also infeasible with more than one data stream per user.

In above example, we started with the assumption that the single data stream allocation does not provide feasibility. Then, we observed that increasing the number of data streams of some user j to two does not lead to feasibility. This result can easily be generalized to any stream allocation, since the data streams of a single user k can be interpreted as the single data streams of virtual users. As long as not more than $\min(M_k, N)$ data streams are allocated to that user, the regularity condition (3) is not violated. So, starting with a stream allocation, where more than one data stream is given to some users, is in fact the scenario considered in above example, where every virtual user has one data stream. This leads to the conclusion that increasing the number of data streams for some user does not lead to feasibility, when the original stream allocation exhibited infeasibility of the rate requirements. Only the case with a single data stream per user must be considered for testing feasibility.

Above results can be summarized in following theorem.

Theorem 3: The feasible rate region in the MIMO BC with linear transceivers and channels fulfilling the regularity condition (3) is constituted by all the rate tuples $R_1, \ldots, R_K$ that are non-negative, i.e., $R_k \geq 0$, and fulfill the condition $\sum_{k=1}^{K} 2^{-R_k} > K - N$. The whole feasible rate region can be achieved by allocating a single data stream per user and cannot be extended by allocating more data streams.

So, the test of feasibility for the MIMO BC is as simple as for the vector BC. If and only if the rate targets $\rho_k\ \forall k$ fulfill

$$\sum_{k=1}^{K} 2^{-\rho_k} > K - N$$

the QoS optimization (1) has a solution.

C. Singular Case

If the channels do not fulfill (3), it is impossible to construct beamformers $\boldsymbol{t}_k\ \forall k$ such that the total channels

$$\boldsymbol{h}_{\text{tot},k} = \boldsymbol{H}_k \boldsymbol{t}_k \quad \forall k$$

fulfill (4). However, for the user groups $\mathcal{I}$ whose channels fulfill $\operatorname{rank}(\boldsymbol{H}_{\mathcal{I}}) \geq \min(|\mathcal{I}|, N)$, beamformers exist such that $\operatorname{rank}(\boldsymbol{H}_{\text{tot},\mathcal{I}}) = \min(|\mathcal{I}|, N)$. In contrast, for the user groups $\mathcal{I}$ violating the condition in (3), the best choice for the beamformers leads to $\operatorname{rank}(\boldsymbol{H}_{\text{tot},\mathcal{I}}) < \min(|\mathcal{I}|, N)$. Consequently, Theorem 2 must be applied. Together with the proof for Theorem 3 (that a single data stream allocation suffices for feasibility), this leads to following theorem.

Theorem 4: The feasible rate region in the MIMO BC with linear transceivers is constituted by all the rate tuples $R_1, \dots, R_K$ that are non-negative, i.e., $R_k \geq 0$, and fulfill the conditions $\sum_{k=1}^{K} 2^{-R_k} > K - \mathrm{rank}(\boldsymbol{H})$ and $\sum_{k\in\mathcal{I}} 2^{-R_k} > |\mathcal{I}| - \mathrm{rank}(\boldsymbol{H}_\mathcal{I})$ for all subsets $\mathcal{I} \subset \{1,\dots,K\}$ for which $\mathrm{rank}(\boldsymbol{H}_\mathcal{I}) < \min(N, |\mathcal{I}|)$. The whole feasible rate region can be achieved by allocating a single data stream per user and cannot be extended by allocating more data streams.

From above theorem, we can infer that also for singular channels violating (3), the test is as simple as for the vector BC. The QoS optimization (1) has a solution, if and only if the rate requirements $\rho_k\ \forall k$ fulfill

$$\sum_{k=1}^{K} 2^{-\rho_k} > K - \mathrm{rank}(\boldsymbol{H})$$

and

$$\sum_{k\in\mathcal{I}}^{K} 2^{-\rho_k} > K - \mathrm{rank}(\boldsymbol{H}_\mathcal{I})$$
$$\forall \mathcal{I} \subset \{1,\dots,K\}: \quad \mathrm{rank}(\boldsymbol{H}_\mathcal{I}) < \min(|\mathcal{I}|, N).$$

Conclusions

Feasibility of the QoS optimization in the vector BC was reviewed and feasibility in the MIMO BC was discussed. It was shown that the transmission of a single data stream per user suffices for feasibility in the MIMO BC and that a larger number of data streams does not improve feasibility. For regular MIMO BC channels, the rate requirements just have to be transformed to MMSE targets. Feasibility can then be checked by a simple inequality for the sum of MMSE targets.

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